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## LOCAL NULL-CONTROLLABILITY OF A TWO-PARABOLIC NONLINEAR SYSTEM WITH COUPLED BOUNDARY CONDITIONS BY A NEUMANN CONTROL

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**ABSTRACT.** This article is concerned with the local boundary null-controllability of a 1-D system of two-parabolic nonlinear equations (often referred as reaction-diffusion system) with coupled boundary conditions by means of a scalar control. The control force is exerted on one of the two state components through a Neumann condition at the left end of the boundary while the other component simply satisfies the homogeneous Neumann condition at that point. On the other hand, at the right end of the boundary, the states are coupled through the so-called  $\delta'$ -type condition. Upon linearization around the stationary point  $(0, 0)$ , we apply the well-known *moments method* to prove the global null-controllability of the associated linearized system with explicit control cost  $Me^{M/T}$  as  $T \rightarrow 0^+$ . Then, we show the local null-controllability of the main system by employing the source term method developed in [29] followed by the Banach fixed point theorem.

### 1. Introduction and main results.

**1.1. The system under consideration.** In this paper, we address the boundary null-controllability result of a  $2 \times 2$  nonlinear parabolic system with coupled boundary conditions by means of one Neumann boundary control. More precisely,

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for given finite time  $T > 0$ , we consider the following system

$$\begin{cases} y_t - y_{xx} = f(y, z, \int_0^1 y, \int_0^1 z), & \text{in } (0, T) \times (0, 1), \\ z_t - z_{xx} = g(y, z, \int_0^1 y, \int_0^1 z), & \text{in } (0, T) \times (0, 1), \\ y_x(t, 0) = q(t), \quad z_x(t, 0) = 0, & \text{in } (0, T), \\ y_x(t, 1) = z_x(t, 1), & \text{in } (0, T), \\ y(t, 1) + z(t, 1) + \alpha y_x(t, 1) = 0, & \text{in } (0, T), \\ y(0, x) = y_0(x), \quad z(0, x) = z_0(x), & \text{in } (0, 1), \end{cases} \quad (1)$$

where  $\alpha \geq 0$  is some real parameter and  $(y_0, z_0)$  is the given initial data which we choose from the space  $[L^2(0, 1)]^2$ .

In the above system, a control function  $q \in L^2(0, T)$  (to be determined) is applied through the Neumann condition of only one state (namely  $y$ ) while the other state  $z$  simply satisfies the homogeneous Neumann boundary condition at the point  $x = 0$ . On the other hand, the states are coupled at the boundary point  $x = 1$  in terms of the “equality condition of their normal derivatives” and a “combined Robin-type condition”. In the literature, this kind of combined conditions (appearing at the point  $x = 1$ ) is typically called the  $\delta'$ -type condition, see for instance [8, p. 26, Chapter 1.4.4] or [18]. In fact, it has been addressed in [18] that the wavefunction of a quantum mechanical particle living on a graph often satisfies the  $\delta'$ -type boundary conditions at the junction points.

The nonlinear functions  $f$  and  $g$  in (1) are given by

$$\begin{cases} f(y, z, \int_0^1 y, \int_0^1 z) &= -yz + ay^2 + bz^2 + r_1(t)y, \\ g(y, z, \int_0^1 y, \int_0^1 z) &= yz + cy^2 + dz^2 + r_2(t)z, \end{cases} \quad (2)$$

where  $a, b, c, d$  are  $L^\infty((0, T) \times (0, 1))$  functions and

$$\begin{cases} r_1(t) = \alpha_1 \int_0^1 (\psi_{1,1}(x)y(t, x) + \psi_{2,1}(x)z(t, x)) dx, \\ r_2(t) = \alpha_2 \int_0^1 (\psi_{1,2}(x)y(t, x) + \psi_{2,2}(x)z(t, x)) dx, \end{cases} \quad (3)$$

with  $\alpha_1, \alpha_2$  are real constants and  $\psi_{1,j}, \psi_{2,j} \in L^\infty(0, 1)$  for  $j = 1, 2$ .

Observe that the nonlinear model (1)–(2) is actually a reaction-diffusion system which often describes several biological phenomenon or chemical reactions. In the literature, such system is commonly known as “Lotka-Volterra” model with diffusion (without any boundary conditions and control for the moment, let say), that sometimes characterize the dynamics of a biological system where two species: *prey* and *predator* interact between each other; see for instance [26, 31, 35]. In our model, we consider that the two species are interacting in the reference domain (through the nonlinear functions  $f, g$ ) as well as at one boundary end (through the coupled conditions at  $x = 1$ ). Then, our goal is to put an external control force only on one species from the other boundary end to locally control both the species at a given time  $T$ . In this regard, we refer the very detailed work [36], where several results concerning the controllability of reaction-diffusion systems in biology and social sciences have been addressed.

**1.2. Bibliographic comments.** The parabolic boundary control systems with less number of control(s) than equations can be a delicate issue in various situations and that there is lack of enough mathematical tools to tackle with these systems. In fact, unlike the scalar problems the boundary controllability for such systems is no longer equivalent with the distributed one, as it has been proven for instance in [22]. Moreover, the very powerful *Carleman technique* is often inefficient in that context. Among some fascinating works on coupled control systems, we point out [22] where the authors have proved a necessary and sufficient condition for boundary null-controllability of some  $2 \times 2$  coupled parabolic system with single Dirichlet control. A more general result regarding the controllability to the trajectories of an  $n \times n$  parabolic system with  $m (< n)$  Dirichlet controls (applied on a part of a boundary) is available in [2]. In those works, the authors actually proved a general Kalman condition which is necessary and sufficient for their controllability results.

To the best of our knowledge, most of the boundary controllability results for a system with less controls than the equations are in 1-D and the reason behind is that the spectral analysis of the associated adjoint elliptic operator helps to deal with the so-called “moments technique” (initially developed by Fattorini and Russell [20, 21]) to construct a control. In this regard, we mention that some multi-D (in cylindrical geometry) results have been developed in [1, 7], which need a sharp estimate of the control cost for the associated 1-D problem and a Lebeau-Robbiano spectral inequality for higher dimensions. We further refer to [3] where the authors made a survey of several recent results concerning the controllability of coupled parabolic systems.

The above references mainly address the parabolic systems with internal couplings. Let us mention that several systems with boundary couplings use to appear when one considers the system of pdes on metric graphs, e.g., [8, 27, 30]. Concerning the controllability issues for such systems, we first address [16, Chapters 6, 8] where the authors have discussed some controllability results of wave, heat and Schrödinger systems in the network when some control(s) is (are) exerted on some of the vertices; see also the survey paper [5]. We also refer the works [4, 12, 13, 14] where several controllability results have been achieved in the setting of metric graph and certainly, in those works, the couplings are arisen in the junction points of the graph. Very recently, the boundary null-controllability of some interior-boundary coupled linear parabolic systems has been addressed in [10] where the boundary coupling is chosen by means of a Kirchhoff-type condition.

In the context of controllability of nonlinear systems, let us first mention [24, Sec. 4, Chap. I] by Fursikov and Imanuvilov where a small-time local null-controllability of semilinear heat equations has been proved using a perturbation argument. In 2000, Barbu [6], independently Fernández-Cara and Zuazua [23] proved the small-time global null-controllability of semilinear heat equations where the nonlinear functions satisfy the growth condition  $|s| \ln^{3/2}(1 + |s|)$ . More recently, the large-time global null-controllability has been established in [28] for the nonlinearities  $F$  growing slower than  $|s| \ln^2(1 + |s|)$  verifying  $sF(s) \geq 0$  and  $\frac{1}{F} \in L^1([0, +\infty))$ . Last but not the least, we mention [25] where the local null-controllability of a nonlocal semilinear heat equation has been intensively investigated along with numerical illustrations.

In the present work, we shall deal with the local null-controllability of the parabolic system (1) and, as far as we know, the  $\delta'$ -type condition has not been treated

in the literature from the control theoretic perspective. Moreover, we consider the nonlocal nonlinearities in this work.

**1.3. Linearized system and functional setting.** For any given boundary parameter  $\alpha \geq 0$ , the linearized system around the equilibrium point  $(0, 0)$  is given by

$$\begin{cases} y_t - y_{xx} = 0, & \text{in } (0, T) \times (0, 1), \\ z_t - z_{xx} = 0, & \text{in } (0, T) \times (0, 1), \\ y_x(t, 0) = q(t), \quad z_x(t, 0) = 0, & \text{in } (0, T), \\ y_x(t, 1) = z_x(t, 1), & \text{in } (0, T), \\ y(t, 1) + z(t, 1) + \alpha y_x(t, 1) = 0, & \text{in } (0, T), \\ y(0, x) = y_0(x), \quad z(0, x) = z_0(x), & \text{in } (0, 1). \end{cases} \quad (4)$$

The free system, that is the set of equations (4) without any control input, can be written in the form of an infinite dimensional system of ordinary differential equations as follows

$$\begin{cases} Y'(t) + \mathcal{A}Y(t) = 0, \\ Y(0) = Y_0, \end{cases} \quad (5)$$

where  $Y := (y, z)$ ,  $Y_0 := (y_0, z_0)$  and the operator

$$\mathcal{A} = \begin{pmatrix} -\partial_{xx} & 0 \\ 0 & -\partial_{xx} \end{pmatrix}, \quad (6)$$

with its domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (u, v) \in [H^2(0, 1)]^2 \mid u'(0) = 0, v'(0) = 0, u'(1) = v'(1), \right. \\ \left. u(1) + v(1) + \alpha u'(1) = 0 \right\}.$$

Observe that the operator  $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$  is self-adjoint in nature but still we denote the adjoint of  $\mathcal{A}$  by  $\mathcal{A}^*$  for more clear presentation.

**1.4. Notations.** Throughout the paper,  $C$  denotes a generic positive constant that may change line to line but does not depend on the time  $T$  or on the initial data  $(y_0, z_0)$ . We also denote the following Lebesgue spaces:

- (i)  $Z := [L^2(0, 1)]^2$ ,
- (ii)  $\mathcal{H} := [H^1(0, 1)]^2$ ,
- (iii)  $\mathcal{H}^*$  = dual of the space  $\mathcal{H}$  with respect to the pivot space  $Z$ ,
- (iv)  $H_{\{a\}}^1(0, 1) = \{u \in H^1(0, 1) : u(a) = 0\}$ , for  $a \in \{0, 1\}$ ,

which shall be intensively used in the present work. The inner product in the space  $Z$  is simply denoted by  $(\cdot, \cdot)_Z$  while we denote the dual product by  $\langle \cdot, \cdot \rangle_{X^*, X}$  between the space  $X$  and its dual  $X^*$ . Sometimes, we write  $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$  to denote the usual inner product in the space  $\mathbb{R}^d$ ,  $d \geq 1$ . The characteristic function will be denoted by  $\chi_{[a, b]}$  in the real interval  $[a, b]$  with  $a < b$ .

**1.5. Main results.** We now write the main results of our present work.

1.5.1. *Local null-controllability of the nonlinear system.* We have the following controllability result for the system (1).

**Theorem 1.1.** *Let  $f$  and  $g$  be given by (2) and  $\alpha \geq 0$ . Then, the nonlinear system (1) is small-time locally null-controllable around the equilibrium  $(0, 0)$ , that is to say, for any given time  $T > 0$ , there is a  $\delta > 0$  such that for chosen initial state  $(y_0, z_0) \in Z$  verifying  $\|(y_0, z_0)\|_Z \leq \delta$ , there exists a solution-control pair  $((y, z), q)$  with  $(y, z) \in C^0([0, T]; Z) \cap L^2(0, T; \mathcal{H})$  and  $q \in L^2(0, T)$  to the system (1) satisfying*

$$(y(T, x), z(T, x)) = (0, 0) \quad \forall x \in (0, 1). \quad (7)$$

The strategy to prove Theorem 1.1 is the following:

- First, we prove the global boundary null-controllability result of the associated linear model (4) by using the method of moments ([20, 21]) with a proper estimation of the control cost, precisely  $Me^{M/T}\|(y_0, z_0)\|_Z$ , where  $M$  is independent in  $T$  and  $(y_0, z_0)$ .
- Next, by applying the source term method introduced in [29], we prove a null-controllability result of the linearized model with additional source terms in  $L^2(0, T; Z)$  which are exponentially decreasing as  $t \rightarrow T^-$ , and in this step, we notably use the precise control cost as prescribed earlier.
- Finally, we use the Banach fixed-point theorem to obtain the local (boundary) null-controllability for our nonlinear system (1).

1.5.2. *Null-controllability of the linear system.* Let us now state the global null-controllability result for the linearized system (4).

**Theorem 1.2.** *Let any  $T > 0$ , initial data  $(y_0, z_0) \in Z$  and parameter  $\alpha \geq 0$  be given. Then, there exists a control  $q \in L^2(0, T)$  such that the solution  $(y, z)$  to the system (4) satisfies  $(y(T, \cdot), z(T, \cdot)) = (0, 0)$ . In addition,  $q$  satisfies the following estimate*

$$\|q\|_{L^2(0, T)} \leq Me^{M/T} \|(y_0, z_0)\|_Z, \quad (8)$$

where the constant  $M > 0$  neither depends on  $T$  nor on  $(y_0, z_0)$ .

## 1.6. Organization of the paper.

- In Section 2, we discuss the required well-posedness results for the linear control problem (4) and its associated adjoint system (without any control input).
- Section 3 is devoted to prove the null-controllability of the linearized system (4). We study the spectral analysis for the associated adjoint operator in subsection 3.1, which is crucial to apply the method of moments to construct a null-control  $q \in L^2(0, T)$  for the system (4) with a precise control cost as introduced earlier (see subsection 3.5).
- In Section 4, we prove the main result of our work, that is, Theorem 1.1.
- Finally, we conclude our paper by mentioning possible extension of this work to a more general internal-boundary coupled parabolic system related to the present paper, see Section 5.

**2. Well-posedness of the linearized system.** This section is devoted to prove the existence and uniqueness of solution to the linear control system (4).

**2.1. Existence of analytic semigroup.** Let us first prove the well-posedness of the following homogeneous system

$$\begin{cases} y_t - y_{xx} = g_1, & \text{in } (0, T) \times (0, 1), \\ z_t - z_{xx} = g_2, & \text{in } (0, T) \times (0, 1), \\ y_x(t, 0) = 0, \quad z_x(t, 0) = 0, & \text{in } (0, T), \\ y_x(t, 1) = z_x(t, 1), & \text{in } (0, T), \\ y(t, 1) + z(t, 1) + \alpha y_x(t, 1) = 0, & \text{in } (0, T), \\ y(0, x) = y_0(x), \quad z(0, x) = z_0(x), & \text{in } (0, 1). \end{cases} \quad (9)$$

with given initial data  $(y_0, z_0) \in Z$  and source term  $(g_1, g_2) \in L^2(0, T; Z)$ . We start by proving the existence of semigroup defined by  $(-\mathcal{A}, \mathcal{D}(\mathcal{A}))$ .

**Proposition 2.1.** *The operator  $(-\mathcal{A}, \mathcal{D}(\mathcal{A}))$  defined in (6) forms an analytic semigroup in the space  $Z$ .*

*Proof.* We shall present the proof for the boundary parameter  $\alpha > 0$ . The case  $\alpha = 0$  is simpler. We prove this result into two steps.

*Step 1.* Let us define the usual norm on  $\mathcal{H}$ , given by

$$\|(u, v)\|_{\mathcal{H}} = \left( \int_0^1 (|u(x)|^2 + |u'(x)|^2) dx + \int_0^1 (|v(x)|^2 + |v'(x)|^2) dx \right)^{\frac{1}{2}},$$

and the sesquilinear map  $h : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$  such that for any  $(u, v), (\varphi, \psi) \in \mathcal{H}$

$$\begin{aligned} h((u, v), (\varphi, \psi)) &= \int_0^1 u'(x)\varphi'(x) dx + \int_0^1 v'(x)\psi'(x) dx \\ &\quad + \frac{1}{\alpha}[u(1) + v(1)][\varphi(1) + \psi(1)]. \end{aligned}$$

It follows that  $h$  is continuous on  $\mathcal{H} \times \mathcal{H}$  with

$$|h((u, v), (\varphi, \psi))| \leq c \|(u, v)\|_{\mathcal{H}} \|(\varphi, \psi)\|_{\mathcal{H}}, \quad \text{for all } (u, v), (\varphi, \psi) \in \mathcal{H},$$

where  $c$  is a positive constant depending on  $\alpha$ . We also have

$$|h((u, v), (u, v))| \geq \|(u, v)\|_{\mathcal{H}}^2 - \|(u, v)\|_Z^2, \quad \text{for all } (u, v) \in \mathcal{H}.$$

Therefore, by [34, Proposition 1.51 & Theorem 1.52], the *negative operator* associated with  $h$  generates an analytic semigroup in  $Z$  of angle  $(\pi/2 - \arctan(c))$ .

It remains to prove that the operator associated to  $h$  is indeed  $\mathcal{A}$  with the domain  $\mathcal{D}(\mathcal{A})$ .

*Step 2.* Let us define the operator  $(\tilde{\mathcal{A}}, \mathcal{D}(\tilde{\mathcal{A}}))$  associated with the map  $h$  as follows.

$$\begin{cases} \mathcal{D}(\tilde{\mathcal{A}}) = \left\{ (\tilde{u}, \tilde{v}) \in \mathcal{H} \mid \exists (f_1, f_2) \in Z \text{ such that} \right. \\ \left. h((\tilde{u}, \tilde{v}), (\varphi, \psi)) = ((f_1, f_2), (\varphi, \psi))_Z, \quad \forall (\varphi, \psi) \in \mathcal{H} \right\}, \\ \tilde{\mathcal{A}}(\tilde{u}, \tilde{v}) := (f_1, f_2). \end{cases}$$

*Part (i).* Here we prove  $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\tilde{\mathcal{A}})$ . Let  $(u, v) \in \mathcal{D}(\mathcal{A})$ . Then, for all  $(\varphi, \psi) \in \mathcal{H}$ , we have

$$\begin{aligned} &h((u, v), (\varphi, \psi)) \\ &= \int_0^1 u'(x)\varphi'(x) dx + \int_0^1 v'(x)\psi'(x) dx + \frac{1}{\alpha}[u(1) + v(1)][\varphi(1) + \psi(1)]. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} h((u, v), (\varphi, \psi)) &= - \int_0^1 u''(x)\varphi(x)dx - \int_0^1 v''(x)\psi(x)dx + u'(1)\varphi(1) \\ &\quad + v'(1)\psi(1) + \frac{1}{\alpha}[u(1) + v(1)][\varphi(1) + \psi(1)]. \end{aligned} \quad (10)$$

We also have that  $u'(1) = v'(1)$  and  $u(1) + v(1) = -\alpha u'(1)$ . Therefore, we get from (10)

$$\begin{aligned} h((u, v), (\varphi, \psi)) &= - \int_0^1 u''(x)\varphi(x)dx - \int_0^1 v''(x)\psi(x)dx \\ &= (\mathcal{A}(u, v), (\varphi, \psi))_Z. \end{aligned}$$

Thus, for given  $(u, v) \in \mathcal{D}(\mathcal{A})$  we found a pair  $(f_1, f_2) = \mathcal{A}(u, v) \in Z$  such that  $h((u, v), (\varphi, \psi)) = ((f_1, f_2), (\varphi, \psi))_Z$  for all  $(\varphi, \psi) \in \mathcal{H}$ . This implies  $(u, v) \in \mathcal{D}(\tilde{\mathcal{A}})$  and consequently,  $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\tilde{\mathcal{A}})$ .

*Part (ii).* We now show that  $\mathcal{D}(\tilde{\mathcal{A}}) \subset \mathcal{D}(\mathcal{A})$ . Let  $(\tilde{u}, \tilde{v}) \in \mathcal{D}(\tilde{\mathcal{A}})$ . Then, there exists  $(f_1, f_2) \in Z$  such that  $h((\tilde{u}, \tilde{v}), (\varphi, \psi)) = ((f_1, f_2), (\varphi, \psi))_Z$ , for all  $(\varphi, \psi) \in \mathcal{H}$  with  $\tilde{\mathcal{A}}(\tilde{u}, \tilde{v}) = (f_1, f_2)$ , and accordingly,

$$\begin{aligned} \int_0^1 \tilde{u}'(x)\varphi'(x)dx + \int_0^1 \tilde{v}'(x)\psi'(x)dx + \frac{1}{\alpha}[\tilde{u}(1) + \tilde{v}(1)][\varphi(1) + \psi(1)] \\ = \int_0^1 f_1(x)\varphi(x)dx + \int_0^1 f_2(x)\psi(x)dx, \end{aligned}$$

for all  $(\varphi, \psi) \in \mathcal{H}$ . Since  $f_1, f_2 \in L^2(0, 1)$ , by elliptic regularity theory, we have  $u, v \in H^2(0, 1)$ . Thus, an integration by parts yields

$$\begin{aligned} - \int_0^1 \tilde{u}''(x)\varphi(x)dx - \int_0^1 \tilde{v}''(x)\psi(x)dx + \tilde{u}'(1)\varphi(1) - \tilde{u}'(0)\varphi(0) + \tilde{v}'(1)\psi(1) \\ - \tilde{v}'(0)\psi(0) + \frac{1}{\alpha}[\tilde{u}(1) + \tilde{v}(1)][\varphi(1) + \psi(1)] \\ = \int_0^1 f_1(x)\varphi(x)dx + \int_0^1 f_2(x)\psi(x)dx, \end{aligned} \quad (11)$$

for all  $(\varphi, \psi) \in \mathcal{H}$ .

Let us first choose any  $(\varphi, \psi) \in [H_0^1(0, 1)]^2 \subset \mathcal{H}$  in (11) and as a result we deduce

$$f_1(x) = -\tilde{u}''(x), \quad f_2(x) = -\tilde{v}''(x), \quad \text{for a.a. } x \in (0, 1).$$

Once we have this, going back to (11), one has

$$\tilde{u}'(1)\varphi(1) - \tilde{u}'(0)\varphi(0) + \tilde{v}'(1)\psi(1) - \tilde{v}'(0)\psi(0) + \frac{1}{\alpha}[\tilde{u}(1) + \tilde{v}(1)][\varphi(1) + \psi(1)] = 0, \quad (12)$$

for all  $(\varphi, \psi) \in \mathcal{H}$ . Now consider any  $(\varphi, \psi) \in H_{\{0\}}^1(0, 1) \times H_0^1(0, 1) \subset \mathcal{H}$ , so that we have

$$\left( \tilde{u}'(1) + \frac{1}{\alpha}[\tilde{u}(1) + \tilde{v}(1)] \right) \varphi(1) = 0,$$

that is,

$$\tilde{u}(1) + \tilde{v}(1) + \alpha \tilde{u}'(1) = 0. \quad (13)$$

Next, by choosing any  $(\varphi, \psi) \in H_{\{1\}}^1(0, 1) \times H_0^1(0, 1) \subset \mathcal{H}$  in (12) we obtain the condition

$$\tilde{u}'(0) = 0, \quad (14)$$



and similarly, the choice of any  $(\varphi, \psi) \in H_0^1(0, 1) \times H_{\{1\}}^1(0, 1) \subset \mathcal{H}$  leads to the condition

$$\tilde{v}'(0) = 0. \quad (15)$$

Finally, by considering any  $(\varphi, \psi) \in \mathcal{H}$  and utilizing the previous boundary conditions (13), (14) and (15), the equality (12) reduces to

$$(\tilde{v}'(1) - \tilde{u}'(1))\psi(1) = 0,$$

for all  $\psi \in H^1(0, 1)$  and this yields

$$\tilde{u}'(1) = \tilde{v}'(1). \quad (16)$$

Therefore  $(\tilde{u}, \tilde{v}) \in \mathcal{D}(\mathcal{A})$ , which proves  $\mathcal{D}(\tilde{\mathcal{A}}) \subset \mathcal{D}(\mathcal{A})$ .

Hence, the operator associated with the sesquilinear form  $h$  is indeed  $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ . This completes the proof.  $\square$

We hereby denote the associated semigroup by  $(e^{-t\mathcal{A}})_{t \geq 0}$  and the following results hold.

**Proposition 2.2.** *Let any parameter  $\alpha \geq 0$  be given. Then, for any  $Y_0 := (y_0, z_0) \in \mathcal{D}(\mathcal{A})$  and  $G := (g_1, g_2) \in \mathcal{C}^1([0, T]; Z)$ , there exists unique strong solution  $Y := (y, z) \in \mathcal{C}^0([0, T]; \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1([0, T]; Z)$  to the system (9), given by*

$$Y(t) = e^{-t\mathcal{A}}Y_0 + \int_0^t e^{-(t-s)\mathcal{A}}G(s) ds. \quad (17)$$

**Proposition 2.3.** *Let any parameter  $\alpha \geq 0$  be given. Then, for any  $(y_0, z_0) \in Z$  and  $(g_1, g_2) \in L^2(0, T; Z)$ , there exists a unique weak solution*

$$(y, z) \in \mathcal{C}^0([0, T]; Z) \cap L^2(0, T; \mathcal{H}) \cap H^1(0, T; \mathcal{H}^*)$$

to the system (9) which satisfies the following energy estimate

$$\begin{aligned} & \| (y, z) \|_{\mathcal{C}^0([0, T]; Z)} + \| (y, z) \|_{L^2(0, T; \mathcal{H})} + \| (y_t, z_t) \|_{L^2(0, T; \mathcal{H}^*)} \\ & \leq C e^{CT} \left( \| (y_0, z_0) \|_Z + \| (g_1, g_2) \|_{L^2(0, T; Z)} \right), \end{aligned} \quad (18)$$

where  $C > 0$  is a constant that does not depend in  $T > 0$ .

*Proof.* For given initial state  $(y_0, z_0) \in Z$  and source term  $(g_1, g_2) \in L^2(0, T; Z)$ , the existence of a unique weak solution  $(y, z) \in \mathcal{C}^0([0, T]; Z)$  can be ensured by applying Proposition 2.1. We just need to prove the energy estimate (18).

- We start with  $(y_0, z_0) \in \mathcal{D}(\mathcal{A})$  and  $(g_1, g_2) \in \mathcal{C}^1([0, T]; Z)$ . Then, the system (9) has a unique strong solution  $(y, z)$  in the space  $\mathcal{C}^0([0, T]; \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1([0, T]; Z)$  as per Proposition 2.2. Taking the inner product in  $Z$  of (9) with  $(y, z)$ , we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \| (y(t), z(t)) \|_Z^2 + (\mathcal{A}(y(t), z(t)), (y(t), z(t)))_Z \\ & = ((g_1(t), g_2(t)), (y(t), z(t)))_Z, \quad \forall t \in [0, T]. \end{aligned}$$

Integrating by parts w.r.t. space and by applying the Cauchy-Schwarz and Young's inequalities, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \| (y(t), z(t)) \|_Z^2 + \| (y(t), z(t)) \|_{\mathcal{H}}^2 + \alpha |y'(t, 1)|^2 \\ & \leq C \left( \| (g_1(t), g_2(t)) \|_Z^2 + \| (y(t), z(t)) \|_Z^2 \right), \quad \forall t \in [0, T]. \end{aligned} \quad (19)$$

Here we recall that  $\alpha \geq 0$ , and then using Gronwall's lemma (see [17, Appendix B.2]) one can obtain the required estimate (18) for the quantity  $\|(y, z)\|_{C^0([0,T];Z)}$ . Then, by integrating (19) over  $[0, T]$  and using the previous estimate, we get the required bound for  $\|(y, z)\|_{L^2(0,T;\mathcal{H})}$ .

- To obtain the estimate for  $(y_t, z_t)$  in  $L^2(0, T; \mathcal{H}^*)$ , we consider any  $(\varphi, \psi) \in \mathcal{H}$  and from (9) we have

$$\begin{aligned} \langle (y_t(t), z_t(t)), (\varphi, \psi) \rangle_{\mathcal{H}^*, \mathcal{H}} + (\mathcal{A}(y(t), z(t)), (\varphi, \psi))_Z \\ = ((g_1(t), g_2(t)), (\varphi, \psi))_Z, \quad \forall t \in [0, T], \end{aligned}$$

which implies

$$\begin{aligned} & |\langle (y_t(t), z_t(t)), (\varphi, \psi) \rangle_{\mathcal{H}^*, \mathcal{H}}| \\ & \leq C \left( \|(y(t), z(t))\|_{\mathcal{H}} + \|(g_1(t), g_2(t))\|_Z \right) \|(\varphi, \psi)\|_{\mathcal{H}}, \quad \forall t \in [0, T], \end{aligned}$$

and this gives the estimation of  $\|(y_t, z_t)\|_{L^2(0,T;\mathcal{H}^*)}$  as stated in (18).

Finally, by applying the usual density argument, we shall obtain the same estimate (18) for given data  $(y_0, z_0) \in Z$  and  $(g_1, g_2) \in L^2(0, T; Z)$ . The proof is finished.  $\square$

**2.2. The homogeneous adjoint system: backward in time.** The adjoint system to the linearized model (9) is given by

$$\begin{cases} -\zeta_t - \zeta_{xx} = 0, & \text{in } (0, T) \times (0, 1), \\ -\theta_t - \theta_{xx} = 0, & \text{in } (0, T) \times (0, 1), \\ \zeta_x(t, 0) = 0, \quad \theta_x(t, 0) = 0, & \text{in } (0, T), \\ \zeta_x(t, 1) = \theta_x(t, 1), & \text{in } (0, T), \\ \zeta(t, 1) + \theta(t, 1) + \alpha \zeta_x(t, 1) = 0, & \text{in } (0, T), \\ \zeta(T, x) = \zeta_T(x), \quad \theta(T, x) = \theta_T(x), & \text{in } (0, 1), \end{cases} \quad (20)$$

with given final data  $(\zeta_T, \theta_T) \in Z$ . In fact, we have the following result.

**Proposition 2.4.** *Let any parameter  $\alpha \geq 0$  and final data  $(\zeta_T, \theta_T) \in Z$  be given. Then, the system (20) possesses a unique weak solution*

$$(\zeta, \theta) \in C^0([0, T]; Z) \cap L^2(0, T; \mathcal{H}) \cap H^1(0, T; \mathcal{H}^*)$$

with the following energy estimate:

$$\begin{aligned} \|(\zeta, \theta)\|_{C^0([0,T];Z)} + \|(\zeta, \theta)\|_{L^2(0,T;\mathcal{H})} + \|(\zeta_t, \theta_t)\|_{L^2(0,T;\mathcal{H}^*)} \\ \leq C e^{CT} \|(\zeta_T, \theta_T)\|_Z, \end{aligned} \quad (21)$$

where  $C > 0$  is a constant independent in  $T > 0$ .

Thanks to Proposition 2.1, the adjoint operator  $(-\mathcal{A}^*, \mathcal{D}(\mathcal{A}^*))$  (which is the same as  $(-\mathcal{A}, \mathcal{D}(\mathcal{A}))$  but we use a different notation for better understanding) defines a strongly continuous semigroup in  $Z$ , which ensures the existence and uniqueness of solution  $(\zeta, \theta) \in C^0([0, T]; Z)$  to (20) and moreover it can be expressed as

$$(\zeta, \theta)(t, x) = e^{-(T-t)\mathcal{A}^*} (\zeta_T, \theta_T)(x), \quad \forall (t, x) \in (0, T) \times (0, 1),$$

where  $(e^{-t\mathcal{A}^*})_{t \geq 0}$  denotes the semigroup defined by  $(-\mathcal{A}^*, \mathcal{D}(\mathcal{A}^*))$ .

Then the energy estimate (21) can be obtained by applying similar technique as described in the proof of Proposition 2.3.

**2.3. The nonhomogeneous linearized system.** We now address the notion of solution to the following nonhomogeneous system (which is forward in time) in the sense of transposition as introduced in [15, 37]. Consider the system

$$\begin{cases} y_t - y_{xx} = g_1, & \text{in } (0, T) \times (0, 1), \\ z_t - z_{xx} = g_2, & \text{in } (0, T) \times (0, 1), \\ y_x(t, 0) = q_1(t), \quad z_x(t, 0) = q_2(t), & \text{in } (0, T), \\ y_x(t, 1) = z_x(t, 1), & \text{in } (0, T), \\ y(t, 1) + z(t, 1) + \alpha y_x(t, 1) = 0, & \text{in } (0, T), \\ y(0, x) = y_0(x), \quad z(0, x) = z_0(x), & \text{in } (0, 1), \end{cases} \quad (22)$$

and we write the following definition.

**Definition 2.5** (Solution by transposition). Let  $\alpha \geq 0$  be a given parameter. Then, for given initial state  $(y_0, z_0) \in Z$ , boundary data  $(q_1, q_2) \in L^2(0, T; \mathbb{R}^2)$  and source term  $(g_1, g_2) \in L^2(0, T; Z)$ , a function  $(y, z) \in \mathcal{C}^0([0, T]; Z)$  is said to be a solution to the system (22), if for any  $t \in [0, T]$  and  $(\zeta_T, \theta_T) \in Z$ , the following relation holds:

$$\begin{aligned} ((y(t), z(t)), (\zeta_T, \theta_T))_Z &= ((y_0, z_0), e^{-tA^*}(\zeta_T, \theta_T))_Z \\ &+ \int_0^t ((g_1(s), g_2(s)), e^{-(t-s)A^*}(\zeta_T, \theta_T))_Z \\ &- \int_0^t \langle (q_1(s), q_2(s)), (e^{-(t-s)A^*}(\zeta_T, \theta_T))(0) \rangle_{\mathbb{R}^2}. \end{aligned} \quad (23)$$

Let us now write the following result.

**Theorem 2.6.** *Let  $\alpha \geq 0$  be a given parameter and  $(y_0, z_0) \in Z$ ,  $(g_1, g_2) \in L^2(0, T; Z)$ ,  $(q_1, q_2) \in L^2(0, T; \mathbb{R}^2)$  be given data. Then the system (22) has a unique solution  $(y, z) \in \mathcal{C}^0([0, T]; Z)$  in the sense of transposition as given by Definition 2.5.*

*Furthermore,  $(y, z) \in L^2(0, T; \mathcal{H}) \cap H^1(0, T; \mathcal{H}^*)$  and it satisfies the natural energy estimate*

$$\begin{aligned} \|(y, z)\|_{\mathcal{C}^0([0, T]; Z)} + \|(y, z)\|_{L^2(0, T; \mathcal{H})} + \|(y_t, z_t)\|_{L^2(0, T; \mathcal{H}^*)} \\ \leq Ce^{CT} \left( \|(y_0, z_0)\|_Z + \|(g_1, g_2)\|_{L^2(0, T; Z)} + \|(q_1, q_2)\|_{L^2(0, T; \mathbb{R}^2)} \right), \end{aligned} \quad (24)$$

where the constant  $C > 0$  does not depend on  $T$ .

The proof for the energy estimate can be done using a similar technique as implemented in the proof of Proposition 2.3. We skip the details.

**Remark 2.7.** For the nonhomogeneous system (22), we can achieve the usual energy estimate (24) since the nonhomogeneous  $L^2(0, T)$ -boundary terms  $q_1, q_2$  appear through the Neumann conditions. This phenomenon has been broadly studied in [32] in the context of parabolic equations with nonhomogeneous Neumann data. We also refer [9, Proposition 2.4] where the usual energy estimate for parabolic equations with nonhomogeneous Robin condition (with  $L^2$  boundary data) has been obtained.

**3. Controllability of the linearized system: the method of moments.** This section is devoted to the proof of null-controllability for our linearized system (4), that is the Theorem 1.2. As mentioned earlier, the method of moments helps us to construct a boundary null-control for our system and as it is well-known, to deal

with this method we first need to study the spectral analysis of the corresponding (adjoint) spatial operator. We discuss about this in the following section.

**3.1. Spectral Analysis of the operator  $\mathcal{A}^*$ .** The eigenvalue problem associated with the operator  $\mathcal{A}^*$  is

$$\mathcal{A}^*U = \lambda U, \quad \text{for } \lambda \in \mathbb{C},$$

with  $U := (u, v)$ , which explicitly looks like

$$\begin{cases} -u''(x) = \lambda u(x), & \text{for } x \in (0, 1), \\ -v''(x) = \lambda v(x), & \text{for } x \in (0, 1), \\ u'(0) = 0, \quad v'(0) = 0, \\ u'(1) = v'(1), \\ u(1) + v(1) + \alpha u'(1) = 0, \quad \alpha \geq 0. \end{cases} \quad (25)$$

We divide the analysis into several parts.

- Observe that the spatial operator (defined by (6)) is self-adjoint and thus, all eigenvalues are real.
- From the set of equations (25), it is clear that  $u = 0 \Leftrightarrow v = 0$  for any  $\lambda \in \mathbb{R}$ .
- $\lambda = 0$  is an eigenvalue of the operator  $\mathcal{A}^*$  associated with the eigenfunction  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ .

We denote this particular eigenfunction by  $\Phi_{\lambda_{0,1}}$  associated with the eigenvalue  $\lambda_{0,1} := 0$  just to be consistent with the notations introduced for the first set of eigenfunctions given by (27).

- Assume now that  $\lambda \neq 0$  and denote  $\mu = \sqrt{\lambda} \in \mathbb{R}^+$ . Thanks to the boundary condition  $u'(0) = v'(0)$ , we expect the solutions to (25) as

$$u(x) = A_1 \cos(\mu x), \quad v(x) = A_2 \cos(\mu x), \quad \forall x \in [0, 1].$$

Then, the boundary conditions  $u'(1) = v'(1)$  and  $u(1) + v(1) + \alpha u'(1) = 0$  respectively gives

$$A_1 \mu \sin \mu = A_2 \mu \sin \mu, \quad (26a)$$

$$A_1 \cos \mu + A_2 \cos \mu - \alpha A_1 \mu \sin \mu = 0. \quad (26b)$$

The case when  $A_1 \neq A_2$ , the equation (26a) yields  $\mu = k\pi$  for any  $k \geq 1$ , since  $\mu \neq 0$ . Using this information in (26b), we deduce  $A_1 = -A_2$ . Therefore, the eigenfunctions of the first family, denote them as  $\Phi_{\lambda_{k,1}}$ , are given by

$$\Phi_{\lambda_{k,1}} := \begin{pmatrix} \cos(k\pi x) \\ -\cos(k\pi x) \end{pmatrix}, \quad (27)$$

associated with the eigenvalues  $\lambda_{k,1} := k^2 \pi^2$  for all  $k \geq 1$ .

In the case when  $\sin \mu \neq 0$ , that is  $A_1 = A_2$  ( $\neq 0$  since we seek for non-trivial  $\mu$ ), we have from (26b) that

$$h(\mu) := 2 \cos \mu - \alpha \mu \sin \mu = 0, \quad \alpha \geq 0. \quad (28)$$

- (i) The case  $\alpha = 0$  is straightforward; we have the eigenfunctions  $\Phi_{\lambda_{k,2}^0}$  as follows:

$$\Phi_{\lambda_{k,2}^0} := \begin{pmatrix} \cos((k + \frac{1}{2})\pi x) \\ \cos((k + \frac{1}{2})\pi x) \end{pmatrix}, \quad (29)$$

associated with the eigenvalues  $\lambda_{k,2}^0 := (k + \frac{1}{2})^2\pi^2$  for all  $k \geq 0$ .

(ii) The case when  $\alpha \neq 0$ , we compute that

$$h(k\pi) = (-1)^k 2 \quad \text{and} \quad h\left(\left(k + \frac{1}{2}\right)\pi\right) = (-1)^{k+1} \alpha \left(k + \frac{1}{2}\right)\pi$$

have different signs which ensures the existence of at least one root of  $h$  in the interval  $(k\pi, (k + \frac{1}{2})\pi)$  for all  $k \geq 0$ .

To prove the uniqueness, we compute

$$h'(\mu) = -(\alpha + 2) \sin \mu - \alpha \mu \cos \mu$$

which has the same sign throughout the interval  $(k\pi, (k + \frac{1}{2})\pi)$  for any  $k \geq 0$  and thus the required claim follows.

We denote this unique root by  $\mu_{k,2}^\alpha$  and the eigenvalues by  $\lambda_{k,2}^\alpha := (\mu_{k,2}^\alpha)^2 \in (k^2\pi^2, (k + \frac{1}{2})^2\pi^2)$  for any  $k \geq 0$ . The associated eigenfunctions will be then

$$\Phi_{\lambda_{k,2}^\alpha} := \begin{pmatrix} \cos(\sqrt{\lambda_{k,2}^\alpha} x) \\ \cos(\sqrt{\lambda_{k,2}^\alpha} x) \end{pmatrix}, \quad \forall k \geq 0. \quad (30)$$

We now write the following lemma concerning the eigen-elements of  $\mathcal{A}^*$ .

**Lemma 3.1.** *Let any  $\alpha \geq 0$  be given. Then, we have the following.*

1. *The spectrum of the operator  $\mathcal{A}^*$  consists of only real simple eigenvalues and it is given by*

$$\Lambda^\alpha := \{\lambda_{k,1}, \lambda_{k,2}^\alpha\}_{k \geq 0}, \quad (31)$$

where

$$\lambda_{k,1} = k^2\pi^2 \quad \text{and} \quad \lambda_{k,2}^\alpha \begin{cases} = (k + \frac{1}{2})^2\pi^2, & \text{when } \alpha = 0, \\ \in (k^2\pi^2, (k + \frac{1}{2})^2\pi^2), & \text{when } \alpha > 0. \end{cases} \quad (32)$$

The associated eigenfunctions are

$$\Phi_{\lambda_{k,1}}(x) = \begin{pmatrix} \cos(k\pi x) \\ -\cos(k\pi x) \end{pmatrix} \quad \text{and} \quad \Phi_{\lambda_{k,2}^\alpha}(x) = \begin{pmatrix} \cos(\sqrt{\lambda_{k,2}^\alpha} x) \\ \cos(\sqrt{\lambda_{k,2}^\alpha} x) \end{pmatrix}, \quad (33)$$

for the eigenvalues  $\lambda_{k,1}$  and  $\lambda_{k,2}^\alpha$  respectively for all  $k \geq 0$ .

2. *Moreover, the set of eigenfunctions  $\{\Phi_{\lambda_{k,1}}, \Phi_{\lambda_{k,2}^\alpha}\}_{k \geq 0}$  forms an orthogonal basis in  $Z = [L^2(0, 1)]^2$ .*

The formal proof of part 1 has been already discussed before the statement of Lemma 3.1. Further, we note that the operator  $\mathcal{A}^*$  is self-adjoint and it can be proved that  $\mathcal{A}^*$  has compact resolvent. Consequently, the result of part 2 follows.

**Lemma 3.2** (Asymptotics of the eigenvalues for  $\alpha > 0$ ). *For each  $\alpha > 0$ , the asymptotic of the second set of eigenvalues  $\lambda_{k,2}^\alpha$  are*

$$\lambda_{k,2}^\alpha = k^2\pi^2 + \frac{4}{\alpha} + O\left(\frac{1}{k^2}\right), \quad \text{for large enough } k \in \mathbb{N}^*. \quad (34)$$

*Proof.* Recall that  $\mu_{k,2}^\alpha \in (k\pi, (k + \frac{1}{2})\pi)$  which uniquely satisfies the equation

$$2 \cos \mu_{k,2}^\alpha - \alpha \mu_{k,2}^\alpha \sin \mu_{k,2}^\alpha = 0, \quad \text{for each } k \geq 0. \quad (35)$$

We set  $\mu_{k,2}^\alpha = k\pi + \delta_k^\alpha$  with  $\delta_k^\alpha \in [0, \frac{\pi}{2}]$ . Then, from (35) we have

$$(-1)^k 2 \cos \delta_k^\alpha - (-1)^k \alpha (k\pi + \delta_k^\alpha) \sin \delta_k^\alpha = 0, \quad (36)$$

$$\Rightarrow \tan \delta_k^\alpha = \frac{2}{\alpha(k\pi + \delta_k^\alpha)} \rightarrow 0 \text{ as } k \rightarrow +\infty$$

$$\Rightarrow \delta_k^\alpha \rightarrow 0 \text{ as } k \rightarrow +\infty. \quad (37)$$

Using the fact (37) in (36), one has

$$\delta_k^\alpha \sim_{+\infty} \frac{2}{\alpha k \pi},$$

and thus,

$$\mu_{k,2}^\alpha \sim_{+\infty} k\pi + \frac{2}{\alpha k \pi}.$$

Thereafter, expressing  $\mu_{k,2}^\alpha = k\pi + \frac{2}{\alpha k \pi} + \tilde{\delta}_k^\alpha$  and substituting this in (35), one can obtain

$$\alpha \tilde{\delta}_k^\alpha k \pi = -\frac{4}{\alpha k^2 \pi^2} - \alpha (\tilde{\delta}_k^\alpha)^2 - \frac{4 \tilde{\delta}_k^\alpha}{k \pi},$$

which asymptotically gives  $\tilde{\delta}_k^\alpha \sim_{+\infty} O(1/k^3)$ . So, finally we have

$$\mu_{k,2}^\alpha = k\pi + \frac{2}{\alpha k \pi} + O\left(\frac{1}{k^3}\right), \text{ for large enough } k \in \mathbb{N}^*,$$

and that the asymptotic expression (34) follows.  $\square$

### 3.2. Formulation of the control problem and approximate controllability.

Let us now present an equivalent criterion for the null-controllability of the linear model (4).

**Proposition 3.3** (Formulation of the control problem). *Let any  $(y_0, z_0) \in Z$ , time  $T > 0$  and parameter  $\alpha \geq 0$  be given. Then a function  $q \in L^2(0, T)$  is said to be a null-control for the system (4) if and only if it satisfies: for any  $(\zeta_T, \theta_T) \in Z$ ,*

$$((y_0, z_0), e^{-T\mathcal{A}^*}(\zeta_T, \theta_T))_Z = \int_0^T q(t) \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (e^{-(T-t)\mathcal{A}^*}(\zeta_T, \theta_T))(0) \right\rangle_{\mathbb{R}^2}. \quad (38)$$

We hereby introduce the observation operator

$$\mathcal{B}^* := \mathbf{1}_{\{x=0\}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} : \mathcal{H} \mapsto \mathbb{R} \quad (39)$$

(recall that  $\mathcal{H} = [H^1(0, 1)]^2$ ) and to this end, we have the following result.

**Proposition 3.4** (Approximate controllability). *Let  $\alpha \geq 0$  be given. Then, the linearized system (4) is approximately controllable at any given time  $T > 0$  in the space  $Z$ .*

*Proof.* Note that  $\mathcal{B}^* \Phi_{\lambda_{k,1}} = \mathcal{B}^* \Phi_{\lambda_{k,2}} = 1$  for all  $\alpha \geq 0$  and  $k \geq 0$ . Then, by applying the Fattorini-Hautus criterion (see [19, 33]), we conclude the proposition.  $\square$

**3.3. The moments problem.** Recall that for any parameter  $\alpha \geq 0$ , the set of eigenfunctions  $\{\Phi_\lambda\}_{\lambda \in \Lambda^\alpha}$  of  $\mathcal{A}^*$  forms an orthogonal basis in  $Z$  (see Lemma 3.1). Thus, it is enough to check the control problem (38) for all  $\Phi_\lambda \in \{\Phi_\lambda\}_{\lambda \in \Lambda^\alpha}$ . This gives us the following.

- For any  $(y_0, z_0) \in Z$  and parameter  $\alpha \geq 0$ , a function  $q \in L^2(0, T)$  is a null-control for the system (4) if and only if we have

$$\int_0^T e^{-\lambda(T-t)} q(t) dt = \frac{e^{-\lambda T}}{\mathcal{B}^* \Phi_\lambda} ((y_0, z_0), \Phi_\lambda)_Z, \quad \text{for all } \lambda \in \Lambda^\alpha. \quad (40)$$

Here, we have used the fact that

$$e^{-t\mathcal{A}^*} \Phi_\lambda = e^{-t\lambda} \Phi_\lambda, \quad \forall \lambda \in \Lambda^\alpha.$$

We also recall that  $\mathcal{B}^* \Phi_\lambda = 1$  for all  $\lambda \in \Lambda^\alpha$  which ensures that the set of moment problems (40) is well-defined and we shall solve those in the next subsections.

**3.4. Existence of bi-orthogonal family.** In the framework of parabolic control theory, the existence of bi-orthogonal families to the family of exponential functions in  $L^2(0, T)$  has been extensively studied from the pioneer work [21] up to the very recent developments. In this paper, we use [11, Theorem V.4.26 & Corollary V.4.27] (which is similar to [7, Theorem 1.5] but with a more general set of assumptions) to establish the following result.

**Lemma 3.5.** *For any  $\alpha \geq 0$  recall the set  $\Lambda^\alpha$  given by (31). Then, there exists a family  $(p_\lambda)_{\lambda \in \Lambda^\alpha} \subset L^2(0, T)$  bi-orthogonal to  $(e^{-\lambda(T-\cdot)})_{\lambda \in \Lambda^\alpha}$ , i.e.,*

$$\int_0^T p_\lambda(t) e^{-\tilde{\lambda}(T-t)} dt = \delta_{\lambda, \tilde{\lambda}}, \quad \text{for any } \lambda, \tilde{\lambda} \in \Lambda^\alpha. \quad (41)$$

In addition, they satisfy the following estimate

$$\|p_\lambda\|_{L^2(0, T)} \leq C e^{\frac{C}{T}} e^{\frac{T}{2}\lambda + C\sqrt{\lambda}}, \quad \forall \lambda \in \Lambda^\alpha, \quad (42)$$

where the constant  $C > 0$  is independent in  $T$ .

**Remark 3.6.** Without loss of generality, we assume that all the eigenvalues are positive. In fact, we can choose some  $c_0 > 0$  such that  $\lambda + c_0 > 0$  for all  $\lambda \in \Lambda^\alpha$ . In what follows, an extra factor  $e^{Tc_0}$  will appear in the estimation of control cost, but without any consequences on our analysis.

Now, as mentioned earlier, we shall use [11, Theorem V.4.26] in order to prove Lemma 3.5, and for that we need to show that the set of eigenvalues  $\Lambda^\alpha$  defined by (31), belongs to some sector of the complex half-plane, satisfies a uniform gap property and some asymptotic conditions on the counting function.

- *The sector condition.* For any  $\nu > 0$ , we define the sector

$$\mathcal{S}_\nu := \{z \in \mathbb{C} \mid \Re z > 0, \text{ and } |\Im z| < (\sinh \nu) \Re z\}.$$

In our case, the set of eigenvalues  $\Lambda^\alpha$  is real and so it is clear that there exists some  $\nu > 0$  such that

$$\Lambda^\alpha \subset \mathcal{S}_\nu, \quad (43)$$

for any  $\alpha \geq 0$ .

- *The gap condition.* Recall the set of eigenvalues given by (31) and the asymptotics of the eigenvalues  $\lambda_{k,2}^\alpha$  for  $\alpha > 0$  from Lemma 3.2. Then it can be seen that there exists some  $c_1 > 0$  such that we have

$$\begin{aligned} |\lambda_{k+1,1} - \lambda_{k,1}| &\geq c_1 k, \quad \forall k \geq 1, \\ |\lambda_{k+1,2}^\alpha - \lambda_{k,2}^\alpha| &\geq c_1 k, \quad \forall k \geq 1 \text{ and } \alpha \geq 0, \end{aligned}$$

and there is some  $k_\alpha \in \mathbb{N}^*$  such that

$$\begin{aligned} |\lambda_{k,2}^0 - \lambda_{k,1}| &\geq c_1 k, \quad \forall k \geq 1, \\ |\lambda_{k,2}^\alpha - \lambda_{k,1}| &\geq \frac{c_1}{\alpha}, \quad \forall k \geq k_\alpha. \end{aligned}$$

**Remark 3.7.** Unlike the case of  $\alpha = 0$ , we note that for  $\alpha > 0$  the gap between  $\lambda_{k,2}^\alpha$  and  $\lambda_{k,1}$  tends to a finite positive number as  $k$  goes to infinity but does not tend to infinity like for the other cases. This is the reason why we needed to compute the precise asymptotic expansions of the eigenvalues  $\lambda_{k,2}^\alpha$  for  $\alpha > 0$ .

Using the above lower bounds of the *differences of eigenvalues* and the fact that the spectrum is discrete, we can say that there is some  $\rho > 0$  such that

$$|\lambda - \tilde{\lambda}| \geq \rho, \quad \text{for any } \lambda, \tilde{\lambda} \in \Lambda^\alpha \text{ with } \lambda \neq \tilde{\lambda}, \quad (44)$$

which is the uniform spectral gap property.

- *The condition on counting function.* Let  $\mathcal{N}_\alpha$  be the counting function associated with the set of eigenvalues  $\Lambda^\alpha$  (for any  $\alpha \geq 0$ ) defined by

$$\mathcal{N}_\alpha(r) := \# \{ \lambda \in \Lambda^\alpha, \text{ s.t. } |\lambda| \leq r \}, \quad \forall r > 0.$$

Our goal is to show that there exists some  $\kappa_0 > 0$  independent in the set of eigenvalues such that

$$\mathcal{N}_\alpha(r) \leq \kappa_0 r^{1/2}, \quad \forall r > 0, \quad (45a)$$

$$|\mathcal{N}_\alpha(r) - \mathcal{N}_\alpha(s)| \leq \kappa_0 \left( 1 + |r - s|^{1/2} \right), \quad \forall r, s > 0. \quad (45b)$$

From (31), we recall that

$$\Lambda^\alpha = \{ \lambda_{k,1}, \lambda_{k,2}^\alpha \}_{k \geq 0}.$$

As it is shown for instance in [11, Lemma V.4.20], it is enough to establish the required results (45) for each of the two sets  $\{ \lambda_{k,1} \}_{k \geq 0}$  and  $\{ \lambda_{k,2}^\alpha \}_{k \geq 0}$ . We shall show this for  $\{ \lambda_{k,2}^\alpha \}_{k \geq 0}$  when  $\alpha > 0$  since the same reasoning will be applicable for the set  $\{ \lambda_{k,2}^0 \}_{k \geq 0}$  or  $\{ \lambda_{k,1} \}_{k \geq 0}$ . We denote the associated counting function by  $\mathcal{N}_{\alpha,2}$ .

- Let  $r > 0$  be fixed. Then,  $\mathcal{N}_{\alpha,2}(r) = k$  ( $k \in \mathbb{N}^*$ ) implies

$$\lambda_{k-1,2}^\alpha \leq r,$$

since  $\{ \lambda_{k,2}^\alpha \}_{k \geq 0}$  is increasing. But we have  $\lambda_{k-1,2}^\alpha \in ((k-1)^2 \pi^2, (k - \frac{1}{2})^2 \pi^2)$  for any  $k \geq 1$ , which gives

$$(k-1)^2 \pi^2 \leq r, \quad \text{i.e., } k \leq 1 + \frac{1}{\pi} \sqrt{r},$$

and the first condition (45a) follows for the counting function.



- Let any  $0 < s < r$  be given. Assume that  $l = \mathcal{N}_{\alpha,2}(s)$  and  $k = \mathcal{N}_{\alpha,2}(r)$  for some  $l, k \in \mathbb{N}^*$  (certainly,  $k > l$ ). Then, using the properties of the set  $\{\lambda_{k,2}^\alpha\}_{k \geq 0}$ , one has

$$(k-1)\pi \leq \sqrt{\lambda_{k-1,2}^\alpha} \leq \sqrt{r}, \quad (l + \frac{1}{2})\pi > \sqrt{\lambda_{l,2}^\alpha} > \sqrt{s},$$

which yields

$$k-l \leq \frac{3}{2} + \frac{1}{\pi}(\sqrt{r} - \sqrt{s}) \leq \frac{3}{2} + \frac{1}{\pi}\sqrt{r-s},$$

and that the second condition (45b) on the counting function is true.

Since the three conditions (43), (44) and (45) are now satisfied, by using [11, Theorem V.4.16], we can ensure the existence of a bi-orthogonal family  $(p_\lambda)_{\lambda \in \Lambda^\alpha} \subset L^2(0, T)$  to  $(e^{-\lambda(T-\cdot)})_{\lambda \in \Lambda^\alpha}$  satisfying the sharp estimate as mentioned in Lemma 3.5.

**3.5. Existence of a boundary null-control.** Now, we are in position to solve the set of moments problem (40) to find a control for the system (4).

*Proof of Theorem 1.2.* For any  $\alpha \geq 0$  and initial data  $(y_0, z_0) \in Z$ , we consider

$$q(t) = \sum_{\lambda \in \Lambda^\alpha} q_\lambda(t), \quad \forall t \in [0, T], \quad (46a)$$

$$\text{with } q_\lambda(t) = \frac{e^{-\lambda T}}{\mathcal{B}^* \Phi_\lambda} ((y_0, z_0), \Phi_\lambda)_Z p_\lambda(t), \quad \forall t \in [0, T], \quad \forall \lambda \in \Lambda^\alpha \quad (46b)$$

where  $p_\lambda$  are given by Lemma 3.5. Observe that, the above choice of function  $q$  formally solves the set of moments problem (40), thanks to the property (41) verified by  $p_\lambda$  for each  $\lambda \in \Lambda^\alpha$ .

Now, recall that  $\mathcal{B}^* \Phi_\lambda = 1$  for all  $\lambda \in \Lambda^\alpha$  (see Proposition 3.4). Also, from the expressions of the eigenfunctions given by (27)–(29)–(30), we have  $\|\Phi_\lambda\|_Z \leq C$  for any  $\lambda \in \Lambda^\alpha$ . Using these and the  $L^2(0, T)$ -estimates of bi-orthogonal family  $(p_\lambda)_{\lambda \in \Lambda^\alpha}$  given by (42), we obtain  $\lambda \in \Lambda^\alpha$ , that

$$\begin{aligned} \|q_\lambda\|_{L^2(0, T)} &\leq C e^{-\lambda T} e^{\frac{C}{T}} e^{\frac{T}{2}\lambda + C\sqrt{\lambda}} \|(y_0, z_0)\|_Z \\ &\leq C e^{\frac{C}{T}} e^{-\frac{T}{2}\lambda} e^{\frac{T}{4}\lambda + \frac{C^2}{T}} \|(y_0, z_0)\|_Z \\ &\leq C e^{\frac{C}{T}} e^{-\frac{T}{4}\lambda} \|(y_0, z_0)\|_Z, \end{aligned} \quad (47)$$

where we have used the Young's inequality

$$C\sqrt{\lambda} \leq \frac{T}{4}\lambda + \frac{C^2}{T}, \quad \forall \lambda \in \Lambda^\alpha.$$

Using (47) we have

$$\begin{aligned} \|q\|_{L^2(0, T)} &\leq \sum_{\lambda \in \Lambda^\alpha} \|q_\lambda\|_{L^2(0, T)} \leq C e^{\frac{C}{T}} \|(y_0, z_0)\|_Z \sum_{\lambda \in \Lambda^\alpha} e^{-\frac{T}{4}\lambda} \\ &\leq M e^{\frac{M}{T}} \|(y_0, z_0)\|_Z, \end{aligned}$$

thanks to the fact that  $\Lambda^\alpha$  is an increasing sequence of order  $k^2$  (see (31)). Moreover, it is clear that the constant  $M > 0$  does not depend on  $T$  or  $(y_0, z_0)$ .

The proof is complete.  $\square$

**4. Local null-controllability of the nonlinear system.** This section is devoted to prove the local null-controllability result for the nonlinear system (1), i.e., Theorem 1.1. The proof will be based on the so-called source term method developed in [29] followed by a Banach fixed point argument and to employ this we shall extensively use the control cost  $Me^{\frac{M}{T}} \|(y_0, z_0)\|_Z$  for the linear system, given by Theorem 1.2.

**4.1. The source term method.** Let us discuss the source term method for our problem. We first consider the following system:

$$\begin{cases} y_t - y_{xx} = \xi, & \text{in } (0, T) \times (0, 1), \\ z_t - z_{xx} = \eta, & \text{in } (0, T) \times (0, 1), \\ y_x(t, 0) = q(t), \quad z_x(t, 0) = 0, & \text{in } (0, T), \\ y_x(t, 1) = z_x(t, 1), & \text{in } (0, T), \\ y(t, 1) + z(t, 1) + \alpha y_x(t, 1) = 0, & \text{in } (0, T), \\ y(0, x) = y_0(x), \quad z(0, x) = z_0(x), & \text{in } (0, 1). \end{cases} \quad (48)$$

Then, our goal is to establish the null-controllability of the above system for any given parameter  $\alpha \geq 0$ , initial data  $(y_0, z_0) \in Z$  and source terms  $(\xi, \eta)$  which belong to some certain weighted  $L^2(0, T; Z)$  space. Let us discuss it at length in the next couple of subsections.

**4.1.1. Construction of weight functions and weighted spaces.** Assume the constants  $\beta > 0, \gamma > 1$  in such a way that

$$1 < \gamma < \sqrt{2}, \quad \text{and} \quad \beta > \frac{\gamma^2}{2 - \gamma^2}. \quad (49)$$

We now define the weight functions

$$\begin{cases} \rho_0(t) = e^{-\frac{\beta M}{(\gamma-1)(T-t)}}, \\ \rho_S(t) = e^{-\frac{(1+\beta)\gamma^2 M}{(\gamma-1)(T-t)}}, \end{cases} \quad \forall t \in \left[ T \left( 1 - \frac{1}{\gamma^2} \right), T \right], \quad (50)$$

and extended them in a constant way in  $\left[ 0, T \left( 1 - \frac{1}{\gamma^2} \right) \right]$  such that they are continuous and non-increasing in  $[0, T]$ . Note that  $\rho_0(T) = \rho_S(T) = 0$  and further, we compute that

$$\frac{\rho_0^2(t)}{\rho_S(t)} = e^{\frac{\gamma^2 M + \beta M(\gamma^2 - 2)}{(\gamma-1)(T-t)}}, \quad \forall t \in \left[ T \left( 1 - \frac{1}{\gamma^2} \right), T \right].$$

Due to the choices of  $\gamma, \beta$  in (49), we have  $M(\gamma^2 + \beta(\gamma^2 - 2)) < 0, (\gamma - 1) > 0$  and therefore we conclude that

$$\frac{\rho_0^2(t)}{\rho_S(t)} \leq 1, \quad \forall t \in [0, T]. \quad (51)$$

Let us now define the following weighted spaces:

$$\mathcal{S} := \left\{ \xi \in L^2(0, T; L^2(0, 1)) \mid \frac{\xi}{\rho_S} \in L^2(0, T; L^2(0, 1)) \right\} \quad (52)$$

$$\mathcal{Y} := \left\{ (y, z) \in L^2(0, T; Z) \mid \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \in L^2(0, T; Z) \right\} \quad (53)$$

$$\mathcal{Q} := \left\{ q \in L^2(0, T) \mid \frac{q}{\rho_0} \in L^2(0, T) \right\}, \quad (54)$$

where the functions  $\rho_0$  and  $\rho_S$  are defined in (50). The inner product on the spaces  $\mathcal{S}$ ,  $\mathcal{Y}$  and  $\mathcal{Q}$  are respectively given by

$$\langle \xi, \tilde{\xi} \rangle_{\mathcal{S}} := \int_0^T \frac{1}{\rho_S^2(t)} \langle \xi(t), \tilde{\xi}(t) \rangle_{L^2(0,1)} dt, \quad \forall \xi, \tilde{\xi} \in \mathcal{S},$$

$$\langle (y, z), (\tilde{y}, \tilde{z}) \rangle_{\mathcal{Y}} := \int_0^T \frac{1}{\rho_0^2(t)} \langle (y(t), z(t)), (\tilde{y}(t), \tilde{z}(t)) \rangle_Z dt, \quad \forall (y, z), (\tilde{y}, \tilde{z}) \in \mathcal{Y},$$

$$\langle q, \tilde{q} \rangle_{\mathcal{Q}} := \int_0^T \frac{1}{\rho_0^2(t)} q(t) \tilde{q}(t) dt, \quad \forall q, \tilde{q} \in \mathcal{Q}.$$

Accordingly, the associated norms on the spaces  $\mathcal{S}$ ,  $\mathcal{Y}$  and  $\mathcal{Q}$  are respectively

$$\|\xi\|_{\mathcal{S}}^2 := \int_0^T \frac{1}{\rho_S^2(t)} \|\xi(t)\|_{L^2(0,1)}^2 dt, \quad \forall \xi \in \mathcal{S}, \quad (55)$$

$$\|(y, z)\|_{\mathcal{Y}}^2 := \int_0^T \frac{1}{\rho_0^2(t)} \|(y(t), z(t))\|_Z^2 dt, \quad \forall (y, z) \in \mathcal{Y}, \quad (56)$$

$$\|q\|_{\mathcal{Q}}^2 := \int_0^T \frac{1}{\rho_0^2(t)} |q(t)|^2 dt, \quad \forall q \in \mathcal{Q}. \quad (57)$$

**4.1.2. Null-controllability of the linearized system with source terms.** Our next result addresses the null-controllability of the inhomogeneous linear system (48) with given source terms  $\xi, \eta$  from the space  $\mathcal{S}$  and by definition of  $\mathcal{S}$ , it is clear that the function  $\xi$  or  $\eta$  vanishes exponentially near  $t = T$ . With the above choice of source functions in hand, and then by utilizing the explicit *control cost*  $Me^{\frac{M}{T}}$  for the homogeneous control system (see Section 3.5), we shall eventually show that there exists a solution-control pair  $((y, z), q)$  in the space  $\mathcal{Y} \times \mathcal{Q}$  to the system (48). Then, by definitions of the space  $\mathcal{Y}$  and weight function  $\rho_0$  (see (53) and (50) resp.), one can conclude that the solution  $(y, z)$  has to be “zero” at  $t = T$ . Precisely we prove the following proposition.

**Proposition 4.1.** *Let any parameter  $\alpha \geq 0$  be given. Then, for any given initial state  $(y_0, z_0) \in Z$  and source terms  $(\xi, \eta) \in L^2(0, T; Z)$ , there exists a linear map  $\mathcal{T} : Z \times L^2(0, T; Z) \rightarrow \mathcal{Y} \times \mathcal{Q}$  such that  $\mathcal{T}((y_0, z_0), (\xi, \eta)) := ((y, z), q)$  solves the system (48).*

*In addition, we have the following estimate*

$$\begin{aligned} & \left\| \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{C^0([0, T]; Z)} + \left\| \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{L^2(0, T; \mathcal{H})} + \left\| \frac{q}{\rho_0} \right\|_{L^2(0, T)} \\ & \leq C e^{CT + \frac{C}{T}} \left( \|(y_0, z_0)\|_Z + \left\| \left( \frac{\xi}{\rho_S}, \frac{\eta}{\rho_S} \right) \right\|_{L^2(0, T; Z)} \right), \end{aligned} \quad (58)$$

for some constant  $C > 0$  that is independent in  $T$ .

*Proof.* For the given time  $T > 0$ , let us define a sequence  $(T_k)_{k \geq 0}$  given by

$$T_k := T - \frac{T}{\gamma^k}, \quad \forall k \geq 0, \quad (59)$$

where  $\gamma$  is introduced in (49), and it can be easily seen that

$$(0, T) = \cup_{k \geq 0} (T_k, T_{k+1}).$$

We also note that with this choice of  $T_k$ , one has

$$\rho_0(T_{k+2}) = e^{\frac{M}{T_{k+2}-T_{k+1}}} \rho_S(T_k), \quad \forall k \geq 0, \quad (60)$$

where  $\rho_0$  and  $\rho_S$  have been defined by (50).

Now, our goal is to decompose (48) in  $(T_k, T_{k+1})$  for each  $k \geq 0$ , into two parts: one is only with forcing terms and zero initial data, and the other one is a homogeneous control system along with the initial data.

• *Inhomogeneous system without control input.* Let us define a sequence  $(a_k)_{k \geq 0}$  such that

$$a_0 := (y_0, z_0) \in Z \quad \text{and} \quad a_{k+1} := (\tilde{y}(T_{k+1}^-), \tilde{z}(T_{k+1}^-)), \quad \forall k \geq 0, \quad (61)$$

where  $(\tilde{y}, \tilde{z})$  is the unique weak solution to the system

$$\begin{cases} \tilde{y}_t - \tilde{y}_{xx} = \xi, & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \tilde{z}_t - \tilde{z}_{xx} = \eta, & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \tilde{y}_x(t, 0) = 0, \quad \tilde{z}_x(t, 0) = 0, & \text{in } (T_k, T_{k+1}), \\ \tilde{y}_x(t, 1) = \tilde{z}_x(t, 1), & \text{in } (T_k, T_{k+1}), \\ \tilde{y}(t, 1) + \tilde{z}(t, 1) + \alpha \tilde{y}_x(t, 1) = 0, & \text{in } (T_k, T_{k+1}), \\ \tilde{y}(T_k^+, \cdot) = 0, \quad \tilde{z}(T_k^+, \cdot) = 0, & \text{in } (0, 1), \end{cases} \quad (62)$$

for all  $k \geq 0$ . Thanks to the estimate (18) in Proposition 2.3, we get

$$\begin{aligned} \|(\tilde{y}, \tilde{z})\|_{C^0([T_k, T_{k+1}]; Z)} + \|(\tilde{y}, \tilde{z})\|_{L^2(T_k, T_{k+1}; \mathcal{H})} \\ \leq C e^{CT} \|(\xi, \eta)\|_{L^2(T_k, T_{k+1}; Z)}, \quad \forall k \geq 0. \end{aligned} \quad (63)$$

In particular, by means of (61), we have

$$\|a_{k+1}\|_Z \leq C e^{CT} \|(\xi, \eta)\|_{L^2(T_k, T_{k+1}; Z)}, \quad \forall k \geq 0. \quad (64)$$

• *Control system without the source terms.* We now consider the following homogeneous control system:

$$\begin{cases} \hat{y}_t - \hat{y}_{xx} = 0, & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \hat{z}_t - \hat{z}_{xx} = 0, & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ \hat{y}_x(t, 0) = \hat{q}_k(t), \quad \hat{z}_x(t, 0) = 0, & \text{in } (T_k, T_{k+1}), \\ \hat{y}_x(t, 1) = \hat{z}_x(t, 1), & \text{in } (T_k, T_{k+1}), \\ \hat{y}(t, 1) + \hat{z}(t, 1) + \alpha \hat{y}_x(t, 1) = 0, & \text{in } (T_k, T_{k+1}), \\ (\hat{y}(T_k^+, \cdot), \hat{z}(T_k^+, \cdot)) = a_k, & \text{in } (0, 1), \end{cases} \quad (65)$$

for all  $k \geq 0$ . Using Theorem 1.2, we have the existence of a control  $\hat{q}_k \in L^2(T_k, T_{k+1})$  with the estimate

$$\|\hat{q}_k\|_{L^2(T_k, T_{k+1})} \leq M e^{\frac{M}{T_{k+1}-T_k}} \|a_k\|_Z, \quad (66)$$

such that the associated solution  $(\hat{y}, \hat{z})$  to (65) satisfies

$$(\hat{y}(T_{k+1}^-, x), \hat{z}(T_{k+1}^-, x)) = (0, 0), \quad \forall x \in (0, 1) \quad \text{and} \quad \forall k \geq 0.$$

Combining (66) with (64), we have

$$\begin{aligned} \|\hat{q}_{k+1}\|_{L^2(T_{k+1}, T_{k+2})} &\leq M e^{\frac{M}{T_{k+2}-T_{k+1}}} \|a_{k+1}\|_Z \\ &\leq C e^{CT} e^{\frac{M}{T_{k+2}-T_{k+1}}} \|(\xi, \eta)\|_{L^2(T_k, T_{k+1}; Z)}, \quad \forall k \geq 0. \end{aligned}$$

But  $\rho_S$  is a non-increasing function in  $[T_k, T_{k+1}]$ ; in what follows we have

$$\|\hat{q}_{k+1}\|_{L^2(T_{k+1}, T_{k+2})} \leq C e^{CT} e^{\frac{M}{T_{k+2}-T_{k+1}}} \rho_S(T_k) \left\| \left( \frac{\xi}{\rho_S}, \frac{\eta}{\rho_S} \right) \right\|_{L^2(T_k, T_{k+1}; Z)}, \quad \forall k \geq 0.$$

Then, using the relation (60) between the weight functions  $\rho_0$  and  $\rho_S$ , we get

$$\|\hat{q}_{k+1}\|_{L^2(T_{k+1}, T_{k+2})} \leq C e^{CT} \rho_0(T_{k+2}) \left\| \left( \frac{\xi}{\rho_S}, \frac{\eta}{\rho_S} \right) \right\|_{L^2(T_k, T_{k+1}; Z)}, \quad \forall k \geq 0. \quad (67)$$

Again, since  $\rho_0$  is non-increasing, we deduce

$$\begin{aligned} \left\| \frac{\hat{q}_{k+1}}{\rho_0} \right\|_{L^2(T_{k+1}, T_{k+2})} &\leq \frac{1}{\rho_0(T_{k+2})} \|\hat{q}_{k+1}\|_{L^2(T_{k+1}, T_{k+2})} \\ &\leq C e^{CT} \left\| \left( \frac{\xi}{\rho_S}, \frac{\eta}{\rho_S} \right) \right\|_{L^2(T_k, T_{k+1}; Z)}, \quad \forall k \geq 0. \end{aligned} \quad (68)$$

We now define the control function  $q$  as follows:

$$q := \sum_{k \geq 0} \hat{q}_k \chi_{(T_k, T_{k+1})} \quad \text{in } (0, T). \quad (69)$$

Recall that we have already established the  $L^2$ -estimates of  $\frac{\hat{q}_k}{\rho_0}$  for all  $k \geq 1$  by (68).

It only remains to find the  $L^2$ -estimate of  $\frac{\hat{q}_0}{\rho_0}$ . But from the bound (66), we get

$$\|\hat{q}_0\|_{L^2(0, T_1)} \leq M e^{\frac{M}{T_1}} \|a_0\|_Z = M e^{\frac{M}{T_1}} \|(y_0, z_0)\|_Z,$$

and then using the fact that  $\rho_0$  is non-increasing, one has

$$\begin{aligned} \left\| \frac{\hat{q}_0}{\rho_0} \right\|_{L^2(0, T_1)} &\leq \frac{1}{|\rho_0(T_1)|} \|\hat{q}_0\| \leq \frac{M}{\rho_0(T_1)} e^{\frac{M}{T_1}} \|(y_0, z_0)\|_Z \\ &= M e^{\frac{M\gamma(1+\beta\gamma)}{(\gamma-1)T}} \|(y_0, z_0)\|_Z, \end{aligned} \quad (70)$$

where in the last inclusion, we have used the fact that  $T_2 = T - \frac{T}{\gamma^2}$  and  $\rho_0(T_1) = \rho_0(T_2) = e^{-\frac{\gamma^2 \beta M}{(\gamma-1)T}}$ . Now, the quantity  $\frac{M\gamma(1+\beta\gamma)}{(\gamma-1)}$  being positive, we eventually obtain (by combining (68) and (70))

$$\left\| \frac{q}{\rho_0} \right\|_{L^2(0, T)} \leq C e^{CT + \frac{C}{T}} \left( \|(y_0, z_0)\|_Z + \left\| \left( \frac{\xi}{\rho_S}, \frac{\eta}{\rho_S} \right) \right\|_{L^2(0, T; Z)} \right), \quad (71)$$

where the constant  $C > 0$  is independent in  $T > 0$ .

- *Control system with the source terms.* We now define

$$(y, z) = (\tilde{y}, \tilde{z}) + (\hat{y}, \hat{z}). \quad (72)$$

Then  $(y, z)$  satisfies the following system

$$\begin{cases} y_t - y_{xx} = \xi, & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ z_t - z_{xx} = \eta, & \text{in } (T_k, T_{k+1}) \times (0, 1), \\ y_x(t, 0) = \hat{q}_k(t), \quad z_x(t, 0) = 0, & \text{in } (T_k, T_{k+1}), \\ y_x(t, 1) = z_x(t, 1), & \text{in } (T_k, T_{k+1}), \\ y(t, 1) + z(t, 1) + \alpha y_x(t, 1) = 0, & \text{in } (T_k, T_{k+1}), \\ (y(T_k, \cdot), z(T_k, \cdot)) = a_k, & \text{in } (0, 1), \end{cases} \quad (73)$$

for all  $k \geq 0$ . Note that, the solution  $(y, z)$  satisfies

$$(y(T_0), z(T_0)) = a_0 = (y_0, z_0),$$

and, for all  $k \geq 0$  we have

$$\begin{aligned} (y(T_{k+1}^-), z(T_{k+1}^-)) &= (\tilde{y}(T_{k+1}^-), \tilde{z}(T_{k+1}^-)) + (\hat{y}(T_{k+1}^-), \hat{z}(T_{k+1}^-)) = a_{k+1}, \\ (y(T_{k+1}^+), z(T_{k+1}^+)) &= (\tilde{y}(T_{k+1}^+), \tilde{z}(T_{k+1}^+)) + (\hat{y}(T_{k+1}^+), \hat{z}(T_{k+1}^+)) = a_{k+1}. \end{aligned}$$

Therefore  $(y, z)$  is continuous at  $T_k$  for all  $k \geq 0$ .

Now, applying the energy estimate (24) for the system (73), and using the estimations for  $a_{k+1}$  from (64) and  $\hat{q}_{k+1}$  from (66), we have

$$\begin{aligned} & \| (y, z) \|_{C^0([T_{k+1}, T_{k+2}]; Z)} + \| (y, z) \|_{L^2(T_{k+1}, T_{k+2}; \mathcal{H})} \\ & \leq C e^{CT} \left( \| a_{k+1} \|_Z + \| (\xi, \eta) \|_{L^2(T_{k+1}, T_{k+2}; Z)} + \| \hat{q}_{k+1} \|_{L^2(T_{k+1}, T_{k+2})} \right) \\ & \leq C e^{CT} \left( \| a_{k+1} \|_Z + \| (\xi, \eta) \|_{L^2(T_{k+1}, T_{k+2}; Z)} + M e^{\frac{M}{T_{k+2} - T_{k+1}}} \| a_{k+1} \|_Z \right) \\ & \leq C e^{CT} \| (\xi, \eta) \|_{L^2(T_k, T_{k+2}; Z)} + C e^{CT} e^{\frac{M}{T_{k+2} - T_{k+1}}} \| (\xi, \eta) \|_{L^2(T_k, T_{k+1}; Z)} \\ & \leq C e^{CT} e^{\frac{M}{T_{k+2} - T_{k+1}}} \| (\xi, \eta) \|_{L^2(T_k, T_{k+2}; Z)}, \end{aligned}$$

for all  $k \geq 0$ .

Since  $\rho_S$  is non-increasing in  $[T_k, T_{k+2}]$ , we obtain from above,

$$\begin{aligned} & \| (y, z) \|_{C^0([T_{k+1}, T_{k+2}]; Z)} + \| (y, z) \|_{L^2(T_{k+1}, T_{k+2}; \mathcal{H})} \\ & \leq C e^{CT} e^{\frac{M}{T_{k+2} - T_{k+1}}} \rho_S(T_k) \left\| \left( \frac{\xi}{\rho_S}, \frac{\eta}{\rho_S} \right) \right\|_{L^2(T_k, T_{k+2}; Z)} \\ & = C e^{CT} \rho_0(T_{k+2}) \left\| \left( \frac{\xi}{\rho_S}, \frac{\eta}{\rho_S} \right) \right\|_{L^2(T_k, T_{k+2}; Z)}, \end{aligned} \quad (74)$$

for all  $k \geq 0$ , since  $\rho_0(T_{k+2}) = e^{\frac{M}{T_{k+2} - T_{k+1}}} \rho_S(T_k)$  (see (60)).

Using the fact that  $\rho_0$  is non-increasing on  $[T_{k+1}, T_{k+2}]$ , we further deduce from (74) that

$$\begin{aligned} & \left\| \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{C^0([T_{k+1}, T_{k+2}]; Z)} + \left\| \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{L^2(T_{k+1}, T_{k+2}; \mathcal{H})} \\ & \leq \frac{1}{\rho_0(T_{k+2})} \left( \| (y, z) \|_{C^0([T_{k+1}, T_{k+2}]; Z)} + \| (y, z) \|_{L^2(T_{k+1}, T_{k+2}; \mathcal{H})} \right) \\ & \leq C e^{CT} \left\| \left( \frac{\xi}{\rho_S}, \frac{\eta}{\rho_S} \right) \right\|_{L^2(T_k, T_{k+2}; Z)}, \end{aligned} \quad (75)$$

for all  $k \geq 0$ .

Now, it remains to find the estimates of  $(y, z)$  in  $[0, T_1]$ . Again, using the energy estimate (24) we find that (also having in mind  $\rho_0(T_1) = e^{-\frac{\gamma\beta M}{(\gamma-1)T}}$ )

$$\begin{aligned} & \|(y, z)\|_{C^0([0, T_1]; Z)} + \|(y, z)\|_{L^2(0, T_1; \mathcal{H})} \\ & \leq Ce^{CT} \left( \|a_0\|_Z + \|\hat{q}_0\|_{L^2(0, T_1)} + \|(\xi, \eta)\|_{L^2(0, T_1; Z)} \right) \\ & \leq Ce^{CT} \left( \|a_0\|_Z + Me^{\frac{M}{T_1}} \|a_0\|_Z + \|(\xi, \eta)\|_{L^2(0, T_1; Z)} \right) \\ & \leq Ce^{CT} e^{\frac{M\gamma(1+\beta)}{(\gamma-1)T}} \rho_0(T_1) \left( \|(y_0, z_0)\|_Z + Me^{\frac{M}{T_1}} \|a_0\|_Z + \|(\xi, \eta)\|_{L^2(0, T_1; Z)} \right). \end{aligned}$$

But,  $\rho_0$  and  $\rho_S$  are non-increasing functions in  $[0, T_1]$  and thus the above estimate follows to:

$$\begin{aligned} & \left\| \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{C^0([0, T_1]; Z)} + \left\| \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{L^2(0, T_1; \mathcal{H})} \\ & \leq Ce^{CT + \frac{C}{T}} \left( \|(y_0, z_0)\|_Z + \left\| \left( \frac{\xi}{\rho_S}, \frac{\eta}{\rho_S} \right) \right\|_{L^2(0, T_1; Z)} \right). \end{aligned} \quad (76)$$

Combining the estimates (75) and (76), we have

$$\begin{aligned} & \left\| \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{C^0([0, T]; Z)} + \left\| \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{L^2(0, T; \mathcal{H})} \\ & \leq Ce^{CT + \frac{C}{T}} \left( \|(y_0, z_0)\|_Z + \left\| \left( \frac{\xi}{\rho_S}, \frac{\eta}{\rho_S} \right) \right\|_{L^2(0, T; Z)} \right), \end{aligned} \quad (77)$$

for some constant  $C > 0$  independent in  $T$ .

The above bound (77) along with (71), we achieve the required estimate (58) of the proposition. This completes the proof.  $\square$

**4.2. Application of Banach fixed point theorem.** This section is devoted to prove the local null-controllability result of our nonlinear system (1), that is Theorem 1.1.

Let any parameter  $\alpha \geq 0$  be given as earlier and assume any initial data  $(y_0, z_0) \in Z$  such that  $\|(y_0, z_0)\|_Z \leq \delta$ , where  $\delta > 0$  will be specified later. We now define the set

$$\mathfrak{S}_\delta := \{(\xi, \eta) \in \mathcal{S} \times \mathcal{S} : \|(\xi, \eta)\|_{\mathcal{S} \times \mathcal{S}} \leq \delta\},$$

where the space  $\mathcal{S}$  is defined in (52).

By Proposition 4.1, we can say that for any given source term  $(\xi, \eta) \in \mathcal{S} \times \mathcal{S}$ , there exists a control  $q \in L^2(0, T)$  such that the corresponding trajectory  $(y, z)$  of the system (48) satisfies the estimate (58). In what follows, we define the map  $\mathfrak{F} : \mathfrak{S}_\delta \rightarrow L^2(0, T; Z)$  by

$$\mathfrak{F}(\xi, \eta) = \begin{pmatrix} f(y, z, \int_0^1 y, \int_0^1 z) \\ g(y, z, \int_0^1 y, \int_0^1 z) \end{pmatrix}, \quad (78)$$

for all  $(\xi, \eta) \in \mathfrak{S}_\delta$ , where we recall that the nonlinear functions  $f$  and  $g$  are given by

$$\begin{cases} f(y, z, \int_0^1 y, \int_0^1 z) & = -yz + ay^2 + bz^2 + r_1(t)y, \\ g(y, z, \int_0^1 y, \int_0^1 z) & = yz + cy^2 + dz^2 + r_2(t)z, \end{cases} \quad (79)$$

where  $a, b, c, d$  are  $L^\infty((0, T) \times (0, 1))$  functions and

$$\begin{cases} r_1(t) = \alpha_1 \int_0^1 (\psi_{1,1}(x)y(t, x) + \psi_{2,1}(x)z(t, x)) dx, \\ r_2(t) = \alpha_2 \int_0^1 (\psi_{1,2}(x)y(t, x) + \psi_{2,2}(x)z(t, x)) dx, \end{cases} \quad (80)$$

with  $\alpha_1, \alpha_2 \in \mathbb{R}$  and  $\psi_{1,j}, \psi_{2,j} \in L^\infty(0, 1)$ ,  $j = 1, 2$ .

Our goal is to prove that there exists some  $\delta > 0$  such that the map  $\mathfrak{F}$  has a unique fixed point in the set  $\mathfrak{S}_\delta$  and to do so, we shall apply the Banach fixed point theorem. We begin with the following lemma.

**Lemma 4.2** (Stability). *There exists some  $\delta > 0$  such that  $\mathfrak{S}_\delta \subset \mathcal{S} \times \mathcal{S}$  is stable under the map  $\mathfrak{F}$ .*

*Proof.* We have for  $(\xi, \eta) \in \mathfrak{S}_\delta$ ,

$$\begin{aligned} \|\mathfrak{F}(\xi, \eta)\|_{\mathcal{S} \times \mathcal{S}}^2 &= \left\| \begin{pmatrix} f(y, z, \int_0^1 y, \int_0^1 z) \\ g(y, z, \int_0^1 y, \int_0^1 z) \end{pmatrix} \right\|_{\mathcal{S} \times \mathcal{S}}^2 \\ &\leq \|-yz + ay^2 + bz^2 + r_1(t)y\|_{\mathcal{S}}^2 + \|yz + cy^2 + dz^2 + r_2(t)z\|_{\mathcal{S}}^2. \end{aligned}$$

Using the definition of norm in  $\mathcal{S}$  (see (55)), we deduce from above that,

$$\begin{aligned} \|\mathfrak{F}(\xi, \eta)\|_{\mathcal{S} \times \mathcal{S}}^2 &\leq C \int_0^T \frac{1}{\rho_{\mathcal{S}}^2(t)} \left( \|y(t)z(t)\|_{L^2(0,1)}^2 + \|y^2(t)\|_{L^2(0,1)}^2 + \|z^2(t)\|_{L^2(0,1)}^2 \right. \\ &\quad \left. + \|r_1(t)y(t)\|_{L^2(0,1)}^2 + \|r_2(t)z(t)\|_{L^2(0,1)}^2 \right) dt, \end{aligned} \quad (81)$$

where  $C := C(\|a\|_{L^\infty}, \|b\|_{L^\infty}, \|c\|_{L^\infty}, \|d\|_{L^\infty}) > 0$ . We now estimate the terms appearing in the r.h.s. of (81). Note that,

$$\|y(t)z(t)\|_{L^2(0,1)}^2 = \int_0^1 |y(t, x)z(t, x)|^2 dx \leq 2 \int_0^1 (|y(t, x)|^4 + |z(t, x)|^4) dx. \quad (82)$$

and

$$\|y^2(t)\|_{L^2(0,1)}^2 = \int_0^1 |y(t, x)|^4 dx, \quad \|z^2(t)\|_{L^2(0,1)}^2 = \int_0^1 |z(t, x)|^4 dx. \quad (83)$$

We also have for  $j = 1, 2$

$$\begin{aligned} \|r_j(t)y(t)\|_{L^2(0,1)}^2 &= |\alpha_j|^2 \int_0^1 \left| y(t, x) \int_0^1 (\psi_{1,j}(x)y(t, x) + \psi_{2,j}(x)z(t, x)) dx \right|^2 dx \\ &\leq C \left( \int_0^1 (|y(t, x)|^2 + |z(t, x)|^2) dx \right) \int_0^1 |y(t, x)|^2 dx, \end{aligned} \quad (84)$$

where  $C := C(|\alpha_1|, |\alpha_2|, \|\psi_{1,1}\|_{L^\infty}, \|\psi_{1,2}\|_{L^\infty}, \|\psi_{2,1}\|_{L^\infty}, \|\psi_{2,2}\|_{L^\infty}) > 0$ .

Combining the above estimates (82), (83) and (84), we obtain from (81),

$$\begin{aligned} \|\mathfrak{F}(\xi, \eta)\|_{\mathcal{S} \times \mathcal{S}}^2 &\leq C \int_0^T \frac{1}{\rho_{\mathcal{S}}^2(t)} \left( \int_0^1 (|y(t, x)|^4 + |z(t, x)|^4) dx \right) dt \\ &= C \int_0^T \int_0^1 \frac{\rho_0^4(t)}{\rho_{\mathcal{S}}^2(t)} \left| \frac{y(t, x)}{\rho_0(t)} \right|^4 dx dt + C \int_0^T \int_0^1 \frac{\rho_0^4(t)}{\rho_{\mathcal{S}}^2(t)} \left| \frac{z(t, x)}{\rho_0(t)} \right|^4 dx dt. \end{aligned} \quad (85)$$



Thanks to the fact (51), we get from (85) that

$$\begin{aligned}
& \|\mathfrak{F}(\xi, \eta)\|_{\mathcal{S} \times \mathcal{S}}^2 \\
& \leq C \int_0^T \left\| \frac{y(t)}{\rho_0(t)} \right\|_{L^\infty(0,1)}^2 \left( \int_0^1 \left| \frac{y(t,x)}{\rho_0(t)} \right|^2 dx \right) dt \\
& \quad + C \int_0^T \left\| \frac{z(t)}{\rho_0(t)} \right\|_{L^\infty(0,1)}^2 \left( \int_0^1 \left| \frac{z(t,x)}{\rho_0(t)} \right|^2 dx \right) dt \\
& \leq C \int_0^T \left( \left\| \frac{y(t)}{\rho_0(t)} \right\|_{H^1(0,1)}^2 \left\| \frac{y(t)}{\rho_0(t)} \right\|_{L^2(0,1)}^2 + \left\| \frac{z(t)}{\rho_0(t)} \right\|_{H^1(0,1)}^2 \left\| \frac{z(t)}{\rho_0(t)} \right\|_{L^2(0,1)}^2 \right) dt \\
& \leq C \left\| \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{C^0([0,T];Z)}^2 \left\| \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{L^2(0,T;\mathcal{H})}^2.
\end{aligned}$$

Using the estimate (58) in above, we finally arrive to the following:

$$\begin{aligned}
\|\mathfrak{F}(\xi, \eta)\|_{\mathcal{S} \times \mathcal{S}} & \leq C \left( \left\| \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{C^0([0,T];Z)} + \left\| \left( \frac{y}{\rho_0}, \frac{z}{\rho_0} \right) \right\|_{L^2(0,T;\mathcal{H})} \right)^2 \\
& \leq C e^{CT + \frac{C}{T}} (\|y_0, z_0\|_Z + \|(\xi, \eta)\|_{\mathcal{S} \times \mathcal{S}})^2 \\
& \leq C e^{CT + \frac{C}{T}} \delta^2,
\end{aligned} \tag{86}$$

due to our choices of initial data  $\|(y_0, z_0)\|_Z \leq \delta$  and source terms  $(\xi, \eta) \in \mathfrak{S}_\delta$ .

Now, one can choose  $\delta > 0$  small enough in (86) so that we have  $\|\mathfrak{F}(\xi, \eta)\|_{\mathcal{S} \times \mathcal{S}} \leq \delta$  for all  $(\xi, \eta) \in \mathfrak{S}_\delta$ . This concludes our lemma.  $\square$

The following lemma shows that  $\mathfrak{F} : \mathfrak{S}_\delta \rightarrow \mathfrak{S}_\delta$  is a contraction map.

**Lemma 4.3** (Contraction). *There exists a  $\delta > 0$  such that the map  $\mathfrak{F}$  defined by (78) is a contraction map on the closed ball  $\mathfrak{S}_\delta$ .*

*Proof.* Consider any two pairs  $(\xi_i, \eta_i) \in \mathfrak{S}_\delta$  for  $i = 1, 2$ . Then, by means of Proposition 4.1, there exist control functions  $q_i \in \mathcal{Q}$  for the system (48) with solutions  $(y_i, z_i) \in \mathcal{Y}$  associated to  $(\xi_i, \eta_i) \in \mathfrak{S}_\delta$  for  $i = 1, 2$ .

Accordingly, we use the notations  $f_i, g_i$  for the nonlinear functions (see (79)–(80)) where

$$\begin{cases} f_i(y_i, z_i, \int_0^1 y_i, \int_0^1 z_i) & = -y_i z_i + ay_i^2 + bz_i^2 + r_{i,1}(t)y_i, \\ g_i(y_i, z_i, \int_0^1 y_i, \int_0^1 z_i) & = y_i z_i + cy_i^2 + dz_i^2 + r_{i,2}(t)z_i, \end{cases}$$

with

$$\begin{cases} r_{i,1}(t) = \alpha_1 \int_0^1 (\psi_{1,1}(x)y_i(t,x) + \psi_{2,1}(x)z_i(t,x)) dx, \\ r_{i,2}(t) = \alpha_2 \int_0^1 (\psi_{1,2}(x)y_i(t,x) + \psi_{2,2}(x)z_i(t,x)) dx, \end{cases}$$

for  $i = 1, 2$ .

Then, we compute

$$\begin{aligned}
& \|\mathfrak{F}(\xi_1, \eta_1) - \mathfrak{F}(\xi_2, \eta_2)\|_{\mathcal{S} \times \mathcal{S}}^2 \\
& = \left\| \begin{pmatrix} -y_1 z_1 + ay_1^2 + bz_1^2 + r_{1,1}(t)y_1 \\ y_1 z_1 + cy_1^2 + dz_1^2 + r_{1,2}(t)z_1 \end{pmatrix} - \begin{pmatrix} -y_2 z_2 + ay_2^2 + bz_2^2 + r_{2,1}(t)y_2 \\ y_2 z_2 + cy_2^2 + dz_2^2 + r_{2,2}(t)z_2 \end{pmatrix} \right\|_{\mathcal{S} \times \mathcal{S}}^2
\end{aligned}$$

$$\begin{aligned}
&= \left\| \begin{pmatrix} -(y_1 z_1 - y_2 z_2) + a(y_1^2 - y_2^2) + b(z_1^2 - z_2^2) + r_{1,1}(t)y_1 - r_{2,1}(t)y_2 \\ y_1 z_1 - y_2 z_2 + c(y_1^2 - y_2^2) + d(z_1^2 - z_2^2) + r_{1,2}(t)z_1 - r_{2,2}(t)z_2 \end{pmatrix} \right\|_{\mathcal{S} \times \mathcal{S}}^2 \\
&\leq C \int_0^T \frac{1}{\rho_{\mathcal{S}}^2(t)} \left( \|y_1(t)z_1(t) - y_2(t)z_2(t)\|_{L^2(0,1)}^2 + \|y_1^2(t) - y_2^2(t)\|_{L^2(0,1)}^2 \right. \\
&\quad + \|z_1^2(t) - z_2^2(t)\|_{L^2(0,1)}^2 + \|r_{1,1}(t)y_1(t) - r_{2,1}(t)y_2(t)\|_{L^2(0,1)}^2 \\
&\quad \left. + \|r_{1,2}(t)z_1(t) - r_{2,2}(t)z_2(t)\|_{L^2(0,1)}^2 \right) dt. \tag{87}
\end{aligned}$$

To this end, we find

$$\begin{aligned}
&\|y_1(t)z_1(t) - y_2(t)z_2(t)\|_{L^2(0,1)}^2 \\
&\leq 2 \left( \|y_1(t)(z_1(t) - z_2(t))\|_{L^2(0,1)}^2 + \|(y_1(t) - y_2(t))z_2(t)\|_{L^2(0,1)}^2 \right) \\
&\leq C \|y_1(t)\|_{L^\infty(0,1)}^2 \|z_1(t) - z_2(t)\|_{L^2(0,1)}^2 \\
&\quad + C \|z_2(t)\|_{L^\infty(0,1)}^2 \|y_1(t) - y_2(t)\|_{L^2(0,1)}^2 \\
&\leq C \|y_1(t)\|_{H^1(0,1)}^2 \|z_1(t) - z_2(t)\|_{L^2(0,1)}^2 \\
&\quad + C \|z_2(t)\|_{H^1(0,1)}^2 \|y_1(t) - y_2(t)\|_{L^2(0,1)}^2. \tag{88}
\end{aligned}$$

A straightforward computation also gives

$$\begin{aligned}
&\|y_1^2(t) - y_2^2(t)\|_{L^2(0,1)}^2 \\
&\leq \left( \|y_1(t)\|_{H^1(0,1)}^2 + \|y_2(t)\|_{H^1(0,1)}^2 \right) \|y_1(t) - y_2(t)\|_{L^2(0,1)}^2, \tag{89}
\end{aligned}$$

and

$$\begin{aligned}
&\|z_1^2(t) - z_2^2(t)\|_{L^2(0,1)}^2 \\
&\leq \left( \|z_1(t)\|_{H^1(0,1)}^2 + \|z_2(t)\|_{H^1(0,1)}^2 \right) \|z_1(t) - z_2(t)\|_{L^2(0,1)}^2. \tag{90}
\end{aligned}$$

Next we look at the remaining terms in (87), we compute

$$\begin{aligned}
&\|r_{1,1}(t)y_1(t) - r_{2,1}(t)y_2(t)\|_{L^2(0,1)}^2 \\
&\leq 2 \int_0^1 |r_{1,1}(t)(y_1(t,x) - y_2(t,x))|^2 dx + 2 \int_0^1 |(r_{1,1}(t) - r_{2,1}(t))y_2(t,x)|^2 dx \\
&\leq 2 |\alpha_1|^2 \left| \int_0^1 (\psi_{1,1}(x)y_1(t,x) + \psi_{2,1}(x)z_1(t,x)) dx \right|^2 \int_0^1 |y_1(t,x) - y_2(t,x)|^2 dx \\
&\quad + 2 \int_0^1 |y_2(t,x)|^2 dx \left| \alpha_1 \int_0^1 (\psi_{1,1}(x)y_1(t,x) + \psi_{2,1}(x)z_1(t,x)) dx \right. \\
&\quad \quad \left. - \alpha_1 \int_0^1 (\psi_{1,1}(x)y_2(t,x) + \psi_{2,1}(x)z_2(t,x)) dx \right|^2 \\
&\leq C \|y_1(t) - y_2(t)\|_{L^2(0,1)}^2 \int_0^1 (|y_1(t,x)|^2 + |z_1(t,x)|^2) dx \\
&\quad + C \|y_2(t)\|_{L^2(0,1)}^2 \times \\
&\quad \int_0^1 \left( |\psi_{1,1}(x)|^2 |y_1(t,x) - y_2(t,x)|^2 + |\psi_{2,1}(x)|^2 |z_1(t,x) - z_2(t,x)|^2 \right) dx
\end{aligned}$$

$$\begin{aligned} &\leq C \left( \|y_1(t)\|_{L^2(0,1)}^2 + \|y_2(t)\|_{L^2(0,1)}^2 + \|z_1(t)\|_{L^2(0,1)}^2 \right) \times \\ &\quad \left( \|y_1(t) - y_2(t)\|_{L^2(0,1)}^2 + \|z_1(t) - z_2(t)\|_{L^2(0,1)}^2 \right). \end{aligned} \quad (91)$$

We similarly obtain

$$\begin{aligned} &\|r_{1,2}(t)z_1(t) - r_{2,2}(t)z_2(t)\|_{L^2(0,1)}^2 \\ &\leq C \left( \|y_1(t)\|_{L^2(0,1)}^2 + \|z_1(t)\|_{L^2(0,1)}^2 + \|z_2(t)\|_{L^2(0,1)}^2 \right) \times \\ &\quad \left( \|y_1(t) - y_2(t)\|_{L^2(0,1)}^2 + \|z_1(t) - z_2(t)\|_{L^2(0,1)}^2 \right). \end{aligned} \quad (92)$$

Combining the estimates (88), (89), (90), (91) and (92), we obtain from (87), that

$$\begin{aligned} &\|\mathfrak{F}(\xi_1, \eta_1) - \mathfrak{F}(\xi_2, \eta_2)\|_{\mathcal{S} \times \mathcal{S}}^2 \\ &\leq C \int_0^T \frac{1}{\rho_S^2(t)} \left[ \|(y_1(t), z_1(t))\|_{\mathcal{H}}^2 + \|(y_2(t), z_2(t))\|_{\mathcal{H}}^2 \right] \times \\ &\quad \|(y_1(t) - y_2(t), z_1(t) - z_2(t))\|_Z^2 dt \\ &\leq C \int_0^T \frac{\rho_0^4(t)}{\rho_S^2(t)} \left[ \left\| \left( \frac{y_1(t)}{\rho_0(t)}, \frac{z_1(t)}{\rho_0(t)} \right) \right\|_{\mathcal{H}}^2 + \left\| \left( \frac{y_2(t)}{\rho_0(t)}, \frac{z_2(t)}{\rho_0(t)} \right) \right\|_{\mathcal{H}}^2 \right] \times \\ &\quad \left\| \left( \frac{y_1(t) - y_2(t)}{\rho_0(t)}, \frac{z_1(t) - z_2(t)}{\rho_0(t)} \right) \right\|_Z^2 dt \\ &\leq C \left\| \left( \frac{y_1}{\rho_0}, \frac{z_1}{\rho_0} \right) - \left( \frac{y_2}{\rho_0}, \frac{z_2}{\rho_0} \right) \right\|_{C^0([0,T];Z)}^2 \times \\ &\quad \left[ \left\| \left( \frac{y_1}{\rho_0}, \frac{z_1}{\rho_0} \right) \right\|_{L^2(0,T;\mathcal{H})}^2 + \left\| \left( \frac{y_2}{\rho_0}, \frac{z_2}{\rho_0} \right) \right\|_{L^2(0,T;\mathcal{H})}^2 \right], \end{aligned} \quad (93)$$

where we have used the fact that  $\frac{\rho_0^2(t)}{\rho_S(t)} \leq 1$  (see (51)).

But, due to the linearity of the solution map (see Proposition 4.1), we have the following estimate (by means of (58))

$$\begin{aligned} &\left\| \left( \frac{y_1}{\rho_0}, \frac{z_1}{\rho_0} \right) - \left( \frac{y_2}{\rho_0}, \frac{z_2}{\rho_0} \right) \right\|_{C^0([0,T];Z)} + \left\| \left( \frac{y_1}{\rho_0}, \frac{z_1}{\rho_0} \right) - \left( \frac{y_2}{\rho_0}, \frac{z_2}{\rho_0} \right) \right\|_{L^2(0,T;\mathcal{H})} \\ &\leq C e^{CT + \frac{C}{T}} \|(\xi_1, \eta_1) - (\xi_2, \eta_2)\|_{\mathcal{S} \times \mathcal{S}}. \end{aligned}$$

Using the above bound and the estimate (58) in (93), we get

$$\begin{aligned} &\|\mathfrak{F}(\xi_1, \eta_1) - \mathfrak{F}(\xi_2, \eta_2)\|_{\mathcal{S} \times \mathcal{S}} \\ &\leq C e^{CT + \frac{C}{T}} \|(\xi_1, \eta_1) - (\xi_2, \eta_2)\|_{\mathcal{S} \times \mathcal{S}} \times \\ &\quad \left[ \|(y_0, z_0)\|_Z + \|(\xi_1, \eta_1)\|_{\mathcal{S} \times \mathcal{S}} + \|(\xi_2, \eta_2)\|_{\mathcal{S} \times \mathcal{S}} \right] \\ &\leq C e^{CT + \frac{C}{T}} \delta \|(\xi_1, \eta_1) - (\xi_2, \eta_2)\|_{\mathcal{S} \times \mathcal{S}} \\ &\leq \frac{1}{2} \|(\xi_1, \eta_1) - (\xi_2, \eta_2)\|_{\mathcal{S} \times \mathcal{S}}, \end{aligned}$$

for chosen  $0 < \delta \leq \frac{1}{2C e^{CT + \frac{C}{T}}}$ .

This proves the contraction property of the map  $\mathfrak{F}$  in the closed ball  $\mathfrak{B}_\delta$  provided we start with initial data  $\|(y_0, z_0)\|_Z \leq \delta$  and source terms in  $\mathfrak{B}_\delta$ .  $\square$

We now conclude the proof of the main result of our work.

*Proof of Theorem 1.1.* Let any boundary parameter  $\alpha \geq 0$  and time  $T > 0$  be given. According to Lemma 4.2 and Lemma 4.3, there exists some  $\delta > 0$  small enough such that if we choose the initial data  $(y_0, z_0) \in Z$  with  $\|(y_0, z_0)\|_Z \leq \delta$ , then by using Banach fixed point theorem we can ensure that the map  $\mathfrak{F} : \mathfrak{S}_\delta \rightarrow \mathfrak{S}_\delta$  (defined by (78)) has a unique fixed point  $(\hat{\xi}, \hat{\eta}) \in \mathfrak{S}_\delta$ .

At this point, by means of Proposition 4.1, there exists a solution-control pair  $((y, z), q) \in \mathcal{Y} \times \mathcal{Q}$  to the system (48) associated with the above source term  $(\hat{\xi}, \hat{\eta}) \in \mathfrak{S}_\delta$ , which in addition satisfy the estimate (58). Then, by construction of the space  $\mathcal{Y}$  (see (53)) and the property  $\lim_{t \rightarrow T^-} \rho_0(t) = 0$  force the solution  $(y, z)$  to satisfy

$$y(T, x) = 0, \quad z(T, x) = 0, \quad \forall x \in (0, 1),$$

which is the required boundary local null-controllability result of our nonlinear system (1).  $\square$

**5. Concluding remarks.** In the present paper, we study the controllability property of a parabolic system where the boundary couplings are posed in terms of the  $\delta'$ -type condition. The linear model of our work (see (4)) simply consists of the aforementioned boundary couplings, and no internal coupling appears. It would be interesting if one could impose internal coupling(s) as well, for instance let us consider the following linear system,

$$\begin{cases} y_t - y_{xx} + k_1 z = 0, & \text{in } (0, T) \times (0, 1), \\ z_t - z_{xx} + k_2 y = 0, & \text{in } (0, T) \times (0, 1), \\ y_x(t, 0) = q(t), \quad z_x(t, 0) = 0, & \text{in } (0, T), \\ y_x(t, 1) = z_x(t, 1), & \text{in } (0, T), \\ y(t, 1) + z(t, 1) + \alpha y_x(t, 1) = 0, & \text{in } (0, T), \\ y(0, x) = y_0(x), \quad z(0, x) = z_0(x), & \text{in } (0, 1), \end{cases} \quad (94)$$

with some constants  $(k_1, k_2) \neq (0, 0)$ . In this regard, we mention the work [10], where the presence of a zeroth order internal coupling in a parabolic system with Kirchhoff boundary condition leads to different controllability results w.r.t. the position of the boundary control (i.e., on  $y$  or, on  $z$ ). To study the controllability of system (94), the main work will be to investigate the spectral properties of the associated adjoint operator, which is not so obvious in the case of  $\delta'$ -type boundary condition, and it needs further care.

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