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Trajectories of two-dimensional harmonic oscillators in rotating frame: application to Foucault pendulum problem and centered trochoid trajectories

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Abstract: The motion of a classical point particle submitted to a central restoring force in a non-inertial frame is studied. Exact trajectories are provided and several classical cases are detailed. Results are applied to the classical case of the Foucault pendulum problem, and centered trochoid trajectories are highlighted. These original solutions are linked to concerns of contemporary physics.

Keywords: Centered trochoid curves; harmonic oscillator; non inertial frame; Simple gravity pendulum;

1. Introduction

It is well known that studies of the trajectories of particles in non-inertial frames are major concerns of physics. Among them, behaviors of harmonic oscillators in such frames are particularly interesting, given the many possible contemporary physical applications. We are thinking, for example, of studies concerning spectroscopy or plasma physics, whether in a classical or quantum framework. Another major application, which will be particularly studied in this article, concerns the Foucault pendulum problem. Indeed, this problem is generally studied in the small-angle approximation, which implies that the pendulum becomes, in this case, a two-dimensional harmonic oscillator.

The purpose of this paper is to contribute to this topic, by presenting an original study of the trajectory of these harmonic oscillators in non-inertial frames, more precisely in time varying rotating frames. Indeed, using the laws of classical mechanics, we will present original solutions, which seem to be known only for certain limit cases. The particular case where the speed of rotation of the non-inertial frame is constant will be highlighted, and we will show that the corresponding trajectories are the centered trochoid curves. In a second part, we will apply these general results to the case of Foucault's pendulum. This will allow us to find certain limiting cases already known, but also to characterize geometrically several possible trajectories and to determine the laws of motion.

There are many reasons for us to apply our results to this particular problem. First of all, it is, of course, one of the most important popular experiment in the history of Physics, which was presented for the first time by the French physicist Foucault to the Parisian public in 1851 [1], and was subsequently reproduced in many museums around the world. Schematically, the experiment consists in setting in motion a long and heavy pendulum and measuring the slow rotation of the plane of oscillation, due to the Earth rotation. It is then possible to highlight this rotation and to deduce, for example, the latitude of the pendulum. Naturally, studies of the behavior of the pendulum has been presented in many physics text books and various solutions have been presented (the reader can consult, for example, the classical references [2, 3, 4]). Unfortunately, these works, and more

particularly the establishment of equations of motion and determination of geometrics characteristics of trajectories, do not seem to be unanimous, and therefore remains an important concern of the contemporary Physics. This explains why research papers are still regularly published on this topic with the aim, in general, to provide the most precise possible laws of motion.

Another reason concerns the possible practical applications of this experiment. For example, it is planned for measurement of the lense-Thirring effect and, consequently to improve our knowledge of the theories of Gravitation and of the space time. Moreover, for the reason the extension of the pendulum principle allows the development of gyroscopic systems, very numerous applications in many fields of applied physics and technology are considered.

Therefore, paper is organized as follows: in a first part, we study the behavior of a two-dimensional harmonic oscillator in all kind of time varying rotating frame. Secondly, we apply our results to the particular case of the Foucault pendulum.

The reader will therefore find in this article progress concerning several areas of the physics: on the one hand, from a geometric point of view, a new approach to trochoid curves. On the other hand, exact solutions of the movement two dimensional harmonic oscillator in time varying rotating frame and application to the Foucault's pendulum problem, in the approximation of small angles. To conclude, we attract his attention that this work has been carried out within the framework of classical mechanics (quantum and relativistic effects have been neglected).

2. Two dimensional harmonic oscillator in time varying rotating frame

2.1 Present the problem

(Note, to simplify the calculation, the masses were, in this part, considered equal to 1, so that the forces and the accelerations are confused)

We begin the study by introducing a Galilean system of coordinate $(O, \vec{X}_0, \vec{Y}_0, \vec{Z}_0)$ where a point particle M moves under the influence of a restoring force centered at O and given with

$$\vec{F} = -k\vec{OM} = -k\vec{r} = -w^2\vec{r} \quad (1)$$

Where w is the pulsation of the pendulum. We consider now a second system of coordinate $(O, \vec{x}, \vec{y}, \vec{z})$. This frame is animated by a rotation movement around the z axis such

$$\vec{\Omega}(t) = \Omega\vec{z}$$

And is consequently non-inertial. As we presented it, goal of the paper is to describes the trajectory of M in this frame.

To obtain the equation of motion we have to consider the fictitious forces, defined, according to the laws of classical mechanics, by

$$\vec{F}_{Fl} = \vec{a}_0 - \vec{\Omega} \times \vec{\Omega} \times \vec{r} - 2\vec{\Omega} \times \frac{d\vec{r}}{dt} - \frac{d\vec{\Omega}}{dt} \times \vec{r} \quad (2)$$

Where we successively recognize the acceleration of translation between the two frames, centrifugal force, the force of Coriolis and of Euler (see for example classical textbooks in references [2, 3]).

Respecting our conditions, the translational acceleration is zero and the equation of the dynamics becomes in this frame

$$\vec{F} = -k\vec{r} - \vec{\Omega} \times \vec{\Omega} \times \vec{r} - 2\vec{\Omega} \times \frac{d\vec{r}}{dt} - \frac{d\vec{\Omega}}{dt} \times \vec{r} \quad (3)$$

2.2 Solving

Equation of motion leads to the system :

$$\begin{cases} \ddot{X} = -w^2X + \Omega^2X + 2\Omega\dot{Y} + Y\dot{\Omega} \\ \ddot{Y} = -w^2Y + \Omega^2Y - 2\Omega\dot{X} - X\dot{\Omega} \end{cases} \quad (4)$$

Again, these equations are not original (see for example [2]). The classical method of solving consists in introducing a complex number such that

$$u = X + jY$$

Differential equation (4) becomes

$$\frac{d^2u}{dt^2} = -(w^2 - \Omega^2)u - 2j\Omega \frac{du}{dt} - j\dot{\Omega}u \quad (5)$$

To solve, we choose to investigate a solution in the original following form:

$$u = r_1 e^{j(wt+f(t))}$$

And (5) becomes

$$j\ddot{f} - (w + \dot{f})^2 = -w^2 + \Omega^2 + 2\Omega(w + \dot{f}) - j\dot{\Omega}$$

Identifying the imaginary and real parts we get the system of equations :

$$\begin{cases} \ddot{f} = -\dot{\Omega} & 6.1 \\ -(w + \dot{f})^2 = -w^2 + \Omega^2 + 2\Omega(w + \dot{f}) & 6.2 \end{cases}$$

By considering the constants of integration as null (6.1) leads to

$$f(t) = - \int \Omega(t) dt \quad (7)$$

Introducing

$$\dot{f} = -\Omega$$

In 6.2 we verify easily the solution. Moreover, we note that a second solution exists, given by

$$u = r_2 e^{j(-wt+f(t))}$$

Because it leads to the same relation (7) and also satisfies real relation, given this time with

$$-(-w + \dot{f})^2 = -w^2 + \Omega^2 + 2\Omega(-w + \dot{f})$$

Whereas these results we obtain thus a general solution such

$$u = r_1 e^{j(wt+f(t))} + r_2 e^{j(-wt+f(t))}$$

Rewriting the expression, we obtain :

$$\begin{cases} X = (r_1 + r_2) \cos wt \cos f(t) - (r_1 - r_2) \sin wt \sin f(t) \\ Y = (r_1 + r_2) \cos wt \sin f(t) + (r_1 - r_2) \sin wt \cos f(t) \end{cases} \quad (8)$$

Now, consider the limiting cases for which we know the solution. This is, of course, the case where $\Omega = 0$. Then, the point particle describes naturally an ellipse of Lissajous-Bowditch, define (with correct initial conditions) by

$$\begin{cases} X = a \cos w t \\ Y = b \sin w t \end{cases}$$

Where (a, b) are the semi major and minor axis of the conic. We deduce that

$$\begin{cases} r_1 + r_2 = a \\ r_1 - r_2 = b \end{cases}$$

Finally the solution can be written in the form of simple trigonometric functions, for example

$$\begin{cases} X = a \cos wt \cos f(t) - b \sin wt \sin f(t) \\ Y = a \cos wt \sin f(t) + b \sin wt \cos f(t) \end{cases} \quad (8)$$

Note that it is possible to verify these results, by calculating the successive derivatives of these solutions and introducing them into the equation of motion (4). Also note that these solutions seems original in general cases.

2.3 Presentation of trajectories

2.3.1 Particular case: Two dimensionnal oscillator in constant speed of rotation : hypotrochoid curves

Solutions becomes simply

$$\begin{cases} x = a \cos wt \cos \Omega t - b \sin wt \sin \Omega t \\ y = a \cos wt \sin \Omega t + b \sin wt \cos \Omega t \end{cases} \quad (9)$$

Note that these trajectories are not unknown but represent a part of the centered trochoid curves (more exactly the hypotrochoid curves) as defined, for example, in reference [5].

Among these trajectories, several particular cases can be noticed : an emblematic case concern the rose trajectory, corresponding on a mono dimensional oscillator ($b = 0$). In this case, the angular speed is constant and simply equal to Ω [6]. Others well known trajectories can be reobtain, such the hypocycloid, etc.. Certains of them will be drawn in the following parts.

2.3.2 General case

It is also possible to study the trajectories of the harmonic oscillator in a time-dependent rotating frame. Consider for example the classic example

$$\vec{\Omega} = \Omega(1 + \beta \cos vt)\vec{z}$$

We obtain using (7) the expression

$$f(t) = -\Omega\left(t + \frac{\beta}{v} \sin vt\right)$$

In figures 2.1 and 2.2 we present the modifications of the trajectories due to the addition of a time-dependent part to the constant rotational speed Ω

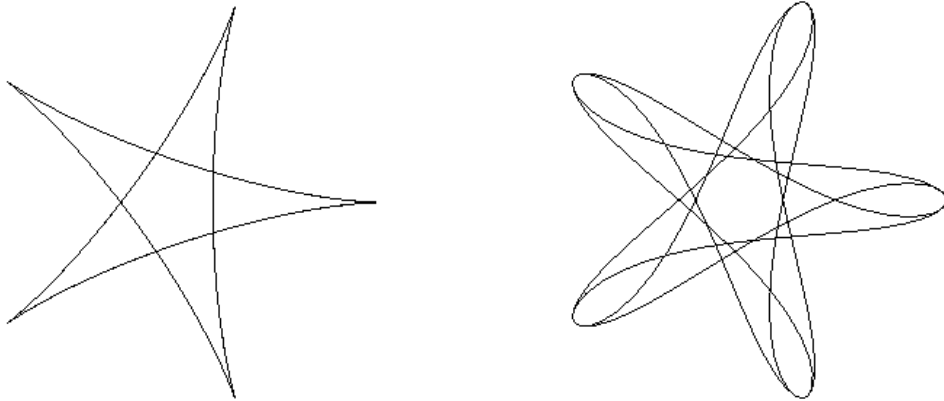


Figure 2.1 . Left : hypocycloid of parameters : $a = 1, b = -1/5, f = t/5$
Right : trajectory under parameter $a = 1, f = 1/5t + 0.2 * \sin(t)$

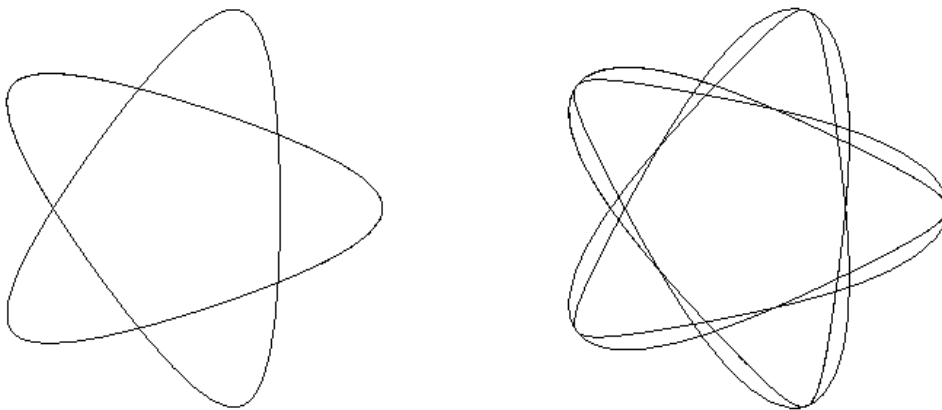


Figure 2.2 . Left : centered trochoid of parameters : $a = 1, b = 0.5, f = -t/5$
Right : trajectory under parameter $a = 1, b = 0.5, f = -1/5(t + 0.8 * \sin(t))$

Consider also the case of the monodimensional oscillator (figure 2.3)

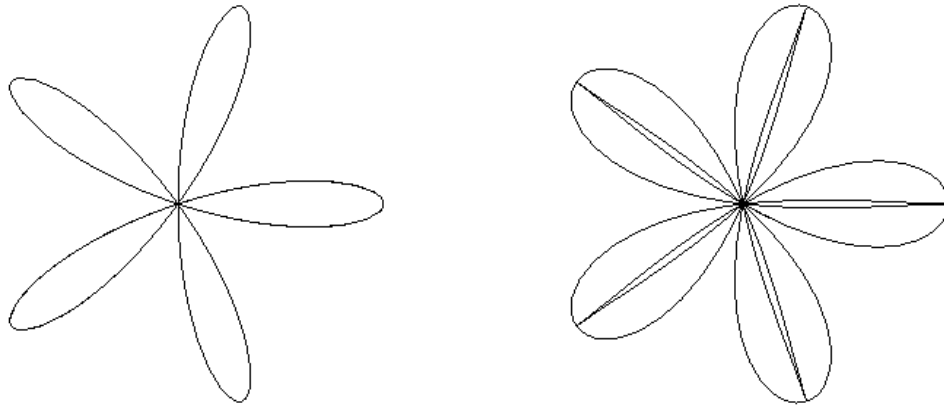


Figure 2.3 . Left : rose curve of parameters : $a = 1, f = t/5$

Right : trajectory under parameter $a = 1, f = 1/5(t + 0.2 * \sin(t))$

Note that trajectories are here defined with

$$\begin{cases} X = a \cos wt \sin f(t) \\ Y = a \cos wt \cos f(t) \end{cases}$$

And are corresponding on a mono dimensional harmonic oscillator. Polar angle is thus simply

$$\theta = f(t) = \int \Omega(t) dt$$

If speed is entirely time varying, such $\vec{\Omega} = \Omega \cos vt \vec{z}$ (figure 2 .4)

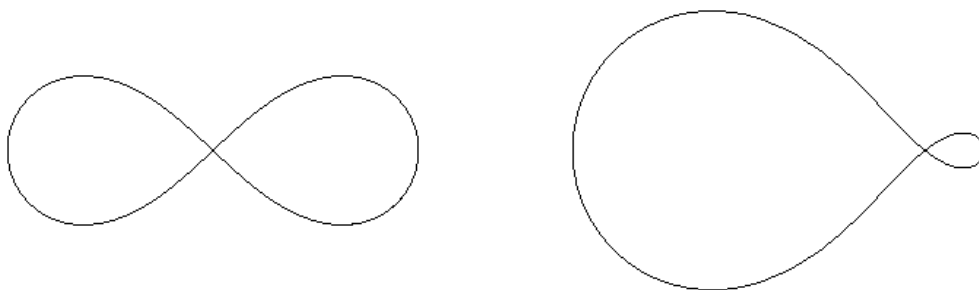


Figure 2.4 . Left : monodimensional oscillator : $a = 1, b = 0, f = -\frac{3}{4} \sin t$

Right : two dimensional oscillator, $a = 1, b = 0.5, f = -\frac{3 \sin t}{4}$

Another classic case concerns a speed of rotation which gradually decreases until it cancels out: Considering for example the case $\Omega = \Omega_0 e^{-\lambda t}$. We obtain

$$f(t) = \frac{\Omega_0}{\lambda} e^{-\lambda t}$$

The trajectories gradually have to become elliptical again (figure 2.4)

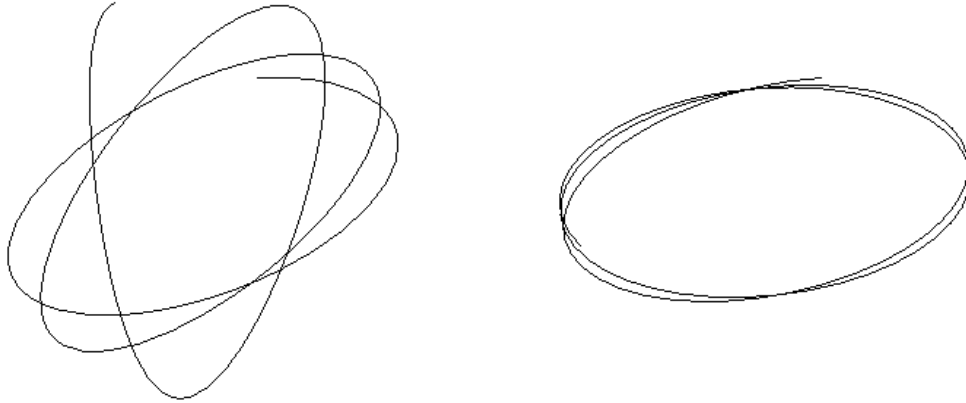


Figure 2.5 $a = 1, b = 0.5, f = 2e^{-0.1t}$. Left $t \in]0, 20]$; Right : $t \in [20, 35]$;

Consider also, for example, the general case

$$\left\{ \begin{array}{l} \Omega(t) = \Omega_0(1 + \alpha e^{-\lambda t}) \\ f(t) = \Omega_0 - \frac{\alpha e^{-\lambda t}}{\lambda} \end{array} \right\}$$

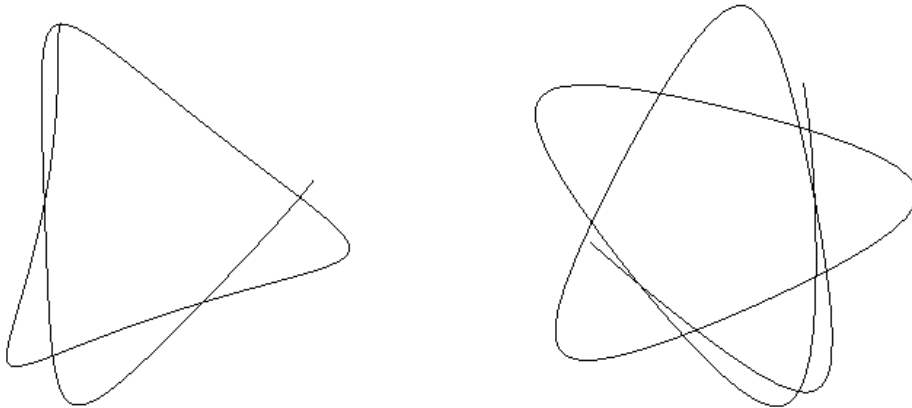


Figure 2.6 $a = 1, b = 0.5, f = 2e^{-0.1t}$. Left $t \in]0, 15]$; Right : $t \in [20, 40]$;

the trajectories this time become the trochoids corresponding to the constant rotation speed.

3. Application to Foucault pendulum problem

As we introduced, we now apply these theoretical results to the case of the Foucault pendulum problem. First, we quickly present the classical case of the simple pendulum in an inertial frame.

3.1 Simple gravity pendulum

Simple pendulum is naturally a well-known oscillating system, which has often be described in classical text books (see for example [2]). In figure 3, the force diagram and the system of coordinate is detailed. Note that friction forces and mass of the rod are neglected, and that length $PM = L$ is considered constant.

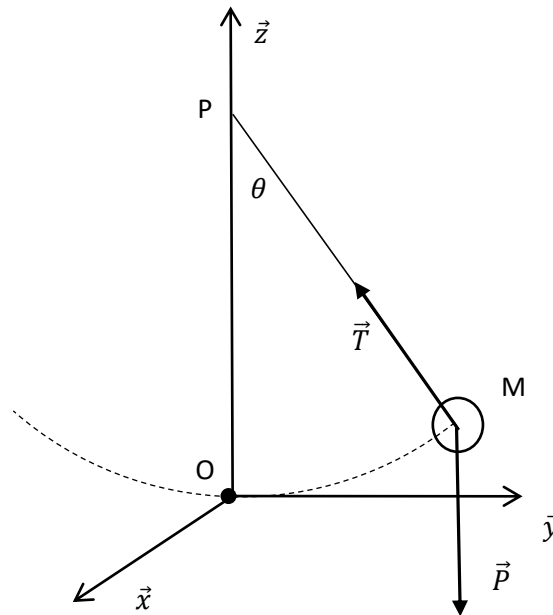


Figure 3.1 Foucault Pendulum

The system of coordinate $(O, \vec{x}, \vec{y}, \vec{z})$ is, in this case, Galilean (because Earth 's rotation is neglected). We can write the second equation of dynamics, which leads to the following relation

$$\vec{P} + \vec{T} = m\vec{a} \quad (10)$$

Where \vec{a} is the acceleration of the point mass $M(m)$. In the small angle approximation, the motion is considered planar and contained in the plane (O, \vec{x}, \vec{y}) . (10) become thus

$$\left\{ \begin{array}{l} -T \sin \theta = m\ddot{y} \\ T \cos \theta - mg_0 = m\ddot{z} \end{array} \right\} \quad (11)$$

In this approximation we can also write following relations:

$$\begin{cases} y = L \sin \theta \approx L\theta \\ z = 0 \end{cases}$$

And (11) becomes

$$\begin{cases} -T\theta = mL\ddot{\theta} \\ T = mg_0 \end{cases}$$

By substitution we obtain the rotational equation of motion

$$\ddot{\theta} + \frac{g_0}{L}\theta = 0$$

i.e. the simple harmonic oscillator equation. Solving leads naturally to the relation

$$\theta = \theta_0 \cos \omega t$$

Where θ_0 is the initial angle and t the time. Pulsation is given with

$$\omega = \sqrt{\frac{g_0}{L}}$$

Problem of this solution is that the Earth rotation is neglected and, consequently, that frame isn't exactly inertial: if we now consider this rotation, we have naturally to modify the equations of motion. This is the subject of the next chapter.

3.2 Physics: fictitious forces and trajectories

Thus consider this time the rotation of the Earth. The system of coordinate $(O, \vec{x}, \vec{y}, \vec{z})$ can no longer be considered Galilean but becomes, due to the Earth's rotation, a non-inertial rotating system frame. To detail corresponding fictitious forces we need to definite more precisely the system of coordinate $(O, \vec{x}, \vec{y}, \vec{z})$ that we use.

Origin O is located at the latitude φ . To simplify the calculation, we choose the x-axis pointing east, y-axis north, and z-axis is vertical. $\vec{v}(\dot{x}, \dot{y}, \dot{z})$ is the speed of the pendulum and vector of rotation of the Earth is $\vec{\Omega}$. In this system, the coordinates of this vector is given by

$$\vec{\Omega} \begin{pmatrix} 0 \\ \Omega \cos \varphi \\ \Omega \sin \varphi \end{pmatrix}$$

We can now deduce the mathematical expression of the fictitious forces:

- In the classical problem, Euler's force is neglected because Earth's rotation speed is nearly constant.

- The centrifugal force (see reference [2]) is directed toward the rotation axis of the Earth and is independent of the motion of the particle in the rotating frame. Its major effect is to modify the gravitational acceleration, which becomes the effective gravitational acceleration given by

$$\vec{g} = \vec{g}_0 - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$$

For this reason, the effect of this force on the pendulum's precession is usually neglected. It is the case in many theoretical Mechanics textbooks.

- The last force (Coriolis force) is thus considered as the only one which acts on the pendulum precession. In our system of coordinate, it can be written

$$\vec{F}_C = -2m\vec{\Omega} \times (\dot{x}\vec{x} + \dot{y}\vec{y} + \dot{z}\vec{z})$$

However, with the small angle approximation, we consider that $\dot{z} = 0$ and we obtain thus the following expression in the plane (O, x, y)

$$F_C \begin{cases} 2m\dot{y}\Omega \sin \varphi \\ -2m\dot{x}\Omega \sin \varphi \end{cases}$$

Finally, the equations of motion can be written, in the rotating frame, by the system

$$\begin{cases} \ddot{x} = -\frac{g}{L}x + 2\Omega \sin \varphi \dot{y} \\ \ddot{y} = -\frac{g}{L}y - 2\Omega \sin \varphi \dot{x} \end{cases} \quad (12)$$

Because of the results presented in the part 2.2, it makes sense to write the solutions in the form

$$\begin{cases} x = a \cos wt \cos ft - b \sin wt \sin ft \\ y = a \cos wt \sin ft + b \sin wt \cos ft \end{cases} \quad (13)$$

Where f is the pulsation given by

$$f = \Omega \sin \varphi$$

And where the frequencies are linked with

$$\frac{g}{L} = w^2 - f^2$$

Trajectories are thus hypotrochoid curves. Depending on initial conditions, we can distinguish several classical cases.

3.3 Hypocycloid trajectories

To carry out the experiment, generally the pendulum is moved away from its equilibrium position on the x-axis, with a small angle θ_0 and released at $t = 0$ without initial velocity. These conditions can be written as follows :

$$\begin{cases} a = L\theta_0 \\ \dot{x} = \dot{y} = 0 \end{cases}$$

Introducing them in equation (13), we deduce the following relation :

$$b = -a \frac{f}{w}$$

To simplify the calculation we introduce the ratio

$$n = \frac{f}{w}$$

And mathematical expression of the solutions are

$$\begin{cases} X = a[\cos wt \cos nwt - n \sin wt \sin nwt] \\ Y = a[\cos wt \sin nwt + n \sin wt \cos nwt] \end{cases}$$

We obtain the relation

$$\begin{cases} r_1 = \frac{a}{2w}(w - f) \\ r_1 = \frac{a}{2w}(w + f) \end{cases}$$

Noting the relation

$$r_1 w_1 = r_2 |w_2|$$

We deduce that the trajectory is one of the hypocycloid curves, among hem astroid, deltoïd, etc.. see for example [7]. An example of these curves is provided in Figure 3.2 :

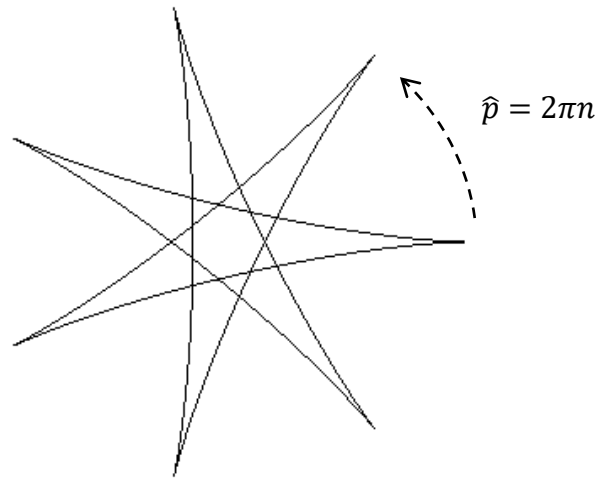


Figure 3.2 $\{a = 1, w = 1, n = 1/7\}$

Mechanical energy is given by

$$E_M = \frac{1}{2} a^2 w^2 (1 - n^2)$$

3.4 Rose trajectories

Consider the case where initial conditions are the following: at the origin of the time the pendulum is located at its equilibrium position, i.e. at the origin of the rotational frame. An impulsion is transmitted along the x-axis, such the initial velocity is $\vec{V}_0(0, \dot{y}_0, 0)$. Equations of position leads thus to

$$a = 0$$

And equations of speed lead to the relation

$$b = \frac{\dot{y}_0}{w}$$

Trajectories are thus given by

$$\begin{cases} X = -b \sin wt \sin ft \\ Y = b \sin wt \cos ft \end{cases}$$

That we can rewrite under a polar form

$$r = b \sin wt$$

That we can rewrite

$$r = \dot{y}_0 \sqrt{\frac{L}{g + Lf^2}} \sin \frac{ft}{n}$$

Trajectory is thus a rose curve. An example is provided in Figure 3.2.

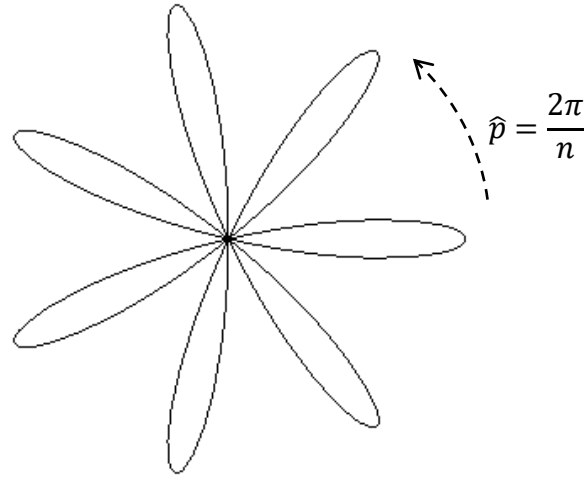


Figure 3.2 Rose curve. $\{a = 1, w = 1, n = 7\}$

As previously, we can obtain the expression of mechanical energy, given by

$$E_M = \frac{1}{2} b^2 w^2$$

The interest of this trajectory is that it is the only one, among the wide variety of hypotrochoid curves, which implies that the pendulum passes through the origin of the system of coordinate, at each oscillation. The time interval between each passage to O is given by the following relation

$$\Delta T = \frac{\pi}{w}$$

Which leads to

$$\Delta T = \frac{\pi}{\sqrt{\frac{g}{L} + f^2}}$$

Pulsation f is thus obtained

$$f = \sqrt{\left[\frac{\pi}{\Delta T}\right]^2 - \frac{g}{L}}$$

Therefore, it might be simpler and more accurate to measure this time interval that to measure the precession angle, as it usually seems to be.

3.5 General case

Consider this time that the pendulum is released such it would describe, if the frame were Galilean, a Lissajous-Bowditch ellipse. In this case, it will be subjected to a precession of its principal axis, and will have one of the orbits presented in the equations (13). Note that it can be interesting to use these initial conditions to improve, in certain cases, the precision of the measurements [10]. The trajectory will then be, in the case of the small angle approximation, one of the hypocotoid curves, depending on the norm of the coriolis force. We present in Figure 3.3 some of them. Note that the curves are closed if n is a rational number.

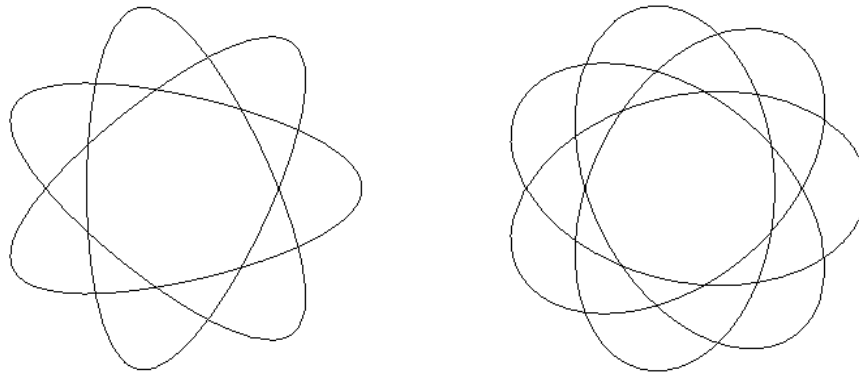


Figure 3.3. Two possible trajectories $n = -1/7$ and $n = 1/7$

Mechanical energy is given here by

$$E_M = \frac{1}{2}w(w(a^2 + b^2) + 2abf)$$

Two particular cases can be noted :

If $a = b$ trajectory is a circle centered at the origin of the frame.

If $w = \pm f$ trajectories are two eccentric circles (but it is here a limiting case) (see Figure 3.4)

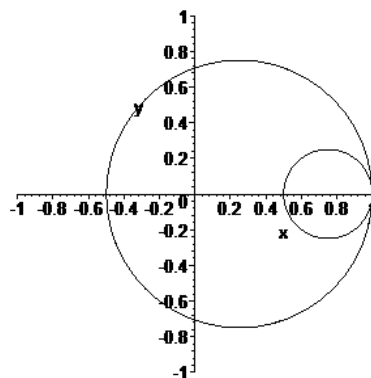


Figure 3.4 : Limiting cases $w = \pm f$

4. Discussion

Consider the general solutions presented in this article, about the behavior of harmonic oscillators in time varying rotating frames. First of all, it seems that these solutions are entirely original. Naturally, these trajectories can be applied to the Foucault pendulum only as a limiting case, if the experiment takes place classically, ie on Earth. Indeed, in this case the Euler force is zero. However, these trajectories can correspond to the movement of such a pendulum located on a time varying rotating frame, theoretically animated by all kinds of variable rotational speeds (we have presented some of them). For this reason, it is possible that these solutions are interesting to improve experiments, for example to verify gravitational effects, as we have introduced. Indeed, we know that the classic experiments would be very long and complex.

Moreover, these results can be applied to other areas, because they describe the theoretical motion of harmonic oscillators in rotating frames. Therefore, it is possible that these solutions can be applied, for example, in the study of gyroscopic movements, or such as those mentioned in the introduction to the article, or others.

5. Conclusion

We have detailed exact trajectories of two-dimensional harmonic oscillator in time varying rotating frame. In a second time, we have applied these results to the Foucault Pendulum problem. In particular, we suggest that trajectories are hypotrochoid curves, that seem a second original result. A simple experiment consists in supplying an impulse in the plane of the oscillations, and in verifying that the trajectory is well a rose curve. Later works could extend these results to other problems, such, for example, gyroscopic motion.

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