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Marc Briane, Juan Casado-Díaz. A divergence-curl result for measures. Application to the two-dimensional ODE's flow. 2023. hal-04088840v2

**HAL Id: hal-04088840**

**<https://hal.science/hal-04088840v2>**

Preprint submitted on 3 Nov 2023

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# A new divergence-curl result for measures. Application to the two-dimensional ODE's flow.

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November 3, 2023

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## Abstract

The paper is devoted to divergence-curl results involving a divergence free measure-valued field  $\sigma = b\nu$ , where  $\nu$  is a signed Radon measure on  $\mathbb{R}^N$  and  $b$  is a non-vanishing regular vector field in  $\mathbb{R}^N$ , and a gradient measure-valued field  $\eta = \nabla u$  on  $\mathbb{R}^N$ ,  $N \geq 2$ . On the one hand, in a non-periodic framework we prove that for any open set  $\Omega$  of  $\mathbb{R}^2$ , the orthogonality condition  $b \cdot \nabla u = 0$  in  $\Omega$  implies the equality  $\text{div}(u\sigma) = 0$  in  $\Omega$ . The key-ingredient of the proof is based on the existence of a representative in  $L_{\text{loc}}^\infty(\Omega)$  of the bounded variation function  $u$  in  $\Omega$ . This result allows us to extend in the setting of ODE's flows, the famous Franks-Misiurewicz theorem which claims that the Herman rotation set of any continuous two-dimensional flow on the torus  $\mathbb{T}^2$  is a closed line segment of a line of  $\mathbb{R}^2$  passing through  $0_{\mathbb{R}^2}$ . Moreover, this non-periodic divergence-curl result can be applied to a finite almost periodic bounded variation function  $u$  and to a finite almost periodic measure-valued field  $\sigma = b\nu$ . On the other hand, in the periodic case with dimension  $N > 2$ , assuming that  $\nu$  is absolutely continuous with respect to Lebesgue's measure on the torus  $\mathbb{T}^N$ , we prove that if the product  $b \cdot \nabla u$  is the zero measure on  $\mathbb{T}^N$ , so is the product of the  $\mathbb{T}^N$ -means  $\overline{b\nu} \cdot \overline{\nabla u}$ .

**Keywords:** Divergence-curl, measure, ODE's flow, invariant measure, rotation set

**Mathematics Subject Classification:** 34E10, 37C10, 37C40, 42B05

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# 1 Introduction

In the spring of 1974, starting from the initial integration by parts

$$\forall \sigma \in C_c^\infty(\mathbb{R}^N)^N, \forall u \in C_c^\infty(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} \sigma(x) \cdot \nabla u(x) dx = - \int_{\mathbb{R}^N} (\operatorname{div} \sigma)(x) u(x) dx,$$

and putting it in the more general context of Fourier analysis, Murat and Tartar proved the following result:

$$\begin{aligned} \left\{ \begin{array}{l} \sigma_\varepsilon \rightharpoonup \sigma \text{ weakly in } L_{\text{loc}}^2(\mathbb{R}^N)^N, \quad \operatorname{div}(\sigma_\varepsilon) \rightarrow \operatorname{div}(\sigma) \text{ strongly in } H_{\text{loc}}^{-1}(\mathbb{R}^N), \\ \eta_\varepsilon \rightharpoonup \eta \text{ weakly in } L_{\text{loc}}^2(\mathbb{R}^N)^N, \quad \operatorname{curl}(\eta_\varepsilon) \rightarrow \operatorname{curl}(\eta) \text{ strongly in } H_{\text{loc}}^{-1}(\mathbb{R}^N)^{N \times N} \end{array} \right. \\ \Downarrow \\ \sigma_\varepsilon \cdot \eta_\varepsilon \rightharpoonup \sigma \cdot \eta \text{ in } \mathcal{D}'(\mathbb{R}^N), \end{aligned} \quad (1.1)$$

called as the *divergence-curl lemma*. So, they inaugurated by result (1.1) the famous compensated compactness method [23, 28] with many fruitful applications to partial differential equations. Historically, the divergence-curl lemma was introduced as the crucial tool in H-convergence theory [24, 25] (see also [32, Chapter 7]), independently from the earlier G-convergence theory due to Spagnolo [27]. There have been many applications of the divergence-curl lemma beyond H-convergence. We cite three other applications due to Tartar: estimates of effective coefficients [29], homogenization of hyperbolic equations [30] and optimal design [31].

Since more than forty years, divergence-curl lemma (1.1) has been the subject of several refinements and various applications. We focus on the main assumptions and applications in the following divergence-curl results:

- In [4, 5] replacing for any open bounded set  $\Omega$  of  $\mathbb{R}^2$ , the  $L^2(\Omega)^2$ -boundedness of  $\sigma_\varepsilon$  by the  $L^2(\Omega)^2$ -boundedness of the sequence  $A_\varepsilon^{-1/2} \sigma_\varepsilon$  where  $A_\varepsilon$  is an equi-coercive sequence of symmetric positive matrix-valued in  $L^\infty(\Omega)^{2 \times 2}$ , but only equi-bounded in  $L^1(\Omega)^{2 \times 2}$ . This divergence-curl result allows us to prove the  $\Gamma$ -convergence compactness of the strongly local Dirichlet form  $u \mapsto \int_\Omega A_\varepsilon \nabla u \cdot \nabla u$  to a similar strongly local one, without the presence of local and nonlocal terms of the classical Beurling-Deny representation of Dirichlet forms [2].
- In [7] the sequences  $\sigma_\varepsilon$  and  $\eta_\varepsilon$  are bounded respectively in  $L^p(\Omega)^N$ ,  $L^q(\Omega)^N$  for an open set  $\Omega$  of  $\mathbb{R}^N$ , and  $\operatorname{div} \sigma_\varepsilon$ ,  $\operatorname{curl} \eta_\varepsilon$  are compact respectively in the dual spaces  $W^{-1,q'}(\Omega)$ ,  $W^{1,p'}(\Omega)^{N \times N}$ , under the condition

$$1 \leq \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{N} \quad \text{for } 1 < p, q < \infty. \quad (1.2)$$

This divergence-curl result allows us to obtain in particular a G-convergence compactness result for non-uniformly bounded monotone operators.

- In [12] the sequences  $\sigma_\varepsilon$ ,  $\eta_\varepsilon$  are bounded respectively in the conjugate spaces  $L^p(\Omega)^N$ ,  $L^{p'}(\Omega)^N$  for an open set  $\Omega$  of  $\mathbb{R}^N$  and for  $1 < p < \infty$ , and  $\operatorname{div} \sigma_\varepsilon$ ,  $\operatorname{curl} \eta_\varepsilon$  are compact sequences respectively in the dual spaces of  $W^{1,\infty}(\Omega)$ ,  $W^{1,\infty}(\Omega)^{N \times N}$  under the extra equi-integrability of  $\sigma_\varepsilon \cdot \eta_\varepsilon$  in  $L^1(\Omega)$ . This divergence-curl result allows them to prove a very weak continuity of the Jacobian determinant in dimension two [12, Corollary].

- In [6] the condition (1.2) of [7] is replaced by the weaker one

$$1 \leq \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{N-1} \quad \text{for } 1 < p, q < \infty. \quad (1.3)$$

under the extra condition that the product  $\sigma_\varepsilon \cdot \eta_\varepsilon$  converges weakly in the dual of  $W^{1,\infty}(\Omega)$ . This divergence-curl result allows us to extend in particular a result due to Brezis, Nguyen [3] on the weak continuity of the Jacobian determinant, under weaker convergences of the Jacobian matrix.

- In connection with the former divergence-curl results, we have to mention the famous result on Hardy spaces [11, Theorem II.2] due to Coifman, Lions, Meyers, Semmes who proved that if  $\sigma \in \mathcal{H}^p(\mathbb{R}^N)^N$  is divergence free and  $\eta \in \mathcal{H}^q(\mathbb{R}^N)^N$  is curl free, with

$$p, q \in (1, \infty) \quad \text{and} \quad \frac{1}{r} := \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{N}, \quad (1.4)$$

then the scalar product  $\sigma \cdot \eta$  has a sense as a distribution in the Hardy space  $\mathcal{H}^r(\mathbb{R}^N)$ . This allowed us to recover (with  $r = 1$ ) the remarkable result of Müller [22] on the regularity  $J(u) \ln J(u) \in L^1_{\text{loc}}(\mathbb{R}^N)$  of any non negative Jacobian determinant  $J(u)$  for  $u$  in  $W^{1,N}_{\text{loc}}(\mathbb{R}^N)^N$ .

In the context of the ODE's flow we also get a specific divergence-curl result involving invariant measures. More precisely, let  $b \in C^{0,1}(\mathbb{R}^N)^N$  be a  $\mathbb{Z}^N$ -periodic Lipschitz continuous vector field in  $\mathbb{R}^N$ , and consider the associated flow  $X(\cdot, x)$  for  $x \in \mathbb{R}^N$ , the solution to the ODE (see, *e.g.*, [17, Section 17.4])

$$\begin{cases} \frac{\partial X}{\partial t}(t, x) = b(X(t, x)), & t \in \mathbb{R} \\ X(0, x) = x. \end{cases} \quad (1.5)$$

A probability measure  $\mu$  on the torus  $\mathbb{T}^N := \mathbb{R}^N / \mathbb{Z}^N$  is said to be invariant for the flow  $X$  if

$$\forall t \in \mathbb{R}, \forall \psi \in C^0_{\#}(\mathbb{T}^N), \quad \int_{\mathbb{T}^N} \psi(X(t, y)) \mu(dy) = \int_{\mathbb{T}^N} \psi(y) \mu(dy), \quad (1.6)$$

or equivalently, by virtue of Liouville's theorem (see, *e.g.*, [26, Lecture 7])

$$\text{div}(b\mu) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \quad (1.7)$$

We easily deduce from (1.7) (see, *e.g.*, [9, Proposition 2.2, Remark 2.2]) the periodic divergence-curl result satisfied by any invariant probability measure  $\mu$  on  $\mathbb{T}^N$  for the flow  $X$ ,

$$\forall \nabla \psi \in C^0_{\#}(\mathbb{T}^N)^N, \quad \int_{\mathbb{T}^N} b(y) \cdot \nabla \psi(y) \mu(dy) = \left( \int_{\mathbb{T}^N} b(y) \mu(dy) \right) \cdot \left( \int_{\mathbb{T}^N} \nabla \psi(y) dy \right), \quad (1.8)$$

where  $\sigma := b\mu$  is a divergence free measure-valued field and  $\eta := \nabla \psi$  is a continuous curl free field.

More generally, in their famous article [15] Franks and Misiurewicz proved that for any two-dimensional continuous flow on the torus  $\mathbb{T}^2$  associated with a lift  $\Phi := (\Phi_t)_{t \in \mathbb{R}}$  to  $\mathbb{R}^2$ , the convex Herman rotation set [16] (see also [21, Remark 2.5]) defined by

$$\rho(\Phi) := \left\{ \int_{\mathbb{T}^2} (\Phi(1, y) - y) \mu(dy) : \mu \text{ invariant probability measure on } \mathbb{T}^2 \text{ for } \Phi \right\}, \quad (1.9)$$

has the remarkable property to be a closed line segment of a line of  $\mathbb{R}^2$  passing through  $0_{\mathbb{R}^2}$ . By virtue of [21, Theorem 2.4, 3.4] and [21, Remark 2.5] (see also [9, Appendix A]) applied in the context of a two-dimensional ODE's flow  $X$  (1.5), it turns out that the rotation set agrees with the set of the cluster points of the sequences  $(X(n, x)/n)_{n \in \mathbb{N}}$  for  $x \in \mathbb{T}^2$ . Therefore, in this framework the Franks-Misiurewicz result [15] means geometrically that the unbounded orbits  $X(\mathbb{R}, x)$  in  $\mathbb{R}^2$ ,  $x \in \mathbb{T}^2$ , have all the same asymptotic direction.

A more precise structure of Herman's rotation set is also derived in [15] depending on the presence or not of incommensurable points in  $\rho(\Phi)$ , and leading to a final conjecture. This conjecture has been widely studied (see, *e.g.*, [20, 19, 13]) with a significative recent progress in [18]. Here, restricting ourselves to the ODE's flow  $X$  (1.5) associated with a vector field  $b \in C^1_{\sharp}(\mathbb{T}^2)^2$  and in view of the properties (1.6) and (1.7), the Franks-Misiurewicz result can be regarded as the following two-dimensional divergence-curl result for measures: for any pair  $(\mu, \nu)$  of probability measures on  $\mathbb{T}^2$ , we have

$$\begin{aligned} \operatorname{div}(b\mu) &= \operatorname{div}(b\nu) = \operatorname{curl}(b\nu^\perp) = 0 \\ &\Downarrow \\ \left( \int_{\mathbb{T}^2} b(y) \cdot \mu(dy) \right) \cdot \left( \int_{\mathbb{T}^2} b(y)^\perp \cdot \nu(dy) \right) &= 0. \end{aligned} \tag{1.10}$$

In the present paper, we extend the divergence-curl result (1.10) associated with the Franks-Misiurewicz theorem [15] both to a non-periodic framework with any pair of signed bounded Radon measures on  $\mathbb{R}^2$  (see Theorem 2.4), as well as a periodic divergence-curl result in dimension  $N > 2$  provided that one of the two measures has a Lebesgue's density in  $L^1_{\sharp}(\mathbb{T}^N)$  (see Theorem 3.1).

The Section 2 of the paper deals with several two-dimensional divergence-curl results. The first result (Theorem 2.4) involves in any open set  $\Omega$  of  $\mathbb{R}^2$ , a non-vanishing vector field  $b$  in  $C^1(\Omega)^2$  (or more generally, according to Remark 2.6,  $b := r\Phi$  where  $r$  is an arbitrary real function in  $C^1(\Omega)$  and  $\Phi$  an arbitrary unit vector field in  $C^1(\Omega)^2$ ) satisfying the divergence-curl result: for any bounded measure  $\nu$  on  $\Omega$  and any bounded variation function  $u$  in  $\Omega$ ,

$$\left. \begin{aligned} \operatorname{div}(b\nu) &= 0 \quad \text{in } \mathcal{D}'(\Omega) \\ b \cdot \nabla u &= 0 \quad \text{in } \mathcal{D}'(\Omega) \end{aligned} \right\} \Rightarrow \operatorname{div}(ub\nu) = 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{1.11}$$

which can be recovered as the equality “ $\operatorname{div}(ub\nu) = b\nu \cdot \nabla u = 0$ ”. Of course, the previous equality is only formal due to the bad-defined product of measures  $\nu \nabla u$ . However, the product  $u\sigma$  in (1.11) has a distributional sense, since we show (see Lemma 2.3) that there exists a neighborhood  $V$  of any point in  $\Omega$ , a bounded variation function  $S$  in  $\mathbb{R}$  and a function  $w$  in  $C^1(V)$  such that  $u = S \circ w$  a.e. in  $V$ . As a consequence, the function  $u$  belongs to  $L^\infty_{\operatorname{loc}}(\Omega)$ .

The second result (Theorem 2.7) provides a divergence-result for a finitely almost periodic vector-valued measure  $b\nu$  and a finitely almost periodic bounded variation function  $u$ . Actually, Theorem 2.7 is not an immediate consequence of Theorem 2.4, since the assumptions of Theorem 2.7 just carry on the function  $u$  which is a finite sum of bounded variation functions  $u_k$  with periodic measure gradients, but not on each of its components  $u_k$ . Then, we need an extra result (Proposition 2.12) which is original by itself, and shows that  $u$  reads as a bounded function plus a linear one. As a by-product of Theorem 2.7, we also obtain (see Corollary 2.18) an extension of the periodic Franks-Misiurewicz result (1.10) to two signed Radon measures on two multiples of the torus  $\mathbb{T}^2$ .

In the higher-dimensional setting of Section 3, the representation of the bounded variation function  $u$  in  $\Omega$  satisfying the orthogonality condition  $b \cdot \nabla u = 0$  on  $\Omega$ , does not allow us to

get that  $u \in L_{\text{loc}}^\infty(\Omega)$ . Then, restricting ourselves to the periodic framework and replacing the divergence free measure-valued field  $\sigma$  by the vector-valued function  $\sigma := f b$  with  $f \in L^1(\mathbb{T}^N)$  and  $b \in W_{\#}^{1,\infty}(\mathbb{T}^N)^N$  (i.e.  $b$  is  $\mathbb{Z}^N$ -periodic and Lipschitz), we prove (see Theorem 3.1) the  $N$ -dimensional periodic divergence-curl result:

$$\left. \begin{array}{l} \operatorname{div}(f b) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \\ b \cdot \nabla u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \end{array} \right\} \Rightarrow \left( \int_{\mathbb{T}^N} f(y) b(y) dy \right) \cdot \left( \int_{\mathbb{T}^N} \nabla u(dy) \right) = 0. \quad (1.12)$$

However, this result has no more connection with any possible  $N$ -dimensional extension of the Franks-Misiurewicz result. Indeed, in dimension  $N > 2$  and contrary to the two-dimensional duality:  $\operatorname{div}(b \nu) = \operatorname{curl}(b \nu^\perp) = 0$ , a divergence free vector field in  $\mathbb{T}^N$  can only read as a curl of a divergence free field in  $\mathbb{T}^N$ . Actually, such an extension cannot hold already in dimension  $N = 3$ , since [10, Theorem 4.1] shows that any convex polyhedron of  $\mathbb{R}^3$  with rational vertices is realizable as a Herman's rotation set for a suitable ODE's flow (1.5).

## Notation

- $(e^1, \dots, e^N)$  denotes the canonical basis of  $\mathbb{R}^N$ , and  $0_{\mathbb{R}^N}$  denotes the null vector of  $\mathbb{R}^N$ .
- $I_N$  denotes the unit matrix of  $\mathbb{R}^{N \times N}$ ,
- $R_\perp$  denotes the  $(2 \times 2)$  rotation matrix  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . For any  $\xi \in \mathbb{R}^2$ ,  $\xi^\perp$  denotes the perpendicular vector  $R_\perp \xi$ .
- “ $\cdot$ ” denotes the scalar product and  $|\cdot|$  the euclidean norm in  $\mathbb{R}^N$ .
- “ $\times$ ” denotes the cross product of two vectors in  $\mathbb{R}^N$  defined by

$$\xi \times \eta := [\xi_i \eta_j - \xi_j \eta_i]_{1 \leq i < j \leq N} \in \mathbb{R}^{\frac{N(N-1)}{2}} \quad \text{for } \xi, \eta \in \mathbb{R}^N,$$

“ $\otimes$ ” the tensor product of two vectors in  $\mathbb{R}^N$  defined by

$$\xi \otimes \eta := [\xi_i \eta_j]_{1 \leq i < j \leq N} \in \mathbb{R}^{N \times N} \quad \text{for } \xi, \eta \in \mathbb{R}^N, \quad (1.13)$$

and “ $\odot$ ” the symmetric tensor product of two vectors in  $\mathbb{R}^N$  defined by

$$\xi \odot \eta := \frac{1}{2} (\xi \otimes \eta + \eta \otimes \xi) \in \mathbb{R}^{N \times N} \quad \text{for } \xi, \eta \in \mathbb{R}^N. \quad (1.14)$$

- $\#A$  denotes the cardinal of the set  $A$ ,  $\mathbb{1}_A$  denotes the characteristic function of the set  $A$ , and  $\bar{f}_A$  denotes the mean value  $\frac{1}{|A|} \int_A$  over the set  $A$ .
- $B(x, R)$  denotes the euclidean open ball of  $\mathbb{R}^N$  centered on  $x \in \mathbb{R}^N$  and of radius  $R > 0$ , and  $B_R$  simply denotes the ball  $B(0_{\mathbb{R}^N}, R)$ .
- $Y_N$  for  $N \geq 2$ , denotes the unit cube  $:= [0, 1]^N$  of  $\mathbb{R}^N$ .
- $\mathbb{T}^N$  for  $N \geq 2$ , denotes the  $N$ -dimensional torus  $\mathbb{R}^N / \mathbb{Z}^N$  which is identified to  $Y_N$  in  $\mathbb{R}^N$ , and  $0_{\mathbb{T}^N}$  denotes the null vector of  $\mathbb{T}^N$ .
- $dx$  or  $dy$  denotes the Lebesgue measure on  $\mathbb{R}^N$ , and  $|A|$  denotes the Lebesgue measure of any measurable set in  $\mathbb{R}^N$  or  $\mathbb{T}^N$ .

- $\mathcal{D}'(\Omega)$  denotes the space of the distributions on an open set  $\Omega$  of  $\mathbb{R}^N$ .
- The Jacobian matrix of a mapping  $\Phi \in L^1_{\text{loc}}(\mathbb{R}^N)^N$  is defined by the matrix-valued distribution  $D\Phi$  with entries

$$[D\Phi]_{i,j} := \frac{\partial \Phi_i}{\partial x_j} \in \mathcal{D}'(\mathbb{R}^N) \quad \text{for } i, j \in \{1, \dots, N\}.$$

The divergence of  $\Phi$  is defined by the distribution

$$\text{div } \Phi := \sum_{i=1}^N \frac{\partial \Phi_i}{\partial x_i} \in \mathcal{D}'(\mathbb{R}^N).$$

The curl of  $\Phi$  is defined by the skew matrix-valued distribution in  $\mathcal{D}'(\mathbb{R}^N)^{N \times N}$  with entries

$$[\text{curl } \Phi]_{i,j} := \frac{\partial \Phi_i}{\partial x_j} - \frac{\partial \Phi_j}{\partial x_i} \in \mathcal{D}'(\mathbb{R}^N) \quad \text{for } i, j \in \{1, \dots, N\}.$$

- The abbreviation “a.e.” for almost anywhere, will be used throughout the paper. The simple mention “a.e.” refers to the Lebesgue measure on  $\mathbb{R}^N$ .
- Let  $(\xi^1, \dots, \xi^N)$  be a basis of  $\mathbb{R}^N$ . We denote the parallelotope spanned by this basis

$$P := \left\{ \sum_{i=1}^N \theta_i \xi^i : \theta_i \in [0, 1) \text{ for } i = 1 \dots N \right\} \quad (1.15)$$

which is associated with the lattice group of  $\mathbb{R}^N$

$$\Lambda_P := \bigoplus_{i=1}^N \mathbb{Z} \xi^i, \quad (1.16)$$

and the torus defined by the quotient group

$$\Pi_P := \mathbb{R}^N / \Lambda_P. \quad (1.17)$$

For example, when  $P = Y_N$  we get that  $\Lambda_P = \mathbb{Z}^N$  and  $\Pi_P = \mathbb{T}^N$ .

- $C_c^k(\mathbb{R}^N)$ ,  $k \in \mathbb{N} \cup \{\infty\}$ , denotes the space of the real-valued functions in  $C^k(\mathbb{R}^N)$  with compact support in  $\mathbb{R}^N$ .
- $C_{\sharp}^k(\Pi_P)$ ,  $k \in \mathbb{N} \cup \{\infty\}$ , denotes the space of the real-valued functions  $f \in C^k(\mathbb{R}^N)$  which are  $\Lambda_P$ -periodic, *i.e.*

$$\forall \kappa \in \Lambda_P, \forall x \in \mathbb{R}^N, \quad f(x + \kappa) = f(x). \quad (1.18)$$

- $C_{\sharp}^{0,\alpha}(\Pi_P)$  for  $\alpha \in [0, 1]$ , denotes the set of the  $\Lambda_P$ -periodic functions which are  $\alpha$ -Hölderian if  $\alpha > 0$ , and the set of the continuous  $\Lambda_P$ -periodic functions if  $\alpha = 0$ .
- For a Borel measure  $\mu$  on the torus  $\Pi_P$ ,  $L_{\sharp}^p(\Pi_P, \mu)$ ,  $p \geq 1$ , denotes the space of the  $\mu$ -measurable functions  $f : \Pi_P \rightarrow \mathbb{C}$  such that

$$\int_{\Pi_P} |f(x)|^p \mu(dx) < \infty.$$

$L_{\sharp}^p(\Pi_P)$  denotes the space of the Lebesgue measurable functions  $f$  in  $L^p_{\text{loc}}(\mathbb{R}^N)$ , which are  $\Lambda_P$ -periodic  $dx$ -a.e. in  $\mathbb{R}^N$ .

- For a Borel measure  $\mu$  on the torus  $\Pi_P$  and for  $f \in L^1_{\sharp}(\Pi_P, \mu)$  which is  $\Lambda_P$ -periodic  $\mu$ -a.e. in  $\mathbb{R}^N$ , we denote the mass of  $f$  with respect to  $\mu$  by

$$\mu(f) := \int_{\Pi_P} f(x) \mu(dx) = \int_P f(x) \mu(dx), \quad (1.19)$$

which is simply denoted by  $\bar{f}$  when  $\mu$  is Lebesgue's measure.

The two formulations of  $\mu(f)$  will be used indifferently along the paper.

Notation (1.19) will be also used for a vector-valued function in  $L^1_{\sharp}(\Pi_P, \mu)^N$ .

- $\mathcal{M}(\Omega)$  denotes the space of the bounded Borel measures on an open set  $\Omega$  of  $\mathbb{R}^N$ .
- $\mathcal{M}_{\text{loc}}(\Omega)$  denotes the space of the Radon measures on  $\Omega$ , *i.e.* the Borel measures which are finite on any compact set of  $\Omega$ .
- $\mathcal{M}_{\sharp}(\Pi_P)$  denotes the space of the Radon measures on the torus  $\Pi_P$ , and  $\mathcal{M}_p(\Pi_P)$  denotes the space of the probability measures on  $\Pi_P$ .
- $BV(\Omega)$  denotes the space of the bounded variation functions on an open set  $\Omega$  of  $\mathbb{R}^N$ , *i.e.* the set of the functions  $u \in L^1(\Omega)$  such that the gradient distribution  $\nabla u$  is in  $\mathcal{M}(\Omega)^N$ .
- $BV_{\sharp}(\Pi_P)$  denotes the space of the functions  $f \in BV_{\text{loc}}(\mathbb{R}^N)$  (*i.e.* with bounded variation locally in  $\mathbb{R}^N$ ) which are  $\Lambda_P$ -periodic a.e. in  $\mathbb{R}^N$ , *i.e.* the set of the functions  $u \in L^{\frac{N}{N-1}}_{\sharp}(\mathbb{T}^N)$  such that  $\nabla u \in \mathcal{M}_{\sharp}(\Pi_P)^N$ .
- $o(\delta) = \delta o_{\delta}(1)$  where  $o_{\delta}(1)$  denotes a term satisfying  $\lim_{\delta \rightarrow 0} o_{\delta}(1) = 0$ .
- $O(\delta) = \delta O_{\delta}(1)$  where  $O_{\delta}(1)$  denotes a bounded term with respect to  $\delta$ .
- $c$  denotes a positive constant which may vary from line to line.

## 2 Two-dimensional results

### 2.1 Some results on bounded variation functions

**Lemma 2.1.** *Let  $\Omega$  be a non-empty open set of  $\mathbb{R}^N$ ,  $N \geq 2$ , and let  $u \in BV(\Omega)$ . Then, for any  $S \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$  with  $S' \in L^{\infty}(\mathbb{R})$ , we have*

$$S(u) := S \circ u \in BV(\Omega) \quad \text{and} \quad \|\nabla(S(u))\|_{\mathcal{M}(\Omega)} \leq \|S'\|_{L^{\infty}(\mathbb{R})} \|Du\|_{\mathcal{M}(\Omega)^N}. \quad (2.1)$$

Moreover, for any vector-valued function  $\Phi \in C^0(\Omega)^N$  satisfying

$$\Phi \cdot \nabla u = 0 \quad \text{in } \mathcal{M}(\Omega) \quad \text{or equivalently} \quad (2.2)$$

we have

$$\Phi \cdot \nabla(S(u)) = 0 \quad \text{in } \mathcal{M}(\Omega). \quad (2.3)$$

*Proof of Lemma 2.1.* For  $\rho$  a radial function in  $C_c^{\infty}(\mathbb{R}^N)$  satisfying

$$\text{supp}(\rho) \subset B(0_{\mathbb{R}^N}, 1), \quad \rho \geq 0 \quad \text{in } \mathbb{R}^N, \quad \int_{\Omega} \rho(y) dy = 1, \quad (2.4)$$

and for  $\varepsilon > 0$ , define the rescaled function

$$\rho_{\varepsilon}(x) = \frac{1}{\varepsilon^N} \rho\left(\frac{x}{\varepsilon}\right) \quad \text{for } x \in \mathbb{R}^N, \quad (2.5)$$



and  $u_\varepsilon := \rho_\varepsilon * u$ . Taking into account that the sequence  $S(u_\varepsilon)$  converges to  $S(u)$  in  $L^1_{\text{loc}}(\Omega)$ ,

$$\nabla(S(u_\varepsilon)) = S'(u_\varepsilon) (\rho_\varepsilon * \nabla u) \quad \text{in } \{x \in \Omega : \text{dist}(x, \partial\Omega) > \varepsilon\},$$

and

$$|\nabla u_\varepsilon| \xrightarrow{*} |\nabla u| \quad \text{in } \mathcal{M}(\Omega),$$

we get that  $S(u) \in BV(\Omega)$  and (2.1).

Let us prove (2.3). For  $\varphi \in C_c^0(\Omega)$  and  $\varepsilon$  small enough, we have by (2.2)

$$\begin{aligned} \int_{\Omega} \varphi(x) \Phi(x) \cdot \nabla(S(u_\varepsilon))(x) dx &= \int_{\Omega} \varphi(x) S'(u_\varepsilon)(x) \left( \int_{\mathbb{R}^N} \rho_\varepsilon(x-y) \Phi(x) \cdot \nabla u(dy) \right) dx \\ &= \int_{\Omega} \varphi(x) S'(u_\varepsilon)(x) \left( \int_{\mathbb{R}^N} \rho_\varepsilon(x-y) (\Phi(x) - \Phi(y)) \cdot \nabla u(dy) \right) dx. \end{aligned}$$

Therefore, setting

$$\omega_\varepsilon := \max_{|y-z| < \varepsilon} |\Phi(y) - \Phi(z)|,$$

for any open set  $\tilde{\Omega} \subset \Omega$  and  $\text{supp}(\varphi) \Subset \tilde{\Omega}$ , we have

$$\begin{aligned} \left| \int_{\Omega} \varphi(x) \Phi(x) \cdot \nabla(S(u_\varepsilon))(x) dx \right| &\leq c \omega_\varepsilon \int_{\tilde{\Omega}} \left( \int_{B(x, \varepsilon)} \rho_\varepsilon(x-y) |\nabla u|(dy) \right) dx \\ &\leq c \frac{\omega_\varepsilon}{\varepsilon^N} \int_{B(0, \varepsilon)} \left( \int_{\Omega} |\nabla u|(dx) \right) dy = c \omega_\varepsilon. \end{aligned}$$

This proves that

$$\int_{\Omega} \varphi \Phi(x) \cdot \nabla(S(u_\varepsilon))(x) dx \xrightarrow{\varepsilon \rightarrow 0} 0.$$

On the other hand, the sequence  $\nabla(S(u_\varepsilon))$  converges in the weak-\* measures sense to  $\nabla(S(u))$ . Therefore, we obtain that

$$\int_{\Omega} \varphi(x) \Phi(x) \cdot \nabla(S(u))(dx) = 0.$$

□

**Lemma 2.2.** Let  $\Theta, \Omega$  be two open sets of  $\mathbb{R}^N$ , let  $F : \Omega \rightarrow \Theta$  be a diffeomorphism of class  $C^1$ . Then, for any  $u \in BV(\Omega)$ , we have  $v := u \circ F^{-1} \in BV(\Theta)$  and

$$\forall \Phi \in C_c^0(\Theta)^N, \quad \int_{\Theta} \Phi(y) \cdot \nabla v(dy) = \int_{\Omega} \left| \det(DF(x)) \right| D(F^{-1})(F(x)) \Phi(F(x)) \cdot \nabla u(dx). \quad (2.6)$$

*Proof of Lemma 2.2.* Let  $\rho_\varepsilon$  be defined by (2.5), let  $u_\varepsilon := \rho_\varepsilon * u$  and  $v_\varepsilon := u_\varepsilon \circ F^{-1}$ . Then, the sequence  $v_\varepsilon$  converges strongly to  $v$  in  $L^1(\Theta)$ . Moreover, for any  $\Phi \in C_c^0(\Theta)^N$ , we have

$$\begin{aligned} \int_{\Theta} \Phi(y) \cdot \nabla v_\varepsilon(y) dy &= \int_{\Theta} D(F^{-1})(y) \Phi(y) \cdot \nabla u_\varepsilon(F^{-1}(y)) dy \\ &= \int_{\Omega} \left| \det(DF(x)) \right| D(F^{-1})(F(x)) \Phi(F(x)) \cdot \nabla u_\varepsilon(x) dx. \end{aligned}$$

Using that  $\nabla u_\varepsilon$  converges in the weak-\* measures sense to  $\nabla u$ , we deduce that  $\nabla v_\varepsilon$  is bounded in  $\mathcal{M}(\Omega)^N$ , so  $v$  belongs to  $BV(\Omega)$  and (2.6) holds. □

**Lemma 2.3.** *Let  $\Omega$  be an open set of  $\mathbb{R}^N$ , let  $u \in BV(\Omega)$  and let  $w \in C^1(\Omega)$ . Assume that there exists  $\mu \in \mathcal{M}(\Omega)$  such that*

$$\nabla u = \nabla w \mu \quad \text{in } \Omega. \quad (2.7)$$

*Then, for any  $x_0 \in \Omega$  with  $\nabla w(x_0) \neq 0_{\mathbb{R}^N}$ , there exist  $\varepsilon > 0$ , an interval  $(a, b) \subset \mathbb{R}$ , and a function  $S \in BV(a, b)$  such that*

$$\forall x \in B(x_0, \varepsilon), \quad w(x) \in (a, b) \quad \text{and} \quad u(x) = S(w(x)) \quad \text{a.e. } x \in B(x_0, \varepsilon).$$

*Proof of Lemma 2.3.* Let  $x_0 \in \Omega$  with  $\nabla w(x_0) \neq 0_{\mathbb{R}^N}$ . By the inverse function theorem there exist an open set  $\Theta$  of  $\mathbb{R}^N$ ,  $\varepsilon > 0$  and a  $C^1$ -diffeomorphism  $F : B(x_0, \varepsilon) \rightarrow \Theta$  such that  $F_1 = w$  in  $B(x_0, \varepsilon)$ . Then, define  $v := u \circ F^{-1}$  in  $\Theta$ . By Lemma 2.2 the function  $v$  is in  $BV(\Theta)$ , and formula (2.6) reads as for any  $\Phi \in C_c^0(\Theta)^N$ ,

$$\begin{aligned} \int_{\Theta} \Phi(y) \cdot \nabla v(dy) &= \int_{B(x_0, \varepsilon)} |\det(DF(x))| \Phi(F(x)) \cdot (D(F^{-1}))^T(F(x)) \nabla w(x) \mu(dx) \\ &= \int_{B(x_0, \varepsilon)} |\det(DF(x))| \Phi(F(x)) \cdot \nabla(w \circ F^{-1})(F(x)) \mu(dx) \\ &= \int_{B(x_0, \varepsilon)} |\det(DF(x))| \Phi_1(F(x)) \mu(dx) \quad (\Phi_1 := \Phi \cdot e^1), \end{aligned}$$

which implies that

$$\forall \Phi = (0, \Phi') \in C_c^0(\Theta)^N, \quad \int_{\Theta} (\operatorname{div} \Phi)(y) v(y) dy = 0.$$

Therefore,  $v$  is a bounded variation function depending only on the variable  $y_1$ , which allows us to conclude.  $\square$

## 2.2 A two-dimensional divergence-curl result

**Theorem 2.4.** *Let  $\Omega$  be an open set of  $\mathbb{R}^2$ , and let  $b$  be a function in  $C^1(\Omega)^2$  with  $b \neq 0_{\mathbb{R}^2}$  in  $\Omega$ . Let  $\sigma := b \nu$  with  $\nu \in \mathcal{M}(\Omega)$ , and let  $u \in BV(\Omega)$  be such that*

$$\operatorname{div} \sigma = 0 \quad \text{in } \mathcal{D}'(\Omega), \quad (2.8)$$

$$b \cdot \nabla u = 0 \quad \text{in } \mathcal{M}(\Omega). \quad (2.9)$$

*Then, the function  $u$  belongs to  $L_{\text{loc}}^\infty(\Omega)$ . Moreover, for any  $y \in \Omega$ , there exist two values  $u_\pm(y)$  satisfying*

$$\lim_{\varepsilon \rightarrow 0} \left( \frac{1}{|B(y, \varepsilon)|} \int_{B(y, \varepsilon) \cap H_\pm(y)} |u(x) - u_\pm(y)| dx \right) = 0, \quad H_\pm(y) := \{ \pm b^\perp(y) \cdot (x - y) > 0 \}, \quad (2.10)$$

*and the representative of  $u$ , still denoted by  $u$ , defined by*

$$u(y) := \frac{1}{2} (u_+(y) + u_-(y)) \quad \text{for } y \in \Omega, \quad (2.11)$$

*satisfies*

$$\operatorname{div}(u \sigma) = 0 \quad \text{in } \mathcal{D}'(\Omega). \quad (2.12)$$

**Remark 2.5.** When the vector field  $b$  is in  $C^1_{\#}(\mathbb{T}^2)^2$ , the divergence free condition (2.8) satisfied by the vector-valued measure  $\sigma := b\nu$  means by virtue of Liouville's theorem (see, *e.g.*, [8, Proposition 2.2]) that the measure  $\nu \in \mathcal{M}_{\#}(\mathbb{T}^2)$  is invariant for the flow  $X$  associated with  $b$  defined by (1.5). More generally, we can replace the vector-valued measure  $b\nu$  in  $\mathcal{M}(\Omega)^2$  by  $b\alpha\nu$  in  $\mathcal{M}(\Omega)^2$  where  $\alpha \in C^1(\Omega)$ , which is equivalent to replace the invariant measure  $\nu \in \mathcal{M}(\Omega)$  by the measure  $\alpha\nu$ , thus keeping the non-vanishing property of the vector field  $b$ .

**Remark 2.6.** Since the vector field  $b \in C^1(\Omega)^2$  does not vanish in  $\Omega$ , by virtue of the local rectification theorem (see, *e.g.*, [1, Chap. 2, § 7.1]), for any  $y \in \Omega$ , there exists an open ball  $V \subset \Omega$  centered on  $y$  and a local diffeomorphism  $\Phi \in C^2(V)^2$  such that  $\nabla\Phi b = e^1$  in  $V$ . Hence, we have  $b \cdot \nabla\Phi_2 = 0$  in  $V$  ( $\Phi_2 := \Phi \cdot e^2$ ), which implies that  $b = \alpha \nabla\Phi_2^\perp$  in  $V$  for some non-vanishing function  $\alpha \in C^1(V)$ . Hence, condition (2.9) yields the condition (2.7) in  $V$ . As a consequence of Lemma 2.3, the product of distributions  $u\sigma$  in equation (2.12) thus has a sense in  $\mathcal{M}_{\text{loc}}(\Omega)$  for a suitable representative of the function  $u \in BV(\Omega)$  (using to this end the local representation of  $u$  and a partition of unity of  $\Omega$ ). More precisely, by virtue of the local representation in Lemma 2.3 of the function  $u \in BV(\Omega)$  satisfying (2.9) in the neighborhood  $V$ , we have  $u = S_V \circ w_V$  where  $S_V$  is a  $BV$  function on an open interval of  $\mathbb{R}$  and  $w_V$  is a function in  $C^1(V)$ . This gives a sense to limits (2.10). Therefore, we will be able to use in the proof of Theorem 2.4 below, the bounded representation (2.11) of the function  $u \in BV(\Omega)$  solution to equations (2.8) and (2.9).

*Proof of Theorem 2.4.* For  $y \in \Omega$ , the characteristic method ensures (see Lemma 2.13 below for details) the existence of  $\hat{\delta} > 0$  and a unique function  $w \in C^1(\overline{B(y, \hat{\delta})})$ , such that

$$b \cdot \nabla w = 0 \text{ in } B(y, \delta) \quad \text{and} \quad w(x) = b^\perp(y) \cdot (x - y) \text{ on } B(y, \delta) \cap \{x \in \mathbb{R}^2 : b(y) \cdot (x - y) = 0\}.$$

In particular, we have  $\nabla w(y) = b^\perp(y) \neq 0$ . Since the function  $u$  satisfies (2.9), we conclude the existence of  $\delta \in (0, \hat{\delta})$  and  $\mu \in \mathcal{M}(B(y, \delta))$  such that

$$\nabla w \neq 0 \text{ and } \nabla u = \nabla w \mu \text{ in } B(y, \delta).$$

By Lemma 2.3 we can choose  $\delta$  small enough to get the existence of an interval  $(a, b) \subset \mathbb{R}$  and a function  $S \in BV(a, b)$  such that for any  $x \in B(y, \delta)$ , we have  $w(x) \in (a, b)$  and  $u(x) = S(w(x))$ . This proves in particular that  $u$  belongs to  $L^\infty(B(y, \delta))$ , and then that  $u$  belongs to  $L^\infty_{\text{loc}}(\Omega)$ . Moreover, defining

$$u_-(y) = \lim_{s \rightarrow w(y)^-} S(s), \quad u_+(y) = \lim_{s \rightarrow w(y)^+} S(s),$$

and reminding that  $\nabla w(y) = b^\perp(y)$ , condition (2.10) is thus satisfied according to [14, Theorem 3, Section 5.9].

Let us now prove that (2.12) holds. For  $\rho_\varepsilon, \varepsilon > 0$ , defined by (2.5), we define the regularized function  $\sigma_\varepsilon := b(\rho_\varepsilon * \nu)$  of  $\sigma$ . Taking  $\varphi \in C_c^\infty(\Omega)$ , we have

$$\begin{aligned} \int_{\Omega} u(x) \sigma_\varepsilon(x) \cdot \nabla \varphi(x) dx &= \int_{\Omega} u(x) \left( \int_{\Omega} \rho_\varepsilon(x - y) \nu(dy) \right) b(x) \cdot \nabla \varphi(x) dx \\ &= \int_{\Omega} \left( \int_{\Omega} \rho_\varepsilon(x - y) u(x) b(x) \cdot \nabla \varphi(x) dx \right) \nu(dy). \end{aligned} \tag{2.13}$$

Integrating by parts and reminding that  $b \cdot \nabla u = 0$ , we get that

$$\begin{aligned} & \int_{\Omega} \left( \int_{\Omega} \rho_{\varepsilon}(x-y) u(x) b(x) \cdot \nabla \varphi(x) dx \right) \nu(dy) \\ &= - \int_{\Omega} \varphi(x) u(x) \left( \int_{\Omega} b(x) \cdot \nabla \rho_{\varepsilon}(x-y) \nu(dy) \right) dx \\ & \quad - \int_{\Omega} \left( \int_{\Omega} \rho_{\varepsilon}(x-y) \varphi(x) u(x) (\operatorname{div} b)(x) dx \right) \nu(dy). \end{aligned} \quad (2.14)$$

On the one hand, by Lebesgue's theorem we have

$$\int_{\Omega} \left( \int_{\Omega} \rho_{\varepsilon}(x-y) \varphi(x) u(x) (\operatorname{div} b)(x) dx \right) \nu(dy) \xrightarrow{\varepsilon \rightarrow 0} \int_{\Omega} \varphi(y) (\operatorname{div} b)(y) u(y) \nu(dy), \quad (2.15)$$

where  $u$  is defined by (2.11). On the other hand, by (2.8) we have

$$\begin{aligned} & \int_{\Omega} \varphi(x) u(x) \left( \int_{\Omega} b(x) \cdot \nabla \rho_{\varepsilon}(x-y) \nu(dy) \right) dx \\ &= \int_{\Omega} \varphi(x) u(x) \left( \int_{\Omega} (b(x) - b(y)) \cdot \nabla \rho_{\varepsilon}(x-y) \nu(dy) \right) dx \\ &= \int_{\Omega} \left( \int_{B(y,\varepsilon)} \varphi(x) u(x) (b(x) - b(y) - \nabla b(x)(x-y)) \cdot \nabla \rho_{\varepsilon}(x-y) dx \right) \nu(dy) \\ & \quad + \int_{\Omega} \left( \int_{B(y,\varepsilon)} \varphi(x) u(x) \nabla b(x)(x-y) \cdot \nabla \rho_{\varepsilon}(x-y) dx \right) \nu(dy). \end{aligned}$$

Due to  $u \in L_{\text{loc}}^{\infty}(\Omega)$  the first term of the right-hand side converges to 0 by Lebesgue's theorem, since the first-order expansion of  $b$  around  $y \in \Omega$  implies that

$$\begin{aligned} & \left| \int_{B(y,\varepsilon)} \varphi(x) u(x) (b(x) - b(y) - \nabla b(x)(x-y)) \cdot \nabla \rho_{\varepsilon}(x-y) dx \right| \\ & \leq \frac{o(1)}{\varepsilon^2} \int_{B(y,\varepsilon)} |\nabla \rho| \left( \frac{x-y}{\varepsilon} \right) dx \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{uniformly with respect to } y \in \Omega. \end{aligned}$$

Similarly, the second term of the right-hand side reads as

$$\begin{aligned} & \int_{\Omega} \left( \int_{B(y,\varepsilon)} \varphi(x) u(x) \nabla b(x)(x-y) \cdot \nabla \rho_{\varepsilon}(x-y) dx \right) \nu(dy) \\ &= \int_{\Omega} \left( \int_{B(y,\varepsilon)} u(x) \nabla b(y)(x-y) \cdot \nabla \rho_{\varepsilon}(x-y) dx \right) \varphi(y) \nu(dy) + o(1). \end{aligned} \quad (2.16)$$

Moreover, by Lebesgue's point estimates (2.10) we have

$$\begin{aligned} & \int_{B(y,\varepsilon)} u(x) \nabla b(y)(x-y) \cdot \nabla \rho_{\varepsilon}(x-y) dx \\ &= u_{+}(y) \int_{B(y,\varepsilon) \cap H_{+}(y)} \nabla b(y)(x-y) \cdot \nabla \rho_{\varepsilon}(x-y) dx \\ & \quad + u_{-}(y) \int_{B(y,\varepsilon) \cap H_{-}(y)} \nabla b(y)(x-y) \cdot \nabla \rho_{\varepsilon}(x-y) dx + o(1). \end{aligned}$$

Then, integrating by parts and noting that

$$\int_{B(y,\varepsilon) \cap H_{\pm}(y)} \rho_{\varepsilon}(x-y) dx = \frac{1}{2} \int_{B(y,\varepsilon)} \rho_{\varepsilon}(x-y) dx = \frac{1}{2} \int_{B(0_{\mathbb{R}^2},1)} \rho(z) dz = \frac{1}{2},$$

we get that

$$\begin{aligned} & \int_{B(y,\varepsilon) \cap H_{\pm}(y)} \nabla b(y) (x-y) \cdot \nabla \rho_{\varepsilon}(x-y) dx \\ &= -(\operatorname{div} b)(y) \int_{B(y,\varepsilon) \cap H_{\pm}(y)} \rho_{\varepsilon}(x-y) dx \mp \int_{B(y,\varepsilon) \cap \partial H_{\pm}(y)} \nabla b(y) (x-y) \cdot b(y) \rho_{\varepsilon}(x-y) \ell(dx) \\ &= -\frac{1}{2} (\operatorname{div} b)(y) \mp \int_{B(y,\varepsilon) \cap \partial H_{\pm}(y)} \nabla b(y) (x-y) \cdot b(y) \rho_{\varepsilon}(x-y) \ell(dx). \end{aligned}$$

However, since the function  $(z \mapsto \nabla b(y) z \cdot b(y) \rho_{\varepsilon}(z))$  is odd in  $B(0_{\mathbb{R}^2}, \varepsilon)$ , the last integral is equal zero, which implies that

$$\forall y \in \Omega, \quad \lim_{\varepsilon \rightarrow 0} \left( \int_{B(y,\varepsilon) \cap H_{\pm}(y)} u(x) \nabla b(y) (x-y) \cdot \nabla \rho_{\varepsilon}(x-y) dx \right) = -u(y) (\operatorname{div} b)(y),$$

where the function  $u$  is pointwise defined by (2.11). Therefore, using Lebesgue's theorem in the right-hand side of (2.16) combined with expression (2.14) and convergence (2.15), we obtain that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \left( \int_{\Omega} \rho_{\varepsilon}(x-y) u(x) b(x) \cdot \nabla \varphi(x) dx \right) = 0.$$

Making  $\varepsilon$  tend to 0 in (2.13) owing to this limit we deduce the desired equation (2.12).  $\square$

## 2.3 The case of finitely almost periodic functions

The following theorem deals with a divergence-curl result involving elementary almost periodic functions (see notations (1.15), (1.16), (1.17)).

**Theorem 2.7.** *Let  $n, m \in \mathbb{N}$ . Let  $Q_k$ ,  $1 \leq k \leq n$ ,  $P_j$ ,  $1 \leq j \leq m$ , be parallelograms in  $\mathbb{R}^2$ , let  $\sigma^k$  be  $n$  vector-valued measures in  $\mathcal{M}_{\sharp}(\Pi_{Q_k})^2$ , and let  $u_j$  be  $m$  functions in  $BV_{\text{loc}}(\mathbb{R}^2)$  such that  $\nabla u_j$  belong to  $\mathcal{M}_{\sharp}(\Pi_{P_j})^2$ . Assume that there exist a vector field  $b$  in  $C^1(\mathbb{R}^2)^2$  with  $b \neq 0_{\mathbb{R}^2}$  in  $\mathbb{R}^2$ , and a measure  $\nu$  in  $\mathcal{M}_{\text{loc}}(\mathbb{R}^2)$  satisfying*

$$\begin{cases} \operatorname{div} \sigma = 0 \text{ in } \mathcal{D}'(\mathbb{R}^2) & \text{where } \sigma := b \nu = \sum_{k=1}^n \sigma^k, \\ b \cdot \nabla u = 0 \text{ in } \mathcal{D}'(\mathbb{R}^2) & \text{where } u := \sum_{j=1}^m u_j. \end{cases} \quad (2.17)$$

Then, we have the formula

$$\sum_{j=1}^m \sum_{k=1}^n \left( \int_{P_j} \nabla u_j(dx) \right) \cdot \left( \int_{Q_k} \sigma^k(dx) \right) = 0. \quad (2.18)$$

As a particular case of Theorem 2.7, we have the following partial (since we do not discuss the commensurability of the rotation vectors) extension of the celebrated Franks-Misiurewicz theorem on two-dimensional continuous torus flows [15] to ODE's flows with invariant signed measures.

**Corollary 2.8.** Let  $b \in C_{\sharp}^1(\mathbb{T}^2)^2$  be a vector field in  $\mathbb{R}^2$  such that  $b = r \Phi$  with  $r \in C_{\sharp}^1(\mathbb{T}^2)$  and  $\Phi$  a unit vector field in  $C_{\sharp}^1(\mathbb{T}^2)^2$ . Let  $P, Q$  be the two rectangles  $P := [0, p]^2$ ,  $Q := [0, q]^2$  for  $p, q \in \mathbb{N}$ , and let  $\mu, \nu$  be two torus measures  $\mu \in \mathcal{M}_{\sharp}(\Pi_P)$ ,  $\nu \in \mathcal{M}_{\sharp}(\Pi_Q)$  (see (1.15) and (1.17)). Assume that the signed measures  $\mu, \nu$  are invariant for the flow (1.5) associated with the vector field  $b$ , i.e. (recall the equivalence between (1.6) and (1.7))

$$\operatorname{div}(b\mu) = \operatorname{div}(b\nu) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \quad (2.19)$$

Then, we have the collinearity property

$$\mu(b) = \int_P b(x) \mu(dx) \parallel \int_Q b(x) \nu(dx) = \nu(b). \quad (2.20)$$

**Remark 2.9.** The Franks-Misiurewicz theorem claims that the Herman rotation set [16] (see also [21, Remark 2.5]) associated with the vector field  $b$  composed of the masses  $\mu(b)$  with respect to invariant probability measures  $\mu$  for the flow (1.5), is a closed line segment of  $\mathbb{R}^2$  supported by a line passing through  $0_{\mathbb{R}^2}$ . Extending Herman's rotation set including the signed measures in  $\mathcal{M}_{\sharp}(\mathbb{T}^2)$ , we get the subspace of  $\mathbb{R}^2$

$$\mathbf{S}(b) := \{\mu(b) : \mu \in \mathcal{M}_{\sharp}(\mathbb{T}^2) \text{ invariant for the flow (1.5)}\}. \quad (2.21)$$

Therefore, property (2.20) means that  $\mathbf{S}(b)$  is either the unit set  $\{0_{\mathbb{R}^2}\}$ , or a line of  $\mathbb{R}^2$  passing through  $0_{\mathbb{R}^2}$ .

**Remark 2.10.** Theorem 2.7 provides a first extension of the Franks-Misiurewicz theorem to the case of almost periodic functions. Similarly to Corollary 2.8, it implies the following result:

Assume that  $b$  is a vector field in  $C^1(\mathbb{R}^2)^2$ , with  $b \neq 0_{\mathbb{R}^2}$  in  $\mathbb{R}^2$ , and that  $\nu, \mu$  are two measures in  $\mathcal{M}_{\text{loc}}(\mathbb{R}^2)$  such that

$$b\nu = \sum_{k=1}^n \sigma^k, \quad b\mu = \sum_{j=1}^m \tau^j, \quad \operatorname{div}(b\nu) = \operatorname{div}(b\mu) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^2),$$

with  $\sigma^k \in \mathcal{M}_{\sharp}(\Pi_{Q_k})^2$ ,  $\tau^j \in \mathcal{M}_{\sharp}(\Pi_{P_j})^2$ ,  $Q_k, P_j$  as in Theorem 2.7. Then, we obtain the collinearity property

$$\lim_{R \rightarrow \infty} \int_{B_R} b(x) \nu(dx) = \sum_{k=1}^n \int_{Q_k} \sigma^k(dx) \parallel \sum_{j=1}^m \int_{P_j} \tau^j(dx) = \lim_{R \rightarrow \infty} \int_{B_R} b(x) \mu(dx).$$

**Remark 2.11.** Proposition A.1 in the appendix below provides both necessary conditions and sufficient conditions for a regular vector field  $b$  in  $\mathbb{R}^2$  to satisfy the regular polar decomposition  $b = r \Phi$  where  $\Phi$  is a unit vector field.

Theorem 2.7 is far to be an easy consequence of Theorem 2.4. Indeed, we cannot apply the boundedness results of Section 2.1 combined with Remark 2.6, to each function  $u_j$  from the decomposition of the finitely almost periodic function  $u$  in (2.17), i.e. to get that

$$\forall j \in \{1, \dots, n\}, \quad \left( x \mapsto u_j(x) - \int_{P_j} \nabla u_j(dx) \cdot x \right) \in L^\infty(\mathbb{R}^2).$$

We only have the assumptions (2.17) on the sum

$$v(x) := \sum_{j=1}^n \left( u_j(x) - \int_{P_j} \nabla u_j(dx) \cdot x \right) = u(x) - M(\nabla u) \cdot x \quad \text{for } x \in \mathbb{R}^2, \quad (2.22)$$

where  $M(\nabla u)$  denotes the mean value of the gradient of  $u$ . Then, we need the following result which allows us to prove that  $v \in L^\infty(\mathbb{R}^2)$ .

**Proposition 2.12.** *Let  $b$  be a vector field in  $C^1(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$  satisfying*

$$|b| \geq \alpha \quad \text{a.e. in } \Omega, \quad (2.23)$$

*and let  $u$  be a function in  $BV_{\text{loc}}(\mathbb{R}^2)$  such that  $\nabla u \cdot b = 0$  in  $\mathbb{R}^2$ . Also assume that  $\nabla u$  is a finite sum of periodic measures (with possibly different periods). Then, we have*

$$(v : x \mapsto u(x) - M(\nabla u) \cdot x) \in L^\infty(\mathbb{R}^2). \quad (2.24)$$

*Proof of Theorem 2.7.* Set

$$\psi^k := \int_{Q_k} \sigma^k(dx) \quad \text{and} \quad \phi^k := \int_{P_k} \nabla u_k(dx) \quad \text{for } k = 1 \dots n.$$

Since by (2.24) the function  $v$  belongs to  $L^\infty(\mathbb{R}^2)$  and the function  $u$  satisfies (2.17), by virtue of Remark 2.6 the product  $u\sigma = v\sigma + M(\nabla u)\sigma$  has a sense in  $\mathcal{M}_{\text{loc}}(\mathbb{R}^2)$ , and by virtue of Theorem 2.4 the vector-valued measure  $u\sigma$  is divergence free in  $\mathcal{D}'(\mathbb{R}^2)$ .

For  $R > 1$ , let  $\theta_n$  be the cut-off function in  $C_c^\infty(\mathbb{R}^2; [0, 1])$  satisfying

$$\theta_n(x) := \begin{cases} 1 & \text{if } |x| \leq R-1 \\ 0 & \text{if } |x| > R. \end{cases} \quad (2.25)$$

Since the function  $v$  and the vector-valued function  $\nabla\theta_n$  (whose support is contained in the annulus  $\{x \in \mathbb{R}^2 : R-1 \leq |x| \leq R\}$ ) are bounded in  $\mathbb{R}^2$  and the measures  $\sigma^k$  of (2.17) are  $\Lambda_{Q_k}$ -periodic (recall (1.15) and (1.16)) for  $k = 1 \dots n$ , we have ( $B_R$  denotes the ball  $B(0_{\mathbb{R}^2}, R)$ )

$$\left| \int_{B_R} v(x) \nabla\theta_n(x) \cdot \sigma(dx) \right| \leq \frac{c}{R} \xrightarrow{R \rightarrow \infty} 0. \quad (2.26)$$

On the other hand, since the vector-valued measure  $u\sigma$  and  $\sigma$  are divergence free, integrating by parts we get that

$$\begin{aligned} & \int_{B_R} v(x) \nabla\theta_n(x) \cdot \sigma(dx) \\ &= \int_{B_R} u(x) \nabla\theta_n(x) \cdot \sigma(dx) - \sum_{j=1}^n \left( \int_{B_R} (\phi^j \cdot x) \nabla\theta_n(x) \cdot \sigma(dx) \right) \\ &= \sum_{j=1}^n \phi^j \cdot \left( \int_{B_R} \theta_n(x) \sigma(dx) \right) = \sum_{j=1}^n \sum_{k=1}^n \phi^j \cdot \left( \int_{B_R} \theta_n(x) \sigma^k(dx) \right). \end{aligned} \quad (2.27)$$

Moreover, due to the property of the cut-off function  $\theta_n$  and the  $\Lambda_{\Pi_{Q_k}}$ -periodicity of the measures  $\sigma^k$  for  $k = 1 \dots n$ , we have

$$\begin{aligned} & \int_{B_R} v(x) \nabla\theta_n(x) \cdot \sigma(dx) = \sum_{j=1}^n \sum_{k=1}^n \phi^j \cdot \left( \int_{B_R} \theta_n(x) \sigma^k(dx) \right) \\ &= \sum_{j=1}^n \sum_{k=1}^n \phi^j \cdot \left( \int_{B_R} \sigma^k(dx) \right) + o(1) \xrightarrow{R \rightarrow \infty} \sum_{j=1}^n \sum_{k=1}^n \phi^j \cdot \psi^k. \end{aligned} \quad (2.28)$$

Finally, this combined with estimate (2.26) yields the desired formula (2.18).  $\square$

*Proof of Corollary 2.8.* Since the vector-valued measure  $b\mu$  is divergence free in  $\mathcal{D}'(\mathbb{R}^2)$ , there exists a function  $u$  in  $BV_{\text{loc}}(\mathbb{R}^2)$  such that

$$b\mu = \nabla u^\perp \in \mathcal{M}_\#(\Pi_P)^2.$$

Hence, the vector-valued measures

$$\sigma := (r\nu)\Phi \in \mathcal{M}_\#(\Pi_Q)^2 \quad \text{and} \quad \nabla u \in \mathcal{M}_\#(\Pi_P)^2$$

satisfy the two conditions of (2.17) with  $k = 1$ . Therefore, applying Theorem 2.7 with the unit vector field  $\Phi$  rather than  $b$ , formula (2.18) is reduced to

$$\begin{aligned} 0 &= \left( \int_Q \Phi(x) (r\nu)(dx) \right) \cdot \left( \int_P \nabla u(dx) \right) \\ &= \left( \int_Q b(x) \nu(dx) \right) \cdot \left( - \int_P b(x) \mu(dx) \right)^\perp = \frac{\mu(b) \cdot \nu(b)^\perp}{|P| |Q|}, \end{aligned}$$

which leads us to the collinearity property (2.20).  $\square$

## 2.4 Proof of Proposition 2.12

The proof of Proposition 2.12 is based on the two following results.

**Lemma 2.13.** *Let  $b$  a vector field in  $C^1(\mathbb{R}^2) \cap W^{1,\infty}(\mathbb{R}^2)$  satisfying (2.23). Then, there exists  $r > 0$  such that for any  $x_0 \in \mathbb{R}^2$  and any  $h \in C^1(B(x_0, r))$ , the problem*

$$\begin{cases} \nabla u(x) \cdot b(x) = 0 & \text{in } B(x_0, r) \\ u(x) = h(x) & \text{on } \{x \in \mathbb{R}^2 : b(x_0) \cdot (x - x_0) = 0\}, \end{cases} \quad (2.29)$$

*has a unique solution  $u$  in  $B(x_0, r)$ . Moreover, the function  $u$  belongs to  $C^1(B(x_0, r))$  provided that  $r$  is chosen small enough.*

**Lemma 2.14.** *Take  $b$  and  $r$  as in Lemma 2.13. Then, there exists a constant  $C > 0$  such that*

$$\begin{cases} \forall x \in B(x_0, r/2), \forall u \in BV(B(x_0, r)) \text{ such that } \nabla u \cdot b = 0 \text{ in } B(x_0, r), \\ |u(x) - u(x_0)| \leq C \int_{B(x_0, r)} |\nabla u|(dx). \end{cases} \quad (2.30)$$

*Proof of Proposition 2.12.* Take  $r$  as in Lemma 2.13 and  $\rho \in C_c^\infty(\mathbb{R}^2)$  such that

$$\text{supp}(\rho) \subset B_{r/2} \quad \text{and} \quad \int_{\mathbb{R}^2} \rho dx = 1. \quad (2.31)$$

Noting that  $M(\nabla u) = M(\nabla(\rho * u))$ , we have the decomposition

$$u(x) - M(\nabla u) \cdot x = u(x) - (\rho * u)(x) + (\rho * u)(x) - M(\nabla(\rho * u)) \cdot x \quad \text{for } x \in \mathbb{R}^2. \quad (2.32)$$

Due to the multiple periodicity of  $\nabla u$  we have

$$\sup_{x_0 \in \mathbb{R}^2} \int_{B(x_0, r)} |\nabla u|(dx) < \infty.$$



which by virtue of Lemma 2.14 (taking  $x_0 := x - y$ ) implies that

$$\begin{aligned} \forall x \in \mathbb{R}^2, \quad |u(x) - (\rho * u)(x)| &\leq \int_{B_{r/2}} |u(x) - u(x - y)| \rho(y) dy \\ &\leq C \sup_{x_0 \in \mathbb{R}^2} \int_{B(x_0, r)} |\nabla u|(dx) < \infty, \end{aligned}$$

which implies that  $u - \rho * u$  is in  $L^\infty(\mathbb{R}^2)$ . Moreover, since  $\nabla(\rho * u) = \rho * \nabla u$  is a finite sum of smooth periodic functions, so is the function  $(x \mapsto (\rho * u)(x) - M(\nabla(\rho * u)) \cdot x)$ . Therefore, it is also in  $L^\infty(\mathbb{R}^2)$ , which by (2.32) implies (2.24).  $\square$

*Proof of Lemma 2.13.* Consider the flow  $X$  associated with the vector field  $b$  by (1.5). Using the characteristics method, it is enough to prove the existence of  $\delta > 0$  such that

$$\forall x \in B(x_0, r), \exists! t_x \in (-\delta, \delta), \quad (X(t_x, x) - x_0) \cdot b(x_0) = 0. \quad (2.33)$$

Indeed, by the uniqueness of the time  $t_x$  and the semi-group property satisfied by the flow  $X$ , we have for any  $x \in B(x_0, r)$ ,

$$\forall s \in \mathbb{R} \text{ with } \begin{cases} X(s, x) \in B(x_0, r) \\ t_x - s \in (-\delta, \delta), \end{cases} \quad t_{X(s, x)} = t_x - s, \quad (2.34)$$

Hence, defining  $u(x) := h(X(t_x, x))$ , it follows that

$$u(X(s, x)) = h(X(t_{X(s, x)}, X(s, x))) = h(X(t_x - s, X(s, x))) = h(X(t_x, x)) = u(x). \quad (2.35)$$

Taking the derivative at  $s = 0$  we get the orthogonality condition of (2.29). Moreover, due to the uniqueness property (2.33) we obtain that

$$b(x_0) \cdot (x - x_0) = b(x_0) \cdot (X(0, x) - x_0) = 0 \Rightarrow t_x = 0 \Rightarrow u(x) = h(X(0, x)) = h(x),$$

which implies the second condition of (2.29). Conversely, if the function  $u$  satisfies (2.29), by (2.33) we get immediately that  $u(x) = h(X(t_x, x))$ , which implies the uniqueness of  $u$ . Therefore, the solution  $u$  to problem (2.29) is given by

$$\forall x \in B(x_0, r), \quad u(x) = h(X(t_x, x)). \quad (2.36)$$

Moreover, we have

$$\forall t \in (-\delta, \delta), \forall x \in B(x_0, r), \quad \frac{\partial}{\partial t} [(X(t, x) - x_0) \cdot b(x_0)] = b(X(t, x)) \cdot b(x_0) > 0,$$

provided that  $r, \delta$  are small enough. Therefore, by virtue of the implicit function theorem we get that the function  $(x \mapsto t_x)$  defined by (2.33) belongs to  $C^1(B(x_0, r))$ , which by definition (2.36) implies that  $u \in C^1(B(x_0, r))$ .

Now, it remains to prove property (2.33). Define

$$K := \|b\|_{L^\infty(\mathbb{R}^2)} \quad \text{and} \quad L := \|\nabla b\|_{L^\infty(\mathbb{R}^2)^2}, \quad (2.37)$$

We have

$$\begin{aligned} |X(t, x) - x - t b(x)| &= \left| \int_0^t (\partial_t X(s, x) - b(x)) ds \right| \leq \int_0^t |b(X(s, x)) - b(x)| ds \\ &\leq \int_0^t |b(X(s, x)) - b(x + s b(x))| ds + \int_0^t |b(x + s b(x)) - b(x)| dt \\ &\leq L \int_0^t |X(s, x) - x - s b(x)| ds + \frac{L}{2} t^2 |b(x)|, \end{aligned}$$

which by Gronwall's inequality (see, *e.g.*, [17, Section 17.3]) implies that

$$|X(t, x) - x - t b(x)| \leq \frac{KL}{2} t^2 e^{Lt}. \quad (2.38)$$

Using inequality (2.38) in

$$\begin{aligned} (X(t, x) - x_0) \cdot b(x_0) &= t |b(x_0)|^2 + (x - x_0 + t b(x) - t b(x_0)) \cdot b(x_0) \\ &\quad + (X(t, x) - x - t b(x)) \cdot b(x_0), \end{aligned}$$

we get that

$$|(X(t, x) - x_0) \cdot b(x_0) - t |b(x_0)|^2| \leq K(1 + L|t|) |x - x_0| + \frac{KL}{2} t^2 e^{Lt}.$$

This combined with (2.23) and (2.38) yields

$$\begin{aligned} \frac{d}{dt} [(X(t, x) - x_0) \cdot b(x_0)] &= b(X(t, x)) \cdot b(x_0) \geq |b(x_0)|^2 - KL |X(t, x) - x_0| \\ &\geq \alpha^2 - KL |X(t, x) - x - t b(x)| - KL |x - x_0| - K^2 L |t| \\ &\geq \alpha^2 - \frac{K^2 L^2}{2} t^2 e^{Lt} - KL |x - x_0| - K^2 L |t|. \end{aligned}$$

From this inequality we deduce the existence of two positive constants  $r, \delta$  which only depend on the constants  $\alpha, K, L$ , such that

$$\forall x \in B(x_0, r), \quad \begin{cases} (X(\delta, x) - x_0) \cdot b(x_0) > 0 & \text{and} & (X(-\delta, x) - x_0) \cdot b(x_0) < 0, \\ \forall t \in [-\delta, \delta], & \frac{d}{dt} [(X(t, x) - x_0) \cdot b(x_0)] > 0. \end{cases}$$

Therefore, property (2.33) holds true for any  $x \in B(x_0, r)$ .  $\square$

*Proof of Lemma 2.14.* Let  $x_0 \in \mathbb{R}^2$  and let  $u \in BV(B(x_0, r))$ , and define the function  $w$  as the solution to

$$\begin{cases} \nabla w(x) \cdot b(x) = 0 & \text{in } B(x_0, r) \\ w(x) = b(x)^\perp \cdot (x - x_0) & \text{on } \{x \in \mathbb{R}^2 : b(x_0) \cdot (x - x_0) = 0\}. \end{cases}$$

According to Remark 2.6 there exist  $\delta > 0$  and  $S \in BV(-\delta, \delta)$  such that  $u = S \circ w$ . For such a function  $S$ , there exists a sequence  $(S_n)$  in  $W^{1,1}(-\delta, \delta)$  which converges to  $S$  in  $L^1(-\delta, \delta)$ , such that  $(S'_n)$  converges weakly  $*$  to  $S'$  in  $\mathcal{M}(-\delta, \delta)$ , and  $\|S'_n\|_{L^1(-\delta, \delta)}$  converges to  $\|S'\|_{\mathcal{M}(-\delta, \delta)}$ . Then, it is enough to prove the result for  $S \in W^{1,1}(-\delta, \delta)$ .

By the coarea formula (see, *e.g.*, [14, Theorem 2, Section 3.4.3]) we have

$$\begin{aligned} \int_{B(x_0, r)} |\nabla u(x)| dx &= \int_{B(x_0, r)} |S'(w(x))| |\nabla w(x)| dx \\ &= \int_{-\delta}^{\delta} |S'(s)| \mathcal{H}_1(\{x \in B(x_0, r) : w(x) = s\}) ds, \end{aligned} \quad (2.39)$$

where  $\mathcal{H}_1$  denotes the one-dimensional Hausdorff measure. Let  $x \in B(x_0, r/2)$ . Then, by (2.38) we have for any  $t \in \mathbb{R}$  with  $|t|$  small enough,

$$|X(t, x) - x_0| \leq |x - x_0| + K \left( |t| + \frac{L}{2} t^2 e^{Lt} \right) \leq \frac{r}{2} + K \left( |t| + \frac{L}{2} t^2 e^{Lt} \right) < r. \quad (2.40)$$

where the constants  $K, L$  are given by (2.37). Noting that the equality  $\nabla w(x) \cdot b(x) = 0$  implies that  $w(x) = w(X(t, x))$  for any  $t \in \mathbb{R}$  with  $X(t, x) \in B(x_0, r)$ , we deduce from estimate (2.40) the existence of a constant  $c > 0$  independent of  $x_0$  (as  $r$  is), satisfying

$$\forall x \in B(x_0, r/2), \quad \mathcal{H}_1(\{y \in B(x_0, r) : w(y) = w(x)\}) \geq c. \quad (2.41)$$

Note that in (2.41) the positive constants  $r, c$  and the function  $w$  are independent of  $x_0$ , while the positive constant  $\delta$  and the function  $S$  do depend on  $x_0$ . Therefore, extending the Borel measure  $S'$  by 0 outside the compact set  $w(B(x_0, r/2))$  and denoting

$$I := \{s \in \mathbb{R} : \exists x \in B(x_0, r/2) \text{ such that } w(x) = s\}$$

which is an open interval, we deduce from (2.39) and (2.41) that

$$c \int_I |S'(s)| ds \leq \int_I |S'(s)| \mathcal{H}_1(\{x \in B(x_0, r) : w(x) = s\}) ds \leq \int_{B(x_0, r)} |\nabla u(x)| dx.$$

Finally, to prove (2.30) we just use that for any  $x \in B(x_0, r/2)$ ,

$$|u(x) - u(x_0)| = |S(w(x)) - S(0)| \leq \left| \int_0^{w(x)} |S'(t)| dt \right| \leq \int_I |S'(t)| dt \leq \frac{1}{c} \int_{B(x_0, r)} |\nabla u(x)| dx.$$

□

### 3 An extension to dimension $N \geq 2$

In the following we extend the divergence-curl result of Theorem 2.4 to any dimension  $N \geq 2$ , but restricting ourselves to a  $\mathbb{Z}^N$ -periodic function  $f$  in  $L^1_\#(\mathbb{T}^N)^N$  rather than a measure  $\nu$  in  $\mathcal{M}_\#(\mathbb{T}^N)^N$ .

**Theorem 3.1.** *Let  $b \in W^{1,\infty}_\#(\mathbb{T}^N)^N$ , let  $f \in L^1_\#(\mathbb{T}^N)$  and let  $u \in BV_{\text{loc}}(\mathbb{R}^N)$  with  $N \geq 2$ . Assume that*

$$\operatorname{div} \sigma = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \quad \text{where } \sigma := f b, \quad (3.1)$$

$$u_\# : \left( y \mapsto u(y) - \int_{\mathbb{T}^N} \nabla u(dy) \right) \in BV_\#(\mathbb{T}^N) \quad \text{and} \quad f u_\# \in L^1(\mathbb{T}^N), \quad (3.2)$$

$$b \cdot \nabla u = 0 \quad \text{in } \mathcal{M}_\#(\mathbb{T}^N). \quad (3.3)$$

Then, we have the formula

$$\left( \int_{\mathbb{T}^N} \sigma(y) dy \right) \cdot \left( \int_{\mathbb{T}^N} \nabla u(dy) \right) = 0. \quad (3.4)$$

*Proof of Theorem 3.1.* For a function  $\rho \in C_c^\infty(\mathbb{R})^N$  satisfying (2.31) in  $\mathbb{R}^N$  and for  $\varepsilon > 0$ , we define  $\rho_\varepsilon$  by (2.5) and the  $\mathbb{Z}^N$ -periodic function  $\sigma_\varepsilon$  by

$$\sigma_\varepsilon = (\rho_\varepsilon * f) b. \quad (3.5)$$

Note that using successively Fubini's theorem, the change of variables  $z = (x - y)/\varepsilon$  and the continuity of the translation operator in  $L^1_\#(\mathbb{T}^N)$  combined with estimate (3.2), we get that

$$\|\sigma_\varepsilon u_\#\|_{L^1(\mathbb{T}^N)} \leq \|b\|_{L^\infty(\mathbb{T}^N)} \int_{\mathbb{R}^N} \rho(z) \left( \int_{\mathbb{T}^N} f(x - \varepsilon z) u_\#(x) dx \right) dz \leq c < \infty. \quad (3.6)$$

*First step:* Strong convergences of  $\sigma_\varepsilon$  and  $\operatorname{div} \sigma_\varepsilon$  in  $L^1_\#(\mathbb{T}^N)$ .

First of all, we have for a.e.  $x \in \mathbb{R}^N$ ,

$$(\operatorname{div} \sigma_\varepsilon)(x) = (\operatorname{div} b)(x) \int_{\mathbb{R}^N} \rho_\varepsilon(x-y) f(y) dy + b(x) \cdot \int_{\mathbb{R}^N} \nabla \rho_\varepsilon(x-y) f(y) dy. \quad (3.7)$$

For the first term on the right-hand side we clearly have

$$(\operatorname{div} b)(x) \int_{\mathbb{R}^N} \rho_\varepsilon(x-y) f(y) dy \xrightarrow{\varepsilon \rightarrow 0} \operatorname{div} b(x) f(x) \quad \text{in } L^1_\#(\mathbb{T}^N). \quad (3.8)$$

For the second one, the divergence free condition (3.1) implies that

$$b(x) \cdot \int_{\mathbb{R}^N} \nabla \rho_\varepsilon(x-y) f(y) dy = \int_{\mathbb{R}^N} (b(x) - b(y)) \cdot \nabla \rho_\varepsilon(x-y) f(y) dy. \quad (3.9)$$

Using that  $b$  is Lipschitz and (2.31), (2.5), equality (3.9) shows in particular the existence of a constant  $C > 0$  independent of  $\varepsilon$ , such that

$$\left| b(x) \cdot \int_{\mathbb{R}^N} \nabla \rho_\varepsilon(x-y) f(y) dy \right| \leq \frac{C}{\varepsilon^N} \int_{B(x,\varepsilon)} |f(y)| dy \quad \text{a.e. } x \in \mathbb{R}^N. \quad (3.10)$$

On the other hand, since  $b$  is Lipschitz, it is derivable a.e. in  $\mathbb{R}^N$ . Then, taking a Lebesgue's point  $x \in \mathbb{R}^N$  for  $f$  such that  $b$  is derivable at  $x$ , we get that

$$\begin{aligned} \int_{\mathbb{R}^N} (b(x) - b(y)) \cdot \nabla \rho_\varepsilon(x-y) f(y) dy &= \int_{\mathbb{R}^N} (b(x) - b(y) + \nabla b(x)(y-x)) \cdot \nabla \rho_\varepsilon(x-y) f(y) dy \\ &- \int_{\mathbb{R}^N} \nabla b(x)(y-x) \cdot \nabla \rho_\varepsilon(x-y) (f(y) - f(x)) dy + \left( \int_{\mathbb{R}^N} \nabla b(x)(x-y) \cdot \nabla \rho_\varepsilon(x-y) dy \right) f(x), \end{aligned}$$

where the two first terms on the right-hand side tend to zero, while in the third one the change of variables  $z = (x-y)/\varepsilon$  gives

$$\begin{aligned} \int_{\mathbb{R}^N} \nabla b(x)(x-y) \cdot \nabla \rho_\varepsilon(x-y) dy &= \int_{B_1} \nabla b(x)z \cdot \nabla \rho(z) dz \\ &= - \int_{B_1} \operatorname{div}_z(\nabla b(x)z) \rho(z) dz = -(\operatorname{div} b)(x). \end{aligned}$$

We have just proved that

$$b(x) \cdot \int_{\mathbb{R}^N} \nabla \rho_\varepsilon(x-y) f(y) dy \xrightarrow{\varepsilon \rightarrow 0} -(\operatorname{div} b)(x) f(x) \quad \text{a.e. } x \in \mathbb{R}^N. \quad (3.11)$$

Also note that the right-hand side of (3.10) may read as a convolution of  $|f|$  by an approximate identity which thus strongly converges to  $|f|$  in  $L^1_\#(\mathbb{T}^N)$ . Therefore, by Lebesgue's theorem (3.10) and (3.11) yield

$$b(x) \cdot \int_{\mathbb{R}^N} \nabla \rho_\varepsilon(x-y) f(y) dy \xrightarrow{\varepsilon \rightarrow 0} -(\operatorname{div} b)(x) f(x) \quad \text{strongly in } L^1_\#(\mathbb{T}^N).$$

This strong convergence combined with (3.7), (3.8) and (3.11) implies that

$$\operatorname{div} \sigma_\varepsilon \rightarrow 0 \quad \text{strongly in } L^1_\#(\mathbb{T}^N). \quad (3.12)$$

Moreover, by definition (3.5) it is clear that

$$\sigma_\varepsilon \rightarrow \sigma \quad \text{strongly in } L^1_{\sharp}(\mathbb{T}^N)^N. \quad (3.13)$$

*Second step:* Proof of equality (3.4).

Consider the truncation function  $T_n$  at size  $n \in \mathbb{N}$ , defined by

$$T_n(t) := (t \wedge n) \vee (-n) \quad \text{for } t \in \mathbb{R}.$$

Then, by virtue of Lemma 2.1,  $T_n(u)$  is a sequence in  $BV_{\text{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  satisfying

$$\nabla T_n(u) \xrightarrow{*} \nabla u \quad \text{in } \mathcal{M}_{\text{loc}}(\mathbb{R}^N)^N \quad \text{and} \quad b \cdot \nabla T_n(u) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^N). \quad (3.14)$$

Set

$$\overline{\sigma_\varepsilon} := \int_{Y_N} \sigma_\varepsilon(y) dy \quad \text{and} \quad \xi := \int_{\mathbb{T}^N} \nabla u(dy). \quad (3.15)$$

Since  $\nabla T_n(u)$  is not  $\mathbb{Z}^N$ -periodic contrary to  $\nabla u$ , we make the integrations by parts in  $\mathbb{R}^N$  rather in the torus  $\mathbb{T}^N$ . Then, recalling the cut off function  $\theta_n$ ,  $n > 1$ , defined by (2.25) we have

$$\begin{aligned} \int_{B_n} \nabla \theta_n(x) \cdot \sigma_\varepsilon(x) T_n(u)(x) dx &= \int_{B_n} \nabla \theta_n(x) \cdot \sigma_\varepsilon(x) T_n(\xi \cdot x) dx \\ &+ \int_{B_n} \nabla \theta_n(x) \cdot \sigma_\varepsilon(x) (T_n(u(x)) - T_n(\xi \cdot x)) dx. \end{aligned} \quad (3.16)$$

On the one hand, integrating by parts the first integral of (3.16), using the second equality of (3.14) and the  $\mathbb{Z}^N$ -periodicity of  $\sigma_\varepsilon$  combined with the strong convergence (3.12), we get that

$$\begin{aligned} \left| \int_{B_n} \nabla \theta_n(x) \cdot \sigma_\varepsilon(x) T_n(u)(x) dx \right| &= \left| \int_{B_n} \theta_n(x) (\operatorname{div} \sigma_\varepsilon)(x) T_n(u)(x) dx \right| \\ &\leq \frac{n}{|B_n|} \int_{B_n} |\operatorname{div} \sigma_\varepsilon|(x) dx \leq c n \|\operatorname{div} \sigma_\varepsilon\|_{L^1(\mathbb{T}^N)} = n o_\varepsilon(1). \end{aligned} \quad (3.17)$$

Let  $K_n$  be the minimal covering of the set  $\{|\xi \cdot x| < n\} \cap B_n$  by the cubes  $(\kappa + Y_N)$  for  $\kappa \in \mathbb{Z}^N$ , and let  $J_n$  be the set of the vectors  $\kappa \in \mathbb{Z}^N$  such that the cubes  $(\kappa + Y_N)$  meet the boundary of the set  $\{|\xi \cdot x| < n\} \cap B_n$ . Since the volume of  $\{|\xi \cdot x| < n\} \cap B_n$  is of order  $n^N$  and the surface of its boundary is of order  $n^{N-1}$ , we have the cardinal estimates

$$\#K_n \approx n^N \quad \text{and} \quad \#J_n \approx n^{N-1}. \quad (3.18)$$

Next, integrating by parts the second integral of (3.16) and using the  $\mathbb{Z}^N$ -periodicity of  $\sigma_\varepsilon$  combined with the two last estimates of (3.18) and again the strong convergence (3.12), we get that

$$\begin{aligned} &\int_{B_n} \nabla \theta_n(x) \cdot \sigma_\varepsilon(x) T_n(\xi \cdot x) dx \\ &= - \int_{B_n} \mathbb{1}_{\{|\xi \cdot x| < n\}}(x) \theta_n(x) \sigma_\varepsilon(x) \cdot \xi dx - \int_{B_n} \theta_n(x) (\operatorname{div} \sigma_\varepsilon)(x) T_n(\xi \cdot x) dx \\ &= - \frac{1}{|B_n|} \int_{\{|\xi \cdot x| < n\} \cap B_n} \sigma_\varepsilon(x) \cdot \xi dx + O(1/n) + n o_\varepsilon(1) \\ &= O(1) \overline{\sigma_\varepsilon} \cdot \xi + O(1/n) + n o_\varepsilon(1). \end{aligned} \quad (3.19)$$

For example to get the estimate

$$\frac{1}{|B_n|} \int_{\{|\xi \cdot x| < n\} \cap B_n} \sigma_\varepsilon(x) \cdot \xi \, dx = O(1) \overline{\sigma}_\varepsilon \cdot \xi, \quad (3.20)$$

we consider a minimal covering of the intersection of the strip  $\{|\xi \cdot x| < n\}$  with the ball  $B_n$  by cubes  $\kappa + Y_N$ ,  $\kappa \in \mathbb{Z}^N$ . The volume of this intersection is of order  $n^N$  so that the number of cubes (of unit volume) of the covering is also of order  $n^N$ . Then, the  $\mathbb{Z}^N$ -periodicity of  $\sigma_\varepsilon$  implies that

$$\int_{\{|\xi \cdot x| < n\} \cap B_n} \sigma_\varepsilon(x) \, dx \approx n^N \overline{\sigma}_\varepsilon,$$

which yields the desired estimate (3.20).

Moreover, using the  $\mathbb{Z}^N$ -periodicity of  $\sigma_\varepsilon u_\#$ , the estimate (3.6) satisfied by  $\sigma_\varepsilon u_\#$  and that  $T_n$  is 1-Lipschitz and the first estimate of (3.18), we also get that

$$\begin{aligned} & \left| \int_{B_n} \nabla \theta_n(x) \cdot \sigma_\varepsilon(x) (T_n(u(x)) - T_n(\xi \cdot x)) \, dx \right| \\ & \leq \frac{1}{|B_n|} \sum_{\kappa \in J_n} \int_{Y_N} |\nabla \theta_n(y + \kappa)| |\sigma_\varepsilon(y)| |u_\#(y)| \, dy \leq c \frac{\#J_n}{|B_n|} \|\sigma_\varepsilon u_\#\|_{L^1(\mathbb{T}^N)} \leq \frac{c}{n}. \end{aligned} \quad (3.21)$$

Now, using equality (3.16) and collecting estimates (3.17), (3.19), (3.21) we deduce the existence of a constant  $c > 0$  such that

$$|\overline{\sigma}_\varepsilon \cdot \xi| \leq \frac{c}{n} + n o_\varepsilon(1). \quad (3.22)$$

Finally, passing successively to the limsup as  $\varepsilon \rightarrow 0$  for a fixed  $n > 1$ , and to  $n \rightarrow \infty$  in (3.22) and (3.15), we obtain the equality  $\overline{\sigma} \cdot \xi = 0$ , or equivalently (3.4), which concludes the proof of Theorem 3.1.  $\square$

**Acknowledgment.** The authors are very grateful for support from the FEDER project PID2020-116809GB-I00 of the *Ministerio de Ciencia e Innovación* of the Spanish Government, the project UAL2020-FQM-B2046 of the *Consejería de Transformación Económica, Industria, Conocimiento y Universidades* of the *Regional Government of Andalusia (Spain)*, and from the French *Institut de Recherche Mathématique de Rennes*, which has funded the visit in Rennes of the second author in January 2023.

**Data availability.** We do not analyse or generate any datasets, because our work proceeds within a theoretical and mathematical approach.

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# A Polar decomposition of a two-dimensional vector field

**Proposition A.1.** *Let  $\Omega$  be a non-empty open set of  $\mathbb{R}^2$ , and  $b$  be a vector field in  $C^2(\Omega)^2$  the roots of which are isolated in  $\Omega$ .*

- *Assume that  $b$  reads as  $b = r \Phi$ , where  $r \in C^2(\Omega)$  and  $\Phi$  is a unit vector field in  $C^2(\Omega)^2$ . Then, any root  $x_0 \in \Omega$  of  $b$  satisfies*

$$\det(\nabla b(x_0)) = 0, \quad (\text{A.1})$$

$$\forall \xi \in \mathbb{R}^2, \forall K \Subset \Omega, \quad \# \{x \in K : \xi \cdot b(x) = 0\} < \infty \Rightarrow (\nabla b(x_0))^T \xi = 0_{\mathbb{R}^2}, \quad (\text{A.2})$$

$$\nabla b(x_0) = 0_{\mathbb{R}^2 \times 2} \Rightarrow \nabla^2 b_1(x_0) \text{ and } \nabla^2 b_2(x_0) \text{ are linearly dependent}, \quad (\text{A.3})$$

where  $\nabla^2 b_i$  denotes the Hessian matrix of  $b_i$ .

- *Conversely, assume that  $b \in C^5(\Omega)^2$  and that for any root  $x_0 \in \Omega$  of  $b$ , the three following conditions hold:*

$$\begin{cases} \nabla b(x_0) = 0_{\mathbb{R}^2 \times 2}, \\ \exists \alpha \in \mathbb{R}, \exists i \in \{1, 2\}, \quad \nabla^2 b_{3-i}(x_0) = \alpha \nabla^2 b_i(x_0), \quad \pm \nabla^2 b_i(x_0) \text{ positive definite}, \\ \nabla^3 b(x_0) = 0_{\mathbb{R}^2 \times 2 \times 2}. \end{cases} \quad (\text{A.4})$$

*Then, for any compact set  $K$  of  $\Omega$ ,  $b$  admits the polar decomposition  $b = |b| \Phi$ , where  $\Phi$  is a unit vector field in  $C^1(K)^2$ .*

*Proof of Proposition A.1.* For the sake of simplicity we deal with the case  $x_0 = 0_{\mathbb{R}^2}$ .

*Proof of (A.1).* First of all, due to  $b(0_{\mathbb{R}^2}) = 0_{\mathbb{R}^2}$  and  $|\Phi(0_{\mathbb{R}^2})| = 1$ , we have  $r(0_{\mathbb{R}^2}) = 0$ . Hence, it follows that (see (1.13))

$$\nabla b(0_{\mathbb{R}^2}) = \nabla(r \Phi)(0_{\mathbb{R}^2}) = r(0_{\mathbb{R}^2}) \nabla \Phi(0_{\mathbb{R}^2}) + \Phi(0_{\mathbb{R}^2}) \otimes \nabla r(0_{\mathbb{R}^2}) = \Phi(0_{\mathbb{R}^2}) \otimes \nabla r(0_{\mathbb{R}^2}) \quad (\text{A.5})$$

which is a matrix of rank  $\leq 1$ . Therefore, equality (A.1) holds.

*Proof of (A.2).* We reason by contraposition. Let  $\xi \in \mathbb{R}^2$ , and assume that

$$0_{\mathbb{R}^2} \neq (\nabla b(0_{\mathbb{R}^2}))^T \xi = \xi_1 \nabla b_1(0_{\mathbb{R}^2}) + \xi_2 \nabla b_2(0_{\mathbb{R}^2}).$$

Then, without loss of generality we may suppose that

$$0_{\mathbb{R}^2} \neq \xi_1 \frac{\partial b_1}{\partial x_1}(0_{\mathbb{R}^2}) + \xi_2 \frac{\partial b_2}{\partial x_1}(0_{\mathbb{R}^2}) = \frac{\partial(\xi \cdot b)}{\partial x_1}(0_{\mathbb{R}^2}).$$

Hence, from the implicit function theorem we deduce the existence of a real function  $f \in C^1(\mathbb{R})$  such for any  $x \in \Omega$  in some neighborhood of  $0_{\mathbb{R}^2}$ ,

$$(\xi \cdot b)(x) = 0 \Leftrightarrow x_1 = f(x_2),$$

which implies that the function  $\xi \cdot b$  has an infinite number of roots in some compact set  $K$  of  $\Omega$ . Therefore, this proves condition (A.2).

*Proof of (A.3).* Let  $i \in \{1, 2\}$ . We have (see (1.14))

$$\begin{aligned} \nabla^2 b_i(0_{\mathbb{R}^2}) &= \nabla(\Phi_i \nabla r)(0_{\mathbb{R}^2}) + \nabla(r \nabla \Phi_i)(0_{\mathbb{R}^2}) \\ &= \Phi_i(0_{\mathbb{R}^2}) \nabla^2 r(0_{\mathbb{R}^2}) + 2 \nabla \Phi_i(0_{\mathbb{R}^2}) \odot \nabla r(0_{\mathbb{R}^2}) + r(0_{\mathbb{R}^2}) \nabla^2 \Phi_i(0_{\mathbb{R}^2}) \\ &= \Phi_i(0_{\mathbb{R}^2}) \nabla^2 r(0_{\mathbb{R}^2}) + 2 \nabla \Phi_i(0_{\mathbb{R}^2}) \odot \nabla r(0_{\mathbb{R}^2}). \end{aligned} \quad (\text{A.6})$$

If  $\nabla b(0_{\mathbb{R}^2}) = 0_{\mathbb{R}^2 \times 2}$ , then from (A.5) we deduce that  $\nabla r(0_{\mathbb{R}^2}) = 0_{\mathbb{R}^2}$ . This combined with (A.6) implies immediately the desired result (A.3).

*Proof of the sufficient condition.* Assume that  $x_0 = 0_{\mathbb{R}^2}$  is a root of  $b$  satisfying (A.4). Without loss of generality we have for some  $\varepsilon = \pm 1$  and  $\alpha \in \mathbb{R}$ ,

$$\varepsilon \nabla^2 b_1(0_{\mathbb{R}^2}) > 0 \quad \text{and} \quad \nabla^2 b_2(0_{\mathbb{R}^2}) = \alpha \nabla^2 b_1(0_{\mathbb{R}^2}).$$

Then, the fourth-order Taylor's formula with integral remainder at  $0_{\mathbb{R}^2}$  applied to the functions  $b_1, b_2$ , yields for any  $x \in \mathbb{R}^2$  close to  $0_{\mathbb{R}^2}$ ,

$$b(x) = \varepsilon q(x) + R(x) \quad \text{where} \quad \begin{cases} q_j(x) := \frac{\varepsilon}{2} \nabla^2 b_j(0_{\mathbb{R}^2}) x \cdot x \\ R_j(x) := \frac{1}{6} \int_0^1 (1-t)^3 \nabla^4 b_j(tx)(x, x, x, x) dt, \end{cases} \quad j = 1, 2.$$

Hence, it follows that for any  $x \neq 0_{\mathbb{R}^2}$  close to  $0_{\mathbb{R}^2}$ ,

$$b(x) = \varepsilon \underbrace{q_1(x)}_{>0} (e_1 + \alpha e_2 + \varepsilon R(x)/q_1(x)) = |b(x)| \Phi(x),$$

where

$$|b(x)| = q_1(x) \left| e_1 + \alpha e_2 + \varepsilon R(x)/q_1(x) \right|, \quad \Phi(x) := \varepsilon \frac{e_1 + \alpha e_2 + \varepsilon R(x)/q_1(x)}{\left| e_1 + \alpha e_2 + \varepsilon R(x)/q_1(x) \right|}. \quad (\text{A.7})$$

Since  $R$  is a vector field in  $C^1(\Omega)^2$  of order  $|x|^4$  around  $0_{\mathbb{R}^2}$  and  $q_1$  is quadratic, the quotient  $R/q_1$  defines a continuous vector-valued function around  $0_{\mathbb{R}^2}$  which satisfies

$$(R/q_1)(0_{\mathbb{R}^2}) = 0_{\mathbb{R}^2} \quad \text{and} \quad \lim_{x \rightarrow 0_{\mathbb{R}^2}} \nabla(R/q_1)(x) = 0_{\mathbb{R}^2 \times 2},$$

so that  $R/q_1$  actually can be extended to a  $C^1$ -regular vector-valued function around  $0_{\mathbb{R}^2}$ . Therefore, the vector field  $\Phi$  is a  $C^1$ -regular unit vector field around  $0_{\mathbb{R}^2}$  such that  $b = |b| \Phi$ . Moreover, the expression of  $|b|$  involving  $R/q_1$  in (A.7) clearly shows that  $|b|$  is also  $C^1$ -regular in the neighborhood of  $0_{\mathbb{R}^2}$ .

Now, consider the  $n \geq 1$  roots  $x^1, \dots, x^n$  of  $b$  in a given compact set  $K$  of  $\Omega$ . By the conditions (A.4) the previous construction at point  $x_0 = 0_{\mathbb{R}^2}$  allows us to define  $n$  vector fields  $\Phi^1, \dots, \Phi^n$  of type  $\Phi$  in (A.7), which are  $C^1$ -regular respectively in two-by-two disjoint closed neighborhoods  $V_1, \dots, V_n$  of the respective points  $x^1, \dots, x^n$  in  $K$ , and such that

$$\varepsilon_k \nabla^2 b_{i_k}(x^k) > 0 \quad \text{and} \quad \nabla^2 b_{3-i_k}(x^k) = \alpha_k \nabla^2 b_{i_k}(x^k), \quad \text{for some } \varepsilon_k = \pm 1, \quad i_k = 1, 2 \text{ and } \alpha_k \in \mathbb{R}.$$

Therefore, it is easy to check that the vector field  $\Phi$  defined by

$$\Phi := \begin{cases} \Phi^k & \text{in } V_k \text{ for } k = 1, \dots, n \\ \frac{b}{|b|} & \text{in } K \setminus (V_1 \cup \dots \cup V_n), \end{cases}$$

belongs to  $C^1(K)^2$  and satisfies the desired decomposition  $b = |b| \Phi$ . □