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# Homogeneous Sobolev and Besov spaces on special Lipschitz domains and their traces <sup>‡</sup>

Anatole GAUDIN<sup>‡</sup>

May 1, 2023

## Abstract

We propose here to garnish the folklore of function spaces on Lipschitz domains. We prove the boundedness of the trace operator for homogeneous Sobolev and Besov spaces on a special Lipschitz domain with sharp regularity. In order to obtain such a result, we also provide appropriate definitions and properties so that our construction of homogeneous Sobolev and Besov spaces on special Lipschitz domains, and their boundary, that are suitable for the treatment of non-linear partial differential equations and boundary value problems. The trace theorem for homogeneous Sobolev and Besov spaces on special Lipschitz domains occurs in range  $s \in (\frac{1}{p}, 1 + \frac{1}{p})$ . While the case of inhomogeneous Sobolev and Besov spaces is very common and well known, the case of homogeneous function spaces seems to be new. This paper uses and improves several arguments exposed by the author in a previous paper for function spaces on the whole and the half-space.

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# 1 Introduction

## 1.1 Issues for realization of homogeneous function spaces

The use of homogeneous Sobolev and Besov spaces is very important in the treatment of global-in-time well-posedness of partial differential equations in unbounded domains. It has been the case especially in the last two decades, see for instance [BCD11,DM09,DM15,DHMT21,OS16,OS21,OS22] and the references therein.

In the scope of non-linear partial differential equations (even on the whole space), one has to be clear about the definition of homogeneous function spaces. Usually, the elements of homogeneous function spaces, such as homogeneous Sobolev and Besov spaces, are given as equivalence classes of tempered distributions modulo polynomials denoted by  $\mathcal{S}'(\mathbb{R}^n) / \mathbb{C}[x]$ , as in [BL76, Chapter 6, Section 6.3], [Tri83, Chapter 5] or [Saw18, Chapter 2].

However, in that case, it is not clear what would be the actual meaning of homogeneous function spaces on general domains (even smooth), since one cannot perform a composition with the change of coordinates, even in the case of a smooth bent half-space.

Indeed, assuming that  $u + P \in \mathcal{S}'(\mathbb{R}^n)$  is a representative of  $[u] \in \mathcal{S}'(\mathbb{R}^n) / \mathbb{C}[x]$ , and  $\Psi$  is a smooth diffeomorphism of  $\mathbb{R}^n$ , the meaning of

$$u \circ \Psi + P \circ \Psi$$

is not clear. Even if it has one, it should not depend on the choice of  $P$  which is again unclear. It is then a major issue to find a way to transfer properties of homogeneous function spaces from the whole and the half space to a bent one by change of coordinates. In particular, investigation of traces at the boundary seems to be tricky.

Therefore the realization of homogeneous function spaces on special Lipschitz domains provided by Costabel, McIntosh and Taggart [CMT13], built on  $\mathcal{S}'(\mathbb{R}^n) / \mathbb{C}[x]$ , seems to be far from being exploitable for linear problems with boundary values, in addition of being not suitable for non-linear problems.

To circumvent this issue, we are going to use the idea of Bahouri, Chemin and Danchin introduced in [BCD11, Chapter 2]. The idea is to introduce a subspace of  $\mathcal{S}'(\mathbb{R}^n)$  such that we get rid of polynomials, see [BCD11, Examples, p.23]. The aforementioned subspace of  $\mathcal{S}'(\mathbb{R}^n)$  is denoted  $\mathcal{S}'_h(\mathbb{R}^n)$ , see the definition at the beginning of Section 1.3.3. With  $\mathcal{S}'_h(\mathbb{R}^n)$  as an ambient space, Bahouri, Chemin and Danchin gave a construction of homogeneous Besov spaces  $\dot{B}_{p,q}^s(\mathbb{R}^n)$  which are complete whenever  $(s, p, q) \in \mathbb{R} \times (1, +\infty) \times [1, +\infty]$  satisfies

$$\left[ s < \frac{n}{p} \right] \text{ or } \left[ q = 1 \text{ and } s \leq \frac{n}{p} \right].$$

Later, this has also led Danchin and Mucha to consider homogeneous Besov spaces on  $\mathbb{R}_+^n$  and on exterior domains, see [DM09,DM15], and Danchin, Hieber, Mucha and Tolksdorf [DHMT21] to consider homogeneous Sobolev spaces  $H^{m,p}$  on  $\mathbb{R}^n$  and  $\mathbb{R}_+^n$ , for  $m \in \mathbb{N}$ ,  $p \in (1, +\infty)$ .

This construction was then extended by the author in the previous work [Gau22] and covers the whole scale of homogeneous Sobolev spaces in the reflexive range on the half space  $\mathbb{R}_+^n$ , with an investigation of interpolation properties, and meaning of traces at the boundary. A review of properties of homogeneous Besov spaces on the half-space has been done too.

In order to be able to give a suitable meaning of traces, we will give a construction of homogeneous Sobolev and Besov spaces on special Lipschitz domains, by improvements of some arguments presented in [Gau22, Section 2]. The structure of extension and projection operators used here induce the consideration of two families of regularity indices:  $(-1 + \frac{1}{p}, 1]$  and  $[0, +\infty)$  with common overlap  $[0, 1]$ . We also note that the naive argument presented for [Gau22, Proposition 2.16] is no longer available for the non-negative family of regularity indices.

## 1.2 Trace theorems

Trace theorems with sharp regularity are fundamental to study boundary value problems in the field of partial differential equations. The usual theorem for traces of Besov or Sobolev functions on  $\mathbb{R}_+^n = \mathbb{R}^{n-1} \times (0, +\infty)$  and on bounded and sufficiently regular domains can be found in [BL76, Subection 6.6], [AF03, Theorem 7.43, Remark 7.45], [Sch10, Theorems 3.16 & 3.19]. A very general result for traces on subsets of  $\mathbb{R}^n$  with minimal geometric assumptions can be found in [JW84, Chapters VI & VII]. The usual trace theorem for Lipschitz domains can be stated as follows.

**Theorem 1.1** *Let  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$ ,  $s \in (\frac{1}{p}, 1 + \frac{1}{p})$ , and  $\Omega$  be either a special or a bounded Lipschitz domain,*

(i) *the trace operator  $[\cdot]_{|\partial\Omega} : H^{s,p}(\Omega) \longrightarrow B_{p,p}^{s-\frac{1}{p}}(\partial\Omega)$  is a bounded surjection, in particular for all  $u \in H^{s,p}(\Omega)$ ,*

$$\|u|_{\partial\Omega}\|_{B_{p,p}^{s-\frac{1}{p}}(\partial\Omega)} \lesssim_{s,p,n} \|u\|_{H^{s,p}(\Omega)};$$

(ii) *the trace operator  $[\cdot]_{|\partial\Omega} : B_{p,q}^s(\Omega) \longrightarrow B_{p,q}^{s-\frac{1}{p}}(\partial\Omega)$  is a bounded surjection, in particular for all  $u \in B_{p,q}^s(\Omega)$ ,*

$$\|u|_{\partial\Omega}\|_{B_{p,q}^{s-\frac{1}{p}}(\partial\Omega)} \lesssim_{s,p,n} \|u\|_{B_{p,q}^s(\Omega)};$$

(iii) *the trace operator  $[\cdot]_{|\partial\Omega} : B_{p,1}^{\frac{1}{p}}(\partial\Omega) \longrightarrow L^p(\partial\Omega)$  is a bounded surjection, in particular for all  $u \in B_{p,1}^{\frac{1}{p}}(\Omega)$ ,*

$$\|u|_{\partial\Omega}\|_{L^p(\partial\Omega)} \lesssim_{p,n} \|u\|_{B_{p,1}^{\frac{1}{p}}(\Omega)};$$

*Moreover, the trace operator  $[\cdot]_{|\partial\Omega}$  admits a right bounded inverse for each of the above cases.*

Roughly speaking, the goal here is, up to technical modifications, to add dots on every H and B symbols in Theorem 1.1 in the case of special Lipschitz domains. We take a focus on special Lipschitz domains for two main reasons. First, on a bounded Lipschitz domain the localisation aspects induce that there are not that much differences between inhomogeneous and homogeneous function spaces, one may think about Poincaré-Wirtinger-Sobolev type inequalities. In a second time, the class of special Lipschitz domains seems to be the only suitable class of domains that admits good extension operators with homogeneous estimates at the moment. For more general unbounded Lipschitz, or smoother, domains one cannot reach homogeneous function spaces by localisation with smooth cut-off since this procedure completely destroys the homogeneity.

The reader must know that in the case of inhomogeneous function spaces, one may find simpler proofs for the existence of traces seeing the trace operator as a (compact) operator with value in  $L^p(\partial\Omega)$  (when  $\Omega$  has compact boundary). Similar results are also available for partial traces of vector fields, even with compactness property in the case of compact boundary, e.g. see [Mon15, Den21] and the references therein. A quite general result for partial traces on bounded Lipschitz domains is achieved by Mitrea, Mitrea and Shaw in [MMS08, Section 4] for differential forms, containing the result for vector fields.

Theorem 1.1 and fine properties of simple and double layer potentials were extensively used to study regularity and well-posedness of elliptic boundary value problems and deduce some functional analytic properties of involved elliptic operators (see e.g. [JK95, FMM98, MM01, MMT01]).

However, if one wants to look at boundary value problems in unbounded domains, such as half-spaces and in particular special Lipschitz domains, we may have a lack of control on derivatives of lower order. In this case, homogeneous estimates are required, then the use of homogeneous Sobolev and Besov spaces,  $\dot{H}^{s,p}(\Omega)$  and  $\dot{B}_{p,q}^s(\Omega)$ . One may see that in the case of the flat upper half-space (or the whole space with trace in a hyper plane), that the trace theorem still holds, see for instance [Jaw78]. But in this case, the usual definition of such function spaces is given by restriction of tempered distributions modulo polynomials, which is not convenient to adapt the usual strategy

which requires point-wise composition in order to flatten the boundary. For a suitable realization of homogeneous function spaces on the flat upper half-space, the expected results are easily obtained by the author in [Gau22, Section 3].

The goal of this paper is to give a proof of the usual trace theorem for scalar-valued homogeneous Sobolev and Besov spaces on special Lipschitz domains. In order to prove the homogeneous version of Theorem 1.1, we aim to follow the strategy exhibited in [Din96], and initially described in [Cos88]. However, the use of the Fourier transform and the overall strategy restrict everything to the case of inhomogeneous  $L^2$ -based Sobolev spaces. The idea we present here is to use the global-in-time  $\dot{H}^{s,p}(L^p)$ -maximal regularity for the Poisson semigroup on  $\mathbb{R}^{n-1}$  and interpolation theory to replace the use of  $L^2$  techniques.

### 1.3 Notations, definitions and review of usual concepts

Throughout this paper the dimension will be  $n \geq 2$ , and  $\mathbb{N}$  will be the set of non-negative integers. For  $a, b \in \mathbb{R}$  with  $a \leq b$ , we write  $\llbracket a, b \rrbracket := [a, b] \cap \mathbb{Z}$ .

For two real numbers  $A, B \in \mathbb{R}$ ,  $A \lesssim_{a,b,c} B$  means that there exists a constant  $C > 0$  depending on  $a, b, c$  such that  $A \leq CB$ . When both  $A \lesssim_{a,b,c} B$  and  $B \lesssim_{a,b,c} A$  are true, we simply write  $A \sim_{a,b,c} B$ . When the number of indices is overloaded, we allow ourselves to write  $A \lesssim_{a,b,c}^{d,e,f} B$  instead of  $A \lesssim_{a,b,c,d,e,f} B$ .

#### 1.3.1 Smooth and measurable functions on open sets

Denote by  $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$  the space of complex valued Schwartz function, and  $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$  its dual called the space of tempered distributions. The Fourier transform on  $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$  is written  $\mathcal{F}$ , and is pointwise defined for any  $f \in L^1(\mathbb{R}^n, \mathbb{C})$  by

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^n.$$

Additionally, for  $p \in [1, +\infty]$ , we will write  $p' = \frac{p}{p-1}$  its *Hölder conjugate*.

For any  $m \in \mathbb{N}$ , the map  $\nabla^m : \mathcal{S}'(\mathbb{R}^n, \mathbb{C}) \rightarrow \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^{n^m})$  is defined as  $\nabla^m u := (\partial^\alpha u)_{|\alpha|=m}$ .

We denote by  $(e^{-t(-\Delta)^{\frac{1}{2}}})_{t \geq 0}$  respectively the Poisson semigroup on  $\mathbb{R}^n$ . We also introduce the operator  $\nabla'$  which stands for the gradient on  $\mathbb{R}^{n-1}$  identified with the  $n-1$  first variables of  $\mathbb{R}^n$ , i.e.  $\nabla' = (\partial_{x_1}, \dots, \partial_{x_{n-1}})$ .

When  $\Omega$  is an open set of  $\mathbb{R}^n$ ,  $C_c^\infty(\Omega, \mathbb{C})$  is the set of smooth compactly supported function in  $\Omega$ , and  $\mathcal{D}'(\Omega, \mathbb{C})$  is its topological dual. For  $p \in [1, +\infty)$ ,  $L^p(\Omega, \mathbb{C})$  is the normed vector space of complex valued (Lebesgue-) measurable functions whose  $p$ -th power is integrable with respect to the Lebesgue measure,  $\mathcal{S}(\overline{\Omega}, \mathbb{C})$  (resp.  $C_c^\infty(\overline{\Omega}, \mathbb{C})$ ) stands for functions which are restrictions on  $\Omega$  of elements of  $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$  (resp.  $C_c^\infty(\mathbb{R}^n, \mathbb{C})$ ). Unless the contrary is explicitly stated, we will always identify  $L^p(\Omega, \mathbb{C})$  (resp.  $C_c^\infty(\Omega, \mathbb{C})$ ) as the subspace of function in  $L^p(\mathbb{R}^n, \mathbb{C})$  (resp.  $C_c^\infty(\mathbb{R}^n, \mathbb{C})$ ) supported in  $\overline{\Omega}$  through the extension by 0 outside  $\Omega$ .  $L^\infty(\Omega, \mathbb{C})$  stands for the space of essentially bounded (Lebesgue-) measurable functions.

For  $s \in \mathbb{R}$ ,  $p \in [1, +\infty)$ ,  $\ell_s^p(\mathbb{Z}, \mathbb{C})$ , stands for the normed vector space of  $p$ -summable sequences of complexes numbers with respect to the counting measure  $2^{ksp} dk$ ;  $\ell_s^\infty(\mathbb{Z}, \mathbb{C})$  stands for sequences  $(x_k)_{k \in \mathbb{Z}}$  such that  $(2^{ks} x_k)_{k \in \mathbb{Z}}$  is bounded. More generally, when  $X$  is a Banach space, for  $p \in [1, +\infty]$ , one may also consider  $L^p(\Omega, X)$  which stands for the space of (Bochner-)measurable functions  $u : \Omega \rightarrow X$ , such that  $t \mapsto \|u(t)\|_X \in L^p(\Omega, \mathbb{R})$ , similarly one may consider  $\ell_s^p(\mathbb{Z}, X)$ . Finally,  $C^0(\Omega, X)$  stands for the space of continuous functions on  $\Omega \subset \mathbb{R}^n$  with values in  $X$ . The subspace  $C_b^0(\mathbb{R}, X)$  is made of uniformly bounded continuous functions and is the set  $C_0^0(\mathbb{R}, X)$  of continuous functions that vanish at infinity. For  $\mathcal{C} \in \{C^0, C_b^0, C_0^0, C_c^\infty\}$ , we set  $\mathcal{C}(\overline{\Omega}, X)$  to be the set of continuous functions on  $\overline{\Omega}$  which are restrictions of elements that belongs to  $\mathcal{C}(\mathbb{R}^n, X)$ .

For  $s \in \mathbb{R}$ ,  $p \in [1, +\infty)$ ,  $\ell_s^p(\mathbb{Z}, \mathbb{C})$ , stands for the normed vector space of  $p$ -summable sequences of complexes numbers with respect to the counting measure  $2^{ksp} dk$ ;  $\ell_s^\infty(\mathbb{Z}, \mathbb{C})$  stands for sequences  $(x_k)_{k \in \mathbb{Z}}$  such that  $(2^{ks} x_k)_{k \in \mathbb{Z}}$  is bounded. More generally, when  $X$  is a Banach space, for  $p \in [1, +\infty]$ , one may also consider  $L^p(\Omega, X)$  which stands for the space of (Bochner-)measurable

functions  $u : \Omega \rightarrow X$ , such that  $t \mapsto \|u(t)\|_X \in L^p(\Omega, \mathbb{R})$ , similarly one may consider  $\ell_s^p(\mathbb{Z}, X)$ . Finally,  $C^0(\Omega, X)$  stands for the space of continuous functions on  $\Omega \subset \mathbb{R}^n$  with values in  $X$ . The subspace  $C_b^0(\mathbb{R}, X)$  is made of uniformly bounded continuous functions and is the set  $C_0^0(\mathbb{R}, X)$  of continuous functions that vanish at infinity. For  $\mathcal{C} \in \{C^0, C_b^0, C_0^0\}$ , we set  $\mathcal{C}(\overline{\Omega}, X)$  to be the set of continuous functions on  $\overline{\Omega}$  which are restrictions of elements that belongs to  $\mathcal{C}(\mathbb{R}^n, X)$ .

For  $\Omega$  an open set of  $\mathbb{R}^n$ , we say that  $\Omega$  is a special Lipschitz domain, if there exists, up to a rotation, a globally Lipschitz function  $\phi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ , such that

$$\Omega = \{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n > \phi(x') \}.$$

In other words, a special Lipschitz domain of  $\mathbb{R}^n$  is the epigraph of real valued Lipschitz function defined on  $\mathbb{R}^{n-1}$ .

### 1.3.2 Interpolation of normed vector spaces

Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be two normed vector spaces. We write  $X \hookrightarrow Y$  to say that  $X$  embeds continuously into  $Y$ . Now let us recall briefly basics of interpolation theory. If there exists a Hausdorff topological vector space  $Z$ , such that  $X, Y \subset Z$ , then  $X \cap Y$  and  $X + Y$  are normed vector spaces with their canonical norms, and one can define the  $K$ -functional of  $z \in X + Y$ , for any  $t > 0$  by

$$K(t, z, X, Y) := \inf_{\substack{(x,y) \in X \times Y, \\ z=x+y}} (\|x\|_X + t\|y\|_Y).$$

This allows us to construct, for any  $\theta \in (0, 1)$ ,  $q \in [1, +\infty]$ , the real interpolation spaces between  $X$  and  $Y$  with indexes  $\theta, q$  as

$$(X, Y)_{\theta, q} := \left\{ x \in X + Y \mid t \mapsto t^{-\theta} K(t, x, X, Y) \in L_*^q(\mathbb{R}_+) \right\},$$

where  $L_*^q(\mathbb{R}_+) := L^q((0, +\infty), dt/t)$ . The interested reader could check [Lun18, Chapter 1], [BL76, Chapter 3] for more informations about real interpolation and its applications.

If moreover we assume that  $X$  and  $Y$  are complex Banach spaces, one can consider  $F(X, Y)$  the set of all continuous functions  $f : \overline{S} \rightarrow X + Y$ ,  $S$  being the strip of complex numbers whose real part is between 0 and 1, with  $f$  holomorphic in  $S$ , and such that

$$t \mapsto f(it) \in C_b^0(\mathbb{R}, X) \quad \text{and} \quad t \mapsto f(1+it) \in C_b^0(\mathbb{R}, Y).$$

We can endow the space  $F(X, Y)$  with the norm

$$\|f\|_{F(X, Y)} := \max \left( \sup_{t \in \mathbb{R}} \|f(it)\|_X, \sup_{t \in \mathbb{R}} \|f(1+it)\|_Y \right),$$

which makes  $F(X, Y)$  a Banach space since it is a closed subspace of  $C^0(\overline{S}, X + Y)$ . Hence for  $\theta \in (0, 1)$ , the normed vector space given by

$$\begin{aligned} [X, Y]_\theta &:= \{ f(\theta) \mid f \in F(X, Y) \}, \\ \|x\|_{[X, Y]_\theta} &:= \inf_{\substack{f \in F(X, Y), \\ f(\theta)=x}} \|f\|_{F(X, Y)}, \end{aligned}$$

is a Banach space called the complex interpolation space between  $X$  and  $Y$  associated with  $\theta$ . Again, the interested reader could check [Lun18, Chapter 2], [BL76, Chapter 4] for more informations about complex interpolation and its applications.

### 1.3.3 Sobolev and Besov spaces on the whole space

To deal with Sobolev and Besov spaces on the whole space, we need to introduce Littlewood-Paley decomposition given by  $\varphi \in C_c^\infty(\mathbb{R}^n)$ , radial, real-valued, non-negative, such that

- $\text{supp } \varphi \subset B(0, 4/3)$ ;
- $\varphi|_{B(0, 3/4)} = 1$ ;

so we define the following functions for any  $j \in \mathbb{Z}$  for all  $\xi \in \mathbb{R}^n$ ,

$$\varphi_j(\xi) := \varphi(2^{-j}\xi), \quad \psi_j(\xi) := \varphi_j(\xi/2) - \varphi_j(\xi),$$

and the family  $(\psi_j)_{j \in \mathbb{Z}}$  has the following properties

- $\text{supp}(\psi_j) \subset \{\xi \in \mathbb{R}^n \mid 3 \cdot 2^{j-2} \leq |\xi| \leq 2^{j+3}/3\}$ ;
- $\forall \xi \in \mathbb{R}^n \setminus \{0\}, \sum_{j=-M}^N \psi_j(\xi) \xrightarrow{N, M \rightarrow +\infty} 1$ .

Such a family  $(\varphi, (\psi_j)_{j \in \mathbb{Z}})$  is called a Littlewood-Paley family. Now, we consider the two following families of operators associated with their Fourier multipliers :

- The **homogeneous** family of Littlewood-Paley dyadic decomposition operators  $(\dot{\Delta}_j)_{j \in \mathbb{Z}}$ , where

$$\dot{\Delta}_j := \mathcal{F}^{-1} \psi_j \mathcal{F},$$

- The **inhomogeneous** family of Littlewood-Paley dyadic decomposition operators  $(\Delta_k)_{k \in \mathbb{Z}}$ , where

$$\Delta_{-1} := \mathcal{F}^{-1} \varphi \mathcal{F},$$

$$\Delta_k := \dot{\Delta}_k \text{ for any } k \geq 0, \text{ and } \Delta_k := 0 \text{ for any } k \leq -2.$$

One may notice, as a direct application of Young's inequality for the convolution, that they are all uniformly bounded families of operators on  $L^p(\mathbb{R}^n)$ ,  $p \in [1, +\infty]$ .

Both family of operators lead for  $s \in \mathbb{R}$ ,  $p, q \in [1, +\infty]$ ,  $u \in \mathcal{S}'(\mathbb{R}^n)$  to the following quantities,

$$\|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} = \left\| (2^{ks} \|\Delta_k u\|_{L^p(\mathbb{R}^n)})_{k \in \mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} \quad \text{and} \quad \|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} = \left\| (2^{js} \|\dot{\Delta}_j u\|_{L^p(\mathbb{R}^n)})_{j \in \mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})},$$

respectively named the inhomogeneous and homogeneous Besov norms, but the homogeneous norm is not really a norm since  $\|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} = 0$  does not imply that  $u = 0$ . Thus, following [BCD11, Chapter 2] and [DHMT21, Chapter 3], we introduce a subspace of tempered distributions such that  $\|\cdot\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)}$  is point-separating, say

$$\mathcal{S}'_h(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \forall \Theta \in C_c^\infty(\mathbb{R}^n), \|\Theta(\lambda \mathcal{D})u\|_{L^\infty(\mathbb{R}^n)} \xrightarrow{\lambda \rightarrow +\infty} 0 \right\},$$

where for  $\lambda > 0$ ,  $\Theta(\lambda \mathcal{D})u = \mathcal{F}^{-1} \Theta(\lambda \cdot) \mathcal{F} u$ . Notice that  $\mathcal{S}'_h(\mathbb{R}^n)$  does not contain any polynomials, and for any  $p \in [1, +\infty)$ ,  $L^p(\mathbb{R}^n) \subset \mathcal{S}'_h(\mathbb{R}^n)$  so that  $\|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} = 0$  does imply that  $u = 0$  when  $u \in \mathcal{S}'_h(\mathbb{R}^n)$ .

One can also define the following quantities called the inhomogeneous and homogeneous Sobolev spaces' potential norms

$$\|u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} := \|(I - \Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^n)} \quad \text{and} \quad \|u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} := \left\| \sum_{j \in \mathbb{Z}} (-\Delta)^{\frac{s}{2}} \dot{\Delta}_j u \right\|_{L^p(\mathbb{R}^n)},$$

where  $(-\Delta)^{\frac{s}{2}}$  is understood on  $u \in \mathcal{S}'_h(\mathbb{R}^n)$  by the action on its dyadic decomposition, *i.e.*

$$(-\Delta)^{\frac{s}{2}} \dot{\Delta}_j u := \mathcal{F}^{-1} |\xi|^s \mathcal{F} \dot{\Delta}_j u,$$

which gives a family of  $C^\infty$  functions with at most polynomial growth.

Hence for any  $p, q \in [1, +\infty]$ ,  $s \in \mathbb{R}$ , we define

- the inhomogeneous and homogeneous Sobolev (Bessel and Riesz potential) spaces,  $\dot{H}^{s,p}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'_h(\mathbb{R}^n) \mid \|u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} < +\infty \right\}$ ;
- and the inhomogeneous and homogeneous Besov spaces,  $\dot{B}_{p,q}^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'_h(\mathbb{R}^n) \mid \|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} < +\infty \right\}$ ,

which are all normed vector spaces.

The treatment of homogeneous Besov spaces  $\dot{B}_{p,q}^s(\mathbb{R}^n)$ ,  $s \in \mathbb{R}$ ,  $p, q \in [1, +\infty]$ , defined on  $\mathcal{S}'_h(\mathbb{R}^n)$  has been done in an extensive manner in [BCD11, Chapter 2]. However, the corresponding construction for homogeneous Sobolev spaces  $\dot{H}^{s,p}(\mathbb{R}^n)$ ,  $s \in \mathbb{R}$ ,  $p \in (1, +\infty)$ . See [BCD11, Chapter 1] for the case  $p = 2$ , [DHMT21, Chapter 3] for the case  $s \in \mathbb{N}$ , [Gau22, Subsection 2.1] for the case  $s \in \mathbb{R}$ .

The following subspace of Schwartz functions, say

$$\mathcal{S}_0(\mathbb{R}^n) := \{ u \in \mathcal{S}(\mathbb{R}^n) \mid 0 \notin \text{supp}(\mathcal{F}f) \},$$

is a nice dense subspace in  $L^p(\mathbb{R}^n)$ ,  $H^{s,p}(\mathbb{R}^n)$ ,  $\dot{H}^{s,p}(\mathbb{R}^n)$ ,  $B_{p,q}^s(\mathbb{R}^n)$  and  $\dot{B}_{p,q}^s(\mathbb{R}^n)$ , for all  $p \in (1, +\infty)$ ,  $q \in [1, +\infty)$ ,  $s \in \mathbb{R}$ .

The inhomogeneous spaces  $L^p(\mathbb{R}^n)$ ,  $H^{s,p}(\mathbb{R}^n)$ , and  $B_{p,q}^s(\mathbb{R}^n)$  are all complete for all  $p, q \in [1, +\infty]$ ,  $s \in \mathbb{R}$ , but in this setting homogenous spaces are no longer always complete (see [BCD11, Proposition 1.34, Remark 2.26]). Indeed, it can be shown (see [BCD11, Theorem 2.25]) that homogeneous Besov spaces  $\dot{B}_{p,q}^s(\mathbb{R}^n)$  are complete whenever  $(s, p, q) \in \mathbb{R} \times (1, +\infty) \times [1, +\infty]$  satisfies

$$\left[ s < \frac{n}{p} \right] \text{ or } \left[ q = 1 \text{ and } s \leq \frac{n}{p} \right], \quad (\mathcal{C}_{s,p,q})$$

From now, and until the end of this paper, we write  $(\mathcal{C}_{s,p})$  for the statement  $(\mathcal{C}_{s,p,p})$ . One may show that, similarly,  $\dot{H}^{s,p}(\mathbb{R}^n)$  is complete whenever  $(\mathcal{C}_{s,p})$  is satisfied, see [Gau22, Proposition 2.4].

We recall that all  $s > 0$ ,  $(p, q) \in (1, +\infty) \times [1, +\infty]$ , we have  $L^p(\mathbb{R}^n) \cap \dot{H}^{s,p}(\mathbb{R}^n) = H^{s,p}(\mathbb{R}^n)$ , and  $L^p(\mathbb{R}^n) \cap \dot{B}_{p,q}^s(\mathbb{R}^n) = B_{p,q}^s(\mathbb{R}^n)$  with equivalent norms, see [BL76, Theorem 6.3.2] for more details.

**Proposition 1.2** ( [Gau22, Proposition 2.11] ) *For any  $s \in \mathbb{R}$ ,  $p \in (1, +\infty)$ ,*

$$\begin{cases} \dot{H}^{s,p} \times \dot{H}^{-s,p'} & \longrightarrow \mathbb{C} \\ (u, v) & \longmapsto \sum_{|j-j'| \leq 1} \langle \dot{\Delta}_j u, \dot{\Delta}_{j'} v \rangle_{\mathbb{R}^n} \end{cases}$$

*defines a continuous bilinear functional on  $\dot{H}^{s,p}(\mathbb{R}^n) \times \dot{H}^{-s,p'}(\mathbb{R}^n)$ . Denote by  $\mathcal{V}^{-s,p'}$  the set of functions  $v \in \mathcal{S}(\mathbb{R}^n) \cap \dot{H}^{-s,p'}(\mathbb{R}^n)$  such that  $\|v\|_{\dot{H}^{-s,p'}(\mathbb{R}^n)} \leq 1$ . If  $u \in \mathcal{S}'_h(\mathbb{R}^n)$ , then we have*

$$\|u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} = \sup_{v \in \mathcal{V}^{-s,p'}} |\langle u, v \rangle_{\mathbb{R}^n}|.$$

*Moreover, if  $(\mathcal{C}_{s,p})$  is satisfied,  $\dot{H}^{s,p}(\mathbb{R}^n)$  is reflexive and we have*

$$(\dot{H}^{-s,p'}(\mathbb{R}^n))' = \dot{H}^{s,p}(\mathbb{R}^n). \quad (1.1)$$

For the same result in the case of homogeneous Besov spaces, see [BCD11, Proposition 2.29].

We recall also the usual interpolation properties.

**Proposition 1.3** ( [Gau22, Proposition 2.10] ) *Let  $(p_0, p_1, p, q, q_0, q_1) \in (1, +\infty)^3 \times [1, +\infty]^3$ ,  $s_0, s_1 \in \mathbb{R}$ , such that  $s_0 \neq s_1$ , and let*

$$\left( s, \frac{1}{p_\theta}, \frac{1}{q_\theta} \right) := (1 - \theta) \left( s_0, \frac{1}{p_0}, \frac{1}{q_0} \right) + \theta \left( s_1, \frac{1}{p_1}, \frac{1}{q_1} \right).$$

*Assuming  $(\mathcal{C}_{s_0,p_0})$  (resp.  $(\mathcal{C}_{s_0,p_0,q_0})$ ), we get the following*

$$(\dot{H}^{s_0,p_0}(\mathbb{R}^n), \dot{H}^{s_1,p_1}(\mathbb{R}^n))_{\theta,q} = (\dot{B}_{p_0,q_0}^{s_0}(\mathbb{R}^n), \dot{B}_{p_1,q_1}^{s_1}(\mathbb{R}^n))_{\theta,q} = \dot{B}_{p,q}^s(\mathbb{R}^n). \quad (1.2)$$

*If moreover  $(\mathcal{C}_{s_0,p_0})$  and  $(\mathcal{C}_{s_1,p_1})$  are true then also is  $(\mathcal{C}_{s,p_\theta})$  and*

$$[\dot{H}^{s_0,p_0}(\mathbb{R}^n), \dot{H}^{s_1,p_1}(\mathbb{R}^n)]_\theta = \dot{H}^{s,p_\theta}(\mathbb{R}^n), \quad (1.3)$$

*and similarly if  $(\mathcal{C}_{s_0,p_0,q_0})$  and  $(\mathcal{C}_{s_1,p_1,q_1})$  are satisfied then  $(\mathcal{C}_{s,p_\theta,q_\theta})$  is also satisfied and*

$$[\dot{B}_{p_0,q_0}^{s_0}(\mathbb{R}^n), \dot{B}_{p_1,q_1}^{s_1}(\mathbb{R}^n)]_\theta = \dot{B}_{p_\theta,q_\theta}^s(\mathbb{R}^n). \quad (1.4)$$

**Proposition 1.4** ( [Gau22, Proposition 2.15] ) *For all  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$ , for all  $s \in$*



$(-1 + \frac{1}{p}, \frac{1}{p})$ , for all  $u \in \dot{H}^{s,p}(\mathbb{R}^n)$  (resp.  $\dot{B}_{p,q}^s(\mathbb{R}^n)$ ),

$$\|1_{\mathbb{R}_+^n} u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \quad (\text{resp. } \|1_{\mathbb{R}_+^n} u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} \lesssim_{s,p,n} \|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)}).$$

The same result still holds with  $(H, B)$  instead of  $(\dot{H}, \dot{B})$ .

We also have the useful and very well known equivalence with the Triebel-Lizorkin norm for Riesz potential spaces.

**Lemma 1.5** ( [Gau22, Lemma 2.7] ) For all  $s \in \mathbb{R}$ ,  $p \in (1, +\infty)$ ,  $u \in \mathcal{S}'_h(\mathbb{R}^n)$ ,

$$\|(\dot{\Delta}_j u)_{j \in \mathbb{Z}}\|_{L^p(\mathbb{R}^n, \ell^2_s(\mathbb{Z}))} \sim_{p,n,s} \|u\|_{\dot{H}^{s,p}(\mathbb{R}^n)}.$$

## 2 Sobolev and Besov spaces on special Lipschitz domains

### 2.1 Function spaces by restriction

Let  $s \in \mathbb{R}$ ,  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$  and  $\Omega$  an open set of  $\mathbb{R}^n$ . For any  $X \in \{B_{p,q}^s, \dot{B}_{p,q}^s, H^{s,p}, \dot{H}^{s,p}\}$ , and we define

$$X(\Omega) := X(\mathbb{R}^n)|_{\Omega},$$

with the quotient norm  $\|u\|_{X(\Omega)} := \inf_{\substack{\tilde{u} \in X(\mathbb{R}^n), \\ \tilde{u}|_{\Omega} = u}} \|\tilde{u}\|_{X(\mathbb{R}^n)}$ . A direct consequence of the definition of those

spaces is the density of  $\mathcal{S}_0(\overline{\Omega}) \subset \mathcal{S}(\overline{\Omega})$  in each of them, and the completeness and reflexivity when their counterpart on  $\mathbb{R}^n$  are complete and reflexive. We can also define

$$X_0(\Omega) := \left\{ u \in X(\mathbb{R}^n) \mid \text{supp } u \subset \overline{\Omega} \right\},$$

with its natural induced norm  $\|u\|_{X_0(\Omega)} := \|u\|_{X(\mathbb{R}^n)}$ . We always have the canonical continuous injection,

$$X_0(\Omega) \hookrightarrow X(\Omega).$$

Since there is a natural embedding  $\mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\Omega)$ , we also have the inclusion

$$X(\Omega) \subset \mathcal{D}'(\Omega),$$

where  $\mathcal{D}'(\Omega) = (C_c^\infty(\Omega))'$  is the topological vector space of distributions on  $\Omega$ .

If  $X$  and  $Y$  are different function spaces

- if one has a continuous embedding

$$Y(\mathbb{R}^n) \hookrightarrow X(\mathbb{R}^n).$$

A direct consequence from the definition is that

$$Y(\Omega) \hookrightarrow X(\Omega),$$

and similarly with  $X_0$  and  $Y_0$ .

- We write  $[X \cap Y](\Omega)$  the restriction of  $X(\mathbb{R}^n) \cap Y(\mathbb{R}^n)$  to  $\Omega$ , in general there is nothing to ensure more than

$$[X \cap Y](\Omega) \hookrightarrow X(\Omega) \cap Y(\Omega).$$

The results corresponding to those obtained for the whole space  $\mathbb{R}^n$  in the previous section are usually carried over by the existence of an appropriate extension operator

$$\mathcal{E} : \mathcal{S}'(\Omega) \longrightarrow \mathcal{S}'(\mathbb{R}^n),$$

bounded from  $X(\Omega)$  to  $X(\mathbb{R}^n)$ .

By mean of Proposition 1.3, as for [DHMT21, Proposition 3.22] and [Gau22, Lemma 2.29], the definition of function spaces by restriction yields the next result.

**Lemma 2.1** Let  $(p, q, q_0, q_1) \in (1, +\infty) \times [1, +\infty]^3$ ,  $s_0, s_1 \in \mathbb{R}$ , such that  $s_0 < s_1$ , and set

$$s := (1 - \theta)s_0 + \theta s_1.$$

If  $(\mathcal{C}_{s_0, p})$  is satisfied, we have continuous embeddings

$$\dot{B}_{p, q}^s(\Omega) \hookrightarrow (\dot{H}^{s_0, p}(\Omega), \dot{H}^{s_1, p}(\Omega))_{\theta, q}, \quad (2.1)$$

$$\dot{B}_{p, q, 0}^s(\Omega) \hookrightarrow (\dot{H}_0^{s_0, p}(\Omega), \dot{H}_0^{s_1, p}(\Omega))_{\theta, q}, \quad (2.2)$$

Similarly if  $(\mathcal{C}_{s_0, p, q_0})$  is satisfied, we also have

$$\dot{B}_{p, q}^s(\Omega) \hookrightarrow (\dot{B}_{p, q_0}^{s_0}(\Omega), \dot{B}_{p, q_1}^{s_1}(\Omega))_{\theta, q}, \quad (2.3)$$

$$\dot{B}_{p, q, 0}^s(\Omega) \hookrightarrow (\dot{B}_{p, q_0, 0}^{s_0}(\Omega), \dot{B}_{p, q_1, 0}^{s_1}(\Omega))_{\theta, q}. \quad (2.4)$$

## 2.2 Quick overview of inhomogeneous function spaces on (special) Lipschitz domains

This subsection is here in order to recall few selected facts about inhomogeneous function spaces on Lipschitz domains. A substantial part of the presented results is used in next sections to carry over the corresponding ones for the homogeneous scales of function spaces. One may also see this subsection as guide lines for the results we aim to reproduce. We follow closely the presentation given in [Gau22, Section 2.2].

For a suitable extension operator in the case of inhomogeneous function spaces on a (special) Lipschitz domain, a notable approach was achieved by Stein in [Ste70, Chapter VI, Section 3], for Sobolev spaces with non-negative index, and Besov spaces of positive index of regularity (this follows by real interpolation). A full and definitive result for the inhomogeneous case on Lipschitz domains, and even in a more general case (allowing  $p, q$  to be less than 1 considering the whole Besov and Triebel-Lizorkin scales), was given by Rychkov in [Ryc99] where the extension operator is known to be universal and to cover even negative regularity index.

The extension operator provided by Rychkov can be used to prove, thanks to [BL76, Theorem 6.4.2], if  $(\mathfrak{h}, \mathfrak{b}) \in \{(H, B), (H_0, B_{\cdot, \cdot, 0})\}$ ,

$$[\mathfrak{h}^{s_0, p_0}(\Omega), \mathfrak{h}^{s_1, p_1}(\Omega)]_{\theta} = \mathfrak{h}^{s, p_{\theta}}(\Omega), \quad (\mathfrak{b}_{p_0, q_0}^{s_0}(\Omega), \mathfrak{b}_{p_1, q_1}^{s_1}(\Omega))_{\theta, q} = \mathfrak{b}_{p, q}^s(\Omega), \quad (2.5)$$

$$(\mathfrak{h}^{s_0, p}(\Omega), \mathfrak{h}^{s_1, p}(\Omega))_{\theta, q} = \mathfrak{b}_{p, q}^s(\Omega), \quad [\mathfrak{b}_{p_0, q_0}^{s_0}(\Omega), \mathfrak{b}_{p_1, q_1}^{s_1}(\Omega)]_{\theta} = \mathfrak{b}_{p_{\theta}, q_{\theta}}^s(\Omega), \quad (2.6)$$

whenever  $(p_0, q_0), (p_1, q_1), (p, q) \in [1, +\infty]^2$  ( $p \neq 1, +\infty$ , when dealing with Sobolev (Bessel potential) spaces),  $\theta \in (0, 1)$ ,  $s_0 \neq s_1$  two real numbers, such that

$$\left(s, \frac{1}{p_{\theta}}, \frac{1}{q_{\theta}}\right) := (1 - \theta) \left(s_0, \frac{1}{p_0}, \frac{1}{q_0}\right) + \theta \left(s_1, \frac{1}{p_1}, \frac{1}{q_1}\right).$$

A nice property is that the description of the boundary yields the following density results, for all  $p \in (1, +\infty)$ ,  $q \in [1, +\infty)$ ,  $s \in \mathbb{R}$ ,

$$H_0^{s, p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^{s, p}(\mathbb{R}^n)}}, \quad \text{and} \quad B_{p, q, 0}^s(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{B_{p, q}^s(\mathbb{R}^n)}}. \quad (2.7)$$

One may check [JK95, Section 2] for the treatment of Sobolev spaces case, the Besov spaces case follows by interpolation argument, see [BL76, Theorem 3.4.2]. As a direct consequence, one has from [JK95, Proposition 2.9] and [BL76, Theorem 3.7.1], that for all  $s \in \mathbb{R}$ ,  $p \in (1, +\infty)$ ,  $q \in [1, +\infty)$ ,

$$(H^{s, p}(\Omega))' = H_0^{-s, p'}(\Omega), \quad (B_{p, q}^s(\Omega))' = B_{p', q', 0}^{-s}(\Omega), \quad (2.8)$$

$$(B_{p, q, 0}^s(\Omega))' = B_{p', q'}^{-s}(\Omega). \quad (2.9)$$

And finally, thanks to a modified version of Proposition 1.4, we also have a particular case of equality of Sobolev spaces, with equivalent norms, for all  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$ ,  $s \in (-1 + \frac{1}{p}, \frac{1}{p})$ ,

$$H^{s, p}(\Omega) = H_0^{s, p}(\Omega), \quad B_{p, q}^s(\Omega) = B_{p, q, 0}^s(\Omega). \quad (2.10)$$

The interested reader may also found an explicit and way more general (and still valid, for the most part of it, in the case of a special Lipschitz domain) treatment for bounded Lipschitz domains

in [KMM07], where the Triebel-Lizorkin scale, including Hardy spaces, and other endpoint function spaces are also treated.

A more recent elementary and accessible exposition is available in [Leo23, Chapter 8 &11]. It deals with inhomogeneous Sobolev-Slobodeckij spaces  $W^{s,p}(\Omega)$ , which coincides with usual Sobolev spaces when  $s \in \mathbb{Z}$ , and with diagonal Besov spaces  $B_{p,p}^s(\Omega)$  when  $s \in \mathbb{R} \setminus \mathbb{Z}$ . The case of indices  $s \in [0, 1]$  is treated in the case of Lipschitz domains, and  $s \in [0, m + 1]$  in the case where  $\Omega$  is a  $C^{m,1}$  domain.

All the results presented above will be used without being mentioned and are assumed to be well known to the reader.

## 2.3 Homogeneous Sobolev and Besov spaces on a special Lipschitz domain.

From now on, and until the end of the paper,  $\Omega$  will be a fixed special Lipschitz domain given by a fixed uniformly Lipschitz function  $\phi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ , *i.e.*,

$$\Omega := \{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n > \phi(x') \}.$$

We also set the following global bi-Lipschitz map of  $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ ,

$$\Psi : (x', x_n) \mapsto (x', x_n + \phi(x')).$$

For which, we have

$$\Psi(\mathbb{R}_+^n) = \Omega, \Psi^{-1}(\Omega) = \mathbb{R}_+^n \text{ and } \det(\nabla\Psi) = \det(\nabla(\Psi^{-1})) = 1. \quad (2.11)$$

As in [Gau22, Section 2.2.2], one may expect to recover similar results for the scale of homogeneous Sobolev and Besov as the one mentioned in Section 2.2. However, it still suffers the same issues as the one raised at the beginning of [Gau22, Section 2.2.2]: the lack of completeness on the scale and it is not known whether Rychkov's extension operator satisfies homogeneous estimates.

The extension method employed in [DHMT21, Chapter 3] and [Gau22, Section 2.2.2], the use of the global change of coordinates and extension operators of higher order reflection around the boundary, will fail for high regularities. Indeed, the global bi-Lipschitz map  $\Psi$  suffers from its own lack of regularity, making it impossible to take its derivatives more than once. Moreover, even if it were a smooth global diffeomorphism, higher order derivatives would produce inhomogeneous parts with lower order terms. However, this method still makes sense for regularity indices  $s \in (-1 + 1/p, 1]$ .

The first idea here is to use Stein's extension operator introduced in [Ste70, Chapter VI], for which we have **homogeneous** estimates for non-negative integers indices of regularity.

The second idea is to fall in a setting so that one just has to use the proofs in [Gau22, Section 2], or at least reproduce it with minor modifications. Indeed those proofs mainly depend on the existence of good extension operator with appropriate homogeneous estimates, and the reflexivity of considered Sobolev spaces.

### 2.3.1 Homogeneous Sobolev spaces

The Sobolev embeddings are a straightforward consequence of the definition of function spaces by restriction, see e.g. [Gau22, Proposition 2.23].

**Proposition 2.2** *Let  $p, q \in (1, +\infty)$ ,  $s \in [0, n)$ , such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{n}.$$

*Then the following inequalities hold,*

$$\begin{aligned} \|u\|_{L^q(\Omega)} &\lesssim_{n,s,p,q} \|u\|_{\dot{H}^{s,p}(\Omega)}, \quad \forall u \in \dot{H}^{s,p}(\Omega), \\ \|u\|_{\dot{H}_0^{-s,q}(\Omega)} &\lesssim_{n,s,p,q} \|u\|_{L^p(\Omega)}, \quad \forall u \in L^p(\Omega). \end{aligned}$$

*Moreover, each underlying embedding is dense.*

In particular, Sobolev embeddings imply that any appropriate extension operator, such as the next one, are already uniquely well defined on any function in homogeneous Sobolev spaces such that  $\dot{H}^{s,p}(\Omega) \subset L^{\frac{pn}{n-sp}}(\Omega)$ , *i.e.* for those who are complete. It remains to show the continuity with respect to homogeneous Sobolev norms.

**Proposition 2.3** *There exists a universal extension operator  $\mathcal{E}$ , such that for all  $p \in (1, +\infty)$ ,  $s \in \mathbb{R}_+$ , all  $u \in H^{s,p}(\Omega)$ ,*

$$\mathcal{E}u|_{\Omega} = u,$$

with the estimate

$$\|\mathcal{E}u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\Omega)} .$$

In particular,  $\mathcal{E} : \dot{H}^{s,p}(\Omega) \longrightarrow \dot{H}^{s,p}(\mathbb{R}^n)$  extends uniquely as a bounded operator whenever  $(\mathcal{C}_{s,p})$  is satisfied.

*Proof.* — Let  $\mathcal{E}$  be Stein's extension operator given in [Ste70, Chapter 6, Section 3, Theorem 5']. For all  $m \in \mathbb{N}$ ,  $p \in (1, +\infty)$ ,  $u \in H^{m,p}(\Omega)$ , we have

$$\|\nabla^m u\|_{L^p(\Omega)} \leq \|u\|_{\dot{H}^{m,p}(\Omega)} = \inf_{\substack{\tilde{u} \in \dot{H}^{m,p}(\mathbb{R}^n), \\ \tilde{u}|_{\Omega} = u}} \|\nabla^m \tilde{u}\|_{L^p(\mathbb{R}^n)} \leq \|\nabla^m(\mathcal{E}u)\|_{L^p(\mathbb{R}^n)} \lesssim_{p,n,m,\partial\Omega} \|\nabla^m u\|_{L^p(\Omega)} .$$

Hence, it satisfies

$$\|\mathcal{E}u\|_{\dot{H}^{m,p}(\mathbb{R}^n)} \lesssim_{p,n,m,\partial\Omega} \|u\|_{\dot{H}^{m,p}(\Omega)} . \quad (2.12)$$

So that  $\mathcal{E} : \dot{H}^{m,p}(\Omega) \longrightarrow \dot{H}^{m,p}(\mathbb{R}^n)$  is bounded on subspace  $H^{m,p}(\Omega)$ , in particular on whole  $\dot{H}^{m,p}(\Omega)$  when it is complete, *i.e.*  $m < \frac{n}{p}$ , since  $\mathcal{S}(\overline{\Omega}) \subset H^{m,p}(\Omega)$  is dense in  $\dot{H}^{m,p}(\Omega)$ .

The estimate (2.12) implies, that for all  $U \in H^{m,p}(\mathbb{R}^n)$ , by the definition of function spaces by restriction

$$\|\mathcal{E}[1_{\Omega}U]\|_{\dot{H}^{m,p}(\mathbb{R}^n)} \lesssim_{p,n,m,\partial\Omega} \|U\|_{\dot{H}^{m,p}(\mathbb{R}^n)} .$$

Therefore, if one uses Lemma 1.5,

$$\|(\dot{\Delta}_j \mathcal{E}[1_{\Omega}U])_{j \in \mathbb{Z}}\|_{L^p(\mathbb{R}^n, \ell_m^2(\mathbb{Z}))} \lesssim_{p,n,m,\partial\Omega} \|(\dot{\Delta}_j U)_{j \in \mathbb{Z}}\|_{L^p(\mathbb{R}^n, \ell_m^2(\mathbb{Z}))} . \quad (2.13)$$

For  $v = (v_j)_{j \in \mathbb{Z}} \in L^p(\mathbb{R}^n, \ell_m^2(\mathbb{Z}))$  with finite support with respect to the discrete variable, we set

$$\Xi_{\Omega} v := \left( \dot{\Delta}_j \mathcal{E} \left[ 1_{\Omega} \left( \sum_{k \in \mathbb{Z}} \dot{\Delta}_k [v_{k-1} + v_k + v_{k+1}] \right) \right] \right)_{j \in \mathbb{Z}}$$

and since  $v$  has finite support with respect to the discrete variable we may define the auxiliary function  $V := \sum_{k \in \mathbb{Z}} \dot{\Delta}_k [v_{k-1} + v_k + v_{k+1}] \in H^{m,p}(\mathbb{R}^n)$ , and we obtain, by [Gra14, Proposition 6.1.4],

$$\|\Xi_{\Omega} v\|_{L^p(\mathbb{R}^n, \ell_m^2(\mathbb{Z}))} \lesssim_{p,n,m,\partial\Omega} \|(\dot{\Delta}_j V)_{j \in \mathbb{Z}}\|_{L^p(\mathbb{R}^n, \ell_m^2(\mathbb{Z}))} \lesssim_{p,n,m,\partial\Omega} \|v\|_{L^p(\mathbb{R}^n, \ell_m^2(\mathbb{Z}))} .$$

It follows that  $\Xi_{\Omega}$  extends uniquely as a bounded linear operator on  $L^p(\mathbb{R}^n, \ell_m^2(\mathbb{Z}))$  for all  $p \in (1, +\infty)$ ,  $m \in \mathbb{N}$ , which is consistent on elements whose support in the discrete variable is finite. It is still consistent on all element of the form  $(\dot{\Delta}_j U)_{j \in \mathbb{Z}}$ , provided  $U \in H^{\alpha,p}(\mathbb{R}^n)$ ,  $\alpha \geq m$  and we have by construction and uniqueness of the extension

$$\Xi_{\Omega}[(\dot{\Delta}_j U)_{j \in \mathbb{Z}}] = (\dot{\Delta}_j \mathcal{E}[1_{\Omega}U])_{j \in \mathbb{Z}} .$$

The complex interpolation of mixed weighted Lebesgue spaces, see [BL76, Theorems 5.1.2 & 5.6.3], yields that

$$\Xi_{\Omega} : L^p(\mathbb{R}^n, \ell_s^2(\mathbb{Z})) \longrightarrow L^p(\mathbb{R}^n, \ell_s^2(\mathbb{Z}))$$

is a well defined bounded linear operator for all  $s \geq 0$ ,  $p \in (1, +\infty)$ . Then, the map

$$\Xi_{\Omega}[(\dot{\Delta}_j [\cdot])_{j \in \mathbb{Z}}] : \dot{H}^{s,p}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n, \ell_s^2(\mathbb{Z})) \quad (2.14)$$

is also well defined and bounded by Lemma 1.5. Provided  $s \geq 0$ ,  $\mathcal{E}u \in H^{s,p}(\mathbb{R}^n)$  is already entirely determined for  $u \in H^{s,p}(\Omega)$  by the boundedness on the inhomogeneous Sobolev scale. Hence, for

$U \in \dot{H}^{s,p}(\mathbb{R}^n)$  such that  $U|_{\Omega} = u$ , by uniqueness of the extension of  $\Xi_{\Omega}$ , we have

$$\Xi_{\Omega}[(\dot{\Delta}_j U)_{j \in \mathbb{Z}}] = (\dot{\Delta}_k \mathcal{E}[1_{\Omega} U])_{k \in \mathbb{Z}} = (\dot{\Delta}_k \mathcal{E}u)_{k \in \mathbb{Z}}.$$

Thus, one may use the estimate (2.14) and Lemma 1.5 to deduce

$$\|\mathcal{E}u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,\partial\Omega} \|U\|_{\dot{H}^{s,p}(\mathbb{R}^n)}.$$

However,  $U$  is an arbitrary  $\dot{H}^{s,p}$ -extension of  $u$ , so that by the definition of function spaces by restriction, it holds that

$$\|\mathcal{E}u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\Omega)}.$$

When  $(\mathcal{C}_{s,p})$  is satisfied,  $\dot{H}^{s,p}(\mathbb{R}^n)$  is complete, and since  $\mathcal{S}_0(\overline{\Omega}) \subset H^{s,p}(\Omega)$  is dense in  $\dot{H}^{s,p}(\Omega)$ , the operator extends uniquely on the whole  $\dot{H}^{s,p}(\Omega)$ .  $\blacksquare$

**Remark 2.4** The method employed here is quite general, and could be adapted to the interpolation of many other kind of linear operators.

The general idea is to lift the operator at a level for which we can take completion without losing any ambient structure information, here at the level of anisotropic Lebesgue spaces  $L^p(\ell_s^2)$ , instead of taking abstract completion of our Sobolev spaces  $\dot{H}^{s,p}$ . From this point, one perform the complex interpolation, then one may hope to get back on a (appropriate dense) subset of those spaces for which we can compute explicitly the operator, which was exactly what we have done.

This is one of the novelty of the present paper.

Proposition 2.3 is already a powerful enough tool to carry many results. However, this Stein's extension operator has its use restricted to non-negative indices of regularity for the Sobolev scale and positive indices of regularity for the Besov scale. It would be of a great interest to be able to look at similar properties for regularity indices  $s \in (-1 + 1/p, 1/p)$ .

We need to carry over the behavior of the global change of coordinates on the homogeneous scale.

For any measurable function  $u$  on either  $\Omega$  or  $\mathbb{R}^n$ , and any measurable function  $v$  on either  $\mathbb{R}_+^n$  or  $\mathbb{R}^n$ , we introduce the maps

$$T_{\phi}u := u \circ \Psi, \quad \text{and} \quad T_{\phi}^{-1}v = v \circ \Psi^{-1}. \quad (2.15)$$

**Proposition 2.5** *Let  $p \in (1, +\infty)$ ,  $s \in [-1, 1]$  and  $\mathcal{T} \in \{T_{\phi}, T_{\phi}^{-1}\}$ . We assume that one of the two following conditions is satisfied*

- (i)  $(\mathcal{C}_{s,p})$  and  $u \in \dot{H}^{s,p}(\mathbb{R}^n)$ ,
- (ii)  $s \geq n/p$  and  $u \in H^{s,p}(\mathbb{R}^n)$ .

Then  $\mathcal{T}u \in \dot{H}^{s,p}(\mathbb{R}^n)$  with the estimate

$$\|\mathcal{T}u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \lesssim_{p,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\mathbb{R}^n)}.$$

*Proof.* — We set  $\mathcal{T} = T_{\phi}$  and  $\mathcal{T}^* = T_{\phi}^{-1}$ . First, we let  $u \in H^{1,p}(\mathbb{R}^n)$  and we recall that the following equalities hold almost everywhere

$$\partial_{x_k}(\mathcal{T}u) = \mathcal{T}(\partial_{x_k}u) + \partial_{x_k}\phi\mathcal{T}(\partial_{x_n}u), \quad \partial_{x_n}(\mathcal{T}u) = \mathcal{T}(\partial_{x_n}u), \quad k \in \llbracket 1, n-1 \rrbracket.$$

We recall that  $\mathcal{T}$  is bounded on  $L^p(\mathbb{R}^n)$ , and that, moreover, the Jacobian determinant of  $\Psi$  is 1, see (2.11). Therefore, we obtain

$$\begin{aligned} \|\nabla\mathcal{T}u\|_{L^p(\mathbb{R}^n)} &\leq \|\partial_{x_n}u\|_{L^p(\mathbb{R}^n)} + \sum_{k=1}^{n-1} \|\partial_{x_k}u\|_{L^p(\mathbb{R}^n)} + \|\partial_{x_k}\phi\|_{L^\infty} \|\partial_{x_n}u\|_{L^p(\mathbb{R}^n)} \\ &\leq (1 + (n-1)\|\nabla'\phi\|_{L^\infty}) \|\nabla u\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Similar computations yield,

$$\|\nabla\mathcal{T}^*u\|_{L^p(\mathbb{R}^n)} \leq (1 + (n-1)\|\nabla'\phi\|_{L^\infty}) \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Now, for  $v \in L^2(\mathbb{R}^n) \cap \dot{H}^{-1,p}(\mathbb{R}^n)$ , by Proposition 1.2, and since the Jacobian determinant of  $\Psi^{-1}$  is 1,

$$\begin{aligned} \|\mathcal{T}v\|_{\dot{H}^{-1,p}(\mathbb{R}^n)} &= \sup_{\substack{u \in \mathcal{S}(\mathbb{R}^n) \\ \|u\|_{\dot{H}^{1,p'}(\mathbb{R}^n)} \leq 1}} \left| \int_{\mathbb{R}^n} \mathcal{T}v(x) u(x) dx \right| \\ &= \sup_{\substack{u \in \mathcal{S}(\mathbb{R}^n) \\ \|u\|_{\dot{H}^{1,p'}(\mathbb{R}^n)} \leq 1}} \left| \int_{\mathbb{R}^n} v(x) \mathcal{T}^*u(x) dx \right| \\ &\leq \|v\|_{\dot{H}^{-1,p}(\mathbb{R}^n)} \left( \sup_{\substack{u \in \mathcal{S}(\mathbb{R}^n) \\ \|u\|_{\dot{H}^{1,p'}(\mathbb{R}^n)} \leq 1}} \|\mathcal{T}^*u\|_{\dot{H}^{1,p'}(\mathbb{R}^n)} \right) \\ &\lesssim_{p,n,\partial\Omega} \|v\|_{\dot{H}^{-1,p}(\mathbb{R}^n)}. \end{aligned}$$

The same goes for  $\mathcal{T}^*$ . Hence,  $\mathcal{T}$  (resp.  $\mathcal{T}^*$ ) extends uniquely as a bounded linear operator on  $\dot{H}^{-1,p}(\mathbb{R}^n)$ . But since  $\mathcal{T}$  (resp.  $\mathcal{T}^*$ ) is known to be bounded on  $L^p(\mathbb{R}^n)$ , by complex interpolation given in Proposition 1.3,  $\mathcal{T}$  (resp.  $\mathcal{T}^*$ ) is then a bounded linear operator on  $\dot{H}^{s,p}(\mathbb{R}^n)$ , for all  $s \in [-1, 0]$ . One may repeat the duality argument, thanks to the boundedness on  $\dot{H}^{-s,p'}(\mathbb{R}^n)$  we just proved, to obtain for  $s \in [0, 1]$ ,  $u \in \dot{H}^{s,p}(\mathbb{R}^n)$ ,

$$\|\mathcal{T}u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\mathbb{R}^n)},$$

and similarly for  $\mathcal{T}^*$ . Finally, when  $(\mathcal{C}_{s,p})$  is satisfied,  $\dot{H}^{s,p}(\mathbb{R}^n)$  is complete, and since  $\dot{H}^{s,p}(\mathbb{R}^n)$  is dense in  $\dot{H}^{s,p}(\mathbb{R}^n)$ , the operator extends uniquely on the whole  $\dot{H}^{s,p}(\mathbb{R}^n)$ . ■

**Remark 2.6** Everything still hold for more general bi-Lipschitz transformation with constant Jacobian determinants. One may probably want to generalize Proposition 2.5 in a way similar to [DM15, Lemma 2.1.1].

We can deduce from Proposition 2.5 many interesting corollaries.

**Corollary 2.7** For all  $p \in (1, +\infty)$ , for all  $s \in (-1 + \frac{1}{p}, \frac{1}{p})$ , for all  $u \in \dot{H}^{s,p}(\mathbb{R}^n)$ ,

$$\|1_\Omega u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\mathbb{R}^n)}$$

The same result still holds with  $\mathbb{H}$  instead of  $\dot{H}$ .

*Proof.* — It suffices to write  $1_\Omega u = T_\phi^{-1} 1_{\mathbb{R}_+^n} T_\phi u$  then to apply Propositions 2.5 and 1.4. ■

**Corollary 2.8** Let  $p \in (1, +\infty)$ ,  $s \in [0, 1]$ . If one of the two following conditions is satisfied

- (i)  $(\mathcal{C}_{s,p})$  and  $u \in \dot{H}^{s,p}(\Omega)$ ,
- (ii)  $s \geq n/p$  and  $u \in \dot{H}^{s,p}(\Omega)$ ,

we have  $T_\phi u \in \dot{H}^{s,p}(\mathbb{R}_+^n)$  with the estimate

$$\|T_\phi u\|_{\dot{H}^{s,p}(\mathbb{R}_+^n)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\Omega)}.$$

In particular,  $T_\phi : \dot{H}^{s,p}(\Omega) \rightarrow \dot{H}^{s,p}(\mathbb{R}_+^n)$  is an isomorphism of Banach spaces whenever  $(\mathcal{C}_{s,p})$  is satisfied. The result still holds if we replace  $(\Omega, \mathbb{R}_+^n, T_\phi)$  by  $(\mathbb{R}_+^n, \Omega, T_\phi^{-1})$ .

*Proof.* — Let  $u \in \dot{H}^{s,p}(\Omega)$ , then  $T_\phi u = [T_\phi \mathcal{E}u]_{|\mathbb{R}_+^n} \in \dot{H}^{s,p}(\mathbb{R}_+^n) \subset \dot{H}^{s,p}(\mathbb{R}_+^n)$ , where  $\mathcal{E}$  is Stein's universal extension operator. Therefore by the definition of function spaces by restriction and Propositions 2.5 and 2.3, we have successively

$$\|T_\phi u\|_{\dot{H}^{s,p}(\mathbb{R}_+^n)} \leq \|T_\phi \mathcal{E}u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,\partial\Omega} \|\mathcal{E}u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\Omega)}.$$

When  $(\mathcal{C}_{s,p})$  is satisfied,  $\dot{H}^{s,p}(\mathbb{R}_+^n)$  is complete, and since  $\mathcal{S}_0(\overline{\Omega}) \subset \dot{H}^{s,p}(\Omega)$  is dense in  $\dot{H}^{s,p}(\Omega)$ , the operator extends uniquely on the whole  $\dot{H}^{s,p}(\Omega)$ . ■

**Corollary 2.9** *Let  $p \in (1, +\infty)$ ,  $s \in (-1 + 1/p, 1]$ . There exists an extension operator  $E$ , such that if one of the two following conditions is satisfied*

- (i)  $(\mathcal{C}_{s,p})$  and  $u \in \dot{H}^{s,p}(\mathbb{R}^n)$ ,
- (ii)  $s \geq n/p$  and  $u \in H^{s,p}(\mathbb{R}^n)$ ,

we have

$$Eu|_{\Omega} = u,$$

with the estimate

$$\|Eu\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\Omega)}.$$

In particular,  $E : \dot{H}^{s,p}(\Omega) \rightarrow \dot{H}^{s,p}(\mathbb{R}^n)$  extends uniquely as a bounded linear operator whenever  $(\mathcal{C}_{s,p})$  is satisfied.

*Proof.* — We introduce the extension operator on the half space by even reflection, for any measurable function  $u : \mathbb{R}_+^n \rightarrow \mathbb{C}$ , and for almost every  $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ ,

$$\tilde{E}u(x', x_n) := \begin{cases} u(x', x_n) & , \text{ if } (x', x_n) \in \mathbb{R}^{n-1} \times (0, +\infty), \\ u(x', -x_n) & , \text{ if } (x', x_n) \in \mathbb{R}^{n-1} \times (-\infty, 0). \end{cases}$$

The operator  $\tilde{E}$  is known to have the desired properties when  $\Omega = \mathbb{R}_+^n$ , see e.g. [Gau22, Proposition 2.16]. It suffices to set

$$E := T_{\phi}^{-1} \tilde{E} T_{\phi}.$$

The boundedness properties follows from Propositions 2.5 and 1.4 when  $s \in (-1 + 1/p, 1/p)$ . When  $s \geq 0$ , it suffices to apply Corollary 2.8.  $\blacksquare$

**Proposition 2.10** *Let  $p \in (1, +\infty)$ ,  $k \in \llbracket 1, +\infty \llbracket$ ,  $s > k - 1 + \frac{1}{p}$ , for all  $u \in H^{s,p}(\Omega)$ ,*

$$\|\nabla^k u\|_{\dot{H}^{s-k,p}(\Omega)} \sim_{s,k,p,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\Omega)}.$$

In particular,  $\|\nabla^k \cdot\|_{\dot{H}^{s-k,p}(\Omega)}$  is an equivalent norm on  $\dot{H}^{s,p}(\Omega)$ , whenever  $(\mathcal{C}_{s-k,p})$  is satisfied.

*Proof.* — For all  $k \in \llbracket 1, +\infty \llbracket$ ,  $p \in (1, +\infty)$ ,  $s \in \mathbb{R}$ ,  $u \in \dot{H}^{s,p}(\Omega)$ , the estimate

$$\|\nabla^k u\|_{\dot{H}^{s-k,p}(\Omega)} \lesssim_{s,k,p,n} \|u\|_{\dot{H}^{s,p}(\Omega)}$$

always holds by the definition of function spaces by restriction. Therefore, it suffices to prove the reverse inequality.

First, we assume  $s \in (1/p, 1]$ . Let  $u \in H^{s,p}(\Omega)$ . In this case, by Corollary 2.9 and the definition of function spaces by restriction,

$$\|u\|_{\dot{H}^{s,p}(\Omega)} \leq \|Eu\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|\nabla Eu\|_{\dot{H}^{s-1,p}(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|\nabla u\|_{\dot{H}^{s-1,p}(\Omega)}.$$

Now, if  $s \geq 1$ , one obtains in a similar way from Proposition 2.3,

$$\|u\|_{\dot{H}^{s,p}(\Omega)} \leq \|\mathcal{E}u\|_{\dot{H}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|\nabla \mathcal{E}u\|_{\dot{H}^{s-1,p}(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|\nabla u\|_{\dot{H}^{s-1,p}(\Omega)}. \quad (2.16)$$

This yields the result when  $s > k - 1 + 1/p$ , and  $k = 1$ .

We mention that the estimate (2.16) is indeed legal, because for  $k \in \llbracket 1, n \llbracket$ , there is a linear operator  $\mathcal{T}_k$  which has exactly the same boundedness properties as  $\mathcal{E}$ , and which satisfies the commutation property  $\partial_{x_k} \mathcal{E} = \mathcal{E}[\partial_{x_k}] + \mathcal{T}_k[\partial_{x_n}]$ , see for instance the argument in the proof of [JK95, Proposition 2.18].

Now, for  $s \geq 2 - 1 + 1/p$ ,  $k = 2$ , by the previous step for  $k = 1$ , we have

$$\|u\|_{\dot{H}^{s,p}(\Omega)} \lesssim_{s,p,n,\partial\Omega} \|\nabla u\|_{\dot{H}^{s-1,p}(\Omega)}.$$

Since  $s - 1 > 1 - 1 + 1/p$ , if  $s \geq 1$  one may use again Stein's extension operator to obtain

$$\|u\|_{\dot{H}^{s,p}(\Omega)} \lesssim_{s,p,n,\partial\Omega} \|\nabla u\|_{\dot{H}^{s-1,p}(\Omega)} \lesssim_{s,p,n,\partial\Omega} \|\nabla \mathcal{E} \nabla u\|_{\dot{H}^{s-2,p}(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|\nabla^2 u\|_{\dot{H}^{s-2,p}(\Omega)}.$$

Otherwise for  $s - 1 \in (1/p, 1]$ , one uses the extension operator given by Corollary 2.9.

For  $k \geq 2$ , one obtains the result by induction, reproducing the steps above.  $\blacksquare$

**Proposition 2.11** *Let  $p_j \in (1, +\infty)$ ,  $s_j > -1 + \frac{1}{p_j}$ ,  $j \in \{0, 1\}$ , such that  $(\mathcal{C}_{s_0, p_0})$  is satisfied. Assume that one of the two following conditions is satisfied*

(i)  $s_0, s_1 \leq 1$  and  $\mathbb{E} = \mathbb{E}$ , given by Corollary 2.9,

(ii)  $s_0, s_1 \geq 0$  and  $\mathbb{E} = \mathcal{E}$ , given by Proposition 2.3.

Then for all  $u \in \dot{H}^{s_0, p_0}(\Omega) \cap \dot{H}^{s_1, p_1}(\Omega)$ , we have  $\mathbb{E}u \in \dot{H}^{s_j, p_j}(\mathbb{R}^n)$ ,  $j \in \{0, 1\}$ , with the estimate

$$\|\mathbb{E}u\|_{\dot{H}^{s_j, p_j}(\mathbb{R}^n)} \lesssim_{s_j, p_j, n} \|u\|_{\dot{H}^{s_j, p_j}(\Omega)}. \quad (2.17)$$

Therefore, the following equality of vector spaces holds with equivalence of norms

$$\dot{H}^{s_0, p_0}(\Omega) \cap \dot{H}^{s_1, p_1}(\Omega) = [\dot{H}^{s_0, p_0} \cap \dot{H}^{s_1, p_1}](\Omega).$$

In particular,  $\dot{H}^{s_0, p_0}(\Omega) \cap \dot{H}^{s_1, p_1}(\Omega)$  is a Banach space which admits  $S_0(\overline{\Omega})$  as a dense subspace.

*Proof.* — Let  $p_j \in (1, +\infty)$ ,  $s_j > -1 + 1/p_j$ , such that  $(\mathcal{C}_{s_0, p_0})$ . We proceed as in [Gau22, Proposition 2.18] for  $\Omega = \mathbb{R}_+^n$ ,  $[\dot{H}^{s_0, p_0} \cap \dot{H}^{s_1, p_1}](\Omega)$  is complete and admits  $S_0(\overline{\Omega})$  as a dense subspace and the following continuous embedding also holds by construction,

$$[\dot{H}^{s_0, p_0} \cap \dot{H}^{s_1, p_1}](\Omega) \hookrightarrow \dot{H}^{s_0, p_0}(\Omega) \cap \dot{H}^{s_1, p_1}(\Omega).$$

One may expect to reproduce the proof of [Gau22, Proposition 2.18], which is only possible when  $\mathbb{E} = \mathbb{E}$ . Indeed, this is not possible for  $\mathbb{E} = \mathcal{E}$ , since the aforementioned proof use the fact that one can take derivatives a finite amount of time to fall in a complete space. When  $p_1$  is large enough  $[0, n/p_1]$  is of size less than one, while  $(-1 + 1/p_1, n/p_1) \cap (-1 + 1/p_1, 1]$  always contains a translation of the interval  $[0, 1]$ .

We let  $\mathbb{E} = \mathcal{E}$  and then we get back on the use of the operator  $\Xi_\Omega$ , as introduced in (2.14). We consider  $u \in \dot{H}^{s_0, p_0}(\Omega) \cap \dot{H}^{s_1, p_1}(\Omega)$ , and let  $U \in \dot{H}^{s_1, p_1}(\mathbb{R}^n)$  such that  $U|_\Omega = u$ . We recall that we have  $\mathcal{E}u \in \dot{H}^{s_0, p_0}(\mathbb{R}^n) \subset \mathcal{S}'_h(\mathbb{R}^n)$  since  $(\mathcal{C}_{s_0, p_0})$  is satisfied. One also has

$$(\dot{\Delta}_k \mathcal{E}u)_{k \in \mathbb{Z}} = (\dot{\Delta}_k \mathcal{E}[1_\Omega U])_{k \in \mathbb{Z}} = \Xi_\Omega[(\dot{\Delta}_k U)_{k \in \mathbb{Z}}] \in L^{p_1}(\mathbb{R}^n, \ell_{s_1}^2(\mathbb{Z})).$$

Therefore, by Lemma 1.5, since  $\mathcal{E}u \in \mathcal{S}'_h(\mathbb{R}^n)$ ,

$$\begin{aligned} \|\mathcal{E}u\|_{\dot{H}^{s_1, p_1}(\mathbb{R}^n)} &\sim_{p_1, s_1, n} \|(\dot{\Delta}_k \mathcal{E}u)_{k \in \mathbb{Z}}\|_{L^{p_1}(\mathbb{R}^n, \ell_{s_1}^2(\mathbb{Z}))} \\ &\sim_{p_1, s_1, n} \|\Xi_\Omega[(\dot{\Delta}_k U)_{k \in \mathbb{Z}}]\|_{L^{p_1}(\mathbb{R}^n, \ell_{s_1}^2(\mathbb{Z}))} \\ &\lesssim_{p_1, s_1, n, \partial\Omega} \|U\|_{\dot{H}^{s_1, p_1}(\mathbb{R}^n)}. \end{aligned}$$

As in the proof of Proposition 2.3, since  $U$  is an arbitrary extension of  $u$  in  $\dot{H}^{s_1, p_1}(\mathbb{R}^n)$ , taking the infimum on all such  $U$  yields

$$\|\mathcal{E}u\|_{\dot{H}^{s_1, p_1}(\mathbb{R}^n)} \lesssim_{p_1, s_1, n, \partial\Omega} \|u\|_{\dot{H}^{s_1, p_1}(\Omega)}.$$

Thus for  $u \in \dot{H}^{s_0, p_0}(\Omega) \cap \dot{H}^{s_1, p_1}(\Omega)$ , and by the definition of restriction spaces,

$$\|u\|_{[\dot{H}^{s_0, p_0} \cap \dot{H}^{s_1, p_1}](\Omega)} \leq \|\mathcal{E}u\|_{\dot{H}^{s_0, p_0}(\mathbb{R}^n)} + \|\mathcal{E}u\|_{\dot{H}^{s_1, p_1}(\mathbb{R}^n)} \lesssim_{s_0, s_1, \partial\Omega}^{p_0, p_1, n} \|u\|_{\dot{H}^{s_0, p_0}(\Omega)} + \|u\|_{\dot{H}^{s_1, p_1}(\Omega)}.$$

This yields the result.  $\blacksquare$

From now on, everything has been set up so that the most part of the remaining proofs in [Gau22, Section 2] could be reproduced *verbatim* (sometimes up to the appropriate technical modifications).

Now, we want to work with homogeneous Sobolev spaces whose elements are supported in  $\overline{\Omega}$ .

**Proposition 2.12** *Let  $p_j \in (1, +\infty)$ ,  $s_j > -1 + \frac{1}{p_j}$ ,  $j \in \{0, 1\}$ , such that  $(\mathcal{C}_{s_0, p_0})$  is satisfied. Then,*

(i) if  $s_0, s_1 \leq 1$ , there exists a linear operator  $\mathcal{P}_0 = \mathcal{P}_0$ ,



(ii) if  $s_0, s_1 \geq 0$ , there exists a linear operator  $\mathcal{P}_0 = \mathcal{P}_0$ ,

such that for all  $u \in \dot{H}^{s_0, p_0}(\mathbb{R}^n) \cap \dot{H}^{s_1, p_1}(\mathbb{R}^n)$ , we have  $\mathcal{P}_0 u \in \dot{H}_0^{s_j, p_j}(\Omega)$ ,  $j \in \{0, 1\}$ , with the estimate

$$\|\mathcal{P}_0 u\|_{\dot{H}^{s_j, p_j}(\mathbb{R}^n)} \lesssim_{s_j, p_j, n, \partial\Omega} \|u\|_{\dot{H}^{s_j, p_j}(\mathbb{R}^n)}.$$

*Proof.* — We notice that  $\overline{\Omega}^c$  is also a special Lipschitz domain. If  $\mathbb{E}$  is an extension operator for  $\Omega$  provided by Proposition 2.11, we denote by  $\mathbb{E}^-$  the extension operator for  $\overline{\Omega}^c$ , and we set for all  $u \in \dot{H}^{s_0, p_0}(\mathbb{R}^n) \cap \dot{H}^{s_1, p_1}(\mathbb{R}^n)$

$$\mathcal{P}_0 u := u - \mathbb{E}^- [1_{\overline{\Omega}^c} u].$$

In this case, the boundedness properties follow from Proposition 2.11.  $\blacksquare$

The next proposition admits a proof similar to the one of [Gau22, Propositions 2.24 & 2.25]. Indeed, all the appropriate tools are available, thanks to Propositions 2.2 and 2.12.

**Proposition 2.13** *Let  $s, s_0, s_1 \in \mathbb{R}$ ,  $p \in (1, +\infty)$ , then the space  $C_c^\infty(\Omega)$  is dense in*

(i)  $\dot{H}_0^{s, p}(\Omega)$ , whenever  $s \in (-n/p', n/p)$ ;

(ii)  $\dot{H}_0^{s_0, p}(\Omega) \cap \dot{H}_0^{s_1, p}(\Omega)$ , if  $s_0, s_1 \geq 0$ , and  $(C_{s_0, p})$  is satisfied.

The next corollary is fundamental for a proper theory of Sobolev spaces involving boundary values. This is a direct combination of Proposition 2.13 and Corollary 2.7.

**Corollary 2.14** *For all  $p \in (1, +\infty)$ ,  $s \in (-1 + \frac{1}{p}, \frac{1}{p})$ ,*

$$\dot{H}_0^{s, p}(\Omega) = \dot{H}^{s, p}(\Omega).$$

*In particular,  $C_c^\infty(\Omega)$  is dense in  $\dot{H}^{s, p}(\Omega)$  for the same range of indices.*

**Proposition 2.15** *Let  $p \in (1, +\infty)$ ,  $s \in (-\frac{n}{p'}, \frac{n}{p})$ , we have*

$$(\dot{H}^{s, p}(\Omega))' = \dot{H}_0^{-s, p'}(\Omega) \text{ and } (\dot{H}_0^{s, p}(\Omega))' = \dot{H}^{-s, p'}(\Omega).$$

*Proof.* — It suffices to reproduce the proof of [Gau22, Proposition 2.27], replacing  $\mathbb{R}_+^n$  by  $\Omega$ .  $\blacksquare$

**Corollary 2.16** *Let  $p \in (1, +\infty)$ ,  $-n/p' < s_0 < s_1 < n/p$ . The space  $C_c^\infty(\Omega)$  is dense in  $\dot{H}_0^{s_0, p}(\Omega) \cap \dot{H}_0^{s_1, p}(\Omega)$ .*

*Proof.* — It suffices to reproduce the proof of [Gau22, Corollary 2.28], replacing  $\mathbb{R}_+^n$  by  $\Omega$ .  $\blacksquare$

**Proposition 2.17** *Let  $p_j \in (1, +\infty)$ ,  $s_j \in (-n/p'_j, n/p_j)$ , for  $j \in \{0, 1\}$  and for  $\theta \in (0, 1)$  we set,*

$$\left(s, \frac{1}{p}\right) := (1 - \theta) \left(s_0, \frac{1}{p_0}\right) + \theta \left(s_1, \frac{1}{p_1}\right).$$

*We assume that one of the two following condition is satisfied*

(i)  $s_j \geq 0$ , for  $j \in \{0, 1\}$ ;

(ii)  $s_j \in (-1 + 1/p_j, 1]$ ,  $j \in \{0, 1\}$ .

*Then for  $\mathfrak{h} \in \{\dot{H}, \dot{H}_0\}$ ,*

$$\begin{aligned} [\mathfrak{h}^{s_0, p_0}(\Omega), \mathfrak{h}^{s_1, p_1}(\Omega)]_\theta &= \mathfrak{h}^{s, p}(\Omega), \\ [\mathfrak{h}^{-s_0, p'_0}(\Omega), \mathfrak{h}^{-s_1, p'_1}(\Omega)]_\theta &= \mathfrak{h}^{-s, p'}(\Omega). \end{aligned}$$

*Proof.* — We prove the first interpolation equality. It suffices to assert that  $\{\dot{h}^{s_0, p_0}(\Omega), \dot{h}^{s_1, p_1}(\Omega)\}$  is a retraction of  $\{\dot{H}^{s_0, p_0}(\mathbb{R}^n), \dot{H}^{s_1, p_1}(\mathbb{R}^n)\}$ , thanks to [BL76, Theorem 6.4.2]. Indeed, both retractions are given by

$$\begin{aligned} \mathbb{E} : \dot{H}^{s_j, p_j}(\Omega) &\longrightarrow \dot{H}^{s_j, p_j}(\mathbb{R}^n) \text{ and } R_\Omega : \dot{H}^{s_j, p_j}(\mathbb{R}^n) \longrightarrow \dot{H}^{s_j, p_j}(\Omega), \\ \iota : \dot{H}_0^{s_j, p_j}(\Omega) &\longrightarrow \dot{H}^{s_j, p_j}(\mathbb{R}^n) \text{ and } \mathcal{P}_0 : \dot{H}^{s_j, p_j}(\mathbb{R}^n) \longrightarrow \dot{H}_0^{s_j, p_j}(\Omega). \end{aligned}$$

where  $\mathbb{E}$  and  $\mathcal{P}_0$  are given by Propositions 2.11 and 2.12, respectively.  $R_\Omega$  stands for the restriction operator, and  $\iota$  for the canonical embedding.

By construction,  $\dot{H}^{s_0, p_0}(\Omega) \cap \dot{H}^{s_1, p_1}(\Omega)$  is dense in  $\dot{H}^{s_j, p_j}(\Omega)$ ,  $j \in \{0, 1\}$ , since it contains  $S_0(\overline{\Omega})$ . Corollary 2.16 gives the density of  $\dot{H}_0^{s_0, p_0}(\Omega) \cap \dot{H}_0^{s_1, p_1}(\Omega)$  in  $\dot{H}_0^{s_j, p_j}(\Omega)$ ,  $j \in \{0, 1\}$ . Since all involved spaces are reflexive, one deduces the second interpolation identity by duality, see [BL76, Corollary 4.5.2].  $\blacksquare$

### 2.3.2 Homogeneous Besov spaces

**Proposition 2.18** *For all  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$ , for all  $s \in (-1 + \frac{1}{p}, \frac{1}{p})$ , for all  $u \in \dot{B}_{p,q}^s(\mathbb{R}^n)$ ,*

$$\|1_\Omega u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)}$$

*The same results still hold with  $B$  instead of  $\dot{B}$ .*

*Proof.* — It suffices to apply real interpolation given by Proposition 1.3 to Corollary 2.7.  $\blacksquare$

**Corollary 2.19** *For all  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$ ,  $s \in (-1 + \frac{1}{p}, \frac{1}{p})$ ,*

$$\dot{B}_{p,q,0}^s(\Omega) = \dot{B}_{p,q}^s(\Omega).$$

*In particular, if  $q < +\infty$ , the space  $C_c^\infty(\Omega)$  is dense in  $\dot{B}_{p,q}^s(\Omega)$  for the same range of indices.*

*Proof.* — The equality of function spaces is straightforward from Proposition 2.18.

We only prove that  $C_c^\infty(\Omega)$  is dense in  $\dot{B}_{p,q}^s(\Omega)$ . We use an interpolation theoretical argument.

By the same equality for homogeneous Sobolev spaces, Corollary 2.14, and the elementary interpolation embeddings (2.1) and (2.2), we obtain the following equality of function spaces with equivalence of norms,

$$(\dot{H}_0^{s_0, p}(\Omega), \dot{H}_0^{s_1, p}(\Omega))_{\frac{s-s_0}{s_1-s_0}, q} = \dot{B}_{p,q,0}^s(\Omega) = \dot{B}_{p,q}^s(\Omega),$$

with  $p \in (1, +\infty)$ ,  $-1 + 1/p < s_0 < s < s_1 < 1/p$ ,  $q \in [1, +\infty]$ . By [BL76, Theorem 3.4.2], when  $q \in [1, +\infty)$ , we have the canonical dense embedding

$$\dot{H}_0^{s_0, p}(\Omega) \cap \dot{H}_0^{s_1, p}(\Omega) \hookrightarrow \dot{B}_{p,q}^s(\Omega).$$

But  $C_c^\infty(\Omega)$  is known to be dense in  $\dot{H}_0^{s_0, p}(\Omega) \cap \dot{H}_0^{s_1, p}(\Omega)$  by Corollary 2.16.  $\blacksquare$

During the proof of the last result, we have shown our first real interpolation identity. In fact it can be improved. In order to prove general real interpolation statements, we want to improve the boundedness of extension and projection operators up to the scale of Besov spaces.

**Proposition 2.20** *Let  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$ ,  $s > -1 + \frac{1}{p}$ . Let us consider the extension operator  $\mathbb{E}$  (resp.  $\mathcal{P}_0$ ) as in Proposition 2.11 (resp. Proposition 2.12). We assume moreover that*

- $s > 0$ , if  $\mathbb{E} = \mathcal{E}$  (resp.  $\mathcal{P}_0 = \mathcal{P}_0$ );
- $s < 1$ , if  $\mathbb{E} = \mathbb{E}$  (resp.  $\mathcal{P}_0 = \mathcal{P}_0$ ).

*If one of the two following assertion is satisfied*

- (i)  $s > 0$  and  $u \in B_{p,q}^s(\Omega)$  (resp.  $u \in B_{p,q}^s(\mathbb{R}^n)$ );
- (ii)  $s \in (-1 + \frac{1}{p}, 0]$  and  $u \in \dot{B}_{p,q}^s(\Omega)$  (resp.  $u \in \dot{B}_{p,q}^s(\mathbb{R}^n)$ ).

*Then  $\mathbb{E}u \in \dot{B}_{p,q}^s(\mathbb{R}^n)$ , (resp.  $\mathcal{P}_0 u \in \dot{B}_{p,q,0}^s(\Omega)$ ) and we have the estimate*

$$\|\mathbb{E}u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|u\|_{\dot{B}_{p,q}^s(\Omega)} \quad (\text{resp. } \|\mathcal{P}_0 u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)}).$$

*In particular,  $\mathbb{E}$  (resp.  $\mathcal{P}_0$ ) is a bounded linear operator from  $\dot{B}_{p,q}^s(\Omega)$  to  $\dot{B}_{p,q}^s(\mathbb{R}^n)$  (resp. from  $\dot{B}_{p,q}^s(\mathbb{R}^n)$  to  $\dot{B}_{p,q,0}^s(\Omega)$ ) whenever  $(C_{s,p,q})$  is satisfied.*

*Proof.* — One just has to reproduce the proof of [Gau22, Corollary 2.30].  $\blacksquare$

**Proposition 2.21** *Let  $p \in (1, +\infty)$ ,  $q \in [1, +\infty)$ ,  $s > -1 + 1/p$ , such that  $(\mathcal{C}_{s,p,q})$  is satisfied. The space  $C_c^\infty(\Omega)$  is dense in  $\dot{B}_{p,q,0}^s(\Omega)$ .*

*Proof.* — For  $s \in (-1 + 1/p, 1/p)$ , the result is proved in Corollary 2.19. Now assume  $s > 0$ . Let  $u \in \dot{B}_{p,q,0}^s(\Omega) \subset \dot{B}_{p,q}^s(\mathbb{R}^n)$ , then for  $(u_k)_{k \in \mathbb{N}}$  a sequence in  $B_{p,q}^s(\mathbb{R}^n)$  which converges to  $u$  in  $\dot{B}_{p,q}^s(\mathbb{R}^n)$ . By Proposition 2.20,  $\mathcal{P}_0 u_k \in B_{p,q,0}^s(\Omega)$  and we have

$$\|u - \mathcal{P}_0 u_k\|_{\dot{B}_{p,q,0}^s(\Omega)} \lesssim_{s,p,q,\partial\Omega} \|u - u_k\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} \xrightarrow{k \rightarrow +\infty} 0.$$

The density of  $C_c^\infty(\Omega)$  in  $B_{p,q,0}^s(\Omega)$  yields the result.  $\blacksquare$

**Proposition 2.22** *Let  $(p_0, p_1, p, q, q_0, q_1) \in (1, +\infty)^3 \times [1, +\infty]^3$ ,  $s_0, s_1 \in \mathbb{R}$ , such that  $s_0 < s_1$ , let  $(\mathfrak{h}, \mathfrak{b}) \in \{(\mathbb{H}, \mathbb{B}), (\mathbb{H}_0, \mathbb{B}_{\cdot, \cdot, 0})\}$ , and set for  $\theta \in (0, 1)$ ,*

$$\left(s, \frac{1}{p_\theta}, \frac{1}{q_\theta}\right) := (1 - \theta) \left(s_0, \frac{1}{p_0}, \frac{1}{q_0}\right) + \theta \left(s_1, \frac{1}{p_1}, \frac{1}{q_1}\right).$$

*If either one of following assertions is satisfied,*

- (i)  $s_j \in (-1 + \frac{1}{p_j}, 1]$ ,  $j \in \{0, 1\}$ ;
- (ii)  $s_j \geq 0$ ,  $j \in \{0, 1\}$ ;

*If  $p_0 = p_1 = p$  and  $(\mathcal{C}_{s,p,q})$  is satisfied, the following equality is true with equivalence of norms*

$$(\dot{\mathfrak{h}}^{s_0,p}(\Omega), \dot{\mathfrak{h}}^{s_1,p}(\Omega))_{\theta,q} = \dot{\mathfrak{b}}_{p,q}^s(\Omega). \quad (2.18)$$

*If  $(\mathcal{C}_{s_j,p_j,q_j})$  is true for  $j \in \{0, 1\}$ , with  $s_j < 1$  in case (i),  $s_j > 0$  in case (ii), then also is  $(\mathcal{C}_{s,p_\theta,q_\theta})$  and*

$$[\dot{\mathfrak{b}}_{p_0,q_0}^{s_0}(\Omega), \dot{\mathfrak{b}}_{p_1,q_1}^{s_1}(\Omega)]_\theta = \dot{\mathfrak{b}}_{p_\theta,q_\theta}^s(\Omega). \quad (2.19)$$

*Proof.* — The proof is *verbatim* the one of [Gau22, Proposition 2.33], even if the statement seems quite confusing due to additional conditions, due to the use of two different extension operators. We propose to reassure the reader by giving the proof anyway.

**Step 1:** First, we assume  $(\mathcal{C}_{s_j,p_j,q_j})$  in the case of Besov space,  $(\mathcal{C}_{s_j,p_j})$  in the case of Sobolev spaces  $j \in \{0, 1\}$ . We have retractions given by

$$\begin{aligned} \mathbb{E} : \dot{H}^{s_j,p_j}(\Omega) &\longrightarrow \dot{H}^{s_j,p_j}(\mathbb{R}^n) \text{ and } R_\Omega : \dot{H}^{s_j,p_j}(\mathbb{R}^n) \longrightarrow \dot{H}^{s_j,p_j}(\Omega), \\ \dot{B}_{p_j,q_j}^{s_j}(\Omega) &\longrightarrow \dot{B}_{p_j,q_j}^{s_j}(\mathbb{R}^n) \quad \dot{B}_{p_j,q_j}^{s_j}(\mathbb{R}^n) \longrightarrow \dot{B}_{p_j,q_j}^{s_j}(\Omega), \\ \iota : \dot{H}_0^{s_j,p_j}(\Omega) &\longrightarrow \dot{H}^{s_j,p_j}(\mathbb{R}^n) \text{ and } \mathcal{P}_0 : \dot{H}^{s_j,p_j}(\mathbb{R}^n) \longrightarrow \dot{H}_0^{s_j,p_j}(\Omega), \\ \dot{B}_{p_j,q_j,0}^{s_j}(\Omega) &\longrightarrow \dot{B}_{p_j,q_j}^{s_j}(\mathbb{R}^n) \quad \dot{B}_{p_j,q_j}^{s_j}(\mathbb{R}^n) \longrightarrow \dot{B}_{p_j,q_j,0}^{s_j}(\Omega). \end{aligned}$$

where  $\mathbb{E}$  and  $\mathcal{P}_0$  are given by Propositions 2.11, 2.12 and 2.20.  $R_\Omega$  stands for the restriction operator, and  $\iota$  for the canonical embedding. In particular, the interpolation identity (2.19) holds.

**Step 2:** The interpolation identity (2.18) for  $(\mathfrak{h}, \mathfrak{b}) = (\mathbb{H}, \mathbb{B})$  when  $q < +\infty$ .

Thanks to (2.1) in Lemma 2.1, we have continuous embedding,

$$\dot{B}_{p,q}^s(\Omega) \hookrightarrow (\dot{H}^{s_0,p}(\Omega), \dot{H}^{s_1,p}(\Omega))_{\theta,q}.$$

Let us prove the reverse embedding,

$$\dot{B}_{p,q}^s(\Omega) \hookleftarrow (\dot{H}^{s_0,p}(\Omega), \dot{H}^{s_1,p}(\Omega))_{\theta,q}.$$

Without loss of generality, we can assume  $s_1 \geq \frac{n}{p}$ . Let  $f \in \mathcal{S}_0(\overline{\Omega}) \subset \dot{B}_{p,q}^s(\Omega)$ , it follows that  $f \in (\dot{H}^{s_0,p}(\Omega), \dot{H}^{s_1,p}(\Omega))_{\theta,q} \subset \dot{H}^{s_0,p}(\Omega) + \dot{H}^{s_1,p}(\Omega)$ . Thus, for all  $(a, b) \in \dot{H}^{s_0,p}(\Omega) \times \dot{H}^{s_1,p}(\Omega)$  such that  $f = a + b$ , we have,

$$b = f - a \in (\mathcal{S}_0(\overline{\Omega}) + \dot{H}^{s_0,p}(\Omega)) \cap \dot{H}^{s_1,p}(\Omega).$$

In particular, we have  $a \in \dot{H}^{s_0,p}(\Omega)$  and  $b \in \dot{H}^{s_0,p}(\Omega) \cap \dot{H}^{s_1,p}(\Omega)$ . Hence, for  $F := \mathbb{E}a + \mathbb{E}b$ , we have  $F|_{\Omega} = f$ ,  $\mathbb{E}a \in \dot{H}^{s_0,p}(\mathbb{R}^n)$  and  $\mathbb{E}b \in \dot{H}^{s_0,p}(\mathbb{R}^n) \cap \dot{H}^{s_1,p}(\mathbb{R}^n)$ , with the estimates, given by Proposition 2.11,

$$\|\mathbb{E}a\|_{\dot{H}^{s_0,p}(\mathbb{R}^n)} \lesssim_{s_0,\partial\Omega,p,n} \|a\|_{\dot{H}^{s_0,p}(\Omega)} \quad \text{and} \quad \|\mathbb{E}b\|_{\dot{H}^{s_1,p}(\mathbb{R}^n)} \lesssim_{s_1,\partial\Omega,p,n} \|b\|_{\dot{H}^{s_1,p}(\Omega)}.$$

Then, one may bound the  $K$ -functional of  $F$ , for  $t > 0$ ,

$$K(t, F, \dot{H}^{s_0,p}(\mathbb{R}^n), \dot{H}^{s_1,p}(\mathbb{R}^n)) \leq \|\mathbb{E}a\|_{\dot{H}^{s_0,p}(\mathbb{R}^n)} + t\|\mathbb{E}b\|_{\dot{H}^{s_1,p}(\mathbb{R}^n)} \lesssim_{s_j,p,n} \|a\|_{\dot{H}^{s_0,p}(\Omega)} + t\|b\|_{\dot{H}^{s_1,p}(\Omega)}$$

Taking the infimum over all such functions  $a$  and  $b$ , we obtain

$$K(t, F, \dot{H}^{s_0,p}(\mathbb{R}^n), \dot{H}^{s_1,p}(\mathbb{R}^n)) \lesssim_{s_j,p,n} K(t, f, \dot{H}^{s_0,p}(\Omega), \dot{H}^{s_1,p}(\Omega)),$$

from which we obtain, after multiplying by  $t^{-\theta}$ , taking the  $L^q_*$ -norm with respect to  $t$ , and applying Proposition 1.3,

$$\|f\|_{\dot{B}^s_{p,q}(\Omega)} \leq \|F\|_{\dot{B}^s_{p,q}(\mathbb{R}^n)} \lesssim_{s,p,n} \|f\|_{(\dot{H}^{s_0,p}(\Omega), \dot{H}^{s_1,p}(\Omega))_{\theta,q}}.$$

Finally, thanks to the first embedding (2.1), we have

$$\|f\|_{\dot{B}^s_{p,q}(\Omega)} \sim_{p,s,n,\partial\Omega} \|f\|_{(\dot{H}^{s_0,p}(\Omega), \dot{H}^{s_1,p}(\Omega))_{\theta,q}}, \quad \forall f \in \mathcal{S}_0(\overline{\Omega}).$$

Since  $q < +\infty$ , we can conclude by density of  $\mathcal{S}_0(\overline{\Omega})$  in both  $\dot{B}^s_{p,q}(\Omega)$  and in the interpolation space  $(\dot{H}^{s_0,p}(\Omega), \dot{H}^{s_1,p}(\Omega))_{\theta,q}$ . Density argument for the later one is carried over by Proposition 2.11 and [BL76, Theorem 3.4.2].

**Step 3:** The interpolation identity (2.18) for  $(\mathfrak{h}, \mathfrak{b}) = (H_0, B_0)$  when  $q < +\infty$ .

Thanks the elementary embedding (2.2) in Lemma 2.1, we have continuous embedding,

$$(\dot{H}_0^{s_0,p}(\Omega), \dot{H}_0^{s_1,p}(\Omega))_{\theta,q} \hookrightarrow \dot{B}^s_{p,q,0}(\Omega).$$

We are going to prove the reverse embedding,

$$(\dot{H}_0^{s_0,p}(\Omega), \dot{H}_0^{s_1,p}(\Omega))_{\theta,q} \hookleftarrow \dot{B}^s_{p,q,0}(\Omega).$$

Again, without loss of generality we can assume  $s_1 \geq \frac{n}{p}$ , otherwise one can go back to **Step 1**. Let us consider  $u \in C_c^\infty(\Omega)$ , then  $u$  belongs to  $\dot{H}^{s_0,p}(\mathbb{R}^n) + \dot{H}^{s_1,p}(\mathbb{R}^n)$ . In particular for  $(a, b) \in \dot{H}^{s_0,p}(\mathbb{R}^n) \times \dot{H}^{s_1,p}(\mathbb{R}^n)$ , such that  $u = a + b$  we have

$$b = u - a \in (C_c^\infty(\Omega) + \dot{H}^{s_0,p}(\mathbb{R}^n)) \cap \dot{H}^{s_1,p}(\mathbb{R}^n).$$

in particular we have  $a \in \dot{H}^{s_0,p}(\mathbb{R}^n)$  and  $b \in \dot{H}^{s_0,p}(\mathbb{R}^n) \cap \dot{H}^{s_1,p}(\mathbb{R}^n)$ .

Consequently, we have  $u = \mathcal{P}_0 u = \mathcal{P}_0 a + \mathcal{P}_0 b$ , with  $\mathcal{P}_0 a \in \dot{H}_0^{s_0,p}(\Omega)$  and  $\mathcal{P}_0 b \in \dot{H}_0^{s_0,p}(\Omega) \cap \dot{H}_0^{s_1,p}(\Omega)$ , with the estimates

$$\|\mathcal{P}_0 a\|_{\dot{H}_0^{s_0,p}(\Omega)} \lesssim_{s_0,p,n,\partial\Omega} \|a\|_{\dot{H}^{s_0,p}(\mathbb{R}^n)} \quad \text{and} \quad \|\mathcal{P}_0 b\|_{\dot{H}_0^{s_1,p}(\Omega)} \lesssim_{s_1,p,n,\partial\Omega} \|b\|_{\dot{H}^{s_1,p}(\mathbb{R}^n)},$$

thanks to Proposition 2.12. Thus, one may follow the lines of **Step 2**, to obtain for all  $u \in C_c^\infty(\Omega)$ ,

$$\|u\|_{\dot{B}^s_{p,q,0}(\Omega)} \sim_{s,p,n,\partial\Omega} \|u\|_{(\dot{H}_0^{s_0,p}(\Omega), \dot{H}_0^{s_1,p}(\Omega))_{\theta,q}}.$$

Again, one can conclude via density arguments since  $q < +\infty$ , and  $C_c^\infty(\Omega)$  is dense in  $\dot{B}^s_{p,q,0}(\Omega)$  thanks to Proposition 2.21.

**Step 4:** The interpolation identity (2.18) when  $q = +\infty$ .

In the case of  $(\mathfrak{h}, \mathfrak{b}) = (H, B)$ , the reiteration theorem [BL76, Theorem 3.5.3] in combination with **Step 1** and **Step 2** yields the identity.

Similarly for the case of  $(\mathfrak{h}, \mathfrak{b}) = (H_0, B_0)$  with **Step 1** and **Step 3**. ■

**Lemma 2.23** *Let  $p_j \in (1, +\infty)$ ,  $q_j \in [1, +\infty]$ ,  $s_j > -1 + \frac{1}{p_j}$ ,  $j \in \{0, 1\}$ , such that  $(\mathcal{C}_{s_0,p_0,q_0})$  is satisfied and consider the extension operator  $\mathbb{E}$  given by Proposition 2.20.*

*Then for all  $u \in \dot{B}^{s_0}_{p_0,q_0}(\Omega) \cap \dot{B}^{s_1}_{p_1,q_1}(\Omega)$ , we have  $\mathbb{E}u \in \dot{B}^{s_j}_{p_j,q_j}(\mathbb{R}^n)$ ,  $j \in \{0, 1\}$ , with the estimate*

$$\|\mathbb{E}u\|_{\dot{B}^{s_j}_{p_j,q_j}(\mathbb{R}^n)} \lesssim_{s_j,p_j,n,\partial\Omega} \|u\|_{\dot{B}^{s_j}_{p_j,q_j}(\Omega)}.$$

*The same result holds replacing  $(\mathbb{E}, \dot{B}^{s_j}_{p_j,q_j}(\Omega), \dot{B}^{s_j}_{p_j,q_j}(\mathbb{R}^n))$  by  $(\mathcal{P}_0, \dot{B}^{s_j}_{p_j,q_j}(\mathbb{R}^n), \dot{B}^{s_j}_{p_j,q_j,0}(\Omega))$ , where  $\mathcal{P}_0$*

is the projection operator given in Proposition 2.20.

*Proof.* — It suffices to apply real interpolation to prove the boundedness

$$\Xi_\Omega : \ell_s^q(\mathbb{Z}, L^p(\mathbb{R}^n)) \longrightarrow \ell_s^q(\mathbb{Z}, L^p(\mathbb{R}^n))$$

from the proof of Proposition 2.3, where  $s > 0$ ,  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$ .

Finally, the result follows reproducing the arguments in the proofs of Propositions 2.3 and 2.11. ■

**Proposition 2.24** *Let  $p_j \in (1, +\infty)$ ,  $q_j \in [1, +\infty]$ ,  $j \in \{0, 1\}$ ,  $-1 + \frac{1}{p} < s_0 < s_1$ , such that  $(\mathcal{C}_{s_0, p_0, q_0})$  is satisfied. We assume moreover that either  $s_0 > 0$ , or  $s_1 < 1$ . Then the following equality of vector spaces holds with equivalence of norms*

$$\dot{B}_{p_0, q_0}^{s_0}(\Omega) \cap \dot{B}_{p_1, q_1}^{s_1}(\Omega) = [\dot{B}_{p_0, q_0}^{s_0} \cap \dot{B}_{p_1, q_1}^{s_1}](\Omega).$$

*In particular,  $\dot{B}_{p_0, q_0}^{s_0}(\Omega) \cap \dot{B}_{p_1, q_1}^{s_1}(\Omega)$  is a Banach space, and it admits  $\mathcal{S}_0(\overline{\Omega})$  as a dense subspace whenever  $q_j < +\infty$ ,  $j \in \{0, 1\}$ .*

*Similarly, the following equality with equivalence of norms holds for all  $s > 0$ ,  $q \in [1, +\infty]$ ,*

$$L^p(\Omega) \cap \dot{B}_{p, q}^s(\Omega) = B_{p, q}^s(\Omega).$$

*Proof.* — This is a direct consequence of Lemma 2.23. ■

**Proposition 2.25** *Let  $p_j \in (1, +\infty)$ ,  $q_j \in [1, +\infty]$ ,  $j \in \{0, 1\}$ , such that  $(\mathcal{C}_{s_0, p_0, q_0})$  is satisfied. For all  $u \in \dot{B}_{p_0, q_0}^{s_0}(\Omega) \cap \dot{B}_{p_1, q_1}^{s_1}(\Omega)$ ,*

$$\|\nabla^{m_j} u\|_{\dot{B}_{p_j, q_j}^{s_j - m_j}(\Omega)} \sim_{s_j, m_j, p_j, n} \|u\|_{\dot{B}_{p_j, q_j}^{s_j}(\Omega)}.$$

*Proof.* — The proof follows the lines of the proof of Proposition 2.10. ■

**Proposition 2.26** *Let  $(p, q, q_0, q_1) \in (1, +\infty) \times [1, +\infty]^3$ ,  $s_0, s_1 \in \mathbb{R}$ , such that  $s_0 < s_1$ .*

*Let  $\mathfrak{b} \in \{\mathbb{B}, \mathbb{B}, \cdot, \cdot, 0\}$ , and set for  $\theta \in (0, 1)$ ,*

$$s := (1 - \theta)s_0 + \theta s_1.$$

*Assume  $q_j < +\infty$  if  $(\mathcal{C}_{s_j, p_j, q_j})$  is **not** true for  $j \in \{0, 1\}$ , and that one of following assertions is satisfied,*

(i)  $s_j \in (-1 + \frac{1}{p_j}, 1)$ ,  $j \in \{0, 1\}$ ;

(ii)  $s_j > 0$ ,  $j \in \{0, 1\}$ ;

*If  $(\mathcal{C}_{s, p, q})$  is satisfied, the following equality is true with equivalence of norms*

$$(\dot{\mathfrak{b}}_{p, q_0}^{s_0}(\Omega), \dot{\mathfrak{b}}_{p, q_1}^{s_1}(\Omega))_{\theta, q} = \dot{\mathfrak{b}}_{p, q}^s(\Omega). \quad (2.20)$$

*Proof.* — The proof follows the lines of the proof of Proposition 2.22. ■

Finally, we claim the density result for  $\dot{B}_{p, \infty}^s$  spaces, whose proof is an exact copy of the same statement [Gau22, Corollary 2.35]. The same goes for the duality result that follows, see [Gau22, Proposition 2.40].

**Proposition 2.27** *Let  $p \in (1, +\infty)$ ,  $s > -1 + 1/p$ , such that  $(\mathcal{C}_{s, p, \infty})$  is satisfied.*

- *The space  $C_c^\infty(\Omega)$  is weak\* dense in  $\dot{B}_{p, \infty, 0}^s(\Omega)$ .*
- *The space  $\mathcal{S}_0(\overline{\Omega})$  is weak\* dense in  $\dot{B}_{p, \infty}^s(\Omega)$ .*

**Proposition 2.28** *Let  $p \in (1, +\infty)$ ,  $q \in (1, +\infty]$ ,  $s > -1 + \frac{1}{p}$ , if  $(\mathcal{C}_{s, p, q})$  is satisfied then the following isomorphisms hold*

$$(\dot{B}_{p', q', 0}^{-s}(\Omega))' = \dot{B}_{p, q}^s(\Omega) \quad \text{and} \quad (\dot{B}_{p', q'}^{-s}(\Omega))' = \dot{B}_{p, q, 0}^s(\Omega).$$

### 3 The trace theorem

In the previous section, an appropriate construction of homogeneous Sobolev and Besov spaces on special Lipschitz domains was given with their interpolation properties. Now, we want to make sense of boundary values in homogeneous function spaces which stand for the main interest of the present paper.

The first subsection is devoted to the construction of function spaces on the boundary.

The second one concerns the transference of properties on the flat upper half-space to the bent one via the global change of coordinates. However, we want to reach the sharp range of regularity  $(1/p, 1 + 1/p)$  for the trace result. The main issue will occur when  $s \in [1, 1 + 1/p)$ , since we do not have more than one full gradient under the action of the global change of coordinate. To circumvent this issue, we introduce an anisotropic trace result inherited from the recent work of the author [Gau23b, Theorem 4.7]. This result is obtained from the  $\dot{H}^{s-1,p}(L^p)$ -maximal regularity for the Poisson semigroup  $(e^{-t(-\Delta')^{1/2}})_{t \geq 0}$  on  $\mathbb{R}^{n-1}$ .

The last section is devoted to the statement of the main theorem, and several straightforward consequences.

#### 3.1 Function spaces on the boundary

To define the trace as in the case of inhomogeneous function spaces, we have to define first (homogeneous) Besov spaces on the boundary  $\partial\Omega$ . To do so, since we have  $\Omega = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid x_n > \phi(x')\}$ , where  $\phi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  is uniformly Lipschitz, we recall that the surface measure on the boundary  $\partial\Omega = \{(x', \phi(x')), x' \in \mathbb{R}^{n-1}\} \subset \mathbb{R}^n$  is defined as

$$\sigma(A) := \int_{\mathbb{R}^{n-1}} 1_A(x', \phi(x')) \sqrt{1 + |\nabla' \phi(x')|^2} dx',$$

where  $A$  is any Lebesgue-measurable set of  $\partial\Omega$ .

We also recall that  $\sigma$  is the unique Borel measure on  $\partial\Omega$  so that we have the integration by parts formula

$$\int_{\Omega} \partial_{x_k} u(x) v(x) dx = - \int_{\Omega} u(x) \partial_{x_k} v(x) dx + \int_{\partial\Omega} u(x) v(x) \nu_k(x) d\sigma_x, \quad k \in \llbracket 1, n \rrbracket, \quad (3.1)$$

provided  $u, v \in C_c^{0,1}(\mathbb{R}^n)$ , the space of complex-valued compactly supported Lipschitz functions. And in (3.1),  $\nu_j$  stands for the  $j$ -th component of the outward unit normal of  $\Omega$ , defined almost everywhere on  $\partial\Omega$  by

$$\nu := \frac{1}{\sqrt{|\nabla' \phi|^2 + 1}} (\nabla' \phi, -1).$$

We introduce the pushforward map from  $\partial\Omega$  to  $\mathbb{R}^{n-1}$  for any measurable function  $u : \partial\Omega \rightarrow \mathbb{C}$ ,

$$S_\phi u(x') := u(x', \phi(x')), \quad x' \in \mathbb{R}^{n-1}. \quad (3.2)$$

We also have the pullback map defined for any measurable function  $v : \mathbb{R}^{n-1} \rightarrow \mathbb{C}$ ,

$$S_\phi^{-1} v(y) := v(y'), \quad y \in \partial\Omega. \quad (3.3)$$

To construct the homogeneous function spaces on the boundary, we are going to follow the ideas given in [DM15, Chapter 2, Section 2.2] and [Din96, Section 2].

**Definition 3.1** For  $p \in [1, +\infty)$ ,  $s \in (0, 1)$ , for any measurable function  $f$  on  $\partial\Omega$ , we define the following quantities

$$\|f\|_{L^p(\partial\Omega)}^p := \int_{\partial\Omega} |f(x)|^p d\sigma_x, \quad \|f\|_{\dot{B}_{p,p}^s(\partial\Omega)}^p := \int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{ps+n-1}} d\sigma_x d\sigma_y,$$

with the usual modification when  $p = +\infty$ . We set, assuming  $p \neq 1, +\infty$  for the last case,

- $L^p(\partial\Omega) := \{u : \partial\Omega \rightarrow \mathbb{C} \text{ meas.} \mid \|u\|_{L^p(\partial\Omega)} < +\infty\}$ ,
- $\dot{B}_{p,p}^s(\partial\Omega) := \{u \in L_{\text{loc}}^1(\partial\Omega) \mid S_\phi u \in S'_h(\mathbb{R}^{n-1}) \text{ \& } \|u\|_{\dot{B}_{p,p}^s(\partial\Omega)} < +\infty\}$

- $\dot{H}^{1,p}(\partial\Omega) := \{ u \in L^1_{\text{loc}}(\partial\Omega) \mid S_\phi u \in \mathcal{S}'_h(\mathbb{R}^{n-1}) \text{ \& } \|u\|_{\dot{H}^{1,p}(\partial\Omega)} := \|S_\phi^{-1}[\nabla' S_\phi u]\|_{L^p(\partial\Omega)} < +\infty \}$ .

The next lemmas justify that these definitions are meaningful.

**Lemma 3.2** *Let  $p \in [1, +\infty]$ . The map*

$$S_\phi : L^p(\partial\Omega) \longrightarrow L^p(\mathbb{R}^{n-1})$$

*is well defined and  $a$  is continuous isomorphism of normed vector spaces. The same result holds with  $L^p_{\text{loc}}$ , instead of  $L^p$ , as complete metric spaces.*

*Proof.* — By direct computations, we obtain for all  $p \in [1, +\infty]$ ,  $u \in L^p(\partial\Omega)$ ,

$$\|S_\phi u\|_{L^p(\mathbb{R}^{n-1})} \leq \|u\|_{L^p(\partial\Omega)} \leq (1 + \|\nabla' \phi\|_{L^\infty}^2)^{\frac{1}{2p}} \|S_\phi u\|_{L^p(\mathbb{R}^{n-1})}.$$

The same goes for  $S_\phi^{-1}$ .

For the  $L^p_{\text{loc}}$  isomorphism property, thanks to the last inequality, it suffices to exhibit two appropriate families of compact subsets.

For  $a, b \in \mathbb{R}^{n-1}$ , with  $a_j < b_j$ ,  $j \in [1, n]$ , we define

$$Q_{(a,b)} := \prod_{j=1}^{n-1} [a_j, b_j] \text{ and } K_{(a,b)} := \{ (x', \phi(x')), x' \in Q_{(a,b)} \}.$$

In this case  $(Q_{(a,b)})_{a < b}$  (resp.  $(K_{(a,b)})_{a < b}$ ) is a family of compact subsets of  $\mathbb{R}^{n-1}$  (resp.  $\partial\Omega$ ), such that

$$(S_\phi 1_{Q_{(a,b)}})_{a < b} = (1_{K_{(a,b)}})_{a < b} \text{ and } (S_\phi^{-1} 1_{K_{(a,b)}})_{a < b} = (1_{Q_{(a,b)}})_{a < b}. \quad \blacksquare$$

The following corollary is a direct consequence of Lemma 3.2.

**Corollary 3.3** *Let  $p \in (1, +\infty)$ . For  $u \in \dot{H}^{1,p}(\partial\Omega)$ , we have  $S_\phi u \in \dot{H}^{1,p}(\mathbb{R}^{n-1}) \cap L^1_{\text{loc}}(\mathbb{R}^{n-1})$ , with the estimate*

$$\|u\|_{\dot{H}^{1,p}(\partial\Omega)} \sim_{s,p,n,\partial\Omega} \|S_\phi u\|_{\dot{H}^{1,p}(\mathbb{R}^{n-1})}.$$

*Conversely, if  $v \in \dot{H}^{1,p}(\mathbb{R}^{n-1}) \cap L^1_{\text{loc}}(\mathbb{R}^{n-1})$ , then  $S_\phi^{-1} v \in \dot{H}^{1,p}(\partial\Omega)$  with the corresponding estimate.*

The idea behind the definition of Besov spaces on the boundary lies in the fact that for all  $u \in L^1_{\text{loc}}(\mathbb{R}^{n-1}) \cap \mathcal{S}'_h(\mathbb{R}^{n-1})$ , when  $s \in (0, 1)$ ,  $p \in [1, +\infty)$ ,

$$\|f\|_{\dot{B}^s_{p,p}(\mathbb{R}^{n-1})}^p \sim_{p,s,n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(x) - f(y)|^p}{|x - y|^{ps+n-1}} dx dy, \quad (3.4)$$

see [BCD11, Theorem 2.36] for a proof. The case  $p = +\infty$  is treated via usual modification with homogeneous Hölder (semi-)norms.

**Lemma 3.4** *Let  $p \in [1, +\infty]$ ,  $s \in (0, 1)$ . For all  $u \in \dot{B}^s_{p,p}(\partial\Omega)$ ,  $S_\phi u \in \dot{B}^s_{p,p}(\mathbb{R}^{n-1}) \cap L^1_{\text{loc}}(\mathbb{R}^{n-1})$  with the estimate*

$$\|u\|_{\dot{B}^s_{p,p}(\partial\Omega)} \sim_{s,p,n,\partial\Omega} \|S_\phi u\|_{\dot{B}^s_{p,p}(\mathbb{R}^{n-1})}$$

*Conversely, for  $v \in \dot{B}^s_{p,p}(\mathbb{R}^{n-1}) \cap L^1_{\text{loc}}(\mathbb{R}^{n-1})$ , one has  $S_\phi^{-1} v \in \dot{B}^s_{p,p}(\partial\Omega)$  with corresponding estimate.*

*Proof.* — For  $u \in L^1_{\text{loc}}(\partial\Omega)$ , if  $p < +\infty$ ,

$$\begin{aligned} \|u\|_{\dot{B}^s_{p,p}(\partial\Omega)}^p &= \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{ps+n-1}} d\sigma_x d\sigma_y \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|u(x', \phi(x')) - u(y', \phi(y'))|^p}{|(x' - y', \phi(x') - \phi(y'))|^{ps+n-1}} \sqrt{|\nabla' \phi(x')|^2 + 1} \sqrt{|\nabla' \phi(y')|^2 + 1} dx' dy' \\ &\leq [1 + \|\nabla' \phi\|_{L^\infty}^2] \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|S_\phi u(x') - S_\phi u(y')|^p}{|x' - y'|^{ps+n-1}} dx' dy' \\ &\lesssim_{p,s,n,\partial\Omega} \|S_\phi u\|_{\dot{B}^s_{p,p}(\mathbb{R}^{n-1})}^p. \end{aligned}$$

The last estimate comes from (3.4).

For the reverse estimate, we start with (3.4) then, we obtain

$$\begin{aligned} \|S_\phi u\|_{\dot{B}_{p,p}^s(\mathbb{R}^{n-1})}^p &\lesssim_{p,s,n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|S_\phi u(x') - S_\phi u(y')|^p}{|x' - y'|^{ps+n-1}} dx' dy' \\ &\lesssim_{p,s,n} [1 + \|\nabla' \phi\|_{L^\infty}^2]^{\frac{(n-1)+ps}{2}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|u(x', \phi(x')) - u(y', \phi(y'))|^p}{|(x' - y', \phi(x') - \phi(y'))|^{ps+n-1}} dx' dy' \\ &\lesssim_{p,s,n,\partial\Omega} \int_{\partial\Omega} \int_{\partial\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{ps+n-1}} d\sigma_x d\sigma_y = \|u\|_{\dot{B}_{p,p}^s(\partial\Omega)}^p. \end{aligned}$$

The case  $p = +\infty$  is similar and left to the reader with

$$\|w\|_{\dot{B}_{\infty,\infty}^s(\partial\Omega)} = \sup_{\substack{(x,y) \in \partial\Omega^2, \\ x \neq y}} \frac{|w(x) - w(y)|}{|x - y|^s}. \quad \blacksquare$$

**Proposition 3.5** *Let  $p \in (1, +\infty)$ ,  $s \in (0, 1)$ . The following equality holds with equivalence of norms*

$$(L^p(\partial\Omega), \dot{H}^{1,p}(\partial\Omega))_{s,p} = \dot{B}_{p,p}^s(\partial\Omega). \quad \blacksquare$$

*Proof.* — Let  $u \in L^p(\partial\Omega) + \dot{H}^{1,p}(\partial\Omega) \subset L_{\text{loc}}^1(\partial\Omega)$ , then for  $(a, b) \in L^p(\partial\Omega) \times \dot{H}^{1,p}(\partial\Omega)$  such that  $u = a + b$ , by Corollary 3.3, we have

$$S_\phi u = S_\phi a + S_\phi b \in L^p(\mathbb{R}^{n-1}) + \dot{H}^{1,p}(\mathbb{R}^{n-1}). \quad (3.5)$$

Therefore, by the definition of the  $K$ -functional and Corollary 3.3, we obtain

$$\begin{aligned} K(t, S_\phi u, L^p(\mathbb{R}^{n-1}), \dot{H}^{1,p}(\mathbb{R}^{n-1})) &\leq \|S_\phi a\|_{L^p(\mathbb{R}^{n-1})} + t \|S_\phi b\|_{\dot{H}^{1,p}(\mathbb{R}^{n-1})} \\ &\lesssim_{p,n,\partial\Omega} \|a\|_{L^p(\partial\Omega)} + t \|b\|_{\dot{H}^{1,p}(\partial\Omega)}. \end{aligned}$$

Looking at the infimum on all such pair  $(a, b)$  yields

$$K(t, S_\phi u, L^p(\mathbb{R}^{n-1}), \dot{H}^{1,p}(\mathbb{R}^{n-1})) \lesssim_{p,n,\partial\Omega} K(t, u, L^p(\partial\Omega), \dot{H}^{1,p}(\partial\Omega)).$$

Now, for the reverse estimate from (3.5), let  $(A, B) \in L^p(\mathbb{R}^{n-1}) \times \dot{H}^{1,p}(\mathbb{R}^{n-1})$ , such that one has  $S_\phi u = A + B$ , since  $S_\phi u, A \in L_{\text{loc}}^1(\mathbb{R}^{n-1})$ , it follows that

$$B = S_\phi u - A \in L_{\text{loc}}^1(\mathbb{R}^{n-1}) \cap \dot{H}^{1,p}(\mathbb{R}^{n-1}).$$

Hence, by Corollary 3.3,

$$u = S_\phi^{-1} A + S_\phi^{-1} B \in L^p(\partial\Omega) + \dot{H}^{1,p}(\partial\Omega).$$

So as before, we obtain,

$$K(t, u, L^p(\partial\Omega), \dot{H}^{1,p}(\partial\Omega)) \lesssim_{p,n,\partial\Omega} K(t, S_\phi u, L^p(\mathbb{R}^{n-1}), \dot{H}^{1,p}(\mathbb{R}^{n-1})).$$

In the end we have obtained for all  $u \in L^p(\partial\Omega) + \dot{H}^{1,p}(\partial\Omega)$  and all  $t > 0$ :

$$K(t, u, L^p(\partial\Omega), \dot{H}^{1,p}(\partial\Omega)) \sim_{p,n,\partial\Omega} K(t, S_\phi u, L^p(\mathbb{R}^{n-1}), \dot{H}^{1,p}(\mathbb{R}^{n-1})). \quad (3.6)$$

Finally, if one multiplies by (3.6) by  $t^{-s}$ , then take its  $L_*^q$ -norm, thanks to (1.2) and Lemma 3.4 we obtain

$$\|u\|_{(L^p(\partial\Omega), \dot{H}^{1,p}(\partial\Omega))_{s,p}} \sim_{s,p,n,\partial\Omega} \|S_\phi u\|_{\dot{B}_{p,p}^s(\mathbb{R}^{n-1})} \sim_{s,p,n,\partial\Omega} \|u\|_{\dot{B}_{p,p}^s(\partial\Omega)}$$

which ends the proof.  $\blacksquare$

Now, we introduce the following definition of homogeneous Besov space on the boundary with third index  $q \neq p$ , consistent with the case  $q = p$ .

**Definition 3.6** For  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$ ,  $s \in (0, 1)$ , we define

$$\dot{B}_{p,q}^s(\partial\Omega) := (L^p(\partial\Omega), \dot{H}^{1,p}(\partial\Omega))_{s,q}.$$



The following results are then a direct consequence of the estimate (3.6) and usual results for homogeneous Sobolev and Besov spaces on  $\mathbb{R}^{n-1}$ .

**Corollary 3.7** *Let  $p \in (1, +\infty)$ ,  $s \in (0, 1)$ ,  $q \in [1, +\infty]$ . For all  $u \in \dot{B}_{p,q}^s(\partial\Omega)$ ,  $S_\phi u \in \dot{B}_{p,q}^s(\mathbb{R}^{n-1}) \cap L_{\text{loc}}^1(\mathbb{R}^{n-1})$  with the estimate*

$$\|u\|_{\dot{B}_{p,q}^s(\partial\Omega)} \sim_{s,p,n,\partial\Omega} \|S_\phi u\|_{\dot{B}_{p,q}^s(\mathbb{R}^{n-1})}$$

*Conversely, for  $v \in \dot{B}_{p,q}^s(\mathbb{R}^{n-1}) \cap L_{\text{loc}}^1(\mathbb{R}^{n-1})$ , one has  $S_\phi^{-1}v \in \dot{B}_{p,q}^s(\partial\Omega)$  with corresponding estimate.*

**Proposition 3.8** *Let  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$ ,  $s \in (0, 1)$ . The following assertions are true.*

(i)  $\dot{B}_{p,q}^s(\partial\Omega)$  is a Banach space whenever  $(\mathcal{C}_{s+\frac{1}{p},p,q})$  is satisfied.

(ii) If  $s \in (0, \frac{n-1}{p})$ , for  $\frac{1}{r} := \frac{1}{p} - \frac{s}{n-1}$ , if  $q \in [1, r]$ , we have the continuous embedding

$$\dot{B}_{p,q}^s(\partial\Omega) \hookrightarrow L^r(\partial\Omega).$$

(iii) When  $p > n - 1$ , we have the continuous embedding

$$\dot{B}_{p,1}^{\frac{n-1}{p}}(\partial\Omega) \hookrightarrow C_0^0(\partial\Omega).$$

**Remark 3.9** One could also check that the intersection space  $\dot{B}_{p_0,q_0}^{s_0}(\partial\Omega) \cap \dot{B}_{p_1,q_1}^{s_1}(\partial\Omega)$  is complete whenever  $(\mathcal{C}_{s_0+\frac{1}{p_0},p_0,q_0})$  is satisfied.

From there, it is straightforward to check that one can recover usual and very well known function spaces  $H^{1,p}(\partial\Omega) = L^p(\partial\Omega) \cap \dot{H}^{1,p}(\partial\Omega)$ ,  $B_{p,q}^s(\partial\Omega) = L^p(\partial\Omega) \cap \dot{B}_{p,q}^s(\partial\Omega)$ ,  $s \in (0, 1)$ ,  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$ .

## 3.2 Preliminary results for the trace theorem

The strategy of the proof will mainly arise from a flattening procedure of the boundary with anisotropic estimates as done in [Din96, Lemma 1, Lemma 2]. For the reader's convenience we recall, from (2.11) and (2.15), that for any measurable function  $u : \Omega \rightarrow \mathbb{C}$ ,

$$T_\phi u(x', x_n) = u(x', x_n + \phi(x')), \quad (x', x_n) \in \mathbb{R}^{n-1} \times [0, +\infty).$$

For  $p \in (1, +\infty)$ ,  $s \in [1, 2]$ , we introduce the function space

$$\mathcal{K}^{s,p}(\mathbb{R}_+^n) := H^{s,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1})) \cap H^{s-1,p}(\mathbb{R}_+, H^{1,p}(\mathbb{R}^{n-1}))$$

with its natural norm. We also introduce the homogeneous semi-norm

$$\|u\|_{\dot{\mathcal{K}}^{s,p}(\mathbb{R}_+^n)} := \|(\partial_{x_n} u, \nabla' u)\|_{\dot{H}^{s-1,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1}))}.$$

We notice that  $\mathcal{K}^{1,p}(\mathbb{R}_+^n) = H^{1,p}(\mathbb{R}_+^n)$ , and  $\|\cdot\|_{\dot{\mathcal{K}}^{1,p}(\mathbb{R}_+^n)} \sim_{p,n} \|\nabla \cdot\|_{L^p(\mathbb{R}_+^n)}$ .

During this section, we will need for a brief moment to use Banach valued (anisotropic) homogeneous Sobolev spaces for non-negative index and with values in a (reflexive) Lebesgue space. See the previous work of the author [Gau23b, Section 3.1] for an elementary construction of homogeneous vector-valued Riesz potential spaces and references therein for a more general review of vector-valued Sobolev (Bessel potential) spaces and their properties.

For  $p \in (1, +\infty)$ , provided  $0 \leq \alpha < 1/p$ , we define  $\frac{1}{r} := \frac{1}{p} - \alpha$ ,

$$\dot{H}^{\alpha,p}(\mathbb{R}, L^p(\mathbb{R}^{n-1})) := \left\{ u \in L^r(\mathbb{R}, L^p(\mathbb{R}^{n-1})) \mid (-\partial_{x_n}^2)^{\frac{\alpha}{2}} u \in L^p(\mathbb{R}, L^p(\mathbb{R}^{n-1})) = L^p(\mathbb{R}^n) \right\}.$$

We also define by restriction, in the sense of distributions, the corresponding space on the half line

$$\dot{H}^{\alpha,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1})) := \dot{H}^{\alpha,p}(\mathbb{R}, L^p(\mathbb{R}^{n-1}))|_{\mathbb{R}_+}.$$

This is a Banach space with respect to the quotient norm

$$\|u\|_{\dot{H}^{\alpha,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1}))} := \inf_{\substack{U|_{\mathbb{R}_+} = u, \\ U \in \dot{H}^{\alpha,p}(\mathbb{R}, L^p(\mathbb{R}^{n-1}))}} \|U\|_{\dot{H}^{\alpha,p}(\mathbb{R}, L^p(\mathbb{R}^{n-1}))}.$$

**Lemma 3.10** Let  $p \in (1, +\infty)$ ,  $\alpha \in [0, 1/p)$ . For all  $u \in \dot{H}^{\alpha,p}(\mathbb{R}_+^n)$ , we have the estimate

$$\|u\|_{\dot{H}_{x_n}^{\alpha,p}(\mathbb{R}_+, L_{x'}^p(\mathbb{R}^{n-1}))} \lesssim_{p,\alpha,n} \|u\|_{\dot{H}^{\alpha,p}(\mathbb{R}_+^n)}.$$

*Proof.* — On  $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ , the result follows from the boundedness of Riesz transforms. The case of the half-space follows from the definition of function spaces by restriction.  $\blacksquare$

**Lemma 3.11** For  $p \in (1, +\infty)$ ,  $s \in [1, 1 + 1/p)$ . Then the linear operator

$$T_\phi : H^{s,p}(\Omega) \longrightarrow \mathcal{K}^{s,p}(\mathbb{R}_+^n)$$

is well defined and bounded.

Moreover, for all  $u \in H^{s,p}(\Omega)$  we have the homogeneous estimate

$$\|T_\phi u\|_{\dot{\mathcal{K}}^{s,p}(\mathbb{R}_+^n)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\Omega)}. \quad (3.7)$$

*Proof.* — For the boundedness of  $T_\phi$  from  $H^{s,p}(\Omega)$  to  $\mathcal{K}^{s,p}(\mathbb{R}_+^n)$ , it suffices to follow the proof of [Din96, Lemma 2]. One may check the boundedness properties

$$\begin{aligned} T_\phi : H^{1,p}(\Omega) &\longrightarrow H^{1,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1})) \\ H^{2,p}(\Omega) &\longrightarrow H^{2,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1})) \end{aligned}$$

which imply by complex interpolation, that

$$T_\phi : H^{s,p}(\Omega) \longrightarrow H^{s,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1})) \quad (3.8)$$

is well defined and bounded for all  $s \in [1, 2]$ . Similarly, from the boundedness

$$\begin{aligned} T_\phi : H^{1,p}(\Omega) &\longrightarrow L^p(\mathbb{R}_+, H^{1,p}(\mathbb{R}^{n-1})) \\ H^{2,p}(\Omega) &\longrightarrow H^{1,p}(\mathbb{R}_+, H^{1,p}(\mathbb{R}^{n-1})), \end{aligned}$$

for  $s \in [1, 2]$ , we deduce that

$$T_\phi : H^{s,p}(\Omega) \longrightarrow H^{s-1,p}(\mathbb{R}_+, H^{1,p}(\mathbb{R}^{n-1})) \quad (3.9)$$

is well defined and bounded. Thus, (3.8) and (3.9) yield the boundedness of  $T_\phi$ . Now, we prove the estimate (3.7). For  $u \in H^{s,p}(\Omega) \subset \dot{H}^{s,p}(\Omega)$ , we have  $T_\phi u \in \mathcal{K}^{s,p}(\mathbb{R}_+^n)$ , and since

$$\partial_{x_k}(T_\phi u) = T_\phi(\partial_{x_k} u) + \partial_{x_k} \phi T_\phi(\partial_{x_n} u), \quad \partial_{x_n}(T_\phi u) = T_\phi(\partial_{x_n} u), \quad k \in \llbracket 1, n-1 \rrbracket,$$

we obtain,

$$\begin{aligned} \|T_\phi u\|_{\dot{\mathcal{K}}^{s,p}(\mathbb{R}_+^n)} &= \|(\partial_{x_n} T_\phi u, \nabla' T_\phi u)\|_{\dot{H}^{s-1,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1}))} \\ &\leq \|T_\phi \partial_{x_n} u\|_{\dot{H}^{s-1,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1}))} + \|\nabla' T_\phi u\|_{\dot{H}^{s-1,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1}))} \\ &\leq (1 + (n-1)\|\nabla' \phi\|_{L^\infty}) \|T_\phi \nabla' u\|_{\dot{H}^{s-1,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1}))} + \|T_\phi \partial_{x_n} u\|_{\dot{H}^{s-1,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1}))} \\ &\lesssim_{p,s,n,\partial\Omega} \|T_\phi \nabla u\|_{\dot{H}^{s-1,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1}))}. \end{aligned}$$

The estimate (3.7) is then a consequence of Lemma 3.10 and Proposition 2.5

$$\begin{aligned} \|T_\phi u\|_{\dot{\mathcal{K}}^{s,p}(\mathbb{R}_+^n)} &\lesssim_{p,s,n,\partial\Omega} \|T_\phi \nabla u\|_{\dot{H}^{s-1,p}(\mathbb{R}_+, L^p(\mathbb{R}^{n-1}))} \\ &\lesssim_{p,s,n,\partial\Omega} \|T_\phi \nabla u\|_{\dot{H}^{s-1,p}(\mathbb{R}_+^n)} \\ &\lesssim_{p,s,n,\partial\Omega} \|\nabla u\|_{\dot{H}^{s-1,p}(\Omega)} \\ &\lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\Omega)}. \quad \blacksquare \end{aligned}$$

One may notice that, for say all  $u \in H^{1,p}(\Omega)$ , we have in  $B_{p,p}^{1-1/p}(\partial\mathbb{R}_+^n)$ , the equality

$$[T_\phi u]_{|\partial\mathbb{R}_+^n} = S_\phi[u|_{\partial\Omega}].$$

**Lemma 3.12** Let  $p \in (1, +\infty)$ ,  $s \in [1, 1 + 1/p)$ . For all  $u \in H^{s,p}(\Omega)$ , we have

$$T_\phi u \in C_0^0(\overline{\mathbb{R}_+^n}, B_{p,p}^{s-1/p}(\mathbb{R}^{n-1})).$$

*Proof.* — For  $u \in \mathbf{H}^{s,p}(\Omega) \subset \mathbf{H}^{1,p}(\Omega)$ , then  $T_\phi u \in \mathbf{H}^{1,p}(\mathbb{R}_+^n) \cap \mathcal{K}^{s,p}(\mathbb{R}_+^n)$ , and by [Gau22, Corollary 3.4],

$$T_\phi u \in \mathbf{C}_0^0(\overline{\mathbb{R}_+^n}, \mathbf{B}_{p,p}^{1-1/p}(\mathbb{R}^{n-1})),$$

and  $[T_\phi u]_{|\partial\mathbb{R}_+^n} = S_\phi[u|_{\partial\Omega}]$  in  $\mathbf{B}_{p,p}^{1-1/p}(\mathbb{R}^{n-1})$ .

Therefore, if we set  $v(t, x') := T_\phi u(x', t)$ ,  $t \geq 0$ ,  $x' \in \mathbb{R}^{n-1}$ , we have

$$F := \partial_t v + (\mathbf{I} - \Delta')^{\frac{1}{2}} v \in \mathbf{H}^{s-1,p}(\mathbb{R}_+, \mathbf{L}^p(\mathbb{R}^{n-1})) \subset \mathbf{L}^p(\mathbb{R}_+, \mathbf{L}^p(\mathbb{R}^{n-1})),$$

$$\text{and } v(0, \cdot) = [T_\phi u]_{|\partial\mathbb{R}_+^n} = S_\phi[u|_{\partial\Omega}] \in \mathbf{B}_{p,p}^{s-1/p}(\mathbb{R}^{n-1}) \subset \mathbf{B}_{p,p}^{1-1/p}(\mathbb{R}^{n-1}) \subset \mathbf{L}^p(\mathbb{R}^{n-1}).$$

By uniqueness of the mild solution, for all  $t \geq 0$

$$v(t) = e^{-t(\mathbf{I} - \Delta')^{\frac{1}{2}}} v(0) + \int_0^t e^{-(t-s)(\mathbf{I} - \Delta')^{\frac{1}{2}}} F(s) \, ds.$$

Therefore, by [Gau23b, Theorem 4.7], since  $(\mathbf{I} - \Delta')^{\frac{1}{2}}$  is invertible on  $\mathbf{L}^p(\mathbb{R}^{n-1})$  with its domain  $\mathbf{D}_p((\mathbf{I} - \Delta')^{\frac{1}{2}}) = \mathbf{H}^{1,p}(\mathbb{R}^{n-1})$ , we have the following maximal regularity estimate

$$\begin{aligned} \|v\|_{\mathbf{L}^\infty(\mathbb{R}_+, \mathbf{B}_{p,p}^{s-1/p}(\mathbb{R}^{n-1}))} &\lesssim_{p,s,n} \|(\partial_t v, (\mathbf{I} - \Delta')^{\frac{1}{2}} v)\|_{\dot{\mathbf{H}}^{s-1,p}(\mathbb{R}_+, \mathbf{L}^p(\mathbb{R}^{n-1}))} \lesssim_{p,s,n} \|F\|_{\dot{\mathbf{H}}^{s-1,p}(\mathbb{R}_+, \mathbf{L}^p(\mathbb{R}^{n-1}))} \\ &\quad + \|v(0)\|_{\mathbf{B}_{p,p}^{s-1/p}(\mathbb{R}^{n-1})}, \end{aligned}$$

and  $v \in \mathbf{C}_0^0(\overline{\mathbb{R}_+^n}, \mathbf{B}_{p,p}^{s-1/p}(\mathbb{R}^{n-1}))$ . ■

**Corollary 3.13** *Let  $p \in (1, +\infty)$ ,  $s \in [1, 1 + 1/p)$ . For all  $u \in \mathbf{H}^{s,p}(\Omega)$ ,*

$$\|T_\phi u\|_{\mathbf{L}^\infty(\mathbb{R}_+, \dot{\mathbf{B}}_{p,p}^{s-1/p}(\mathbb{R}^{n-1}))} \lesssim_{p,s,n,\partial\Omega} \|T_\phi u\|_{\dot{\mathcal{K}}^{s,p}(\mathbb{R}_+^n)}.$$

*Proof.* — By Lemma 3.12,

$$T_\phi u \in \mathbf{C}_0^0(\overline{\mathbb{R}_+^n}, \mathbf{B}_{p,p}^{s-1/p}(\mathbb{R}^{n-1})) \subset \mathbf{C}_0^0(\overline{\mathbb{R}_+^n}, \dot{\mathbf{B}}_{p,p}^{s-1/p}(\mathbb{R}^{n-1})).$$

As in the proof of Lemma 3.12, for  $v(t, x') := T_\phi u(x', t)$ ,  $x' \in \mathbb{R}^{n-1}$  and  $t \geq 0$ , we have

$$f := \partial_t v + (-\Delta')^{\frac{1}{2}} v \in \mathbf{H}^{s-1,p}(\mathbb{R}_+, \mathbf{L}^p(\mathbb{R}^{n-1})) \subset \dot{\mathbf{H}}^{s-1,p}(\mathbb{R}_+, \mathbf{L}^p(\mathbb{R}^{n-1})),$$

$$\text{and } v(0, \cdot) = [T_\phi u]_{|\partial\mathbb{R}_+^n} = S_\phi[u|_{\partial\Omega}] \in \mathbf{B}_{p,p}^{s-1/p}(\mathbb{R}^{n-1}) \subset \dot{\mathbf{B}}_{p,p}^{s-1/p}(\mathbb{R}^{n-1}).$$

Therefore, by [Gau23b, Theorem 4.7], since the operator  $(-\Delta')^{\frac{1}{2}}$  on  $\mathbf{L}^p(\mathbb{R}^{n-1})$  has homogeneous domain  $\mathbf{D}_p((-\Delta')^{\frac{1}{2}}) = \dot{\mathbf{H}}^{1,p}(\mathbb{R}^{n-1})$ , we have the following estimate maximal regularity estimate

$$\|v\|_{\mathbf{L}^\infty(\mathbb{R}_+, \dot{\mathbf{B}}_{p,p}^{s-1/p}(\mathbb{R}^{n-1}))} \lesssim_{p,s,n} \|(\partial_t v, (-\Delta')^{\frac{1}{2}} v)\|_{\dot{\mathbf{H}}^{s-1,p}(\mathbb{R}_+, \mathbf{L}^p(\mathbb{R}^{n-1}))} \sim_{p,s,n} \|v\|_{\dot{\mathcal{K}}^{s,p}(\mathbb{R}_+^n)}. \quad \blacksquare$$

We may need a Besov counterpart of Corollary 2.8 to carryover the trace result, or more generally to transfer properties from the half-space to Besov spaces on special Lipschitz domains by global change of coordinates. However, our strategy will be a bit different, so that the result will only be used in the very specific case  $s = \frac{1}{p}, q = 1$ , but it still has its own interest.

**Lemma 3.14** *Let  $p \in (1, +\infty)$ ,  $s \in (-1 + 1/p, 1)$ ,  $q \in [1, +\infty]$ . If one of the two following conditions is satisfied*

$$(i) \quad (\mathcal{C}_{s,p,q}) \text{ and } u \in \dot{\mathbf{B}}_{p,q}^s(\Omega),$$

$$(ii) \quad s \geq n/p \text{ and } u \in \mathbf{B}_{p,q}^s(\Omega),$$

we have  $T_\phi u \in \dot{\mathbf{B}}_{p,q}^s(\mathbb{R}_+^n)$  with the estimate

$$\|T_\phi u\|_{\dot{\mathbf{B}}_{p,q}^s(\mathbb{R}_+^n)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{\mathbf{B}}_{p,q}^s(\Omega)}.$$

In particular,  $T_\phi : \dot{\mathbf{B}}_{p,q}^s(\Omega) \rightarrow \dot{\mathbf{B}}_{p,q}^s(\mathbb{R}_+^n)$  is an isomorphism of Banach spaces whenever  $(\mathcal{C}_{s,p,q})$  is satisfied. The result still holds if we replace  $(\Omega, \mathbb{R}_+^n, T_\phi)$  by  $(\mathbb{R}_+^n, \Omega, T_\phi^{-1})$ .

*Proof.* — For  $s \in (-1 + 1/p, 1/p)$ , the result follows from real interpolation, by Corollary 2.8 and Proposition 2.22. Therefore, without loss of generality, we can assume  $s \in [1/p, 1)$ . We are going to proceed via a manual real interpolation scheme. Let  $u \in B_{p,q}^s(\Omega)$ , then we have

$$u \in L^p(\Omega) + H^{1,p}(\Omega) \subset L^p(\Omega) + \dot{H}^{1,p}(\Omega).$$

Let  $(a, b) \in L^p(\Omega) \times \dot{H}^{1,p}(\Omega)$ , such that  $u = a + b$ ,

$$b = u - a \in (B_{p,q}^s(\Omega) + L^p(\Omega)) \cap \dot{H}^{1,p}(\Omega) \subset H^{1,p}(\Omega).$$

We have  $T_\phi u \in B_{p,q}^s(\mathbb{R}_+^n) \subset \dot{B}_{p,q}^s(\mathbb{R}_+^n)$ , and  $T_\phi a \in L^p(\mathbb{R}_+^n)$ ,  $T_\phi b \in H^{1,p}(\mathbb{R}_+^n)$ ,

$$T_\phi u = T_\phi a + T_\phi b \in L^p(\mathbb{R}_+^n) + H^{1,p}(\mathbb{R}_+^n) \subset L^p(\mathbb{R}_+^n) + \dot{H}^{1,p}(\mathbb{R}_+^n).$$

For  $E$ , the operator extension from  $\mathbb{R}_+^n$  to  $\mathbb{R}^n$  by reflection given in the proof of Corollary 2.9, we have  $ET_\phi u \in B_{p,q}^s(\mathbb{R}_+^n)$ , and

$$ET_\phi u = ET_\phi a + ET_\phi b \in L^p(\mathbb{R}^n) + H^{1,p}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) + \dot{H}^{1,p}(\mathbb{R}^n).$$

Now, for  $t > 0$ , by definition of the  $K$ -functional and homogeneous estimates given by Corollaries 2.8 and 2.9, we obtain

$$\begin{aligned} K(t, ET_\phi u, L^p(\mathbb{R}^n), \dot{H}^{1,p}(\mathbb{R}^n)) &\leq \|ET_\phi a\|_{L^p(\mathbb{R}^n)} + t \|ET_\phi b\|_{\dot{H}^{1,p}(\mathbb{R}^n)} \\ &\lesssim_{p,n,\partial\Omega} \|a\|_{L^p(\Omega)} + t \|b\|_{\dot{H}^{1,p}(\Omega)}. \end{aligned}$$

Looking at the infimum over such pairs  $(a, b)$  yields for  $t > 0$ ,

$$K(t, ET_\phi u, L^p(\mathbb{R}^n), \dot{H}^{1,p}(\mathbb{R}^n)) \lesssim_{p,n,\partial\Omega} K(t, u, L^p(\Omega), \dot{H}^{1,p}(\Omega)).$$

One multiply by  $t^{-s}$ , and take the  $L_*^q$ -norm, so that as a consequence of Lemma 2.1, Proposition 1.3 and the definition of function spaces by restriction

$$\begin{aligned} \|T_\phi u\|_{\dot{B}_{p,q}^s(\mathbb{R}_+^n)} &\leq \|ET_\phi u\|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} \sim_{p,s,n} \|ET_\phi u\|_{(L^p(\mathbb{R}^n), \dot{H}^{1,p}(\mathbb{R}^n))_{s,q}} \\ &\lesssim_{p,s,n,\partial\Omega} \|u\|_{(L^p(\Omega), \dot{H}^{1,p}(\Omega))_{s,q}} \\ &\lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{B}_{p,q}^s(\Omega)}. \end{aligned}$$

If  $q < +\infty$  and  $(\mathcal{C}_{s,p,q})$  is satisfied, the result follows by density argument. For  $q = +\infty$ , if  $(\mathcal{C}_{s,p,q})$  is satisfied, the result follows from real interpolation by the mean of Proposition 2.26 and the reiteration theorem [BL76, Theorem 3.5.3]. One may reproduce a similar proof for  $T_\phi^{-1}$ .  $\blacksquare$

### 3.3 The trace theorem and related results

**Theorem 3.15** *Let  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$ ,  $s \in (1/p, 1 + 1/p)$ . The following statements are true:*

(i) *for all  $u \in H^{s,p}(\Omega)$ ,*

$$\|u|_{\partial\Omega}\|_{\dot{B}_{p,p}^{s-\frac{1}{p}}(\partial\Omega)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{H^{s,p}(\Omega)};$$

*In particular, the trace operator  $[\cdot]|_{\partial\Omega}$  extends uniquely as a bounded linear operator from  $\dot{H}^{s,p}(\Omega)$  to  $\dot{B}_{p,p}^{s-\frac{1}{p}}(\partial\Omega)$  whenever  $(\mathcal{C}_{s,p})$  is satisfied.*

(ii) *for all  $u \in B_{p,q}^s(\Omega)$ ,*

$$\|u|_{\partial\Omega}\|_{\dot{B}_{p,q}^{s-\frac{1}{p}}(\partial\Omega)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{B_{p,q}^s(\Omega)};$$

*In particular, the trace operator  $[\cdot]|_{\partial\Omega}$  extends uniquely as a bounded linear operator from  $\dot{B}_{p,q}^s(\Omega)$  to  $\dot{B}_{p,q}^{s-\frac{1}{p}}(\partial\Omega)$  whenever  $(\mathcal{C}_{s,p,q})$  is satisfied.*

(iii) for all  $u \in \dot{B}_{p,1}^{\frac{1}{p}}(\Omega)$ ,

$$\|u|_{\partial\Omega}\|_{L^p(\partial\Omega)} \lesssim_{p,n,\partial\Omega} \|u\|_{\dot{B}_{p,1}^{\frac{1}{p}}(\Omega)};$$

(iv) for all  $u \in \dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega)$ ,

$$\|u|_{\partial\Omega}\|_{\dot{H}^{1,p}(\partial\Omega)} \lesssim_{p,n,\partial\Omega} \|u\|_{\dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega)};$$

In particular, the trace operator  $[\cdot]_{\partial\Omega}$  extends uniquely as a bounded linear operator from  $\dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega)$  to  $\dot{H}^{1,p}(\partial\Omega)$  whenever  $p < n - 1$  is satisfied.

*Proof.* — We are going to cut the proof into three parts.

(i) **First part:** homogeneous Sobolev spaces.

Let  $u \in H^{s,p}(\Omega)$ . We assume first that  $s \in (\frac{1}{p}, 1]$ . By Proposition 2.5, we have  $T_\phi u \in H^{s,p}(\mathbb{R}_+^n)$ . The standard trace theorem with homogeneous estimates [Gau22, Theorem 3.5] yields that,  $x_n \mapsto T_\phi u(\cdot, x_n) \in C_b^0(\mathbb{R}_+, \dot{B}_{p,p}^{s-1/p}(\mathbb{R}^{n-1}))$ , with the estimates

$$\|T_\phi u(\cdot, 0)\|_{\dot{B}_{p,p}^{s-\frac{1}{p}}(\mathbb{R}^{n-1})} \lesssim_{s,p,n} \|T_\phi u\|_{\dot{H}^{s,p}(\mathbb{R}_+^n)} \lesssim_{s,p,n} \|u\|_{\dot{H}^{s,p}(\Omega)}.$$

But for almost every  $x' \in \mathbb{R}^{n-1}$ , we recall recall that

$$T_\phi u(x', 0) = u(x', 0 + \phi(x')) = S_\phi[u|_{\partial\Omega}](x').$$

Thus, one may apply Lemma 3.4 :

$$\|u|_{\partial\Omega}\|_{\dot{B}_{p,p}^{s-\frac{1}{p}}(\partial\Omega)} \lesssim_{p,s,n,\partial\Omega} \|S_\phi[u|_{\partial\Omega}]\|_{\dot{B}_{p,p}^{s-\frac{1}{p}}(\mathbb{R}^{n-1})} \lesssim_{s,p,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\Omega)}.$$

Now, we consider  $u \in H^{s,p}(\Omega)$ , with  $s \in [1, 1 + \frac{1}{p})$  it follows from the successive use of Lemma 3.10 and Corollary 3.13, that,

$$\|T_\phi u(\cdot, 0)\|_{\dot{B}_{p,p}^{s-\frac{1}{p}}(\mathbb{R}^{n-1})} \lesssim_{s,p,n} \|T_\phi u\|_{\dot{K}^{s,p}(\mathbb{R}_+^n)} \lesssim_{s,p,n,\partial\Omega} \|u\|_{\dot{H}^{s,p}(\Omega)}.$$

Therefore one may ends similarly as in the case  $s \in (\frac{1}{p}, 1]$ .

Hence, if moreover, we assume  $(\mathcal{C}_{s,p})$ , then every involved space is complete and one may conclude by density argument.

(ii) **Second part:** homogeneous Besov spaces with traces in Lebesgue/Sobolev spaces.

For  $u \in \dot{B}_{p,1}^{\frac{1}{p}}(\Omega)$ , by Lemma 3.14 and [Gau22, Theorem 3.5],

$$T_\phi u \in \dot{B}_{p,1}^{\frac{1}{p}}(\mathbb{R}_+^n) \subset C_0^0(\overline{\mathbb{R}_+^n}, L^p(\mathbb{R}^{n-1}))$$

with the estimates

$$\|u|_{\partial\Omega}\|_{L^p(\partial\Omega)} \sim_{p,n,\partial\Omega} \|T_\phi u(\cdot, 0)\|_{L^p(\mathbb{R}^{n-1})} \lesssim_{p,n,\partial\Omega} \|T_\phi u\|_{\dot{B}_{p,1}^{\frac{1}{p}}(\mathbb{R}_+^n)} \lesssim_{p,n,\partial\Omega} \|u\|_{\dot{B}_{p,1}^{\frac{1}{p}}(\Omega)}. \quad (3.10)$$

Now, for  $u \in \dot{B}_{p,1}^{\frac{1}{p}}(\Omega) \cap \dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega)$ , since  $\nabla u \in \dot{B}_{p,1}^{\frac{1}{p}}(\Omega)$ , we may use the estimate (3.10),

$$\|[\nabla u]|_{\partial\Omega}\|_{L^p(\partial\Omega)} \lesssim_{p,n,\partial\Omega} \|\nabla u\|_{\dot{B}_{p,1}^{\frac{1}{p}}(\Omega)} \lesssim_{p,n,\partial\Omega} \|u\|_{\dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega)}.$$

But one may check that we have

$$\|u|_{\partial\Omega}\|_{\dot{H}^{1,p}(\partial\Omega)} = \|S_\phi^{-1} \nabla' [S_\phi u|_{\partial\Omega}]\|_{L^p(\partial\Omega)} \lesssim_{p,n,\partial\Omega} \|[\nabla u]|_{\partial\Omega}\|_{L^p(\partial\Omega)} \lesssim_{p,n,\partial\Omega} \|u\|_{\dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega)}. \quad (3.11)$$

(iii) **Third part:** homogeneous Besov spaces.

Let  $s \in (1/p, 1 + 1/p)$ ,  $q \in [1, +\infty]$ , and let  $u \in B_{p,q}^s(\Omega)$ .

$$u \in B_{p,q}^s(\Omega) \subset B_{p,1}^{\frac{1}{p}}(\Omega) + B_{p,1}^{1+\frac{1}{p}}(\Omega) \subset \dot{B}_{p,1}^{\frac{1}{p}}(\Omega) + \dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega).$$

Let  $(a, b) \in \dot{B}_{p,1}^{\frac{1}{p}}(\Omega) \times \dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega)$ , such that  $u = a + b$ ,

$$b = a - u \in (\dot{B}_{p,1}^{\frac{1}{p}}(\Omega) + B_{p,q}^s(\Omega)) \cap \dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega) \subset \dot{B}_{p,1}^{\frac{1}{p}}(\Omega) \cap \dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega).$$

Therefore  $u|_{\partial\Omega} \in B_{p,q}^{s-\frac{1}{p}}(\partial\Omega) \subset \dot{B}_{p,q}^{s-\frac{1}{p}}(\partial\Omega)$  and  $(a|_{\partial\Omega}, b|_{\partial\Omega}) \in L^p(\partial\Omega) \times H^{1,p}(\partial\Omega)$ , so that by definition of the  $K$ -functional and homogeneous estimates (3.10) and (3.11), we obtain

$$\begin{aligned} K(t, u|_{\partial\Omega}, L^p(\partial\Omega), \dot{H}^{1,p}(\partial\Omega)) &\leq \|a|_{\partial\Omega}\|_{L^p(\partial\Omega)} + t \|b|_{\partial\Omega}\|_{\dot{H}^{1,p}(\partial\Omega)} \\ &\lesssim_{p,n,\partial\Omega} \|a\|_{\dot{B}_{p,1}^{\frac{1}{p}}(\Omega)} + t \|b\|_{\dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega)}. \end{aligned}$$

Looking at the infimum over all such pairs  $(a, b)$  yields for all  $t > 0$ ,

$$K(t, u|_{\partial\Omega}, L^p(\partial\Omega), \dot{H}^{1,p}(\partial\Omega)) \lesssim_{p,n,\partial\Omega} K(t, u, \dot{B}_{p,1}^{\frac{1}{p}}(\Omega), \dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega)).$$

Multiplying by  $t^{-(s-\frac{1}{p})}$ , then taking the  $L^q$ -norm, by definition of homogeneous Besov spaces on the boundary (see Definition 3.6, and Lemma 2.1), it holds that

$$\begin{aligned} \|u|_{\partial\Omega}\|_{\dot{B}_{p,q}^{s-\frac{1}{p}}(\partial\Omega)} &\sim_{p,s,n} \|u|_{\partial\Omega}\|_{(L^p(\partial\Omega), \dot{H}^{1,p}(\partial\Omega))_{s-\frac{1}{p},q}} \lesssim_{p,s,n,\partial\Omega} \|u\|_{(\dot{B}_{p,1}^{\frac{1}{p}}(\Omega), \dot{B}_{p,1}^{1+\frac{1}{p}}(\Omega))_{s-\frac{1}{p},q}} \\ &\lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{B}_{p,q}^s(\Omega)}. \end{aligned}$$

If  $q < +\infty$  and  $(\mathcal{C}_{s,p,q})$  is satisfied, the result follows by density argument. For  $q = +\infty$ , if  $(\mathcal{C}_{s,p,q})$  is satisfied, the result follows from real interpolation by the mean of Proposition 2.26, Definition 3.6 and the reiteration theorem [BL76, Theorem 3.5.3].  $\blacksquare$

We state interesting consequences concerning regularity and integrability of traces in the case of intersection spaces, and identification of function spaces for functions that vanish on the boundary. The proofs are very similar to the proofs one can find at the end of [Gau22, Section 3] for  $\Omega$  being the half-space, therefore we present shortened proofs and the full ones are omitted.

**Proposition 3.16** *Let  $p \in (1, +\infty)$ ,  $q \in [1, +\infty)$ , and  $\theta \in (0, 1)$ ,  $-1 + \frac{1}{p} < s_0 < \frac{1}{p} < s_1 < 1 + \frac{1}{p}$  such that*

$$\frac{1}{p} = (1 - \theta)s_0 + \theta s_1.$$

*Then the following assertions hold.*

(i) *For all  $u \in \dot{H}^{s_0,p}(\Omega) \cap \dot{H}^{s_1,p}(\Omega)$ , we have  $u|_{\partial\Omega} \in B_{p,p}^{s_1-\frac{1}{p}}(\partial\Omega)$ , with the estimate*

$$\|u|_{\partial\Omega}\|_{B_{p,p}^{s_1-\frac{1}{p}}(\partial\Omega)} \lesssim_{s_0,s_1,p,n,\partial\Omega} \|u\|_{\dot{H}^{s_0,p}(\Omega)}^{1-\theta} \|u\|_{\dot{H}^{s_1,p}(\Omega)}^{\theta} + \|u\|_{\dot{H}^{s_1,p}(\Omega)}.$$

*We also have*

$$\|u|_{\partial\Omega}\|_{B_{p,p}^{s_1-\frac{1}{p}}(\partial\Omega)} \lesssim_{s_0,s_1,p,n,\partial\Omega} \|u\|_{\dot{H}^{s_1,p}(\Omega)};$$

(ii) *For all  $u \in \dot{B}_{p,q}^{s_0}(\Omega) \cap \dot{B}_{p,q}^{s_1}(\Omega)$ , we have  $u|_{\partial\Omega} \in B_{p,q}^{s_1-\frac{1}{p}}(\partial\Omega)$ , with the estimate*

$$\|u|_{\partial\Omega}\|_{B_{p,q}^{s_1-\frac{1}{p}}(\partial\Omega)} \lesssim_{s_0,s_1,p,n,\partial\Omega} \|u\|_{\dot{B}_{p,q}^{s_0}(\Omega)}^{1-\theta} \|u\|_{\dot{B}_{p,q}^{s_1}(\Omega)}^{\theta} + \|u\|_{\dot{B}_{p,q}^{s_1}(\Omega)}.$$

*We also have*

$$\|u|_{\partial\Omega}\|_{B_{p,q}^{s_1-\frac{1}{p}}(\partial\Omega)} \lesssim_{s_0,s_1,p,n,\partial\Omega} \|u\|_{\dot{B}_{p,q}^{s_1}(\Omega)};$$

(iii) For all  $u \in \dot{B}_{p,\infty}^{s_0}(\Omega) \cap \dot{B}_{p,\infty}^{s_1}(\Omega)$ , we have  $u|_{\partial\Omega} \in L^p(\partial\Omega)$ , with the estimate

$$\|u|_{\partial\Omega}\|_{L^p(\partial\Omega)} \lesssim_{s_0,s_1,p,n,\partial\Omega} \|u\|_{\dot{B}_{p,\infty}^{s_0}(\Omega)}^{1-\theta} \|u\|_{\dot{B}_{p,\infty}^{s_1}(\Omega)}^\theta.$$

*Proof.* — We mention the result [Gau22, Proposition 3.6] for the case  $\Omega = \mathbb{R}_+^n$ , where the proof only relies on good interpolation inequalities and the appropriate trace estimates. Everything has been made in order to recover the same interpolation inequalities, the result then follows from Theorem 3.15 points (i) and (iii) for the case of Sobolev spaces, and from points (ii) and (iii) for the case of Besov spaces. ■

**Proposition 3.17** *Let  $p_j \in (1, +\infty)$ ,  $q_j \in [1, +\infty)$ ,  $s_j > 1/p_j$ ,  $j \in \{0, 1\}$ , such that  $(\mathcal{C}_{s_0,p_0})$  (resp.  $(\mathcal{C}_{s_0,p_0,q_0})$ ) is satisfied. Then,*

(i) For all  $u \in [\dot{H}^{s_0,p_0} \cap \dot{H}^{s_1,p_1}](\Omega)$ , we have  $u|_{\partial\Omega} \in \dot{B}_{p_j,p_j}^{s_j - \frac{1}{p_j}}(\partial\Omega)$ ,  $j \in \{0, 1\}$ , with the estimate

$$\|u|_{\partial\Omega}\|_{\dot{B}_{p_j,p_j}^{s_j - \frac{1}{p_j}}(\partial\Omega)} \lesssim_{s_j,p,n} \|u\|_{\dot{H}^{s_j,p_j}(\Omega)};$$

(ii) For all  $u \in [\dot{B}_{p_0,q_0}^{s_0} \cap \dot{B}_{p_1,q_1}^{s_1}](\Omega)$ , we have  $u|_{\partial\Omega} \in \dot{B}_{p_j,q_j}^{s_j - \frac{1}{p_j}}(\partial\Omega)$ ,  $j \in \{0, 1\}$ , with the estimate

$$\|u|_{\partial\Omega}\|_{\dot{B}_{p_j,q_j}^{s_j - \frac{1}{p_j}}(\partial\Omega)} \lesssim_{s_0,s_1,p,n} \|u\|_{\dot{B}_{p_j,q_j}^{s_j}(\Omega)};$$

*Proof.* — This is a direct consequence of density results provided by Propositions 2.11 and 2.24, and the trace theorem Theorem 3.15. ■

**Lemma 3.18** *Let  $p_j \in (1, +\infty)$ ,  $s_j \in (1/p_j, 1 + 1/p_j)$ ,  $j \in \{0, 1\}$  such that  $(\mathcal{C}_{s_0,p_0})$  is satisfied. We have the following canonical isomorphism of Banach spaces*

$$\{u \in [\dot{H}^{s_0,p_0} \cap \dot{H}^{s_1,p_1}](\Omega) \mid u|_{\partial\Omega} = 0\} \simeq [\dot{H}_0^{s_0,p_0} \cap \dot{H}_0^{s_1,p_1}](\Omega).$$

*The result still holds replacing  $\dot{H}^{s_j,p_j}$  by  $\dot{B}_{p_j,q_j}^{s_j}$ ,  $q_j \in [1, +\infty)$ ,  $j \in \{0, 1\}$  assuming that  $(\mathcal{C}_{s_0,p_0,q_0})$  is satisfied.*

*Proof.* — We mention the results [Gau22, Lemma 3.10 & Corollary 3.11] where the proofs only rely on integration by parts, the meaning of traces, and the possibility to extend elements of  $\dot{H}^{s,p}$  from  $\Omega$  to the whole space  $\mathbb{R}^n$  by 0 boundedly, whenever  $s \in (-1 + 1/p, 1/p)$ . The boundedness of the extension by 0 from  $\Omega$  to the whole space  $\mathbb{R}^n$  is a direct consequence of Corollary 2.7. The same holds for Besov spaces with Proposition 2.18. ■

We conclude with few remarks.

- In Theorem 3.15, we don't give any claim about a right bounded inverse. The naive composition with Poisson's extension only yields right bounded inverse for regularity indices between  $1/p$  and 1 for Theorem 3.15.
- For a right bounded inverse regularity indices between  $1/p$  and  $1 + 1/p$  in Theorem 3.15, personal discussions with Patrick Tolksdorf and Moritz Egert persuaded the author that it should be possible to adapt Jonsson and Wallin's extension operator [JW84, Chapter VII, Theorem 3] from the boundary from the whole domain in a way so that it preserves homogeneous norms.
- One may also use the same Jonsson and Wallin's (usual) extension operator to reprove exactly the same way [Gau23a, Theorem A.2] making sense of weak partial traces of differentials forms, replacing  $\mathbb{R}_+^n$  by  $\Omega$  a special Lipschitz domain. The result is still far from being optimal by the way, for instance one may see [MMS08] for the case of inhomogeneous function spaces on bounded Lipschitz domains.

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