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Duality of Codes over Non-unital Rings of Order Four

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ABSTRACT In this paper, we present a basic theory of the duality of linear codes over three of the non-unital rings of order four; namely I , E , and H as denoted in (Fine, 1993). A new notion of duality is introduced in the case of the non-commutative ring E . The notion of self-dual codes with respect to this duality coincides with that of quasi self-dual codes over E as introduced in (Alahmadi *et al*, 2022). We characterize self-dual codes and LCD codes over the three rings, and investigate the properties of their corresponding additive codes over \mathbb{F}_4 . We study the connection between the dual of any linear code over these rings and the dual of its associated binary codes. A MacWilliams formula is established for linear codes over E .

INDEX TERMS Additive codes, LCD codes, non-unital rings, self-dual codes

I. INTRODUCTION

There are, up to isomorphism, exactly eleven rings of order four [9], [18]. The only unital ones amongst these are $\mathbb{F}_4, \mathbb{Z}_4, \mathbb{F}_2 \times \mathbb{F}_2$, and $\mathbb{F}_2 + u\mathbb{F}_2$. Before [4], these were the only rings of order four used as alphabets in Coding Theory [19]. In a series of papers [2]–[4], self-orthogonal codes over three of the non-unital rings among the eleven rings were investigated; namely I , E , and H , as per the notation of Fine [9]. Let \mathcal{R} be one of the rings I , E , or H . Throughout this paper, if the statement does not depend on which ring we are using, we shall denote the ring by \mathcal{R} . The goal of this paper is to lay down the foundations of the study of duality of linear codes over these three rings. In particular, the classes of self-dual codes and LCD codes are considered.

Self-dual codes have been given much attention in Coding Theory and have been widely studied for codes over finite fields and codes over rings [11, Chapter 4], [16, Chapter 3], [12], [14]. Due to technical hurdles, the study of self-dual codes over \mathcal{R} was replaced by that of Quasi Self-Dual codes (QSD codes) in the series of papers mentioned above. In the present paper, we initiate the study of self-dual codes over \mathcal{R} . In the case of the alphabet E , the new notion of self-dual codes coincides with that of QSD codes.

We then consider another class of codes that can be described in terms of their relationship with their dual. More precisely, it is the class of Linear codes with Complementary

Dual (LCD codes). The notion of LCD codes was introduced by Massey in [13] on codes over finite fields. It was the object of much attention in recent years due to its application in Boolean masking, a powerful countermeasure for cryptographic algorithms [8]. The study of LCD codes over non-unital rings first appeared in [20] where the authors investigated left LCD codes over E . We study LCD codes over E and H , and explain why no nontrivial LCD codes over I exist.

We show that self-dual codes and LCD codes over \mathcal{R} can be characterized in terms of their associated binary codes. Moreover, we study the duality of the corresponding additive codes over \mathbb{F}_4 with respect to the trace inner product.

To discuss the notions of self-dual codes and LCD codes, we address general properties of the dual of linear codes over \mathcal{R} . Through our investigation of duality, we prove a MacWilliams formula [12], which relates the weight enumerator of a linear code to that of its dual, for codes over the non-commutative non-unital ring E , where the dual is our two-sided dual.

The paper consists of seven sections. Section II recalls some background material on binary codes and additive codes over \mathbb{F}_4 as well as general terminologies on linear codes over \mathcal{R} . Sections III, IV, and V are devoted to studying codes over I , E , and H , respectively. As a preparation for the study of the main topic, we begin each of these three

sections by taking a closer look at the structure of linear codes over each particular ring. Then we proceed to study the duality of codes and prove various specific results on self-dual codes and LCD codes. Section VI summarizes the relationships among different classes of codes over \mathcal{R} . Section VII concludes the article.

II. DEFINITIONS AND NOTATIONS

A. RINGS OF ORDER FOUR

We describe the main properties of the rings I , E , and H of order four. These rings are defined by relations on two generators a and b . We shall write $c = a + b$ for all three rings.

The ring I is defined by

$$I = \langle a, b \mid 2a = 2b = 0, a^2 = b, ab = 0 \rangle.$$

It is a non-unital commutative ring with characteristic two. The ring is local with maximal ideal $\{0, b\}$. Its multiplication table is given in Table 1.

TABLE 1. Multiplication table for the ring I

\times	0	a	b	c
0	0	0	0	0
a	0	b	0	b
b	0	0	0	0
c	0	b	0	b

The ring E is defined by

$$E = \langle a, b \mid 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b \rangle.$$

It is a non-unital non-commutative ring with characteristic two. The ring is local with maximal ideal $\{0, c\}$. Its multiplication table is given in Table 2.

TABLE 2. Multiplication table for the ring E

\times	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	b	b	0
c	0	c	c	0

The ring H is defined by

$$H = \langle a, b \mid 2a = 2b = 0, a^2 = 0, b^2 = b, ab = ba = 0 \rangle.$$

It is a non-unital commutative ring with characteristic two. The ring is semi-local with two maximal ideals $\{0, a\}$ and $\{0, b\}$. Its multiplication table is given in Table 3.

TABLE 3. Multiplication table for the ring H

\times	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	b	b
c	0	0	b	b

For further details on the properties of \mathcal{R} , we refer the reader to [2]–[4].

B. CODES

We recall some preliminary notions and terminologies of binary codes, additive codes over \mathbb{F}_4 , and codes over \mathcal{R} .

1) Binary linear codes

An $[n, k]$ binary code C of length n and dimension k is a subspace of \mathbb{F}_2^n . The (Hamming) *weight* $\text{wt}(\mathbf{x})$ of $\mathbf{x} \in C$ is the number of nonzero coordinates in \mathbf{x} . The *dual* C^\perp of C is an $[n, n - k]$ code defined as

$$C^\perp = \{\mathbf{y} \in \mathbb{F}_2^n \mid \mathbf{x} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{x} \in C\}$$

where $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$ denotes the standard inner product in \mathbb{F}_2^n . A binary linear code C is *self-orthogonal* if $C \subseteq C^\perp$ and *self-dual* if $C = C^\perp$. The length n of a self-dual code is even and its dimension is $n/2$. A binary code C is *linear with complementary dual (LCD)* if $C \cap C^\perp = \{0\}$. Two binary codes are *permutation equivalent* if there is a permutation of coordinates that maps one to the other.

2) Additive codes over \mathbb{F}_4

Consider the finite field \mathbb{F}_4 consisting of the four elements $\{0, 1, \omega, \omega^2\}$ where $\omega^2 = 1 + \omega$. An $(n, 2^k)$ *additive code* over \mathbb{F}_4 of length n and size 2^k is an additive subgroup of \mathbb{F}_4^n . The *trace inner product* $\langle \mathbf{u}, \mathbf{v} \rangle_T$ of vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}_4^n$ is defined as

$$\langle \mathbf{u}, \mathbf{v} \rangle_T = \text{Tr}(\mathbf{u} \cdot \mathbf{v}^2) = \text{Tr} \left(\sum_{i=1}^n u_i v_i^2 \right)$$

where $\text{Tr}: \mathbb{F}_4 \rightarrow \mathbb{F}_2$ is the trace map defined by $\text{Tr}(u) = u + u^2$. The *trace dual* C^{\perp_T} of an additive code C of length n over \mathbb{F}_4 is defined as

$$C^{\perp_T} = \{\mathbf{v} \in \mathbb{F}_4^n \mid \langle \mathbf{u}, \mathbf{v} \rangle_T = 0 \text{ for all } \mathbf{u} \in C\}.$$

If C is an $(n, 2^k)$ additive code over \mathbb{F}_4 , then C^{\perp_T} is an $(n, 2^{2n-k})$ additive code over \mathbb{F}_4 . An additive code C over \mathbb{F}_4 is *trace self-orthogonal* if $C \subseteq C^{\perp_T}$ and *trace self-dual* if $C = C^{\perp_T}$. An additive code C over \mathbb{F}_4 is *additive with complementary dual (ACD)* if $C \cap C^{\perp_T} = \{0\}$.

Remark 2.1. Any $[n, k]$ binary code can be regarded as an $(n, 2^k)$ additive code over \mathbb{F}_4 since \mathbb{F}_2^n is an additive subgroup of \mathbb{F}_4^n .

3) Codes over \mathcal{R}

A linear code of length n over \mathcal{R} is a left \mathcal{R} -submodule of \mathcal{R}^n . The (Hamming) weight $\text{wt}(\mathbf{x})$ of $\mathbf{x} \in \mathcal{R}^n$ is the number of nonzero coordinates in \mathbf{x} . The inner product of $\mathbf{x} = x_1x_2 \dots x_n$ and $\mathbf{y} = y_1y_2 \dots y_n$ in \mathcal{R}^n is defined by $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i$.

The left dual C^{\perp_L} of a linear code C is the left module defined by

$$C^{\perp_L} = \{\mathbf{y} \in \mathcal{R}^n \mid \mathbf{y} \cdot \mathbf{x} = 0 \text{ for all } \mathbf{x} \in C\}.$$

The right dual C^{\perp_R} of a linear code C is the right module defined by

$$C^{\perp_R} = \{\mathbf{y} \in \mathcal{R}^n \mid \mathbf{x} \cdot \mathbf{y} = 0 \text{ for all } \mathbf{x} \in C\}.$$

A linear code C is self-orthogonal if for any $\mathbf{x}, \mathbf{y} \in C$, $\mathbf{x} \cdot \mathbf{y} = 0$. Thus, any self-orthogonal code C satisfies the inclusion $C \subseteq C^{\perp_L} \cap C^{\perp_R}$. A linear code of length n is quasi self-dual (QSD) if it is self-orthogonal and of size 2^n . A QSD code where all of its codewords have even weight is called Type IV. A linear code C is left self-dual (respectively, right self-dual) if $C = C^{\perp_L}$ (respectively, $C = C^{\perp_R}$). A linear code C of length n is left nice (respectively, right nice) if $|C||C^{\perp_L}| = 4^n$ (respectively, $|C||C^{\perp_R}| = 4^n$).

When \mathcal{R} is commutative, $C^{\perp_R} = C^{\perp_L}$ and thus we omit the adjectives left and right and simply say dual and denote it by C^\perp . We do the same for the notions of self-dual and nice.

Two linear codes over \mathcal{R} are permutation equivalent if there is a permutation of coordinates that maps one to the other.

We note that in the upcoming sections, \oplus denotes the direct sum of vector spaces over \mathbb{F}_2 . This concept is used to represent linear codes over \mathcal{R} as additive codes over \mathcal{R} ; where an additive code of length n over \mathcal{R} is an additive subgroup of \mathcal{R}^n .

III. RESULTS ON LINEAR CODES OVER I

We begin this section by summarizing facts and notions essential to our study for linear codes over I . A detailed introduction on such codes can be found in [3].

To every linear code C of length n over I , there is an additive code $\phi_I(C)$ over \mathbb{F}_4 such that ϕ_I is defined by the alphabet substitution

$$0 \rightarrow 0, a \rightarrow \omega, b \rightarrow 1, c \rightarrow \omega^2,$$

extended in the natural way to a map from C to \mathbb{F}_4^n .

There are two binary linear codes of length n associated canonically with every linear code C of length n over I :

- 1) the residue code $\text{res}(C)$ defined by

$$\text{res}(C) = \{\alpha(\mathbf{y}) \mid \mathbf{y} \in C\}$$

where $\alpha : I \rightarrow \mathbb{F}_2$ is the map defined by $\alpha(0) = \alpha(b) = 0$ and $\alpha(a) = \alpha(c) = 1$, extended componentwise from C to \mathbb{F}_2^n ,

- 2) the torsion code $\text{tor}(C)$ defined by

$$\text{tor}(C) = \{\mathbf{x} \in \mathbb{F}_2^n \mid b\mathbf{x} \in C\}.$$

The two binary codes satisfy the inclusion $\text{res}(C) \subseteq \text{tor}(C)$ and their sizes are related to the size of C by $|C| = |\text{res}(C)||\text{tor}(C)|$. Throughout this section, we let $k_1 = \dim(\text{res}(C))$ and $k_2 = \dim(\text{tor}(C)) - k_1$. The linear code C is said to be of type (k_1, k_2) . We say that a linear code is free if and only if $k_2 = 0$. Equivalently, C is free if and only if $\text{res}(C) = \text{tor}(C)$. A QSD code C is quasi Type IV if $\text{tor}(C)$ is even.

A. STRUCTURE OF LINEAR CODES

As noted in [3, Section 4], two distinct linear codes over I may share the same residue and torsion codes. This means that codes over I do not have a unique algebraic representation via their two associated binary codes. Nevertheless, these two binary codes are useful when studying the structure of codes over I and their dual.

The following theorem gives a connection between any linear code over I and its residue code.

Theorem 3.1. *If C is a linear code of length n over I , then the following hold:*

- 1) Every codeword $\mathbf{c} \in C$ can be written as $\mathbf{c} = \mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v}$ for some $\mathbf{u} \in \text{res}(C)$ and $\mathbf{v} \in \mathbb{F}_2^n$.
- 2) If $\mathbf{u} \in \text{res}(C)$, then $\mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v}$ is a codeword in C for some $\mathbf{v} \in \mathbb{F}_2^n$.

Proof. Let $\mathbf{c} \in C$. We can write \mathbf{c} in a b -adic decomposition form as $\mathbf{c} = \mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^n$. Since $\alpha(\mathbf{c}) = \alpha(\mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v}) = \mathbf{u}$, $\mathbf{u} \in \text{res}(C)$. This proves (1).

Now let $\mathbf{u} \in \text{res}(C)$. Then there exists $\mathbf{c} \in C$ such that $\alpha(\mathbf{c}) = \mathbf{u}$. We can write \mathbf{c} in a b -adic decomposition form as $\mathbf{c} = \mathbf{a}\mathbf{w} + \mathbf{b}\mathbf{v}$ where $\mathbf{w}, \mathbf{v} \in \mathbb{F}_2^n$. Observe that $\alpha(\mathbf{a}\mathbf{w} + \mathbf{b}\mathbf{v}) = \mathbf{w}$. On the other hand, $\alpha(\mathbf{a}\mathbf{w} + \mathbf{b}\mathbf{v}) = \alpha(\mathbf{c}) = \mathbf{u}$. Hence, $\mathbf{w} = \mathbf{u}$ and so $\mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v}$ is a codeword in C . This proves (2). \square

The following corollary determines, from the residue code, when a linear code is self-orthogonal.

Corollary 3.2. *A linear code C of length n over I is self-orthogonal if and only if $\text{res}(C)$ is a binary self-orthogonal code.*

Proof. For any $\mathbf{u}_1, \mathbf{u}_2 \in \text{res}(C)$ and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{F}_2^n$,

$$(\mathbf{a}\mathbf{u}_1 + \mathbf{b}\mathbf{v}_1) \cdot (\mathbf{a}\mathbf{u}_2 + \mathbf{b}\mathbf{v}_2) = \mathbf{b}(\mathbf{u}_1 \cdot \mathbf{u}_2).$$

Due to this and the relation between C and $\text{res}(C)$ given in Theorem 3.1, it follows that C is self-orthogonal if and only if $\text{res}(C)$ is self-orthogonal. \square

Now we study the close connection between the minimum distance of any linear code over I and that of its torsion code.

Theorem 3.3. *If C is a nonzero linear code over I , then the minimum distance of C equals the minimum distance of $\text{tor}(C)$.*

Proof. Let d be the minimum distance of C and let d_t be the minimum distance of $\text{tor}(C)$. Then there exists a nonzero

$\mathbf{t} \in \text{tor}(C)$ such that $\text{wt}(\mathbf{t}) = d_t$. Since $b\text{tor}(C) \subseteq C$ and $\text{wt}(b\mathbf{t}) = \text{wt}(\mathbf{t}) = d_t, d \leq d_t$.

Now we prove that $d \geq d_t$. Let $\mathbf{x} \in C$ such that $\text{wt}(\mathbf{x}) = d$. By Theorem 3.1, $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$ where $\mathbf{u} \in \text{res}(C)$ and $\mathbf{v} \in \mathbb{F}_2^n$. Since C is nonzero, we have the following three cases depending on \mathbf{u} and \mathbf{v} :

- If $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$, then we have $\mathbf{v} \in \text{tor}(C)$ and $\text{wt}(\mathbf{x}) = \text{wt}(b\mathbf{v}) = \text{wt}(\mathbf{v}) \geq d_t$.
- If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} = \mathbf{0}$, then $\text{wt}(\mathbf{x}) = \text{wt}(a\mathbf{u}) = \text{wt}(\mathbf{u})$.
- If $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, then $\text{wt}(\mathbf{x}) \geq \text{wt}(a\mathbf{x}) = \text{wt}(b\mathbf{u}) = \text{wt}(\mathbf{u})$.

Since $\mathbf{u} \in \text{res}(C) \subseteq \text{tor}(C)$, it follows that $\text{wt}(\mathbf{u}) \geq d_t$. Thus, in all cases, $d = \text{wt}(\mathbf{x}) \geq d_t$.

Since $d \leq d_t$ and $d \geq d_t$, it follows that $d = d_t$. \square

B. DUALITY

The following theorem presents properties of the residue and torsion codes of the dual of linear codes over I .

Theorem 3.4. *If C is a linear code of length n over I , then the following hold:*

- 1) $\text{res}(C^\perp) = \text{res}(C)^\perp$.
- 2) $\text{tor}(C^\perp) = \mathbb{F}_2^n$.

Proof. To prove (1), let $\mathbf{u} \in \text{res}(C^\perp)$. By Theorem 3.1, $a\mathbf{u} + b\mathbf{v}$ is a codeword in C^\perp for some $\mathbf{v} \in \mathbb{F}_2^n$. Let $\mathbf{x} \in \text{res}(C)$. By Theorem 3.1, $a\mathbf{x} + b\mathbf{y}$ is a codeword in C for some $\mathbf{y} \in \mathbb{F}_2^n$. By definition of C^\perp ,

$$0 = (a\mathbf{u} + b\mathbf{v}) \cdot (a\mathbf{x} + b\mathbf{y}) = b(\mathbf{u} \cdot \mathbf{x}).$$

Hence, $\mathbf{u} \cdot \mathbf{x} = 0$ which implies that $\mathbf{u} \in \text{res}(C)^\perp$. Therefore, $\text{res}(C^\perp) \subseteq \text{res}(C)^\perp$.

Now assume $\mathbf{u} \in \text{res}(C)^\perp$. Let $\mathbf{c} \in C$. By Theorem 3.1, $\mathbf{c} = a\mathbf{x} + b\mathbf{y}$ where $\mathbf{x} \in \text{res}(C)$ and $\mathbf{y} \in \mathbb{F}_2^n$. Observe that

$$a\mathbf{u} \cdot \mathbf{c} = a\mathbf{u} \cdot (a\mathbf{x} + b\mathbf{y}) = b(\mathbf{u} \cdot \mathbf{x}) = 0.$$

Hence, $a\mathbf{u} \in C^\perp$ and $a(a\mathbf{u}) = \mathbf{u}$ which yields $\mathbf{u} \in \text{res}(C^\perp)$. Therefore, $\text{res}(C)^\perp \subseteq \text{res}(C^\perp)$. This proves (1).

To prove (2), we will show that $\mathbb{F}_2^n \subseteq \text{tor}(C^\perp)$. Let $\mathbf{u} \in \mathbb{F}_2^n$ and let $\mathbf{c} \in C$. By Theorem 3.1, $\mathbf{c} = a\mathbf{x} + b\mathbf{y}$ where $\mathbf{x} \in \text{res}(C)$ and $\mathbf{y} \in \mathbb{F}_2^n$. Observe that

$$\mathbf{c} \cdot b\mathbf{u} = (a\mathbf{x} + b\mathbf{y}) \cdot b\mathbf{u} = 0.$$

Hence, $b\mathbf{u} \in C^\perp$ which gives $\mathbf{u} \in \text{tor}(C^\perp)$. Therefore, $\mathbb{F}_2^n = \text{tor}(C^\perp)$. This proves (2). \square

The dual of a linear code of length n over I can be written uniquely in terms of the dual of its residue code and the binary vector space \mathbb{F}_2^n as the following theorem shows.

Theorem 3.5. *If C is a linear code of length n over I , then $C^\perp = a\text{res}(C)^\perp \oplus b\mathbb{F}_2^n$.*

Proof. Let $\mathbf{z} \in C^\perp$. Applying Theorems 3.1 and 3.4, we have $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$ where $\mathbf{x} \in \text{res}(C^\perp) = \text{res}(C)^\perp$ and $\mathbf{y} \in \mathbb{F}_2^n$. This proves that $C^\perp \subseteq a\text{res}(C)^\perp + b\mathbb{F}_2^n$.

Now assume that $\mathbf{w} := a\mathbf{u} + b\mathbf{v} \in a\text{res}(C)^\perp + b\mathbb{F}_2^n$. Let

$\mathbf{c} \in C$. By Theorem 3.1, $\mathbf{c} = a\mathbf{r} + b\mathbf{s}$ where $\mathbf{r} \in \text{res}(C)$ and $\mathbf{s} \in \mathbb{F}_2^n$. Observe that

$$\mathbf{w} \cdot \mathbf{c} = (a\mathbf{u} + b\mathbf{v}) \cdot (a\mathbf{r} + b\mathbf{s}) = b(\mathbf{u} \cdot \mathbf{r}) = 0.$$

Hence, $\mathbf{w} \in C^\perp$. Therefore, $a\text{res}(C)^\perp + b\mathbb{F}_2^n \subseteq C^\perp$. This proves that $C^\perp = a\text{res}(C)^\perp + b\mathbb{F}_2^n$. Since

$$|C^\perp| = |\text{res}(C^\perp)| |\text{tor}(C^\perp)| = |\text{res}(C)^\perp| |\mathbb{F}_2^n|$$

by Theorem 3.4, the sum is direct. \square

The size of the dual of any type (k_1, k_2) linear code of length n over I equals 2^{2n-k_1} . Thus, we have the following result.

Proposition 3.6. *The only nice code over I is the zero code.*

Proof. Suppose C is a type (k_1, k_2) nice code of length n . By the definition of nice codes and Theorem 3.5,

$$4^n = |C| |C^\perp| = 2^{2n+k_1+k_2}$$

which holds if and only if $k_1 + k_2 = 0$. Since $k_1, k_2 \geq 0$, it follows that C is nice if and only if $k_1 = k_2 = 0$. Hence, the only nice code over I is the zero code. \square

An interesting fact about the families of QSD codes and self-dual codes of length n over I is the following.

Proposition 3.7. *Let \mathcal{Q} be the family of all QSD codes of length n over I and let \mathcal{S} be the family of all self-dual codes of length n over I . Then $\mathcal{Q} \cap \mathcal{S} = \emptyset$.*

Proof. Suppose that a linear code C over I is QSD and self-dual. Then C is nonzero and $|C| = |C^\perp| = 2^n$ which implies that C is nice. By Proposition 3.6, no such codes exist. This means that a linear code over I can never simultaneously be both QSD and self-dual. \square

1) Self-dual codes

Self-dual codes over I are characterized by means of their residue and torsion codes as the following theorem shows.

Theorem 3.8. *A linear code C of length n over I is self-dual if and only if the following two conditions are satisfied:*

- 1) $\text{res}(C)$ is a self-dual binary code,
- 2) $\text{tor}(C) = \mathbb{F}_2^n$.

Proof. Suppose that C is self-dual. Then $C = C^\perp$. Consequently, $\text{res}(C) = \text{res}(C^\perp)$ and $\text{tor}(C) = \text{tor}(C^\perp)$. By Theorem 3.4, $\text{res}(C) = \text{res}(C)^\perp$ and $\text{tor}(C) = \mathbb{F}_2^n$. Conversely, suppose that $\text{res}(C)$ is self-dual and $\text{tor}(C) = \mathbb{F}_2^n$. By Theorems 3.1 and 3.5,

$$C \subseteq a\text{res}(C) + b\mathbb{F}_2^n = a\text{res}(C)^\perp + b\mathbb{F}_2^n = C^\perp.$$

As $|C| = |\text{res}(C)| |\text{tor}(C)| = |\text{res}(C)^\perp| |\mathbb{F}_2^n| = |C^\perp|$, it follows that $C = C^\perp$ and hence C is self-dual. \square

Corollary 3.9. *If B is a self-dual binary code of length n , then B is a residue code of a self-dual code over I .*

Proof. Since B is self-dual and $B \subseteq \mathbb{F}_2^n$, by [3, Theorem 4], the linear code C defined by $C = aB + b\mathbb{F}_2^n$ is a self-orthogonal code over I with $\text{res}(C) = B$ and $\text{tor}(C) = \mathbb{F}_2^n$. By Theorem 3.8 and the self-duality of B , it follows that C is self-dual. \square

By Theorem 3.8 and Corollary 3.9, self-dual codes over I exist only for even lengths and there are as many type $(n/2, n/2)$ self-dual codes of length n over I as there are $[n, n/2]$ binary self-dual codes.

Theorem 3.10. *Two self-dual codes over I are permutation equivalent if and only if their residue codes are permutation equivalent.*

Proof. Let C and C' be two permutation equivalent codes over I . Then there is a permutation matrix P such that $C' = CP$. Since $\alpha(C') = \alpha(CP) = \alpha(C)P$, it follows that $\text{res}(C)$ and $\text{res}(C')$ are permutation equivalent. Conversely, suppose that C and C' are self-dual codes over I where $\text{res}(C)$ and $\text{res}(C')$ are permutation equivalent. Then there is a permutation matrix P such that $\text{res}(C') = \text{res}(C)P$. As $\mathbb{F}_2^n = \mathbb{F}_2^n P$, we have

$$a \text{res}(C') + b\mathbb{F}_2^n = a \text{res}(C)P + b\mathbb{F}_2^n P. \quad (1)$$

Since C and C' are self-dual, by Theorems 3.5 and 3.8, we have $C = a \text{res}(C) \oplus b\mathbb{F}_2^n$ and $C' = a \text{res}(C') \oplus b\mathbb{F}_2^n$. By (1), we obtain $C' = CP$, proving that C and C' are permutation equivalent. \square

The following example shows that Theorem 3.10 may not hold if the codes over I are not self-dual.

Example 3.11. The linear codes C and C' with generator matrices

$$\begin{pmatrix} a & a & b \\ 0 & b & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & a & 0 \\ 0 & b & b \end{pmatrix},$$

respectively, have the same residue code. In particular, $\text{res}(C) = \text{res}(C') = \{000, 110\}$. However, C and C' are not permutation equivalent as shown in the classification of QSD codes in [3, Section 6]. Note that C and C' are not self-dual, by Proposition 3.7.

From the results of this subsection, we see that there is a one-to-one correspondence between inequivalent self-dual binary codes and inequivalent self-dual codes over I of the same length. In other words, classifying self-dual codes over I , up to permutation equivalence, is equivalent to classifying self-dual binary codes, up to equivalence. All binary self-dual codes have been classified, up to equivalence, for length n with $2 \leq n \leq 32$ [10], [15], [17]. Using this classification along with Theorem 3.8 and Corollary 3.9, the classification of all self-dual codes over I of the same lengths is immediate. We remark that by Theorems 3.3 and 3.8, all self-dual codes over I have minimum distance equals 1.

To conclude this subsection, we note that the image of any self-dual code over I under the map ϕ_I is never an additive

trace self-dual code over \mathbb{F}_4 . However, the trace dual of this image is an additive trace self-orthogonal code over \mathbb{F}_4 .

Proposition 3.12. *If C is a self-dual code of length $2n$ over I , then $\phi_I(C)^{\perp_T}$ is trace self-orthogonal of size 2^n ; in particular, $\phi_I(C)$ is not trace self-dual.*

Proof. By Theorems 3.5 and 3.8, the self-duality of C implies that $C = a \text{res}(C) \oplus b \text{tor}(C)$ with $|\text{res}(C)| = 2^n$ and $|\text{tor}(C)| = |\mathbb{F}_2^{2n}| = 2^{2n}$. Then, $|\phi_I(C)| = |C| = 2^{3n}$ and $|\phi_I(C)^{\perp_T}| = 2^n$. Comparing cardinalities, we see that $\phi_I(C) \neq \phi_I(C)^{\perp_T}$ which shows that $\phi_I(C)$ is not trace self-dual.

We claim that $\text{res}(C) = \phi_I(C)^{\perp_T}$. Let $\mathbf{u} \in \text{res}(C)$ and let $\mathbf{x} \in \phi_I(C)$. Then, there exists a codeword $a\mathbf{r} + b\mathbf{t}$ in C where $\mathbf{r} \in \text{res}(C)$ and $\mathbf{t} \in \text{tor}(C)$ such that $\mathbf{x} = \phi_I(a\mathbf{r} + b\mathbf{t}) = \omega\mathbf{r} + \mathbf{t}$. Observe that

$$\begin{aligned} \langle \mathbf{x}, \mathbf{u} \rangle_T &= \langle \omega\mathbf{r} + \mathbf{t}, \mathbf{u} \rangle_T \\ &= \langle \omega\mathbf{r}, \mathbf{u} \rangle_T + \langle \mathbf{t}, \mathbf{u} \rangle_T \\ &= \text{Tr}(\omega\mathbf{r} \cdot \mathbf{u}) + \text{Tr}(\mathbf{t} \cdot \mathbf{u}). \end{aligned}$$

Since C is self-dual, by Theorem 3.8, $\text{res}(C)$ is self-dual and therefore $\mathbf{r} \cdot \mathbf{u} = 0$ which gives $\text{Tr}(\omega\mathbf{r} \cdot \mathbf{u}) = 0$. As $\mathbf{t} \cdot \mathbf{u} \in \{0, 1\}$, $\text{Tr}(\mathbf{t} \cdot \mathbf{u}) = 0$. Thus, $\langle \mathbf{x}, \mathbf{u} \rangle_T = 0$ proving that $\mathbf{u} \in \phi_I(C)^{\perp_T}$ and consequently $\text{res}(C) \subseteq \phi_I(C)^{\perp_T}$. The fact that $|\text{res}(C)| = 2^n = |\phi_I(C)^{\perp_T}|$ implies that $\text{res}(C) = \phi_I(C)^{\perp_T}$ as claimed. Now observe that since $\text{res}(C) \subseteq \text{tor}(C)$, for any $\mathbf{v} \in \text{res}(C)$, $b\mathbf{v} \in C$ and thus $\mathbf{v} = \phi_I(b\mathbf{v}) \in \phi_I(C)$. Hence, we obtain $\text{res}(C) \subseteq \phi_I(C)$. In particular, $\phi_I(C)^{\perp_T} \subseteq \phi_I(C)$ which proves that $\phi_I(C)^{\perp_T}$ is trace self-orthogonal. \square

2) LCD codes

Based on the following proposition, nontrivial LCD codes over I do not exist.

Proposition 3.13. *If C is a nonzero linear code of length n over I , then $C \cap C^\perp \neq \{0\}$.*

Proof. Suppose that \mathbf{x} is a nonzero codeword in C . By Theorem 3.1, $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$ where $\mathbf{u} \in \text{res}(C)$ and $\mathbf{v} \in \mathbb{F}_2^n$. We have two cases depending on \mathbf{u} .

- If $\mathbf{u} = 0$, then $\mathbf{x} = b\mathbf{v}$. Since $b\mathbb{F}_2^n \subseteq C^\perp$ and $\mathbf{x} \in C$, it follows that $\mathbf{x} \in C \cap C^\perp$.
- If $\mathbf{u} \neq 0$, then $a\mathbf{x} = b\mathbf{u}$ is a nonzero codeword in C . Since $b\mathbb{F}_2^n \subseteq C^\perp$ and $a\mathbf{x} \in C$, we have $a\mathbf{x} \in C \cap C^\perp$.

This proves that $C \cap C^\perp \neq \{0\}$. \square

IV. RESULTS ON LINEAR CODES OVER E

We begin this section by summarizing facts and notions essential to our study for linear codes over E . A detailed introduction on such codes can be found in [4].

To every linear code C of length n over E , there is an additive code $\phi_E(C)$ over \mathbb{F}_4 such that ϕ_E is defined by the alphabet substitution

$$0 \rightarrow 0, \quad a \rightarrow \omega, \quad b \rightarrow \omega^2, \quad c \rightarrow 1,$$

extended in the natural way to a map from C to \mathbb{F}_4^n .

There are two binary linear codes of length n associated canonically with every linear code C of length n over E :

- 1) the *residue code* $\text{res}(C)$ defined by

$$\text{res}(C) = \{\alpha(\mathbf{y}) \mid \mathbf{y} \in C\}$$

where $\alpha : E \rightarrow \mathbb{F}_2$ is the map defined by $\alpha(0) = \alpha(c) = 0$ and $\alpha(a) = \alpha(b) = 1$, extended componentwise from C to \mathbb{F}_2^n ,

- 2) the *torsion code* $\text{tor}(C)$ defined by

$$\text{tor}(C) = \{\mathbf{x} \in \mathbb{F}_2^n \mid c\mathbf{x} \in C\}.$$

The two binary codes satisfy the inclusion $\text{res}(C) \subseteq \text{tor}(C)$ and their sizes are related to the size of C by $|C| = |\text{res}(C)||\text{tor}(C)|$. Throughout this section, we let $k_1 = \dim(\text{res}(C))$ and $k_2 = \dim(\text{tor}(C)) - k_1$. The linear code C is said to be of *type* (k_1, k_2) . We say that a linear code is *free* if and only if $k_2 = 0$. Equivalently, C is free if and only if $\text{res}(C) = \text{tor}(C)$.

A. STRUCTURE OF LINEAR CODES

The following two results improve Lemma 3 and Theorem 6 of [4] by removing the QSD requirement from their statements.

Lemma 4.1. *If C is a linear code of length n over E , then $a \text{res}(C) \subseteq C$.*

Proof. Let $\mathbf{u} \in \text{res}(C)$. Then there exists $\mathbf{c} \in C$ such that $\alpha(\mathbf{c}) = \mathbf{u}$. We can write \mathbf{c} in a c -adic decomposition form as $\mathbf{c} = a\mathbf{x} + c\mathbf{y}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{F}_2^n$. Now observe that $\mathbf{u} = \alpha(\mathbf{c}) = \alpha(a\mathbf{x} + c\mathbf{y}) = \mathbf{x}$. Hence, $\mathbf{c} = a\mathbf{u} + c\mathbf{y}$. By linearity of C , we have $a\mathbf{c} \in C$ and thus $a\mathbf{u} \in C$. Therefore, $a \text{res}(C) \subseteq C$. \square

Theorem 4.2. *If C is a linear code of length n over E , then $C = a \text{res}(C) \oplus c \text{tor}(C)$.*

Proof. Let $\mathbf{c} \in C$. We can write \mathbf{c} in a c -adic decomposition form as $\mathbf{c} = a\mathbf{x} + c\mathbf{y}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{F}_2^n$. Since $\alpha(\mathbf{c}) = \alpha(a\mathbf{x} + c\mathbf{y}) = \mathbf{x}$, it follows that $\mathbf{x} \in \text{res}(C)$. By Lemma 4.1, $a\mathbf{x} \in C$. By linearity of C , $c\mathbf{y} \in C$ and hence $\mathbf{y} \in \text{tor}(C)$. This proves that $C \subseteq a \text{res}(C) + c \text{tor}(C)$. The inclusion $a \text{res}(C) + c \text{tor}(C) \subseteq C$ follows from the linearity of C together with the facts that $a \text{res}(C) \subseteq C$ and $c \text{tor}(C) \subseteq C$. Hence, $C = a \text{res}(C) + c \text{tor}(C)$. The sum is direct since $|C| = |\text{res}(C)||\text{tor}(C)|$. \square

Corollary 4.3. *A linear code C over E is self-orthogonal if and only if $\text{res}(C) \subseteq \text{tor}(C)^\perp$.*

Proof. From [4, Lemma 2], if C is self-orthogonal, then $\text{res}(C) \subseteq \text{tor}(C)^\perp$.

For the converse suppose that $\text{res}(C) \subseteq \text{tor}(C)^\perp$ and let $\mathbf{x}, \mathbf{y} \in C$. Then, by Theorem 4.2, $\mathbf{x} = a\mathbf{r}_1 + c\mathbf{t}_1$ and $\mathbf{y} = a\mathbf{r}_2 + c\mathbf{t}_2$ where $\mathbf{r}_1, \mathbf{r}_2 \in \text{res}(C)$ and $\mathbf{t}_1, \mathbf{t}_2 \in \text{tor}(C)$. Observe that

$$\mathbf{x} \cdot \mathbf{y} = (a\mathbf{r}_1 + c\mathbf{t}_1) \cdot (a\mathbf{r}_2 + c\mathbf{t}_2) = a(\mathbf{r}_1 \cdot \mathbf{r}_2) + c(\mathbf{t}_1 \cdot \mathbf{r}_2).$$

Since $\text{res}(C) \subseteq \text{tor}(C)^\perp \subseteq \text{res}(C)^\perp$, it follows that $(\mathbf{r}_1 \cdot \mathbf{r}_2) = (\mathbf{t}_1 \cdot \mathbf{r}_2) = 0$ and thus $\mathbf{x} \cdot \mathbf{y} = 0$. This proves that C is self-orthogonal. \square

The following theorem is the analogue of Theorem 3.3.

Theorem 4.4. *If C is a nonzero linear code over E , then the minimum distance of C equals the minimum distance of $\text{tor}(C)$.*

Proof. Let d be the minimum distance of C and let d_t be the minimum distance of $\text{tor}(C)$. Then there exists a nonzero $\mathbf{t} \in \text{tor}(C)$ such that $\text{wt}(\mathbf{t}) = d_t$. Since $c \text{tor}(C) \subseteq C$ and $\text{wt}(c\mathbf{t}) = \text{wt}(\mathbf{t}) = d_t$, $d \leq d_t$.

Now we prove that $d \geq d_t$. Let $\mathbf{x} \in C$ such that $\text{wt}(\mathbf{x}) = d$. By Theorem 4.2, $\mathbf{x} = a\mathbf{u} + c\mathbf{v}$ where $\mathbf{u} \in \text{res}(C)$ and $\mathbf{v} \in \text{tor}(C)$. Since C is nonzero, we have the following three cases depending on \mathbf{u} and \mathbf{v} :

- If $\mathbf{u} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$, then $\text{wt}(\mathbf{x}) = \text{wt}(c\mathbf{v}) = \text{wt}(\mathbf{v})$.
- If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} = \mathbf{0}$, then $\text{wt}(\mathbf{x}) = \text{wt}(a\mathbf{u}) = \text{wt}(\mathbf{u})$.
- If $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$, then $\text{wt}(\mathbf{x}) \geq \text{wt}(a\mathbf{u}) = \text{wt}(\mathbf{u})$.

Since $\mathbf{u} \in \text{res}(C) \subseteq \text{tor}(C)$ and $\mathbf{v} \in \text{tor}(C)$, it follows that $\text{wt}(\mathbf{u}), \text{wt}(\mathbf{v}) \geq d_t$. Therefore, in all cases $d = \text{wt}(\mathbf{x}) \geq d_t$. Since $d \leq d_t$ and $d \geq d_t$, it follows that $d = d_t$. \square

B. DUALITY

The following theorem presents properties of the residue and torsion codes of the one-sided duals of linear codes over E .

Theorem 4.5. *If C is a linear code of length n over E , then the following hold:*

- 1) $\text{res}(C^{\perp L}) = \text{tor}(C^{\perp L}) = \text{res}(C)^\perp$.
- 2) $\text{res}(C^{\perp R}) = \text{tor}(C)^\perp$.
- 3) $\text{tor}(C^{\perp R}) = \mathbb{F}_2^n$.

Proof. To prove (1), it suffices to show that

$$\text{tor}(C^{\perp L}) \subseteq \text{res}(C)^\perp \subseteq \text{res}(C^{\perp L}).$$

Let $\mathbf{v} \in \text{tor}(C^{\perp L})$. Then, $c\mathbf{v} \in C^{\perp L}$. Let $\mathbf{x} \in \text{res}(C)$. By Lemma 4.1, $a\mathbf{x} \in C$. By definition of $C^{\perp L}$,

$$0 = c\mathbf{v} \cdot a\mathbf{x} = c(\mathbf{v} \cdot \mathbf{x}).$$

Hence, $\mathbf{v} \cdot \mathbf{x} = 0$ which implies that $\mathbf{v} \in \text{res}(C)^\perp$, proving that

$$\text{tor}(C^{\perp L}) \subseteq \text{res}(C)^\perp.$$

Now assume $\mathbf{u} \in \text{res}(C)^\perp$. Let $\mathbf{c} \in C$. By Theorem 4.2, $\mathbf{c} = a\mathbf{r} + c\mathbf{t}$ where $\mathbf{r} \in \text{res}(C)$ and $\mathbf{t} \in \text{tor}(C)$. Observe that

$$a\mathbf{u} \cdot \mathbf{c} = a\mathbf{u} \cdot (a\mathbf{r} + c\mathbf{t}) = a(\mathbf{u} \cdot \mathbf{r}) = 0.$$

So, $a\mathbf{u} \in C^{\perp L}$. Since $\alpha(a\mathbf{u}) = \mathbf{u}$, $\mathbf{u} \in \text{res}(C^{\perp L})$. Hence,

$$\text{res}(C)^\perp \subseteq \text{res}(C^{\perp L}).$$

Thus, $\text{tor}(C^{\perp L}) \subseteq \text{res}(C)^\perp \subseteq \text{res}(C^{\perp L})$. This, together with the fact that $\text{res}(C^{\perp L}) \subseteq \text{tor}(C^{\perp L})$ yield

$$\text{res}(C^{\perp L}) = \text{tor}(C^{\perp L}) = \text{res}(C)^\perp.$$

To prove (2), assume $\mathbf{u} \in \text{res}(C^{\perp_R})$. By Lemma 4.1, $a\mathbf{u} \in C^{\perp_R}$. Let $\mathbf{x} \in \text{tor}(C)$. Then, $c\mathbf{x} \in C$. By definition of C^{\perp_R} , $0 = c\mathbf{x} \cdot a\mathbf{u} = c(\mathbf{x} \cdot \mathbf{u})$. Hence, $\mathbf{x} \cdot \mathbf{u} = 0$ which implies that $\mathbf{u} \in \text{tor}(C)^\perp$. Therefore, $\text{res}(C^{\perp_R}) \subseteq \text{tor}(C)^\perp$. Now assume $\mathbf{v} \in \text{tor}(C)^\perp$. Let $\mathbf{c} \in C$. By Theorem 4.2, $\mathbf{c} = a\mathbf{r} + c\mathbf{t}$ where $\mathbf{r} \in \text{res}(C) \subseteq \text{tor}(C)$ and $\mathbf{t} \in \text{tor}(C)$. Observe that

$$\mathbf{c} \cdot a\mathbf{v} = (a\mathbf{r} + c\mathbf{t}) \cdot a\mathbf{v} = a(\mathbf{r} \cdot \mathbf{v}) + c(\mathbf{t} \cdot \mathbf{v}) = 0$$

So, $a\mathbf{v} \in C^{\perp_R}$. Since $\alpha(a\mathbf{v}) = \mathbf{v}$, $\mathbf{v} \in \text{res}(C^{\perp_R})$. Hence, $\text{tor}(C)^\perp \subseteq \text{res}(C^{\perp_R})$. Thus we obtain the equality $\text{res}(C^{\perp_R}) = \text{tor}(C)^\perp$.

To prove (3), we need to show that $\mathbb{F}_2^n \subseteq \text{tor}(C^{\perp_R})$. Let $\mathbf{u} \in \mathbb{F}_2^n$ and $\mathbf{c} \in C$. By Theorem 4.2, $\mathbf{c} = a\mathbf{x} + c\mathbf{y}$ where $\mathbf{x} \in \text{res}(C)$ and $\mathbf{y} \in \text{tor}(C)$. Observe that

$$\mathbf{c} \cdot c\mathbf{u} = (a\mathbf{x} + c\mathbf{y}) \cdot c\mathbf{u} = 0.$$

Hence, $c\mathbf{u} \in C^{\perp_R}$ and so $\mathbf{u} \in \text{tor}(C^{\perp_R})$. Therefore, we have $\mathbb{F}_2^n = \text{tor}(C^{\perp_R})$. \square

Corollary 4.6. *If C is a linear code of length n over E , then the following hold:*

- 1) $C^{\perp_L} = a \text{res}(C)^\perp \oplus c \text{res}(C)^\perp$.
- 2) $C^{\perp_R} = a \text{tor}(C)^\perp \oplus c \mathbb{F}_2^n$.

Proof. The result follows immediately from Theorems 4.2 and 4.5. \square

As a consequence of this corollary, we obtain the following two propositions.

Proposition 4.7. *A linear code C of length n over E is both left nice and right nice if and only if C is zero.*

Proof. Suppose that C is a type (k_1, k_2) code of length n over E . By Corollary 4.6,

$$|C||C^{\perp_L}| = 2^{2n+k_2} \quad \text{and} \quad |C||C^{\perp_R}| = 2^{2n+k_1}.$$

Hence, C is both left nice and right nice if and only if $k_1 = k_2 = 0$. Equivalently, C is both left nice and right nice if and only if C is zero. \square

Proposition 4.8. *If C is a nonzero linear code of length n over E , then $C^{\perp_R} \neq C^{\perp_L}$.*

Proof. Suppose that $C^{\perp_R} = C^{\perp_L}$. By Corollary 4.6, we have $\text{tor}(C)^\perp = \text{res}(C)^\perp = \mathbb{F}_2^n$ which implies that $\text{tor}(C) = \text{res}(C) = \{\mathbf{0}\}$ and so C is zero. \square

This shows that no self-dual codes nor nontrivial nice codes over E , as defined in [4], exist. This motivates us to modify the conditions of these classes of codes. Thus we define the two-sided dual of a code over E and redefine the self-duality and the nice property accordingly as follows:

Definition 4.9. Let C be a linear code of length n over E .

- The *two-sided dual* of C , denoted by C^\perp , is defined as $C^\perp = C^{\perp_L} \cap C^{\perp_R}$.
- C is *self-dual* provided that $C = C^\perp$.
- C is *nice* provided that $|C||C^\perp| = 4^n$.

Similar to Theorem 4.5, the following theorem presents properties of the residue and torsion codes of the two-sided dual of linear codes over E .

Theorem 4.10. *If C is a linear code of length n over E , then the following hold:*

- 1) $\text{res}(C^\perp) = \text{tor}(C)^\perp$.
- 2) $\text{tor}(C^\perp) = \text{res}(C)^\perp$.

Proof. By Theorem 4.5 and the fact that $\text{res}(C) \subseteq \text{tor}(C)$, it follows that

$$\begin{aligned} \text{res}(C^\perp) &= \text{res}(C^{\perp_L} \cap C^{\perp_R}) \\ &= \text{res}(C^{\perp_L}) \cap \text{res}(C^{\perp_R}) \\ &= \text{res}(C)^\perp \cap \text{tor}(C)^\perp \\ &= \text{tor}(C)^\perp. \end{aligned}$$

Also by Theorem 4.5, we have

$$\begin{aligned} \text{tor}(C^\perp) &= \text{tor}(C^{\perp_L} \cap C^{\perp_R}) \\ &= \text{tor}(C^{\perp_L}) \cap \text{tor}(C^{\perp_R}) \\ &= \text{res}(C)^\perp \cap \mathbb{F}_2^n \\ &= \text{res}(C)^\perp. \end{aligned}$$

\square

Corollary 4.11. *If C is a linear code over E , then $C^\perp = a \text{tor}(C)^\perp \oplus c \text{res}(C)^\perp$.*

Proof. By Theorems 4.2 and 4.10,

$$C^\perp = a \text{res}(C^\perp) \oplus c \text{tor}(C^\perp) = a \text{tor}(C)^\perp \oplus c \text{res}(C)^\perp.$$

\square

Corollary 4.12. *Let C be a linear code of length n over E . The following are equivalent:*

- 1) C is free.
- 2) C is left nice.
- 3) $C^\perp = C^{\perp_L}$.

Proof. By Theorem 4.2 and Corollary 4.6, we have $|C||C^{\perp_L}| = 2^{2n+k_2}$. Hence, C is free if and only if C is left nice; proving that (1) and (2) are equivalent. By Corollaries 4.6 and 4.11, $C^\perp = C^{\perp_L}$ if and only if $\text{tor}(C)^\perp = \text{res}(C)^\perp$ or equivalently C is free; hence (1) and (3) are equivalent. \square

Corollary 4.13. *Let C be a linear code of length n over E . The following are equivalent:*

- 1) $\text{res}(C) = \{\mathbf{0}\}$.
- 2) C is right nice.
- 3) $C^\perp = C^{\perp_R}$.

Proof. By Theorem 4.2 and Corollary 4.6, we have $|C||C^{\perp_R}| = 2^{2n+k_1}$. Hence, C is right nice if and only if $k_1 = 0$ or equivalently $\text{res}(C) = \{\mathbf{0}\}$; proving that (1) and (2) are equivalent. By Corollaries 4.6 and 4.11, $C^\perp = C^{\perp_R}$ if and only if $\text{res}(C)^\perp = \mathbb{F}_2^n$ or equivalently $\text{res}(C) = \{\mathbf{0}\}$; hence (1) and (3) are equivalent. \square

Corollary 4.14. *Every right nice code over E is self-orthogonal.*

Proof. Suppose that C is right nice. By Corollary 4.13, $\text{res}(C) = \{\mathbf{0}\}$. Hence, $\text{res}(C) \subseteq \text{tor}(C)^\perp$. By Corollary 4.3, C is self-orthogonal. \square

The following two identities do not hold in general for codes over other non-unital rings of order four.

Corollary 4.15. *If C is a linear code over E , then $(C^\perp)^\perp = C$.*

Proof. By Theorem 4.10 and Corollary 4.11, $(C^\perp)^\perp = a \text{tor}(C^\perp)^\perp \oplus c \text{res}(C^\perp)^\perp = a \text{res}(C) \oplus c \text{tor}(C) = C$. \square

Corollary 4.16. *If C is a linear code of length n over E , then C is nice.*

Proof. By Theorem 4.2 and Corollary 4.11,

$$|C||C^\perp| = |\text{res}(C)||\text{tor}(C)||\text{res}(C)^\perp||\text{tor}(C)^\perp| = 4^n.$$

By definition, C is nice. \square

Remark 4.17. In view of Definition 4.9, the notions of QSD codes and self-dual codes over E are equivalent. To see this, suppose that C is a QSD code of length n . Then $C \subseteq C^\perp$ and $|C| = 2^n$. By Corollary 4.16, $|C^\perp| = 4^n/2^n = 2^n = |C|$. Hence, $C = C^\perp$ and therefore C is self-dual. Conversely, if C is a self-dual code of length n , then $C = C^\perp$ and $|C| = |C^\perp|$. By Corollary 4.16, $|C|^2 = 4^n$. Hence $|C| = 2^n$ and therefore C is QSD.

To prepare for investigating the MacWilliams formula for linear codes over E , we recall from [7], [12] that the *weight enumerator* of any linear or additive code C is the polynomial $W(x, y) = \sum_{i=0}^n A_i x^{n-i} y^i$ where the sequence A_0, \dots, A_n is the *weight distribution* of C . That is, A_i is the number of codewords in C of weight i . We state the following useful theorem from [7] without proof.

Theorem 4.18. [7, Theorem 5]. *If C is an $(n, 2^k)$ additive code over \mathbb{F}_4 with weight enumerator $W(x, y)$, the weight enumerator of the trace dual code $C^{\perp T}$ is given by $2^{-k}W(x + 3y, x - y)$.*

To establish the MacWilliams formula for linear codes over E , we also need the following identity.

Theorem 4.19. *If C is a linear code of length n over E , then $\phi_E(C^\perp) = \phi_E(C)^{\perp T}$.*

Proof. Let $\phi_E(\mathbf{y}) \in \phi_E(C^\perp)$ and $\phi_E(\mathbf{x}) \in \phi_E(C)$. By Corollary 4.11 and Theorem 4.2, $\mathbf{y} = a\mathbf{u} + c\mathbf{v}$ and

$\mathbf{x} = a\mathbf{r} + c\mathbf{t}$ such that $\mathbf{u} \in \text{tor}(C)^\perp$, $\mathbf{v} \in \text{res}(C)^\perp$, $\mathbf{r} \in \text{res}(C)$, and $\mathbf{t} \in \text{tor}(C)$. Observe that

$$\begin{aligned} \langle \phi_E(\mathbf{x}), \phi_E(\mathbf{y}) \rangle_T &= \text{Tr}(\phi_E(\mathbf{x}) \cdot (\phi_E(\mathbf{y}))^2) \\ &= \text{Tr}(\phi_E(a\mathbf{r} + c\mathbf{t}) \cdot (\phi_E(a\mathbf{u} + c\mathbf{v}))^2) \\ &= \text{Tr}((\omega\mathbf{r} + \mathbf{t}) \cdot (\omega\mathbf{u} + \mathbf{v})^2) \\ &= \text{Tr}((\omega\mathbf{r} + \mathbf{t}) \cdot (\omega^2\mathbf{u} + \mathbf{v})) \\ &= \text{Tr}(\mathbf{r} \cdot \mathbf{u} + \omega\mathbf{r} \cdot \mathbf{v} + \omega^2\mathbf{t} \cdot \mathbf{u} + \mathbf{t} \cdot \mathbf{v}) \\ &= \mathbf{r} \cdot \mathbf{v} + \mathbf{t} \cdot \mathbf{u} \\ &= 0. \end{aligned}$$

This proves that $\phi_E(C^\perp) \subseteq \phi_E(C)^{\perp T}$. Since

$$|\phi_E(C^\perp)| = |C^\perp| = 2^{2n-(2k_1+k_2)} = |\phi_E(C)^{\perp T}|,$$

it follows that $\phi_E(C^\perp) = \phi_E(C)^{\perp T}$. \square

Theorem 4.20. *If C is a linear code of type (k_1, k_2) over E with weight enumerator $W_C(x, y)$, then the weight enumerator of the dual code C^\perp is given by*

$$W_{C^\perp}(x, y) = \frac{1}{2^{(2k_1+k_2)}} W_C(x + 3y, x - y).$$

Proof. Since C is a linear code of type (k_1, k_2) over E , $\phi_E(C)$ is an $(n, 2^{2k_1+k_2})$ additive code over \mathbb{F}_4 with

$$W_C(x, y) = W_{\phi_E(C)}(x, y)$$

and

$$W_{C^\perp}(x, y) = W_{\phi_E(C^\perp)}(x, y) = W_{\phi_E(C)^{\perp T}}(x, y),$$

by Theorem 4.19. Hence, by Theorem 4.18,

$$W_{C^\perp}(x, y) = \frac{1}{2^{(2k_1+k_2)}} W_C(x + 3y, x - y). \quad \square$$

1) Self-dual codes

The following two theorems characterize (one-sided) self-dual codes over E .

Theorem 4.21. *If C is a linear code of length n over E , then the following hold:*

- 1) C is left self-dual if and only if C is free and $\text{res}(C)$ is self-dual.
- 2) C is right self-dual if and only if C is of type $(0, n)$.

Proof. We use Theorem 4.2 and Corollary 4.6 to establish the results. Observe that $C = C^{\perp L}$ if and only if $\text{res}(C) = \text{res}(C)^\perp = \text{tor}(C)$. Thus (1) holds. Now observe that $C = C^{\perp R}$ if and only if $\text{res}(C) = \text{tor}(C)^\perp$ and $\text{tor}(C) = \mathbb{F}_2^n$. Equivalently, $C = C^{\perp R}$ if and only if $\text{res}(C) = \{\mathbf{0}\}$ and $\text{tor}(C) = \mathbb{F}_2^n$. Thus (2) now follows. \square

Theorem 4.22. *A linear code C over E is self-dual if and only if $\text{res}(C) = \text{tor}(C)^\perp$.*

Proof. The result follows immediately from Theorem 4.2 and Corollary 4.11. \square

Corollary 4.23. *Let C be a linear code of length n over E . If C is either left self-dual or right self-dual, then C is self-dual.*

Proof. If C is left self-dual, then by Theorem 4.21, $\text{res}(C) = \text{res}(C)^\perp = \text{tor}(C)$. In particular, $\text{res}(C) = \text{tor}(C)^\perp$. By Theorem 4.22, C is self-dual.

If C is right self-dual, then by Theorem 4.21, $\text{res}(C) = \{0\}$ and $\text{tor}(C) = \mathbb{F}_2^n$ which imply that $\text{res}(C) = \text{tor}(C)^\perp$. By Theorem 4.22, C is self-dual. \square

The converse of Corollary 4.23 is not true in general as the following examples show.

Example 4.24. The repetition code of length 2 defined by

$$C = \{00, aa, bb, cc\}$$

is self-dual and left self-dual but not right self-dual, since

- $C^{\perp L} = C$,
- $C^{\perp R} = \{00, aa, bb, cc, ab, ba, 0c, c0\}$,
- $C^\perp = C$.

Example 4.25. The linear code defined by

$$C = \{00, 0c, c0, cc\}$$

is self-dual and right self-dual but not left self-dual, since

- $C^{\perp L} = E^2$,
- $C^{\perp R} = C$,
- $C^\perp = C$.

Example 4.26. The linear code defined by

$$C = \{000, a0a, b0b, c0c, 0c0, ccc, aca, bcb\}$$

is self-dual but neither left self-dual nor right self-dual, since

- $C^{\perp L} = \{000, a0a, 0a0, aaa, b0b, 0b0, bbb, c0c, 0c0, ccc, aba, aca, bcb, bab, cac, cbc\}$,
- $C^{\perp R} = \{000, a0a, b0b, c00, 0c0, 00c, cc0, c0c, 0cc, ccc, b0a, aca, a0b, bca, acb, bcb\}$,
- $C^\perp = C$.

Corollary 4.27. *Every left self-dual code over E is Type IV.*

Proof. Let C be a left self-dual code. By Corollary 4.23, C is self-dual and thus C is QSD by Remark 4.17. From Theorem 4.21, $\text{res}(C)$ is self-dual. Hence, $\text{res}(C)$ contains the all-one codeword. By [4, Theorem 4], C is Type IV. \square

From Theorem 4.21 it follows that for each positive integer n , the linear code $c\mathbb{F}_2^n$ is the unique right self-dual code of length n over E . To classify left self-dual codes, we need the following theorem.

Theorem 4.28. *Two free codes over E are permutation equivalent if and only if their residue codes are permutation equivalent.*

Proof. Let C and C' be two permutation equivalent codes over E . Then there is a permutation matrix P such that $C' = CP$. Since $\alpha(C') = \alpha(CP) = \alpha(C)P$, it follows

that $\text{res}(C)$ and $\text{res}(C')$ are permutation equivalent.

Conversely, suppose that C and C' are free codes over E where $\text{res}(C)$ and $\text{res}(C')$ are permutation equivalent. By Theorem 4.2 and the freeness of the codes, $C = a\text{res}(C) \oplus c\text{res}(C)$ and $C' = a\text{res}(C') \oplus c\text{res}(C')$. As $\text{res}(C)$ and $\text{res}(C')$ are permutation equivalent, there is a permutation matrix P such that $\text{res}(C') = \text{res}(C)P$. Thus, we have

$$C' = a\text{res}(C') \oplus c\text{res}(C') = a\text{res}(C)P \oplus c\text{res}(C)P = CP$$

which proves that C and C' are permutation equivalent. \square

The following example shows that Theorem 4.28 may not hold if the codes are not free.

Example 4.29. The linear codes C and C' with generator matrices

$$\begin{pmatrix} a & a & 0 \\ 0 & c & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & 0 & a \\ 0 & c & 0 \end{pmatrix},$$

respectively, have residue codes $\text{res}(C) = \{000, 110\}$ and $\text{res}(C') = \{000, 101\}$ which are permutation equivalent. However, C and C' are not permutation equivalent as they have weight distributions $[\langle 0, 1 \rangle, \langle 1, 2 \rangle, \langle 2, 5 \rangle]$ and $[\langle 0, 1 \rangle, \langle 1, 1 \rangle, \langle 2, 3 \rangle, \langle 3, 3 \rangle]$, respectively. Note that C and C' are not free.

As all left-self dual codes over E are necessarily free codes by Theorem 4.21, the following corollary is a special case of Theorem 4.28.

Corollary 4.30. *Two left self-dual codes over E are permutation equivalent if and only if their residue codes are permutation equivalent.*

Similar to the case of self-dual codes over I , using the classification of self-dual binary codes along with Theorem 4.21, the classification of all left self-dual codes over E of the same lengths is immediate.

To conclude this subsection, we note that the image of any self-dual or one-sided self-dual code over E under the map ϕ_E is an additive trace self-dual code over \mathbb{F}_4 .

Corollary 4.31. *If C is a self-dual code over E , then $\phi_E(C)$ is trace self-dual.*

Proof. The result follows immediately from Theorem 4.19. \square

Corollary 4.32. *If C is a left self-dual code over E , then $\phi_E(C)$ is trace self-dual.*

Proof. By Theorem 4.21, C is free. By Corollary 4.12, $C^\perp = C^{\perp L}$ and thus C is self-dual. By Corollary 4.31, $\phi_E(C)$ is trace self-dual. \square

Corollary 4.33. *If C is a right self-dual code of length n over E , then $\phi_E(C)$ is trace self-dual.*

Proof. By Theorem 4.21, $C = c\mathbb{F}_2^n$ and thus $\phi_E(C) = \mathbb{F}_2^n$ which is an additive trace self-dual code over \mathbb{F}_4 . \square

The converse of the preceding three corollaries is not true in general. The $(12, 2^{12}, 6)$ dodecacode D is trace self-dual [7] but $\phi_E^{-1}(D)$ is not a linear code over E [4, Example 2].

2) LCD codes

The study of LCD codes over non-unital rings first appeared in [20] where the authors investigated left LCD codes over E and defined this notion as follows:

Definition 4.34. A code C over E is *left linear with complementary dual (left LCD)* if it is left nice and $C \cap C^{\perp L} = \{0\}$.

We define LCD codes over E where $C^{\perp} = C^{\perp L} \cap C^{\perp R}$ as follows:

Definition 4.35. A code C over E is *linear with complementary dual (LCD)* if $C \cap C^{\perp} = \{0\}$.

LCD codes over E can be characterized via their residue and torsion codes as in the following theorem.

Theorem 4.36. *Let C be a linear code over E . Then the following hold:*

- 1) *If C is LCD, then $\text{res}(C)$ and $\text{tor}(C)$ are binary LCD codes.*
- 2) *If C is free and $\text{res}(C)$ is a binary LCD code, then C is LCD.*

Proof. First assume that C is an LCD code over E . By definition, $C \cap C^{\perp} = \{0\}$. By Theorem 4.2 and Corollary 4.11, it follows that $\text{res}(C) \cap \text{tor}(C)^{\perp} = \{0\}$ and $\text{tor}(C) \cap \text{res}(C)^{\perp} = \{0\}$. Suppose $\mathbf{x} \in \text{res}(C) \cap \text{res}(C)^{\perp}$. Since $\text{res}(C) \subseteq \text{tor}(C)$, $\mathbf{x} \in \text{tor}(C) \cap \text{res}(C)^{\perp}$. Hence, $\mathbf{x} = 0$ which implies that $\text{res}(C)$ is LCD. Similarly, suppose that $\mathbf{x} \in \text{tor}(C) \cap \text{tor}(C)^{\perp}$. Since $\text{tor}(C)^{\perp} \subseteq \text{res}(C)^{\perp}$, $\mathbf{x} \in \text{tor}(C) \cap \text{res}(C)^{\perp}$. Hence, $\mathbf{x} = 0$ which implies that $\text{tor}(C)$ is LCD. This proves (1).

Now assume that $\text{res}(C)$ is LCD and C is free. Then we have $\text{res}(C) \cap \text{res}(C)^{\perp} = \{0\}$ and $\text{res}(C) = \text{tor}(C)$. In particular, $\text{res}(C) \cap \text{tor}(C)^{\perp} = \{0\}$ and $\text{tor}(C) \cap \text{res}(C)^{\perp} = \{0\}$. By Theorem 4.2 and Corollary 4.11, $C \cap C^{\perp} = \{0\}$ and so C is LCD. This proves (2). \square

For free codes over E , there is no distinction between LCD and left LCD codes.

Theorem 4.37. *A linear code over E is left LCD if and only if it is LCD and free.*

Proof. Suppose that C is left LCD. By definition, C is left nice and $C \cap C^{\perp L} = \{0\}$. By Corollary 4.12, C is free. By definition of C^{\perp} , $C \cap C^{\perp} \subseteq C \cap C^{\perp L}$. This implies that C is LCD.

For the converse, suppose that C is LCD and free. By Corollary 4.12, C is left nice and $C \cap C^{\perp L} = C \cap C^{\perp} = \{0\}$. Hence C is left LCD. \square

The following simple examples illustrate Theorems 4.36 and 4.37.

Example 4.38. Let C be the linear code defined by $C = \{00, a0, b0, c0\}$. Then $C^{\perp} = C^{\perp L} = \{00, 0a, 0b, 0c\}$. Therefore, $C \cap C^{\perp} = C \cap C^{\perp L} = \{0\}$ and C is left-nice. Hence, C is LCD and left LCD. Note that the binary code $\text{res}(C) = \{00, 10\}$ is LCD and C is free as $\text{tor}(C) = \text{res}(C)$.

Example 4.39. Let C be the linear code defined by $C = \{00, a0, b0, c0, 0c, cc, bc, ac\}$. Then $C^{\perp} = \{00, 0c\}$ and $C^{\perp L} = \{00, 0a, 0b, 0c\}$. Hence, C is neither LCD nor left LCD. Note that the binary codes $\text{res}(C) = \{00, 10\}$ and $\text{tor}(C) = \mathbb{F}_2^2$ are LCD. However, C is not free.

In the next results we investigate the LCD property of the dual of LCD codes over E .

Corollary 4.40. *If C is an LCD code over E , then C^{\perp} is LCD.*

Proof. The result follows immediately from Corollary 4.15 and the definition of LCD codes. \square

Corollary 4.41. *If C is a free LCD code over E , then $C^{\perp L}$ is LCD.*

Proof. By Corollary 4.12, since C is free, C is left nice and $C^{\perp L} = C^{\perp}$. Since C is LCD, C^{\perp} is LCD by Corollary 4.40. Thus, $C^{\perp L}$ is LCD. \square

Corollary 4.42. *If C is a nonzero linear code of length n over E , then $C^{\perp R}$ is not LCD.*

Proof. By Theorem 4.5 and Corollary 4.11, we have $(C^{\perp R})^{\perp} = a \text{tor}(C^{\perp R})^{\perp} \oplus c \text{res}(C^{\perp R})^{\perp} = c \text{tor}(C) \subseteq c \mathbb{F}_2^n \subseteq C^{\perp R}$ where the last inclusion follows from Corollary 4.6. Since C is nonzero, $\text{tor}(C)$ must also be nonzero and thus $C^{\perp R} \cap (C^{\perp R})^{\perp} \neq \{0\}$. This proves that $C^{\perp R}$ is not LCD. \square

Recall that an additive code D over \mathbb{F}_4 is ACD if $D \cap D^{\perp T} = \{0\}$. The image of any LCD or left LCD code over E under the map ϕ_E is ACD.

Corollary 4.43. *If C is an LCD code over E , then $\phi_E(C)$ is ACD.*

Proof. Since C is an LCD code over E and ϕ_E is a bijective map, we have $\{0\} = \phi_E(\{0\}) = \phi_E(C \cap C^{\perp}) = \phi_E(C) \cap \phi_E(C^{\perp}) = \phi_E(C) \cap \phi_E(C)^{\perp T}$ where the last equality follows from Theorem 4.19. \square

Corollary 4.44. *If C is a left LCD code over E , then $\phi_E(C)$ is ACD.*

Proof. By Theorem 4.37, C is LCD. By Corollary 4.43, $\phi_E(C)$ is ACD. \square

The converse of Corollaries 4.43 and 4.44 are not true in general as the next example shows.

Example 4.45. Let D be the $(4, 2^4)$ additive code over \mathbb{F}_4 with generator matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \omega & \omega & 0 & \omega \\ 0 & \omega & \omega & \omega \end{pmatrix}.$$

Then D is ACD as shown in [21, Example 2]. The residue and the torsion codes of C over E where $\phi_E(C) = D$ are $\text{res}(C) = \{0000, 1101, 0111, 1010\}$ and $\text{tor}(C) = \{0000, 1100, 0011, 1111\}$. Since $\text{res}(C) \not\subseteq \text{tor}(C)$, C is not linear and thus not LCD.

We state the following theorem from [21] without proof.

Theorem 4.46. [21, Theorem 3]. *If C is an $(n, 2^k)$ ACD code over \mathbb{F}_4 , then k is even.*

Corollary 4.47. *If C is a type (k_1, k_2) LCD code over E , then k_2 is even.*

Proof. By Corollary 4.43, $\phi_E(C)$ is an $(n, 2^{2k_1+k_2})$ ACD code over \mathbb{F}_4 . By Theorem 4.46, $2k_1+k_2$ is even which yields that k_2 is even. \square

Remark 4.48. We restrict our investigation in this section on LCD and left LCD codes over E without mentioning right LCD codes since there are no such codes over E , as shown in [20, Remark 1].

V. RESULTS ON LINEAR CODES OVER H

We begin this section by summarizing facts and notions essential to our study for linear codes over H . A detailed introduction on such codes can be found in [2].

To every linear code C of length n over H , there is an additive code $\phi_H(C)$ over \mathbb{F}_4 such that ϕ_H is defined by the alphabet substitution

$$0 \rightarrow 0, a \rightarrow \omega, b \rightarrow 1, c \rightarrow \omega^2,$$

extended in the natural way to a map from C to \mathbb{F}_4^n .

There are two binary linear codes, namely C_a and C_b , of length n associated canonically with every linear code C of length n over H ;

- 1) $C_a = \alpha_b(C)$ where $\alpha_b : H \rightarrow \mathbb{F}_2$ is the map defined by $\alpha_b(0) = \alpha_b(b) = 0$ and $\alpha_b(a) = \alpha_b(c) = 1$, extended componentwise from C to \mathbb{F}_2^n ,
- 2) $C_b = \alpha_a(C)$ where $\alpha_a : H \rightarrow \mathbb{F}_2$ is the map defined by $\alpha_a(0) = \alpha_a(a) = 0$ and $\alpha_a(b) = \alpha_a(c) = 1$, extended componentwise from C to \mathbb{F}_2^n .

Any linear code C over H can be written as $C = aC_a \oplus bC_b$. The code C is self-orthogonal if and only if C_b is self-orthogonal [2, Lemma 1]. A QSD code C is *quasi Type IV* if C_a is even.

A. STRUCTURE OF LINEAR CODES

The minimum distance of a linear code over H does not exceed the minimum distance of either of its associated binary codes.

Theorem 5.1. *Let $C = aC_a \oplus bC_b$ be a linear code over H where C_a and C_b are nonzero binary codes. The minimum distance d of C is $d = \min\{d_1, d_2\}$ where d_1 and d_2 are the minimum distances of C_a and C_b , respectively.*

Proof. Since d_1 and d_2 are the minimum distances of C_a and C_b , respectively, there exist nonzero binary vectors $\mathbf{u} \in C_a$ and $\mathbf{v} \in C_b$ such that $\text{wt}(\mathbf{u}) = d_1$ and $\text{wt}(\mathbf{v}) = d_2$. Since $aC_a \subseteq C$ and $bC_b \subseteq C$, it follows that $a\mathbf{u}, b\mathbf{v} \in C$ with $\text{wt}(a\mathbf{u}) = \text{wt}(\mathbf{u}) = d_1$ and $\text{wt}(b\mathbf{v}) = \text{wt}(\mathbf{v}) = d_2$. This means that $d \leq \min\{d_1, d_2\}$.

Now we prove that $d \geq \min\{d_1, d_2\}$. Let $\mathbf{w} \in C$ such that $\text{wt}(\mathbf{w}) = d$. Then, $\mathbf{w} = a\mathbf{x} + b\mathbf{y}$ where $\mathbf{x} \in C_a$ and $\mathbf{y} \in C_b$. Since C is nonzero, we have the following three cases depending on \mathbf{x} and \mathbf{y} :

- If $\mathbf{x} = \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$, then

$$\text{wt}(\mathbf{w}) = \text{wt}(b\mathbf{y}) = \text{wt}(\mathbf{y}) \geq d_2.$$

- If $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} = \mathbf{0}$, then

$$\text{wt}(\mathbf{w}) = \text{wt}(a\mathbf{x}) = \text{wt}(\mathbf{x}) \geq d_1.$$

- If $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$, then

$$\text{wt}(\mathbf{w}) \geq \text{wt}(b\mathbf{w}) = \text{wt}(b\mathbf{y}) = \text{wt}(\mathbf{y}) \geq d_2.$$

In all cases, $d = \text{wt}(\mathbf{w}) \geq \min\{d_1, d_2\}$.

Since $d \leq \min\{d_1, d_2\}$ and $d \geq \min\{d_1, d_2\}$, it follows that $d = \min\{d_1, d_2\}$. \square

The following result shows the relationship between the permutation equivalence of two linear codes over H and that of their constituents.

Theorem 5.2. *Let $C = aC_a \oplus bC_b$ and $C' = aC'_a \oplus bC'_b$ be two linear codes over H . Then C and C' are permutation equivalent if and only if there is a permutation which sends (C_a, C_b) to (C'_a, C'_b) .*

Proof. Let C and C' be two permutation equivalent codes over H . Then there is a permutation matrix P such that $C' = CP$. Since $\alpha_a(C') = \alpha_a(CP) = \alpha_a(C)P$ and $\alpha_b(C') = \alpha_b(CP) = \alpha_b(C)P$, it follows that P sends (C_a, C_b) to (C'_a, C'_b) .

Conversely, suppose that P is a permutation matrix which sends (C_a, C_b) to (C'_a, C'_b) . Then,

$$aC'_a \oplus bC'_b = aC_a P \oplus bC_b P$$

and thus $C' = CP$, proving that C and C' are permutation equivalent. \square

B. DUALITY

To prepare for the study of self-dual and LCD codes over H , we need the following theorem.

Theorem 5.3. *If $C = aC_a \oplus bC_b$ is a linear code of length n over H , then $C^\perp = a\mathbb{F}_2^n \oplus bC_b^\perp$.*

Proof. Let $\mathbf{c} \in C^\perp$. We can write \mathbf{c} as $\mathbf{c} = \mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in \mathbb{F}_2^n$. To prove that $\mathbf{v} \in C_b^\perp$, let $\mathbf{t} \in C_b$. As $bC_b \subseteq C$, $\mathbf{b}\mathbf{t} \in C$. By definition,

$$0 = \mathbf{c} \cdot \mathbf{b}\mathbf{t} = (\mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v}) \cdot \mathbf{b}\mathbf{t} = b(\mathbf{v} \cdot \mathbf{t}).$$

Hence, $\mathbf{v} \cdot \mathbf{t} = 0$ which implies that $\mathbf{v} \in C_b^\perp$. Thus, $\mathbf{c} \in a\mathbb{F}_2^n + bC_b^\perp$. Therefore, $C^\perp \subseteq a\mathbb{F}_2^n + bC_b^\perp$. Now assume that $\mathbf{c} = \mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v} \in a\mathbb{F}_2^n + bC_b^\perp$. Let $\mathbf{w} \in C$. Then, $\mathbf{w} = \mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y}$ where $\mathbf{x} \in C_a$ and $\mathbf{y} \in C_b$. Observe that

$$\mathbf{c} \cdot \mathbf{w} = (\mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v}) \cdot (\mathbf{a}\mathbf{x} + \mathbf{b}\mathbf{y}) = b(\mathbf{v} \cdot \mathbf{y}) = 0.$$

Hence, $\mathbf{c} \in C^\perp$. Therefore, $a\mathbb{F}_2^n + bC_b^\perp \subseteq C^\perp$. This proves the equality $C^\perp = a\mathbb{F}_2^n + bC_b^\perp$. The sum is direct since $a\mathbb{F}_2^n$ and bC_b^\perp have a trivial intersection. \square

Corollary 5.4. Let $C = aC_a \oplus bC_b$ be a linear code of length n over H . Then, $(C^\perp)^\perp = C$ if and only if $C_a = \mathbb{F}_2^n$.

Proof. By Theorem 5.3, $(C^\perp)^\perp = a\mathbb{F}_2^n \oplus bC_b$. Hence, $(C^\perp)^\perp = C$ if and only if $C_a = \mathbb{F}_2^n$. \square

Corollary 5.5. Let $C = aC_a \oplus bC_b$ be a linear code of length n over H . Then, C is nice if and only if $C_a = \{\mathbf{0}\}$.

Proof. Let k_a and k_b denote the dimensions of C_a and C_b , respectively. Then we have $|C| = |C_a||C_b| = 2^{k_a+k_b}$ and $|C^\perp| = |\mathbb{F}_2^n| |C_b^\perp| = 2^{2n-k_b}$ from Theorem 5.3. Therefore, $|C||C^\perp| = 2^{2n+k_a}$. Hence, $|C||C^\perp| = 4^n$ if and only if $k_a = 0$. Equivalently, C is nice if and only if $C_a = \{\mathbf{0}\}$. \square

1) Self-dual codes

We characterize self-dual codes over H in terms of their associated binary codes.

Theorem 5.6. A linear code $C = aC_a \oplus bC_b$ of length n over H is self-dual if and only if the following two conditions are satisfied:

- 1) C_b is a self-dual binary code,
- 2) $C_a = \mathbb{F}_2^n$.

Proof. The result follows from Theorem 5.3 and the definition of self-dual codes. \square

By Theorem 5.6, self-dual codes over H exist only for even lengths and there are as many self-dual codes of length n over H as there are $\lfloor n/2 \rfloor$ binary self-dual codes.

Corollary 5.7. Let $C = aC_a \oplus bC_b$ and $C' = aC'_a \oplus bC'_b$ be two self-dual codes over H . Then C and C' are permutation equivalent if and only if C_b and C'_b are permutation equivalent.

Proof. The self-duality of C and C' imply that $C_a = C'_a = \mathbb{F}_2^n$ by Theorem 5.6. Thus, by Theorem 5.2, C and C' are permutation equivalent if and only if C_b and C'_b are permutation equivalent. \square

From the preceding results, we see that there is a one-to-one correspondence between inequivalent self-dual binary codes and inequivalent self-dual codes over H of the same

length. Similar to the case of self-dual codes over I , using the classification of self-dual binary codes along with Theorem 5.6, the classification of all self-dual codes over H of the same lengths is immediate. We remark that by Theorems 5.1 and 5.6, all self-dual codes over H have minimum distance equals 1.

The same observations on self-dual codes over I in Propositions 3.7 and 3.12 apply for self-dual codes over H as well.

Proposition 5.8. Let \mathcal{Q} be the family of all QSD codes of length n over H and let \mathcal{S} be the family of all self-dual codes of length n over H . Then $\mathcal{Q} \cap \mathcal{S} = \emptyset$.

Proof. Suppose that a linear code C over H is QSD and self-dual. Then $|C| = |C^\perp| = 2^n$ which implies that $|C||C^\perp| = 4^n$. By Corollary 5.5, $C_a = \{\mathbf{0}\}$ contradicting part (2) of Theorem 5.6. This means that a linear code over H can never simultaneously be both QSD and self-dual. \square

Proposition 5.9. If $C = aC_a \oplus bC_b$ is a self-dual code of length $2n$ over H , then $\phi_H(C)^{\perp T}$ is trace self-orthogonal of size 2^n ; in particular, $\phi_H(C)$ is not trace self-dual.

Proof. By Theorems 5.3 and 5.6, the self-duality of C implies that $|C_a| = |\mathbb{F}_2^{2n}| = 2^{2n}$ and $|C_b| = 2^n$. Then, $|\phi_H(C)| = |C| = 2^{3n}$ and $|\phi_H(C)^{\perp T}| = 2^n$. Comparing cardinalities, we see that $\phi_H(C) \neq \phi_H(C)^{\perp T}$ which shows that $\phi_H(C)$ is not trace self-dual.

We claim that $\omega C_b = \phi_H(C)^{\perp T}$. Let $\mathbf{w} \in C_b$ and let $\mathbf{x} \in \phi_H(C)$. Then, $\omega\mathbf{w} \in \omega C_b$ and there exists a codeword $\mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v}$ in C where $\mathbf{u} \in C_a$ and $\mathbf{v} \in C_b$ such that $\mathbf{x} = \phi_H(\mathbf{a}\mathbf{u} + \mathbf{b}\mathbf{v}) = \omega\mathbf{u} + \mathbf{v}$. Observe that

$$\begin{aligned} \langle \mathbf{x}, \omega\mathbf{w} \rangle_T &= \langle \omega\mathbf{u} + \mathbf{v}, \omega\mathbf{w} \rangle_T \\ &= \langle \omega\mathbf{u}, \omega\mathbf{w} \rangle_T + \langle \mathbf{v}, \omega\mathbf{w} \rangle_T \\ &= \text{Tr}(\mathbf{u} \cdot \mathbf{w}) + \text{Tr}(\omega^2 \mathbf{v} \cdot \mathbf{w}). \end{aligned}$$

Since C is self-dual, by Theorem 5.6, C_b is self-dual and therefore $\mathbf{v} \cdot \mathbf{w} = 0$ which gives $\text{Tr}(\omega^2 \mathbf{v} \cdot \mathbf{w}) = 0$. As $\mathbf{u} \cdot \mathbf{w} \in \{0, 1\}$, $\text{Tr}(\mathbf{u} \cdot \mathbf{w}) = 0$. Thus, we obtain $\langle \mathbf{x}, \omega\mathbf{w} \rangle_T = 0$ proving that $\omega\mathbf{w} \in \phi_H(C)^{\perp T}$ and consequently $\omega C_b \subseteq \phi_H(C)^{\perp T}$. The fact that $|C_b| = 2^n = |\phi_H(C)^{\perp T}|$ implies that $\omega C_b = \phi_H(C)^{\perp T}$ as claimed. Now observe that since $C_b \subseteq \mathbb{F}_2^{2n} = C_a$, we have $aC_b \subseteq aC_a \subseteq C$ and thus $\omega C_b = \phi_H(aC_b) \subseteq \phi_H(C)$. In particular, $\phi_H(C)^{\perp T} \subseteq \phi_H(C)$ which proves that $\phi_H(C)^{\perp T}$ is trace self-orthogonal. \square

2) LCD codes

We define LCD codes over H as follows:

Definition 5.10. A code C over H is linear with complementary dual (LCD) if $C \cap C^\perp = \{\mathbf{0}\}$.

The following theorem provides a characterization of LCD codes over H .

Theorem 5.11. A linear code $C = aC_a \oplus bC_b$ of length n over H is LCD if and only if C is nice and C_b is LCD.

Proof. Suppose that C is LCD. By definition, $C \cap C^\perp = \{0\}$. By Theorem 5.3, $C_a \cap \mathbb{F}_2^n = \{0\}$ and $C_b \cap C_b^\perp = \{0\}$, proving that C_a is zero and C_b is LCD. By Corollary 5.5, C is nice and C_b is LCD.

Conversely, suppose that C is nice and C_b is LCD. By Corollary 5.5, C_a is zero. Thus, $C_a \cap \mathbb{F}_2^n = \{0\}$ and $C_b \cap C_b^\perp = \{0\}$. By Theorem 5.3, $C \cap C^\perp = \{0\}$, proving that C is LCD. \square

Corollary 5.12. Let $C = aC_a \oplus bC_b$ and $C' = aC'_a \oplus bC'_b$ be two LCD codes over H . Then C and C' are permutation equivalent if and only if C_b and C'_b are permutation equivalent.

Proof. Since C and C' are LCD, by Theorem 5.11, $C_a = C'_a = \{0\}$. Thus, by Theorem 5.2, C and C' are permutation equivalent if and only if C_b and C'_b are permutation equivalent. \square

The classification of LCD codes over H reduces to that of LCD binary codes. A complete classification of binary LCD codes was done in [6] for lengths up to 13. Using this classification along with Theorem 5.11, the classification of all LCD codes over H of the same lengths is immediate.

To conclude this subsection, we note that the image of any LCD code over H under the map ϕ_H is never an ACD. However, it is an additive trace self-orthogonal code over \mathbb{F}_4 .

Proposition 5.13. If $C = aC_a \oplus bC_b$ is a nonzero LCD code of length n over H , then $\phi_H(C)$ is trace self-orthogonal; in particular, $\phi_H(C)$ is not ACD.

Proof. By Theorem 5.11 and Corollary 5.5, $C = bC_b$. Then, $\phi_H(C) = \phi_H(bC_b) = C_b$ and so $\phi_H(C)^{\perp_T} = C_b^{\perp_T}$. Observe that for any $\mathbf{x}, \mathbf{y} \in C_b$, we have

$$\langle \mathbf{x}, \mathbf{y} \rangle_T = \text{Tr}(\mathbf{x} \cdot \mathbf{y}^2) = \text{Tr}(\mathbf{x} \cdot \mathbf{y}) = 0.$$

This proves that $C_b \subseteq C_b^{\perp_T}$ and $\phi_H(C) \subseteq \phi_H(C)^{\perp_T}$. Hence, $\phi_H(C)$ is trace self-orthogonal. Since C is nonzero, $\phi_H(C) \cap \phi_H(C)^{\perp_T} \neq \{0\}$, proving that $\phi_H(C)$ is not ACD. \square

VI. RELATIONS DIAGRAMS

In this section, we summarize the relationships among different classes of codes over \mathcal{R} . These relations are illustrated via diagrams in Figures 1, 2, 3. They give a general picture of the intersections of the classes of codes over each ring. The symbol \equiv denotes an equivalence of two notions.

Also, we list in Tables 4, 5, 6 the main properties of each class of codes over a particular ring along with the connections between different classes. We gather these facts from the preceding sections as well as [2]–[5], [20].

A. CODES OVER THE RING I

In Table 4 assume that C is a linear code of length n over I . We use this table to draw the relations diagram in Figure 1.

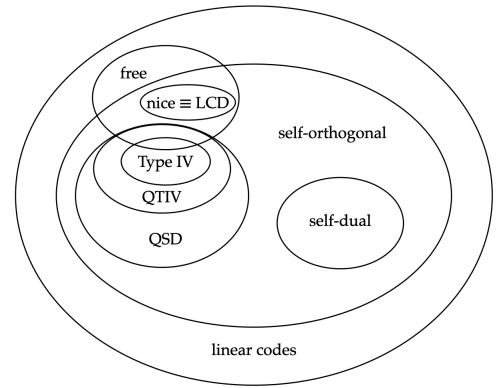


FIGURE 1. Classes of linear codes over the ring I

B. CODES OVER THE RING E

In Table 5 assume that C is a linear code of length n over E . We use this table to draw the relations diagram in Figure 2.

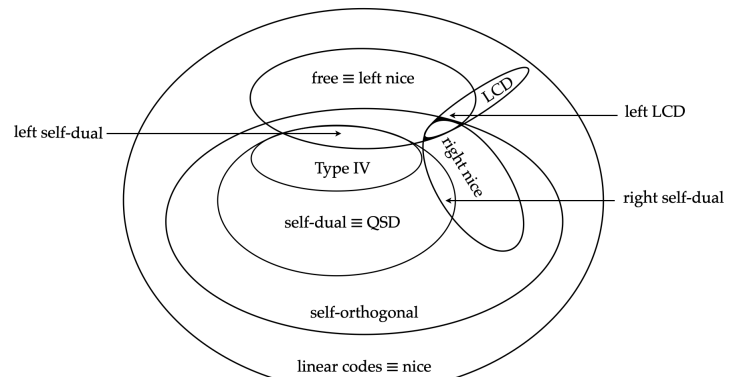


FIGURE 2. Classes of linear codes over the ring E

C. CODES OVER THE RING H

In Table 6 assume that $C = aC_a \oplus bC_b$ is a linear code of length n over H . We use this table to draw the relations diagram in Figure 3.

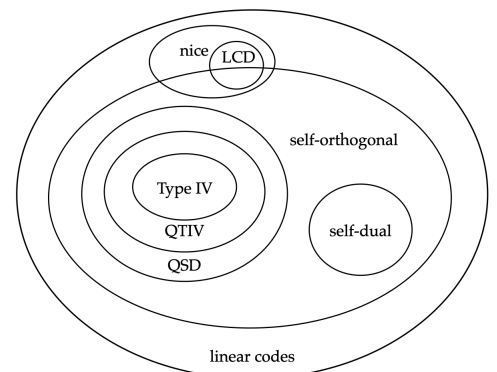


FIGURE 3. Classes of linear codes over the ring H

TABLE 4. Properties of different classes of linear codes over the ring I

	Property	Reference
(1)	C is self-orthogonal if and only if $\text{res}(C)$ is self-orthogonal	Corollary 3.2
(2)	C is nice if and only if $C = \{\mathbf{0}\}$	Proposition 3.6
(3)	C is free if and only if $\text{res}(C) = \text{tor}(C)$	[3, Definition]
(4)	C is QSD if and only if C is self-orthogonal and $ C = 2^n$	[3, Definition]
(5)	C is quasi Type IV if and only if C is QSD and $\text{tor}(C)$ is even	[3, Definition]
(6)	C is Type IV if and only if C is QSD and even	[3, Definition]
(7)	Every Type IV code is quasi Type IV	[3, Remark]
(8)	Every free QSD code is quasi Type IV	[5, Corollary 1]
(9)	C is self-dual if and only if $\text{res}(C)$ is self-dual and $\text{tor}(C) = \mathbb{F}_2^n$	Theorem 3.8
(10)	C can never be self-dual and QSD at the same time	Proposition 3.7
(11)	C can never be self-dual and free at the same time	Properties (3) and (9)
(12)	C is LCD if and only if $C = \{\mathbf{0}\}$	Proposition 3.13

TABLE 5. Properties of different classes of linear codes over the ring E

	Property	Reference
(1)	C is nice	Corollary 4.16
(2)	C is self-orthogonal if and only if $\text{res}(C) \subseteq \text{tor}(C)^\perp$	Corollary 4.3
(3)	C is self-dual if and only if $\text{res}(C) = \text{tor}(C)^\perp$	Theorem 4.22
(4)	C is QSD if and only if C is self-dual	Remark 4.17
(5)	C is Type IV if and only if C is QSD and even	[4, Definition]
(6)	C is free if and only if $\text{res}(C) = \text{tor}(C)$	[4, Definition]
(7)	C is left nice if and only if C is free	Corollary 4.12
(8)	C is right nice if and only if $\text{res}(C) = \{\mathbf{0}\}$	Corollary 4.13
(9)	C is left self-dual if and only if $\text{res}(C)$ is self dual and C is free	Theorem 4.21
(10)	C is left self-dual if and only if C is free self-dual	Properties (3), (6), (9)
(11)	Every left self-dual code is Type IV	Corollary 4.27
(12)	C is right self-dual if and only if C is of type $(0, n)$	Theorem 4.21
(13)	C is right self-dual if and only if C is right nice and self-dual	Properties (3), (8), (12)
(14)	C can never be both left self-dual and right self-dual	Proposition 4.8
(15)	C is both left nice and right nice if and only if $C = \{\mathbf{0}\}$	Proposition 4.7
(16)	Every right nice code is self-orthogonal	Corollary 4.14
(17)	C is left LCD if and only if it is left nice and $C \cap C^{\perp L} = \{\mathbf{0}\}$	[20, Definition 3]
(18)	If C is free and $\text{res}(C)$ is a binary LCD code, then C is LCD	Theorem 4.36
(19)	C is left LCD if and only if it is LCD and free	Theorem 4.37

TABLE 6. Properties of different classes of linear codes over the ring H

	Property	Reference
(1)	C is self-orthogonal if and only if C_b is self-orthogonal	[2, Lemma 1]
(2)	C is QSD if and only if C_b is self-orthogonal and $\dim(C_a) = n - \dim(C_b)$	[2, Lemma 1]
(3)	C is quasi Type IV if and only if C is QSD and C_a is even	[2, Definition]
(4)	C is Type IV if and only if C is QSD and even	[2, Definition]
(5)	Every Type IV code is quasi Type IV	[2, Remark]
(6)	C is self-dual if and only if C_b is self-dual and $C_a = \mathbb{F}_2^n$	Theorem 5.6
(7)	C is nice if and only if $C_a = \{\mathbf{0}\}$	Corollary 5.5
(8)	C can never be self-dual and QSD at the same time	Proposition 5.8
(9)	C can never be self-dual and nice at the same time	Properties (6) and (7)
(10)	C can never be QSD and nice at the same time	Properties (2) and (7)
(11)	C is LCD if and only if C is nice and C_b is LCD	Theorem 5.11

VII. CONCLUSION

In the present paper, we have aimed to lay down the theoretical foundation of the duality of codes over three non-unital rings of order four. The classes of self-orthogonal, self-dual, quasi self-dual, and LCD codes have been considered for each ring in turn. The properties of their associated binary codes, as well as that of their quaternary images have been established.

The main direction opened by this study is to extend these results to non-unital rings of higher order. In particular self-orthogonal codes over certain non-unital rings of order six have been studied in [1]. This is a concrete motivation for such an extension.

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