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# Non-decreasing martingale couplings

Benjamin Jourdain\*      Kexin Shao†

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## Abstract

For many examples of couples  $(\mu, \nu)$  of probability measures on the real line in the convex order, we observe numerically that the Hobson and Neuberger martingale coupling, which maximizes for  $\rho = 1$  the integral of  $|y - x|^\rho$  with respect to any martingale coupling between  $\mu$  and  $\nu$ , is still a maximizer for  $\rho \in (0, 2)$  and a minimizer for  $\rho > 2$ . We investigate the theoretical validity of this numerical observation and give rather restrictive sufficient conditions for the property to hold. We also exhibit couples  $(\mu, \nu)$  such that it does not hold. The support of the Hobson and Neuberger coupling is known to satisfy some monotonicity property which we call non-decreasing. We check that the non-decreasing property is preserved for maximizers when  $\rho \in (0, 1]$ . In general, there exist distinct non-decreasing martingale couplings, and we find some decomposition of  $\nu$  which is in one-to-one correspondence with martingale couplings non-decreasing in a generalized sense.

## 1 Introduction

In this paper, for  $\mu, \nu$  in the set  $\mathcal{P}_1(\mathbb{R})$  of probability measures on the real line with a finite first order moment, we are interested in martingale couplings between  $\mu$  and  $\nu$  that attain  $\overline{\mathcal{M}}_\rho(\mu, \nu)$  and  $\underline{\mathcal{M}}_\rho(\mu, \nu)$  defined for  $\rho > 0$  by

$$\overline{\mathcal{M}}_\rho(\mu, \nu) = \sup_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^\rho \pi(dx, dy) \text{ and } \underline{\mathcal{M}}_\rho(\mu, \nu) = \inf_{\pi \in \Pi_M(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^\rho \pi(dx, dy).$$

Here  $\Pi_M(\mu, \nu)$  denotes the subset of

$$\Pi(\mu, \nu) = \left\{ \pi \text{ probability measure on } \mathbb{R} \times \mathbb{R} \mid \int_{y \in \mathbb{R}} \pi(dx, dy) = \mu(dx), \int_{x \in \mathbb{R}} \pi(dx, dy) = \nu(dy) \right\}$$

consisting in martingale couplings :

$$\Pi_M(\mu, \nu) = \left\{ \pi(dx, dy) = \mu(dx) \pi_x(dy) \in \Pi(\mu, \nu) \mid \mu(dx)\text{-a.e.}, \int_{y \in \mathbb{R}} y \pi_x(dy) = x \right\}.$$

By Strassen's theorem [14],

$$\Pi_M(\mu, \nu) \neq \emptyset \iff \mu \leq_{cx} \nu,$$

where  $\mu \leq_{cx} \nu$  means that  $\mu$  is smaller than  $\nu$  in the convex order :

$$\forall \varphi : \mathbb{R} \rightarrow \mathbb{R} \text{ convex}, \int_{\mathbb{R}} \varphi(x) \mu(dx) \leq \int_{\mathbb{R}} \varphi(y) \nu(dy).$$

When  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$  are such that  $\mu \leq_{cx} \nu$ , then  $\underline{\mathcal{M}}_\rho(\mu, \nu) \leq \overline{\mathcal{M}}_\rho(\mu, \nu) < +\infty$  for  $\rho \in (0, 1]$ . For  $\rho > 1$ , the finiteness remains true when  $\mu, \nu$  belong to the set  $\mathcal{P}_\rho(\mathbb{R})$  of probability measures on the real

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line with a finite moment of order  $\rho$ . For  $\rho \geq 1$ , a martingale Wasserstein inequality is investigated in [9]: it is proved that there exists a finite constant  $K_\rho$  such that for  $\mu, \nu \in \mathcal{P}_\rho(\mathbb{R})$  with  $\mu \leq_{cx} \nu$ ,

$$\underline{\mathcal{M}}_\rho^\rho(\mu, \nu) \leq K_\rho \left( \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^\rho \pi(dx, dy) \right)^{\frac{1}{\rho}} \min_{z \in \mathbb{R}} \left( \int_{\mathbb{R}} |z - y|^\rho \nu(dy) \right)^{\frac{\rho-1}{\rho}}.$$

The couplings attaining  $\overline{\mathcal{M}}_1(\mu, \nu)$  and  $\underline{\mathcal{M}}_1(\mu, \nu)$  were first investigated in the literature. Motivated by the robust pricing and hedging of forward start straddle options, Hobson and Neuberger state in Theorem 8.2 [7] that for the cost function  $|x - y|$ , there exists a maximizing martingale coupling with the form

$$\pi^{\text{HN}} = \int_0^1 \left( \frac{r(u) - q(u)}{r(u) - p(u)} \delta_{(q(u), p(u))} + \frac{q(u) - p(u)}{r(u) - p(u)} \delta_{(q(u), r(u))} \right) du$$

with  $p, q, r$  non-decreasing on  $(0, 1)$  and such that  $p \leq q \leq r$  (by convention the integrand is equal to  $\delta_{(q(u), q(u))}$  when  $p(u) = q(u) = r(u)$ ). This provides an example of a martingale coupling non-decreasing in the sense of Definition 2.3 that we give below. The study of such non-decreasing martingale couplings is one of main contributions of the present paper. The necessary optimality criterion given by Beiglböck and Juillet in Lemma 1.11 [3] ensures that any maximizing martingale coupling is non-decreasing. When  $\mu$  does not weight points, we have the equality

$$\pi^{\text{HN}} = \int_{\mathbb{R}} \left( \frac{g(x) - x}{g(x) - f(x)} \delta_{(x, f(x))} + \frac{x - f(x)}{g(x) - f(x)} \delta_{(x, g(x))} \right) \mu(dx)$$

for non-decreasing functions  $f$  and  $g$  such that  $\forall x \in \mathbb{R}, f(x) \leq x \leq g(x)$  and, according to Theorem 7.3 [3], uniqueness of maximizing martingale couplings holds. Under the dispersion assumption that there exists a finite interval  $I$  such that  $(\nu - \mu)^+(I) = 0$  and  $(\mu - \nu)^+(I) = (\mu - \nu)^+(\mathbb{R})$ , Hobson and Klimmek state in Theorem [6] that for the cost function  $|x - y|$ , there is a minimizing martingale coupling of the form

$$\pi^{\text{HK}} = \int_{\mathbb{R}} \delta_{(z, z)} \mu \wedge \nu(dz) + (1 - \mu \wedge \nu(\mathbb{R})) \int_0^1 \left( \frac{r(u) - q(u)}{r(u) - p(u)} \delta_{(q(u), p(u))} + \frac{q(u) - p(u)}{r(u) - p(u)} \delta_{(q(u), r(u))} \right) du$$

for a non-decreasing function  $q$  and non-increasing functions  $p, r$  such that  $p \leq q \leq r$ . When  $\mu \neq \nu$  the integral over  $u$  provides an example of a coupling in  $\Pi_M \left( \frac{(\mu - \nu)^+}{1 - \mu \wedge \nu(\mathbb{R})}, \frac{(\nu - \mu)^+}{1 - \mu \wedge \nu(\mathbb{R})} \right)$  non-increasing in the sense of Definition 2.3 below. Without the dispersion assumption, according to Theorem 7.4 [3], there is a unique minimizing coupling when  $\mu$  does not weight points and this coupling writes

$$\int_{\mathbb{R}} \delta_{(z, z)} \mu \wedge \nu(dz) + \int_{\mathbb{R}} \left( \frac{g(x) - x}{g(x) - f(x)} \delta_{(x, f(x))} + \frac{x - f(x)}{g(x) - f(x)} \delta_{(x, g(x))} \right) (\mu - \nu)^+(dx)$$

with  $f$  and  $g$  such that  $\forall x \in \mathbb{R}, f(x) \leq x \leq g(x)$ .

For many choices of  $\mu, \nu \in \mathcal{P}_{\rho \vee 1}(\mathbb{R})$  such that  $\mu \leq_{cx} \nu$ , when solving the linear programming problem for the cost function  $|x - y|^\rho$  obtained by approximating  $\mu$  and  $\nu$  by finitely supported probability measures still in the convex order, it turns out that the coupling  $\pi^{\text{HN}}$  maximizing when  $\rho = 1$  still maximizes when  $\rho \in (0, 2)$  and minimizes when  $\rho > 2$ : see the numerical results in Section 2.1. It is not surprising that 2 appears as a threshold for the power  $\rho$  since the martingale property ensures that when  $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$ ,

$$\forall \pi \in \Pi_M(\mu, \nu), \int_{\mathbb{R}^2} |x - y|^2 \pi(dx, dy) = \int_{\mathbb{R}} y^2 \nu(dy) - \int_{\mathbb{R}} x^2 \mu(dx) = \underline{\mathcal{M}}_2^2(\mu, \nu) = \overline{\mathcal{M}}_2^2(\mu, \nu).$$

With the complete lattice structure of  $\{\eta \in \mathcal{P}_1(\mathbb{R}) : \int_{\mathbb{R}} z \eta(dz) = \int_{\mathbb{R}} y^2 \nu(dy) - \int_{\mathbb{R}} x^2 \mu(dx)\}$  for the convex order stated in [10], we deduce in Proposition 2.1 below that  $\{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \nu)\}$  where  $\text{sq}\#\pi$  denotes the image of the measure  $\pi$  by the square cost function  $\text{sq}(x, y) = (y - x)^2$  admits an infimum and a supremum for this order. When there exists  $\pi \in \Pi_M(\mu, \nu)$  such that  $\text{sq}\#\pi$  is equal to the infimum, then  $\int_{\mathbb{R}^2} |x - y|^\rho \pi(dx, dy)$  is equal to  $\overline{\mathcal{M}}_\rho^\rho(\mu, \nu)$  when  $\rho \in (0, 2)$  and to  $\underline{\mathcal{M}}_\rho^\rho(\mu, \nu)$  when  $\rho > 2$ . In particular, when  $\nu(dy) = \int_{x \in \mathbb{R}} \frac{1}{2} (\delta_{x-a}(dy) + \delta_{x+a}(dy)) \mu(dx)$  for some  $a \in \mathbb{R}$ , the

infimum is equal to  $\delta_{a^2}$  and is attained with  $\underline{\pi}(dx, dy) = \frac{1}{2}(\delta_{x-a}(dy) + \delta_{x+a}(dy))\mu(dx)$ . When there exists  $\bar{\pi} \in \Pi_M(\mu, \nu)$  such that  $\text{sq}\#\bar{\pi}$  is equal to the supremum, then  $\int_{\mathbb{R}^2} |x-y|^\rho \bar{\pi}(dx, dy)$  is equal to  $\underline{\mathcal{M}}_\rho^\rho(\mu, \nu)$  when  $\rho \in (0, 2)$  and to  $\overline{\mathcal{M}}_\rho^\rho(\mu, \nu)$  when  $\rho > 2$ . When  $\mu$  does not weight points, the uniqueness of the maximizing and minimizing couplings for  $\rho = 1$  stated in Theorems 7.3 and 7.4 [3] implies that  $\pi^{\text{HN}}$  (resp.  $\pi^{\text{HK}}$ ) is the only possible value for  $\underline{\pi}$  (resp.  $\bar{\pi}$ ). Unfortunately, even if we are able to exhibit in Proposition 2.21 a simple situation where the infimum and supremum are attained, we also provide examples where they are not attained (see Propositions 2.10 and 2.23 where some of the above consequences of attainment do not hold). That is why we directly study couplings that attain  $\overline{\mathcal{M}}_\rho^\rho(\mu, \nu)$  and  $\underline{\mathcal{M}}_\rho^\rho(\mu, \nu)$ . It turns out that, the non-decreasing property of maximizing couplings as well as their uniqueness when  $\mu$  does not weight points can be extended from the specific value  $\rho = 1$  to the case  $\rho \in (0, 1]$ . But the optimizers may be distinct for distinct values of  $\rho$ . This motivates our investigation of non-decreasing martingale couplings and we find the decomposition of  $\nu$  into  $(\int_{x \in \mathbb{R}} \mathbb{1}_{\{y < x\}} \pi(dx, dy), \int_{x \in \mathbb{R}} \mathbb{1}_{\{y = x\}} \pi(dx, dy), \int_{x \in \mathbb{R}} \mathbb{1}_{\{y > x\}} \pi(dx, dy))$  for  $\pi \in \Pi_M(\mu, \nu)$  is in one to one correspondence with martingale couplings in  $\Pi_M(\mu, \nu)$  non-decreasing in a generalized sense. We also show that the existence of a non-increasing martingale coupling is equivalent to a restrictive nested supports condition between  $\mu$  and  $\nu$ , under which there is a unique non-increasing and a unique non-decreasing martingale couplings. Even under the nested supports condition, the image  $\text{sq}\#\pi^{\text{HN}}$  of the unique non-decreasing martingale coupling  $\pi^{\text{HN}}$  is not necessarily the infimum of  $\{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \nu)\}$  for the convex order. But under a more and more restrictive supports condition as  $\rho$  grows, this coupling  $\pi^{\text{HN}}$  attains  $\overline{\mathcal{M}}_\rho^\rho(\mu, \nu)$  when  $\rho \in (1, 2)$  and  $\underline{\mathcal{M}}_\rho^\rho(\mu, \nu)$  when  $\rho > 2$ .

Our results are stated in Section 2 while their proofs are given in Section 3.

The proofs rely on several properties of the quantile function, which we list below. We denote the cumulative distribution function of  $\eta \in \mathcal{P}(\mathbb{R})$  by  $F_\eta : \mathbb{R} \ni x \mapsto \eta(-\infty, x] \in [0, 1]$ , and its quantile function by  $F_\eta^{-1} : (0, 1) \ni u \mapsto \inf \{x \in \mathbb{R} : u \leq F_\eta(x)\} \in \mathbb{R}$ . The following properties hold :

- (a)  $F_\eta$  is right-continuous with left-hand limits,  $F_\eta^{-1}$  is left-continuous with right-hand limits;
- (b) For all  $u \in (0, 1)$  and  $x \in \mathbb{R}$ ,

$$F_\eta^{-1}(u) \leq x \iff u \leq F_\eta(x), \quad (1)$$

denoting  $F_\eta(y-)$  the left-hand limit of  $F_\eta$  at  $y \in \mathbb{R}$ ,

$$F_\eta(x-) < u \leq F_\eta(x) \implies x = F_\eta^{-1}(u) \quad \text{and} \quad F_\eta(F_\eta^{-1}(u)-) \leq u \leq F_\eta(F_\eta^{-1}(u)); \quad (2)$$

- (c) For  $\eta(dx)$ -almost every  $x \in \mathbb{R}$ ,

$$F_\eta(x) > 0, F_\eta(x-) < 1, \quad \text{and} \quad F_\eta^{-1}(F_\eta(x)) = x; \quad (3)$$

- (d) By the inverse transform sampling, for  $f : \mathbb{R} \rightarrow \mathbb{R}$  measurable and bounded,

$$\int_{\mathbb{R}} f(x) \eta(dx) = \int_0^1 f(F_\eta^{-1}(v)) dv \quad (4)$$

We also denote by  $u_\eta : \mathbb{R} \ni x \mapsto \int_{\mathbb{R}} |y-x| \eta(dy) \in \mathbb{R}_+$  the potential function of  $\eta \in \mathcal{P}_1(\mathbb{R})$ . For  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ , we have

$$\mu \leq_{cx} \nu \iff \forall x \in \mathbb{R}, u_\mu(x) \leq u_\nu(x).$$

We say that  $\mu$  is smaller than  $\nu$  in the stochastic order and denote  $\mu \leq_{st} \nu$  if there exists  $\pi \in \Pi(\mu, \nu)$  such that  $\pi(\{(x, y) \in \mathbb{R}^2 : x \leq y\}) = 1$ . We have  $\mu \leq_{st} \nu \iff F_\mu \geq F_\nu \iff F_\mu^{-1} \leq F_\nu^{-1}$ .

## 2 Main results

## 2.1 Numerical experiments

For  $\mu, \nu$  such that  $\mu \leq_{cx} \nu$ , we consider  $(X_i)_{1 \leq i \leq I}$  (resp.  $(Y_j)_{1 \leq j \leq J}$ ) independent and identically distributed according to  $\mu$  (resp.  $\nu$ ). The empirical measures

$$\hat{\mu}_I = \frac{1}{I} \sum_{i=1}^I \delta_{X_i - \bar{X}_I} \text{ with } \bar{X}_I = \frac{1}{I} \sum_{i=1}^I X_i \quad \text{and} \quad \hat{\nu}_J = \frac{1}{J} \sum_{j=1}^J \delta_{Y_j - \bar{Y}_J} \text{ with } \bar{Y}_J = \frac{1}{J} \sum_{j=1}^J Y_j,$$

are both centred and respectively approximate the respective images  $\hat{\mu}$  and  $\hat{\nu}$  of  $\mu$  and  $\nu$  by  $x \mapsto x - \int_{\mathbb{R}} y \nu(dy)$ . Applying [1, Algorithm 1], we compute  $\hat{\mu}_I \vee_{cx} \hat{\nu}_J$ . The computation of  $\overline{\mathcal{M}}_\rho(\hat{\mu}_I, \hat{\mu}_I \vee_{cx} \hat{\nu}_J)$  (resp.  $\underline{\mathcal{M}}_\rho(\hat{\mu}_I, \hat{\mu}_I \vee_{cx} \hat{\nu}_J)$ ) is a linear programming problem that we solve using the CVXPY package [4] in python for  $I = J = 100$ . We call  $\pi^{\text{HN}}$  the obtained maximizer for the cost  $|y - x|$ . When  $\rho \in (0, 1) \cup (1, 2)$  (resp.  $\rho > 2$ ), we compare the optimal value  $\overline{\mathcal{M}}_\rho(\hat{\mu}_I, \hat{\mu}_I \vee_{cx} \hat{\nu}_J)$  (resp.  $\underline{\mathcal{M}}_\rho(\hat{\mu}_I, \hat{\mu}_I \vee_{cx} \hat{\nu}_J)$ ) and corresponding maximizer  $\bar{\pi}$  (resp. minimizer  $\underline{\pi}$ ) computed by the solver with  $\mathcal{I}_\rho^{\pi^{\text{HN}}} := \int |x - y|^\rho \pi^{\text{HN}}(dx, dy)$  and  $\pi^{\text{HN}}$  respectively. It turns out that the values always coincide while the couplings  $\bar{\pi}$  and  $\underline{\pi}$  are distinct from  $\pi^{\text{HN}}$  when  $\mu$  and  $\nu$  are continuous distributions.

Normal distributions : $\mu = \mathcal{N}(0, 0.24)$ and $\nu = \mathcal{N}(0, 0.28)$							
$\rho$	$\overline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\text{HN}}}$	$\ \bar{\pi} - \pi^{\text{HN}}\ $	$\rho$	$\underline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\text{HN}}}$	$\ \underline{\pi} - \pi^{\text{HN}}\ $
0.3	0.586396	0.586391	0.022987	2.1	0.025338	0.025338	0.027715
0.7	0.289132	0.289131	0.012609	2.3	0.017985	0.017987	0.034845
1.0	0.170704	0.170704	0.0	2.5	0.012783	0.012785	0.037657
1.4	0.084928	0.084927	0.021816	3.0	0.005473	0.005477	0.044496
1.9	0.035740	0.035740	0.026054	5.0	0.000199	0.000202	0.055856

Log-normal distributions : $\mu = \exp \# \mathcal{N}(0, 0.24)$ and $\nu = \exp \# \mathcal{N}(-0.0104, 0.28)$							
$\rho$	$\overline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\text{HN}}}$	$\ \bar{\pi} - \pi^{\text{HN}}\ $	$\rho$	$\underline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\text{HN}}}$	$\ \underline{\pi} - \pi^{\text{HN}}\ $
0.3	0.607721	0.607720	0.017452	2.1	0.036152	0.036153	0.030068
0.7	0.316957	0.316957	0.008263	2.3	0.026842	0.026843	0.033392
1.0	0.196361	0.196361	0.0	2.5	0.019985	0.019988	0.034966
1.4	0.104947	0.104946	0.020343	3.0	0.009673	0.009678	0.041519
1.9	0.048834	0.048833	0.027026	5.0	0.000614	0.000619	0.062155

Exponential distributions : $\mu = \mathcal{L}(Y - 1)$ $\nu = \mathcal{L}(X - 2)$ with $X \sim \mathcal{E}(1)$ and $Y \sim \mathcal{E}(0.5)$							
$\rho$	$\overline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\text{HN}}}$	$\ \bar{\pi} - \pi^{\text{HN}}\ $	$\rho$	$\underline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\text{HN}}}$	$\ \underline{\pi} - \pi^{\text{HN}}\ $
0.3	0.768373	0.768293	0.039204	2.1	0.196164	0.196179	0.050821
0.7	0.550068	0.550059	0.023933	2.3	0.171925	0.171980	0.060709
1.0	0.433394	0.433394	0.0	2.5	0.151178	0.151284	0.062783
1.4	0.320305	0.320281	0.032152	3.0	0.111067	0.111337	0.067680
1.9	0.224602	0.224591	0.043663	5.0	0.037085	0.038208	0.078178

Binomial distributions : $\mu = \mathcal{L}(X - 5)$ , $\nu = \mathcal{L}(Y - 20)$ with $X \sim \mathcal{B}(10, 0.5)$ , $Y \sim \mathcal{B}(40, 0.5)$							
$\rho$	$\overline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\text{HN}}}$	$\ \bar{\pi} - \pi^{\text{HN}}\ $	$\rho$	$\underline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\text{HN}}}$	$\ \underline{\pi} - \pi^{\text{HN}}\ $
0.3	1.366952	1.366793	5.6e-03	2.1	10.025562	10.025562	3.9e-09
0.7	2.090477	2.090467	2.9e-03	2.3	12.680593	12.680593	4.0e-09
1.0	2.893664	2.893664	0.0	2.5	16.085421	16.085421	3.3e-09
1.4	4.504064	4.504064	4.2e-09	3.0	29.533935	29.533935	4.5e-09
1.9	7.949088	7.949088	3.2e-09	5.0	407.035561	407.347830	2.7e-03

Poisson distributions : $\mu = \mathcal{L}(X - 1)$ , $\nu = \mathcal{L}(Y - 4)$ with $X \sim \mathcal{P}(1)$ , $Y \sim \mathcal{P}(4)$							
$\rho$	$\overline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\text{HN}}}$	$\ \overline{\pi} - \pi^{\text{HN}}\ $	$\rho$	$\underline{\mathcal{M}}_\rho$	$\mathcal{I}_\rho^{\pi^{\text{HN}}}$	$\ \underline{\pi} - \pi^{\text{HN}}\ $
0.3	1.116329	1.116329	1.2e-09	2.1	2.776364	2.776364	7.0e-09
0.7	1.317531	1.317531	2.0e-09	2.3	3.148135	3.148135	1.4e-09
1.0	1.513291	1.513291	0.0	2.5	3.584910	3.584910	4.8e-09
1.4	1.854353	1.854353	1.1e-09	3.0	5.047828	5.047828	1.1e-09
1.9	2.459404	2.459404	1.2e-08	5.0	24.477011	24.477011	1.2e-09

## 2.2 Infimum and supremum of $\{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \nu)\}$ for the convex order

**Proposition 2.1.** *Let  $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$  be such that  $\mu \leq_{cx} \nu$ . Then the set  $\{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \nu)\}$  admits an infimum and a supremum for the convex order.*

The conclusion follows from the next lemma and the complete lattice structure of the set  $\{\eta \in \mathcal{P}_1(\mathbb{R}) : \int_{\mathbb{R}} z\eta(dz) = \int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx)\}$  with minimal element  $\delta_{\int_{\mathbb{R}} y^2\nu(dy) - \int_{\mathbb{R}} x^2\mu(dx)}$  stated in [10].

**Lemma 2.2.** *Let  $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$  be such that  $\mu \leq_{cx} \nu$ . Then the set  $\{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \nu)\}$  is bounded from above in the convex order.*

## 2.3 Definition and optimality of non-decreasing and non-increasing martingale couplings

**Definition 2.3.** *Let  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$  be such that  $\mu \leq_{cx} \nu$ . A martingale coupling  $\pi(dx, dy) \in \Pi_M(\mu, \nu)$  is called **non-decreasing** if there exists a Borel set  $\Gamma \subseteq \mathbb{R} \times \mathbb{R}$  such that  $\pi(\Gamma) = 1$  and*

- (a) *if  $(x_-, y_-), (x_+, y_+) \in \Gamma$  with  $y_- \leq x_-, y_+ \leq x_+$ , and  $x_- < x_+$ , then  $y_- \leq y_+$*
- (b) *if  $(x_-, z_-), (x_+, z_+) \in \Gamma$  with  $x_- \leq z_-, x_+ \leq z_+$ , and  $x_- < x_+$ , then  $z_- \leq z_+$ .*

Respectively,  $\pi(dx, dy)$  is called **non-increasing** if there exists a Borel set  $\Gamma \subseteq \mathbb{R} \times \mathbb{R}$  such that  $\pi(\Gamma) = 1$  and

- (c) *if  $(x_-, y_+), (x_+, y_-) \in \Gamma$  with  $y_+ \leq x_-, y_- \leq x_+$ , and  $x_- < x_+$ , then  $y_- \leq y_+$*
- (d) *if  $(x_-, z_+), (x_+, z_-) \in \Gamma$  with  $x_- \leq z_+, x_+ \leq z_-$ , and  $x_- < x_+$ , then  $z_- \leq z_+$ .*

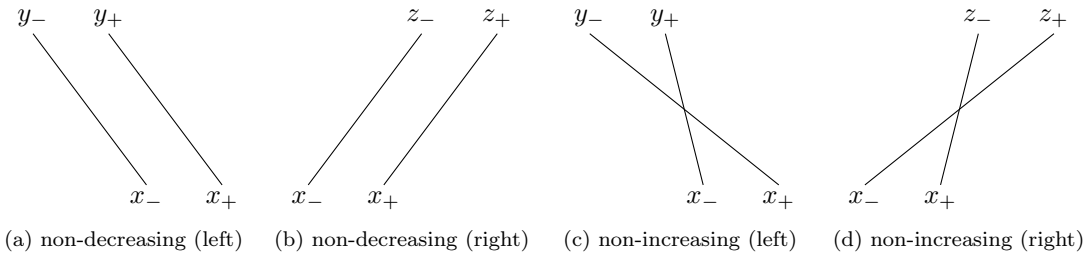


Figure 1: Non-decreasing / non-increasing coupling

A different notion of monotone martingale couplings is introduced by Beiglböck and Juillet in [3] where they prove that there exists a unique martingale coupling  $\pi$  left-monotone (resp. right-monotone) in the sense that  $\pi(\Gamma) = 1$  for some Borel subset  $\Gamma$  of  $\mathbb{R}^2$  such that  $(x_-, y_-), (x_-, z_-)$  and  $(x_+, w) \in \Gamma$  with  $x_- < x_+$  and  $y_- < z_-$  implies that  $w \notin (y_-, z_-)$  (resp.  $(x_+, y_+), (x_+, z_+)$  and  $(x_-, w) \in \Gamma$  with  $x_- < x_+$  and  $y_+ < z_+$  implies that  $w \notin (y_+, z_+)$ ). According to [5], the left-monotone (resp. right-monotone) martingale coupling minimizes (resp. maximizes) over  $\Pi_M(\mu, \nu)$  the integral of smooth cost functions  $c(x, y)$  satisfying the martingale Spence-Mirrlees condition  $\partial_{xyy}^3 c \leq 0$ .

In the beginning of the proof of Theorem 7.3 [3], combining the necessary optimality condition that they give in Lemma 1.11 together with some specific properties of the cost function  $\mathbb{R}^2 \ni (x, y) \mapsto |x - y|$  stated in Lemma 7.5, Beiglböck and Juillet show that any coupling maximizing  $\int_{\mathbb{R}^2} |x - y| \pi(dx, dy)$  over  $\pi \in \Pi_M(\mu, \nu)$  is non-decreasing. Replacing Lemma 7.5 [3] by Lemma 3.4 below, we generalize this result to costs  $\varphi(|x - y|)$  where  $\mathbb{R}_+ \ni z \mapsto \varphi(z) \in \mathbb{R}$  is increasing and concave. To check that, on the other hand, any coupling minimizing  $\int_{\mathbb{R}^2} \varphi(|x - y|) \pi(dx, dy)$  for such a function is non-increasing, we need  $\mu$  and  $\nu$  to satisfy the following nested supports condition.

**Definition 2.4.** *We say that  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$  satisfy the nested supports condition if there exist  $-\infty < a \leq b < +\infty$  such that  $\mu([a, b]) = 1$  and  $\nu((a, b)) = 0$ .*

The nested supports condition is slightly weaker than the Reinforced Dispersion Assumption 7.1 considered by Hobson and Klimmek [6] and which amounts to the existence of some interval  $I$  such that  $\mu(I) = 1$  and  $\nu(I) = 0$ . Indeed, under the nested supports condition, it may happen that  $\mu(\{a\})\nu(\{a\}) > 0$  or  $\mu(\{b\})\nu(\{b\}) > 0$ .

**Proposition 2.5.** *Let  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$  be such that  $\mu \leq_{cx} \nu$  and  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$  be some increasing concave function.*

- (i) *If  $\tilde{\pi} \in \Pi_M(\mu, \nu)$  maximizes  $\int \varphi(|x - y|) \pi(dx, dy)$  over  $\pi \in \Pi_M(\mu, \nu)$ , then  $\tilde{\pi}$  is non-decreasing and, when  $\varphi$  is continuous, there exists such a maximizing coupling.*
- (ii) *There exists  $\tilde{\pi} \in \Pi_M(\mu, \nu)$  which minimizes  $\int \varphi(|x - y|) \pi(dx, dy)$  over  $\pi \in \Pi_M(\mu, \nu)$  and when  $\mu$  and  $\nu$  satisfy the nested supports condition, then any such coupling  $\tilde{M}$  is non-increasing.*

**Remark 2.6.** *Let  $\lambda \in [\varphi'_r(1), \varphi'_l(1)]$  where  $\varphi'_r(1)$  and  $\varphi'_l(1)$  are the respective right-hand and left-hand derivatives of the increasing concave function  $\varphi$  at point 1. We have  $\forall z \in \mathbb{R}_+, \varphi(0) \leq \varphi(z) \leq \varphi(1) + \lambda(z - 1)$  so that for  $\pi \in \Pi_M(\mu, \nu)$ ,*

$$\begin{aligned} \varphi(0) &\leq \int_{\mathbb{R}^2} \varphi(|x - y|) \pi(dx, dy) \leq \varphi(1) + \lambda \int_{\mathbb{R}^2} |x - y| \pi(dx, dy) \\ &\leq \varphi(1) + \lambda \left( \int_{\mathbb{R}} |x| \mu(dx) + \int_{\mathbb{R}} |y| \nu(dy) \right), \end{aligned}$$

where the right-hand side is finite for  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$ . The concavity of the function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$  implies that this function is continuous on  $(0, +\infty)$ . It may have a positive jump  $\lim_{x \rightarrow 0^+} \varphi(x) - \varphi(0)$  at the origin and then  $\mathbb{R}^2 \ni (x, y) \mapsto \varphi(|x - y|)$  is lower semi-continuous but not upper-semicontinuous.

**Remark 2.7.** *For  $\rho \in (0, 1]$ , since  $\mathbb{R}_+ \ni z \mapsto z^\rho$  is concave and increasing, a martingale coupling that maximizes  $\int |x - y|^\rho \pi(dx, dy)$  is non-decreasing.*

It turns out that the nested supports condition is also necessary for the existence of a non-increasing coupling in  $\Pi_M(\mu, \nu)$  and that, under this condition there is exists a unique non-increasing coupling and also a unique non-decreasing coupling in  $\Pi_M(\mu, \nu)$ .

**Proposition 2.8.** *Let  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$  be such that  $\mu \leq_{cx} \nu$ . The following assertions are equivalent*

- (i) *there exists a non-increasing martingale coupling in  $\Pi_M(\mu, \nu)$ ,*
- (ii)  *$\mu$  and  $\nu$  satisfy the nested supports condition :  $\exists -\infty < a \leq b < +\infty$  s.t.  $\mu([a, b]) = 1$  and  $\nu((a, b)) = 0$ ,*
- (iii) *there exists a unique non-increasing martingale coupling  $\pi^\downarrow$  in  $\Pi_M(\mu, \nu)$  and  $\pi^\downarrow(\{(a, a)\}) = \mu(\{a\}) \wedge \nu(\{a\})$ ,  $\pi^\downarrow(\{(b, b)\}) = \mu(\{b\}) \wedge \nu(\{b\})$ .*

Moreover, under the nested supports condition, there exists a unique non-decreasing martingale coupling  $\pi^\uparrow \in \Pi_M(\mu, \nu)$  and, when  $u_\nu(a) > u_\mu(a)$  (resp.  $u_\nu(b) > u_\mu(b)$ ), then  $\pi^\uparrow(\{(a, a)\}) = (\mu(\{a\}) - p_-(a) - p_+(a))^+$  (resp.  $\pi^\uparrow(\{(b, b)\}) = (\mu(\{b\}) - p_-(b) - p_+(b))^+$ ) where  $p_-$  and  $p_+$  are defined in Proposition 2.14 below.

The next proposition generalizes to continuous, increasing and concave functions  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$  the statement in Theorem 7.3 [3] which deals with the case when  $\varphi$  is the identity function .

**Proposition 2.9.** *Let  $\mu, \nu \in \mathcal{P}(\mathbb{R})$  be such that  $\mu \leq_{cx} \nu$  and that  $\mu$  does not weight points. Let  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$  be some continuous increasing concave function. There exists a unique non-decreasing martingale coupling  $\tilde{\pi}$  that maximizes  $\int \varphi(|x - y|)\pi(dx, dy)$ . Moreover, there exist two non-decreasing functions  $T_1, T_2 : \mathbb{R} \rightarrow \mathbb{R}$  such that  $T_1(x) \leq x \leq T_2(x)$  and  $\tilde{\pi}$  is concentrated on the graphs of these functions.*

Notice that the example in Section 7.3.1 [3] shows that for general continuous, increasing and concave functions  $\varphi$ , the uniqueness of maximizers may fail without the continuity assumption on  $\mu$ . The next proposition illustrates that unique maximizers may be distinct for distinct power functions  $\varphi$ . Since those maximizers are non-decreasing couplings, this motivates the investigation of the set of non-decreasing couplings in the next subsection.

**Proposition 2.10.** *Let  $0 < \rho' < \rho \leq 1$ ,  $y < \frac{y+z}{2} < m < z$ ,*

$$\beta \in \left(0, 2\rho \left(\frac{m-y}{z-y}(z-m)^{\rho-1} - \frac{z-m}{z-y}(m-y)^{\rho-1}\right) \wedge \frac{2m-y-z}{z-m}\right), \text{ and}$$

$$\begin{aligned} \mu_\varepsilon &= \frac{1}{1 + (1 + \beta\varepsilon^{1-\rho})^{\frac{1}{\rho}}} \delta_{m - (1 + \beta\varepsilon^{1-\rho})^{\frac{1}{\rho}} \varepsilon} + \frac{(1 + \beta\varepsilon^{1-\rho})^{\frac{1}{\rho}}}{1 + (1 + \beta\varepsilon^{1-\rho})^{\frac{1}{\rho}}} \delta_{m+\varepsilon}, \\ \nu &= \frac{1}{3} \delta_m + \frac{2(z-m)}{3(z-y)} \delta_y + \frac{2(m-y)}{3(z-y)} \delta_z. \end{aligned}$$

For  $\varepsilon < (z-m) \wedge \frac{m-y}{(1+\beta)^{\frac{1}{\rho}}} \wedge \frac{2(z-m)(m-y)}{3(z-y)} \wedge 1$ ,  $\Pi_M(\mu_\varepsilon, \nu)$  contains distinct martingale couplings and any element of  $\Pi_M(\mu_\varepsilon, \nu)$  is non-decreasing. For  $\varepsilon$  small enough, the non-decreasing martingale coupling that attains  $\overline{\mathcal{M}}_{\rho'}(\mu_\varepsilon, \nu)$  is different from the one that attains  $\overline{\mathcal{M}}_\rho(\mu_\varepsilon, \nu)$ .

## 2.4 Directionally decomposed non-decreasing martingale couplings

The uniqueness of non-decreasing couplings under the nested supports condition which determines the parts of the measure  $\nu$  attained in the left direction and in the right direction in any martingale coupling motivates the introduction of the following decomposition. For  $\pi \in \Pi_M(\mu, \nu)$ , we denote by

$$\begin{aligned} \nu_l^\pi(dy) &= \int_{x \in \mathbb{R}} \mathbf{1}_{\{y < x\}} \pi(dx, dy) = \int_{x \in \mathbb{R}} \mathbf{1}_{\{y < x\}} \pi_x(dy) \mu(dx), \\ \nu_0^\pi(dy) &= \int_{x \in \mathbb{R}} \mathbf{1}_{\{y = x\}} \pi(dx, dy) = \int_{x \in \mathbb{R}} \mathbf{1}_{\{y = x\}} \pi_x(dy) \mu(dx), \\ \nu_r^\pi(dy) &= \int_{x \in \mathbb{R}} \mathbf{1}_{\{y > x\}} \pi(dx, dy) = \int_{x \in \mathbb{R}} \mathbf{1}_{\{y > x\}} \pi_x(dy) \mu(dx). \end{aligned}$$

Note that  $\nu_l^\pi + \nu_0^\pi + \nu_r^\pi = \nu$ . For  $(\nu_l, \nu_r)$  a couple of non-negative measures such that  $\nu_l + \nu_r \leq \nu$ , we denote

$$\Pi_M(\mu, \nu, \nu_l, \nu_r) = \{\pi \in \Pi_M(\mu, \nu) : (\nu_l^\pi, \nu_0^\pi, \nu_r^\pi) = (\nu_l, \nu - \nu_l - \nu_r, \nu_r)\}.$$

Our main result, stated in the next theorem and in assertion (i) of the corollary that follows is that  $\Pi_M(\mu, \nu, \nu_l, \nu_r)$  is not empty if and only if there exists a coupling in this set which is non-decreasing away from the support of  $\nu - \nu_l - \nu_r$  where it goes straight.

**Theorem 2.11.** *Let  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$  be such that  $\mu \leq_{cx} \nu$  and  $\mu \neq \nu$ . Let  $(\nu_l, \nu_r)$  a couple of non-negative measures such that  $\nu_l + \nu_r = \nu$ . We have*

$$\Pi_M(\mu, \nu, \nu_l, \nu_r) \neq \emptyset \iff \exists! \pi^\uparrow \in \Pi_M(\mu, \nu, \nu_l, \nu_r) \text{ non-increasing,}$$

and then, for each continuous increasing concave function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ ,

$$\forall \pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r) \setminus \{\pi^\uparrow\}, \int_{\mathbb{R}^2} \varphi(|y - x|) \pi(dx, dy) < \int_{\mathbb{R}^2} \varphi(|y - x|) \pi^\uparrow(dx, dy). \quad (5)$$



**Corollary 2.12.** Let  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$  be such that  $\mu \leq_{cx} \nu$  and  $\mu \neq \nu$ . For  $(\nu_l, \nu_r)$  a couple of non-negative measures such that  $\nu_l + \nu_r \leq \nu$ , we have that

$$(i) \Pi_M(\mu, \nu, \nu_l, \nu_r) \neq \emptyset \iff \exists! \pi^\uparrow \in \Pi_M(\mu, \nu, \nu_l, \nu_r) \text{ s.t. } \frac{\pi^\uparrow(dx, dy) + (\nu_l + \nu_r - \nu)(dx)\delta_x(dy)}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}$$

is non-decreasing

$$\text{and then } \forall \pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r) \setminus \{\pi^\uparrow\}, \int_{\mathbb{R}^2} \varphi(|y-x|)\pi(dx, dy) < \int_{\mathbb{R}^2} \varphi(|y-x|)\pi^\uparrow(dx, dy) \quad (6)$$

for each continuous increasing concave function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ ,

(ii)  $\exists \pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$  non-decreasing

$$\implies \nu - \nu_l - \nu_r = \mathbb{1}_{\{u_\mu = u_\nu\}}\mu + \sum_{\substack{x \in \mathbb{R} \\ \mu \wedge \nu(\{x\}) \times (u_\nu(x) - u_\mu(x)) > 0}} p(x)\delta_x \text{ for some } p(x) \in [0, \mu \wedge \nu(\{x\})],$$

(iii) There is at most one non-decreasing coupling  $\pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$ .

According to the next proposition, some restrictive structure on the supports of  $\mu$  and  $\nu$  is needed to ensure the existence of a martingale coupling in  $\Pi_M(\mu, \nu, \nu_l, \nu_r)$  non-increasing away from the support of  $\nu - \nu_l - \nu_r$  where it goes straight.

**Proposition 2.13.** Let  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$  be such that  $\mu \leq_{cx} \nu$  and  $\mu \neq \nu$ . For  $(\nu_l, \nu_r)$  a couple of non-negative measures such that  $\nu_l + \nu_r \leq \nu$ , the existence of  $\pi^\downarrow \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$  such that  $\frac{\pi^\downarrow(dx, dy) + (\nu_l + \nu_r - \nu)(dx)\delta_x(dy)}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}$  is non-increasing is equivalent to the Dispersion Assumption 2.1 in [6] i.e. the existence of a non-empty finite interval  $I$  with ends  $a \leq b$  such that  $(\mu - \nu)^+(I) = (\mu - \nu)^+(\mathbb{R})$  and  $(\nu - \mu)^+(I) = 0$  combined with  $\nu_l(dy) = \mathbb{1}_{\{y \leq a\}}(\nu - \mu)^+(dy)$  and  $\nu_r(dy) = \mathbb{1}_{\{y \geq b\}}(\nu - \mu)^+(dy)$ .

In view of statement (ii) in Corollary 2.12, to better characterize the couples  $(\nu_l, \nu_r)$  such that  $\nu_l + \nu_r \leq \nu$  and there exists a truly non-decreasing coupling in  $\Pi_M(\mu, \nu, \nu_l, \nu_r)$ , it is useful to study the points  $x \in \mathbb{R}$  such that  $\inf_{\pi \in \Pi_M(\mu, \nu)} \pi(\{(x, x)\}) > 0$ .

**Proposition 2.14.** Let  $\mu \leq_{cx} \nu$  and  $x \in \mathbb{R}$  be such that  $u_\nu(x) > u_\mu(x)$ . Then there exists a unique couple  $(p_-(x), p_+(x)) \in (0, F_\nu(x-)] \times (0, 1 - F_\nu(x)]$  such that

$$\int_{\mathbb{R}} (z-x)^+ \nu(dz) = \int_{\mathbb{R}} (y-x)^+ \mu(dy) + \int_{F_\nu(x)}^{F_\nu(x)+p_+(x)} (F_\nu^{-1}(v) - x) dv. \quad (7)$$

$$\text{and } \int_{\mathbb{R}} (x-z)^+ \nu(dz) = \int_{\mathbb{R}} (x-y)^+ \mu(dy) + \int_{F_\nu(x-)-p_-(x)}^{F_\nu(x-)} (x - F_\nu^{-1}(v)) dv \quad (8)$$

Moreover,  $\inf_{\pi \in \Pi_M(\mu, \nu)} \pi(\{(x, x)\}) = (\mu(\{x\}) - p_-(x) - p_+(x))^+$  where the infimum is attained. When  $\mu(\{x\}) > p_-(x) + p_+(x)$ , then any  $\pi \in \Pi_M(\mu, \nu)$  satisfying  $\pi(\{(x, x)\}) = (\mu(\{x\}) - p_-(x) - p_+(x))^+$  is such that  $\pi(\{(-\infty, x) \times (x, +\infty)\} \cup \{(x, +\infty) \times (-\infty, x)\}) = 0$  and  $\pi_x$  is equal to

$$\eta_x = \left(1 - \frac{p_-(x) + p_+(x)}{\mu(\{x\})}\right) \delta_x + \frac{1}{\mu(\{x\})} \int_{F_\nu(x-)-p_-(x)}^{F_\nu(x-)} \delta_{F_\nu^{-1}(v)} dv + \frac{1}{\mu(\{x\})} \int_{F_\nu(x)}^{F_\nu(x)+p_+(x)} \delta_{F_\nu^{-1}(v)} dv. \quad (9)$$

When  $\mu(\{x\}) \in (0, p_-(x) + p_+(x)]$ , there exists a unique  $q(x) \in [(\mu(\{x\}) - p_+(x))^+, p_-(x) \wedge \mu(\{x\})]$  such that  $\int_{F_\nu(x-)-q(x)}^{F_\nu(x-)} F_\nu^{-1}(v) dv + \int_{F_\nu(x)}^{F_\nu(x)+\mu(\{x\})-q(x)} F_\nu^{-1}(v) dv = \mu(\{x\})x$  and there exists  $\pi \in \Pi_M(\mu, \nu)$  such that  $\pi_x$  is equal to

$$\eta_x = \frac{1}{\mu(\{x\})} \int_{F_\nu(x-)-q(x)}^{F_\nu(x-)} \delta_{F_\nu^{-1}(v)} dv + \frac{1}{\mu(\{x\})} \int_{F_\nu(x)}^{F_\nu(x)+\mu(\{x\})-q(x)} \delta_{F_\nu^{-1}(v)} dv. \quad (10)$$

**Remark 2.15.** The case when  $\mu = \delta_x$  and  $\nu \in \mathcal{P}_1(\mathbb{R})$  is such that  $\int_{\mathbb{R}} z\nu(dz) = x$  and  $\nu(\{x\}) \in (0, 1)$  provides an example where  $p_-(x) = F_\nu(x-)$ ,  $p_+(x) = 1 - F_\nu(x)$  so that  $p_-(x) + p_+(x) = 1 - \nu(\{x\}) < \mu(\{x\})$ . The only element  $\pi(dy, dz) = \delta_x(dy)\nu(dz)$  of  $\Pi(\mu, \nu)$  and  $\Pi_M(\mu, \nu)$  satisfies  $\pi(\{(x, x)\}) = \nu(\{x\}) = 1 - F_\nu(x-) - (1 - F_\nu(x)) = \mu(\{x\}) - p_-(x) - p_+(x)$ .

**Corollary 2.16.** Let  $\mu \leq_{cx} \nu$  and  $\mathcal{X}_0 = \{x \in \mathbb{R} : u_\nu(x) > u_\mu(x) \text{ and } \mu(\{x\}) > p_-(x) + p_+(x)\}$ . Then for each  $\pi \in \Pi_M(\mu, \nu)$ , one has

$$\mathbb{1}_{\{u_\mu(y)=u_\nu(y)\}}\mu(dy) + \sum_{x \in \mathcal{X}_0} (\mu(\{x\}) - p_-(x) - p_+(x)) \delta_x(dy) \leq \nu_0^\pi(dy) \leq \mu \wedge \nu(dy),$$

where the bound from above is attained but the bound from below may not be attained. Moreover, when  $\mu \neq \nu$  and  $(\nu_l, \nu_r)$  is a couple of non-negative measures such that  $\nu - \nu_l - \nu_r = \mathbb{1}_{\{u_\mu(y)=u_\nu(y)\}}\mu + \sum_{x \in \mathcal{X}_0} (\mu(\{x\}) - p_-(x) - p_+(x)) \delta_x$ , then

$$\Pi_M(\mu, \nu, \nu_l, \nu_r) \neq \emptyset \iff \exists! \pi^\uparrow \in \Pi_M(\mu, \nu, \nu_l, \nu_r) \text{ non-decreasing.}$$

In the next example, the bound from below is not attained.

**Example 2.17.** Let  $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$  and

$$\nu(dy) = \frac{1}{4}(\delta_{-1}(dy) + \delta_1(dy)) + \frac{1}{6}(\mathbb{1}_{[-2,-1]}(y) + \mathbb{1}_{[1,2]}(y)) dy + \frac{1}{12}\mathbb{1}_{[-1,1]}(y)dy.$$

Then  $u_\mu(-1) = 1 = u_\mu(1)$ ,  $u_\nu(-1) = \frac{7}{6} = u_\nu(1)$ ,  $p_-(-1) = \frac{1}{6} = p_+(1)$  and  $p_+(-1) = \frac{\sqrt{2}}{12} = p_-(-1)$ . By Proposition 2.14, there exists  $\pi \in \Pi_M(\mu, \nu)$  such that  $\pi_{-1}(\{-1\}) = 1 - \frac{p_-(-1)+p_+(-1)}{\mu(\{-1\})} = \frac{4-\sqrt{2}}{6}$  and then  $\pi_{-1} = \eta_{-1}$  with  $\eta_{-1}$  given by (9) and  $\pi_1 = 2\nu - \eta_{-1}$ . Therefore the unique coupling  $\pi \in \Pi_M(\mu, \nu)$  such that  $\pi_{-1}(\{-1\}) = 1 - \frac{p_-(-1)+p_+(-1)}{\mu(\{-1\})} = \frac{4-\sqrt{2}}{6}$  is

$$\begin{aligned} \pi(dx, dy) &= \frac{1}{2}\delta_{-1}(dx) \left( \frac{1}{3}\mathbb{1}_{[-2,-1]}(y)dy + \frac{4-\sqrt{2}}{6}\delta_{-1}(dy) + \frac{1}{6}\mathbb{1}_{[-1,-1+\sqrt{2}]}(y)dy \right) \\ &+ \frac{1}{2}\delta_1(dx) \left( \frac{\sqrt{2}-1}{6}\delta_{-1}(dy) + \frac{1}{6}\mathbb{1}_{[-1+\sqrt{2},1]}(y)dy + \frac{1}{2}\delta_1(dy) + \frac{1}{3}\mathbb{1}_{[1,2]}(y)dy \right) \end{aligned}$$

and satisfies  $\pi_1(\{1\}) = \frac{1}{2} > \frac{4-\sqrt{2}}{6} = 1 - \frac{p_-(-1)+p_+(-1)}{\mu(\{-1\})}$ . In a symmetric way, the only coupling  $\pi \in \Pi_M(\mu, \nu)$  such that  $\pi_1(\{1\}) = 1 - \frac{p_-(-1)+p_+(-1)}{\mu(\{-1\})}$  satisfies  $\pi_{-1}(\{-1\}) = \frac{1}{2} > \frac{4-\sqrt{2}}{6} = 1 - \frac{p_-(-1)+p_+(-1)}{\mu(\{-1\})}$ .

The next example shows that one cannot restrict the summation over  $\{x \in \mathbb{R} : \mu \wedge \nu(\{x\}) \times (u_\nu(x) - u_\mu(x)) > 0\}$  to a summation over  $\{x \in \mathbb{R} : \inf_{\pi \in \Pi_M(\mu, \nu)} \pi(\{x, x\}) > 0\}$  in Assertion (ii) of Corollary 2.12.

**Example 2.18.** Let for  $\alpha \in [0, 1]$ ,  $\nu = \frac{1}{6}((2-\alpha)\delta_{-4} + \alpha\delta_{-1} + 2\delta_0 + \alpha\delta_1 + (2-\alpha)\delta_4)$  and  $\mu = \frac{1}{3}(\delta_{-2} + \delta_0 + \delta_2)$ . Then  $u_\nu(0) = \frac{8-3\alpha}{3} = u_\mu(0) + \frac{4-3\alpha}{3}$ ,  $p_-(0) = \frac{1}{6} = p_+(0)$ ,  $q(0) = \frac{1}{6}$  and

$$\pi := \frac{1}{6}(\delta_{(-2,-4)} + \delta_{(-2,0)} + (1-\alpha)\delta_{(0,-4)} + \alpha\delta_{(0,-1)} + \alpha\delta_{(0,1)} + (1-\alpha)\delta_{(0,4)} + \delta_{(2,0)} + \delta_{(2,4)})$$

is a non-decreasing martingale coupling such that  $\pi_0 = \eta_0$  given by (10) and which does not weight  $(0, 0)$ . On the other hand,

$$\frac{4-\alpha}{24}(\delta_{(-2,-4)} + \delta_{(2,4)}) + \frac{\alpha}{6}(\delta_{(-2,-1)} + \delta_{(2,1)}) + \frac{4-3\alpha}{24}(\delta_{(-2,0)} + \delta_{(0,-4)} + \delta_{(0,4)} + \delta_{(2,0)}) + \frac{\alpha}{4}\delta_{(0,0)}$$

(and any strict convex combination with the former coupling) is a non-decreasing martingale coupling which weights  $(0, 0)$  when  $\alpha > 0$ .

## 2.5 Do $\pi^\uparrow$ and $\pi^\downarrow$ optimize the cost function $|x - y|^\rho$ for $\rho > 1$ under the nested supports condition?

In this section, we suppose that the probability measures  $\mu$  and  $\nu$  in the convex order satisfy the nested supports condition so that, by Proposition 2.8, there exist a unique non-decreasing coupling  $\pi^\uparrow$  and a unique non-increasing coupling  $\pi^\downarrow$  in  $\Pi_M(\mu, \nu)$ . Then, by Proposition 2.5,  $\pi^\uparrow$  maximises

(resp.  $\pi^\downarrow$  minimizes)  $\int_{\mathbb{R}^2} c(x, y)\pi(dx, dy)$  over  $\pi \in \Pi_M(\mu, \nu)$  for cost functions  $c(x, y) = \varphi(|x - y|)$  with  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$  continuous, increasing and concave and in particular for  $c(x, y) = |x - y|^\rho$  with  $\rho \in (0, 1]$ . We investigate whether  $\pi^\uparrow$  (resp.  $\pi^\downarrow$ ) still maximises (resp. minimises) for  $c(x, y) = |x - y|^\rho$  with  $\rho \in (1, 2)$  and minimises (resp. maximises) for  $c(x, y) = |x - y|^\rho$  with  $\rho > 2$ . It turns out that this is the case under a reinforced support condition with  $\mu$  and  $\nu$  not weighting some non empty intervals at the left of  $a$  and at the right of  $b$ . The larger  $\rho$ , the larger these gaps should be chosen. When  $\mu$  only weights two points and  $\nu$  weights two points to the left of the support of  $\mu$  and two points to the right, we exhibit conditions ensuring that  $\text{sq}\#\pi^\uparrow$  is equal to the infimum  $\inf_{cx} \{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \nu)\}$  and  $\text{sq}\#\pi^\downarrow$  is equal to the supremum  $\sup_{cx} \{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \nu)\}$  for the convex order. Last, we check that, still with such finite supports, optimality may fail when the gaps are too small.

### 2.5.1 Preservation of the optimality

**Proposition 2.19.** *Let  $\mu \leq_{cx} \nu$  be such that there exist  $\underline{y} < \bar{y} < \underline{x} < \bar{x} < \underline{z} < \bar{z}$  with  $\mu([\underline{x}, \bar{x}]) = 1$  and  $\nu([\underline{y}, \bar{y}] \cup [\underline{z}, \bar{z}]) = 1$ . If*

$$\begin{aligned} \underline{x} - \bar{y} &\geq \alpha_\rho(\bar{z} - \bar{y}), \\ \underline{z} - \bar{x} &\geq \alpha_\rho(\underline{z} - \underline{y}), \end{aligned}$$

where  $\alpha_\rho \in (0, \frac{1}{2})$  is the unique solution of  $\psi_\rho(\alpha) = 0$  with  $\psi_\rho : (0, 1] \rightarrow \mathbb{R}$  defined by

$$\psi_\rho(\alpha) = \alpha + \alpha^{2-\rho}(1-\alpha)^{\rho-1} + 1 - \rho.$$

Then, the unique non-decreasing coupling  $\pi^\uparrow$  (resp. non-increasing coupling  $\pi^\downarrow$ ) in  $\Pi_M(\mu, \nu)$  is the unique optimal coupling in  $\Pi_M(\mu, \nu)$  that attains  $\overline{\mathcal{M}}_\rho(\mu, \nu)$  (resp.  $\underline{\mathcal{M}}_\rho(\mu, \nu)$ ) when  $\rho \in (1, 2)$  and  $\underline{\mathcal{M}}_\rho(\mu, \nu)$  (resp.  $\overline{\mathcal{M}}_\rho(\mu, \nu)$ ) when  $\rho > 2$ .

**Remark 2.20.** *Let  $\rho \in (1, 2) \cup (2, +\infty)$  and  $\bar{y} < \underline{z}$ . Since, by the fifth and seventh assertions in Lemma 3.10 below,  $\alpha_\rho < \frac{1}{2}$ , it is possible to find  $\underline{x}, \bar{x} \in (\bar{y} + \alpha_\rho(\underline{z} - \bar{y}), \underline{z} - \alpha_\rho(\underline{z} - \bar{y}))$  such that  $\bar{x} - \underline{x} > 0$ . Then for any  $\underline{y} \in [\frac{1}{\alpha_\rho}(\bar{x} - (1 - \alpha_\rho)\underline{z}), \bar{y}]$  and any  $\bar{z} \in (\underline{z}, \frac{1}{\alpha_\rho}(\underline{x} - (1 - \alpha_\rho)\bar{y}))$ , the conditions on the points  $\underline{y}, \bar{y}, \underline{x}, \bar{x}, \underline{z}, \bar{z}$  in Proposition 2.19 are satisfied.*

**Proposition 2.21.** *Let  $p \in (0, 1)$  and  $y_- < y_+ < x_- < x_+ < z_- < z_+$  be such that*

$$\begin{aligned} x_+ - y_- &\geq z_- - x_-, \quad z_+ - x_- \geq x_+ - y_+ \text{ and} \\ (x_- - y_-) \wedge (x_+ - y_+) \wedge (z_- - x_-) \wedge (z_+ - x_+) &\geq (x_- - y_+) \vee (z_- - x_+). \end{aligned} \quad (11)$$

We set  $\mu = p\delta_{x_-} + (1-p)\delta_{x_+}$ ,

$$\begin{aligned} \underline{\nu} &= p \frac{z_- - x_-}{z_- - y_-} \delta_{y_-} + p \frac{x_- - y_-}{z_- - y_-} \delta_{z_-} + (1-p) \frac{z_+ - x_+}{z_+ - y_+} \delta_{y_+} + (1-p) \frac{x_+ - y_+}{z_+ - y_+} \delta_{z_+}, \\ \pi^\uparrow &= p \left( \frac{z_- - x_-}{z_- - y_-} \delta_{(x_-, y_-)} + \frac{x_- - y_-}{z_- - y_-} \delta_{(x_-, z_-)} \right) + (1-p) \left( \frac{z_+ - x_+}{z_+ - y_+} \delta_{(x_+, y_+)} + \frac{x_+ - y_+}{z_+ - y_+} \delta_{(x_+, z_+)} \right), \\ \bar{\nu} &= p \frac{z_+ - x_-}{z_+ - y_+} \delta_{y_+} + p \frac{x_- - y_+}{z_+ - y_+} \delta_{z_+} + (1-p) \frac{z_- - x_+}{z_- - y_-} \delta_{y_-} + (1-p) \frac{x_+ - y_-}{z_- - y_-} \delta_{z_-}, \\ \pi^\downarrow &= p \left( \frac{z_+ - x_-}{z_+ - y_+} \delta_{(x_-, y_+)} + \frac{x_- - y_+}{z_+ - y_+} \delta_{(x_-, z_+)} \right) + (1-p) \left( \frac{z_- - x_+}{z_- - y_-} \delta_{(x_+, y_-)} + \frac{x_+ - y_-}{z_- - y_-} \delta_{(x_+, z_-)} \right). \end{aligned}$$

One has  $\pi^\uparrow \in \Pi_M(\mu, \underline{\nu})$ ,  $\text{sq}\#\pi^\uparrow(dx, dy) = \inf_{cx} \{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \underline{\nu})\}$  and  $\pi^\downarrow \in \Pi_M(\mu, \bar{\nu})$ ,  $\text{sq}\#\pi^\downarrow(dx, dy) = \sup_{cx} \{\text{sq}\#\pi(dx, dy) : \pi \in \Pi_M(\mu, \bar{\nu})\}$ .

**Remark 2.22.** • *If  $x_- - y_+ = z_- - x_+$ , then inequality (11) is satisfied. So for  $y_+ < x_- < x_+ < z_-$  such that  $x_- - y_+ = z_- - x_+$ , we may choose  $z_+$  large enough and  $y_-$  small enough so that the conditions in Proposition 2.21 are met.*

- We deduce that for any convex function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$

$$\int_{\mathbb{R}^2} \varphi(|y-x|^2) \pi^\uparrow(dx, dy) = \inf_{\pi \in \Pi_M(\mu, \underline{\nu})} \int_{\mathbb{R}^2} \varphi(|y-x|^2) \pi(dx, dy) \text{ and}$$

$$\int_{\mathbb{R}^2} \varphi(|y-x|^2) \pi^\downarrow(dx, dy) = \sup_{\pi \in \Pi_M(\mu, \bar{\nu})} \int_{\mathbb{R}^2} \varphi(|y-x|^2) \pi(dx, dy).$$

Indeed the function  $\varphi(z) = \mathbf{1}_{\{z=0\}} \lim_{w \rightarrow 0^+} \varphi(w) + \mathbf{1}_{\{z>0\}} \varphi(z)$  is the uniform limit of the restriction to  $\mathbb{R}_+$  of the non-decreasing sequence  $(\varphi_n)_{n \geq 1}$  of convex functions on  $\mathbb{R}$  defined by  $\varphi_n(z) = \mathbf{1}_{\{z < \frac{1}{n}\}} (\varphi(\frac{1}{n}) + \varphi'_l(\frac{1}{n})(z - \frac{1}{n})) + \mathbf{1}_{\{z \geq \frac{1}{n}\}} \varphi(z)$  where  $\varphi'_l$  denotes the left-hand derivative of  $\varphi$ . Moreover,  $\varphi(0) - \lim_{w \rightarrow 0^+} \varphi(w) \geq 0$  and for probability measures  $\theta, \vartheta \in \mathcal{P}_1(\mathbb{R})$  such that  $\theta(\mathbb{R}_+) = 1 = \vartheta(\mathbb{R}_+)$ ,  $\theta \leq_{cx} \vartheta \Rightarrow \theta(\{0\}) \leq \vartheta(\{0\})$ .

In particular since  $\mathbb{R}_+ \ni z \mapsto z^{\rho/2}$  is concave for  $\rho \in (0, 2)$  and convex for  $\rho > 2$ ,

$$\forall \rho > 0, (\rho - 2) \int_{\mathbb{R}^2} |y-x|^\rho \pi^\uparrow(dx, dy) = \inf_{\pi \in \Pi_M(\mu, \underline{\nu})} (\rho - 2) \int_{\mathbb{R}^2} |y-x|^\rho \pi(dx, dy) \text{ and}$$

$$(\rho - 2) \int_{\mathbb{R}^2} |y-x|^\rho \pi^\downarrow(dx, dy) = \sup_{\pi \in \Pi_M(\mu, \bar{\nu})} (\rho - 2) \int_{\mathbb{R}^2} |y-x|^\rho \pi(dx, dy).$$

## 2.5.2 Non-preservation of the optimality

For  $p \in (0, 1)$  and  $y_- < y_+ < x_- < x_+ < z_- < z_+$ , we set

$$\begin{aligned} \mu &= p \delta_{x_-} + (1-p) \delta_{x_+}, \\ \bar{\nu} &= p \frac{z_+ - x_-}{z_+ - y_-} \delta_{y_-} + p \frac{x_- - y_-}{z_+ - y_-} \delta_{z_+} + (1-p) \frac{z_- - x_+}{z_- - y_+} \delta_{y_+} + (1-p) \frac{x_+ - y_+}{z_- - y_+} \delta_{z_-}, \\ \underline{\nu} &= p \frac{z_- - x_-}{z_- - y_+} \delta_{y_+} + p \frac{x_- - y_+}{z_- - y_+} \delta_{z_-} + (1-p) \frac{z_+ - x_+}{z_+ - y_-} \delta_{y_-} + (1-p) \frac{x_+ - y_-}{z_+ - y_-} \delta_{z_+}. \end{aligned}$$

The couplings

$$\begin{aligned} \bar{\pi}^* &= p \left( \frac{z_+ - x_-}{z_+ - y_-} \delta_{(x_-, y_-)} + \frac{x_- - y_-}{z_+ - y_-} \delta_{(x_-, z_+)} \right) + (1-p) \left( \frac{z_- - x_+}{z_- - y_+} \delta_{(x_+, y_+)} + \frac{x_+ - y_+}{z_- - y_+} \delta_{(x_+, z_-)} \right), \\ \underline{\pi}^* &= p \left( \frac{z_- - x_-}{z_- - y_+} \delta_{(x_-, y_+)} + \frac{x_- - y_+}{z_- - y_+} \delta_{(x_-, z_-)} \right) + (1-p) \left( \frac{z_+ - x_+}{z_+ - y_-} \delta_{(x_+, y_-)} + \frac{x_+ - y_-}{z_+ - y_-} \delta_{(x_+, z_+)} \right), \end{aligned}$$

respectively belong to  $\Pi_M(\mu, \bar{\nu})$  and  $\Pi_M(\mu, \underline{\nu})$ . Since  $(x_-, z_+)$  and  $(x_+, z_-)$  (resp.  $(x_-, y_+)$  and  $(x_+, y_-)$ ) belong to any Borel subset  $\Gamma$  of  $\mathbb{R}^2$  such that  $\bar{\pi}^*(\Gamma) = 1$  (resp.  $\underline{\pi}^*(\Gamma) = 1$ ),  $\bar{\pi}^*$  and  $\underline{\pi}^*$  are not non-decreasing. Since  $(x_-, y_-)$  and  $(x_+, y_+)$  (resp.  $(x_-, z_-)$  and  $(x_+, z_+)$ ) belong to any Borel subset  $\Gamma$  of  $\mathbb{R}^2$  such that  $\bar{\pi}^*(\Gamma) = 1$  (resp.  $\underline{\pi}^*(\Gamma) = 1$ ),  $\bar{\pi}^*$  and  $\underline{\pi}^*$  are not non-increasing.

**Proposition 2.23.** *Let  $\rho \in (1, 2) \cup (2, +\infty)$  and  $y_- < y_+ < z_- < z_+$  be such that  $z_- - y_- > z_+ - z_-$  and  $z_- - y_+ \geq (\rho - 1)^{\frac{1}{2-\rho}} (z_+ - z_-)$ . For  $\rho \in (1, 2)$  (resp.  $\rho > 2$ ), there exists  $x_\rho \in (y_+, z_-)$  such that for  $x_\rho < x_- < x_+ < z_-$ ,  $\bar{\pi}^*$  is the unique optimal coupling that attains  $\bar{\mathcal{M}}_\rho(\mu, \bar{\nu})$  (resp.  $\underline{\mathcal{M}}_\rho(\mu, \bar{\nu})$ ) and  $\underline{\pi}^*$  is the unique optimal coupling that attains  $\underline{\mathcal{M}}_\rho(\mu, \underline{\nu})$  (resp.  $\bar{\mathcal{M}}_\rho(\mu, \underline{\nu})$ ).*

**Remark 2.24.** *For  $\rho \in (1, 2) \cup (2, +\infty)$ ,  $(\rho - 1)^{\frac{1}{2-\rho}} < 1$  so that when  $y_- < y_+ < z_- < z_+$ , it is enough that  $z_- - y_+ \geq z_+ - z_-$  to ensure  $z_- - y_- > z_+ - z_-$  and  $z_- - y_+ \geq (\rho - 1)^{\frac{1}{2-\rho}} (z_+ - z_-)$ .*

**Remark 2.25.** *Let  $y_- < y_+ < x_- < x_+ < z_- < z_+$  be such that  $z_- - y_+ \geq z_+ - z_-$  and  $\beta = \frac{z_- - y_+}{z_+ + z_- - y_+ - y_-}$ . Since  $\beta \in (0, \frac{1}{2})$ , by Lemma 3.10, the equation  $\alpha_\rho = \beta$  admits a unique root  $\rho_\beta > 1$ . Since  $(1, +\infty) \ni \rho \mapsto \alpha_\rho$  is increasing by the fifth item in Lemma 3.10, for  $\rho \in (1, \rho_\beta)$ ,  $\alpha_\rho < \beta$  and setting  $x_+ = (1 - \alpha_\rho)z_- + \alpha_\rho y_-$ , we have*

$$\begin{aligned} x_+ - y_+ &= z_- - y_+ + \alpha_\rho (y_- - z_-) = z_- - y_+ - \alpha_\rho (z_+ + z_- - y_+ - y_-) + \alpha_\rho (z_+ - y_+) \\ &> z_- - y_+ - \beta (z_+ + z_- - y_+ - y_-) + \alpha_\rho (z_+ - y_+) = \alpha_\rho (z_+ - y_+). \end{aligned}$$

We may choose  $x_- < x_+$  close to  $x_+$  such that  $x_- - y_+ \geq \alpha_\rho(z_+ - y_+)$  and then, by Proposition 2.19, the only martingale coupling in  $\Pi_M(\mu, \bar{\nu})$  that maximizes (resp. minimizes)  $\int |x-y|^\rho \pi(dx, dy)$  when  $\rho \in (1, 2 \wedge \rho_\beta)$  (resp.  $\rho \in (2, \rho_\beta)$ ) is non-decreasing. Hence for  $\rho \in (1, 2) \cup (2, +\infty)$  such that  $\rho < \rho_\beta$ , one has  $x_\rho \geq (1 - \alpha_\rho)z_- + \alpha_\rho y_-$ . In particular, since  $\lim_{\rho \rightarrow 1+} \alpha_\rho = 0$  by the sixth item in Lemma 3.10, we have  $\lim_{\rho \rightarrow 1+} x_\rho = z_-$ .

**Remark 2.26.** By Propositions 2.5 and 2.8, the unique non-decreasing coupling in  $\Pi_M(\mu, \bar{\nu})$  (resp. non-increasing coupling in  $\Pi_M(\mu, \underline{\nu})$ ) is the unique optimal coupling that attains  $\bar{\mathcal{M}}_\rho(\mu, \bar{\nu})$  (resp.  $\underline{\mathcal{M}}_\rho(\mu, \underline{\nu})$ ) for  $\rho \in (0, 1]$ . When  $\bar{\pi}^*$  (resp.  $\underline{\pi}^*$ ) is the unique optimal coupling that attains  $\bar{\mathcal{M}}_\rho(\mu, \bar{\nu})$  (resp.  $\underline{\mathcal{M}}_\rho(\mu, \underline{\nu})$ ) for  $\rho \in (1, 2)$  and  $\bar{\mathcal{M}}_\rho(\mu, \bar{\nu})$  (resp.  $\underline{\mathcal{M}}_\rho(\mu, \underline{\nu})$ ) for  $\rho > 2$ , then  $\inf_{\text{cx}} \{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \bar{\nu})\}$  (resp.  $\sup_{\text{cx}} \{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \underline{\nu})\}$ ) does not belong to  $\{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \bar{\nu})\}$  (resp.  $\{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \underline{\nu})\}$ ).

### 3 Proofs

To recall the necessary optimality condition stated by Beiglböck and Juillet in Lemma 1.11 [3], we first recall their definition of competitors :

**Definition 3.1** (Competitor, taken from Definition 1.10 [3]). Let  $\alpha$  be a measure on  $\mathbb{R} \times \mathbb{R}$  with finite first moment in the second variable. We say that  $\alpha'$ , a measure on the same space, is a competitor of  $\alpha$  if  $\alpha'$  has the same marginals as  $\alpha$  and

$$\int_{y \in \mathbb{R}} \alpha(dx, dy) \text{ a.e.}, \int y \alpha_x(dy) = \int y \alpha'_x(dy).$$

Now, we state a version of Lemma 1.11 [3] slightly reinforced by also taking into account Lemma 3.3 just below.

**Lemma 3.2.** Let  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$  be such that  $\mu \leq_{\text{cx}} \nu$  and  $c : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a measurable cost function such that  $\forall (x, y) \in \mathbb{R}^2$ ,  $c(x, y) \geq a(x) + b(y)$  for measurable functions  $a$  and  $b$  integrable with respect to  $\mu$  and  $\nu$  respectively. When  $\pi \in \Pi_M(\mu, \nu)$  is a minimizing coupling leading to finite cost, there exists a Borel set  $\Gamma \subset \mathbb{R}^2$  such that  $\pi(\Gamma) = 1$ ,

$$\forall x \in \mathbb{R}, \exists y < x \text{ s.t. } (x, y) \in \Gamma \iff \exists z > x \text{ s.t. } (x, z) \in \Gamma, \quad (12)$$

and if  $\alpha$  is a measure finitely supported on  $\Gamma$ , then  $\int c(x, y) \alpha(dx, dy) \leq \int c(x, y) \alpha'(dx, dy)$  for each competitor  $\alpha'$  of  $\alpha$ .

**Lemma 3.3.** Let  $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$  be such that  $\mu \leq_{\text{cx}} \nu$  and  $\pi \in \Pi_M(\mu, \nu)$ . For any Borel subset  $\Gamma$  of  $\mathbb{R}^2$  such that  $\pi(\Gamma) = 1$ , we may find a Borel subset  $\tilde{\Gamma}$  of  $\mathbb{R}^2$  such that  $\tilde{\Gamma} \subset \Gamma$ ,  $\pi(\tilde{\Gamma}) = 1$  and

$$\forall x \in \mathbb{R}, \exists y < x \text{ s.t. } (x, y) \in \tilde{\Gamma} \iff \exists z > x \text{ s.t. } (x, z) \in \tilde{\Gamma}.$$

*Proof of Lemma 3.3.* For  $x \in \mathbb{R}$ , let  $\Gamma_x = \{y \in \mathbb{R} : (x, y) \in \Gamma\}$ . We set

$$A = \{x \in \mathbb{R} : \pi_x(\Gamma_x \cap (-\infty, x)) \times \pi_x(\Gamma_x \cap (x, +\infty)) = 0\},$$

and

$$\tilde{\Gamma} = \Gamma \setminus \bigcup_{x \in A} \{x\} \times \{\Gamma_x \cap \{(-\infty, x) \cup (x, +\infty)\}\}.$$

Since  $\mu(dx)$  almost everywhere,  $\pi_x(\Gamma_x) = 1$  and  $\int_{\Gamma_x} y \pi_x(dy) = \int_{\mathbb{R}} y \pi_x(dy) = x$ ,

$$\mu(dx) \text{ a.e.}, x \in A \Rightarrow \pi_x(\Gamma_x \cap \{(-\infty, x) \cup (x, +\infty)\}) = 0.$$

Therefore

$$\pi \left( \bigcup_{x \in A} \{x\} \times \{\Gamma_x \cap \{(-\infty, x) \cup (x, +\infty)\}\} \right) = \int_{x \in A} \pi_x(\Gamma_x \cap \{(-\infty, x) \cup (x, +\infty)\}) \mu(dx) = 0,$$

which implies  $\pi(\tilde{\Gamma}) = \pi(\Gamma) = 1$ . Let  $\tilde{\Gamma}_x = \{y \in \mathbb{R} : (x, y) \in \tilde{\Gamma}\}$  for  $x \in \mathbb{R}$ . For  $x \in A$ ,  $\tilde{\Gamma}_x \subset \{x\}$ .

For  $x \notin A$ ,  $\tilde{\Gamma}_x \cap (-\infty, x) = \Gamma_x \cap (-\infty, x) \neq \emptyset$  and  $\tilde{\Gamma}_x \cap (x, +\infty) = \Gamma_x \cap (x, +\infty) \neq \emptyset$ , which concludes the proof.  $\square$

### 3.1 Proof of Lemma 2.2

When  $\nu = \mu$ , the conclusion holds since the set under consideration is  $\{\delta_0\}$ . We now suppose that  $\nu \neq \mu$ , which implies that  $\int_{\mathbb{R}} y^2 \nu(dy) > \int_{\mathbb{R}} x^2 \mu(dx)$ . Since  $2(x^2 + y^2) \geq (y - x)^2$ , we have that  $(y - x)^2 > z$  implies  $4x^2 > z$  or  $4y^2 > z$  so that, for  $\pi \in \Pi(\mu, \nu)$  and  $z \geq 0$ ,

$$\pi(\{(x, y) : (y - x)^2 > z\}) \leq f(z) \text{ where } f(z) = (\mu(\{x : 4x^2 > z\}) + \nu(\{y : 4y^2 > z\})) \wedge 1. \quad (13)$$

The non-increasing function  $f$  is right-continuous on  $[0, +\infty)$  and so is  $[0, +\infty) \ni z \mapsto zf(z) + \int_z^{+\infty} f(w)dw$ . We have

$$zf(z) + \int_z^{+\infty} f(w)dw \leq 4 \left( \int \mathbf{1}_{\{x > \sqrt{z}/2\}} x^2 \mu(dx) + \int \mathbf{1}_{\{y > \sqrt{z}/2\}} y^2 \nu(dy) \right) \xrightarrow{z \rightarrow +\infty} 0.$$

On the other hand

$$\int_0^{+\infty} f(w)dw \geq \int_0^{+\infty} \nu(\{y : 4y^2 \geq w\})dw = 4 \int_{\mathbb{R}} y^2 \nu(dy) > \int_{\mathbb{R}} y^2 \nu(dy) - \int_{\mathbb{R}} x^2 \mu(dx).$$

As a consequence  $\bar{z} := \sup\{z > 0 : zf(z) + \int_z^{+\infty} f(w)dw \geq \int_{\mathbb{R}} y^2 \nu(dy) - \int_{\mathbb{R}} x^2 \mu(dx)\}$  belongs to  $(0, +\infty)$  and is such that  $\bar{z} \lim_{z \rightarrow \bar{z}^-} f(z) + \int_{\bar{z}}^{+\infty} f(w)dw \geq \int_{\mathbb{R}} y^2 \nu(dy) - \int_{\mathbb{R}} x^2 \mu(dx) \geq \bar{z}f(\bar{z}) + \int_{\bar{z}}^{+\infty} f(w)dw$ . Therefore we may find  $p \in [f(\bar{z}), \lim_{z \rightarrow \bar{z}^-} f(z)] \subset [0, 1]$  such that  $\bar{z}p + \int_{\bar{z}}^{+\infty} f(w)dw = \int_{\mathbb{R}} y^2 \nu(dy) - \int_{\mathbb{R}} x^2 \mu(dx)$ . The probability measure  $\bar{\eta}$  with cumulative distribution function  $F_{\bar{\eta}}(z) = \mathbf{1}_{\{0 \leq z \leq \bar{z}\}}(z)(1-p) + \mathbf{1}_{\{z > \bar{z}\}}(1-f(z))$  is such that  $\int_{\mathbb{R}} z \bar{\eta}(dz) = \int_0^{+\infty} (1 - F_{\bar{\eta}}(z))dz = \int_{\mathbb{R}} y^2 \nu(dy) - \int_{\mathbb{R}} x^2 \mu(dx)$ . Let  $\pi \in \Pi_M(\mu, \nu)$ ,  $\eta = \text{sq}\#\pi$  and  $G(z) = \int_z^{+\infty} (1 - F_{\bar{\eta}}(w))dw - \int_z^{+\infty} (1 - F_{\eta}(w))dw$  for  $z \in \mathbb{R}$ . By (13), we have  $\forall z > \bar{z}$ ,  $1 - F_{\eta}(z) \leq f(z) = 1 - F_{\bar{\eta}}(z)$  so that  $G(z) \geq 0$  for all  $z \in [\bar{z}, +\infty)$ . Since  $\int_0^{+\infty} (1 - F_{\eta}(w))dw = \int_{\mathbb{R}} z \eta(dz) = \int_{\mathbb{R}} y^2 \nu(dy) - \int_{\mathbb{R}} x^2 \mu(dx)$ ,  $G(0) = 0$  and since  $F_{\bar{\eta}}(z) = F_{\eta}(z) = 0$  for  $z \in (-\infty, 0)$ ,  $G(z) = 0$  for  $z \in (-\infty, 0]$ . The function  $G$  being concave on  $[0, \bar{z}]$  by constancy of  $F_{\bar{\eta}}$  on this interval, we conclude that  $G(z) \geq 0$  for all  $z \in \mathbb{R}$ . By Theorem 3.A.1 (a) [13], this together with the equality of means implies that  $\eta \leq_{cx} \bar{\eta}$ . Therefore  $\{\text{sq}\#\pi : \pi \in \Pi_M(\mu, \nu)\}$  is bounded by  $\bar{\eta}$  for the convex order.

### 3.2 Proofs of the optimality results in Section 2.3 but Proposition 2.8

The proof of Proposition 2.8, which relies on the one of Theorem 2.11, is postponed to the next section.

**Lemma 3.4.** *Let  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$  be concave and increasing. Let  $y < m < z$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined by*

$$f(x) = \frac{z - m}{z - y} \varphi(|x - y|) + \frac{m - y}{z - y} \varphi(|z - x|) - \varphi(|x - m|).$$

*Then,  $f$  is increasing on  $[y, m]$  and decreasing on  $[m, z]$ .*

*Proof of Lemma 3.4.* To check the monotonicity on  $[y, m]$ , we choose  $x_-$  and  $x_+$  such that  $y \leq x_- < x_+ \leq m$ ,

$$\begin{aligned} f(x_+) - f(x_-) &= \frac{z - m}{z - y} (\varphi(x_+ - y) - \varphi(x_- - y) - \varphi(m - x_+) + \varphi(m - x_-)) \\ &\quad + \frac{m - y}{z - y} (\varphi(z - x_+) - \varphi(z - x_-) - \varphi(m - x_+) + \varphi(m - x_-)). \end{aligned}$$

Since  $x_+ - y > x_- - y$  and  $m - x_- > m - x_+$ , we have  $\varphi(x_+ - y) > \varphi(x_- - y)$  and  $\varphi(m - x_-) > \varphi(m - x_+)$  so that the first term in the right-hand side is positive. On the other hand, we know that

$$m - x_+ \leq (z - x_+) \wedge (m - x_-) \leq (z - x_+) \vee (m - x_-) \leq z - x_-.$$

By the concavity of  $\varphi$ , we have

$$\begin{aligned}\varphi(z - x_+) &\geq \frac{z - m}{z - m + x_+ - x_-} \varphi(z - x_-) + \frac{x_+ - x_-}{z - m + x_+ - x_-} \varphi(m - x_+), \\ \varphi(m - x_-) &\geq \frac{z - m}{z - m + x_+ - x_-} \varphi(m - x_+) + \frac{x_+ - x_-}{z - m + x_+ - x_-} \varphi(z - x_-),\end{aligned}$$

which implies  $\varphi(z - x_+) + \varphi(m - x_-) \geq \varphi(z - x_-) + \varphi(m - x_+)$ . Therefore,  $f(x_+) - f(x_-) > 0$  and  $f$  is increasing on  $[y, m]$ .

Similarly, to check the monotonicity on  $[m, z]$ , we choose  $x_-$  and  $x_+$  such that  $m \leq x_- < x_+ \leq z$ ,

$$\begin{aligned}f(x_+) - f(x_-) &= \frac{z - m}{z - y} (\varphi(x_+ - y) - \varphi(x_- - y) - \varphi(x_+ - m) + \varphi(x_- - m)) \\ &\quad + \frac{m - y}{z - y} (\varphi(z - x_+) - \varphi(z - x_-) - \varphi(x_+ - m) + \varphi(x_- - m)).\end{aligned}$$

Since  $z - x_+ < z - x_-$  and  $x_- - m < x_+ - m$ , we have  $\varphi(z - x_+) < \varphi(z - x_-)$  and  $\varphi(x_- - m) < \varphi(x_+ - m)$  so that the second term in the right-hand side is negative. On the other hand, we know that

$$x_- - m \leq (x_+ - m) \wedge (x_- - y) \leq (x_+ - m) \vee (x_- - y) \leq x_+ - y.$$

By the concavity of  $\varphi$ , we have

$$\begin{aligned}\varphi(x_+ - m) &\geq \frac{m - y}{m - y + x_+ - x_-} \varphi(x_- - m) + \frac{x_+ - x_-}{m - y + x_+ - x_-} \varphi(x_+ - y), \\ \varphi(x_- - y) &\geq \frac{m - y}{m - y + x_+ - x_-} \varphi(x_+ - y) + \frac{x_+ - x_-}{m - y + x_+ - x_-} \varphi(x_- - m),\end{aligned}$$

which implies  $\varphi(x_+ - y) + \varphi(x_- - m) \leq \varphi(x_- - y) + \varphi(x_+ - m)$ . Therefore,  $f(x_+) - f(x_-) < 0$  and  $f$  is decreasing on  $[m, z]$ .  $\square$

*Proof of Proposition 2.5.* The cost function  $\mathbb{R}^2 \ni (x, y) \mapsto \varphi(|x - y|)$  is lower-semicontinuous and even continuous when  $\varphi$  is continuous. Since it satisfies the bounds in Remark 2.6, the existence of optimizers follows from Theorem 1 [2]. Let us establish the monotonicity properties. By Lemma 3.2, if  $\tilde{\pi} \in \Pi_M(\mu, \nu)$  is a maximizing (resp. minimizing) martingale coupling which leads to finite costs, then there exists a Borel set  $\Gamma \subseteq \mathbb{R} \times \mathbb{R}$  (resp.  $\Gamma \subseteq [a, b] \times (-\infty, a] \cup [b, +\infty)$ ) with  $\tilde{\pi}(\Gamma) = 1$  such that

$$\forall x \in \mathbb{R}, \exists y < x \text{ s.t. } (x, y) \in \Gamma \iff \exists z > x \text{ s.t. } (x, z) \in \Gamma, \quad (14)$$

and for a measure  $\alpha$  which is finitely supported on  $\Gamma$ , we have

$$\int \varphi(|x - y|) \alpha'(dx, dy) \stackrel{(\text{resp. } \geq)}{\leq} \int \varphi(|x - y|) \alpha(dx, dy), \quad (15)$$

for every competitor  $\alpha'$  of  $\alpha$ . Assume that  $(x_-, y_-), (x_+, y_+) \in \Gamma$  (resp.  $(x_-, y_+), (x_+, y_-) \in \Gamma$ ) with  $y_- \leq x_-, y_+ \leq x_+$  (resp.  $y_+ \leq x_-, y_- \leq x_+$ ) and  $x_- < x_+$ . Let us assume  $y_+ < y_-$  and obtain a contradiction against (15). Then  $y_+ < y_- \leq x_- < x_+$  (resp.  $y_+ < y_- \leq a \leq x_- < x_+$ ). By (14), there exists  $z > x_+$  such that  $(x_+, z) \in \Gamma$  (resp.  $z > x_-$  such that  $(x_-, z) \in \Gamma$ ) and then  $y_+ < y_- \leq a \leq x_- < x_+ \leq b \leq z$ . By Lemma 3.4 applied with  $(y, m, z) = (y_+, y_-, z)$ , the function  $f$  defined by

$$f(x) = \frac{z - y_-}{z - y_+} \varphi(|x - y_+|) + \frac{y_- - y_+}{z - y_+} \varphi(|z - x|) - \varphi(|x - y_-|)$$

is decreasing on  $[y_-, z]$ . We deduce that for

$$\begin{aligned}\beta(dx, dy) &= \delta_{(x_-, y_-)} + \frac{z - y_-}{z - y_+} \delta_{(x_+, y_+)} + \frac{y_- - y_+}{z - y_+} \delta_{(x_+, z)}, \\ \gamma(dx, dy) &= \delta_{(x_+, y_-)} + \frac{z - y_-}{z - y_+} \delta_{(x_-, y_+)} + \frac{y_- - y_+}{z - y_+} \delta_{(x_-, z)},\end{aligned}$$

$$\int \varphi(|x-y|) \beta(dx, dy) - \int \varphi(|x-y|) \gamma(dx, dy) = f(x_+) - f(x_-) < 0.$$

This contradicts inequality (15) since  $\gamma$  (resp.  $\beta$ ) is a competitor of  $\beta$  (resp.  $\gamma$ ), which is finitely supported on  $\Gamma$ . Therefore, we deduce that  $y_- \leq y_+$ .

On the other hand, assume that  $(x_-, z_-), (x_+, z_+) \in \Gamma$  (resp.  $(x_-, z_+), (x_+, z_-) \in \Gamma$ ) with  $x_- \leq z_-, x_+ \leq z_+$  (resp.  $x_- \leq z_+, x_+ \leq z_-$ ) and  $x_- < x_+$ . Let us assume that  $z_+ < z_-$  and obtain a contradiction against (15). Then  $x_- < x_+ \leq z_+ < z_-$  (resp.  $x_- < x_+ \leq b \leq z_+ < z_-$ ). By (14), there exists  $y < x_-$  such that  $(x_-, y) \in \Gamma$  (resp.  $y < x_+$  such that  $(x_+, y) \in \Gamma$ ) and then  $y \leq a \leq x_- < x_+ \leq b \leq z_+ < z_-$ . By Lemma 3.4 applied with  $(y, m, z) = (y, z_+, z_-)$ , the function  $g$  defined by

$$g(x) = \frac{z_- - z_+}{z_- - y} \varphi(|x-y|) + \frac{z_+ - y}{z_- - y} \varphi(|z_- - x|) - \varphi(|x - z_+|)$$

is increasing on  $[y, z_+]$ . We deduce that for

$$\begin{aligned} \beta(dx, dy) &= \delta_{(x_+, z_+)} + \frac{z_- - z_+}{z_- - y} \delta_{(x_-, y)} + \frac{z_+ - y}{z_- - y} \delta_{(x_-, z_-)}, \\ \gamma(dx, dy) &= \delta_{(x_-, z_+)} + \frac{z_- - z_+}{z_- - y} \delta_{(x_+, y)} + \frac{z_+ - y}{z_- - y} \delta_{(x_+, z_-)}, \end{aligned}$$

$$\int |x-y| \beta(dx, dy) - \int |x-y| \gamma(dx, dy) = g(x_-) - g(x_+) < 0.$$

This contradicts inequality (15) since  $\gamma$  (resp.  $\beta$ ) is a competitor of  $\beta$  (resp.  $\gamma$ ), which is finitely supported on  $\Gamma$ . Therefore, we deduce that  $z_- \leq z_+$ .  $\square$

**Remark 3.5.** *The support assumption  $\mu([a, b]) = 1 = 1 - \nu((a, b))$  made in Proposition 2.5 to ensure that any minimizing martingale coupling for the cost  $\varphi(|y-x|)$  is non-increasing is crucial to avoid in the proof the situations  $(x_-, y_+), (x_+, y_-) \in \Gamma$  with  $y_+ \leq x_- < y_- \leq x_+$  and  $(x_-, z_+), (x_+, z_-) \in \Gamma$  with  $x_- \leq z_+ < x_+ \leq z_-$  where we could not derive a contradiction.*

*Proof of Proposition 2.9.* The existence follows from Proposition 2.5. The proof for uniqueness is the same as the one of Theorem 7.3 [3]. Since, from Lemma 3.4, we only know that  $f$  is increasing on  $[y, m]$  and decreasing on  $[m, z]$ , the forbidden cases in Equation (23) [3] should be replaced by

$$y' \leq x' < x \leq y^+ \text{ or } y^- \leq x < x' \leq y'. \quad (16)$$

Actually, in the case  $\varphi(u) = u$  considered in Theorem 7.3 [3],  $f$  is constant on  $(-\infty, y]$  and on  $[z, +\infty)$  and the forbidden cases should be  $y' \leq x' < x \wedge y^+$  or  $y^- \vee x < x' \leq y'$  instead of the more general condition in Equation (23) [3]. Nevertheless, in view of Lemma 3.3, in the second paragraph of the proof of Theorem 7.3 [3], we can assume that  $b^- < a < b^+$  so that we end up with a situation in the restricted set of forbidden cases (16).  $\square$

*Proof of Proposition 2.10.* We set  $x_- = m - (1 + \beta\varepsilon^{1-\rho})^{\frac{1}{\rho}} \varepsilon$  and  $x_+ = m + \varepsilon$ . Let us suppose that  $\varepsilon < (z-m) \wedge \frac{m-y}{(1+\beta)^{\frac{1}{\rho}}} \wedge 1$ . This ensures that  $y < x_-$  and  $x_+ < z$ . For  $x \in (-\infty, x_-] \cup [x_+, +\infty)$ ,  $u_{\mu_\varepsilon}(x) = |x-m| \leq u_\nu(x)$ . The function  $u_{\mu_\varepsilon}$  is affine on  $[x_-, x_+]$ . Since  $y < x_- < m < x_+ < z$ , the function  $u_\nu$  is affine on  $[x_-, m]$  and on  $[m, x_+]$ . The inequality  $\varepsilon < \frac{2(z-m)(m-y)}{3(z-y)}$  in our assumptions implies that  $u_{\mu_\varepsilon}(m) = \frac{2(1+\beta\varepsilon^{1-\rho})^{\frac{1}{\rho}}}{1+(1+\beta\varepsilon^{1-\rho})^{\frac{1}{\rho}}} \varepsilon < \frac{4(z-m)(m-y)}{3(z-y)} = u_\nu(m)$ . Then,

$$\forall x \in \mathbb{R}, \quad u_{\mu_\varepsilon}(x) \leq u_\nu(x) \quad \text{and} \quad \mu_\varepsilon \leq_{cx} \nu.$$

Since  $y < x_- < m < x_+ < z$ , the set  $\Gamma = \{(x_-, y), (x_-, m), (x_-, z), (x_+, y), (x_+, m), (x_+, z)\}$  is such that  $\Gamma \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq x_1\} = \{(x_-, y), (x_+, y), (x_+, m)\}$  and  $\Gamma \cap \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq x_2\} = \{(x_-, m), (x_-, z), (x_+, z)\}$  so that  $\Gamma$  satisfies conditions (a)(b) in Definition 2.3. Since any martingale coupling gives full weight to  $\Gamma$ , it is non-decreasing.



Then, let  $\pi(dx, dy) = \mu_\varepsilon(dx) \pi_x(dy) \in \Pi_M(\mu_\varepsilon, \nu)$ . Since  $u_{\mu_\varepsilon}(m) < u_\nu(m)$  and

$$\begin{aligned} u_{\mu_\varepsilon}(m) &= \mu_\varepsilon(\{x_-\})(m - x_-) + \mu_\varepsilon(\{x_+\})(x_+ - m) \\ u_\nu(m) &= \mu_\varepsilon(\{x_-\}) \int |m - w| \pi_{x_-}(dw) + \mu_\varepsilon(\{x_+\}) \int |m - w| \pi_{x_+}(dw), \end{aligned}$$

we have either  $m - x_- < \int |m - w| \pi_{x_-}(dw)$  so that  $\pi_{x_-}(\{y\})\pi_{x_-}(\{z\}) > 0$  or  $x_+ - m < \int |m - w| \pi_{x_+}(dw)$  so that  $\pi_{x_+}(\{y\})\pi_{x_+}(\{z\}) > 0$ . We deduce that when  $\pi_{x_-}(\{m\})\pi_{x_+}(\{m\}) > 0$ , then either  $\pi_{x_-}(\{m\})\pi_{x_+}(\{y\})\pi_{x_+}(\{z\}) > 0$  or  $\pi_{x_+}(\{m\})\pi_{x_-}(\{y\})\pi_{x_-}(\{z\}) > 0$ . If  $\pi_{x_+}(\{m\}) = 0$  (resp.  $\pi_{x_-}(\{m\}) = 0$ ), then  $\pi_{x_+}(\{y\})\pi_{x_+}(\{z\}) > 0$  (resp.  $\pi_{x_-}(\{y\})\pi_{x_-}(\{z\}) > 0$ ) by the martingale property, and since  $\nu(\{m\}) > 0$ , we have  $\pi_{x_-}(\{m\}) > 0$  (resp.  $\pi_{x_+}(\{m\}) > 0$ ). Therefore, we always have

$$\pi_{x_-}(\{m\})\pi_{x_+}(\{y\})\pi_{x_+}(\{z\}) > 0 \quad \text{or} \quad \pi_{x_-}(\{y\})\pi_{x_-}(\{z\})\pi_{x_+}(\{m\}) > 0,$$

and there exists more than one martingale coupling between  $\mu_\varepsilon$  and  $\nu$  since for  $\alpha$  positive and small enough in the first case and  $-\alpha$  positive and small enough in the second case, adding

$$\alpha \left( \delta_{(x_+, m)} + \frac{z - m}{z - y} \delta_{(x_-, y)} + \frac{m - y}{z - y} \delta_{(x_-, z)} - \delta_{(x_-, m)} - \frac{z - m}{z - y} \delta_{(x_+, y)} + \frac{m - y}{z - y} \delta_{(x_+, z)} \right)$$

to  $\pi$  leads to a distinct coupling in  $\Pi_M(\mu_\varepsilon, \nu)$ . To further parametrize the coupling  $\pi$ , let us apply the martingale property again. For  $x \in \{x_-, x_+\}$ , we have

$$\begin{cases} \pi_x(\{y\}) + \pi_x(\{m\}) + \pi_x(\{z\}) = 1 \\ y \pi_x(\{y\}) + m \pi_x(\{m\}) + z \pi_x(\{z\}) = x \end{cases} \implies \begin{cases} \pi_x(\{y\}) = \frac{z - x - (z - m)\pi_x(\{m\})}{z - y}, \\ \pi_x(\{z\}) = \frac{x - y - (m - y)\pi_x(\{m\})}{z - y}. \end{cases}$$

Since  $\mu_\varepsilon(\{x_+\})\pi_{x_+}(\{m\}) = \nu(\{m\}) - \mu_\varepsilon(\{x_-\})\pi_{x_-}(\{m\})$ , the coupling  $\pi$  can be parameterized with respect to the variable  $\pi_{x_-}(\{m\})$  only as follows:

$$\begin{aligned} \pi &= \gamma + \mu_\varepsilon(\{x_-\})\pi_{x_-}(\{m\}) \left( \frac{z - m}{z - y} \delta_{(x_+, y)} + \frac{m - y}{z - y} \delta_{(x_+, z)} - \delta_{(x_+, m)} \right. \\ &\quad \left. - \frac{z - m}{z - y} \delta_{(x_-, y)} - \frac{m - y}{z - y} \delta_{(x_-, z)} + \delta_{(x_-, m)} \right) \text{ with} \\ \gamma &= \mu_\varepsilon(\{x_-\}) \left( \frac{z - x_-}{z - y} \delta_{(x_-, y)} + \frac{x_- - y}{z - y} \delta_{(x_-, z)} \right) + \mu_\varepsilon(\{x_+\}) \left( \frac{z - x_+}{z - y} \delta_{(x_+, y)} + \frac{x_+ - y}{z - y} \delta_{(x_+, z)} \right) \\ &\quad + \nu(\{m\}) \left( \delta_{(x_+, m)} - \frac{z - m}{z - y} \delta_{(x_+, y)} - \frac{m - y}{z - y} \delta_{(x_+, z)} \right). \end{aligned}$$

For  $\tilde{\rho} \in (0, 1]$ , defining the cost function  $f_{\tilde{\rho}} : [y, z] \rightarrow \mathbb{R}$  by

$$f_{\tilde{\rho}}(x) = \frac{z - m}{z - y} (x - y)^{\tilde{\rho}} + \frac{m - y}{z - y} (z - x)^{\tilde{\rho}} - |x - m|^{\tilde{\rho}},$$

we have

$$\int |x - y|^{\tilde{\rho}} \pi(dx, dy) = \int |x - y|^{\tilde{\rho}} \gamma(dx, dy) + \mu_\varepsilon(\{x_-\})\pi_{x_-}(\{m\})(f_{\tilde{\rho}}(x_+) - f_{\tilde{\rho}}(x_-)).$$

To maximize  $\int |x - y|^{\tilde{\rho}} \pi(dx, dy)$ , we need to maximize  $\pi_{x_-}(\{m\})$  when  $f_{\tilde{\rho}}(x_+) > f_{\tilde{\rho}}(x_-)$  and minimize  $\pi_{x_-}(\{m\})$  when  $f_{\tilde{\rho}}(x_+) < f_{\tilde{\rho}}(x_-)$ . Applying the first order Taylor expansion at  $x = m$  to the first two terms of the function  $f_{\tilde{\rho}}$  with  $\tilde{\rho} \in (0, 1]$ , we get

$$\begin{aligned} f_{\tilde{\rho}}(x_+) - f_{\tilde{\rho}}(x_-) &= \tilde{\rho} \left( \frac{z - m}{z - y} (m - y)^{\tilde{\rho}-1} - \frac{m - y}{z - y} (z - m)^{\tilde{\rho}-1} \right) (x_+ - x_-) + o(x_+ - x_-) \\ &\quad + (m - x_-)^{\tilde{\rho}} - (x_+ - m)^{\tilde{\rho}}. \end{aligned} \tag{17}$$

Let us suppose that  $\rho \in (0, 1)$ . We have that  $(m - x_-)^\rho - (x_+ - m)^\rho = \beta\varepsilon$ . When  $\varepsilon \rightarrow 0+$ ,  $(1 + \beta\varepsilon^{1-\rho})^{\frac{1}{\rho}} \rightarrow 1$  so that  $x_+ - x_- \sim 2\varepsilon$  and

$$f_\rho(x_+) - f_\rho(x_-) = \left( 2\rho \left( \frac{z-m}{z-y} (m-y)^{\rho-1} - \frac{m-y}{z-y} (z-m)^{\rho-1} \right) + \beta \right) \varepsilon + o(\varepsilon).$$

Since  $\beta \in \left( 0, 2\rho \left( \frac{m-y}{z-y} (z-m)^{\rho-1} - \frac{z-m}{z-y} (m-y)^{\rho-1} \right) \right)$  by assumption, for  $\varepsilon$  positive small enough,  $f_\rho(x_+) - f_\rho(x_-) < 0$ . On the other hand, for  $\rho' \in (0, \rho)$ , we have  $(1 + \beta\varepsilon^{1-\rho})^{\frac{\rho'}{\rho}} - 1 \sim \frac{\rho'}{\rho} \beta \varepsilon^{1-\rho}$  so that  $(m - x_-)^{\rho'} - (x_+ - m)^{\rho'} \sim \frac{\rho'}{\rho} \beta \varepsilon^{1-\rho+\rho'}$ . Hence, by (17) for  $\tilde{\rho} = \rho'$ ,  $f_{\rho'}(x_+) - f_{\rho'}(x_-) \sim \frac{\rho'}{\rho} \beta \varepsilon^{1-\rho+\rho'} > 0$ .

Finally, let us suppose that  $\rho = 1$ . We have

$$f_1(x_+) - f_1(x_-) = \frac{2\varepsilon(\beta(z-m) - (2m-y-z))}{z-y}.$$

Since  $y < \frac{y+z}{2} < m < z$  and  $\beta < \frac{2m-y-z}{z-m}$  by assumption, we have  $f_1(x_+) - f_1(x_-) < 0$ . For  $\rho' \in (0, 1)$ ,  $(m - x_-)^{\rho'} - (x_+ - m)^{\rho'} \sim \left( (1 + \beta)^{\rho'} - 1 \right) \varepsilon^{\rho'}$  and, since  $x_+ - x_- = (2 + \beta)\varepsilon$ , by (17) applied with  $\tilde{\rho} = \rho'$ ,  $f_{\rho'}(x_+) - f_{\rho'}(x_-) \sim \left( (1 + \beta)^{\rho'} - 1 \right) \varepsilon^{\rho'} > 0$ .

Therefore, for  $0 < \rho' < \rho \leq 1$  and  $\varepsilon$  positive small enough, to maximize  $\int |x-y|^{\rho'} \pi(dx, dy)$ , we need to maximize  $\pi_{x_-}(\{m\})$  and to maximize  $\int |x-y|^\rho \pi(dx, dy)$ , we need to minimize  $\pi_{x_-}(\{m\})$ .  $\square$

### 3.3 Proof of the results in Section 2.4 and Proposition 2.8

*Proof of Corollary 2.12.* (i) As  $\mu \neq \nu$ , the non vacuity of  $\pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$  implies that  $\nu_l + \nu_r$  is not the zero measure. Since

$$\begin{aligned} \Pi_M(\mu, \nu, \nu_l, \nu_r) &\ni \pi \mapsto \frac{\pi(dx, dy) + (\nu_l + \nu_r - \nu)(dx)\delta_x(dy)}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})} \\ &\in \Pi_M \left( \frac{\mu + \nu_l + \nu_r - \nu}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}, \frac{\nu_l + \nu_r}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}, \frac{\nu_l}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}, \frac{\nu_r}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})} \right) \end{aligned}$$

is a bijection, the statement follows from Theorem 2.11 (concerning (6), this relies on the fact that the corresponding mapping between the integrals is affine with a positive factor).

(ii) Let  $\pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$  be non-decreasing and let  $\Gamma$  denote a Borel subset of  $\mathbb{R} \times \mathbb{R}$  satisfying the properties in Definition 2.3. Since  $\pi(\{(x, x)\}) \leq \mu(\{x\}) \wedge \nu(\{x\})$ , we remove  $\{(x, x) : \mu(\{x\}) > 0 \text{ and } \nu(\{x\}) = 0\} \cap \Gamma$  from  $\Gamma$ , while preserving these properties. Since  $\pi(\Gamma) = 1$ , we have

$$\nu_0^\pi(\mathbb{R}) = \pi(\{(x, x) : x \in \mathbb{R}\}) = \pi(\{(x, x) : x \in \mathbb{R}\} \cap \Gamma) = \nu_0^\pi(\{x \in \mathbb{R} : (x, x) \in \Gamma\}).$$

Let  $x \in \mathbb{R}$  be such that  $(x, x) \in \Gamma$ . Then,  $\Gamma \subset \{(-\infty, x] \times (-\infty, x]\} \cup \{[x, +\infty) \times [x, +\infty)\}$  and either  $\mu(\{x\}) > 0$  or

$$\begin{aligned} u_\nu(x) &= \int_{\mathbb{R}^2} |z-x| \pi(dy, dz) = \int_{(y,z) \in (-\infty, x] \times \mathbb{R}} (x-z) \pi_y(dz) \mu(dy) + \int_{(y,z) \in (x, +\infty) \times \mathbb{R}} (z-x) \pi_y(dz) \mu(dy) \\ &= \int_{(-\infty, x)} (x-y) \mu(dy) + \int_{(x, +\infty)} (y-x) \mu(dy) = u_\mu(x). \end{aligned}$$

Hence

$$\nu_0^\pi(\mathbb{R}) = \nu_0^\pi(\{x \in \mathbb{R} : u_\nu(x) = u_\mu(x) \text{ or } (u_\nu(x) - u_\mu(x)) \times (\mu(\{x\}) \wedge \nu(\{x\})) > 0\}).$$

The conclusion follows since  $\nu_0^\pi \leq \mu \wedge \nu$  by definition and  $\nu_0^\pi(dx) \geq \mathbb{1}_{\{u_\mu(x) = u_\nu(x)\}} \mu(dx)$  from the decomposition in irreducible components stated in [3, Theorem A.4].

(iii) Since for any non-decreasing coupling  $\pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$ ,  $\frac{\pi(dx, dy) + (\nu_l + \nu_r - \nu)(dx)\delta_x(dy)}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}$  is non-decreasing, the uniqueness is ensured by (i).  $\square$

The proof of Theorem 2.11 relies on the following lemmas.

**Lemma 3.6.** *Let  $\eta \in \mathcal{P}(\mathbb{R})$  and  $\hat{\eta}$  be a non-negative measure on the real line such that  $\hat{\eta} \leq \eta$ . For  $f : \mathbb{R} \rightarrow \mathbb{R}$  non-decreasing such that  $\int_{\mathbb{R}} |f(x)|\eta(dx) < +\infty$ , one has  $\int_{\mathbb{R}} |f(x)|\hat{\eta}(dx) < +\infty$  and*

$$\int_0^{\hat{\eta}(\mathbb{R})} f(F_{\hat{\eta}}^{-1}(u))du \leq \int_{\mathbb{R}} f(x)\hat{\eta}(dx) \leq \int_{1-\hat{\eta}(\mathbb{R})}^1 f(F_{\eta}^{-1}(u))du,$$

and if  $\hat{\eta}(\mathbb{R}) > 0$ ,  $\frac{1}{\hat{\eta}(\mathbb{R})} \int_0^{\hat{\eta}(\mathbb{R})} \delta_{F_{\hat{\eta}}^{-1}(u)} du \leq_{st} \frac{\hat{\eta}}{\hat{\eta}(\mathbb{R})} \leq_{st} \frac{1}{\hat{\eta}(\mathbb{R})} \int_{1-\hat{\eta}(\mathbb{R})}^1 \delta_{F_{\eta}^{-1}(u)} du.$

**Lemma 3.7.** *For  $\mu, \nu \in \mathcal{P}(\mathbb{R})$ ,*

$$\exists \pi \in \Pi(\mu, \nu) \text{ s.t. } \pi(\{(x, y) \in \mathbb{R}^2 : x > y\}) = 1 \iff du \text{ a.e. on } (0, 1), F_{\mu}^{-1}(u) > F_{\nu}^{-1}(u).$$

*Proof of Lemma 3.6.* Since the inequalities are obvious when  $\hat{\eta}(\mathbb{R}) = 0$ , we suppose that  $\hat{\eta}(\mathbb{R}) > 0$  and denote  $\tilde{\eta} = \frac{\hat{\eta}}{\hat{\eta}(\mathbb{R})}$ . Since  $\hat{\eta} \leq \eta$ ,  $\int_{\mathbb{R}} |f(x)|\hat{\eta}(dx) \leq \int_{\mathbb{R}} |f(x)|\eta(dx) < +\infty$ . Moreover, by the inverse transform sampling,

$$\int_{\mathbb{R}} f(x)\hat{\eta}(dx) = \hat{\eta}(\mathbb{R}) \int_{\mathbb{R}} f(x)\tilde{\eta}(dx) = \hat{\eta}(\mathbb{R}) \int_0^1 f(F_{\tilde{\eta}}^{-1}(u))du.$$

For  $u \in (0, 1)$ , using (2) for the first inequality then  $\hat{\eta}(\mathbb{R})\tilde{\eta} = \hat{\eta} \leq \eta$  for the second, we have

$$\hat{\eta}(\mathbb{R})u \leq \hat{\eta}(\mathbb{R})F_{\tilde{\eta}}(F_{\tilde{\eta}}^{-1}(u)) \leq F_{\eta}(F_{\tilde{\eta}}^{-1}(u))$$

so that  $F_{\tilde{\eta}}^{-1}(u) \geq F_{\eta}^{-1}(\hat{\eta}(\mathbb{R})u)$ . With the monotonicity of  $f$ , we deduce that

$$\int_{\mathbb{R}} f(x)\hat{\eta}(dx) \geq \hat{\eta}(\mathbb{R}) \int_0^1 f(F_{\eta}^{-1}(\hat{\eta}(\mathbb{R})u))du = \int_0^{\hat{\eta}(\mathbb{R})} f(F_{\eta}^{-1}(v))dv.$$

On the other hand, still for  $u \in (0, 1)$ , using  $\tilde{\eta} \leq \frac{\eta}{\hat{\eta}(\mathbb{R})}$  for the first inequality and (2) for the second,

$$\begin{aligned} 1 - F_{\tilde{\eta}}(F_{\tilde{\eta}}^{-1}(1 - \hat{\eta}(\mathbb{R})u)) &= \tilde{\eta}((F_{\tilde{\eta}}^{-1}(1 - \hat{\eta}(\mathbb{R})u), +\infty)) \leq \frac{\eta((F_{\eta}^{-1}(1 - \hat{\eta}(\mathbb{R})u), +\infty))}{\hat{\eta}(\mathbb{R})} \\ &= \frac{1 - F_{\eta}(F_{\eta}^{-1}(1 - \hat{\eta}(\mathbb{R})u))}{\hat{\eta}(\mathbb{R})} \leq u \end{aligned}$$

so that  $F_{\tilde{\eta}}^{-1}(1 - u) \leq F_{\eta}^{-1}(1 - \hat{\eta}(\mathbb{R})u)$ . We conclude that

$$\int_{\mathbb{R}} f(x)\hat{\eta}(dx) = \hat{\eta}(\mathbb{R}) \int_0^1 f(F_{\tilde{\eta}}^{-1}(1 - u))du \leq \hat{\eta}(\mathbb{R}) \int_0^1 f(F_{\eta}^{-1}(1 - \hat{\eta}(\mathbb{R})u))du = \int_{1-\hat{\eta}(\mathbb{R})}^1 f(F_{\eta}^{-1}(v))dv. \quad \square$$

*Proof of Lemma 3.7.* Since the image of the Lebesgue measure on  $(0, 1)$  by  $(F_{\mu}^{-1}, F_{\nu}^{-1})$  belongs to  $\Pi(\mu, \nu)$ , the sufficient condition is obvious. To prove the necessary condition, we let  $(X, Y)$  be distributed according to  $\pi \in \Pi(\mu, \nu)$  such that  $\pi(\{(x, y) \in \mathbb{R}^2 : x > y\}) = 1$ . We have  $\mathbb{P}(X > Y) = 1$  and, since  $\mathbb{Q}$  is dense in  $\mathbb{R}$ ,  $\{X > Y\} = \bigcup_{q \in \mathbb{Q}} \{X \geq q > Y\}$ . Since  $\mathbb{Q}$  is countable, there exists a bijection  $\mathbb{N} \ni n \mapsto q_n \in \mathbb{Q}$ . For  $n \in \mathbb{N}$ , let  $A_n = \{X \geq q_n > Y\} \cap \left\{ \bigcup_{k=0}^{n-1} \{X \geq q_k > Y\} \right\}^c$ . The events  $(A_n)_{n \in \mathbb{N}}$  are disjoint and such that  $\sum_{n \in \mathbb{N}} \mathbb{P}(A_n) = \mathbb{P}(X > Y) = 1$ . The discrete random variable  $Z = \sum_{n \in \mathbb{N}} q_n \mathbf{1}_{A_n}$  is such that  $\mathbb{P}(X \geq Z > Y) = 1$  so that  $\nu \leq_{st} \eta \leq_{st} \mu$ , where  $\eta$  denotes the distribution of  $Z$ . For  $n \in \mathbb{N}$ , we have

$$F_{\eta}(q_n) = \mathbb{P}(Z \leq q_n) = \mathbb{P}(Y < Z \leq q_n) \leq \mathbb{P}(Y < q_n) = F_{\nu}(q_n -).$$

We deduce that for all  $u \in (F_{\eta}(q_n -), F_{\eta}(q_n))$ ,  $F_{\nu}^{-1}(u) < q_n = F_{\eta}^{-1}(u)$ . Since

$$\int_0^1 \mathbf{1}_{\bigcup_{n \in \mathbb{N}} (F_{\eta}(q_n -), F_{\eta}(q_n))}(u)du = \sum_{n \in \mathbb{N}} \mathbb{P}(Z = q_n) = 1,$$

and  $F_{\eta}^{-1} \leq F_{\mu}^{-1}$  as  $\eta \leq_{st} \mu$ , the statement follows.  $\square$

*Proof of Theorem 2.11.* Since  $\mu \neq \nu$ , the non vacuity of  $\pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$  implies that  $\nu_l + \nu_r$  is not the zero measure. Since the equality of the means of  $\mu$  and  $\nu$  implies that  $0 = \int_{\mathbb{R}^2} (y - x) \mathbb{1}_{\{y < x\}} \pi_x(dy) \mu(dx) + \int_{\mathbb{R}^2} (y - x) \mathbb{1}_{\{y > x\}} \pi_x(dy) \mu(dx)$ , then  $\nu_l \neq 0$  and  $\nu_r \neq 0$ .

Let  $\pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$ . Since for  $x < x'$ ,

$$0 \leq \int_{\mathbb{R}^2} \mathbb{1}_{\{z < y \leq x'\}} \pi(dy, dz) - \int_{\mathbb{R}^2} \mathbb{1}_{\{z < y \leq x\}} \pi(dy, dz) \leq \int_{\mathbb{R}^2} \mathbb{1}_{\{x < y \leq x'\}} \pi(dy, dz) = F_\mu(x') - F_\mu(x),$$

there exists a function  $\phi_\pi$  such that

$$\phi_\pi(F_\mu(x)) = \int_{\mathbb{R}^2} \mathbb{1}_{\{z < y \leq x\}} \pi(dy, dz) \in [0, \nu_l(\mathbb{R})]. \quad (18)$$

One has, using  $\nu_0^\pi(\mathbb{R}) = 0$ ,

$$\nu_r(\mathbb{R}) \geq \int_{\mathbb{R}^2} \mathbb{1}_{\{y \leq x, y < z\}} \pi(dy, dz) = \int_{\mathbb{R}^2} \mathbb{1}_{\{y \leq x\}} \pi(dy, dz) - \int_{\mathbb{R}^2} \mathbb{1}_{\{z < y \leq x\}} \pi(dy, dz) = F_\mu(x) - \phi_\pi(F_\mu(x)).$$

Let  $\tilde{\nu}_l = \frac{\nu_l}{\nu_l(\mathbb{R})}$ ,  $\tilde{\nu}_r = \frac{\nu_r}{\nu_r(\mathbb{R})}$  and for  $u \in [0, 1]$  and  $v \in [(u - \nu_r(\mathbb{R}))^+, u \wedge \nu_l(\mathbb{R})]$ ,

$$G(u, v) = \int_0^v F_{\tilde{\nu}_l}^{-1} \left( \frac{w}{\nu_l(\mathbb{R})} \right) dw + \int_0^{u-v} F_{\tilde{\nu}_r}^{-1} \left( \frac{w}{\nu_r(\mathbb{R})} \right) dw.$$

We are going to prove the existence of a non-decreasing function  $\phi_\uparrow : [0, 1] \rightarrow [0, \nu_l(\mathbb{R})]$  such that  $u \mapsto u - \phi_\uparrow(u)$  is also non-decreasing and

$$\forall u \in [0, 1], \phi_\uparrow(u) \in [(u - \nu_r(\mathbb{R}))^+, u \wedge \nu_l(\mathbb{R})] \text{ and } \int_0^u F_\mu^{-1}(w) dw = G(u, \phi_\uparrow(u)). \quad (19)$$

The existence of  $\phi_\uparrow(F_\mu(x)) \in [(F_\mu(x) - \nu_r(\mathbb{R}))^+, \phi_\pi(F_\mu(x))]$  for  $x \in \mathbb{R}$  is addressed in **Step 1** below. **Step 2** deals with the monotonicity of  $x \mapsto \phi_\uparrow(F_\mu(x))$  and  $x \mapsto F_\mu(x) - \phi_\uparrow(F_\mu(x))$ . The existence of  $\phi_\uparrow(u)$  for  $u \in \bigcup_{x \in \mathbb{R}} [F_\mu(x-), F_\mu(x))$ , and the monotonicity of  $u \mapsto \phi_\uparrow(u)$  and  $u \mapsto u - \phi_\uparrow(u)$  on  $(0, 1)$  are addressed in **Step 3**, while (5) is proved in **Step 4**. When  $F_\mu(x) > 0$  for each  $x \in \mathbb{R}$  (resp.  $F_\mu(x) < 1$  for each  $x \in \mathbb{R}$ ), the function  $\phi_\uparrow$  is continuously and monotonically extended by  $\phi_\uparrow(0) = 0$  since  $\lim_{u \rightarrow 0^+} (u - \nu_r(\mathbb{R}))^+ = 0 = \lim_{u \rightarrow 0^+} u \wedge \nu_l(\mathbb{R})$  (resp.  $\phi_\uparrow(1) = \nu_l(\mathbb{R})$  since  $\lim_{u \rightarrow 1^-} (u - \nu_r(\mathbb{R}))^+ = \nu_l(\mathbb{R}) = \lim_{u \rightarrow 1^-} u \wedge \nu_l(\mathbb{R})$ ) so that (19) holds for  $u = 0$  (resp.  $u = 1$ ). We are now going to prove the existence of a non-decreasing coupling  $\pi^\uparrow \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$  such that  $\phi_{\pi^\uparrow} = \phi_\uparrow$ . Because of the monotonicity of  $[0, 1] \ni u \mapsto \phi_\uparrow(u)$  and  $[0, 1] \ni u \mapsto u - \phi_\uparrow(u)$ , these two functions are 1-Lipschitz and therefore absolutely continuous. Up to modifying the derivative  $\phi_\uparrow'$  of  $\phi_\uparrow$  in the sense of distributions on a Lebesgue negligible subset of  $(0, 1)$ , we may suppose that it is  $[0, 1]$ -valued. The functions  $[0, 1] \ni u \mapsto \int_0^{\phi_\uparrow(u)} F_{\tilde{\nu}_l}^{-1} \left( \frac{w}{\nu_l(\mathbb{R})} \right) dw$  and  $[0, 1] \ni u \mapsto \int_0^{u - \phi_\uparrow(u)} F_{\tilde{\nu}_r}^{-1} \left( \frac{w}{\nu_r(\mathbb{R})} \right) dw$  are convex and continuous and therefore absolutely continuous. Using [11, Lemma 1.2] to differentiate (19), we obtain

$$F_\mu^{-1}(u) = \phi_\uparrow'(u) F_{\tilde{\nu}_l}^{-1} \left( \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})} \right) + (1 - \phi_\uparrow'(u)) F_{\tilde{\nu}_r}^{-1} \left( \frac{u - \phi_\uparrow(u)}{\nu_r(\mathbb{R})} \right), \quad du \text{ a.e. on } (0, 1), \quad (20)$$

with the convention that the first (resp. second) product in the right-hand side is 0 when  $\phi_\uparrow'(u) = 0$  (resp.  $\phi_\uparrow'(u) = 1$ ). We set

$$\begin{aligned} \pi^\uparrow(dx, dy) = & \int_0^1 \left( \phi_\uparrow'(u) \delta_{(F_\mu^{-1}(u), F_{\tilde{\nu}_l}^{-1}(\frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})}))} \right. \\ & \left. + (1 - \phi_\uparrow'(u)) \delta_{(F_\mu^{-1}(u), F_{\tilde{\nu}_r}^{-1}(\frac{u - \phi_\uparrow(u)}{\nu_r(\mathbb{R})}))} \right) (dx, dy) du. \end{aligned}$$

We have  $\int_{y \in \mathbb{R}} \pi^\uparrow(dx, dy) = \int_0^1 \delta_{F_\mu^{-1}(u)}(dx) du$ , so that, by the inverse transform sampling, the first marginal of  $\pi^\uparrow$  is  $\mu$ . Using Lemma 2.6 in [8] like in the proof of Proposition 2.3 in [8], one checks

that  $\pi^\uparrow$  is a martingale coupling. We are next going to check that

$$\phi'_\uparrow(u)du \text{ a.e. on } (0, 1), F_{\tilde{\nu}_l}^{-1} \left( \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})} \right) < F_\mu^{-1}(u) \quad (21)$$

$$\text{and } (1 - \phi'_\uparrow(u))du \text{ a.e. on } (0, 1), F_\mu^{-1}(u) < F_{\tilde{\nu}_r}^{-1} \left( \frac{u - \phi_\uparrow(u)}{\nu_r(\mathbb{R})} \right). \quad (22)$$

With the definition of  $\pi^\uparrow$ , we deduce that  $\nu_l^{\pi^\uparrow} = \int_0^1 \phi'_\uparrow(u) \delta_{F_{\tilde{\nu}_l}^{-1} \left( \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})} \right)} du$ . For  $g : [0, 1] \rightarrow \mathbb{R}$  measurable and bounded, by Proposition 4.9 [12] and the remark just above applied with  $f(u) = g \left( \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})} \right)$  and  $A_u = \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})}$ ,

$$\int_0^1 g \left( \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})} \right) \frac{\phi'_\uparrow(u)}{\nu_l(\mathbb{R})} du = \int_0^1 g(v) dv. \quad (23)$$

Hence  $\nu_l^{\pi^\uparrow} = \nu_l(\mathbb{R}) \int_0^1 \delta_{F_{\tilde{\nu}_l}^{-1}(w)} dw = \nu_l(\mathbb{R}) \tilde{\nu}_l = \nu_l$ . In the same way,  $\nu_r^{\pi^\uparrow} = \int_0^1 (1 - \phi'_\uparrow(u)) \delta_{F_{\tilde{\nu}_r}^{-1} \left( \frac{u - \phi_\uparrow(u)}{\nu_r(\mathbb{R})} \right)} du = \nu_r$ , so that, since  $\nu = \nu_l + \nu_r$ ,  $\pi^\uparrow \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$ . Let

$$\begin{aligned} \Gamma = & \left\{ \left\{ \left( F_\mu^{-1}(u), F_{\tilde{\nu}_l}^{-1} \left( \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})} \right) \right) : u \in \mathbb{R} \right\} \cap \{(x, y) \in \mathbb{R}^2 : y < x\} \right\} \\ & \cup \left\{ \left\{ \left( F_\mu^{-1}(u), F_{\tilde{\nu}_r}^{-1} \left( \frac{u - \phi_\uparrow(u)}{\nu_r(\mathbb{R})} \right) \right) : u \in \mathbb{R} \right\} \cap \{(x, y) \in \mathbb{R}^2 : x < y\} \right\}. \end{aligned}$$

The definition of  $\pi^\uparrow$  combined with (21) and (22) ensures that  $\pi^\uparrow(\Gamma) = 1$ . By monotonicity of  $u \mapsto F_\mu^{-1}(u)$ ,  $u \mapsto F_{\tilde{\nu}_l}^{-1} \left( \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})} \right)$  and  $u \mapsto F_{\tilde{\nu}_r}^{-1} \left( \frac{u - \phi_\uparrow(u)}{\nu_r(\mathbb{R})} \right)$ ,  $\Gamma$  satisfies conditions (a) and (b) in Definition 2.3. Hence  $\pi^\uparrow$  is non-decreasing.

To check (21), we introduce  $\tilde{\mu}_l(dy) = \frac{1}{\nu_l(\mathbb{R})} \int_{z \in \mathbb{R}} \mathbb{1}_{\{z < y\}} \pi(dy, dz)$ . As  $\frac{\mathbb{1}_{\{z < y\}}}{\nu_l(\mathbb{R})} \pi(dy, dz) \in \Pi(\tilde{\mu}_l, \tilde{\nu}_l)$ , Lemma 3.7 implies that  $du$  a.e.,  $F_{\tilde{\nu}_l}^{-1}(u) < F_{\tilde{\mu}_l}^{-1}(u)$ . With (23), we deduce that

$$\phi'_\uparrow(u)du \text{ a.e. on } (0, 1), F_{\tilde{\nu}_l}^{-1} \left( \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})} \right) < F_{\tilde{\mu}_l}^{-1} \left( \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})} \right). \quad (24)$$

For  $x \in \mathbb{R}$ , by (18),  $F_{\tilde{\mu}_l}(x) = \frac{\phi_\pi(F_\mu(x))}{\nu_l(\mathbb{R})}$  and since  $F_\mu(F_\mu^{-1}(F_\mu(x))) = F_\mu(x)$ ,  $\phi_\pi(F_\mu(F_\mu^{-1}(F_\mu(x)))) = \phi_\pi(F_\mu(x))$  and  $F_{\tilde{\mu}_l}^{-1} \left( \frac{\phi_\pi(F_\mu(x))}{\nu_l(\mathbb{R})} \right) \leq F_\mu^{-1}(F_\mu(x))$ . Since, by **Step 1**,  $\phi_\uparrow(F_\mu(x)) \leq \phi_\pi(F_\mu(x))$  and  $F_{\tilde{\mu}_l}^{-1}$  is non-increasing, we deduce that  $F_{\tilde{\mu}_l}^{-1} \left( \frac{\phi_\uparrow(F_\mu(x))}{\nu_l(\mathbb{R})} \right) \leq F_\mu^{-1}(F_\mu(x))$ . As  $(0, 1) \cap \left\{ \bigcup_{x \in \mathbb{R}: \mu(\{x\}) > 0} [F_\mu(x-), F_\mu(x)] \right\}^c$  is included in the range of  $F_\mu$ , with (24) we conclude that

$$\phi'_\uparrow(u)du \text{ a.e. on } (0, 1) \cap \left\{ \bigcup_{x \in \mathbb{R}: \mu(\{x\}) > 0} [F_\mu(x-), F_\mu(x)] \right\}^c, F_{\tilde{\nu}_l}^{-1} \left( \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})} \right) < F_\mu^{-1}(u).$$

If  $\mu(\{x\}) > 0$ , then by **Step 3** below,  $\phi'_\uparrow(u)du$  a.e. on  $(F_\mu(x-), F_\mu(x))$ ,  $F_{\tilde{\nu}_l}^{-1} \left( \frac{\phi_\uparrow(u)}{\nu_l(\mathbb{R})} \right) < F_\mu^{-1}(u)$  and therefore (21) holds.

In a symmetric way, since  $\frac{\mathbb{1}_{\{y < z\}}}{\nu_r(\mathbb{R})} \pi(dy, dz) \in \Pi(\tilde{\mu}_r, \tilde{\nu}_r)$  where  $\tilde{\mu}_r(dy) = \frac{1}{\nu_r(\mathbb{R})} \int_{z \in \mathbb{R}} \mathbb{1}_{\{y < z\}} \pi(dy, dz)$ , Lemma 3.7 implies that  $du$  a.e.,  $F_{\tilde{\nu}_r}^{-1}(u) > F_{\tilde{\mu}_r}^{-1}(u)$ . For  $x \in \mathbb{R}$ ,  $F_{\tilde{\mu}_r}(x) = \frac{F_\mu(x) - \phi_\pi(F_\mu(x))}{\nu_r(\mathbb{R})}$ . Hence for  $w > \frac{F_\mu(x-) - \lim_{y \rightarrow x-} \phi_\pi(F_\mu(y))}{\nu_r(\mathbb{R})}$ ,  $F_{\tilde{\mu}_r}^{-1}(w) \geq x$ . Since  $\phi_\uparrow(F_\mu(x-)) \leq \lim_{y \rightarrow x-} \phi_\pi(F_\mu(y))$ , we deduce that when  $\mu(\{x\}) > 0$ ,  $F_{\tilde{\nu}_r}^{-1}(w) > x$  for  $w > \frac{F_\mu(x-) - \phi_\uparrow(F_\mu(x-))}{\nu_r(\mathbb{R})}$ . With this argument replacing the above reference to **Step 3** when  $\mu(\{x\}) > 0$ , we check that (22) holds.

**Uniqueness :** Let  $\hat{\pi}^\uparrow \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$  be non-decreasing. Let  $\Gamma$  be a Borel subset of  $\mathbb{R}^2$  such that  $\hat{\pi}^\uparrow(\Gamma) = 1$  and properties (a)(b) in Definition 2.3 are satisfied. Let  $x \in \mathbb{R}$  be such that  $F_\mu(x) > 0$ . The set  $\{z \in (-\infty, x) : \exists y \in (z, x] \text{ s.t. } (y, z) \in \Gamma\}$  is not empty since

$$\int_{\mathbb{R}^2} \mathbb{1}_\Gamma(y, z) \mathbb{1}_{\{z < y \leq x\}} \hat{\pi}^\uparrow(dy, dz) = \int_{\mathbb{R}^2} \mathbb{1}_{\{z < y \leq x\}} \hat{\pi}^\uparrow(dy, dz) = \int_{(-\infty, x]} \hat{\pi}_y^\uparrow((-\infty, y)) \mu(dy) > 0,$$

as  $\nu_0^{\hat{\pi}^\uparrow} = 0$  implies that  $\mu(dy)$  a.e.  $\hat{\pi}_y^\uparrow((-\infty, y)) > 0$ . Therefore  $l(x) := \sup\{z \in (-\infty, x) : \exists y \in (z, x] \text{ s.t. } (y, z) \in \Gamma\}$  belongs to  $(-\infty, x]$ . Since  $\Gamma \cap \{(y, z) \in \mathbb{R}^2 : z < y \leq x \text{ and } z > l(x)\} = \emptyset$ ,

$$0 = \int_{\mathbb{R}^2} \mathbb{1}_\Gamma(y, z) \mathbb{1}_{\{z < y \leq x\}} \mathbb{1}_{\{z > l(x)\}} \hat{\pi}^\uparrow(dy, dz) = \int_{\mathbb{R}^2} \mathbb{1}_{\{z < y \leq x\}} \mathbb{1}_{\{z > l(x)\}} \hat{\pi}^\uparrow(dy, dz).$$

On the other hand, by property (a) in Definition 2.3,  $\Gamma \cap (x, +\infty) \times (-\infty, l(x)) = \emptyset$  so that

$$0 = \int_{\mathbb{R}^2} \mathbb{1}_\Gamma(y, z) \mathbb{1}_{\{y > x\}} \mathbb{1}_{\{z < l(x)\}} \hat{\pi}^\uparrow(dy, dz) = \int_{\mathbb{R}^2} \mathbb{1}_{\{y > x\}} \mathbb{1}_{\{z < l(x)\}} \hat{\pi}^\uparrow(dy, dz).$$

With the equality,  $\frac{1}{\nu_l(\mathbb{R})} \int_{y \in \mathbb{R}} \mathbb{1}_{\{z < y\}} \hat{\pi}^\uparrow(dy, dz) = \tilde{\nu}_l(dz)$ , we deduce that

$$\mathbb{1}_{\{z < l(x)\}} \tilde{\nu}_l(dz) \leq \frac{1}{\nu_l(\mathbb{R})} \int_{y \in (-\infty, x]} \mathbb{1}_{\{z < y\}} \hat{\pi}^\uparrow(dy, dz) \leq \mathbb{1}_{\{z \leq l(x)\}} \tilde{\nu}_l(dz).$$

Therefore  $\frac{\phi_{\hat{\pi}^\uparrow}(F_\mu(x))}{\nu_l(\mathbb{R})} \in [F_{\tilde{\nu}_l}^{-1}(l(x)-), F_{\tilde{\nu}_l}^{-1}(l(x))]$  and

$$\begin{aligned} \frac{1}{\nu_l(\mathbb{R})} \int_{y \in (-\infty, x]} \mathbb{1}_{\{z < y\}} \hat{\pi}^\uparrow(dy, dz) &= \mathbb{1}_{\{z < l(x)\}} \tilde{\nu}_l(dz) + \left( \frac{\phi_{\hat{\pi}^\uparrow}(F_\mu(x))}{\nu_l(\mathbb{R})} - F_{\tilde{\nu}_l}^{-1}(l(x)-) \right) \delta_{l(x)}(dz) \\ &= \int_0^{\frac{\phi_{\hat{\pi}^\uparrow}(F_\mu(x))}{\nu_l(\mathbb{R})}} \delta_{F_{\tilde{\nu}_l}^{-1}(u)}(dz) du = \frac{1}{\nu_l(\mathbb{R})} \int_0^{\phi_{\hat{\pi}^\uparrow}(F_\mu(x))} \delta_{F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right)}(dz) dw. \end{aligned} \quad (25)$$

In a symmetric way, with property (b) in Definition 2.3, we check that

$$\int_{y \in (-\infty, x]} \mathbb{1}_{\{z > y\}} \hat{\pi}^\uparrow(dy, dz) = \int_0^{F_\mu(x) - \phi_{\hat{\pi}^\uparrow}(F_\mu(x))} \delta_{F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right)}(dz) dw. \quad (26)$$

Using the martingale property of  $\hat{\pi}^\uparrow$  for the second equality and  $\nu_0^{\hat{\pi}^\uparrow} = 0$  for the fourth, we deduce that

$$\begin{aligned} \int_0^{F_\mu(x)} F_\mu^{-1}(u) du &= \int_{y \in (-\infty, x]} y \mu(dy) = \int_{y \in (-\infty, x]} \int_{z \in \mathbb{R}} z \hat{\pi}_y^\uparrow(dz) \mu(dy) = \int_{(-\infty, x] \times \mathbb{R}} z \hat{\pi}^\uparrow(dy, dz) \\ &= \int_{(-\infty, x] \times \mathbb{R}} z \mathbb{1}_{\{z < y\}} \hat{\pi}^\uparrow(dy, dz) + \int_{(-\infty, x] \times \mathbb{R}} z \mathbb{1}_{\{z > y\}} \hat{\pi}^\uparrow(dy, dz) \\ &= \int_0^{\phi_{\hat{\pi}^\uparrow}(F_\mu(x))} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) dw + \int_0^{F_\mu(x) - \phi_{\hat{\pi}^\uparrow}(F_\mu(x))} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw \\ &= G(F_\mu(x), \phi_{\hat{\pi}^\uparrow}(F_\mu(x))). \end{aligned}$$

Equations (25) and (26) also hold with  $\hat{\pi}^\uparrow$  replaced by  $\pi^\uparrow$  so that  $G(F_\mu(x), \phi_{\pi^\uparrow}(F_\mu(x))) = \int_0^{F_\mu(x)} F_\mu^{-1}(u) du$ . Note that the equalities

$$\begin{aligned} \int_{y \in (-\infty, x]} \mathbb{1}_{\{z < y\}} \pi^\uparrow(dy, dz) &= \int_0^{\phi_{\pi^\uparrow}(F_\mu(x))} \delta_{F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right)}(dz) dw \\ \int_{y \in (-\infty, x]} \mathbb{1}_{\{z > y\}} \pi^\uparrow(dy, dz) &= \int_0^{F_\mu(x) - \phi_{\pi^\uparrow}(F_\mu(x))} \delta_{F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right)}(dz) dw, \end{aligned}$$

together with  $\phi_{\pi^\uparrow}(F_\mu(x)) = \int_0^{F_\mu(x)} \phi'_\uparrow(u) du = \phi_\uparrow(F_\mu(x))$  can also be derived from the definition of  $\pi^\uparrow$  combined with (21), (22), (1), Proposition 4.9 [12] and the remark just above applied like in the above derivation of (23).

By **Step 1** below applied with  $\pi = \hat{\pi}^\uparrow$  and  $\pi = \pi^\uparrow$ , there exists a unique  $v \in [(F_\mu(x) - \nu_r(\mathbb{R}))^+, \phi_{\hat{\pi}^\uparrow}(F_\mu(x)) \vee \phi_{\pi^\uparrow}(F_\mu(x))]$  such that  $G(F_\mu(x), v) = \int_0^{F_\mu(x)} F_\mu^{-1}(w) dw$  so that  $\phi_{\hat{\pi}^\uparrow}(F_\mu(x)) = \phi_{\pi^\uparrow}(F_\mu(x))$  for each  $x \in \mathbb{R}$  and

$$\begin{aligned} \int_{y \in (-\infty, x]} \hat{\pi}^\uparrow(dy, dz) &= \int_0^{\phi_{\hat{\pi}^\uparrow}(F_\mu(x))} \delta_{F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right)}(dz) dw + \int_0^{F_\mu(x) - \phi_{\hat{\pi}^\uparrow}(F_\mu(x))} \delta_{F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right)}(dz) dw \\ &= \int_{y \in (-\infty, x]} \pi^\uparrow(dy, dz). \end{aligned}$$

Therefore the set

$$\mathcal{M} := \left\{ A \in \mathcal{B}(\mathbb{R}) \mid \int_A \pi_y^\uparrow \mu(dy) = \int_A \hat{\pi}_y^\uparrow \mu(dy) \right\}$$

contains  $\{(-\infty, x] : x \in \mathbb{R}\}$ . One easily checks that if  $A, B \in \mathcal{M}$  are such that  $B \subset A$ , then  $A \setminus B \in \mathcal{M}$ . Now let  $(A_n)_{n \in \mathbb{N}}$  be a sequence of elements in  $\mathcal{M}$  such that  $A_n \subseteq A_{n+1}$ . For  $B \in \mathcal{B}(\mathbb{R})$ , since for all  $n \in \mathbb{N}$ ,  $\int_{A_n} (\hat{\pi}_y^\uparrow(B) - \pi_y^\uparrow(B)) \mu(dy) = 0$  and  $|\hat{\pi}_y^\uparrow(B) - \pi_y^\uparrow(B)| \leq 1$ , by Lebesgue's dominated convergence theorem,

$$\int_{\bigcup_{n \in \mathbb{N}} A_n} (\hat{\pi}_y^\uparrow(B) - \pi_y^\uparrow(B)) \mu(dy) = 0.$$

We deduce that  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{M}$  and  $\mathcal{M}$  is a monotone class. Since  $\mathcal{M}$  contains the class  $\mathcal{C} = \{(-\infty, x] : x \in \mathbb{R}\}$ , which is stable by finite intersections, by the monotone class theorem, we have  $\sigma(\mathcal{C}) \subset \mathcal{M}$ . Since  $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R})$  and  $\mathcal{M} \subseteq \mathcal{B}(\mathbb{R})$ , we conclude that  $\mathcal{M} = \mathcal{B}(\mathbb{R})$ . In particular, for any  $B \in \mathcal{B}(\mathbb{R})$ ,

$$\mu(\{y \in \mathbb{R} : \hat{\pi}_y^\uparrow(B) > \pi_y^\uparrow(B)\}) = 0 = \mu(\{y \in \mathbb{R} : \hat{\pi}_y^\uparrow(B) < \pi_y^\uparrow(B)\}).$$

We deduce that

$$\mu(\{y \in \mathbb{R} : \forall x \in \mathbb{Q}, \hat{\pi}_y^\uparrow((-\infty, x]) = \pi_y^\uparrow((-\infty, x])\}) = 1 = \mu(\{y \in \mathbb{R} : \hat{\pi}_y^\uparrow = \pi_y^\uparrow\}).$$

**Step 1 :** For  $x \in \mathbb{R}$ , we are going to check that  $G(F_\mu(x), \phi_\pi(F_\mu(x))) \leq \int_0^{F_\mu(x)} F_\mu^{-1}(w) dw$  (**Step 1.1** below),  $G(F_\mu(x), (F_\mu(x) - \nu_r(\mathbb{R}))^+) \geq \int_0^{F_\mu(x)} F_\mu^{-1}(w) dw$  (**Step 1.2** below) and that  $v \mapsto G(F_\mu(x), v)$  is decreasing on  $[(F_\mu(x) - \nu_r(\mathbb{R}))^+, \phi_\pi(F_\mu(x))]$  (**Step 1.3** below). Since this function is also continuous, this implies the existence of a unique  $\phi_\uparrow(F_\mu(x)) \in [(F_\mu(x) - \nu_r(\mathbb{R}))^+, \phi_\pi(F_\mu(x))]$  such that  $G(F_\mu(x), \phi_\uparrow(F_\mu(x))) = \int_0^{F_\mu(x)} F_\mu^{-1}(v) dv$ , where the notation  $\phi_\uparrow(F_\mu(x))$  is justified since this point clearly only depends on  $x$  through  $F_\mu(x)$ .

**Step 1.1 :** By Lemma 3.6, applied with  $\eta = \tilde{\nu}_l$  and  $\hat{\eta}(dz) = \frac{1}{\nu_l(\mathbb{R})} \int_{y \in (-\infty, x]} \mathbb{1}_{\{z < y\}} \pi(dy, dz)$  such that  $\hat{\eta}(\mathbb{R}) = \frac{\phi_\pi(F_\mu(x))}{\nu_l(\mathbb{R})}$ ,

$$\frac{1}{\nu_l(\mathbb{R})} \int_{\mathbb{R}^2} z \mathbb{1}_{\{z < y \leq x\}} \pi(dy, dz) \geq \int_0^{\frac{\phi_\pi(F_\mu(x))}{\nu_l(\mathbb{R})}} F_{\tilde{\nu}_l}^{-1}(w) dw = \frac{1}{\nu_l(\mathbb{R})} \int_0^{\phi_\pi(F_\mu(x))} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) dw$$

and, with  $\eta = \tilde{\nu}_r$ ,  $\hat{\eta}(dz) = \frac{1}{\nu_r(\mathbb{R})} \int_{y \in (-\infty, x]} \mathbb{1}_{\{z > y\}} \pi(dy, dz) \leq \tilde{\nu}_r(dz)$  such that  $\hat{\eta}(\mathbb{R}) = \frac{F_\mu(x) - \phi_\pi(F_\mu(x))}{\nu_r(\mathbb{R})}$ ,

$$\frac{1}{\nu_r(\mathbb{R})} \int_{\mathbb{R}^2} z \mathbb{1}_{\{y \leq x, y < z\}} \pi(dy, dz) \geq \int_0^{\frac{F_\mu(x) - \phi_\pi(F_\mu(x))}{\nu_r(\mathbb{R})}} F_{\tilde{\nu}_r}^{-1}(w) dw = \frac{1}{\nu_r(\mathbb{R})} \int_0^{F_\mu(x) - \phi_\pi(F_\mu(x))} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw.$$

Using (1) for the first equality, the inverse transform sampling for the second and the martingale property of  $\pi$ , and  $\nu_l^\pi + \nu_r^\pi = \nu$  for the third, we deduce that

$$\begin{aligned} \int_0^{F_\mu(x)} F_\mu^{-1}(w) dw &= \int_0^1 F_\mu^{-1}(w) \mathbb{1}_{\{F_\mu^{-1}(w) \leq x\}} dw = \int_{\mathbb{R}} y \mathbb{1}_{\{y \leq x\}} \mu(dy) \\ &= \int_{\mathbb{R}^2} z \mathbb{1}_{\{z < y \leq x\}} \pi(dy, dz) + \int_{\mathbb{R}^2} z \mathbb{1}_{\{y \leq x, y < z\}} \pi(dy, dz) \geq G(F_\mu(x), \phi_\pi(F_\mu(x))). \end{aligned}$$

**Step 1.2 :** If  $F_\mu(x) < \nu_r(\mathbb{R})$ , setting

$$\psi(z) = \mathbb{1}_{\{z < F_{\tilde{\nu}_r}^{-1}(\frac{F_\mu(x)}{\nu_r(\mathbb{R})})\}} + \mathbb{1}_{\{z = F_{\tilde{\nu}_r}^{-1}(\frac{F_\mu(x)}{\nu_r(\mathbb{R})})\}} \mathbb{1}_{\{\tilde{\nu}_r(\{F_{\tilde{\nu}_r}^{-1}(\frac{F_\mu(x)}{\nu_r(\mathbb{R})})\}) > 0\}} \frac{\frac{F_\mu(x)}{\nu_r(\mathbb{R})} - \tilde{\nu}_r((-\infty, F_{\tilde{\nu}_r}^{-1}(\frac{F_\mu(x)}{\nu_r(\mathbb{R})})))}{\tilde{\nu}_r(\{F_{\tilde{\nu}_r}^{-1}(\frac{F_\mu(x)}{\nu_r(\mathbb{R})})\})},$$

we have that

$$\begin{aligned} \int_0^{F_\mu(x)} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw &= \nu_r(\mathbb{R}) \int_{\mathbb{R}} z \psi(z) \tilde{\nu}_r(dz) = \int_{\mathbb{R} \times \mathbb{R}} z \psi(z) \mathbb{1}_{\{y < z\}} \pi(dy, dz) \\ &\geq \int_{\mathbb{R} \times \mathbb{R}} y \psi(z) \mathbb{1}_{\{y < z\}} \pi(dy, dz) = \int_{\mathbb{R}} y \hat{\mu}(dy) \text{ where} \end{aligned}$$

$\hat{\mu}(dy) := \left( \int_{z \in \mathbb{R}} \psi(z) \mathbf{1}_{\{y < z\}} \pi_y(dz) \right) \mu(dy) \leq \mu(dy)$  since  $\int_{z \in \mathbb{R}} \psi(z) \mathbf{1}_{\{y < z\}} \pi_y(dz) \in [0, 1]$  and satisfies

$$\hat{\mu}(\mathbb{R}) = \int_{\mathbb{R} \times \mathbb{R}} \psi(z) \mathbf{1}_{\{y < z\}} \pi(dy, dz) = \nu_r(\mathbb{R}) \int_{\mathbb{R}} \psi(z) \tilde{\nu}_r(dz) = F_\mu(x).$$

By Lemma 3.6 applied with  $(\eta, \hat{\eta}) = (\mu, \hat{\mu})$ , this ensures that  $\int_{\mathbb{R}} y \hat{\mu}(dy) \geq \int_0^{F_\mu(x)} F_\mu^{-1}(v) dv$  so that

$$\int_0^{F_\mu(x)} F_\mu^{-1}(v) dv \leq \int_0^{F_\mu(x)} F_{\tilde{\nu}_r}^{-1} \left( \frac{w}{\nu_r(\mathbb{R})} \right) dw = G(F_\mu(x), 0).$$

If  $F_\mu(x) \geq \nu_r(\mathbb{R})$ , one has using  $\int_0^{\nu_r(\mathbb{R})} F_{\tilde{\nu}_r}^{-1} \left( \frac{w}{\nu_r(\mathbb{R})} \right) dw = \int_{\mathbb{R}} y \nu_r(dy) = \int_{\mathbb{R}} y \mu(dy) - \int_{\mathbb{R}} y \nu_l(dy)$  and  $\nu_r(\mathbb{R}) = 1 - \nu_l(\mathbb{R})$  for the second equality,

$$\begin{aligned} G(F_\mu(x), F_\mu(x) - \nu_r(\mathbb{R})) &= \int_0^{F_\mu(x) - \nu_r(\mathbb{R})} F_{\tilde{\nu}_l}^{-1} \left( \frac{w}{\nu_l(\mathbb{R})} \right) dw + \int_0^{\nu_r(\mathbb{R})} F_{\tilde{\nu}_r}^{-1} \left( \frac{w}{\nu_r(\mathbb{R})} \right) dw \\ &= \int_0^{\nu_l(\mathbb{R}) - (1 - F_\mu(x))} F_{\tilde{\nu}_l}^{-1} \left( \frac{w}{\nu_l(\mathbb{R})} \right) dw + \int_{\mathbb{R}} y \mu(dy) - \int_{\mathbb{R}} y \nu_l(dy) \\ &= \int_0^{F_\mu(x)} F_\mu^{-1}(w) dw + \int_{F_\mu(x)}^1 F_\mu^{-1}(w) dw - \int_{\nu_l(\mathbb{R}) - (1 - F_\mu(x))}^{\nu_l(\mathbb{R})} F_{\tilde{\nu}_l}^{-1} \left( \frac{w}{\nu_l(\mathbb{R})} \right) dw. \end{aligned}$$

By a reasoning similar to the above derivation of  $\int_0^{F_\mu(x)} F_\mu^{-1}(v) dv \leq \int_0^{F_\mu(x)} F_{\tilde{\nu}_r}^{-1} \left( \frac{w}{\nu_r(\mathbb{R})} \right) dw$ , we check that the sum of the last two terms in the right-hand side is non-negative. Combining the two cases, we obtain that  $G(F_\mu(x), (F_\mu(x) - \nu_r(\mathbb{R}))^+) \geq \int_0^{F_\mu(x)} F_\mu^{-1}(w) dw$ .

**Step 1.3 :** To prove that  $v \mapsto G(F_\mu(x), v)$  is decreasing on  $[(F_\mu(x) - \nu_r(\mathbb{R}))^+, \phi_\pi(F_\mu(x))]$ , it is enough to check that

$$\forall w \in \left( 0, \frac{\phi_\pi(F_\mu(x))}{\nu_l(\mathbb{R})} \right), F_{\tilde{\nu}_l}^{-1}(w) < x \text{ and } \forall w \in \left( \frac{F_\mu(x) - \phi_\pi(F_\mu(x))}{\nu_r(\mathbb{R})}, 1 \right), x < F_{\tilde{\nu}_r}^{-1}(w). \quad (27)$$

Let  $x$  be such that  $\phi_\pi(F_\mu(x)) > 0$ . Then  $F_\mu(x) > 0$  and  $F_\mu(x) - \phi_\pi(F_\mu(x)) > 0$ . Since, by definition of  $\phi_\pi(F_\mu(x))$  and  $\nu_l^\pi = \nu_l$ ,

$$\begin{aligned} \phi_\pi(F_\mu(x)) &= \int_{\mathbb{R}^2} \mathbf{1}_{\{z < y \leq x\}} \pi(dy, dz) = \int_{\mathbb{R}^2} \mathbf{1}_{\{z < y \wedge x\}} \pi(dy, dz) - \int_{\mathbb{R}^2} \mathbf{1}_{\{z < x < y\}} \pi(dy, dz) \\ &= \nu_l(\mathbb{R}) \tilde{\nu}_l((-\infty, x)) - \int_{\mathbb{R}^2} \mathbf{1}_{\{z < x < y\}} \pi(dy, dz), \end{aligned}$$

we have  $\tilde{\nu}_l((-\infty, x)) \geq \frac{\phi_\pi(F_\mu(x))}{\nu_l(\mathbb{R})}$ , from which we deduce the first part in (27). In a symmetric way and since  $\nu_0^\pi = \nu_0 = 0$ ,

$$\begin{aligned} F_\mu(x) - \phi_\pi(F_\mu(x)) &= \int_{\mathbb{R}^2} \mathbf{1}_{\{y \leq x\}} \pi(dy, dz) - \int_{\mathbb{R}^2} \mathbf{1}_{\{z < y \leq x\}} \pi(dy, dz) \\ &= \int_{\mathbb{R}^2} \mathbf{1}_{\{y < z \leq x\}} \pi(dy, dz) + \int_{\mathbb{R}^2} \mathbf{1}_{\{y \leq x < z\}} \pi(dy, dz) \\ &= \nu_r(\mathbb{R}) F_{\tilde{\nu}_r}(x) + \int_{\mathbb{R}^2} \mathbf{1}_{\{y \leq x < z\}} \pi(dy, dz), \end{aligned}$$

so that  $F_{\tilde{\nu}_r}(x) \leq \frac{F_\mu(x) - \phi_\pi(F_\mu(x))}{\nu_r(\mathbb{R})}$ . With the right-continuity of  $F_{\tilde{\nu}_r}$ , we deduce the other part in (27).

**Step 2 :** monotonicity of  $x \mapsto \phi_\uparrow(F_\mu(x))$  and  $x \mapsto F_\mu(x) - \phi_\uparrow(F_\mu(x))$ .

**Step 2.1 :** Let us first check that when  $F_\mu(x) < F_\mu(x')$ ,  $F_\mu(x) - \phi_\uparrow(F_\mu(x)) \leq F_\mu(x') - \phi_\uparrow(F_\mu(x'))$ . If  $\phi_\pi(F_\mu(x')) \leq \phi_\uparrow(F_\mu(x)) + F_\mu(x') - F_\mu(x)$ , then we conclude with the inequality



$\phi_{\uparrow}(F_{\mu}(x')) \leq \phi_{\pi}(F_{\mu}(x'))$  established in **step 1**. Let us now suppose that  $\phi_{\uparrow}(F_{\mu}(x)) + F_{\mu}(x') - F_{\mu}(x) \leq \phi_{\pi}(F_{\mu}(x'))$ . Then, by the monotonicity of  $F_{\tilde{\nu}_l}^{-1}$ ,

$$\int_{\phi_{\uparrow}(F_{\mu}(x))}^{\phi_{\uparrow}(F_{\mu}(x)) + F_{\mu}(x') - F_{\mu}(x)} F_{\tilde{\nu}_l}^{-1} \left( \frac{w}{\nu_l(\mathbb{R})} \right) dw \leq \int_{\phi_{\pi}(F_{\mu}(x')) + F_{\mu}(x) - F_{\mu}(x')}^{\phi_{\pi}(F_{\mu}(x'))} F_{\tilde{\nu}_l}^{-1} \left( \frac{w}{\nu_l(\mathbb{R})} \right) dw.$$

We are going to prove that

$$\int_{\phi_{\pi}(F_{\mu}(x')) + F_{\mu}(x) - F_{\mu}(x')}^{\phi_{\pi}(F_{\mu}(x'))} F_{\tilde{\nu}_l}^{-1} \left( \frac{w}{\nu_l(\mathbb{R})} \right) dw \leq \int_{F_{\mu}(x)}^{F_{\mu}(x')} F_{\mu}^{-1}(w) dw. \quad (28)$$

Adding the two inequalities to  $G(F_{\mu}(x), \phi_{\uparrow}(F_{\mu}(x))) = \int_0^{F_{\mu}(x)} F_{\mu}^{-1}(w) dw$ , we obtain that

$$G(F_{\mu}(x'), \phi_{\uparrow}(F_{\mu}(x)) + F_{\mu}(x') - F_{\mu}(x)) \leq \int_0^{F_{\mu}(x')} F_{\mu}^{-1}(w) dw.$$

Since by **Step 1.3**,  $v \mapsto G(F_{\mu}(x'), v)$  is decreasing on  $[(F_{\mu}(x') - \nu_r(\mathbb{R}))^+, \phi_{\pi}(F_{\mu}(x'))]$ , we deduce that  $\phi_{\uparrow}(F_{\mu}(x')) \leq \phi_{\uparrow}(F_{\mu}(x)) + F_{\mu}(x') - F_{\mu}(x)$ .

To prove (28), we first remark that since

$$F_{\mu}(x') - F_{\mu}(x) \leq \phi_{\pi}(F_{\mu}(x')) = \int_0^{F_{\mu}(x')} \pi_{F_{\mu}^{-1}(w)}((-\infty, F_{\mu}^{-1}(w))) dw,$$

where we used the inverse transform sampling and (1) for the equality,

$$\exists u \in [0, F_{\mu}(x')] \text{ s.t. } \int_u^{F_{\mu}(x')} \pi_{F_{\mu}^{-1}(w)}((-\infty, F_{\mu}^{-1}(w))) dw = F_{\mu}(x') - F_{\mu}(x). \quad (29)$$

By Lemma 3.6, applied with  $\eta = \frac{1}{F_{\mu}(x')} \int_0^{F_{\mu}(x')} \delta_{F_{\mu}^{-1}(w)} dw$  such that  $F_{\eta}^{-1}(w) = F_{\mu}^{-1}(F_{\mu}(x')w)$ ,  $\hat{\eta} = \frac{1}{F_{\mu}(x')} \int_u^{F_{\mu}(x')} \pi_{F_{\mu}^{-1}(w)}((-\infty, F_{\mu}^{-1}(w))) \delta_{F_{\mu}^{-1}(w)} dw$  with mass  $\hat{\eta}(\mathbb{R}) = \frac{F_{\mu}(x') - F_{\mu}(x)}{F_{\mu}(x')}$ ,

$$\frac{1}{F_{\mu}(x')} \int_{F_{\mu}(x)}^{F_{\mu}(x')} F_{\mu}^{-1}(w) dw \geq \frac{1}{F_{\mu}(x')} \int_u^{F_{\mu}(x')} \pi_{F_{\mu}^{-1}(w)}((-\infty, F_{\mu}^{-1}(w))) F_{\mu}^{-1}(w) dw. \quad (30)$$

For the desintegrations  $\pi(dy, dz) = \mu(dy)\pi_y(dz) = \nu(dz) \overleftarrow{\pi}_z(dy)$ , let

$$\begin{aligned} \sigma(dy) &= \frac{\mathbb{1}_{\{y \leq x'\}}}{\phi_{\pi}(F_{\mu}(x'))} \pi_y((-\infty, y)) \mu(dy) \text{ and} \\ \theta(dz) &= \mathbb{1}_{\{z < x'\}} \overleftarrow{\pi}_z((z, x']) \frac{\nu(dz)}{\phi_{\pi}(F_{\mu}(x'))} = \mathbb{1}_{\{z < x'\}} \overleftarrow{\pi}_z((z, +\infty)) \times \frac{\nu_l(dz)}{\phi_{\pi}(F_{\mu}(x'))}. \end{aligned}$$

The coupling  $\frac{\mathbb{1}_{\{z \leq y \leq x'\}}}{\phi_{\pi}(F_{\mu}(x'))} \pi(dy, dz) \in \Pi(\sigma, \theta)$  giving full weight to  $\{(y, z) \in \mathbb{R}^2 : y > z\}$ , one has  $\sigma \geq_{st} \theta$ . On the other hand, Lemma 3.6 applied with  $\eta = \tilde{\nu}_l$  and  $\hat{\eta} = \frac{\phi_{\pi}(F_{\mu}(x'))}{\nu_l(\mathbb{R})} \theta$  such that  $\hat{\eta}(\mathbb{R}) = \frac{\phi_{\pi}(F_{\mu}(x'))}{\nu_l(\mathbb{R})}$  implies that

$$\theta \geq_{st} \vartheta := \frac{\nu_l(\mathbb{R})}{\phi_{\pi}(F_{\mu}(x'))} \int_0^{\frac{\phi_{\pi}(F_{\mu}(x'))}{\nu_l(\mathbb{R})}} \delta_{F_{\tilde{\nu}_l}^{-1}(w)} dw.$$

Hence  $\sigma \geq_{st} \vartheta$  so that  $F_{\sigma}^{-1} \geq F_{\vartheta}^{-1}$  where  $F_{\vartheta}^{-1}(w) = F_{\tilde{\nu}_l}^{-1} \left( \frac{\phi_{\pi}(F_{\mu}(x'))}{\nu_l(\mathbb{R})} w \right)$  and

$$\begin{aligned} \int_{1 - \frac{F_{\mu}(x') - F_{\mu}(x)}{\phi_{\pi}(F_{\mu}(x'))}}^1 F_{\sigma}^{-1}(w) dw &\geq \int_{1 - \frac{F_{\mu}(x') - F_{\mu}(x)}{\phi_{\pi}(F_{\mu}(x'))}}^1 F_{\vartheta}^{-1}(w) dw \\ &= \frac{1}{\phi_{\pi}(F_{\mu}(x'))} \int_{\phi_{\pi}(F_{\mu}(x')) + F_{\mu}(x) - F_{\mu}(x')}^{\phi_{\pi}(F_{\mu}(x'))} F_{\tilde{\nu}_l}^{-1} \left( \frac{w}{\nu_l(\mathbb{R})} \right) dw. \end{aligned}$$

By the inverse transform sampling and (1),  $\sigma = \frac{1}{\phi_\pi(F_\mu(x'))} \int_0^{F_\mu(x')} \pi_{F_\mu^{-1}(w)}((-\infty, F_\mu^{-1}(w))) \delta_{F_\mu^{-1}(w)} dw$  so that, with (29),  $\int_{1-\frac{F_\mu(x')-F_\mu(x)}{\phi_\pi(F_\mu(x'))}}^1 F_\sigma^{-1}(w) dw = \frac{1}{\phi_\pi(F_\mu(x'))} \int_u^{F_\mu(x')} \pi_{F_\mu^{-1}(w)}((-\infty, F_\mu^{-1}(w))) F_\mu^{-1}(w) dw$ .

Hence

$$\int_u^{F_\mu(x')} \pi_{F_\mu^{-1}(w)}((-\infty, F_\mu^{-1}(w))) F_\mu^{-1}(w) dw \geq \int_{\phi_\pi(F_\mu(x'))+F_\mu(x)-F_\mu(x')}^{\phi_\pi(F_\mu(x'))} F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) dw.$$

Combining this inequality with (30), we obtain (28).

**Step 2.2 :** Let us now check that when  $F_\mu(x) < F_\mu(x')$ ,  $\phi_\uparrow(F_\mu(x)) \leq \phi_\uparrow(F_\mu(x'))$ . If  $\phi_\uparrow(F_\mu(x)) \leq F_\mu(x') - \nu_r(\mathbb{R})$ , then we have  $\phi_\uparrow(F_\mu(x)) \leq \phi_\uparrow(F_\mu(x'))$ , since  $(F_\mu(x') - \nu_r(\mathbb{R}))^+ \leq \phi_\uparrow(F_\mu(x'))$  by **Step 1**. Let us now suppose that  $F_\mu(x') - \phi_\uparrow(F_\mu(x)) \leq \nu_r(\mathbb{R})$ . Since, by **Step 1**,  $\phi_\uparrow(F_\mu(x)) \leq \phi_\pi(F_\mu(x))$  and  $F_{\tilde{\nu}_r}^{-1}$  is non-decreasing, we have, using a reasoning analogous to the above derivation of (28) for the second inequality,

$$\int_{F_\mu(x)-\phi_\uparrow(F_\mu(x))}^{F_\mu(x')-\phi_\uparrow(F_\mu(x))} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw \geq \int_{F_\mu(x)-\phi_\pi(F_\mu(x))}^{F_\mu(x')-\phi_\pi(F_\mu(x))} F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) dw \geq \int_{F_\mu(x)}^{F_\mu(x')} F_\mu^{-1}(w) dw.$$

Adding this inequality to  $G(F_\mu(x), \phi_\uparrow(F_\mu(x))) = \int_0^{F_\mu(x)} F_\mu^{-1}(w) dw$ , we obtain that

$$G(F_\mu(x'), \phi_\uparrow(F_\mu(x))) \geq \int_0^{F_\mu(x')} F_\mu^{-1}(w) dw.$$

Since  $\phi_\uparrow(F_\mu(x)) \leq \phi_\pi(F_\mu(x)) \leq \phi_\pi(F_\mu(x'))$  and, by **Step 1.3**,  $v \mapsto G(F_\mu(x'), v)$  is decreasing on  $[(F_\mu(x') - \nu_r(\mathbb{R}))^+, \phi_\pi(F_\mu(x'))]$ , we deduce that  $\phi_\uparrow(F_\mu(x')) \geq \phi_\uparrow(F_\mu(x))$ .

**Step 3 :** Let  $x \in \mathbb{R}$  be such that  $\mu(\{x\}) > 0$ . The monotonicity proved in **Step 2** ensures the existence of the left-hand limit  $\lim_{y \rightarrow x-} \phi_\uparrow(F_\mu(y))$  such that  $F_\mu(x-) - \lim_{y \rightarrow x-} \phi_\uparrow(F_\mu(y)) \leq F_\mu(x) - \phi_\uparrow(F_\mu(x))$ . Taking the limit  $y \rightarrow x-$  in the inequality  $(F_\mu(y) - \nu_r(\mathbb{R}))^+ \leq \phi_\uparrow(F_\mu(y)) \leq F_\mu(y) \wedge \nu_l(\mathbb{R})$  implies that  $(F_\mu(x-) - \nu_r(\mathbb{R}))^+ \leq \lim_{y \rightarrow x-} \phi_\uparrow(F_\mu(y)) \leq F_\mu(x-) \wedge \nu_l(\mathbb{R})$ . By continuity of  $(u, v) \mapsto G(u, v)$  and  $u \mapsto \int_0^u F_\mu^{-1}(w) dw$ , taking the limit  $y \rightarrow x-$  in the equality  $G(F_\mu(y), \phi_\uparrow(F_\mu(y))) = \int_0^{F_\mu(y)} F_\mu^{-1}(w) dw$ , we obtain that

$$G\left(F_\mu(x-), \lim_{y \rightarrow x-} \phi_\uparrow(F_\mu(y))\right) = \int_0^{F_\mu(x-)} F_\mu^{-1}(w) dw, \quad (31)$$

so that we can set  $\phi_\uparrow(F_\mu(x-)) = \lim_{y \rightarrow x-} \phi_\uparrow(F_\mu(y))$ . Subtracting (31) from  $G(F_\mu(x), \phi_\uparrow(F_\mu(x))) = \int_0^{F_\mu(x)} F_\mu^{-1}(w) dw$  and using that, according to (2),  $F_\mu^{-1}(w) = x$  for  $w \in (F_\mu(x-), F_\mu(x)]$ , we obtain

$$\int_{\phi_\uparrow(F_\mu(x-))}^{\phi_\uparrow(F_\mu(x))} \left(F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) - x\right) dw + \int_{F_\mu(x-)-\phi_\uparrow(F_\mu(x-))}^{F_\mu(x)-\phi_\uparrow(F_\mu(x))} \left(F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) - x\right) dw = 0. \quad (32)$$

By (27) and  $\phi_\uparrow(F_\mu(x)) \leq \phi_\pi(F_\mu(x))$ ,  $F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) - x < 0$  for  $w \in (\phi_\uparrow(F_\mu(x-)), \phi_\uparrow(F_\mu(x)))$ . Since, for  $y < x$ , by (27) and  $\phi_\uparrow(F_\mu(y)) \leq \phi_\pi(F_\mu(y))$ ,  $F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) - y > 0$  for  $w \in (F_\mu(y) - \phi_\uparrow(F_\mu(y)), \nu_r(\mathbb{R}))$ , one has  $F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) - x \geq 0$  for  $w \in (F_\mu(x-) - \phi_\uparrow(F_\mu(x-)), F_\mu(x) - \phi_\uparrow(F_\mu(x)))$ . Now let  $u \in (F_\mu(x-), F_\mu(x))$  and, for  $v \in [\phi_\uparrow(F_\mu(x-)) \vee (u + \phi_\uparrow(F_\mu(x)) - F_\mu(x)), \phi_\uparrow(F_\mu(x)) \wedge (u + \phi_\uparrow(F_\mu(x-)) - F_\mu(x-))]$ ,

$$H(u, v) = \int_{\phi_\uparrow(F_\mu(x-))}^v \left(F_{\tilde{\nu}_l}^{-1}\left(\frac{w}{\nu_l(\mathbb{R})}\right) - x\right) dw + \int_{F_\mu(x-)-\phi_\uparrow(F_\mu(x-))}^{u-v} \left(F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) - x\right) dw.$$

If  $\phi_\uparrow(F_\mu(x-)) \geq u + \phi_\uparrow(F_\mu(x)) - F_\mu(x)$ ,

$$H(u, \phi_\uparrow(F_\mu(x-))) = \int_{F_\mu(x-)-\phi_\uparrow(F_\mu(x-))}^{u-\phi_\uparrow(F_\mu(x-))} \left(F_{\tilde{\nu}_r}^{-1}\left(\frac{w}{\nu_r(\mathbb{R})}\right) - x\right) dw \geq 0$$

while if  $\phi_{\uparrow}(F_{\mu}(x-)) \leq u + \phi_{\uparrow}(F_{\mu}(x)) - F_{\mu}(x)$ , using (32) for the equality,

$$H(u, u + \phi_{\uparrow}(F_{\mu}(x)) - F_{\mu}(x)) = \int_{u + \phi_{\uparrow}(F_{\mu}(x)) - F_{\mu}(x)}^{\phi_{\uparrow}(F_{\mu}(x))} \left( x - F_{\nu_l}^{-1} \left( \frac{w}{\nu_l(\mathbb{R})} \right) \right) dw \geq 0,$$

so that  $H(u, \phi_{\uparrow}(F_{\mu}(x-)) \vee (u + \phi_{\uparrow}(F_{\mu}(x)) - F_{\mu}(x))) \geq 0$ . In a similar way,  $H(u, \phi_{\uparrow}(F_{\mu}(x)) \wedge (u + \phi_{\uparrow}(F_{\mu}(x-)) - F_{\mu}(x-))) \leq 0$  and since  $v \mapsto H(u, v)$  is decreasing on the interval, there exists a unique  $\phi_{\uparrow}(u)$  such that  $\phi_{\uparrow}(F_{\mu}(x-)) \leq \phi_{\uparrow}(u) \leq \phi_{\uparrow}(F_{\mu}(x))$ ,  $F_{\mu}(x-) - \phi_{\uparrow}(F_{\mu}(x-)) \leq u - \phi_{\uparrow}(u) \leq F_{\mu}(x) - \phi_{\uparrow}(F_{\mu}(x))$  and  $H(u, \phi_{\uparrow}(u)) = 0$ . With (31), this implies that  $G(u, \phi_{\uparrow}(u)) = 0$ . Repeating the above reasoning with the equality  $H(u, \phi_{\uparrow}(u)) = 0$  replacing (32), we obtain for  $u' \in (F_{\mu}(x-), u)$  the existence of a unique  $\phi_{\uparrow}(u') \in [\phi_{\uparrow}(F_{\mu}(x-)) \vee (u' + \phi_{\uparrow}(u) - u), \phi_{\uparrow}(u) \wedge (u' + \phi_{\uparrow}(F_{\mu}(x-)) - F_{\mu}(x-))]$  solving  $H(u', \phi_{\uparrow}(u')) = 0$ , which implies that  $u' + \phi_{\uparrow}(u) - u \leq \phi_{\uparrow}(u')$  and  $\phi_{\uparrow}(u') \leq \phi_{\uparrow}(u)$ . Hence  $\phi_{\uparrow}$  and  $u \mapsto u - \phi_{\uparrow}(u)$  are non-decreasing on  $(0, 1)$ . Moreover,  $\phi'_{\uparrow}(u) du$  a.e. on  $(F_{\mu}(x-), F_{\mu}(x))$ ,  $\phi_{\uparrow}(u) < \phi_{\uparrow}(F_{\mu}(x))$  and  $F_{\nu_l}^{-1} \left( \frac{\phi_{\uparrow}(u)}{\nu_l(\mathbb{R})} \right) < x < F_{\nu_r}^{-1} \left( \frac{u - \phi_{\uparrow}(u)}{\nu_r(\mathbb{R})} \right)$ .

**Step 4 :** Let us now prove (5). Since  $\pi^{\uparrow}$  is the only non-decreasing coupling in  $\Pi_M(\mu, \nu, \nu_l, \nu_r)$ , it is enough to check that any coupling maximizing  $\int_{\mathbb{R}^2} \varphi(|x - y|) \pi(dx, dy)$  over  $\pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$  is non-decreasing. To do so, we are going to adapt the proof of Lemma 1.11 [3]. We modify the definition of the set  $M$  in this proof into  $M = \bigcup_{\tau \in \mathcal{S}_3} \{M_{l,\tau} \cup M_{r,\tau}\}$ , where for  $\tau$  in the set  $\mathcal{S}_3$  of permutations of  $\{1, 2, 3\}$ ,  $M_{l,\tau}$  and  $M_{r,\tau}$  are the respective images of

$$M_l = \{((x_-, y_-), (x_+, y_+), (x_+, z)) \in \{\mathbb{R}^2\}^3 : y_+ < y_- < x_- < x_+ < z\}$$

$$\text{and } M_r = \{((x_-, y), (x_-, z_-), (x_+, z_+)) \in \{\mathbb{R}^2\}^3 : y < x_- < x_+ < z_+ < z_-\}.$$

by  $\{\mathbb{R}^2\}^3 \ni (\chi_1, \chi_2, \chi_3) \mapsto (\chi_{\tau(1)}, \chi_{\tau(2)}, \chi_{\tau(3)}) \in \{\mathbb{R}^2\}^3$ . Applying Theorem 3.1 [3] like in the proof of Lemma 1.11 [3], we obtain in case (1) of this theorem a Borel subset  $\hat{\Gamma}$  of  $\mathbb{R}^2$  such that  $\pi(\hat{\Gamma}) = 1$  and  $M_l \cap \hat{\Gamma}^3 = \emptyset = M_r \cap \hat{\Gamma}^3$ . By Lemma 3.3, we may suppose that

$$\forall x \in \mathbb{R}, \exists y < x \text{ s.t. } (x, y) \in \hat{\Gamma} \Leftrightarrow \exists z > x \text{ s.t. } (x, z) \in \hat{\Gamma}.$$

We deduce that

$$\{((x_-, y_-), (x_+, y_+)) \in \{\mathbb{R}^2\}^2 : y_+ < y_- < x_- < x_+\} \cap \hat{\Gamma}^2 = \emptyset$$

$$\text{and } \{((x_-, z_-), (x_+, z_+)) \in \{\mathbb{R}^2\}^2 : x_- < x_+ < z_+ < z_-\} \cap \hat{\Gamma}^2 = \emptyset.$$

Setting  $\Gamma = \hat{\Gamma} \cap \{(x, y) \in \mathbb{R}^2 : y \neq x\}$ , we have

$$\{((x_-, y_-), (x_+, y_+)) \in \{\mathbb{R}^2\}^2 : y_+ < y_- \leq x_- < x_+\} \cap \Gamma^2 = \emptyset$$

$$\text{and } \{((x_-, z_-), (x_+, z_+)) \in \{\mathbb{R}^2\}^2 : x_- < x_+ \leq z_+ < z_-\} \cap \Gamma^2 = \emptyset.$$

Conditions (a)(b) in Definition 2.3 hold and since

$$\begin{aligned} \pi(\{(x, y) \in \mathbb{R}^2 : y \neq x\}) &= \pi(\{(x, y) \in \mathbb{R}^2 : y < x\}) + \pi(\{(x, y) \in \mathbb{R}^2 : y > x\}) \\ &= \nu_l(\mathbb{R}) + \nu_r(\mathbb{R}) = 1, \end{aligned}$$

$\pi(\Gamma) = 1$  so that  $\pi$  is non-decreasing. Like in the proof of Lemma 1.11 [3], we still need to check that case (2) in Theorem 3.1 [3] cannot occur. In the notation of the proof of the lemma, we need to construct a competitor  $\omega'$  of  $\omega$  such that  $\pi - \omega + \omega' \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$  and

$$\int_{\mathbb{R}^2} \varphi(|x - y|) \omega'(dx, dy) > \int_{\mathbb{R}^2} \varphi(|x - y|) \omega(dx, dy). \quad (33)$$

This is done by choosing

$$\alpha'_p = \frac{1}{3} \left( \frac{z - y_-}{z - y_+} (\delta_{(x_-, y_+)} + \delta_{(x_+, z)}) + \frac{y_- - y_+}{z - y_+} (\delta_{(x_-, z)} + \delta_{(x_+, y_+)}) + \delta_{(x_+, y_-)} \right)$$

when  $\alpha_p = \frac{1}{3} (\delta_{(x_-, y_-)} + \delta_{(x_+, y_+)} + \delta_{(x_+, z)})$  for some  $y_+ < y_- < x_- < x_+ < z$  and by choosing

$$\alpha'_p = \frac{1}{3} \left( \frac{z_- - z_+}{z_- - y} (\delta_{(x_-, z_-)} + \delta_{(x_+, y)}) + \frac{z_+ - y}{z_- - y} (\delta_{(x_-, y)} + \delta_{(x_+, z_-)}) + \delta_{(x_-, z_+)} \right)$$

when  $\alpha_p = \frac{1}{3} (\delta_{(x_-, y)} + \delta_{(x_-, z_-)} + \delta_{(x_+, z_+)})$  for some  $y < x_- < x_+ < z_+ < z_-$ . Note that, in the first case,

$$\int_{v \in \mathbb{R}} \mathbf{1}_{\{w < v\}} \alpha_p(dv, dw) = \frac{1}{3} (\delta_{y_-}(dw) + \delta_{y_+}(dw)) = \int_{v \in \mathbb{R}} \mathbf{1}_{\{w < v\}} \alpha'_p(dv, dw)$$

and  $\int_{v \in \mathbb{R}} \mathbf{1}_{\{w > v\}} \alpha_p(dv, dw) = \frac{1}{3} \delta_z(dw) = \int_{v \in \mathbb{R}} \mathbf{1}_{\{w > v\}} \alpha'_p(dv, dw),$

which contributes to ensuring that  $\pi - \omega + \omega' \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$ . Moreover, by Lemma 3.4 applied with  $(y, m, z) = (y_+, y_-, z)$ , the function  $f$  defined by

$$f(x) = \frac{z - y_-}{z - y_+} \varphi(|x - y_+|) + \frac{y_- - y_+}{z - y_+} \varphi(|z - x|) - \varphi(|x - y_-|)$$

is decreasing on  $[y_-, z]$ . We deduce that, still in the first case,

$$\int_{\mathbb{R}^2} \varphi(|v - w|) \alpha'_p(dv, dw) - \int_{\mathbb{R}^2} \varphi(|v - w|) \alpha_p(dv, dw) = \frac{1}{3} (f(x_-) - f(x_+)) > 0,$$

which contributes to ensuring (33). One checks in the same way that the two properties still hold in the second case.

Let us now give an alternative simple argument in the particular case when  $\varphi$  is the identity function. For  $\pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$ , we have  $\int_{\mathbb{R}^2} (y - z) \mathbf{1}_{\{z < y\}} \pi(dy, dz) - \int_{\mathbb{R}^2} (z - y) \mathbf{1}_{\{z > y\}} \pi(dy, dz) = 0$ . Since  $\nu_l^\pi = \nu_l$ , we deduce with Fubini's theorem and (18) that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^2} |y - z| \pi(dy, dz) &= \int_{\mathbb{R}^2} (y - z) \mathbf{1}_{\{z < y\}} \pi(dy, dz) \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}_+} \mathbf{1}_{\{y > x\}} dx \mathbf{1}_{\{z < y\}} \pi(dy, dz) - \int_{\mathbb{R}^2} \int_{\mathbb{R}_-} \mathbf{1}_{\{y \leq x\}} dx \mathbf{1}_{\{z < y\}} \pi(dy, dz) - \int_{\mathbb{R}} z \nu_l(dz) \\ &= \int_{\mathbb{R}_+} (\nu_l(\mathbb{R}) - \phi_\pi(F_\mu(x))) dx - \int_{\mathbb{R}_-} \phi_\pi(F_\mu(x)) dx - \int_{\mathbb{R}} z \nu_l(dz). \end{aligned}$$

Since, according to **Step 1**,  $\forall x \in \mathbb{R}, \phi_\uparrow(F_\mu(x)) \leq \phi_\pi(F_\mu(x))$ , we conclude that

$$\int_{\mathbb{R}^2} |y - z| \pi^\uparrow(dy, dz) = \sup_{\pi \in \Pi_M(\mu, \nu, \nu_l, \nu_r)} \int_{\mathbb{R}^2} |y - z| \pi(dy, dz).$$

□

*Proof of Proposition 2.13.* Let us suppose that there exists  $\pi^\downarrow \in \Pi_M(\mu, \nu, \nu_l, \nu_r)$  such that the coupling  $\pi(dx, dy) := \frac{\pi^\downarrow(dx, dy) + (\nu_l + \nu_r - \nu)(dx) \delta_x(dy)}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})} \in \Pi_M\left(\frac{\mu + \nu_l + \nu_r - \nu}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}, \frac{\nu_l + \nu_r}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}, \frac{\nu_l}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}, \frac{\nu_r}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}\right)$  is non-increasing. Then, by Proposition 2.8, there exist  $-\infty < a \leq b < +\infty$  such that  $(\mu + \nu_l + \nu_r - \nu)([a, b]) = (\nu_l + \nu_r)(\mathbb{R}), (\nu_l + \nu_r)((a, b)) = 0$  and  $\pi(\{(x, x)\}) = \frac{(\mu + \nu_l + \nu_r - \nu)(\{x\})}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})} \wedge \frac{(\nu_l + \nu_r)(\{x\})}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}$  for  $x \in \{a, b\}$ . Since  $\pi(\{(x, x) : x \in \mathbb{R}\}) = 0$ , we deduce that we may choose a closed or semi-open or open interval  $I$  with ends  $a$  and  $b$  such that  $\frac{\mu + \nu_l + \nu_r - \nu}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}(I) = 1$  and  $\frac{\nu_l + \nu_r}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}(I) = 0$ . Since  $\nu - (\nu_l + \nu_r) = \nu_0^\downarrow \leq \mu \wedge \nu$ , we have that  $\mu + \nu_l + \nu_r - \nu - (\mu - \nu)^+ = \nu_l + \nu_r - (\nu - \mu)^+$  is a non-negative measure and we deduce that  $\nu_l + \nu_r = (\nu - \mu)^+$  and  $\mu + \nu_l + \nu_r - \nu = (\mu - \nu)^+$ . Hence the Dispersion Assumption is satisfied. Moreover, since  $\pi(\{(-\infty, a) \cup (b, +\infty)\} \times \mathbb{R}) \leq \frac{(\mu + \nu_l + \nu_r - \nu)(\mathbb{R} \setminus I)}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})} = 0$ ,  $\pi(\mathbb{R} \times (a, b)) \leq \frac{\nu_l + \nu_r}{\nu_l(\mathbb{R}) + \nu_r(\mathbb{R})}(I) = 0$  and  $\pi(\{a, a\}) = 0$ , we have

$$\begin{aligned} \nu_l(dy) &= \nu_l^\pi(dy) = \int_{x \in \mathbb{R}} \mathbf{1}_{\{y < x\}} \pi(dx, dy) = \int_{x \in \mathbb{R}} \mathbf{1}_{\{y < x \leq b\}} \pi(dx, dy) \\ &= \int_{x \in \mathbb{R}} \mathbf{1}_{\{y < x, y \leq a\}} \pi(dx, dy) = \int_{x \in \mathbb{R}} \mathbf{1}_{\{y \leq a\}} \pi(dx, dy) = \mathbf{1}_{\{y \leq a\}} (\nu - \mu)^+(dy). \end{aligned}$$

and, in a symmetric way,  $\nu_r(dy) = \mathbf{1}_{\{y \geq b\}} (\nu - \mu)^+(dy)$  (notice that, when  $a = b$ , then  $\mu = \delta_a$  and  $(\nu - \mu)^+(\{a\}) = 0$ ).

In the other direction, under the Dispersion Assumption, denoting by  $a \leq b$  the ends of the interval  $I$ , we have  $(\mu - \nu)^+([a, b]) = (\mu - \nu)^+(\mathbb{R})$  and  $(\nu - \mu)^+((a, b)) = 0$  so that Proposition 2.8 ensures the existence of a unique non-increasing coupling  $\pi \in \Pi_M \left( \frac{(\mu - \nu)^+}{(\mu - \nu)^+(\mathbb{R})}, \frac{(\nu - \mu)^+}{(\mu - \nu)^+(\mathbb{R})} \right)$  such that

$$\pi(\{(a, a)\}) = \frac{(\mu - \nu)^+(\{a\})}{(\mu - \nu)^+(\mathbb{R})} \wedge \frac{(\nu - \mu)^+(\{a\})}{(\mu - \nu)^+(\mathbb{R})} = 0 = \pi(\{(b, b)\}).$$

Hence  $\int_{x \in \mathbb{R}} \mathbb{1}_{\{y < x\}} \pi(dx, dy) = \mathbb{1}_{\{y \leq a\}} \frac{(\nu - \mu)^+(dy)}{(\mu - \nu)^+(\mathbb{R})}$  and  $\int_{x \in \mathbb{R}} \mathbb{1}_{\{y > x\}} \pi(dx, dy) = \mathbb{1}_{\{y \geq b\}} \frac{(\nu - \mu)^+(dy)}{(\mu - \nu)^+(\mathbb{R})}$ . The coupling  $(\mu - \nu)^+(\mathbb{R}) \pi(dx, dy) + \mu \wedge \nu(dx) \delta_x(dy)$  belongs to  $\Pi_M(\mu, \nu, \mathbb{1}_{\{y \leq a\}}(\nu - \mu)^+(dy), \mathbb{1}_{\{y \geq b\}}(\nu - \mu)^+(dy))$ .  $\square$

The proof of Proposition 2.14 relies on the following lemma.

**Lemma 3.8.** *Let  $\eta \in \mathcal{P}_1(\mathbb{R})$  be such that  $\int_{\mathbb{R}} z \eta(dz) = y$ . For  $x > y$ , if  $\eta((x, +\infty)) > 0$  then there exists a non-negative measure  $\hat{\eta}$  such that  $\hat{\eta}(dz) \leq \mathbb{1}_{\{z < x\}} \eta(dz)$  and  $\int_{\mathbb{R}} (z - x) \hat{\eta}(dz) + \int_{(x, +\infty)} (z - x) \eta(dz) = 0$ .*

*Proof of Lemma 3.8.* The function  $[0, F_\eta(x-)] \ni u \mapsto G(u) = \int_0^u (F_\eta^{-1}(v) - x) dv + \int_{(x, +\infty)} (z - x) \eta(dz)$  is continuous, decreasing and such that  $G(0) = \int_{(x, +\infty)} (z - x) \eta(dz) > 0$  and  $G(F_\eta(x-)) = \int_{\mathbb{R}} (z - x) \eta(dz) = y - x < 0$ . Hence there exists a unique  $\hat{u} \in (0, F_\eta(x-))$  such that  $G(\hat{u}) = 0$ . We set  $\hat{\eta}(dz) = \int_0^{\hat{u}} \delta_{F_\eta^{-1}(v)}(dz) dv \leq \int_0^{F_\eta(x-)} \delta_{F_\eta^{-1}(v)}(dz) dv = \mathbb{1}_{\{z < x\}} \eta(dz)$ .  $\square$

*Proof of Proposition 2.14.* Since  $\int_{\mathbb{R}} (z - x)^+ \nu(dz) - \int_{\mathbb{R}} (x - z)^+ \nu(dz) = \int_{\mathbb{R}} z \nu(dz) - x = \int_{\mathbb{R}} y \mu(dy) - x = \int_{\mathbb{R}} (y - x)^+ \mu(dy) - \int_{\mathbb{R}} (x - y)^+ \mu(dy)$ , we have that

$$\int_{\mathbb{R}} (z - x)^+ \nu(dz) - \int_{\mathbb{R}} (y - x)^+ \mu(dy) = \int_{\mathbb{R}} (x - z)^+ \nu(dz) - \int_{\mathbb{R}} (x - y)^+ \mu(dy) = \frac{1}{2} (u_\nu(x) - u_\mu(x)). \quad (34)$$

From now on, we suppose that  $u_\nu(x) > u_\mu(x)$ . Then  $\nu((-\infty, x)) = F_\nu(x-) > 0$  and  $\nu((x, +\infty)) = 1 - F_\nu(x) > 0$ . Since  $F_\nu^{-1}(v) - x > 0$  for  $v \in (F_\nu(x), 1)$ , the function  $[0, 1 - F_\nu(x)] \ni u \mapsto G(u) = \int_{F_\nu(x)}^{F_\nu(x)+u} (F_\nu^{-1}(v) - x) dv$  is increasing. It also is continuous and such that  $G(0) = 0$  and  $G(1 - F_\nu(x)) = \int_{\mathbb{R}} (z - x)^+ \nu(dz)$ . We deduce the existence of a unique  $p_+(x) \in (0, 1 - F_\nu(x))$  such that (7) holds. In a similar way, there exists a unique  $p_-(x) \in (0, F_\nu(x-))$  such that (8) holds.

The proof of the other statements relies on three steps. In the first step, we are going to check that

$$\inf_{\pi \in \Pi_M(\mu, \nu)} \pi(\{(x, x)\}) \geq (\mu(\{x\}) - p_-(x) - p_+(x))^+, \quad (35)$$

and that when  $\mu(\{x\}) > p_-(x) + p_+(x)$ , if  $\pi \in \Pi_M(\mu, \nu)$  is such that  $\pi(\{(x, x)\}) = \mu(\{x\}) - p_-(x) - p_+(x)$  then  $\pi_x = \eta_x$  with  $\eta_x$  given by (9) and  $\pi(\{(-\infty, x) \times (x, +\infty)\} \cup \{(x, +\infty) \times (-\infty, x)\}) = 0$ . In the second step, we will check that if  $\inf_{\pi \in \Pi_M(\mu, \nu)} \pi(\{(x, x)\}) > 0$ , then there exists  $\pi \in \Pi_M(\mu, \nu)$  such that  $\pi_x = \eta_x$  where  $\eta_x$  is given by (9) so that  $\pi(\{(x, x)\}) = \mu(\{x\}) - p_-(x) - p_+(x)$  and, by (35),  $\inf_{\pi \in \Pi_M(\mu, \nu)} \pi(\{(x, x)\}) = \mu(\{x\}) - p_-(x) - p_+(x)$ . As a consequence,  $\inf_{\pi \in \Pi_M(\mu, \nu)} \pi(\{(x, x)\}) = 0$  when  $\mu(\{x\}) - p_-(x) - p_+(x) \leq 0$  and (35) is an equality. In the last step, we prove that when  $\mu(\{x\}) \in (0, p_-(x) + p_+(x)]$ , there exists  $\pi \in \Pi_M(\mu, \nu)$  such that  $\pi_x = \eta_x$  where  $\eta_x$  is given by (10).

**Step 1 :** Let  $\pi \in \Pi_M(\mu, \nu)$ . Since, by the martingale property and Jensen's inequality,  $\mu(dy)$  a.e.,  $\int_{\mathbb{R}} |z - x| \pi_y(dz) \geq |y - x|$ , one has

$$\begin{aligned} \mu(\{x\}) \int_{\mathbb{R}} |z - x| \pi_x(dz) &\leq \int_{\mathbb{R} \times \mathbb{R}} (|z - x| - |y - x|) \pi_y(dz) \mu(dy) = u_\nu(x) - u_\mu(x) \\ &= \int_{F_\nu(x-)-p_-(x)}^{F_\nu(x-)} (x - F_\nu^{-1}(v)) dv + \int_{F_\nu(x)}^{F_\nu(x)+p_+(x)} (F_\nu^{-1}(v) - x) dv. \end{aligned} \quad (36)$$

Moreover, by the martingale property for the first equality and by (7), (8) and (34) for the second equality,

$$\begin{aligned} \mu(\{x\}) \int_{\mathbb{R}} (z-x)^+ \pi_x(dz) - \mu(\{x\}) \int_{\mathbb{R}} (x-z)^+ \pi_x(dz) &= 0 \\ &= \int_{F_\nu(x)}^{F_\nu(x)+p_+(x)} (F_\nu^{-1}(v) - x) dv - \int_{F_\nu(x)-p_-(x)}^{F_\nu(x-)} (x - F_\nu^{-1}(v)) dv. \end{aligned}$$

Hence,

$$\begin{aligned} \mu(\{x\}) \int_{\mathbb{R}} (z-x)^+ \pi_x(dz) &\leq \int_{F_\nu(x)}^{F_\nu(x)+p_+(x)} (F_\nu^{-1}(v) - x) dv, \\ \mu(\{x\}) \int_{\mathbb{R}} (x-z)^+ \pi_x(dz) &\leq \int_{F_\nu(x)-p_-(x)}^{F_\nu(x-)} (x - F_\nu^{-1}(v)) dv. \end{aligned} \quad (37)$$

We have  $\mu(\{x\}) \mathbb{1}_{\{x < z\}} \pi_x(dz) \leq \mathbb{1}_{\{x < z\}} \nu(dz)$  so that, by Lemma 3.6 applied with  $f(z) = z - x$ ,  $\hat{\eta}(dz) = \frac{\mu(\{x\}) \mathbb{1}_{\{x < z\}} \pi_x(dz)}{1 - F_\nu(x)}$  and  $\eta(dz) = \frac{\mathbb{1}_{\{x < z\}} \nu(dz)}{1 - F_\nu(x)}$  such that  $\hat{\eta}(\mathbb{R}) = \frac{\mu(\{x\}) \pi_x((x, +\infty))}{1 - F_\nu(x)}$ ,  $F_\eta(z) = \frac{(F_\nu(z) - F_\nu(x))^+}{1 - F_\nu(x)}$  and  $F_\eta^{-1}(u) = F_\nu^{-1}(F_\nu(x) + (1 - F_\nu(x))u)$ ,

$$\int_{F_\nu(x)}^{F_\nu(x) + \mu(\{x\}) \pi_x((x, +\infty))} (F_\nu^{-1}(v) - x) dv = (1 - F_\nu(x)) \int_0^{\hat{\eta}(\mathbb{R})} (F_\eta^{-1}(v) - x) dv \leq \mu(\{x\}) \int_{\mathbb{R}} (z-x)^+ \pi_x(dz),$$

and when  $\mu(\{x\}) \pi_x((x, +\infty)) > 0$ ,

$$\frac{\mathbb{1}_{\{x < z\}} \pi_x(dz)}{\pi_x((x, +\infty))} \geq_{st} \frac{1}{\mu(\{x\}) \pi_x((x, +\infty))} \int_{F_\nu(x)}^{F_\nu(x) + \mu(\{x\}) \pi_x((x, +\infty))} \delta_{F_\nu^{-1}(u)} du. \quad (38)$$

With (37), the first inequality implies that

$$\int_{F_\nu(x)}^{F_\nu(x) + \mu(\{x\}) \pi_x((x, +\infty))} (F_\nu^{-1}(v) - x) dv \leq \int_{F_\nu(x)}^{F_\nu(x) + p_+(x)} (F_\nu^{-1}(v) - x) dv.$$

Since  $F_\nu^{-1}(v) - x > 0$  for  $v \in (F_\nu(x), 1)$ , we deduce that  $\mu(\{x\}) \pi_x((x, +\infty)) \leq p_+(x)$ . In a symmetric way,  $\mu(\{x\}) \pi_x((-\infty, x)) \leq p_-(x)$ . Since  $\pi(\{(x, x)\}) = \mu(\{x\}) (1 - \pi_x((-\infty, x)) - \pi_x((x, +\infty)))$ , we conclude that (35) holds.

Let us now suppose that  $\mu(\{x\}) > p_-(x) + p_+(x)$  and that  $\pi \in \Pi_M(\mu, \nu)$  is such that  $\pi(\{(x, x)\}) = \mu(\{x\}) - p_-(x) - p_+(x)$ . Then  $\mu(\{x\}) \pi_x((x, +\infty)) = p_+(x)$  and  $\mu(\{x\}) \pi_x((-\infty, x)) = p_-(x)$ . Moreover, by (38) and (37),  $\mathbb{1}_{\{x < z\}} \pi_x(dz) = \frac{1}{\mu(\{x\})} \int_{F_\nu(x)}^{F_\nu(x) + p_+(x)} \delta_{F_\nu^{-1}(u)} du$ . In a symmetric way,  $\mathbb{1}_{\{z < x\}} \pi_x(dz) = \frac{1}{\mu(\{x\})} \int_{F_\nu(x) - p_-(x)}^{F_\nu(x-)} \delta_{F_\nu^{-1}(u)} du$  so that  $\pi_x = \eta_x$  with  $\eta_x$  given by (9). Moreover, the inequality in (36) becomes an equality, so that, with the martingale constraint,

$$\begin{aligned} 0 &= \int_{\mathbb{R}^2} (\mathbb{1}_{\{y < x\}} (x - z + 2(z - x)^+ - x + y) + \mathbb{1}_{\{y > x\}} (z - x + 2(x - z)^+ - y + x)) \pi_y(dz) \mu(dy) \\ &= \int_{\mathbb{R}^2} 2 (\mathbb{1}_{\{y < x\}} (z - x)^+ + \mathbb{1}_{\{y > x\}} (x - z)^+) \pi_y(dz) \mu(dy). \end{aligned}$$

Hence  $\pi(\{(-\infty, x) \times (x, +\infty)\} \cup \{(x, +\infty) \times (-\infty, x)\}) = 0$ .

**Step 2 :** Let us suppose that  $\inf_{\pi \in \Pi_M(\mu, \nu)} \pi(\{(x, x)\}) > 0$ , which implies that  $\mu(\{x\}) > 0$ . Let  $\tilde{\pi} \in \Pi_M(\mu, \nu)$ . We are now going to modify  $\tilde{\pi}$  into  $\pi \in \Pi_M(\mu, \nu)$  such that  $\pi_x = \eta_x$  where  $\eta_x$  is given by (9). By lemma 3.8, we have that  $\mu(dy)$  a.e. on  $(-\infty, x)$ ,  $\tilde{\pi}_y((x, +\infty)) > 0$  implies the existence of  $\hat{\pi}_y(dz) \leq \mathbb{1}_{\{z < x\}} \tilde{\pi}_y(dz)$  such that  $\int_{\mathbb{R}} z \hat{\pi}_y(dz) + \int_{(x, +\infty)} z \tilde{\pi}_y(dz) = x (\hat{\pi}_y(\mathbb{R}) + \tilde{\pi}_y((x, +\infty)))$ . In the same way  $\mu(dy)$  a.e. on  $(x, +\infty)$ ,  $\tilde{\pi}_y((-\infty, x)) > 0$  implies the existence of  $\hat{\pi}_y(dz) \leq \mathbb{1}_{\{z > x\}} \tilde{\pi}_y(dz)$

such that  $\int_{\mathbb{R}} z \hat{\pi}_y(dz) + \int_{(-\infty, x)} z \tilde{\pi}_y(dz) = x(\hat{\pi}_y(\mathbb{R}) + \tilde{\pi}_y((-\infty, x)))$ . We now set for  $u \in [0, 1]$

$$\begin{aligned} \bar{\pi}(u, dy, dz) &= \tilde{\pi}(dy, dz) + u \mathbb{1}_{\{y < x\}} \mu(dy) ((\tilde{\pi}_y((x, +\infty)) + \hat{\pi}_y(\mathbb{R})) \delta_x(dz) - \mathbb{1}_{\{z > x\}} \tilde{\pi}_y(dz) - \hat{\pi}_y(dz)) \\ &\quad + u \delta_x(dy) \int_{w \in (-\infty, x)} (\mathbb{1}_{\{z > x\}} \tilde{\pi}_w(dz) + \hat{\pi}_w(dz) - (\tilde{\pi}_w((x, +\infty)) + \hat{\pi}_w(\mathbb{R})) \delta_x(dz)) \mu(dw) \\ &\quad + u \mathbb{1}_{\{y > x\}} \mu(dy) ((\tilde{\pi}_y((-\infty, x)) + \hat{\pi}_y(\mathbb{R})) \delta_x(dz) - \mathbb{1}_{\{z < x\}} \tilde{\pi}_y(dz) - \hat{\pi}_y(dz)) \\ &\quad + u \delta_x(dy) \int_{w \in (x, +\infty)} (\mathbb{1}_{\{z < x\}} \tilde{\pi}_w(dz) + \hat{\pi}_w(dz) - (\tilde{\pi}_w((-\infty, x)) + \hat{\pi}_w(\mathbb{R})) \delta_x(dz)) \mu(dw). \end{aligned}$$

Since  $\mu(dy)$  a.e. on  $(-\infty, x)$  (resp.  $(x, +\infty)$ ),  $\tilde{\pi}_y(dz) - u \mathbb{1}_{\{z > x\}} \tilde{\pi}_y(dz) - u \hat{\pi}_y(dz) + u(\tilde{\pi}_y((x, +\infty)) + \hat{\pi}_y(\mathbb{R})) \delta_x(dz)$  (resp.  $\tilde{\pi}_y(dz) - u \mathbb{1}_{\{z < x\}} \tilde{\pi}_y(dz) - u \hat{\pi}_y(dz) + u(\tilde{\pi}_y((-\infty, x)) + \hat{\pi}_y(\mathbb{R})) \delta_x(dz)$ ) is a probability measure with expectation  $y$ , as long as

$$u \left( \int_{(-\infty, x)} (\tilde{\pi}_w((x, +\infty)) + \hat{\pi}_w(\mathbb{R})) \mu(dw) + \int_{(x, +\infty)} (\tilde{\pi}_w((-\infty, x)) + \hat{\pi}_w(\mathbb{R})) \mu(dw) \right) \leq \tilde{\pi}(\{(x, x)\}), \quad (39)$$

$\bar{\pi}(u, dy, dz) \in \Pi_M(\mu, \nu)$  and  $\bar{\pi}(u, \{(x, x)\}) > 0$ , so that the inequality in (39) is strict. Therefore (39) holds with strict inequality for  $u \in [0, 1]$ , and  $\bar{\pi}(1, dy, dz) \in \Pi_M(\mu, \nu)$ . Denoting for simplicity  $\bar{\pi}(dy, dz) = \bar{\pi}(1, dy, dz)$ , we have that  $\bar{\pi}(\{(-\infty, x) \times (x, +\infty)\} \cup \{(x, +\infty) \times (-\infty, x)\}) = 0$ . As a consequence,

$$\int_{\mathbb{R}^2} \mathbb{1}_{\{y \neq x\}} (|z - x| - |y - x|) \bar{\pi}(dy, dz) = 0$$

so that  $\mu(\{x\}) \int_{\mathbb{R}} |z - x| \bar{\pi}_x(dz) = u_\nu(x) - u_\mu(x)$ . With the martingale property, this implies that

$$\mu(\{x\}) \int_{\mathbb{R}} (x - z) \mathbb{1}_{\{z < x\}} \bar{\pi}_x(dz) = \frac{1}{2}(u_\nu(x) - u_\mu(x)) = \int_{F_\nu(x-) - p_-(x)}^{F_\nu(x-)} (x - F_\nu^{-1}(v)) dv. \quad (40)$$

Let  $w = F_\nu^{-1}(F_\nu(x-) - p_-(x))$  and  $r = F_\nu(w) - F_\nu(x-) + p_-(x) \in [0, \nu(\{w\})]$ . We set

$$\gamma(dz) = \int_{F_\nu(x-) - p_-(x)}^{F_\nu(x-)} \delta_{F_\nu^{-1}(v)}(dz) dv - \mu(\{x\}) \mathbb{1}_{\{w < z < x\}} \bar{\pi}_x(dz) - (\bar{\pi}(\{(x, w)\}) \wedge r) \delta_w(dz)$$

$$\text{and } \sigma(dz) = \mu(\{x\}) \mathbb{1}_{\{z < w\}} \bar{\pi}_x(dz) + (\bar{\pi}(\{(x, w)\}) - r)^+ \delta_w(dz).$$

Since  $\int_{F_\nu(x-) - p_-(x)}^{F_\nu(x-)} \delta_{F_\nu^{-1}(v)}(dz) dv = \mathbb{1}_{\{w < z < x\}} \nu(dz) + r \delta_w(dz)$  and  $\bar{\pi}((x, +\infty) \times (-\infty, x)) = 0$ ,  $\int_{F_\nu(x-) - p_-(x)}^{F_\nu(x-)} \delta_{F_\nu^{-1}(v)}(dz) dv = \int_{(-\infty, x]} \mathbb{1}_{\{w < z < x\}} \bar{\pi}_y(dz) \mu(dy) + r \delta_w(dz)$  and  $\nu(\{w\}) - \bar{\pi}(\{(x, w)\}) = \int_{(-\infty, x)} \bar{\pi}_y(\{w\}) \mu(dy)$  so that setting  $\alpha = \mathbb{1}_{\{\nu(\{w\}) > \bar{\pi}(\{(x, w)\})\}} \frac{(r - \bar{\pi}(\{(x, w)\}))^+}{\nu(\{w\}) - \bar{\pi}(\{(x, w)\})} \in [0, 1]$  and  $f(z) = \alpha \mathbb{1}_{\{z = w\}} + \mathbb{1}_{\{w < z < x\}}$ ,

$$\begin{aligned} \gamma(dz) &= \int_{(-\infty, x)} \mathbb{1}_{\{w < z < x\}} \bar{\pi}_y(dz) \mu(dy) + (r - \bar{\pi}(\{(x, w)\}))^+ \delta_w(dz) \\ &= \int_{(-\infty, x)} \mathbb{1}_{\{w < z < x\}} \bar{\pi}_y(dz) \mu(dy) + \alpha \int_{(-\infty, x)} \bar{\pi}_y(\{w\}) \delta_w(dz) \mu(dy) = \int_{(-\infty, x)} f(z) \bar{\pi}_y(dz) \mu(dy). \end{aligned} \quad (41)$$

Since

$$\gamma(dz) - \sigma(dz) = \int_{F_\nu(x-) - p_-(x)}^{F_\nu(x-)} \delta_{F_\nu^{-1}(v)}(dz) dv - \mu(\{x\}) \mathbb{1}_{\{z < x\}} \bar{\pi}_x(dz), \quad (42)$$

(40) implies that  $\int_{\mathbb{R}} (x - z) \gamma(dz) = \int_{\mathbb{R}} (x - z) \sigma(dz)$ . Since  $\sigma$  (resp.  $\gamma$ ) only weights  $(-\infty, w]$  (resp.  $[w, x)$ ), this implies that  $\sigma(\mathbb{R}) \leq \gamma(\mathbb{R})$ . If  $\gamma(\mathbb{R}) = 0$ , then  $\sigma(\mathbb{R}) = 0$  so that, by (42),  $\mu(\{x\}) \mathbb{1}_{\{z < x\}} \bar{\pi}_x(dz) = \int_{F_\nu(x-) - p_-(x)}^{F_\nu(x-)} \delta_{F_\nu^{-1}(v)}(dz) dv$  and we set  $\tilde{\pi} = \bar{\pi}$ . Otherwise, since  $\int_{\mathbb{R}} z \gamma(dz) = (\gamma(\mathbb{R}) - \sigma(\mathbb{R}))x + \int_{\mathbb{R}} z \sigma(dz)$ , the probability measures  $\frac{\gamma}{\gamma(\mathbb{R})}$  and  $\frac{1}{\gamma(\mathbb{R})} ((\gamma(\mathbb{R}) - \sigma(\mathbb{R})) \delta_x + \sigma)$  share

the same mean and, by comparison of their supports, satisfy  $\frac{\gamma}{\gamma(\mathbb{R})} \leq_{cx} \frac{1}{\gamma(\mathbb{R})} ((\gamma(\mathbb{R}) - \sigma(\mathbb{R}))\delta_x + \sigma)$ . By Strassen's theorem, there exists  $\theta \in \Pi_M(\frac{\gamma}{\gamma(\mathbb{R})}, \frac{1}{\gamma(\mathbb{R})} ((\gamma(\mathbb{R}) - \sigma(\mathbb{R}))\delta_x + \sigma))$ . We set

$$\begin{aligned} \tilde{\pi}(dy, dz) &= \mathbf{1}_{\{y > x\}} \bar{\pi}(dy, dz) + \mathbf{1}_{\{y < x\}} \mu(dy) \left( (1 - f(z)) \bar{\pi}_y(dz) + \int f(v) \theta_v(dz) \bar{\pi}_y(dv) \right) \\ &+ \mu(\{x\}) \delta_x(dy) \left( \bar{\pi}_x(dz) + \frac{1}{\mu(\{x\})} \int_{t \in (-\infty, x)} \left( f(z) \bar{\pi}_t(dz) - \int f(v) \theta_v(dz) \bar{\pi}_t(dv) \right) \mu(dt) \right). \end{aligned}$$

For  $\mu(dy)$  a.e.  $y \in (-\infty, x)$ , since  $f(v) \bar{\pi}_y(dv)$  a.e.  $\int_{\mathbb{R}} z \theta_v(dz) = v$  by (41) and the martingale property of  $\theta$ , we have

$$\begin{aligned} \int_{\mathbb{R}} z \tilde{\pi}_y(dz) &= \int_{\mathbb{R}} z(1 - f(z)) \bar{\pi}_y(dz) + \int_{\mathbb{R}} f(v) \int_{\mathbb{R}} z \theta_v(dz) \bar{\pi}_y(dv) \\ &= \int_{\mathbb{R}} z(1 - f(z)) \bar{\pi}_y(dz) + \int_{\mathbb{R}} v f(v) \bar{\pi}_y(dv) = \int_{\mathbb{R}} z \bar{\pi}_y(dz) = y. \end{aligned}$$

Using (41), the definition of  $\theta$  and (42), we obtain

$$\begin{aligned} \mu(\{x\}) \tilde{\pi}_x(dz) &= \mu(\{x\}) \bar{\pi}_x(dz) + \gamma(dz) - \int_{v \in \mathbb{R}} \theta_v(dz) \gamma(dv) \\ &= \mu(\{x\}) \bar{\pi}_x(dz) + \gamma(dz) - \sigma(dz) - (\gamma(\mathbb{R}) - \sigma(\mathbb{R})) \delta_x(dz) \\ &= \mu(\{x\}) \mathbf{1}_{\{z \geq x\}} \bar{\pi}_x(dz) + \int_{F_\nu(x-) - p_-(x)}^{F_\nu(x-)} \delta_{F_\nu^{-1}(v)}(dz) dv - (\gamma(\mathbb{R}) - \sigma(\mathbb{R})) \delta_x(dz). \end{aligned}$$

The positivity of  $\tilde{\pi}(\{(x, x)\}) = \bar{\pi}(\{(x, x)\}) - (\gamma(\mathbb{R}) - \sigma(\mathbb{R}))$  can be checked by considering  $(1 - u) \bar{\pi} + u \tilde{\pi}$  for  $u \in [0, 1]$  as we did with  $\bar{\pi}(u, dy, dz) = (1 - u) \tilde{\pi}(dy, dz) + u \bar{\pi}(dy, dz)$  in **Step 1**. Hence  $\tilde{\pi} \in \Pi_M(\mu, \nu)$ . With a symmetric reasoning, we construct from  $\tilde{\pi}$  a coupling  $\pi \in \Pi_M(\mu, \nu)$  such that  $\pi_x = \eta_x$ .

**Step 3 :** Let us now suppose that  $\mu(\{x\}) \in (0, p_-(x) + p_+(x)]$ , so that, by **Step 2**,

$$\inf_{\pi \in \Pi_M(\mu, \nu)} \pi(\{(x, x)\}) = 0.$$

Let  $(\pi^n)_{n \in \mathbb{N}}$  denote a  $\Pi_M(\mu, \nu)$ -valued sequence such that  $\lim_{n \rightarrow \infty} \pi^n(\{(x, x)\}) = 0$ . Since  $\Pi_M(\mu, \nu)$  is compact for the weak convergence topology, we may extract a subsequence that we still index by  $n$  for notational simplicity and which converges weakly to  $\pi^\infty \in \Pi_M(\mu, \nu)$ . For  $\varepsilon \in (0, F_\nu(x-) \wedge (1 - F_\nu(x)))$ , we have

$$\begin{aligned} \pi^n(\{x\} \times \{(-\infty, F_\nu^{-1}(F_\nu(x-) - \varepsilon)] \cup [F_\nu^{-1}(F_\nu(x) + \varepsilon), +\infty)\}) \\ \geq \pi^n(\{x\} \times \mathbb{R}) - \pi^n(\{(x, x)\}) - \nu((F_\nu^{-1}(F_\nu(x-) - \varepsilon), x) \cup (x, F_\nu^{-1}(F_\nu(x) + \varepsilon))) \\ \geq \mu(\{x\}) - \pi^n(\{(x, x)\}) - 2\varepsilon. \end{aligned}$$

Taking the limit  $n \rightarrow \infty$  in this inequality and using the closedness of  $\{x\} \times \{(-\infty, F_\nu^{-1}(F_\nu(x-) - \varepsilon)] \cup [F_\nu^{-1}(F_\nu(x) + \varepsilon), +\infty)\}$  together with the Portmanteau theorem for the left-hand side, we obtain

$$\pi^\infty(\{x\} \times \{(-\infty, F_\nu^{-1}(F_\nu(x-) - \varepsilon)] \cup [F_\nu^{-1}(F_\nu(x) + \varepsilon), +\infty)\}) \geq \mu(\{x\}) - 2\varepsilon.$$

so that  $\pi^\infty(\{(x, x)\}) \leq 2\varepsilon$ . Letting  $\varepsilon \rightarrow 0+$ , we conclude that  $\pi^\infty(\{(x, x)\}) = 0$ .

The function  $\left[ (\mu(\{x\}) - p_+(x))^+, p_-(x) \wedge \mu(\{x\}) \right] \ni q \mapsto G(q) = \int_{F_\nu(x-) - q}^{F_\nu(x-)} (x - F_\nu^{-1}(v)) dv + \int_{F_\nu(x)}^{F_\nu(x) + \mu(\{x\}) - q} (x - F_\nu^{-1}(v)) dv$  is continuous and increasing. If  $\mu(\{x\}) \geq p_+(x)$  then, using (7), (8) and (34) for the equality and  $-p_-(x) \leq p_+(x) - \mu(\{x\})$  for the inequality, we get

$$G(\mu(\{x\}) - p_+(x)) = \int_{F_\nu(x-) - p_-(x)}^{F_\nu(x-) + p_+(x) - \mu(\{x\})} (F_\nu^{-1}(v) - x) dv \leq 0$$



while otherwise  $G(0) = \int_{F_\nu(x)}^{F_\nu(x)+\mu(\{x\})} (x - F_\nu^{-1}(v)) dv \leq 0$  so that  $G((\mu(\{x\}) - p_+(x))^+) \leq 0$ . Since, in a symmetric way,  $G(p_-(x) \wedge \mu(\{x\})) \geq 0$ , there exists a unique  $q(x)$  in the interval such that  $G(q(x)) = 0$ . If  $\mu(\{x\}) = 1$ , then  $p_-(x) + p_+(x) = 1$  by (35),  $F_\nu(x-) = F_\nu(x) = q(x)$  and  $\eta_x$  given by (10) is such that  $\eta_x = \nu$  so that the unique element of  $\Pi_M(\mu, \nu)$  is  $\delta_x(dy)\eta_x(dz)$ . Let us now suppose that  $\mu(\{x\}) \in (0, 1)$  and check that the probability measure  $\eta_x$  given by (10) is such that  $\eta_x \leq_{cx} \pi_x^\infty$ . Since  $\pi^\infty(\{(x, x)\}) = 0$ , one has  $\pi_x^\infty \leq \frac{1}{\mu(\{x\})} \left( \int_0^{F_\nu(x-)} \delta_{F_\nu^{-1}(v)} dv + \int_{F_\nu(x)}^1 \delta_{F_\nu^{-1}(v)} dv \right)$  so that  $\pi_x^\infty = \frac{1}{\mu(\{x\})} \left( \int_0^{F_\nu(x-)} p(v) \delta_{F_\nu^{-1}(v)} dv + \int_{F_\nu(x)}^1 p(v) \delta_{F_\nu^{-1}(v)} dv \right)$  for some measurable function  $p : [0, 1] \rightarrow [0, 1]$  such that  $\int_0^{F_\nu(x-)} p(v) dv + \int_{F_\nu(x)}^1 p(v) dv = \mu(\{x\})$ . Let  $C = \int_{F_\nu(x-)-q(x)}^{F_\nu(x-)} (1 - p(v)) dv + \int_{F_\nu(x)}^1 (1 - p(v)) dv$ . One has  $C = \int_0^{F_\nu(x-)-q(x)} p(v) dv + \int_{F_\nu(x)+\mu(\{x\})-q(x)}^1 p(v) dv$  and either  $C = 0$  and then  $\pi_x^\infty = \eta_x$  or  $C > 0$ , which we now suppose. Then

$$\theta := \frac{1}{C} \int_{F_\nu(x-)-q(x)}^{F_\nu(x-)} (1 - p(v)) \delta_{F_\nu^{-1}(v)} dv + \int_{F_\nu(x)}^{F_\nu(x)+\mu(\{x\})-q(x)} (1 - p(v)) \delta_{F_\nu^{-1}(v)} dv$$

$$\text{and } \vartheta := \frac{1}{C} \int_0^{F_\nu(x-)-q(x)} p(v) \delta_{F_\nu^{-1}(v)} dv + \int_{F_\nu(x)+\mu(\{x\})-q(x)}^1 p(v) \delta_{F_\nu^{-1}(v)} dv$$

are two probability measures such that  $\vartheta - \theta = \frac{\mu(\{x\})}{C} (\pi_x^\infty - \eta_x)$ . Since  $\pi_x^\infty$  and  $\eta_x$  have common mean, so do  $\vartheta$  and  $\theta$ . Moreover,

$$\theta([F_\nu^{-1}(F_\nu(x-) - q(x)), F_\nu^{-1}(F_\nu(x) + \mu(\{x\}) - q(x))]) = 1$$

$$= \vartheta((-\infty, F_\nu^{-1}(F_\nu(x-) - q(x))) \cup [F_\nu^{-1}(F_\nu(x) + \mu(\{x\}) - q(x)), +\infty)).$$

Either  $\vartheta((-\infty, F_\nu^{-1}(F_\nu(x-) - q(x)))) = 0$  so that  $\theta \leq_{st} \vartheta$  and, with the equality of means,  $\theta = \vartheta$  or  $\vartheta((-\infty, F_\nu^{-1}(F_\nu(x-) - q(x)))) = 1$  so that  $\vartheta \leq_{st} \theta$  and, with the equality of means,  $\theta = \vartheta$  or  $\vartheta((-\infty, F_\nu^{-1}(F_\nu(x-) - q(x)))) \vartheta([F_\nu^{-1}(F_\nu(x) + \mu(\{x\}) - q(x)), +\infty)) > 0$ . In all cases,  $\theta \leq_{cx} \vartheta$  and  $\eta_x \leq_{cx} \pi_x^\infty$ . As a consequence  $\frac{\mu - \mu(\{x\})\delta_x}{1 - \mu(\{x\})} \leq_{cx} \frac{\nu - \mu(\{x\})\pi_x^\infty}{1 - \mu(\{x\})} \leq_{cx} \frac{\nu - \mu(\{x\})\eta_x}{1 - \mu(\{x\})}$ . Let  $\tilde{\pi} \in \Pi_M\left(\frac{\mu - \mu(\{x\})\delta_x}{1 - \mu(\{x\})}, \frac{\nu - \mu(\{x\})\eta_x}{1 - \mu(\{x\})}\right)$ . Then,  $\pi := (1 - \mu(\{x\}))\tilde{\pi}(dy, dz) + \mu(\{x\})\delta_x(dy)\eta_x(dz)$  belongs to  $\Pi_M(\mu, \nu)$  and satisfies  $\pi_x = \eta_x$ .  $\square$

*Proof of Corollary 2.16.* Let us first deal with the bound from above. Since for each  $\pi \in \Pi(\mu, \nu)$ ,  $\mu(dx)\pi_x(\{x\}) \leq \mu \wedge \nu(dx)$ , we have  $\nu_0^\pi(dx) \leq \mu \wedge \nu(dx)$  for each  $\pi \in \Pi_M(\mu, \nu)$ . Moreover, if  $\mu \wedge \nu(\mathbb{R}) = 1$ , then  $\mu = \nu$  and  $\mu(dx)\delta_x(dy)$  is the only element of  $\Pi_M(\mu, \nu)$ . If  $\mu \wedge \nu(\mathbb{R}) < 1$ , then  $u_\mu \leq u_\nu$  implies that  $u \frac{\mu - \mu \wedge \nu}{1 - \mu \wedge \nu(\mathbb{R})} \leq u \frac{\nu - \mu \wedge \nu}{1 - \mu \wedge \nu(\mathbb{R})}$  so that there exists  $\hat{\pi} \in \Pi_M\left(\frac{\mu - \mu \wedge \nu}{1 - \mu \wedge \nu(\mathbb{R})}, \frac{\nu - \mu \wedge \nu}{1 - \mu \wedge \nu(\mathbb{R})}\right)$  and  $\pi(dx, dy) = \mu \wedge \nu(dx)\delta_x(dy) + (1 - \mu \wedge \nu(\mathbb{R}))\hat{\pi}(dx, dy)$  is such that  $\pi \in \Pi_M(\mu, \nu)$  and  $\nu_0^\pi = \mu \wedge \nu$ .

From the decomposition in irreducible components stated in [3, Theorem A.4], we have  $\nu_0^\pi(dy) \geq \mathbb{1}_{\{u_\mu(y) = u_\nu(y)\}} \mu(dy)$  for each  $\pi \in \Pi_M(\mu, \nu)$ . With Proposition 2.14, we deduce the bound from below for  $\nu_0^\pi$ . In Example 2.17, this bound from below is not attained.

Let us now suppose that  $\mu \neq \nu$  and that  $(\nu_l, \nu_r)$  is a couple of non-negative measures such that  $\nu - \nu_l - \nu_r = \nu_0 := \mathbb{1}_{\{u_\mu = u_\nu\}} \mu + \sum_{x \in \mathcal{X}_0} (\mu(\{x\}) - p_-(x) - p_+(x)) \delta_x$ . By Corollary 2.12 (iii), there is at most one non-decreasing coupling in  $\Pi_M(\mu, \nu, \nu_l, \nu_r)$ . We now assume that  $\Pi_M(\mu, \nu, \nu_l, \nu_r) \neq \emptyset$  and check that the coupling  $\pi^\uparrow$  given by Corollary 2.12 (i) is non-decreasing. Let  $\Gamma$  be a set associated, in the sense of Definition 2.3, to  $\tilde{\pi}(dx, dy) = \frac{\pi^\uparrow(dx, dy) - \nu_0(dx)\delta_x(dy)}{1 - \nu_0(\mathbb{R})}$  which is non-decreasing. We denote by  $((a_n, b_n))_{n \in \mathbb{N}}$  the irreducible components for  $(\mu, \nu)$ . Since  $\pi^\uparrow(\bigcup_{n \in \mathbb{N}} (a_n, b_n) \times [a_n, b_n]) = 1 - \mu(\{x \in \mathbb{R} : u_\mu(x) = u_\nu(x)\})$  and

$$\int_{\mathbb{R}^2} \mathbb{1}_{\bigcup_{n \in \mathbb{N}} (a_n, b_n) \times [a_n, b_n]}(x, y) \nu_0(dx)\delta_x(dy) = \sum_{x \in \mathcal{X}_0} (\mu(\{x\}) - p_-(x) - p_+(x)),$$

we have  $\tilde{\pi}(\bigcup_{n \in \mathbb{N}} (a_n, b_n) \times [a_n, b_n]) = 1$  and  $\tilde{\Gamma} = \Gamma \cup \bigcup_{n \in \mathbb{N}} (a_n, b_n) \times [a_n, b_n]$  is such that  $\tilde{\pi}(\tilde{\Gamma}) = 1$ . Setting  $\hat{\Gamma} = \tilde{\Gamma} \cup \{(x, x) : x \in \mathbb{R} \text{ s.t. } u_\mu(x) = u_\nu(x) \text{ or } x \in \mathcal{X}_0\}$ , we have  $\pi^\uparrow(\hat{\Gamma}) = 1$ . For  $x$  in the at most countable set  $\mathcal{X}_0$ , we have  $\pi^\uparrow(\{(-\infty, x) \times (x, +\infty)\} \cup \{(x, +\infty) \times (-\infty, x)\}) = 0$  by

Proposition 2.14, so that  $\pi^\uparrow(\bigcup_{x \in \mathcal{X}_0} \{(-\infty, x) \times (x, +\infty)\} \cup \{(x, +\infty) \times (-\infty, x)\}) = 0$ . Hence  $\Gamma^\uparrow = \hat{\Gamma} \cap \bigcap_{x \in \mathcal{X}_0} \{(-\infty, x] \times (-\infty, x] \cup \{[x, +\infty) \times [x, +\infty)\}\}$  is such that  $\pi^\uparrow(\Gamma^\uparrow) = 1$ .

Let  $(x_-, y_-), (x_+, y_+) \in \Gamma^\uparrow$  with  $y_- \leq x_-, y_+ \leq x_+$ , and  $x_- < x_+$ . Either  $y_+ = x_+$  and then  $y_- < y_+$  or there exists  $n_{x_+} \in N$  such that  $x_+ \in (a_{n_{x_+}}, b_{n_{x_+}})$  and then  $y_+ \in [a_{n_{x_+}}, b_{n_{x_+}})$ . In the latter case, if  $u_\mu(x_-) = u_\nu(x_-)$ , then  $y_- = x_- \leq a_{n_{x_+}} \leq y_+$  and if there exists  $n_{x_-} \in N$  such that  $x_- \in (a_{n_{x_-}}, b_{n_{x_-}})$ , then either  $(x_-, y_-), (x_+, y_+) \in \Gamma$  so that  $y_- \leq y_+$  or  $x_- \in \mathcal{X}_0$  and  $y_- = x_-$  so that, since  $\Gamma^\uparrow \cap \{(x_-, +\infty) \times (-\infty, x_-)\} = \emptyset$ ,  $y_+ \geq x_- = y_-$ . A similar reasoning ensures that if  $(x_-, z_-), (x_+, z_+) \in \Gamma^\uparrow$  with  $x_- \leq z_-, x_+ \leq z_+$ , and  $x_- < x_+$ , then  $z_- \leq z_+$ . So Definition 2.3 is satisfied by  $\pi^\uparrow$  with set  $\Gamma^\uparrow$ .  $\square$

*Proof of Proposition 2.8.* Since, clearly, (iii)  $\Rightarrow$  (i), to prove that (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii), it is enough to check that (i)  $\Rightarrow$  (ii) and (ii)  $\Rightarrow$  (iii), which we do now.

**Proof of (i)  $\Rightarrow$  (ii) :** Let  $\pi$  be a non-increasing coupling in  $\Pi_M(\mu, \nu)$  and  $\Gamma$  be a Borel subset of  $\mathbb{R}^2$  such that  $\pi(\Gamma) = 1$  and conditions (c)(d) in Definition 2.3 hold. Let for  $x \in \mathbb{R}$ ,  $\Gamma_x = \{y : (x, y) \in \Gamma\}$ . We have  $\mu(dx)$  a.e.  $\pi_x(\Gamma_x) = 1$ . With the martingale property, this implies

$$\mu(dx) \text{ a.e.}, \int_{\Gamma_x} y \pi_x(dy) = x \text{ so that } \pi_x(\Gamma_x \cap (-\infty, x]) \wedge \pi_x(\Gamma_x \cap [x, +\infty)) > 0. \quad (43)$$

Therefore  $\{(x, y) \in \Gamma : y \leq x\} \neq \emptyset$  and  $\{(x, z) \in \Gamma : z \geq x\} \neq \emptyset$ . Let

$$a = \inf \{x \in \mathbb{R} : \exists y \leq x, (x, y) \in \Gamma\} \quad \text{and} \quad b = \sup \{x \in \mathbb{R} : \exists z \geq x, (x, z) \in \Gamma\}.$$

Since when  $x < a$ ,  $\Gamma_x \cap (-\infty, x] = \emptyset$  and when  $x > b$ ,  $\Gamma_x \cap [x, +\infty) = \emptyset$ , (43) implies that

$$\mu((-\infty, a) \cup (b, +\infty)) = 0.$$

Let  $(x_0, y_0) \in \Gamma$  be such that  $y_0 \leq x_0$ . By condition (c) in Definition 2.3, for any  $(x, y) \in \Gamma$  with  $y \leq x < x_0$ , we have  $y_0 \leq y$  so that  $y_0 \leq x$ . Hence  $y_0 \leq a$ ,  $-\infty < a$  and  $\Gamma \cap \{(x, y) : y \leq x\} \subset [a, +\infty) \times (-\infty, a]$ . In a symmetric way, condition (d) in Definition 2.3 implies that  $b < +\infty$  and  $\Gamma \cap \{(x, z) : z \geq x\} \subset (-\infty, b] \times [b, +\infty)$ . Therefore  $\Gamma \subset \mathbb{R} \times \{(-\infty, a] \cup [b, +\infty)\}$  and

$$\nu((-\infty, a] \cup [b, +\infty)) = \pi(\mathbb{R} \times \{(-\infty, a] \cup [b, +\infty)\}) \geq \pi(\Gamma) = 1.$$

**Proof of (ii)  $\Rightarrow$  (iii) :** By Proposition 2.5, there exists a coupling  $\pi^* \in \Pi_M(\mu, \nu)$  minimizing  $\int_{\mathbb{R}^2} |x - y| \pi(dx, dy)$  over  $\pi \in \Pi_M(\mu, \nu)$  and this coupling is non-increasing. It remains to check that  $\pi^*$  is the only non-increasing coupling in  $\Pi_M(\mu, \nu)$  and that  $\pi^*(\{(a, a)\}) = \mu(\{a\}) \wedge \nu(\{a\})$  and  $\pi^*(\{(b, b)\}) = \mu(\{b\}) \wedge \nu(\{b\})$ . This is clear when  $a = b$  since then  $\delta_a(dx)\nu(dy)$  is the only element of  $\Pi(\mu, \nu)$ . We thus suppose that  $a < b$ . Let  $\pi^\downarrow \in \Pi_M(\mu, \nu)$  be non-increasing,  $x \in \mathbb{R}$  and  $\psi_{\pi^\downarrow}(x) = \pi^\downarrow((-\infty, x] \times (-\infty, a])$ . Since  $\psi_{\pi^\downarrow}(x) \leq \pi^\downarrow(\mathbb{R} \times (-\infty, a]) = F_\nu(a)$  and

$$\begin{aligned} 0 \leq F_\mu(x) - \psi_{\pi^\downarrow}(x) &= \pi^\downarrow((-\infty, x] \times \mathbb{R}) - \pi^\downarrow((-\infty, x] \times (-\infty, a]) \\ &= \pi^\downarrow((-\infty, x] \times [b, +\infty)) \leq \nu([b, +\infty)) = 1 - F_\nu(a), \end{aligned}$$

we have  $\psi_{\pi^\downarrow}(x) \in [(F_\nu(a) + F_\mu(x) - 1)^+, F_\nu(a) \wedge F_\mu(x)]$ . Using properties (c)(d) in Definition 2.3, we check like in the derivation of (25) and (26) in the uniqueness part of the proof of Theorem 2.11 that

$$\begin{aligned} \int_{y \in (-\infty, x]} \mathbb{1}_{\{z \leq a\}} \pi^\downarrow(dy, dz) &= \int_{F_\nu(a) - \psi_{\pi^\downarrow}(x)}^{F_\nu(a)} \delta_{F_\nu^{-1}(u)}(dz) du \\ \int_{y \in (-\infty, x]} \mathbb{1}_{\{z \geq b\}} \pi^\downarrow(dy, dz) &= \int_{1 + \psi_{\pi^\downarrow}(x) - F_\mu(x)}^1 \delta_{F_\nu^{-1}(u)}(dz) du. \end{aligned}$$

With the martingale property, this ensures that

$$\begin{aligned} \int_{(-\infty, x]} y \mu(dy) &= \int_{(-\infty, x]} \int_{\mathbb{R}} z \pi^\downarrow_y(dz) \mu(dy) \\ &= \int_{(-\infty, x] \times \mathbb{R}} \mathbb{1}_{\{z \leq a\}} z \pi^\downarrow(dz, dy) + \int_{(-\infty, x] \times \mathbb{R}} \mathbb{1}_{\{z \geq b\}} z \pi^\downarrow(dz, dy) \\ &= \int_{F_\nu(a) - \psi_{\pi^\downarrow}(x)}^{F_\nu(a)} F_\nu^{-1}(v) dv + \int_{1 + \psi_{\pi^\downarrow}(x) - F_\mu(x)}^1 F_\nu^{-1}(v) dv = G(\psi_{\pi^\downarrow}(x)), \end{aligned}$$

where  $[(F_\nu(a)+F_\mu(x)-1)^+, F_\nu(a)\wedge F_\mu(x)] \ni u \mapsto G_x(u) = \int_{F_\nu(a)-u}^{F_\nu(a)} F_\nu^{-1}(v)dv + \int_{1+u-F_\mu(x)}^1 F_\nu^{-1}(v)dv$ . The three last equalities also hold with  $\pi^\downarrow$  and  $\psi_{\pi^\downarrow}(x)$  replaced by  $\pi^*$  (the coupling minimizing  $\int_{\mathbb{R}^2} |x-y|\pi(dx, dy)$  over  $\pi \in \Pi_M(\mu, \nu)$ ) and  $\psi_{\pi^*}(x) = \pi^*((-\infty, x] \times (-\infty, a])$ . For  $(F_\nu(a)+F_\mu(x)-1)^+ \leq u' \leq u \leq F_\nu(a) \wedge F_\mu(x)$ , we have

$$G_x(u) - G_x(u') = \int_{F_\nu(a)-u}^{F_\nu(a)-u'} F_\nu^{-1}(v) dv - \int_{1+u'-F_\mu(x)}^{1+u-F_\mu(x)} F_\nu^{-1}(v) dv \leq (u-u')a - (u-u')b.$$

Therefore  $G_x$  is a decreasing function and the equality  $G_x(\psi_{\pi^\downarrow}(x)) = \int_{(-\infty, x]} y\mu(dy) = G_x(\psi_{\pi^*}(x))$  implies that  $\psi_{\pi^\downarrow}(x) = \psi_{\pi^*}(x)$ . Hence

$$\begin{aligned} \forall x \in \mathbb{R}, \int_{y \in (-\infty, x]} \mathbb{1}_{\{z \leq a\}} \pi^\downarrow(dy, dz) &= \int_{y \in (-\infty, x]} \mathbb{1}_{\{z \leq a\}} \pi^*(dy, dz) \\ \int_{y \in (-\infty, x]} \mathbb{1}_{\{z \geq b\}} \pi^\downarrow(dy, dz) &= \int_{y \in (-\infty, x]} \mathbb{1}_{\{z \geq b\}} \pi^*(dy, dz) \\ \text{and } \int_{y \in (-\infty, x]} \pi^\downarrow_y \mu(dy) &= \int_{y \in (-\infty, x]} \pi_y^* \mu(dy). \end{aligned}$$

We conclude that  $\pi^\downarrow = \pi^*$  like in the end of the uniqueness part of the proof of Theorem 2.11. Finally, let us check that  $\pi^*({(a, a)}) = \mu({a}) \wedge \nu({a})$ , the equality  $\pi^*({(b, b)}) = \mu({b}) \wedge \nu({b})$  being derived in a symmetric way. We have  $F_\mu(a) = \mu({a})$ . When  $\mu({a}) \leq \nu({a})$ , then  $G_a(\mu({a})) = a\mu({a})$  so that  $\psi_{\pi^*}(a) = \mu({a})$ . When  $\mu({a}) > \nu({a})$ , then  $G_a(\nu({a})) \geq a\nu({a}) + b(\mu({a}) - \nu({a})) > a\mu({a})$  so that  $\psi_{\pi^*}(a) > \nu({a})$ . Since  $\mu({a})\mathbb{1}_{\{z \leq a\}}\pi_a^*(dz) = \int_{F_\nu(a)-\psi_{\pi^*}(a)}^{F_\nu(a)} \delta_{F_\nu^{-1}(u)}(dz)du$ , we deduce that  $\pi^*({(a, a)}) = \mu({a}) \wedge \nu({a})$ .  $\square$

### 3.4 Proofs of the results in Section 2.5 concerning optimality of $\pi^\uparrow$ and $\pi^\downarrow$ for the cost function $|x-y|^\rho$ with $\rho > 1$ under the nested supports condition

*Proof of Proposition 2.19.* Since, by Proposition 2.8 applied with  $a = \underline{x}$  and  $b = \bar{x}$ , there exists a unique non-decreasing (resp. non-increasing) coupling in  $\Pi_M(\mu, \nu)$ , it is enough to check that any coupling  $\pi \in \Pi_M(\mu, \nu)$  that attains  $\overline{\mathcal{M}}_\rho(\mu, \nu)$  (resp.  $\underline{\mathcal{M}}_\rho(\mu, \nu)$ ) when  $\rho \in (1, 2)$  or  $\underline{\mathcal{M}}_\rho(\mu, \nu)$  (resp.  $\overline{\mathcal{M}}_\rho(\mu, \nu)$ ) when  $\rho > 2$  is non-decreasing (resp. non-increasing). Let  $\Gamma$  be a set associated with  $\pi$  by Lemma 3.2. Since  $\pi([\underline{x}, \bar{x}] \times \{[\underline{y}, \bar{y}] \cup [\underline{z}, \bar{z}]\}) = 1$ , we suppose without restriction that  $\Gamma \subset [\underline{x}, \bar{x}] \times \{[\underline{y}, \bar{y}] \cup [\underline{z}, \bar{z}]\}$ . Let us suppose the existence of  $(x_-, y_-), (x_+, y_+) \in \Gamma$  (resp.  $(x_-, y_+), (x_+, y_-) \in \Gamma$ ) such that  $x_- < x_+, y_- \leq x_-, y_+ \leq x_+, y_+ < y_-$  and obtain a contradiction. By (12), there exists  $z > x_+$  such that  $(x_+, z) \in \Gamma$  (resp.  $z > x_-$  such that  $(x_-, z) \in \Gamma$ ). We have

$$\underline{y} \leq y_+ < y_- \leq \bar{y} < \underline{x} \leq x_- < x_+ \leq \bar{x} < \underline{z} \leq z \leq \bar{z}.$$

Let us define the function  $f_\rho : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f_\rho(x) = \frac{z-y_-}{z-y_+} |x-y_+|^\rho + \frac{y_- - y_+}{z-y_+} |z-x|^\rho - |x-y_-|^\rho.$$

Since  $(1-\alpha_\rho)y_- + \alpha_\rho z \leq (1-\alpha_\rho)\bar{y} + \alpha_\rho \bar{z} \leq \underline{x}$ , by Lemma 3.9 just below applied with  $(y, m, z) = (y_+, y_-, z)$ ,  $f_\rho$  is decreasing on  $[\underline{x}, \bar{x}]$  when  $\rho \in (1, 2)$  and increasing on  $[\underline{x}, \bar{x}]$  when  $\rho > 2$ . We deduce the desired contradiction like in the proof of Proposition 2.5.

In a symmetric way, the existence of  $(x_-, z_-), (x_+, z_+) \in \Gamma$  (resp.  $(x_-, z_+), (x_+, z_-) \in \Gamma$ ) such that  $x_- < x_+, x_- \leq z_-, x_+ \leq z_+, z_+ < z_-$  implies the existence of  $y < x_-$  such that  $(x_-, y) \in \Gamma$  (resp.  $y < x_+$  such that  $(x_+, y) \in \Gamma$ ) and

$$\underline{y} \leq y \leq \bar{y} < \underline{x} \leq x_- < x_+ \leq \bar{x} < \underline{z} \leq z_+ < z_- \leq \bar{z}.$$

Since  $(1-\alpha_\rho)z_+ + \alpha_\rho y \geq (1-\alpha_\rho)\underline{z} + \alpha_\rho \underline{y} \geq \bar{x}$ , by Lemma 3.9 just below applied with  $(y, m, z) = (y, z_+, z_-)$ , the function  $\mathbb{R} \ni x \mapsto \frac{z_- - z_+}{z_- - y} |x-y|^\rho + \frac{z_+ - y}{z_- - y} |z_- - x|^\rho - |z_+ - x|^\rho$  is increasing on  $[\underline{x}, \bar{x}]$  when  $\rho \in (1, 2)$  and decreasing on  $[\underline{x}, \bar{x}]$  when  $\rho > 2$ , yielding again a contradiction.  $\square$

**Lemma 3.9.** Let  $y < m < z$  and  $f_\rho : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f_\rho(x) = \frac{z-m}{z-y}|x-y|^\rho + \frac{m-y}{z-y}|z-x|^\rho - |x-m|^\rho.$$

When  $\rho \in (1, 2)$  (resp.  $\rho > 2$ ),  $f_\rho$  is increasing (resp. decreasing) on  $[y, (1 - \alpha_\rho)m + \alpha_\rho y]$  and decreasing (resp. increasing) on  $[(1 - \alpha_\rho)m + \alpha_\rho z, z]$ , where  $\alpha_\rho \in (0, \frac{1}{2})$  is defined in Lemma 3.10 just below.

On the other hand, for any  $\rho \in (1, 2) \cup (2, +\infty)$  and any  $\alpha \in (0, \alpha_\rho)$ , we may find  $y < m < z$  such that  $(2 - \rho)f'_\rho((1 - \alpha)m + \alpha y) < 0$  and  $y < m < z$  such that  $(2 - \rho)f'_\rho((1 - \alpha)m + \alpha z) > 0$ .

**Lemma 3.10.** For any  $\rho > 1$ , let  $\psi_\rho : (0, 1] \rightarrow \mathbb{R}$  be the function defined by

$$\psi_\rho(\alpha) = \alpha + \alpha^{2-\rho}(1 - \alpha)^{\rho-1} + 1 - \rho, \quad (44)$$

the following statements hold:

- for all  $\rho \in (1, 2) \cup (2, +\infty)$ , there exists a unique  $\alpha_\rho \in (0, 1)$  such that  $\psi_\rho(\alpha_\rho) = 0$ ,
- if  $\alpha \in (\alpha_\rho, 1]$ , then  $\psi_\rho(\alpha) > 0$  when  $\rho \in (1, 2)$  and  $\psi_\rho(\alpha) < 0$  when  $\rho > 2$ ,
- if  $\alpha \in (0, \alpha_\rho)$ , then  $\psi_\rho(\alpha) < 0$  when  $\rho \in (1, 2)$  and  $\psi_\rho(\alpha) > 0$  when  $\rho > 2$ ,
- $\rho \mapsto \alpha_\rho$  (represented by Figure 2 below) is continuously differentiable on  $(1, 2) \cup (2, \infty)$  and can be extended by continuity at  $\rho = 2$  with the value  $\alpha_2$  which is the unique solution to  $h(\alpha) = 0$  where  $(0, 1) \ni \alpha \mapsto h(\alpha) = 1 + (1 - \alpha) \ln \frac{\alpha}{1-\alpha}$ ,
- the extended function  $\rho \in (1, \infty) \mapsto \alpha_\rho \in (0, 1]$  is increasing,
- $\alpha_\rho < \frac{\rho-1}{2}$  when  $\rho \in (1, 2)$ ,
- $\lim_{\rho \rightarrow \infty} \alpha_\rho = \frac{1}{2}$ .

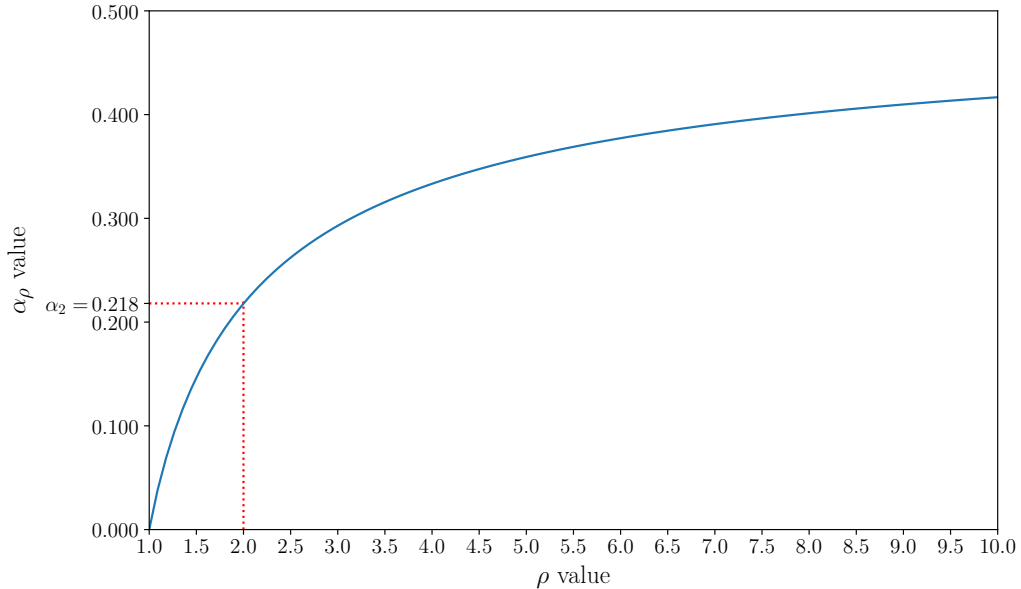


Figure 2: The function  $\rho \mapsto \alpha_\rho$  (with  $\alpha_\rho$  computed by a root finding algorithm).

The proof of Lemma 3.10 is postponed after the one of Lemma 3.9 which relies on the next lemma.

**Lemma 3.11.** For  $\rho \in (1, +\infty) \setminus \{2\}$ , let us define the function  $\varphi_\rho : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}$  by

$$\varphi_\rho(\alpha, w) = (\alpha w)^\rho - \frac{w}{1+w} (1 + \alpha w)^\rho - \frac{w^\rho}{1+w} (1 - \alpha)^\rho.$$

Then, we have

$$\begin{aligned} \forall \alpha \in [\alpha_\rho, 1], \forall w > 0, \quad (2 - \rho) \partial_\alpha \varphi_\rho(\alpha, w) > 0, \\ \forall \alpha \in [0, \alpha_\rho), \text{ for } w \text{ large enough,} \quad (2 - \rho) \partial_\alpha \varphi_\rho(\alpha, w) < 0. \end{aligned}$$

where  $\alpha_\rho$  is defined in Lemma 3.10.

The proof of Lemma 3.11 is postponed after the one of Lemma 3.9.

*Proof of Lemma 3.9.*

When  $x \in [y, m]$ , we have

$$\begin{aligned} f_\rho(x) &= \frac{z-m}{z-y} (x-y)^\rho + \frac{m-y}{z-y} (z-x)^\rho - (m-x)^\rho \\ &= (z-m)^\rho \left[ \frac{z-m}{z-y} \times \left( \frac{m-y}{z-m} \right)^\rho \times \left( 1 - \frac{m-x}{m-y} \right)^\rho + \frac{m-y}{z-y} \times \left( 1 + \frac{m-x}{z-m} \right)^\rho - \left( \frac{m-x}{z-m} \right)^\rho \right] \\ &= -(z-m)^\rho \varphi_\rho \left( \frac{m-x}{m-y}, \frac{m-y}{z-m} \right), \end{aligned}$$

By Lemma 3.11, since  $x \in [y, (1 - \alpha_\rho)m + \alpha_\rho y] \Leftrightarrow \frac{m-x}{m-y} \in [\alpha_\rho, 1]$ , we deduce that for  $\rho \in (1, 2)$  (resp.  $\rho > 2$ ),  $f_\rho$  is increasing (resp. decreasing) on  $[y, (1 - \alpha_\rho)m + \alpha_\rho y]$ . Moreover, for any  $\alpha \in (0, \alpha_\rho)$ , when  $\frac{m-y}{z-m}$  is large enough,  $(2 - \rho) f'_\rho((1 - \alpha)m + \alpha y) < 0$ .

When  $x \in [m, z]$ , we have

$$\begin{aligned} f_\rho(x) &= \frac{z-m}{z-y} (x-y)^\rho + \frac{m-y}{z-y} (z-x)^\rho - (x-m)^\rho \\ &= (m-y)^\rho \left[ \frac{z-m}{z-y} \times \left( 1 + \frac{x-m}{m-y} \right)^\rho + \frac{m-y}{z-y} \times \left( \frac{z-m}{m-y} \right)^\rho \times \left( 1 - \frac{x-m}{z-m} \right)^\rho - \left( \frac{x-m}{m-y} \right)^\rho \right]. \\ &= -(m-y)^\rho \varphi_\rho \left( \frac{x-m}{z-m}, \frac{z-m}{m-y} \right). \end{aligned}$$

By Lemma 3.11 again, since  $x \in [(1 - \alpha_\rho)m + \alpha_\rho z, z] \Leftrightarrow \frac{x-m}{z-m} \in [\alpha_\rho, 1]$ , we deduce that for  $\rho \in (1, 2)$  (resp.  $\rho > 2$ ),  $f_\rho$  is decreasing (resp. increasing) on  $[(1 - \alpha_\rho)m + \alpha_\rho z, z]$ . Moreover, for any  $\alpha \in (0, \alpha_\rho)$ , when  $\frac{z-m}{m-y}$  is large enough,  $(2 - \rho) f'_\rho((1 - \alpha)m + \alpha z) > 0$ . □

*Proof of Lemma 3.11.* Taking the derivative of  $\varphi_\rho(\alpha, w)$  with respect to  $\alpha$ , we have

$$\frac{1}{\rho} \partial_\alpha \varphi_\rho(\alpha, w) = \alpha^{\rho-1} w^\rho - \frac{w^2}{1+w} (1 + \alpha w)^{\rho-1} + \frac{w^\rho}{1+w} (1 - \alpha)^{\rho-1}$$

so that  $\frac{1}{\rho} \partial_\alpha \varphi_\rho(0, w) = \frac{w^{\rho-2}}{1+w}$  is negative (resp. positive) when  $w > 1$  and  $\rho \in (1, 2)$  (resp.  $\rho > 2$ ). Let us now suppose that  $\alpha \in (0, 1)$ . Then

$$\begin{aligned} \frac{1}{\rho} \partial_\alpha \varphi_\rho(\alpha, w) &= \frac{\alpha^{\rho-2} w^\rho}{1+w} (\alpha + \alpha^{2-\rho} (1 - \alpha)^{\rho-1} - (\rho - 1)) \\ &\quad + \frac{w^2}{1+w} \left( (\rho - 1) (\alpha w)^{\rho-2} + (\alpha w)^{\rho-1} - (1 + \alpha w)^{\rho-1} \right) \\ &= \frac{\alpha^{\rho-2} w^\rho}{1+w} \psi_\rho(\alpha) + \frac{w^2 (\rho - 1)}{1+w} \int_{\alpha w}^{1+\alpha w} ((\alpha w)^{\rho-2} - v^{\rho-2}) dv \\ &= \frac{\alpha^{\rho-2} w^\rho}{1+w} \left( \psi_\rho(\alpha) + (\rho - 1) \int_{\alpha w}^{1+\alpha w} \left( 1 - \left( \frac{\alpha w}{v} \right)^{2-\rho} \right) dv \right). \end{aligned}$$

Since

$$\begin{aligned} 0 &< \int_{\alpha w}^{1+\alpha w} \left(1 - \left(\frac{\alpha w}{v}\right)^{2-\rho}\right) dv \leq 1 - \left(\frac{\alpha w}{1+\alpha w}\right)^{2-\rho} && \text{when } \rho \in (1, 2), \\ 0 &> \int_{\alpha w}^{1+\alpha w} \left(1 - \left(\frac{\alpha w}{v}\right)^{2-\rho}\right) dv \geq 1 - \left(\frac{1+\alpha w}{\alpha w}\right)^{\rho-2} && \text{when } \rho > 2, \end{aligned}$$

by the first and second items in Lemma 3.10, we deduce that for  $\alpha \in [\alpha_\rho, 1]$ , we have  $\partial_\alpha \varphi_\rho(\alpha, w) > 0$  when  $\rho \in (1, 2)$  and  $\partial_\alpha \varphi_\rho(\alpha, w) < 0$  when  $\rho > 2$ .

For  $\alpha \in (0, \alpha_\rho)$ , since  $\lim_{w \rightarrow \infty} \int_{\alpha w}^{1+\alpha w} \left(1 - \left(\frac{\alpha w}{v}\right)^{2-\rho}\right) dv = 0$ , by the third item in Lemma 3.10, we may find  $w$  large enough so that  $\partial_\alpha \varphi_\rho(\alpha, w) < 0$  when  $\rho \in (1, 2)$  and  $\partial_\alpha \varphi_\rho(\alpha, w) > 0$  when  $\rho > 2$ .  $\square$

*Proof of Lemma 3.10.*

Taking the derivative of  $\psi_\rho$ , we get

$$\psi'_\rho(\alpha) = 1 + (2 - \rho) \left(\frac{1 - \alpha}{\alpha}\right)^{\rho-1} - (\rho - 1) \left(\frac{\alpha}{1 - \alpha}\right)^{2-\rho}.$$

It appears that

- when  $1 < \rho < 2$ ,
  - (1)  $\psi'_\rho$  is a decreasing function, so  $\psi_\rho$  is strictly concave,
  - (2)  $\lim_{\alpha \rightarrow 0^+} \psi_\rho(\alpha) = 1 - \rho < 0$  and  $\psi_\rho(1) = 2 - \rho > 0$ .
- when  $\rho > 2$ ,
  - (1)  $\psi'_\rho$  is an increasing function, and  $\psi_\rho$  is strictly convex,
  - (2)  $\lim_{\alpha \rightarrow 0^+} \psi_\rho(\alpha) = +\infty$  and  $\psi_\rho(1) = 2 - \rho < 0$ .

With the continuity of  $\psi_\rho$ , we deduce that there exists a unique  $\alpha_\rho \in (0, 1)$  such that  $\psi_\rho(\alpha_\rho) = 0$ . Moreover, for  $\alpha \in (\alpha_\rho, 1]$ ,  $\psi_\rho(\alpha) > 0$  when  $\rho \in (1, 2)$  and  $\psi_\rho(\alpha) < 0$  when  $\rho > 2$ . And for  $\alpha \in (0, \alpha_\rho)$ ,  $\psi_\rho(\alpha) < 0$  when  $\rho \in (1, 2)$  and  $\psi_\rho(\alpha) > 0$  when  $\rho > 2$ .

Since  $\psi_\rho(\alpha_\rho) = 0$ , we have  $\left(\frac{1-\alpha_\rho}{\alpha_\rho}\right)^{\rho-2} = \frac{\rho-1-\alpha_\rho}{1-\alpha_\rho}$ ,

$$\psi'_\rho(\alpha_\rho) = 1 + (2 - \rho) \frac{\rho - 1 - \alpha_\rho}{\alpha_\rho} - (\rho - 1) \frac{\rho - 1 - \alpha_\rho}{1 - \alpha_\rho} = \frac{(\rho - 1)(2 - \rho)}{\alpha_\rho(1 - \alpha_\rho)},$$

so that  $\psi'_\rho(\alpha_\rho) > 0$  for  $\rho \in (1, 2)$  and  $\psi'_\rho(\alpha_\rho) < 0$  for  $\rho > 2$ . Furthermore, by the implicit function theorem, as the mapping  $(\rho, \alpha) \in \{(1, 2) \cup (2, \infty)\} \times (0, 1] \mapsto \psi_\rho(\alpha) \in \mathbb{R}$  is  $C^1$ , it follows that the function  $\rho \mapsto \alpha_\rho$  is also  $C^1$  on  $(1, 2) \cup (2, \infty)$  and its derivative writes

$$\partial_\rho \alpha_\rho = \frac{1 - \alpha_\rho^{2-\rho}(1 - \alpha_\rho)^{\rho-1} \ln \frac{1-\alpha_\rho}{\alpha_\rho}}{\psi'_\rho(\alpha_\rho)}.$$

Since  $\psi_\rho(\alpha_\rho) = 0$ , we have

$$(\rho - 2) \ln \frac{1 - \alpha_\rho}{\alpha_\rho} = \ln \frac{\rho - 1 - \alpha_\rho}{1 - \alpha_\rho} = (\rho - 2) \int_0^1 \frac{du}{1 - \alpha_\rho + (\rho - 2)u}.$$

Then, since  $\alpha_\rho^{2-\rho}(1 - \alpha_\rho)^{\rho-1} = \rho - 1 - \alpha_\rho$ , the numerator of the formula of  $\partial_\rho \alpha_\rho$  can be written as

$$1 - \alpha_\rho^{2-\rho}(1 - \alpha_\rho)^{\rho-1} \ln \frac{1 - \alpha_\rho}{\alpha_\rho} = 1 - \int_0^1 \frac{\rho - 1 - \alpha_\rho}{1 - \alpha_\rho + (\rho - 2)u} du.$$

When  $\rho \in (1, 2)$ ,

$$\psi_\rho \left( \frac{\rho-1}{2} \right) = \frac{\rho-1}{2} + \frac{\rho-1}{2} \left( \frac{3-\rho}{\rho-1} \right)^{\rho-1} + 1 - \rho = \frac{\rho-1}{2} \left( \left( \frac{3-\rho}{\rho-1} \right)^{\rho-1} - 1 \right) > 0,$$

so that  $\alpha_\rho < \frac{\rho-1}{2}$ , and

$$\int_0^1 \frac{\rho-1-\alpha_\rho}{1-\alpha_\rho+(\rho-2)u} du < \int_0^1 \frac{\rho-1-\alpha_\rho}{1-\alpha_\rho+(\rho-2)} du = 1.$$

When  $\rho > 2$ ,

$$\int_0^1 \frac{\rho-1-\alpha_\rho}{1-\alpha_\rho+(\rho-2)u} du > \int_0^1 \frac{\rho-1-\alpha_\rho}{1-\alpha_\rho+(\rho-2)} du = 1.$$

Hence, we can deduce that

- $1 - \alpha_\rho^{2-\rho}(1 - \alpha_\rho)^{\rho-1} \ln \frac{1-\alpha_\rho}{\alpha_\rho} > 0$  when  $\rho \in (1, 2)$ ,
- $1 - \alpha_\rho^{2-\rho}(1 - \alpha_\rho)^{\rho-1} \ln \frac{1-\alpha_\rho}{\alpha_\rho} < 0$  when  $\rho > 2$ .

We conclude that  $\partial_\rho \alpha_\rho > 0$  on  $(1, 2) \cup (2, +\infty)$ .

Since  $h'(\alpha) = \frac{1}{\alpha} - \ln \frac{\alpha}{1-\alpha}$  is a decreasing function,  $h$  is strictly concave. We have  $\lim_{\alpha \rightarrow 0^+} h(\alpha) = -\infty$  and  $\lim_{\alpha \rightarrow 1^-} h(\alpha) = 1$ . With the continuity of  $h$  and  $h(\frac{1}{2}) = 1$ , we deduce that there exists a unique  $\alpha_2 \in (0, \frac{1}{2})$  such that  $h(\alpha_2) = 0$ . Furthermore, by the fact that  $e^x \geq 1 + x$  with strict inequality when  $x \neq 0$ , we have

$$\begin{aligned} \psi_\rho(\alpha) &= \alpha + (1-\alpha)e^{(2-\rho)\ln \frac{\alpha}{1-\alpha}} + 1 - \rho \\ &\geq \alpha + (1-\alpha) \left( 1 + (2-\rho) \ln \frac{\alpha}{1-\alpha} \right) + 1 - \rho \\ &= (2-\rho) \left( 1 + (1-\alpha) \ln \frac{\alpha}{1-\alpha} \right) = (2-\rho)h(\alpha), \end{aligned}$$

with strict inequality when  $\ln \frac{\alpha}{1-\alpha} \neq 0$ . Hence  $\psi_\rho(\alpha_2) > (2-\rho)h(\alpha_2) = 0$ . We deduce that

- for  $\rho \in (1, 2)$ , we have  $\alpha_\rho < \alpha_2$  and  $h(\alpha_\rho) < 0$ ,
- for  $\rho \in (2, +\infty)$ , we have  $\alpha_\rho > \alpha_2$  and  $h(\alpha_\rho) > 0$ .

On the other hand, applying Taylor's theorem to the term  $e^{(2-\rho)\ln \frac{\alpha}{1-\alpha}}$ , we have

$$\psi_\rho(\alpha) = \alpha + (1-\alpha) \left( 1 + (2-\rho) \ln \frac{\alpha}{1-\alpha} \right) + 1 - \rho + \mathcal{O}((2-\rho)^2) = (2-\rho)(h(\alpha) + \mathcal{O}(2-\rho)).$$

For  $\alpha \in (0, \alpha_2)$ , we have  $h(\alpha) < 0$  and therefore for  $\rho < 2$  close enough to 2,  $\psi_\rho(\alpha) < 0$ , which implies that  $\alpha < \alpha_\rho < \alpha_2$ . Hence  $\lim_{\rho \rightarrow 2^-} \alpha_\rho = \alpha_2$ . In the same way, we can prove that  $\lim_{\rho \rightarrow 2^+} \alpha_\rho = \alpha_2$ .

We have  $\psi_\rho(\frac{1}{2}) = 2 - \rho$ , so that, when  $\rho > 2$ ,  $\psi_\rho(\frac{1}{2}) < 0$  and  $\alpha_\rho < \frac{1}{2}$ . On the other hand, for  $\alpha < \frac{1}{2}$ ,  $\frac{1-\alpha}{\alpha} > 1$  and  $\lim_{\rho \rightarrow +\infty} \psi_\rho(\alpha) = +\infty$  so that for large values of  $\rho$ , we have  $\psi_\rho(\alpha) > 0$  and  $\alpha < \alpha_\rho$ . We deduce that

$$\lim_{\rho \rightarrow +\infty} \alpha_\rho = \frac{1}{2}.$$

□

*Proof of Proposition 2.21.* For  $\pi \in \Pi_M(\mu, \nu)$ , we have

$$\begin{aligned} \pi_{x_-}(\{y_+\}) &\geq 0 = \pi_{x_-}^\uparrow(\{y_+\}), & \pi_{x_-}(\{z_+\}) &\geq 0 = \pi_{x_-}^\uparrow(\{z_+\}), \\ \pi_{x_-}^\uparrow(\{y_-\}) &= \frac{\nu(\{y_-\})}{\mu(\{x_-\})} \geq \pi_{x_-}(\{y_-\}), & \pi_{x_-}^\uparrow(\{z_-\}) &= \frac{\nu(\{z_-\})}{\mu(\{x_-\})} \geq \pi_{x_-}(\{z_-\}). \end{aligned}$$

Now suppose that  $\pi \neq \pi^\uparrow$  and define

$$\hat{\tau} = \frac{\pi_{x_-}(\{y_+\})\delta_{y_+} + \pi_{x_-}(\{z_+\})\delta_{z_+}}{\pi_{x_-}(\{y_+\}) + \pi_{x_-}(\{z_+\})},$$

$$\check{\tau} = \frac{\left(\pi_{x_-}^\uparrow(\{y_-\}) - \pi_{x_-}(\{y_-\})\right)\delta_{y_-} + \left(\pi_{x_-}^\uparrow(\{z_-\}) - \pi_{x_-}(\{z_-\})\right)\delta_{z_-}}{\pi_{x_-}^\uparrow(\{y_-\}) - \pi_{x_-}(\{y_-\}) + \pi_{x_-}^\uparrow(\{z_-\}) - \pi_{x_-}(\{z_-\})}.$$

Since

$$\pi_{x_-}(\{y_-\}) + \pi_{x_-}(\{y_+\}) + \pi_{x_-}(\{z_-\}) + \pi_{x_-}(\{z_+\}) = 1 = \pi_{x_-}^\uparrow(\{y_-\}) + \pi_{x_-}^\uparrow(\{z_-\}),$$

the denominators of the formulas defining  $\hat{\tau}$  and  $\check{\tau}$  are equal. Let us denote the common value by  $s$ . Clearly  $s > 0$  and  $\pi_{x_-} - \pi_{x_-}^\uparrow = s(\hat{\tau} - \check{\tau})$ . Moreover, since  $p\pi_{x_-} + (1-p)\pi_{x_+} = \nu = p\pi_{x_-}^\uparrow + (1-p)\pi_{x_+}^\uparrow$ ,  $\pi_{x_+} - \pi_{x_+}^\uparrow = \frac{ps}{1-p}(\check{\tau} - \hat{\tau})$ .

Since

$$\int y \pi_{x_-}(dy) = x_- = \int y \pi_{x_-}^\uparrow(dy),$$

$\hat{\tau}$  and  $\check{\tau}$  have a common expectation  $m$  and  $m \in [y_+, z_+] \cap [y_-, z_-] = [y_+, z_-]$  and we have

$$\hat{\tau} = \frac{z_+ - m}{z_+ - y_+} \delta_{y_+} + \frac{m - y_+}{z_+ - y_+} \delta_{z_+}, \quad \check{\tau} = \frac{z_- - m}{z_- - y_-} \delta_{y_-} + \frac{m - y_-}{z_- - y_-} \delta_{z_-}. \quad (45)$$

The difference between  $\pi$  and  $\pi^\uparrow$  can be expressed as  $\pi - \pi^\uparrow = ps \delta_{x_-} \otimes (\hat{\tau} - \check{\tau}) + ps \delta_{x_+} \otimes (\check{\tau} - \hat{\tau})$ . Let us denote

$$\underline{\gamma}_m = \frac{1}{2} \left( \frac{z_- - m}{z_- - y_-} \delta_{(x_- - y_-)^2} + \frac{m - y_-}{z_- - y_-} \delta_{(z_- - x_-)^2} + \frac{z_+ - m}{z_+ - y_+} \delta_{(x_+ - y_+)^2} + \frac{m - y_+}{z_+ - y_+} \delta_{(z_+ - x_+)^2} \right),$$

$$\bar{\gamma}_m = \frac{1}{2} \left( \frac{z_+ - m}{z_+ - y_+} \delta_{(x_- - y_+)^2} + \frac{m - y_+}{z_+ - y_+} \delta_{(z_+ - x_-)^2} + \frac{z_- - m}{z_- - y_-} \delta_{(x_+ - y_-)^2} + \frac{m - y_-}{z_- - y_-} \delta_{(z_- - x_+)^2} \right).$$

We have

$$\text{sq}\#\pi - \text{sq}\#\pi^\uparrow = 2ps(\bar{\gamma}_m - \underline{\gamma}_m). \quad (46)$$

Let us check that for any  $m \in [y_+, z_-]$ ,  $\underline{\gamma}_m \leq_{cx} \bar{\gamma}_m$ . Since  $\underline{\gamma}_m$  and  $\bar{\gamma}_m$  linearly depend on  $m$ , it is equivalent to check that  $\underline{\gamma}_{y_+} \leq_{cx} \bar{\gamma}_{y_+}$  and  $\underline{\gamma}_{z_-} \leq_{cx} \bar{\gamma}_{z_-}$ .

Since

$$(x_- - y_+)^2 \vee (z_- - x_+)^2 \leq (x_+ - y_+)^2 \wedge (x_- - y_-)^2 \wedge (z_- - x_-)^2$$

$$\leq (x_+ - y_+)^2 \vee (x_- - y_-)^2 \vee (z_- - x_-)^2 \leq (x_+ - y_-)^2,$$

we have

$$\underline{\gamma}_{y_+} = \frac{1}{2} \left( \delta_{(x_+ - y_+)^2} + \frac{z_- - y_+}{z_- - y_-} \delta_{(x_- - y_-)^2} + \frac{y_+ - y_-}{z_- - y_-} \delta_{(z_- - x_-)^2} \right)$$

$$\leq_{cx} \frac{1}{2} \left( \delta_{(x_- - y_+)^2} + \frac{z_- - y_+}{z_- - y_-} \delta_{(x_+ - y_-)^2} + \frac{y_+ - y_-}{z_- - y_-} \delta_{(z_- - x_+)^2} \right) = \bar{\gamma}_{y_+}.$$

In a symmetric way, since

$$(x_- - y_+)^2 \vee (z_- - x_+)^2 \leq (x_+ - y_+)^2 \wedge (z_+ - x_+)^2 \wedge (z_- - x_-)^2$$

$$\leq (x_+ - y_+)^2 \vee (z_+ - x_+)^2 \vee (z_- - x_-)^2 \leq (z_+ - x_-)^2,$$

we have  $\underline{\gamma}_{z_-} \leq_{cx} \bar{\gamma}_{z_-}$ . Combining the two cases together, we conclude that for any  $m \in [y_+, z_-]$ ,

$\underline{\gamma}_m \leq_{cx} \bar{\gamma}_m$ , so that, by (46),  $|y - x|^2 \#\pi^\uparrow(dx, dy) \leq_{cx} |y - x|^2 \#\pi(dx, dy)$ .

For  $\pi \in \Pi_M(\mu, \bar{\nu}) \setminus \{\pi^\downarrow\}$ , we check in the same way that

$$\pi - \pi^\downarrow = ps \delta_{x_-} \otimes (\check{\tau} - \hat{\tau}) + ps \delta_{x_+} \otimes (\hat{\tau} - \check{\tau})$$

with  $\hat{\tau}$  and  $\check{\tau}$  satisfying (45) for some  $m \in [y_+, z_-]$  and

$$s = \pi_{x_-}^\downarrow(\{y_+\}) - \pi_{x_-}(\{y_+\}) + \pi_{x_-}^\downarrow(\{z_+\}) - \pi_{x_-}(\{z_+\}) = \pi_{x_-}(\{y_-\}) + \pi_{x_-}(\{z_-\}) > 0.$$

As a consequence,  $\text{sq}\#\pi - \text{sq}\#\pi^\downarrow = 2ps(\underline{\gamma}_m - \bar{\gamma}_m)$  so that  $\text{sq}\#\pi \leq_{cx} \text{sq}\#\pi^\downarrow$ .  $\square$



*Proof of Proposition 2.23.* Reasoning like in the beginning of the proof of Proposition 2.21, we check that for  $\pi \in \Pi_M(\mu, \bar{\nu}) \setminus \{\bar{\pi}^*\}$ ,  $\pi = \bar{\pi}^* + ps\delta_{x_-} \otimes (\hat{\tau} - \check{\tau}) + ps\delta_{x_+} \otimes (\check{\tau} - \hat{\tau})$  where

$$\begin{aligned} \hat{\tau} &= \frac{z_- - m}{z_- - y_+} \delta_{y_+} + \frac{m - y_+}{z_- - y_+} \delta_{z_-}, & \check{\tau} &= \frac{z_+ - m}{z_+ - y_-} \delta_{y_-} + \frac{m - y_-}{z_+ - y_-} \delta_{z_+} \\ s &= \pi_{x_-}(\{y_+\}) + \pi_{x_-}(\{z_-\}) = \bar{\pi}_{x_-}^*(\{y_-\}) - \pi_{x_-}(\{y_-\}) + \bar{\pi}_{x_-}^*(\{z_+\}) - \pi_{x_-}(\{z_+\}) > 0. \end{aligned} \quad (47)$$

Therefore the difference between the costs of  $\pi$  and  $\bar{\pi}^*$  can be calculated as

$$\int |x - y|^\rho \pi(dx, dy) - \int |x - y|^\rho \bar{\pi}^*(dx, dy) = ps(f(x_-, m) - f(x_+, m)), \quad (48)$$

where for  $x \in [y_+, z_-]$ ,

$$f(x, m) = \frac{z_- - m}{z_- - y_+} (x - y_+)^{\rho} + \frac{m - y_+}{z_- - y_+} (z_- - x)^{\rho} - \frac{z_+ - m}{z_+ - y_-} (x - y_-)^{\rho} - \frac{m - y_-}{z_+ - y_-} (z_+ - x)^{\rho}.$$

In the same way, we can check that any  $\pi \in \Pi_M(\mu, \underline{\nu})$  distinct from  $\underline{\pi}^*$  can be expressed as

$$\pi = \underline{\pi}^* + ps\delta_{x_-} \otimes (\check{\tau} - \hat{\tau}) + ps\delta_{x_+} \otimes (\hat{\tau} - \check{\tau}),$$

for  $\hat{\tau}$  and  $\check{\tau}$  satisfying (47) for some  $m \in [y_+, z_-]$  and

$$s = \pi_{x_-}(\{y_-\}) + \pi_{x_-}(\{z_+\}) = \underline{\pi}_{x_-}^*(\{y_+\}) - \pi_{x_-}(\{y_+\}) + \underline{\pi}_{x_-}^*(\{z_-\}) - \pi_{x_-}(\{z_-\}) > 0.$$

Hence the difference between the costs of  $\pi$  and  $\underline{\pi}^*$  can be calculated as

$$\int |x - y|^\rho \pi(dx, dy) - \int |x - y|^\rho \underline{\pi}^*(dx, dy) = ps(f(x_+, m) - f(x_-, m)).$$

Checking the optimality of  $\bar{\pi}^*$  and  $\underline{\pi}^*$  is equivalent to check when  $1 < \rho < 2$  (resp.  $\rho > 2$ ) that for every  $m \in [y_+, z_-]$ ,  $f(x_-, m) < f(x_+, m)$  (resp.  $f(x_-, m) > f(x_+, m)$ ).

The function  $m \mapsto f(x, m)$  being affine, it is also equivalent to show  $f(x_-, y_+) < f(x_+, y_+)$  and  $f(x_-, z_-) < f(x_+, z_-)$  for  $1 < \rho < 2$  (resp.  $f(x_-, y_+) > f(x_+, y_+)$  and  $f(x_-, z_-) > f(x_+, z_-)$  for  $\rho > 2$ ), which we are now going to demonstrate. We have

$$\begin{aligned} f(x, y_+) &= (x - y_+)^{\rho} - \frac{z_+ - y_+}{z_+ - y_-} (x - y_-)^{\rho} - \frac{y_+ - y_-}{z_+ - y_-} (z_+ - x)^{\rho}, \\ \frac{1}{\rho} \partial_x f(x, y_+) &= (x - y_+)^{\rho-1} - \frac{z_+ - y_+}{z_+ - y_-} (x - y_-)^{\rho-1} + \frac{y_+ - y_-}{z_+ - y_-} (z_+ - x)^{\rho-1}, \\ f(x, z_-) &= (z_- - x)^{\rho} - \frac{z_+ - z_-}{z_+ - y_-} (x - y_-)^{\rho} - \frac{z_- - y_-}{z_+ - y_-} (z_+ - x)^{\rho}, \\ \frac{1}{\rho} \partial_x f(x, z_-) &= -(z_- - x)^{\rho-1} - \frac{z_+ - z_-}{z_+ - y_-} (x - y_-)^{\rho-1} + \frac{z_- - y_-}{z_+ - y_-} (z_+ - x)^{\rho-1}. \end{aligned}$$

We are going to check that  $(2 - \rho)\partial_x f(z_-, z_-) > 0$  and  $(2 - \rho)\partial_x f(z_-, y_+) > 0$ , so that by continuity of  $x \mapsto (\partial_x f(x, z_-), \partial_x f(x, y_+))$ , there exists  $x_\rho \in (y_+, z_-)$  such that for all  $x \in [x_\rho, z_-]$ ,  $(2 - \rho)\partial_x f(x, z_-) > 0$ ,  $(2 - \rho)\partial_x f(x, y_+) > 0$  and for all  $x_-, x_+$  such that  $x_\rho < x_- < x_+ < z_-$ ,  $(2 - \rho)(f(x_+, z_-) - f(x_-, z_-)) > 0$  and  $(2 - \rho)(f(x_+, y_+) - f(x_-, y_+)) > 0$ .

On the one hand, we have

$$\begin{aligned} \frac{1}{\rho} \partial_x f(z_-, z_-) &= -\frac{z_+ - z_-}{z_+ - y_-} (z_- - y_-)^{\rho-1} + \frac{z_- - y_-}{z_+ - y_-} (z_+ - z_-)^{\rho-1} \\ &= \frac{z_+ - z_-}{z_+ - y_-} (z_- - y_-)^{\rho-1} \left( \left( \frac{z_- - y_-}{z_+ - z_-} \right)^{2-\rho} - 1 \right). \end{aligned}$$

Since, by assumption  $z_- - y_- > z_+ - z_-$ , we deduce that  $\partial_x f(z_-, z_-) > 0$  when  $1 < \rho < 2$  and  $\partial_x f(z_-, z_-) < 0$  when  $\rho > 2$ . On the other hand,

$$\begin{aligned} \frac{(z_+ - y_-)}{\rho} \partial_x f(z_-, y_+) &= (z_+ - y_-)(z_- - y_+)^{\rho-1} - (z_+ - y_+)(z_- - y_-)^{\rho-1} + (y_+ - y_-)(z_+ - z_-)^{\rho-1} \\ &= g(y_+ - y_-, z_- - y_+, z_+ - z_-), \end{aligned}$$

where, for  $a, b, c \geq 0$ ,

$$g(a, b, c) = (a + b + c) b^{\rho-1} - (b + c)(a + b)^{\rho-1} + ac^{\rho-1}.$$

When  $\rho \in (1, 2)$  and  $b > 0$ , by concavity of  $\mathbb{R}_+ \ni x \mapsto x^{\rho-1}$ , we have

$$\begin{aligned} g(a, b, c) &\geq (a + b + c) b^{\rho-1} - (b + c)(b^{\rho-1} + (\rho - 1)b^{\rho-2}a) + ac^{\rho-1} \\ &= (2 - \rho)ab^{\rho-1} + ac \left( c^{\rho-2} - ((\rho - 1)^{\frac{1}{\rho-2}} b)^{\rho-2} \right). \end{aligned}$$

and the inequality is reversed when  $\rho > 2$  by convexity of  $\mathbb{R}_+ \ni x \mapsto x^{\rho-1}$ . We deduce that

$$\forall a, b > 0, \forall c \in [0, (\rho - 1)^{\frac{1}{\rho-2}} b], g(a, b, c) > 0 \text{ when } \rho \in (1, 2) \text{ and } g(a, b, c) < 0 \text{ when } \rho > 2. \quad (49)$$

Since  $y_+ - y_- > 0$ ,  $z_- - y_+ > 0$  and  $z_- - y_+ \geq (\rho - 1)^{\frac{1}{2-\rho}}(z_+ - z_-)$ , we have  $\partial_x f(z_-, y_+) > 0$  when  $\rho \in (1, 2)$  and  $\partial_x f(z_-, y_+) < 0$  when  $\rho > 2$ .  $\square$

## References

- [1] A. Alfonsi, J. Corbetta, and B. Jourdain. Sampling of one-dimensional probability measures in the convex order and computation of robust option price bounds. *International Journal of Theoretical and Applied Finance*, 22(3), 2019.
- [2] M. Beiglböck, P. Henry-Labordère, and F. Penkner. Model-independent bounds for option prices: A mass transport approach. *Finance and Stochastics*, 17(3):477–501, 2013.
- [3] M. Beiglböck and N. Juillet. On a problem of optimal transport under marginal martingale constraints. *Annals of Probability*, 44(1):42–106, 2016.
- [4] S. Diamond and S. Boyd. CVXPY: A Python-embedded modeling language for convex optimization. *Journal of Machine Learning Research*, 17(83):1–5, 2016.
- [5] P. Henry-Labordère and N. Touzi. An explicit martingale version of the one-dimensional Brenier theorem. *Finance and Stochastics*, 20(3):635–668, 2016.
- [6] D. Hobson and M. Klimmek. Robust price bounds for the forward starting straddle. *Finance and Stochastics*, 19(1):189–214, 2015.
- [7] D. Hobson and A. Neuberger. Robust Bounds for Forward Start Options. *Mathematical Finance*, 22(1):31–56, 2012.
- [8] B. Jourdain and W. Margheriti. A new family of one dimensional martingale couplings. *Electronic Journal of Probability*, 25, 2020.
- [9] B. Jourdain and W. Margheriti. Martingale Wasserstein inequality for probability measures in the convex order. *Bernoulli*, 28(2):830–858, 2022.
- [10] R. P. Kertz and U. Rösler. Complete lattices of probability measures with applications to martingale theory. In *Game theory, optimal stopping, probability and statistics*, volume 35 of *IMS Lecture Notes Monogr. Ser.*, pages 153–177. Inst. Math. Statist., Beachwood, OH, 2000.
- [11] M. Marcus and V. J. Mizel. Absolute continuity on tracks and mappings of Sobolev spaces. *Arch. Rational Mech. Anal.*, 45:294–320, 1972.
- [12] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin Heidelberg, 3 edition, 1999.
- [13] M. Shaked and G. Shanthikumar. *Stochastic Orders*. Springer Series in Statistics. Springer-Verlag, New York, 2007.
- [14] V. Strassen. The existence of probability measures with given marginals. *Annals of Mathematical Statistics*, 36(2):423–439, 1965.