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Is the Velocity Always in Phase with the Wave Excitation Force in Constrained Optimal Control of Wave Energy Converters?

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Abstract: The resonance condition for maximum absorption of the energy carried by ocean waves incident to an oscillating wave energy converter (WEC) is one of the most well known results of the WEC control research. It was shown that the maximum occurs when the WEC velocity is in phase with the wave excitation force. The condition was obtained with the assumption that the WEC frees to oscillate with whatever amplitude was necessary, and that the power take-off (PTO) system can deliver the unlimited force. In practical WEC implementations, this assumption does not hold. The purpose of this paper is twofold. First, to obtain the frequency response of the closed-loop WEC motion, and of the optimal control law in terms of the hydrodynamic coefficients with limited PTO force and limited motion amplitudes. Secondly, the obtained results are used to investigate the resonance condition of the WEC velocity with the incoming wave in the presence of constraints.

Keywords: Optimal Control, WEC, Force and Motion Amplitudes Constraints, Linear Matrix Inequalities, Complex-Conjugate Control,

1. INTRODUCTION

Since modern research on the WEC topic emerged in the early 1970s, the question of optimal control of the WEC systems has always been an important issue. The conditions for maximum wave energy absorption soon emerged by using the frequency domain approach. It was proved in Evans [1981b], Faldes [2002] that the maximum occurs when the controller has the correct value of resistance, and that its intrinsic impedance cancels the WEC intrinsic impedance. The terms complex-conjugate or impedance matching control were adopted to describe this situation of optimal operation, see Nebel [1992], Salter et al. [1976]. As a consequence of this result, the closed-loop WEC velocity is in phase with the incident wave excitation force. The results were derived in the absence of the PTO force constraints as well as the WEC position and velocity constraints. When implementing a control strategy for a real device, one must take into account these physical limitations in order to exploit the full absorption potential of an installed WEC. It is also for safety reasons.

To the best of the author's knowledge, Evans [1981a] is the first paper that studies the WEC control under the constraints. The method of Lagrange multipliers was considered to optimize the absorbed power subject to a global motion constraint of a multi-degree-of-freedom WEC system. In Pizer [1993], the results were extended that allow to take into account the individual motion constraint for each controlled degree-of-freedom. In Cotten and Forehand [2020], the uncontrolled degrees-of-freedom were studied.

However in these works, the results were obtained with the complex-conjugate control. No PTO force constraints were considered. In addition, the results are not optimal since the control law is pre-defined. Recently, the line of research Evans [1981a] was also expanded in Korde [2019]. In this work, PTO force constraints were considered. However, the complex-conjugate control law was still used.

The aim of this paper is twofold. First, to provide a method to calculate the frequency response of the optimal control law as well as of the closed-loop WEC system. The main features of the solution are

- PTO force and WEC motion constraints are taken input account.
- No complex-conjugate or impedance matching control were assumed. Hence the solution is globally optimal.
- It is shown that the problem of maximizing the harvested power can be converted into a convex optimization problem, for which the optimal solution is unique.

Secondly, once the optimal solution is obtained, the results are used to verify the resonance condition of the optimal WEC velocity with the incoming wave excitation force.

The paper is organized as follows. Section 2 describes the problem formulation and some preliminaries concerning semi-definite program. A procedure to calculate the real and imaginary parts of the frequency response of the closed-loop WEC system, and of the optimal control law is presented in Section 3. In Section 4, the results are applied

to the Wavestar WEC system. Finally, some conclusions are drawn in Section 5.

Notation: For symmetric matrices, the symbol $(*)$ denotes each of its symmetric blocks. For any real symmetric matrix X , the notation $X \succeq 0$ indicates that all eigenvalues of X are nonnegative.

2. PROBLEM FORMULATION

2.1 Hydrodynamic Characteristics of the WEC

We consider a single body floating system as in Fig. 1. The energy is extracted from the relative motion with the sea bottom through a generic power take-off (PTO) mechanism. The system is constrained to move in the heave direction only. The system is axis-symmetric so that the energy conversion rate is the same for waves coming from all directions.

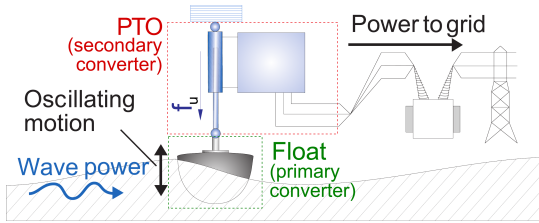


Fig. 1. One-degree-of-freedom floating system for wave energy conversion.

The external forces acting on the WEC are the wave excitation force and the control force produced by the PTO. Neglecting viscosity, and other losses, and under the assumption of linearity, the motion of the WEC system can be described in the frequency domain by

$$\left(j\omega M + \frac{K_{hd}}{j\omega} + Z_r(j\omega) \right) V(j\omega) = F_{ex}(j\omega) - F_u(j\omega) \quad (1)$$

where M is the WEC mass, $V(j\omega)$ is the heaving velocity, $F_{ex}(j\omega)$ is the wave excitation force, and $F_u(j\omega)$ is the PTO force. The hydrostatic force gives stiffness force as deviation from hydrostatic equilibrium. It is modeled by the coefficient K_{hd} . The radiation force due to radiated waves is expressed through the radiation impedance $Z_r(j\omega)$. In general, $Z_r(j\omega)$ is decomposed as

$$Z_r(j\omega) = B(\omega) + j\omega [M_\infty + M_a(\omega)] \quad (2)$$

where $B(\omega)$ is the radiation resistance, M_∞ is the added mass at infinitely high frequency, and $M_a(\omega)$ is the added mass after M_∞ is removed. $B(\omega)$ is an even, real and positive function.

Using (1), (2), the motion response of the WEC system is given by the following model

$$V(j\omega) = \frac{1}{Z_i(j\omega)} [F_{ex}(j\omega) - F_u(j\omega)] \quad (3)$$

where $Z_i(j\omega)$ is the intrinsic impedance of the system

$$Z_i(j\omega) = B(\omega) + j\omega \left[M - \frac{K_{hd}}{\omega^2} + M_\infty + M_a(\omega) \right] \quad (4)$$

Equations (3), (4) will serve as a basis for the results in the next sections.

2.2 Control Objective

The objective is to choose the input signal $f_u(t)$ to maximize the harvested average power P_a

$$P_a = \frac{1}{T} \int_0^T P(t) dt \quad (5)$$

where $P(t) = f_u(t)v(t)$ is the instantaneous power.

The PTO input $f_u(t)$, the WEC velocity $v(t)$, and the WEC position $s(t)$ are subject to the constraints

$$\begin{cases} -F_{lim} \leq f_u(t) \leq F_{lim}, \\ -V_{lim} \leq v(t) \leq V_{lim}, \\ -S_{lim} \leq s(t) \leq S_{lim} \end{cases} \quad (6)$$

2.3 Earlier Works in the Literature

In the absence of the constraints (6), it was shown in Evans [1981b], Falnes [2002] that P_a reaches maximum when the PTO input is given as

$$F_u(j\omega) = Z_i^*(j\omega)V(j\omega) \quad (7)$$

where $Z_i^*(j\omega)$ is the complex conjugate of $Z_i(j\omega)$, i.e.,

$$Z_i^*(j\omega) = B(\omega) - j\omega \left[M - \frac{K_{hd}}{\omega^2} + M_\infty + M_a(\omega) \right] \quad (8)$$

Using (3), (7), one gets

$$V(j\omega) = \frac{1}{Z_i(j\omega)} [F_{ex}(j\omega) - Z_i^*(j\omega)V(j\omega)]$$

or equivalently

$$V(j\omega) = \frac{F_{ex}(j\omega)}{Z_i(j\omega) + Z_i^*(j\omega)} = \frac{F_{ex}(j\omega)}{2B(\omega)} \quad (9)$$

Now it is assumed that the wave excitation force is of the form

$$f_{ex}(t) = A \sin(\omega t) \quad (10)$$

where A, ω are, respectively, the amplitude, and the angular wave frequency of $f_{ex}(t)$. Note that without loss of generality, the phase angle is set to zero in (10).

Recall that $B(\omega)$ is an even, real and positive function. Using (9), (10), one gets

$$v(t) = \frac{f_{ex}(t)}{2B(\omega)} = \frac{A}{2B(\omega)} \sin(\omega t) \quad (11)$$

Hence maximum energy transfer from the waves to the WEC system is obtained when the velocity is in phase with the wave excitation force.

The control law (7), (8) is called complex-conjugate control. The results (7), (9), (11) were obtained in the frequency domain in the absence of the constraints (6). The objective of this paper is to provide the relationship between $F_u(j\omega), V(j\omega)$ and $F_{ex}(j\omega)$ in terms of hydrodynamics coefficient in the presence of (6). Especially, we would like to know if the WEC velocity is always in phase with the wave excitation force in the constrained optimal control framework.

2.4 Preliminaries - Semidefinite Programming

In the following, some basic definitions of semidefinite programming (SDP) are recalled. They will be used to calculate the optimal frequency response of the WEC motions and of the optimal control law.

A linear matrix inequality (LMI) is a condition of the type

$$F(x) \succeq 0$$

where $x \in \mathbb{R}^n$ is a decision variable, and the matrix $F(x)$ is affine in x , i.e.,

$$F(x) = F_0 + \sum_{i=1}^n F_i x_i$$

where the matrices $F_i \in \mathbb{R}^{m \times m}$ are symmetric.

LMIs can either be understood as constraints for optimization problems or as feasibility conditions. Optimization of a linear function over LMI constraints is called SDP, which is a generalization of linear programming, see Boyd and Vandenberghe [2004], Nguyen [2013]. The main advantage of using the SDP formulation is that for solving a SDP problem, several polynomial time algorithms were developed and implemented in free available software packages, such as CVX Grant and Boyd [2014], YALMIP Lofberg [2004], etc.

The Schur complement is a very useful tool for matrix inequalities. It allows to reformulate equivalently certain nonlinear matrix inequalities as a higher dimensional LMI condition. Using the Schur complement, the following nonlinear condition

$$\begin{cases} U(x) \succeq 0, \\ P(x) - Q(x)^T U(x)^{-1} Q(x) \succeq 0 \end{cases}$$

can be equivalently written in the LMI form

$$\begin{bmatrix} P(x) & Q(x)^T \\ Q(x) & U(x) \end{bmatrix} \succeq 0$$

3. MAIN RESULTS

Following Evans [1981b], Falnes [2002], the control input is assumed to be a linear function of the velocity, i.e.,

$$F_u(jw) = Z_c(jw)V(jw) \quad (12)$$

where $Z_c(jw)$ is a design variable.

Combining (3), (12), one obtains

$$V(jw) = \frac{1}{Z_i(jw)} (F_{ex}(jw) - Z_c(jw)V(jw))$$

thus

$$V(jw) = \frac{1}{Z_i(jw) + Z_c(jw)} F_{ex}(jw) \quad (13)$$

Define

$$G(jw) = \frac{1}{Z_i(jw) + Z_c(jw)} \quad (14)$$

thus

$$V(jw) = G(jw)F_{ex}(jw) \quad (15)$$

$G(jw)$ is the closed-loop transfer function with the input $F_{ex}(jw)$, and with the output $V(jw)$. For a given frequency w , $G(jw)$ is a complex number. It can be decomposed as

$$G(jw) = R_g(w) + jX_g(w) \quad (16)$$

where $R_g(w), X_g(w)$ are the real part and the imaginary part of $G(jw)$, respectively.

Define

$$\begin{cases} R_i(w) = B(w), \\ X_i(w) = M - \frac{K_{hd}}{w^2} + M_\infty + M_a(w) \end{cases} \quad (17)$$

i.e., $R_i(w), X_i(w)$ are, respectively, the real part and the imaginary parts of $Z_i(jw)$.

For simplicity, the argument w will be omitted from R_i, X_i, R_g, X_g whenever possible. The following theorem holds.

Theorem 1: Given the wave excitation force (10), and the control input (12), the average power is computed as

$$P_a = \frac{A^2}{2} (R_g - R_i R_g^2 - R_i X_g^2) \quad (18)$$

Proof: Combining (12), (14), one obtains

$$F_u(jw) = \left(\frac{1}{G(jw)} - Z_i(jw) \right) V(jw)$$

thus, with (15)

$$F_u(jw) = (1 - Z_i(jw)G(jw)) F_{ex}(jw) \quad (19)$$

Because (19) is a linear system, one gets, for the given wave excitation force (10),

$$f_u(t) = A_u A \sin(\omega t + \theta_u) \quad (20)$$

where

$$\begin{cases} A_u = |1 - Z_i(jw)G(jw)|, \\ \theta_u = \angle(1 - Z_i(jw)G(jw)) \end{cases}$$

are the amplitude and the phase of $(1 - Z_i(jw)G(jw))$, evaluated at frequency w , respectively.

Analogously, since the system (15) is linear, one obtains

$$v(t) = A_g A \sin(\omega t + \theta_g) \quad (21)$$

where $A_g = |G(jw)|$ and $\theta_g = \angle G(jw)$ are, respectively, the amplitude and the phase of $G(jw)$.

Using (20), (21), the instantaneous power $P(t)$ is computed by

$$P(t) = v(t)f_u(t) = A_g A_u A^2 \sin(\omega t + \theta_g) \sin(\omega t + \theta_u) \quad (22)$$

and the average power

$$P_a = \frac{1}{T} \int_0^T P(t) dt$$

It is clear that it suffices to calculate P_a for one period $T = \frac{2\pi}{w}$, since the wave (10) is periodic. Hence

$$P_a = \frac{w}{2\pi} A_g A_u A^2 \int_0^{\frac{2\pi}{w}} \sin(\omega t + \theta_g) \sin(\omega t + \theta_u) dt \quad (23)$$

Because

$$\sin(\alpha) \sin(\beta) = \frac{1}{2} (\cos(\alpha - \beta) - \cos(\alpha + \beta))$$

one gets from (23)

$$\begin{aligned} P_a &= \frac{w}{4\pi} A_g A_u A^2 \int_0^{\frac{2\pi}{w}} \cos(\theta_g - \theta_u) dt \\ &\quad - \frac{w}{4\pi} A_g A_u A^2 \int_0^{\frac{2\pi}{w}} \cos(2\omega t + \theta_g + \theta_u) dt \\ &= \frac{w}{4\pi} A_g A_u A^2 \left(\cos(\theta_g - \theta_u) t - \frac{\sin(2\omega t + \theta_g + \theta_u)}{2w} \right) \Big|_0^{\frac{2\pi}{w}} \end{aligned}$$

It follows

$$P_a = \frac{1}{2} A_g A_u A^2 \cos(\theta_g - \theta_u) \quad (24)$$

By using the following product-to-sum trigonometric identity

$$\cos(\theta_g - \theta_u) = \cos(\theta_g) \cos(\theta_u) + \sin(\theta_g) \sin(\theta_u)$$

and thus, with (24)

$$P_a = \frac{1}{2} A_g A_u A^2 (\cos(\theta_g) \cos(\theta_u) + \sin(\theta_g) \sin(\theta_u)) \quad (25)$$

The following equations hold

$$\begin{cases} A_g \cos(\theta_g) = R_g, \\ A_g \sin(\theta_g) = X_g, \\ A_u \cos(\theta_u) = 1 - R_i R_g + X_i X_g, \\ A_u \sin(\theta_u) = -R_i X_g - X_i R_g \end{cases} \quad (26)$$

Hence, using (25), one obtains

$$P_a = \frac{A^2}{2} (R_g(1 - R_i R_g + X_i X_g) - X_g(R_i X_g + X_i R_g))$$

or equivalently

$$P_a = \frac{A^2}{2} (R_g - R_i R_g^2 - R_i X_g^2)$$

The proof is complete. \square

Theorem 1 gives the analytical expression of the average power in terms of the hydrodynamic coefficients. For the input constraint (6), using (20), (26), one gets

$$\begin{aligned} f_u(t) &= A_u A \sin(\omega t + \theta_u) \\ &= \sqrt{(1 - R_i R_g + X_i X_g)^2 + (R_i X_g + X_i R_g)^2} A \sin(\omega t + \theta_u) \end{aligned}$$

Hence $-F_{lim} \leq f_u(t) \leq F_{lim}$ if and only if

$$(1 - R_i R_g + X_i X_g)^2 + (R_i X_g + X_i R_g)^2 \leq \left(\frac{F_{lim}}{A}\right)^2 \quad (27)$$

For the velocity constraint (6), using (21), (26), one obtains

$$v(t) = A_g A \sin(\omega t + \theta_g) = \sqrt{R_g^2 + X_g^2} A \sin(\omega t + \theta_g)$$

Hence the velocity constraint is satisfied if and only if

$$R_g^2 + X_g^2 \leq \left(\frac{V_{lim}}{A}\right)^2 \quad (28)$$

Concerning the position constraint, note that

$$S(j\omega) = \frac{1}{j\omega} V(j\omega)$$

thus, with (15),

$$S(j\omega) = \frac{G(j\omega)}{j\omega} F_{ex}(j\omega)$$

For the given wave excitation force (10), it follows that

$$\begin{aligned} s(t) &= \frac{A_g}{w} A \sin(\omega t + \theta_g - \frac{\pi}{2}) \\ &= \frac{\sqrt{R_g^2 + X_g^2}}{w} A \sin(\omega t + \theta_g - \frac{\pi}{2}) \end{aligned}$$

Therefore the position constraint is satisfied if and only if

$$R_g^2 + X_g^2 \leq \left(\frac{w S_{lim}}{A}\right)^2 \quad (29)$$

Using (28), (29), if $w S_{lim} \leq V_{lim}$, then the position and velocity constraints are satisfied if and only if (29) hold. Otherwise, if $w S_{lim} \geq V_{lim}$, then (28) should hold. For simplicity, only the position constraint (29) is considered in the rest of the paper.

Combining (18), (27), (29), the problem of maximizing the average power can be recast as

$$\begin{aligned} &\max_{R_g, X_g} \{R_g - R_i R_g^2 - R_i X_g^2\}, \\ &\text{s.t.} \begin{cases} (1 - R_i R_g + X_i X_g)^2 + (R_i X_g + X_i R_g)^2 \leq \left(\frac{F_{lim}}{A}\right)^2 \\ R_g^2 + X_g^2 \leq \left(\frac{w S_{lim}}{A}\right)^2 \end{cases} \end{aligned}$$

or equivalently

$$\begin{aligned} &\min_{R_g, X_g} \{R_i(R_g^2 + X_g^2) - R_g\}, \\ &\text{s.t.} \begin{cases} (1 - R_i R_g + X_i X_g)^2 + (R_i X_g + X_i R_g)^2 \leq \left(\frac{F_{lim}}{A}\right)^2 \\ R_g^2 + X_g^2 \leq \left(\frac{w S_{lim}}{A}\right)^2 \end{cases} \end{aligned} \quad (30)$$

Recall that $R_i = B(w)$ is positive. Hence the cost function and the constraints are convex. Consequently (30) is a convex optimization problem.

Remark: In the absence of the PTO force constraints and WEC position constraints, (30) has a closed-form solution

$$R_g = \frac{1}{2R_i}, X_g = 0 \quad (31)$$

In this case $G(j\omega) = \frac{1}{2R_i} = \frac{1}{2B(w)}$. Therefore

$$V(j\omega) = \frac{1}{2B(w)} F_{ex}(j\omega) \quad (32)$$

One recovers the result in Evans [1981b], Falnes [2002]. Hence their result is a particular case of Theorem 1.

The optimal solution to (30) is unique because the cost (30) is strongly convex. The solution can be found by using any numerical procedure for convex optimization such as interior point or gradient projection methods, see Boyd and Vandenberghe [2004]. Here we show how to reformulate (30) as a SDP problem, which is a subclass of a convex optimization problem.

Using the epigraph method, see Boyd and Vandenberghe [2004], the cost function (30) is equivalently rewritten as

$$\begin{aligned} &\min_{g, R_g, X_g} \{g\} \\ &\text{s.t.} R_i(R_g^2 + X_g^2) - R_g \leq g \end{aligned}$$

thus, with the Schur complement

$$\min_{g, R_g, X_g} \{g\} \quad (33)$$

$$\text{s.t.} \begin{bmatrix} g + R_g & R_g & X_g \\ (*) & \frac{1}{R_i} & 0 \\ (*) & (*) & \frac{1}{R_i} \end{bmatrix} \succeq 0 \quad (34)$$

Analogously, using the Schur complement, the constraints (30) are equivalently rewritten as

$$\begin{bmatrix} \left(\frac{F_{lim}}{A}\right)^2 & 1 - R_i R_g + X_i X_g & R_i X_g + X_i R_g \\ (*) & 1 & 0 \\ (*) & (*) & 1 \\ \left(\frac{w S_{lim}}{A}\right)^2 & R_g & X_g \\ (*) & 1 & 0 \\ (*) & (*) & 1 \end{bmatrix} \succeq 0, \quad (35)$$

Combining (33), (34), (35), the problem of maximizing the harvested average power can be formulated as the following SDP problem

$$\begin{aligned} &\min_{g, R_g, X_g} \{g\}, \\ &\text{s.t.} (34), (35) \end{aligned} \quad (36)$$

The optimal values of R_g^*, X_g^* are computed as follows

Algorithm 1: Computation of R_g^*, X_g^*

- 1: Select a reasonable finite but large set of frequency samples $0 < w_1 < \dots < w_N$.
- 2: Select a reasonable finite but large set of amplitude $0 < A_1 < \dots < A_L$.
- 3: For each frequency $w_l, \forall l = \overline{1, N}$, and each amplitude $A_k, \forall k = \overline{1, L}$ in the chosen set.
 - 3.1: Solve the SDP problem (36).
 - 3.2: The optimal value of R_g^*, X_g^* are

$$R_g^* = R_g, X_g^* = X_g$$

Note that R_g^*, X_g^* are functions of both frequency w and of amplitude A .

Once R_g^*, X_g^* are computed, the optimal frequency response of the control block are given by

$$\begin{aligned} R_c &= \frac{R_g^*}{(X_g^*)^2 + (R_g^*)^2} - R_i, \\ X_c &= -\frac{X_g^*}{(X_g^*)^2 + (R_g^*)^2} - X_i \end{aligned} \quad (37)$$

where R_c, X_c are, respectively, the real part and the imaginary part of $Z_c(jw)$.

For a given amplitude $A = 6$, using Algorithm 1, Fig. 2 and Fig. 3 show the optimal values R_g^*, X_g^* (solid blue) as a function w , for $1 \leq w \leq 10$. The parameters of the WEC system and the constraints are given in Section 4. Fig. 2 also shows the unconstrained optimal solution (31) (dashed red).

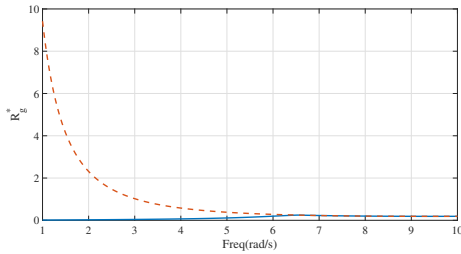


Fig. 2. Optimal R_g^* for a given amplitude A .

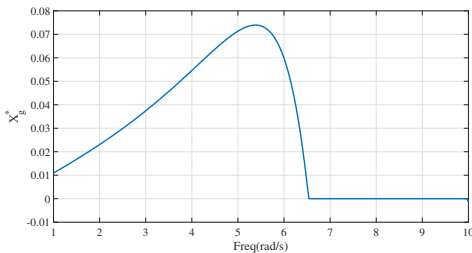


Fig. 3. Optimal X_g^* for a given amplitude A .

Using Fig. 2, and Fig. 3, the following remarks are made

- $X_g^* \neq 0$ for all frequencies, let us say, less than $6.5(\text{rad/s})$. It follows that in the constrained case, the optimal velocity is generally not in phase with the wave excitation force.
- The optimal R_g^* in the unconstrained case is generally much greater than that in the constrained case.

4. APPLICATION TO THE WAVESTAR WEC MACHINE

4.1 WEC Parameters

To study the results in Section 3, we use a laboratory prototype of a point absorber WEC on a 1:20 scale with respect to the well-known Wavestar machine installed near Hanstholm in Denmark from 2009 to 2013, see Zurkinden et al. [2012], Nguyen [2022]. The WEC parameters are given as $M = 1.44$, $K_{hd} = 93$, and

$$Z_r(w) = \frac{a_4(jw)^4 + a_3(jw)^3 + a_2(jw)^2 + a_1(jw) + a_0}{b_5(jw)^5 + b_4(jw)^4 + b_3(jw)^3 + b_2(jw)^2 + b_1(jw) + b_0} \quad (38)$$

whose coefficients are given in Table 1.

Numerator	Denominator
$a_4 = 30.8619$	$b_5 = 1$
$a_3 = 6.4642 \times 10^3$	$b_4 = 208.5959$
$a_2 = 2.6016 \times 10^6$	$b_3 = 8.5834 \times 10^4$
$a_1 = 2.6550 \times 10^8$	$b_2 = 8.8994 \times 10^6$
$a_0 = -7.0967 \times 10^5$	$b_1 = 1.0740 \times 10^8$
	$b_0 = 7.0311 \times 10^8$

Table 1. Radiation model coefficients

Fig. 4 shows the Bode plot of the WaveStar WEC system. Note that this Bode plot is very typical for a WEC system of the point-absorber type.

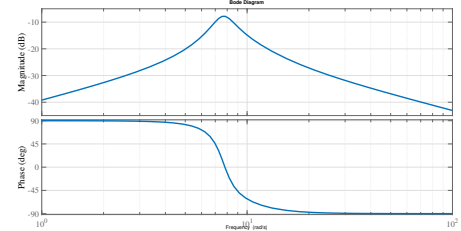


Fig. 4. Bode plot of the WaveStar-WEC system.

Fig. 5 presents the real part $R_i(w)$ and the imaginary part $X_i(w)$ of the considered WEC intrinsic impedance $Z_i(w)$

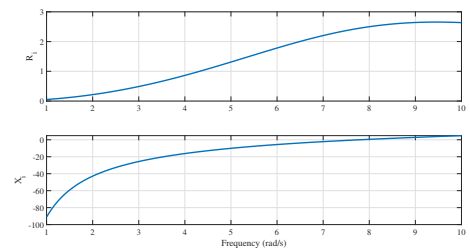


Fig. 5. Real part and imaginary part of the intrinsic impedance of the WaveStar-WEC system.

For simplicity, only input constraints are considered, see Nguyen and Tona [2020]

$$-6.25 \leq f_u(t) \leq 6.25 \quad (39)$$

4.2 Numerical Results

The WaveStar-WEC system was installed in September, 2009 near Hanstholm at the Western coast of Denmark, see Kramer et al. [2011]. A number of ocean waves were recorded. For the considered system, interesting wave frequencies are from 1rad/s to 10 rad/s with the maximum amplitude of 15N/m. Note that the given bounds are for the 1:20 scale.

Using Algorithm 1 for $1 \leq w \leq 10$ and $1 \leq A \leq 15$, Fig. 6 and Fig. 7 present, respectively, the optimal real part R_g^* and the optimal imaginary part X_g^* of the closed-loop transfer function $G(w)$.

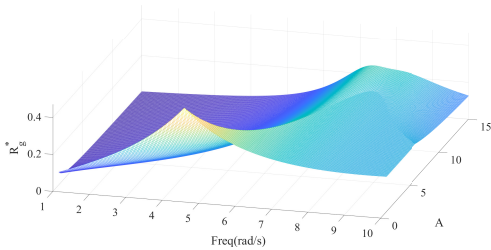


Fig. 6. Optimal R_g^* for the WaveStar-WEC system.

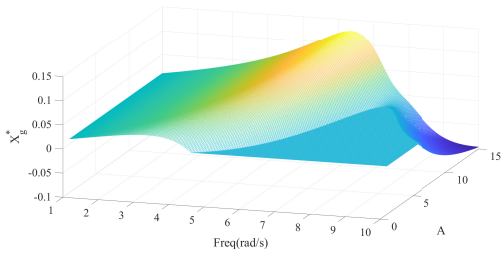


Fig. 7. Optimal X_g^* for the WaveStar - WEC system.

As can be seen from Fig. 6 and Fig. 7, the optimal R_g^* , X_g^* are a function of two variables w and A in the constrained case. Recall that in the literature, the optimal R_g^* , X_g^* are only a function of w . Note that $X_g^* \neq 0$ or equivalently the optimal velocity is not in phase with the wave excitation force for small w and for big A where the wave has the most energy.

5. CONCLUSION

This paper provides an algorithm to compute the optimal frequency response of the closed-loop WEC system, and of the control law. The main distinguished feature of the proposed approach compared with the existing results in the literature is that the input and WEC motions constraints are taken into account. The results are obtained by solving a convex optimization problem, for which the optimal solution is unique. It is shown that in the presence of constraints, the well-known resonance condition, i.e., the optimal velocity is in phase with the wave excitation force, does not generally hold.

We will in the future, exploit the result for the control design. We are also interested in extending the technique to the multi-float configuration case.

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