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SOME CHARACTERISTIC PROPERTIES OF THE TIME DISTRIBUTION OF STANDARD BROWNIAN MOTION

L.K. ROY and M.T. WASAN

1. Introduction.

In a paper to appear elsewhere Roy and Wasan [9] considered the estimation of the parameter of the time distribution of standard Brownian motion whose probability density function is defined as

$$g_X(x; \lambda) = \left(\frac{\lambda}{2\pi x^3} \right)^{1/2} e^{-\lambda/2x} \quad \text{for } x > 0 \quad (1)$$

$$= 0 \quad \text{otherwise}$$

(see Ito and McKean [5]) with parameter $\lambda > 0$. Its characteristic function is

$$\phi_X(\theta) = \exp \{ -(-2i\lambda\theta)^{1/2} \} \quad (2)$$

for all real θ .

In the present paper we discuss certain properties of the distribution described by (1) and then proceed to present three characterizations: the first through a differential equation involving the characteristic function $\phi_X(\theta)$, the second through a differential equation involving $r(\lambda/x) = xg_X(x; \lambda)$, and the third using the mixture Y of the random variable X with a random variable T . Finally we discuss the problem of a dam with infinite capacity having a continuous random content, when $g_X(x; \lambda)$ is used to describe the input. We are able to derive closed results for the transient behavior of the dam and in particular we find that the probability density function of the time of first emptiness of the dam is Inverse Gaussian (see Tweedie [12]).

2. Properties .

PROPERTY 1.— The distribution described by (1) is a stable law and hence is infinitely divisible (see Levy [7]).

PROPERTY 2.— The distribution described by (1) is a Pearson Type V distribution.

PROPERTY 3.— The probability density function $g_X(x; \lambda)$ is a solution of Euler's functional equation

$$F(xz, yz) = z^k F(x, y) \quad (z \neq 0)$$

with $y = \lambda$, for the special case $k = -1$. (See Aczel [1] for the general solution of this functional equation).

PROPERTY 4.— If $f_X(x; \mu, \lambda)$ is the probability density function of an Inverse Gaussian distribution with parameters $\mu > 0$ and $\lambda > 0$, (as defined by Tweedie [12]) then in the limit as μ approaches ∞ , $f_X(x; \mu, \lambda)$ becomes $g_X(x; \lambda)$ (see page 221 of Cox and Miller [2]).

PROPERTY 5.— Let X_0, X_1, X_2 be three independent random variables and

$$\text{let } W_1 = \frac{1}{2} \frac{X_0 - X_1}{\sqrt{X_0 X_1}}, \quad W_2 = \frac{1}{2} \frac{X_0 - X_2}{\sqrt{X_0 X_2}}. \quad \text{The necessary and suf-}$$

ficient condition that X_0, X_1, X_2 be identically distributed with probability density function $g_X(x; \lambda)$ is that (W_1, W_2) have the joint distribution

$$f(w_1, w_2) = \frac{2}{\pi} \frac{(w_1 + \sqrt{1+w_1^2})(w_2 + \sqrt{1+w_2^2})}{\{(1+w_1)^2(1+w_2)^2[1+(w_1 + \sqrt{1+w_1^2})^2 + (w_2 + \sqrt{1+w_2^2})^2]^3\}^{1/2}}$$

(The above property is an adaptation of a theorem presented by Kotlarski [6]).

3. Characterizations.

THEOREM 1. — Let X be a random variable with characteristic function $\phi(\theta)$. Suppose $\phi'(\theta)$ exists for all real θ . Then the density function of X is $g_X(x; \lambda)$ if and only if

$$\ln \phi(\theta) = -i\lambda \frac{\phi(\theta)}{\phi'(\theta)}.$$

PROOF. — Suppose

$$\ln \phi = i\lambda \frac{\phi}{\phi'}$$

then

$$\frac{\ln \phi}{\phi} d\phi = -i\lambda d\theta.$$

Integrating both sides gives

$$\frac{1}{2} (\ln \phi)^2 = -i\lambda \theta + c$$

where c is a constant. The condition $\phi(0) = 1$ yields $c = 0$ so that

$$\ln \phi = \pm \sqrt{-2i\lambda\theta}$$

or

$$\phi = e^{\pm \sqrt{-2i\lambda\theta}}$$

Since ϕ is a characteristic function we obtain

$$\phi(\theta) = e^{-\sqrt{-2i\lambda\theta}}$$

which is the characteristic function of a random variable with probability density $g_X(x; \lambda)$.

THEOREM 2. — Let X be a random variable with probability density function $\frac{1}{x} r(y)$ where $y = \lambda/x$ and $0 < x < \infty$, $\lambda > 0$, and suppose that

$\frac{dr}{dy}$ exists for all positive y . Then X has probability density function $g_X(x; \lambda)$ if and only if

$$2y \frac{dr}{dy} = (1-y)r.$$

PROOF. — Suppose $2y \frac{dr}{dy} = (1-y)r$ then

$$\frac{1}{r} dr = \left(\frac{1}{2y} - \frac{1}{2} \right) dy.$$

Integrating both sides gives

$$\ln r = \frac{1}{2} \ln y - \frac{1}{2} y + C$$

where C is a constant. Therefore

$$r(y) = D \sqrt{y} e^{-y/2}$$

where $D = e^C$. Using the fact that $y = \lambda/x$ we obtain the probability density function of X to be

$$\frac{1}{x} r(y) = D \left(\frac{\lambda}{x^3} \right)^{1/2} e^{-\lambda/2x}.$$

The condition that

$$\int_0^\infty \frac{1}{x} r(y) dx = 1$$

implies that $D = \frac{1}{\sqrt{2\pi}}$. Hence X has probability density function

$$g_X(x; \lambda).$$

THEOREM 3. — Let Y be a random variable with probability density function given by $f_1(y)$ and $f_2(y)$ respectively, where

$$f_1(y) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}+\frac{1}{2})} \frac{y^{\frac{n-1}{2}}}{(1+y)^{\frac{n+2}{2}}} \quad n = 0, 1, 2, \dots \quad y > 0$$

$$= 0 \quad \text{otherwise}$$

$$f_2(y) = \frac{1}{\sqrt{2}} y^{-3/2} e^{-\sqrt{2}/\sqrt{y}} \quad y > 0$$

$$= 0 \quad \text{otherwise.}$$

If Y is the mixture of a random variable X with a random variable T , where X is distributed according to the unknown, additively closed density function $g_X(x;t)$, and T is distributed according to $k_1(t)$ and $k_2(t)$ respectively, where

$$k_1(t) = \frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n+1}{2})} t^n e^{-t^2/2} \quad n = 0, 1, 2, \dots \quad t > 0$$

$$= 0 \quad \text{otherwise}$$

$$k_2(t) = \frac{2}{t^3} e^{-1/t^2} \quad t > 0$$

$$= 0 \quad \text{otherwise}$$

then $g_X(x;t)$ is the density described by equation (1) where $t = \sqrt{\lambda}$. These results are special cases of a theorem by Teicher [11].

PROOF. — Suppose that

$$\frac{1}{\sqrt{\pi}} \Gamma(\frac{n}{2}+1) \frac{x^{\frac{n-1}{2}}}{(1+x)^{\frac{n+2}{2}}} = \int_0^\infty g_X(x;t) \frac{t^n e^{-t^2/2}}{2^{\frac{n-1}{2}}} dt.$$

Then

$$\int_0^\infty \frac{1}{\sqrt{\pi}} e^{i\theta x} \Gamma\left(\frac{n}{2}+1\right) \frac{x^{\frac{n-1}{2}} dx}{(1+x)^{\frac{n}{2}+1}} = \int_0^\infty \int_0^\infty e^{i\theta x} g_X(x;t) \frac{t^n e^{-t^2/2}}{2^{\frac{n-1}{2}}} dt \quad (3)$$

The left hand side of equation (3) becomes

$$\frac{1}{\sqrt{\pi}} 2^{\frac{n+1}{2}} \Gamma\left(\frac{n}{2}+1\right) \Gamma\left(\frac{n+1}{2}\right) e^{-\frac{i\theta}{2}} D_{-(n+1)}(\sqrt{-2i\theta}) \quad (4)$$

by formula 5.216.1 of Ryshik and Gradstein [10]. Now consider the right hand side of equation (3). Using the assumption that $g_X(x;t)$ is additively closed we obtain

$$\int_0^\infty a^t \frac{t^n e^{-t^2/2}}{2^{\frac{n-1}{2}}} dt$$

where a^t is the characteristic function of $g_X(x;t)$. Using the fact that $a^t = e^{t \ln a}$, we obtain

$$\int_0^\infty e^{t \ln a} \frac{t^n e^{-t^2/2}}{2^{\frac{n-1}{2}}} dt$$

Formula 5.221.3 of [10] gives this to be

$$\begin{aligned} & \frac{1}{2^{\frac{n-1}{2}}} \Gamma(n+1) e^{\frac{(\ln a)^2}{4}} D_{-(n+1)}(-\ln a) \\ &= \frac{1}{\sqrt{\pi}} 2^{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \Gamma\left(\frac{n}{2}+1\right) e^{\frac{(\ln a)^2}{4}} D_{-(n+1)}(-\ln a) \quad (5) \end{aligned}$$

by the multiplication theorem for gamma functions (formula 6.335.1 of [10]). Comparing equations (5) and (6) gives the result that

$$\ln a = -\sqrt{-2i\theta}$$

that is, the characteristic function of $g_X(x;t)$ is

$$a^t = e^{-t\sqrt{-2i\theta}}$$

which is the required result.

The other part of the theorem may be proven in a similar manner.

4. Application to the Problem of Storage.

Let a dam of infinite capacity have a continuous random content $Z(t) \geq 0$ at time $t \geq 0$ and for this period let the input $X(t) \geq 0$ follow an infinitely divisible distribution. Let the dam be subject to a steady release at constant unit rate except if it is empty, in which case the release ceases. Gani and Prabhu [3] have presented a general theory of storage for dams subject to a steady release. Hasofer [4] has pointed out that the procedure of Gani and Prabhu cannot be carried out in closed form for the case of a Gamma input studied by Moran [8]. By instead choosing an Inverse Gaussian model and using the theory of Gani and Prabhu, he derived closed expressions for the probability of emptiness, etc. In particular he found that if z_0 is the initial content of the dam, the density function $h(z_0, T)$ of the time of first emptiness of the dam takes the form

$$h(z_0, T) = \frac{z_0}{\sigma\sqrt{2\pi}} \left(\frac{\rho}{T-z_0}\right)^{3/2} \exp\left\{-\frac{\rho}{2\sigma^2(T-z_0)}[\rho T - (T-z_0)]^2\right\}$$

$$\text{for } T \geq z_0 > 0$$

$$= 0 \quad \text{for } z_0 > T$$

where ρ is the mean and σ^2 the variance per unit time of the input $X(t)$. Thus $h(z_0, T)$ is also an Inverse Gaussian density function.

It is our intention to use a similar method to derive closed results for the transient behaviour of the dam when $g_X(x;\lambda)$ is used as the proba-

bility density function of the input $X(t)$. It will be convenient to let $\lambda = \gamma^2$ in the following calculations.

The Laplace transform of $g_X(x; \gamma)$ is of the form $e^{-v(\theta)} \gamma$ where $v(\theta) = \sqrt{2\theta}$. The function $v(\theta)$ may be represented as follows

$$v(\theta) = \int_0^\infty (1 - e^{-\theta u}) \alpha(u) du$$

where

$$\alpha(u) = \frac{1}{\sqrt{2\pi}} u^{-3/2}.$$

The transient behavior of the dam.

The transient behavior of the dam is governed by the function $\eta(s)$ which satisfies the following equation

$$\eta(s) = s + v(\eta(s))$$

and is such that $\eta(s) > 0$ when s is real and positive. In the case under consideration we obtain

$$\eta(s) = s + \sqrt{2\eta(s)}.$$

This equation reduces to a quadratic of which only one solution satisfies the condition $\eta(s) > 0$ when s is real and positive, namely

$$\eta(s) = s + 1 + \sqrt{1+2s}.$$

If z_0 is the initial content of the dam, one can calculate the density function $h(z_0, t)$ of the time of first emptiness by inverting its Laplace transform

$$h^*(z_0, s) = \exp \{ -z_0 [s + 1 + \sqrt{1+2s}] \}.$$

Thus

$$h(z_0, T) = \frac{z_0 e^{-z_0/2}}{\sqrt{2\pi(T-z_0)^3}} \exp \left\{ -\frac{(T-z_0)}{2} - \frac{z_0^2}{2(T-z_0)} \right\} \quad \text{for } T \geq z_0$$

$$= 0 \quad \text{for } z_0 > T.$$

Therefore $h(z_0, T)$ is an Inverse Gaussian density function.

The probability of emptiness when $z_0 = 0$.

Let $F(0, t)$ be the probability of the dam being empty, assuming that $z_0 = 0$. The Laplace transform of $F(0, t)$ is

$$F^*(0, s) = \frac{1}{\eta(s)} = \frac{1}{s + 1 + \sqrt{2s+1}}.$$

On inverting we obtain

$$F(0, t) = \frac{e^{-t/2}}{2\sqrt{\pi t^3}} \int_0^\infty u^2 e^{-\frac{u^2}{4t} - \frac{u}{\sqrt{2}}} du$$

$$= \frac{4}{\sqrt{\pi}} \int_{\sqrt{t/2}}^\infty u^2 e^{-u^2} du - 2\sqrt{\frac{2t}{\pi}} e^{-t^2/2} + \frac{2t}{\sqrt{\pi}} \int_{\sqrt{t/2}}^\infty e^{-u^2} du.$$

Bibliographie

- [1] ACZEL (J.). — *Lectures on functional equations and their applications*, Academic Press, New York (1966).
- [2] COX (D.R.) & MILLER (H.D.). — *The theory of stochastic processes*, John Wiley, New York (1965).
- [3] GANI (J.) & PRABHU (N.U.). — A storage model with continuous infinitely divisible inputs, *Proc. Cambridge Philos. Soc.*, 59 (1963), 417-430.
- [4] HASOFER (A.M.). — A dam with inverse Gaussian input, *Proc. Cambridge Philos. Soc.*, 60 (1964), 931-933.
- [5] ITO (KUJOSI) & Mc KEAN (H.P. Jr.). — *Diffusion processes and their sample paths*, Springer-Verlag, Berlin (1965).
- [6] KOTLARSKI (Ignacy). — On characterizing the Chi-Square distribution by the Student law, *J. Amer. Statist. Assoc.*, 61 (1966), 976-981.
- [7] LEVY (Paul). — Sur certains processus stochastiques homogènes, *Compositio Math.*, 7 (1940), 283-339.
- [8] MORAN (P.A.P.). — Probability theory of a dam with continuous release, *Quart. J. Math. Oxford Ser.*, 27 (1956), 130-137.
- [9] ROY (L.K.) & WASAN (M.T.). — Properties of the time distribution of standard Brownian motion, *Trabajos Estadist.*, 19, I-II (1968), 67-82.
- [10] RYSHIK (I.M.) & GRADSTEIN (I.S.). — *Tables of series, products, and integrals*, Veb Deutscher Verlag Der Wissenschaften, Berlin (1957).
- [11] TEICHER (Henry). — Identifiability of mixtures, *Ann. Math. Statist.*, 32 (1961), 244-248.
- [12] TWEEDIE (M.C.K.). — Statistical properties of inverse Gaussian distributions, I, *Ann. Math. Statist.*, 28 (1957), 362-377.

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