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SYMPLECTIC NON-CONVEXITY OF TORIC DOMAINS

JULIEN DARDENNES, JEAN GUTT, AND JUN ZHANG

ABSTRACT. We investigate the convexity up to symplectomorphism (called symplectic convexity) of star-shaped toric domains in \mathbb{R}^4 . In particular, based on the criterion from Chaidez-Edtmair in [4] via Ruelle invariant and systolic ratio of the boundary of star-shaped toric domains, we provide elementary operations on domains that can kill the symplectic convexity. These operations only result in small perturbations in terms of domains' volume. Moreover, one of the operations is a systematic way to produce examples of dynamically convex but not symplectically convex toric domains. Finally, we are able to provide concrete bounds for the constants that appear in Chaidez-Edtmair's criterion.

1. INTRODUCTION

A toric domain X_Ω in $\mathbb{R}^4 (\simeq \mathbb{C}^2)$ is a symplectic manifold with boundary, defined via the moment map $\mu : \mathbb{C}^2 \rightarrow \mathbb{R}_{\geq 0}^2$ as follows,

$$(1) \quad X_\Omega := \mu^{-1}(\Omega) \quad \text{where} \quad \mu(z_1, z_2) := (\pi|z_1|^2, \pi|z_2|^2)$$

and $\Omega \subset \mathbb{R}_{\geq 0}^2$ is a domain, called moment image, containing a neighborhood of the origin of \mathbb{R}^2 . A toric domain X_Ω is called star-shaped if the radial vector field of \mathbb{R}^4 is transverse to the boundary ∂X_Ω and intersects ∂X_Ω only once. This is equivalent to the condition, in terms of the moment image, that all the rays in $\mathbb{R}_{\geq 0}^2$ starting from the origin are transverse to $\partial\Omega$. The following Figure 1 provides an example (of the moment image Ω) of a toric star-shaped domain X_Ω . In this paper, we always assume

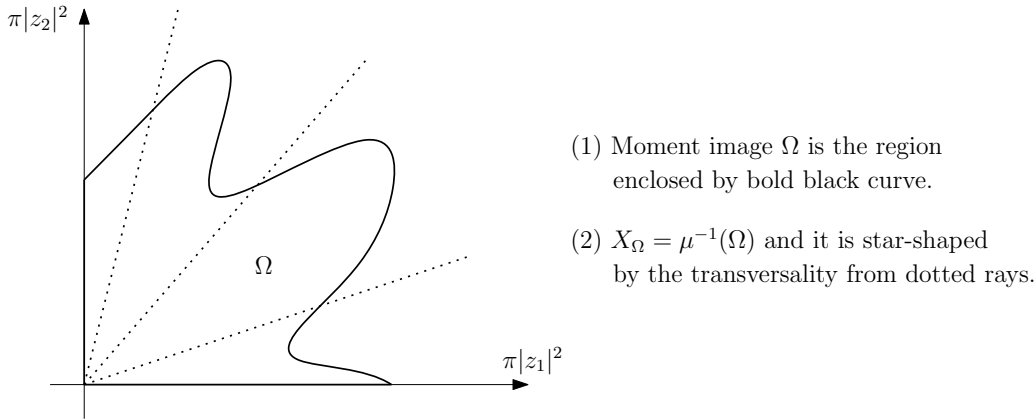


FIGURE 1. An example of a toric star-shaped domain.

that $\partial\Omega$ is smooth. It is well-known that for a (toric) star-shaped domain $X(= X_\Omega)$ of \mathbb{C}^2 , its boundary ∂X is a contact manifold with the contact structure given by the kernel of the contact 1-form $\lambda = \lambda_{\text{std}}|_{\partial X}$, where

$$\lambda_{\text{std}} = \frac{1}{2} \left(\sum x_i dy_i - y_i dx_i \right).$$

The study of the convexity of a symplectic toric domain X_Ω has a long history. As (geometric) convexity is not invariant under symplectomorphisms of \mathbb{R}^4 , various substitutive notions, of symplectic nature, have been introduced. The most fruitful one is dynamical convexity, introduced in [8], which requires the minimal Conley-Zehnder index of the Reeb orbits on ∂X_Ω has to be sufficiently positive. This serves as a dynamical hypothesis for many interesting problems (see [2, 13, 5, 9]). In this paper, we are mainly interested in the convexity of the toric domain up to symplectomorphism, for brevity, called *symplectic convexity*. A toric domain X_Ω is called *symplectically convex* if $\phi(X_\Omega)$ is (geometrically) convex for some $\phi \in \text{Symp}(\mathbb{R}^4)$. A fundamental result in [8] says that any symplectically convex domain is dynamically convex. Interestingly, it has recently been confirmed that

$$(2) \quad \text{dynamical convexity} \neq \text{symplectic convexity on } S^3.$$

This is the main result, Theorem 1.8, of ([4]).

Remark 1.1. *Following [8] rigorously, up to symplectomorphism, only a strictly convex and non-degenerate domain is dynamically convex. Since both strict convexity and non-degeneracy can be obtained by generic perturbations, we will always consider an arbitrary convex domain as a natural extension from strictly convex and non-degenerate domains, more precisely, a geometric limit of a sequence of such domains. In this way, symplectic convexity as above is defined over any convex domain*

Verifying that a given toric domain X_Ω is symplectically convex is difficult since it requires one to provide an explicit $\phi \in \text{Symp}(\mathbb{R}^4)$ and to confirm that the image $\phi(X_\Omega)$ is indeed convex. In this paper, we work towards a somewhat opposite direction: try to carry out a minimal amount of operations on a toric domain X so that the resulting domain is *not* symplectically convex. To this end, a good characterization of a necessary condition for a toric domain to be symplectically convex is needed. We use the following one, which applies to any star-shaped domain, not necessarily toric.

Proposition 1.1 (Proposition 3.1 in [4]). *Given a star-shaped domain X , denote $Y := \partial X$. If X is symplectically convex, then*

$$(3) \quad c \leq \text{ru}(Y, \lambda) \cdot \text{sys}(Y, \lambda)^{\frac{1}{2}} \leq C$$

where c and C are positive constants, independent of the input (Y, λ) . The contact form λ is the restriction of the standard contact form on the boundary: $\lambda = \lambda_{\text{std}}|_Y$.

The quantities $\text{ru}(Y, \lambda)$ and $\text{sys}(Y, \lambda)$ in Proposition 1.1 are the *Ruelle ratio* and the *systolic ratio* of (Y, λ) respectively. They are defined by

$$\text{ru}(Y, \lambda) := \frac{\text{Ru}(Y, \lambda)}{\text{Vol}(Y, \lambda)^{\frac{1}{2}}}, \quad \text{and} \quad \text{sys}(Y, \lambda) = \frac{(\text{min period of a Reeb orbit})^2}{\text{Vol}(Y, \lambda)},$$

where $\text{Ru}(Y, \lambda)$ is the Ruelle invariant of the contact manifold (Y, λ) . For more details (and a complete definition), we refer to Section 2.

It is important to emphasize that even though the ratios ru and sys are defined only on contact manifolds (here, the boundary of a star-shaped domain), they are in fact invariant under symplectomorphisms of the bounding domains. Indeed, by Lemma 3.5 in [4], it is readily verified that if star-shaped domains X and X' are symplectomorphic, then $\text{ru}(\partial X, \lambda) = \text{ru}(\partial X', \lambda)$ and $\text{sys}(\partial, \lambda) = \text{sys}(X', \lambda)$. We can, and we shall, therefore talk about the Ruelle ratio and the systolic ratio of a toric domain, that is,

$$\text{ru}(X_\Omega) := \text{ru}(\partial X_\Omega, \lambda) \quad \text{and} \quad \text{sys}(X_\Omega) := \text{sys}(\partial X_\Omega, \lambda).$$

The applications of the Ruelle invariant, originally defined in [11], in symplectic and contact geometry were initiated by [10].

Remark 1.2. *It is worth mentioning that for star-shaped domains, even though $\text{Ru}(X)$, under Definition 2.1, is invariant under symplectomorphisms, it cannot provide symplectic embedding obstructions in general. For instance, the ellipsoid $E(1, 4)$ symplectically embeds into the ball $B^4(2)$, but by Proposition 1.4 we have $\text{Ru}(E(1, 4)) = 5$ which is bigger than $\text{Ru}(B^4(2)) = 4$. In fact, Ruelle invariant behaves fundamentally differently from symplectic capacities (cf. [10, Corollary 1.13]).*

The main result of this paper is the following:

Theorem 1.3. *Given any star-shaped toric domain X_Ω in \mathbb{R}^4 , there exist small perturbations of X_Ω , in terms of the volume, such that the resulting domains $X_{\hat{\Omega}}$ are still star-shaped but the product $\text{ru}(X_{\hat{\Omega}}) \cdot \text{sys}(X_{\hat{\Omega}})^{\frac{1}{2}}$ can be arbitrarily small or arbitrarily large. In particular, the resulting domains $X_{\hat{\Omega}}$ are not symplectically convex.*

The notation $X_{\hat{\Omega}}$ indicates that the perturbations of X_Ω promised in Theorem 1.3 can be carried on directly on the moment image Ω . Indeed, the proof of Theorem 1.3 in Section 3 provides two explicit constructions of such perturbations - strangulation operation and strain operation. A schematic picture below, Figure 2, illustrate these two operations on the level of moment images. More explicitly, given a moment image Ω of a toric starshaped domain X_Ω , the strangulation operation removes a small part (blue shaded region) along a ray from Ω , while the strain operation adds a thin triangle (red shaded region) to Ω . Refined pictures of these operations with precise parameters will show up in Figure 3 and Figure 4. The resulting new toric domain is denoted by $X_{\hat{\Omega}}$ with its moment image $\hat{\Omega}$.

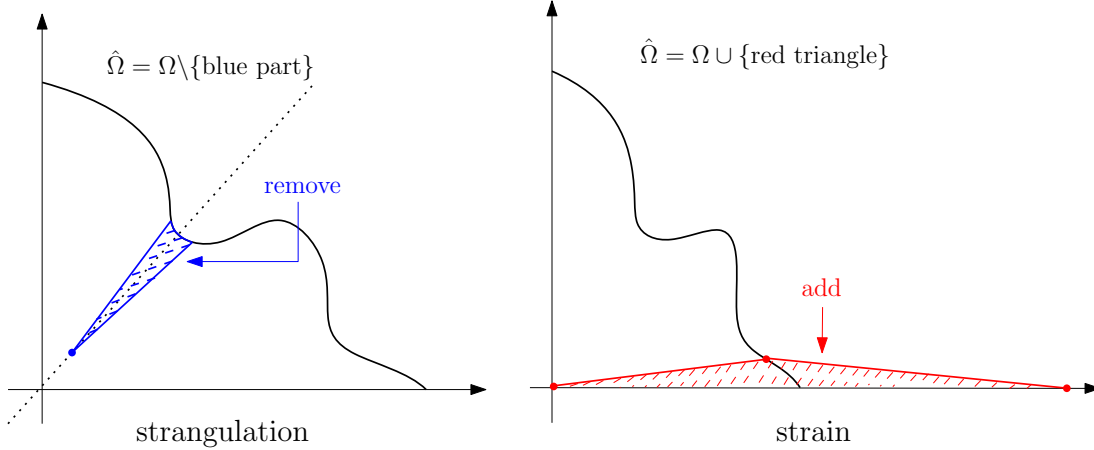


FIGURE 2. Strangulation operation and strain operation.

We will apply Proposition 1.1 to deduce that $X_{\hat{\Omega}}$ is not symplectically convex by deforming Ω until passing below the lower bound c or over the upper bound C , even though we don't know explicitly how big c and C are in general. As expected, there will be non-trivial estimations of the ratios

$$\text{ru}(X_{\hat{\Omega}}) \quad \text{and} \quad \text{sys}(X_{\hat{\Omega}}).$$

This is based on results from Section 2 and Section 3. Explicitly, for the Ruelle ratio ru (in fact, the Ruelle invariant Ru), we have the following computational result. Its proof occupies Section 2.

Proposition 1.4. *Let X_{Ω} be any 4-dimensional toric star-shaped domain. Then its Ruelle invariant is given by*

$$\text{Ru}(X_{\Omega}) := \text{Ru}(\partial X, \lambda_{\text{std}}|_{\partial X}) = a(\Omega) + b(\Omega)$$

where $a(\Omega)$ and $b(\Omega)$ are the w_1 -intercept and w_2 -intercept, respectively, of the moment image Ω in $\mathbb{R}_{\geq 0}^2$, in (w_1, w_2) -coordinate.

Remark 1.5. *Proposition 1.4 generalizes [10, Proposition 1.11] since there is no hypothesis that the profile curve, as the boundary $\partial\Omega \cap \mathbb{R}_{>0}^2$, has slopes everywhere negative (cf. [10, footnote on page 6]).*

On the other hand, for the systolic ratio, or more generally, the (minimal) action of the Reeb orbits, the discussion at the beginning of Section 3 provides a general formula (15), which is a well-known result in symplectic toric geometry. Then one verifies that the strangulation operation yields *only* the systolic ratio to be arbitrarily small, while the strain operation yields *only* the Ruelle ratio to be arbitrarily large.

Remark 1.6 (Related to capacity). *The operations in Theorem 1.3 do not always result in small perturbations in terms of symplectic capacities. It is possible to verify that $X_{\hat{\Omega}}$ from method two - strain in subsection 3.2 - satisfies $X_{\Omega} \hookrightarrow X_{\hat{\Omega}} \hookrightarrow hX_{\Omega}$ for a*

rescaling h close to 1, via symplectic folding technique (see [12]), where \hookrightarrow represents a symplectic embedding. However, the method one - strangulation in subsection 3.1 - is believed to change the symplectic capacities dramatically. How to describe such changes quantitatively in terms of capacities will be explored in further work.

If one applies results in [8], symplectic convexity can be killed by breaking the dynamical convexity. In fact, this can be achieved as well by a small perturbation also in the sense of the volume. More explicitly, one simply modifies the profile curve of Ω in $\mathbb{R}_{\geq 0}^2$ near the intersection points on the two axes so that the Conley-Zehnder indices of the Reeb orbits corresponding to these intersection points are always negative. We leave the details to interested readers.

Here, we emphasize that our operations in Theorem 1.3 are fundamentally different. In particular, method one - strangulation in subsection 3.1 - can be distinguished with the operation elaborated above via symplectic capacities (for instance, the minimal action), even though it always goes beyond the category of dynamically convex toric domains due to [6, Proposition 1.8]. Method two - strain in subsection 3.2 - can be carried out even within the category of dynamically convex domains. In particular, we have the following useful result.

Corollary 1.7. *For any dynamically convex toric domain X_Ω in \mathbb{R}^4 , there exists a small perturbation in terms of the volume such that the resulting domain $X_{\hat{\Omega}}$ is still dynamically convex but not symplectically convex.*

Proof. This directly comes from the construction of strain operation in subsection 3.2, Corollary 3.4, and [6, Proposition 1.8]. \square

Note that Corollary 1.7 provides a variety of examples that support (2). In sharp contrast to the example produced in subsection 1.5 in ([4]) (which is closely related to the one invented in [1]), Corollary 1.7 above is to our best knowledge the first systematic way to produce *toric* examples to confirm (2) and it is more direct and much simpler than the one in [4].

As the proof of Theorem 1.3 is essentially fighting against the constant c and C appearing in (3), one may be curious about how small or large these constants are. In general, due to the complexity of the proof of (3) in [4], it seems difficult to read off any bounds for c and C directly. However, for a special family of star-shaped domains, called *monotone* toric domains (introduced in [6]), we are able to estimate the product $\text{ru}(X_\Omega) \cdot \text{sys}(X_\Omega)^{\frac{1}{2}}$ by the following result, which yields concrete bounds for c and C in (3) (cf [4, Remark 1.11]).

Theorem 1.8. *For any monotone toric domain X_Ω , we have*

$$\text{ru}(X_\Omega) \cdot \text{sys}(X_\Omega)^{\frac{1}{2}} \geq \frac{1}{2}.$$

If, furthermore, X_Ω is (geometrically) convex in \mathbb{R}^4 , then

$$\text{ru}(X_\Omega) \cdot \text{sys}(X_\Omega)^{\frac{1}{2}} \leq 3.$$

In particular, the constant c and C in the criterion (3) satisfies $c \leq \frac{1}{2}$ and $C \geq 3$, respectively.

Here, monotone means that the outward normal vectors $\nu = (\nu_1, \nu_2)$ at any point of the boundary component $\partial_+ \Omega := \partial \Omega \cap \mathbb{R}_{>0}^2$ have non-negative components ($\nu_1 \geq 0, \nu_2 \geq 0$). This is more general than the classical notion of convex or concave toric domains and admit many nice properties. For instance, by [6, Proposition 1.8], the category of (strictly) monotone toric domains coincides with the category of domains that are both toric and dynamically convex. For another instance, due to [6, Theorem 1.7], all normalized symplectic capacities agree on strictly monotone toric domains.

Remark 1.9. *In the proof of Theorem 1.3, strain operation in subsection 3.2, shows that within the category of strict monotone toric domains, there is no upper bound for the constant C in (3), see Corollary 3.4. The constants in the conclusion of Theorem 1.8) are sharp. The upper bound is realised by the toric domain whose moment image is given by the blue curve of Figure 6. Moreover, the lower bound $\frac{1}{2}$ can be arbitrarily approximated by the family of polydisks $P(a, b)$ (its definition given by (23)) with $b \rightarrow \infty$, we have*

$$\text{ru}(P(a, b)) \cdot \text{sys}(P(a, b))^{\frac{1}{2}} = \frac{(a+b)}{\sqrt{2ab}} \cdot \frac{a}{\sqrt{2ab}} = \frac{(a+b)}{2b} \xrightarrow{b \rightarrow +\infty} \frac{1}{2}$$

For more details, see subsection 4.

Remark 1.10. *Soon after our paper was completed, Chaidez-Edtmair uploaded a new preprint [3], where it generalizes the criterion (3) to higher-dimensional domains. Moreover, inspired by our strain operation (see Figure 4), they provide higher-dimensional examples that are dynamically convex but not symplectically convex.*

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2. RUELLE INVARIANT

For simplicity, let us assume our closed contact 3-manifold (Y, λ) is a homology 3-sphere. To the contact structure $\xi = \ker \lambda$, we can associate a real number $\text{Ru}(Y, \lambda)$ as follows. Observe that the Reeb flow $\phi_{R_\alpha}^t$ preserves the contact form λ . In particular, it preserves the contact structure ξ . Then, for any time t and any fixed point $y \in Y$, the

pushforward, or linearization, $(\phi_{R_\alpha}^t)_*$ brings the contact 2-plane ξ_y to the contact 2-plane $\xi_{\phi_{R_\alpha}^t(y)}$. Under a given trivialization τ of the contact structure, this linearization can be regarded as a linear transformation of \mathbb{R}^2 , denoted by $\Phi_{y,t}^\tau$. For any real $T \geq 0$, the path $\Phi = \{\Phi_{y,t}^\tau\}_{t \in [0,T]}$ defines an element of the universal cover of the symplectic group $\widetilde{\text{Sp}}(2)$. Together with the rotation number, $\rho : \widetilde{\text{Sp}}(2) \rightarrow \mathbb{R}$ (see subsection 1.2 in [10] or subsection 2.1 in [4]), this yields a real number $\rho(y, T, \tau) := \rho(\{\Phi_{y,t}^\tau\}_{t \in [0,T]})$ and the following limit

$$(4) \quad \text{rot}(y) = \lim_{T \rightarrow +\infty} \frac{\rho(y, T, \tau)}{T}$$

is well-defined. In particular, $\text{rot}(y)$ is independent of the trivialization τ , since here, Y has a unique trivialization up to homotopy. In general, $\text{rot}(y)$ only depends on the homotopy class of a trivialization. As elaborated by Hutchings in [10], subsection 1.2, right above Definition 1.3 (and which traces back to Ruelle in [11]), we have that $\text{rot}(y)$ is an integrable function. Proposition 2.13 in [4] also proves these properties.

Definition 2.1. *Suppose the closed contact 3-manifold (Y, λ) is a homology 3-sphere, then its Ruelle invariant is defined by*

$$\text{Ru}(Y, \lambda) := \int_Y \text{rot}(y) \lambda \wedge d\lambda.$$

In particular, if X is a star-shaped domain in \mathbb{R}^4 , then we define

$$\text{Ru}(X) := \text{Ru}(\partial X, \lambda_{\text{std}}|_{\partial X})$$

2.1. Linearized Reeb flow. Denote by $\partial_+\Omega := \partial\Omega \cap \mathbb{R}_{>0}^2$. For any point $p = (w_1, w_2) \in \partial_+\Omega$, consider the polar coordinate $(w_1, \theta_1, w_2, \theta_2)$. Then (recall that μ is the moment map) for any $z \in \mu^{-1}(p)$, one can verify that the Reeb vector field R is

$$(5) \quad R(z) = \frac{2\pi}{\nu_1(p)w_1 + \nu_2(p)w_2} \left(\nu_1(p) \frac{\partial}{\partial \theta_1} + \nu_2(p) \frac{\partial}{\partial \theta_2} \right)$$

where $(\nu_1(p), \nu_2(p))$ is the unit normal vector of $\partial_+\Omega$ at point p , pointing outward of Ω . Note that $\nu_1(p)w_1 + \nu_2(p)w_2 > 0$ for any $p \in \partial_+\Omega$ due to our hypothesis that X_Ω is star-shaped. Moreover, the contact 2-plane at z is given by,

$$(6) \quad \xi_z = \left\{ a_1 \frac{\partial}{\partial w_1} + b_1 \frac{\partial}{\partial \theta_1} + a_2 \frac{\partial}{\partial w_2} + b_2 \frac{\partial}{\partial \theta_2} \mid \begin{array}{l} \nu_1(p)a_1 + \nu_2(p)a_2 = 0 \\ w_1b_1 + w_2b_2 = 0 \end{array} \right\},$$

and one can choose a basis of ξ_z as follows,

$$(7) \quad e_1(p) = -\nu_2(p) \frac{\partial}{\partial w_1} + \nu_1(p) \frac{\partial}{\partial w_2} \quad \text{and} \quad e_2(p) = -w_2 \frac{\partial}{\partial \theta_1} + w_1 \frac{\partial}{\partial \theta_2}.$$

Note that $(e_1(p), e_2(p))$ is an *ordered* basis in that $(\omega_{\text{std}})_z(e_1(p), e_2(p)) > 0$. Using this basis, along any Reeb trajectory $\gamma = (\gamma(t))_{t \in [0,T]}$, one can chose a trivialization

$\tau : \gamma^*\xi \rightarrow \gamma \times \mathbb{R}^2$ explicitly defined as follows. For any $(z, v) \in (\gamma^*\xi)_z$ where $z \in \gamma$ and $v \in \xi_z$,

$$(8) \quad \tau(p)((z, v)) = (z, (v_R, v_\theta)) \quad \text{where } v = v_R e_1(p) + v_\theta e_2(p).$$

Moreover, under this trivialization, the differentials of the Reeb flow along the trajectory γ form a path in $\text{Sp}(2)$, denoted by Φ . The following lemma is crucial.

Lemma 2.2. *With respect to the trivialization given in (8), along the Reeb trajectory $\gamma = (\gamma(t))_{t \in [0, T]}$ the resulting path Φ in $\text{Sp}(2)$ from the differentials of the Reeb flow is*

$$\Phi = \left\{ \begin{pmatrix} 1 & 0 \\ f(t) & 1 \end{pmatrix} \middle| t \in [0, T] \right\}$$

where $f(t)$ is a linear function of t depending only on $\gamma(0)$ and $e_1(\mu(\gamma(0)))$ in (7). In particular, $\text{rot}_\tau(\gamma(0), T) = 0$.

Proof. Suppose $\gamma(0) \in \mu^{-1}(p)$ for some $p \in \partial_+\Omega$. For $v \in \xi_{\gamma(0)}$ and any $t \in [0, T]$, to compute the differential $(d\phi_R^t|_{\xi_{\gamma(0)}})(v)$, we need to take a locally defined smooth path $r(s) : (-\epsilon, \epsilon) \rightarrow \partial X_\Omega$ for ϵ sufficiently small such that $r(0) = \gamma(0)$ and $r'(0) = v$. Denote for brevity $r(s) = (w_1(s), \theta_1(s), w_2(s), \theta_2(s))$ where $w_1(0) = w_1$ and $w_2(0) = w_2$. For any $s \in (-\epsilon, \epsilon)$, by (7) and (8),

$$\begin{aligned} r(s) &= r(0) + sv + o(s) \\ &= (w_1 - s\nu_2(p)v_R, \theta_1 - sw_2v_\theta, w_2 + s\nu_1(p)v_R, \theta_2 + sw_1v_\theta) + o(s). \end{aligned}$$

Note that the approximation term $o(s)$ exist to guarantee that $r(s) \in \partial X_\Omega$. Then, by (5), we have

$$\begin{aligned} \phi_R^t(r(s)) &= \phi_{R_\alpha}^t(w_1(s), \theta_1(s), w_2(s), \theta_2(s)) \\ &= (w_1(s), \theta_1(s) + \Theta_1(s) \cdot t, w_2(s), \theta_2(s) + \Theta_2(s) \cdot t) \end{aligned}$$

where

$$\Theta_1(s) = \frac{2\pi\nu_1(p(s))}{\nu_1(p(s))w_1(s) + \nu_2(p(s))w_2(s)} \quad \text{and} \quad \Theta_2(s) = \frac{2\pi\nu_2(p(s))}{\nu_1(p(s))w_1(s) + \nu_2(p(s))w_2(s)},$$

where by notation $p(s) = (w_1(s), w_2(s))$. Observe that the denominator of $\Theta_1(s)$ and $\Theta_2(s)$ can be simplified as follows,

$$\nu_1(p(s))w_1(s) + \nu_2(p(s))w_2(s) = \nu_1(p(s))w_1 + \nu_2(p(s))w_2 + o(s).$$

In particular, it converges to $\nu_1(p)w_1 + \nu_2(p)w_2$ as $s \rightarrow 0$. Then, by the definition of a differential and computations above,

$$\begin{aligned} (d\phi_R^t)(v) &= \lim_{s \rightarrow 0} \frac{\phi_R^t(r(s)) - \phi_R^t(r(0))}{s} \\ &= (-\nu_2(p)v_R, -w_2v_\theta, \nu_1(p)v_R, w_1v_\theta) \\ &\quad + \left(0, \lim_{s \rightarrow 0} \frac{\Theta_1(s) - \frac{2\pi\nu_1(p)}{\nu_1(p)w_1 + \nu_2(p)w_2}}{s} \cdot t, 0, \lim_{s \rightarrow 0} \frac{\Theta_2(s) - \frac{2\pi\nu_2(p)}{\nu_1(p)w_1 + \nu_2(p)w_2}}{s} \cdot t \right). \end{aligned}$$

Meanwhile, further simplifications yield

$$(9) \quad \lim_{s \rightarrow 0} \frac{\Theta_1(s) - \frac{2\pi\nu_1(p)}{\nu_1(p)w_1 + \nu_2(p)w_2}}{s} = 2\pi \cdot \frac{(\nu_1(p)\nu_2(p(s))'|_{s=0} - \nu_1(p(s))'|_{s=0}\nu_2(p))(-w_2)}{(\nu_1(p)w_1 + \nu_2(p)w_2)^2},$$

and similarly,

$$(10) \quad \lim_{s \rightarrow 0} \frac{\Theta_2(s) - \frac{2\pi\nu_2(p)}{\nu_1(p)w_1 + \nu_2(p)w_2}}{s} = 2\pi \cdot \frac{(\nu_1(p)\nu_2(p(s))'|_{s=0} - \nu_1(p(s))'|_{s=0}\nu_2(p))w_1}{(\nu_1(p)w_1 + \nu_2(p)w_2)^2},$$

where the $\nu_i(p(s))'|_{s=0}$ denotes the derivative with respect to the variable s and then evaluated at $s = 0$. For brevity, denote by

$$A(p; v) := 2\pi \cdot \frac{\nu_1(p)\nu_2(p(s))'|_{s=0} - \nu_1(p(s))'|_{s=0}\nu_2(p)}{(\nu_1(p)w_1 + \nu_2(p)w_2)^2},$$

the common factor in (9) and (10). Then

$$(11) \quad (d\phi_{R_\alpha}^t)(v) = (-\nu_2(p)v_R, (A(p; v) + v_\theta)(-w_2)t, \nu_1(p)v_R, (A(p; v) + v_\theta)w_1t).$$

In particular,

$$d\phi_R^t(e_1(p)) = e_1(p) + (A(p; e_1(p))t)e_2(p) \quad \text{and} \quad d\phi_R^t(v) = e_2(p).$$

Representing this by a matrix with respect to the basis $(e_1(p), e_2(p))$, one gets that

$$(12) \quad d\phi_R^t|_{\xi_{\gamma(0)}} = \begin{pmatrix} 1 & 0 \\ A(p; e_1(p))t & 1 \end{pmatrix}.$$

Thus we prove the first conclusion by setting $f(t) := A(p; e_1(p))t$. Moreover, the second conclusion is straightforward, since each matrix representation of the differential $d\phi_R^t|_{\xi_z}$ as in (12) is similar to a shear matrix, which does not contribute any rotations. \square

2.2. Proof of Proposition 1.4. Note that the trivialization in (8) does not extend to the entire ∂X_Ω (since the polar coordinate is not well-defined for the points where $w_1 = 0$ or $w_2 = 0$). For any globally defined trivialization $\bar{\tau}$, compared with the trivialization via the polar coordinate, the only difference of the rotation number at point $z \in \mu^{-1}(p)$ for $p \in \partial_+ \Omega$ comes from how much the function $\theta_1 + \theta_2$ changes along the Reeb flow. Indeed, as explained in the proof of Lemma 2.2 in [10], moving along the circle given by a rotation of either θ_1 or θ_2 results in the desired change of the factor v_R in (8), with respect to the trivialization $\bar{\tau}$. Then Lemma 2.2 yields

$$\rho(z) = \lim_{T \rightarrow \infty} \frac{\text{rot}_{\bar{\tau}}(z, T)}{T} = 0 + \frac{(d\theta_1 + d\theta_2)(R_\alpha(z))}{2\pi} = \frac{\nu_1(p) + \nu_2(p)}{\nu_1(p)w_1 + \nu_2(p)w_2}$$

where $p = (w_1, w_2)$. Note that $\rho(z)$ is in fact a function of $p \in \partial_+\Omega$. Then by the definition of Ruelle invariant,

$$\begin{aligned} \text{Ru}(X_\Omega) &= \int_{\partial X_\Omega} \rho(z) \alpha \wedge d\alpha \\ &= \int_{\partial_+\Omega} \frac{\nu_1(p) + \nu_2(p)}{\nu_1(p)w_1 + \nu_2(p)w_2} (w_1 dw_2 - w_2 dw_1) \end{aligned}$$

where the second equality comes from a change of variable via the moment map μ defined in (1) (and restricted to ∂X_Ω). Suppose the profile curve $\overline{\partial_+\Omega}$ is parametrized by $\{(w_1(s), w_2(s))\}_{s \in [0,1]}$ such that

$$w_1(0) = a(\Omega), \quad w_1(1) = 0 \quad \text{and} \quad w_2(0) = 0, \quad w_2(1) = b(\Omega),$$

where $a(\Omega)$ and $b(\Omega)$ are the w_1 -intercept and w_2 -intercept. We may assume after a change of parametrization that

$$w_1 dw_2 - w_2 dw_1 = (w_1(s)w_2'(s) - w_2(s)w_1'(s))ds.$$

Meanwhile, observe that

(13)

$$(\nu_1(p), \nu_2(p)) = (\nu_1(s), \nu_2(s)) = \left(\frac{-w_2'(s)}{\sqrt{|w_1'(s)|^2 + |w_2'(s)|^2}}, \frac{w_1'(s)}{\sqrt{|w_1'(s)|^2 + |w_2'(s)|^2}} \right).$$

Therefore, by (13),

$$\begin{aligned} \text{Ru}(X_\Omega) &= \int_0^1 \frac{\nu_1(s) + \nu_2(s)}{\nu_1(s)w_1(s) + \nu_2(s)w_2(s)} (w_1(s)w_2'(s) - w_2(s)w_1'(s))ds \\ &= \int_0^1 \frac{-w_2'(s) + w_1'(s)}{-w_2'(s)w_1(s) + w_1'(s)w_2(s)} (w_1(s)w_2'(s) - w_2(s)w_1'(s))ds \\ &= \int_0^1 w_2'(s) - w_1'(s)ds \\ &= (w_2(1) - w_2(0)) - (w_1(1) - w_1(0)) = b(\Omega) + a(\Omega). \end{aligned}$$

Therefore, we obtain the desired conclusion.

3. PROOF OF THEOREM 1.3

In this section, we will give the proof of the main result, Theorem 1.3 in the introduction. Let us start from the following elementary observation on the closed Reeb orbit on a star-shaped domain $(\partial X, \lambda)$. By (5), for a point $p = (w_1, w_2) \in \partial_+\Omega = \partial\Omega \cap \mathbb{R}_{>0}^2$, we know that

$$(14) \quad \text{a Reeb trajectory at } \mu^{-1}(p) \text{ is closed} \quad \text{if and only if} \quad \left| \frac{\nu_2(p)}{\nu_1(p)} \right| \in \mathbb{Q}.$$

Assume that $\left| \frac{\nu_2(p)}{\nu_1(p)} \right| \in \mathbb{Q}$. Denote by $h_p \in \mathbb{R}_{>0}$ the unique non-zero positive scalar such that

- (i) $(h_p\nu_1(p), h_p\nu_2(p)) \in \mathbb{Z}^2$;
- (ii) $h_p\nu_1(p), h_p\nu_2(p)$ are coprime.

For brevity, denote $(m_p, n_p) := (h_p\nu_1(p), h_p\nu_2(p))$. The second condition (ii) above guarantees that the corresponding closed Reeb orbit, denoted by $\gamma_{(m_p, n_p)}$ is primitive (that is, not a multiple cover of another closed Reeb orbit). Note that the period of $\gamma_{(m_p, n_p)}$ is, by definition, the action $\mathcal{A}(\gamma_{(m_p, n_p)})$. Hence,

$$(15) \quad \mathcal{A}(\gamma_{(m_p, n_p)}) = (\nu_1(p)w_1 + \nu_2(p)w_2) \cdot h_p = m_p w_1 + n_p w_2.$$

This can be viewed as the inner product of the (integer-normalized) normal vector (m_p, n_p) and the position vector (w_1, w_2) (for point p). Due to our hypothesis that X_Ω is star-shaped, the action $\mathcal{A}(\gamma_{(m_p, n_p)})$ is always positive, even though the vector (m_p, n_p) does not have both of its components positive.

3.1. Method one: strangulation. Since X_Ω is a star-shaped domain, we can assume, for its moment image Ω , without loss of generality, that the diagonal of $\mathbb{R}_{>0}^2$ intersects $\partial\Omega$ at point (w_*, w_*) such that a neighborhood within $\mathbb{R}_{>0}^2$ of the subset $\{(w, w) \in \mathbb{R}_{>0}^2 \mid 0 \leq w < w_*\}$ lies in the interior of Ω . In general, there always exists some ray in $\mathbb{R}_{\geq 0}^2$ satisfying this condition. By our assumption, since $\overline{\partial_+\Omega}$ is smooth, for any given $\epsilon > 0$, there exists some angle $\theta(\epsilon)$ such that the unbounded sector with vertex (ϵ, ϵ) , divided in half by the diagonal, and angle equal to $2\theta(\epsilon)$, intersects Ω in a closed region $\mathcal{S}(\epsilon)$ with points $(w_1, w_2) \in \partial\Omega \cap \mathcal{S}$ satisfying

$$|w_1 - w_*| \leq \epsilon \quad \text{and} \quad |w_2 - w_*| \leq \epsilon.$$

Now, carry on the following *strangulation* operation on Ω , that is, define

$$\hat{\Omega} := \Omega \setminus (\text{int}(\mathcal{S}(\epsilon)) \cup \text{int}(\partial\Omega \cap \mathcal{S}(\epsilon))).$$

For a picture of this operation, see Figure 3.

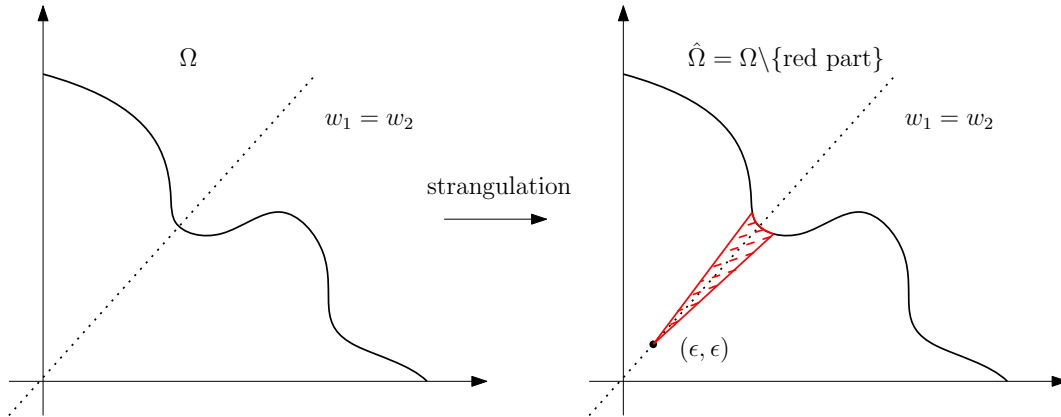


FIGURE 3. Strangulation operation.

After smoothing all singularities of $\hat{\Omega}$, we have that the resulting domain, still denoted by $\hat{\Omega}$, is again a closed domain in $\mathbb{R}_{\geq 0}^2$ with its pre-image under the moment

map $\mu^{-1}(\widehat{\Omega}) =: X_{\widehat{\Omega}}$ being a star-shaped domain. Moreover, $X_{\widehat{\Omega}}$ satisfies the following quantitative properties.

Lemma 3.1. *The strangulation operation on X_{Ω} results in a star-shaped domain $X_{\widehat{\Omega}}$ which satisfies*

- (1) $\text{Vol}(X_{\widehat{\Omega}}) = \text{Vol}(X_{\Omega}) + O(\epsilon)$.
- (2) $\text{sys}(X_{\widehat{\Omega}}) \leq O(\epsilon)$.
- (3) $\text{Ru}(X_{\widehat{\Omega}}) = \text{Ru}(X_{\Omega})$.

Here, $O(\epsilon)$ represents a constant, proportional to ϵ in Figure 3, that can be arbitrarily small.

Assuming Lemma 3.1, we continue the proof of Theorem 1.3 via the strangulation operation. First, we have the following computations on volumes from Stokes' theorem,

$$\begin{aligned} \text{Vol}(\partial X_{\widehat{\Omega}}, \lambda) &= \int_{\partial X_{\widehat{\Omega}}} \lambda \wedge d\lambda = \int_{X_{\widehat{\Omega}}} d(\lambda \wedge d\lambda) \\ &= \int_{X_{\widehat{\Omega}}} d\lambda \wedge d\lambda = 2\text{Vol}(X_{\widehat{\Omega}}). \end{aligned}$$

Second, suppose ϵ is sufficiently small so that $O(\epsilon) < \frac{\text{Vol}(X_{\Omega})}{2}$. Then (1) and (2) in Lemma 3.1 imply that

$$\text{ru}(X_{\widehat{\Omega}})^2 = \frac{\text{Ru}(X_{\widehat{\Omega}})^2}{2\text{Vol}(X_{\widehat{\Omega}})} = \frac{\text{Ru}(X_{\Omega})^2}{2\text{Vol}(X_{\Omega}) - 2O(\epsilon)} \leq 2 \cdot \text{ru}(X_{\Omega})^2,$$

where, in particular, the upper bound $2 \cdot \text{ru}(X_{\Omega})^2$ is finite. Third, (3) in Lemma 3.1 implies that

$$\text{ru}(X_{\widehat{\Omega}}) \cdot \text{sys}(X_{\widehat{\Omega}})^{\frac{1}{2}} \leq \sqrt{2} \cdot \text{ru}(X_{\Omega}) \cdot \sqrt{O(\epsilon)} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Therefore, the product $\text{ru}(X_{\widehat{\Omega}}) \cdot \text{sys}(X_{\widehat{\Omega}})^{\frac{1}{2}}$ will be lower than the constant c appearing in criterion (3), whenever ϵ is sufficiently small. In conclusion, the domain $X_{\widehat{\Omega}}$ is not symplectically convex.

Proof of Lemma 3.1. When $\epsilon < w_*$, the 4-dimensional volume of X_{Ω} and $X_{\widehat{\Omega}}$ (with respect to the standard symplectic structure on \mathbb{R}^4) satisfy

$$(16) \quad |\text{Vol}(X_{\widehat{\Omega}}) - \text{Vol}(X_{\Omega})| \leq \pi \cdot 2(w_* + \epsilon)^2 \cdot \frac{\theta(\epsilon)}{\pi} = 8w_*^2 \cdot \theta(\epsilon)$$

which goes to 0 as ϵ goes to 0 (since $\theta(\epsilon)$ goes to 0). Therefore, $X_{\widehat{\Omega}}$ is indeed a small perturbation in terms of the volume of X_{Ω} . This proves (1).

By the discussion above on the closed Reeb orbits in (14), applied to the new domain $\widehat{\Omega}$, there exists a closed Reeb orbit at $p = (\epsilon, \epsilon)$ corresponding to the normal

vector $(1, 1)$. In particular, by (15) its action is $1 \cdot \epsilon + 1 \cdot \epsilon = 2\epsilon$. Denoting by \widehat{T}_{\min} the minimal period of a closed Reeb orbit of $\partial X_{\widehat{\Omega}}$, we have

$$\widehat{T}_{\min} \leq 2\epsilon$$

Therefore,

$$\text{sys}(X_{\widehat{\Omega}}) \leq \frac{4\epsilon^2}{\text{Vol}(\partial X_{\widehat{\Omega}}, \lambda)} = \frac{4\epsilon^2}{2\text{Vol}(X_{\widehat{\Omega}})} \leq \frac{4\epsilon^2}{2\text{Vol}(X_{\Omega}) - 16w_*^2 \cdot \theta(\epsilon)}$$

where the second inequality comes from (16). Hence, when ϵ is sufficiently small so that $w_*^2 \cdot \theta(\epsilon) < \frac{\text{Vol}(X_{\Omega})}{16}$, we have

$$(17) \quad \text{sys}(X_{\widehat{\Omega}}) \leq \frac{4\epsilon^2}{\text{Vol}(X_{\Omega})} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

This proves (2).

Finally, since the strangulation operation does not change the w_1 or w_2 -intercepts of the original domain Ω , due to Proposition 1.4, the Ruelle invariant does not change, that is, $\text{Ru}(X_{\widehat{\Omega}}) = \text{Ru}(X_{\Omega})$. This proves (3). \square

Remark 3.2. *It is not necessary to carry out the strangulation operation along the diagonal, as we did above. In general, most rays starting from the origin work in a similar way. An extreme case is to carry out such an operation along the w_1 -axis or w_2 -axis. The only difference is that the Ruelle invariant will change but gets smaller (so we still obtain the result that the product of ratios $\text{ru} \cdot \text{sys}^{\frac{1}{2}}$ will be eventually smaller than the constant c in the criterion in (3)). In fact, such an operation has been investigated in [14] on ellipsoids, called truncated ellipsoid.*

3.2. Method two: strain. Given a star-shaped domain X_{Ω} , suppose that the w_1 -intercept of $\overline{\partial_+ \Omega}$ is $a > 0$. Consider a generic small perturbation of Ω near $(a, 0)$ but with the w_1 -intercept a fixed, which also results in a small perturbation of X_{Ω} in terms of the volume, such that in a neighborhood N of $(a, 0)$, the boundary $\overline{\partial_+ \Omega}$ has a constant slope k , either positive or negative (but not equal to $\pm\infty$). This can be achieved due to our hypothesis that $\overline{\partial_+ \Omega}$ is smooth, and we can consider N sufficiently small so that the minimal period of the Reeb orbit of ∂X_{Ω} changes in an arbitrarily small way. For brevity, we still denote the domain after this perturbation by Ω .

Next, for any $\epsilon > 0$, sufficiently small so that the (unique) point $(w_*(\epsilon), \epsilon) \in \partial_+ \Omega$ for some $w_* > 0$ lies in the neighborhood N above, we have $\frac{\epsilon - 0}{w_*(\epsilon) - a} = k$, that is,

$$(18) \quad w_*(\epsilon) = \frac{\epsilon}{k} + a.$$

Consider the following triangle

$$\mathcal{T}(\epsilon) := \text{the triangle determined by vertices } (0, 0), (w_*(\epsilon), \epsilon), \text{ and } \left(\frac{1}{\sqrt{\epsilon}}, 0\right)$$

where ϵ is sufficiently small so that

$$(19) \quad \frac{-\epsilon}{\frac{1}{\sqrt{\epsilon}} - w_*(\epsilon)} > k \quad \text{if } k < 0.$$

This can be achieved since (19) is equal to $k(a - \frac{1}{\sqrt{\epsilon}}) > 0$, so when $\epsilon \rightarrow 0$, we have $a - \frac{1}{\sqrt{\epsilon}} < 0$ (since $k < 0$). Then consider the following *strain* operation on Ω , that is,

$$\hat{\Omega} := \Omega \cup \mathcal{T}(\epsilon).$$

For a picture of this operation, see Figure 4.

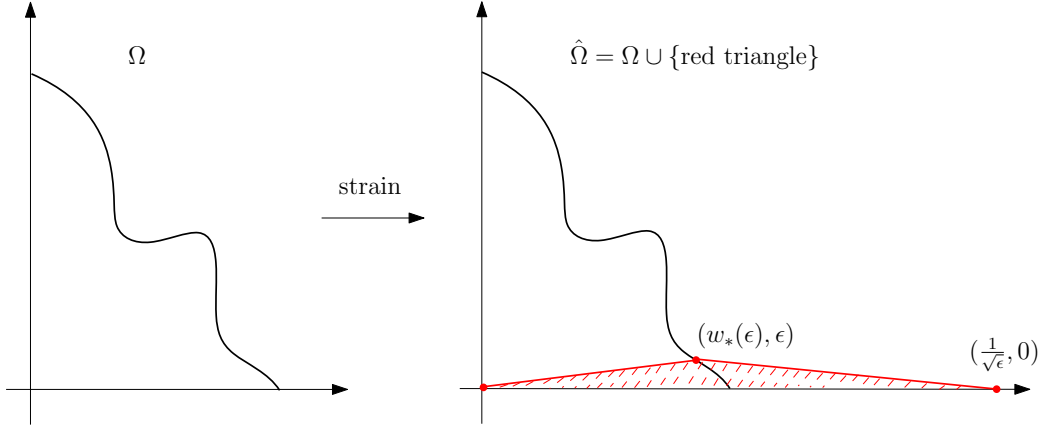


FIGURE 4. Strain operation.

Observe that condition (19) together with the hypothesis that X_Ω is star-shaped, implies that $\Omega \subset \hat{\Omega}$ and the pre-image $X_{\hat{\Omega}} = \mu^{-1}(\hat{\Omega})$ is again star-shaped. In particular, X_Ω being star-shaped is used to deal with the case when $k > 0$. Similarly to Lemma 3.1, we have the following quantitative result for $X_{\hat{\Omega}}$.

Lemma 3.3. *The strain operation on X_Ω results in a star-shaped domain $X_{\hat{\Omega}}$ that satisfies*

- (1) $\text{Vol}(X_{\hat{\Omega}}) = \text{Vol}(X_\Omega) + O(\epsilon)$.
- (2) $\text{sys}(X_{\hat{\Omega}}) \geq A$, where A is a constant independent of ϵ .
- (3) $\text{Ru}(X_{\hat{\Omega}}) \geq \frac{1}{O(\epsilon)}$.

Here, $O(\epsilon)$ represents a constant, proportional to ϵ in Figure 3, that can be chosen arbitrarily small.

Assuming Lemma 3.3, we continue the proof of Theorem 1.3 via the strain operation. The conclusions (1) and (3) in Lemma 3.3 imply that, if $O(\epsilon) < \text{Vol}(X_\Omega)$, we have

$$\text{ru}(X_{\hat{\Omega}})^2 = \frac{\text{Ru}(X_{\hat{\Omega}})^2}{2\text{Vol}(X_{\hat{\Omega}})} = \frac{\text{Ru}(X_{\hat{\Omega}})^2}{2\text{Vol}(X_\Omega) + 2O(\epsilon)} \geq \frac{1}{O(\epsilon)^2} \cdot \frac{1}{4\text{Vol}(X_\Omega)}.$$

Then we have

$$(20) \quad \text{ru}(X_{\widehat{\Omega}}) \cdot \text{sys}(X_{\widehat{\Omega}})^{\frac{1}{2}} \geq \frac{1}{O(\epsilon)^2} \cdot \frac{A}{4\text{Vol}(X_{\Omega})} \rightarrow +\infty \text{ as } \epsilon \rightarrow 0.$$

Hence, the product of the ratios will be larger than the constant C appearing in criterion (3). Therefore, the domain $X_{\widehat{\Omega}}$ is not symplectically convex.

Proof of Lemma 3.3. Comparing the difference of the volume in \mathbb{R}^4 , we have

$$(21) \quad |\text{Vol}(X_{\widehat{\Omega}}) - \text{Vol}(X_{\Omega})| \leq \frac{\epsilon \cdot \frac{1}{\sqrt{\epsilon}}}{2} = \frac{\sqrt{\epsilon}}{2}$$

which goes to 0 as ϵ goes to 0. This proves (1).

This operation possibly introduces various new closed Reeb orbits. Besides the one corresponding to the w_1 -intercept point $(\frac{1}{\sqrt{\epsilon}}, 0)$ with large action, others will concentrate only near the point $p = (w_*(\epsilon), \epsilon)$, after smoothing $\widehat{\Omega}$ at p . By (14), these closed Reeb orbits correspond to the pairs of integers,

$$(m_p, n_p) \in \mathbb{Z}_{>0} \times \mathbb{Z} \text{ with } \min\left\{-\frac{1}{k}, 0\right\} \leq \frac{n_p}{m_p} \leq \frac{\sqrt{\epsilon} - w_*(\epsilon)}{\epsilon}.$$

Concerning their action, we have by (15),

$$\begin{aligned} \mathcal{A}(\gamma_{(m_p, n_p)}) &= m_p w_*(\epsilon) + n_p \epsilon \\ &= m_p \left(w_*(\epsilon) + \frac{n_p}{m_p} \epsilon \right) \\ &\geq m_p \left(\frac{\epsilon}{k} + a + \min\left\{-\frac{\epsilon}{k}, 0\right\} \right) \geq \frac{a}{2}, \end{aligned}$$

when ϵ is sufficiently small. We denote as above, T_{\min} the minimal period of a closed Reeb orbit on ∂X_{Ω} and \widehat{T}_{\min} the minimal period of a closed Reeb orbit on $\partial X_{\widehat{\Omega}}$. If $T_{\min} < \frac{a}{2}$, then obviously $\widehat{T}_{\min} = T_{\min}$. If, on the other hand, $T_{\min} \geq \frac{a}{2}$, then $\widehat{T}_{\min} \geq \frac{a}{2}$. Meanwhile, by assumption, $T_{\min} \leq a$. Therefore, in either case, we have

$$\widehat{T}_{\min} \geq \frac{T_{\min}}{2}.$$

In particular, the lower bound $\frac{T_{\min}}{2}$ is independent of the parameter ϵ .

Now, for the ratios in discussion, by (21),

$$\text{sys}(X_{\widehat{\Omega}}) \geq \frac{T_{\min}^2}{4\text{Vol}(\partial X_{\widehat{\Omega}}, \lambda)} = \frac{T_{\min}^2}{8\text{Vol}(X_{\widehat{\Omega}})} \geq \frac{T_{\min}^2}{8\text{Vol}(X_{\Omega}) + 4\sqrt{\epsilon}}$$

When ϵ is sufficiently small, say $\sqrt{\epsilon} < \text{Vol}(X_{\Omega})$, we have

$$(22) \quad \text{sys}(X_{\widehat{\Omega}}) \geq \frac{1}{12} \cdot \frac{T_{\min}^2}{\text{Vol}(X_{\Omega})} (:= A) \text{ which is independent of } \epsilon.$$

This proves (2).

Finally, the strain operation results in an essential change of the Ruelle invariant. By Proposition 1.4,

$$\text{Ru}(X_{\widehat{\Omega}}) = (w_2\text{-intercept of } \overline{\partial_+\Omega}) + \frac{1}{\sqrt{\epsilon}} \geq \frac{1}{\sqrt{\epsilon}}.$$

This proves (3). \square

As an immediate consequence from the strain operation, we have the following.

Corollary 3.4. *The product of ratios $\text{ru} \cdot \text{sys}^{\frac{1}{2}}$ is unbounded on the category of monotone toric domains.*

Proof. Note that the strain operation is closed within the category of strictly monotone toric domains. Hence, it is closed within the category of dynamically convex toric domains by [6, Proposition 1.8], since by the definition of a monotone toric domain, near the w_1 -intercept the corresponding slope k is always negative. Then the desired conclusion follows from the computation (20). \square

4. ESTIMATE OF CONSTANTS

Recall that a strictly monotone toric domain is a star-shaped domain such that the outward normal vectors along the boundary component $\partial_+\Omega = \partial\Omega \cap \mathbb{R}_{>0}^2$ all have both components positive. In contrast with Corollary 3.4, in this section, we give a proof of Theorem 1.8, which provides a uniform bounds of the product of ratios $\text{ru} \cdot \text{sys}^{\frac{1}{2}}$ for such domains (when they are geometrically convex in \mathbb{R}^4). In terms of the notation, $P(a, b)$ denotes the polydisk in \mathbb{C}^2 defined as

$$(23) \quad P(a, b) = \{(z_1, z_2) \in \mathbb{C}^2 \mid \pi|z_1|^2 \leq a, \pi|z_2|^2 \leq b\}.$$

Let us start from the following useful result. Denote by $c_{\text{Gr}}(X_\Omega)$ the Gromov width of a toric domain X_Ω , measuring the largest $B^4(a)$ that can be symplectically embedded into X_Ω .

Lemma 4.1. *Let X_Ω be a monotone toric domain where the w_1 -intercept and w_2 -intercept of $\overline{\partial_+\Omega}$ are $(a, 0)$ and $(0, b)$ respectively. Suppose that $b \geq a$, then*

$$\text{Vol}(X_\Omega) \leq b \cdot c_{\text{Gr}}(X_\Omega).$$

Proof. By the proof of [6, Theorem 1.11], we know that $c_{\text{Gr}}(X_\Omega)$ is equal to the largest $L > 0$ such that the right triangle with vertices $(0, L)$, $(L, 0)$ and $(0, 0)$ is contained in Ω . Denote by (s, t) one of these intersection points. Then observe that X_Ω being monotone implies that $X_\Omega \subset P(s, b) \cup P(a, t)$. Therefore, we have

$$\begin{aligned} \text{Vol}(X_\Omega) &\leq sb + at = sb + a(-s + c_{\text{Gr}}(X_\Omega)) \\ &= s(b - a) + ac_{\text{Gr}}(X_\Omega) \leq b \cdot c_{\text{Gr}}(X_\Omega) \end{aligned}$$

where the last inequality comes from $s \leq c_{\text{Gr}}(X_\Omega)$. \square

Now, we are ready to give the proof of Theorem 1.8.

Proof of Theorem 1.8. For a monotone toric domain X_Ω , [6, Theorem 1.7] shows that all normalized symplectic capacities coincide. In particular, the minimal period of a Reeb orbit is equal to $c_{\text{Gr}}(X_\Omega)$. Without loss of generality, assume $b \geq a$. Then by Proposition 1.4 and Lemma 4.1, we have

$$\begin{aligned} \text{ru}(X_\Omega)^2 \cdot \text{sys}(X_\Omega) &= \frac{(a+b)^2}{4} \cdot \frac{c_{\text{Gr}}(X_\Omega)^2}{\text{Vol}(X_\Omega)^2} \\ &\geq \frac{(a+b)^2}{4} \frac{c_{\text{Gr}}(X_\Omega)^2}{(b \cdot c_{\text{Gr}}(X_\Omega))^2} \\ &\geq \frac{(a+b)^2}{4b^2} = \frac{1}{4} \left(1 + \frac{a}{b}\right)^2 \geq \frac{1}{4}. \end{aligned}$$

Thus, we complete the proof of the first conclusion.

Now, suppose furthermore that X_Ω is geometrically convex in \mathbb{R}^4 . Up to a rescaling, assume the w_1 -intercept of Ω is 1 while the w_2 -intercept of Ω is still b . Up to a reflection between w_1 and w_2 , we can assume that $b \geq 1$. Therefore, we have $\text{Ru}(X_\Omega) = 1 + b$ for any such domain and so

$$\text{ru}(X_\Omega)^2 \cdot \text{sys}(X_\Omega) = \frac{(1+b)^2 c_{\text{Gr}}(X_\Omega)^2}{4 \text{Vol}(X_\Omega)^2}$$

depends only on $\frac{c_{\text{Gr}}(X_\Omega)}{\text{Vol}(X_\Omega)}$. We thus aim to bound above this quantity among monotone toric domains which are geometrically convex.

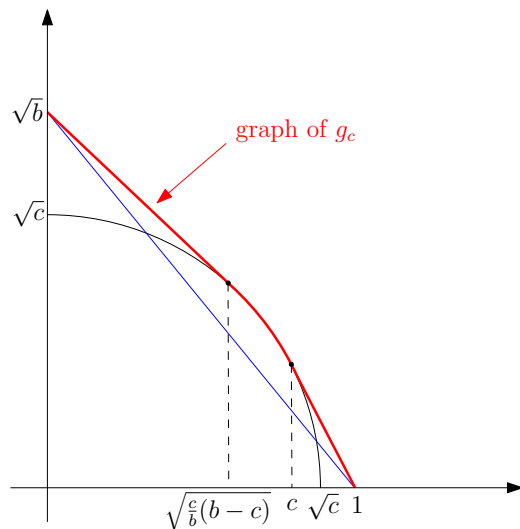
By [6, Proposition 2.3], the following subset

$$\tilde{\Omega} := \{(\mu_1, \mu_2) \in \mathbb{R}^2 \mid (\pi|\mu_1|^2, \pi|\mu_2|^2) \in \Omega\}$$

is a convex subset in \mathbb{R}^2 . In particular, when restricted to $\mathbb{R}_{\geq 0}^2$, the boundary $\partial\tilde{\Omega}$ can be written as a decreasing concave function $\mu_2 = g(\mu_1)$. Since g is concave, we have $g(\mu_1) \geq \sqrt{b}(1 - \mu_1)$ for all $\mu_1 \in [0, 1]$. Meanwhile, if we fix $c = c_{\text{Gr}}(X_\Omega)$, we also have $g(\mu_1) \geq \sqrt{c - \mu_1^2}$ for all $\mu_1 \in [0, 1]$ (since $B^4(c) \subset X_\Omega$). Therefore, g is above the broken curve consisting of the two previous curves. Hence, among these g , the one whose domain maximizes $\frac{c_{\text{Gr}}}{\text{Vol}}$ is the one minimizing the volume i.e. the convex hull of this broken curve, see Figure 5. It has the following boundary:

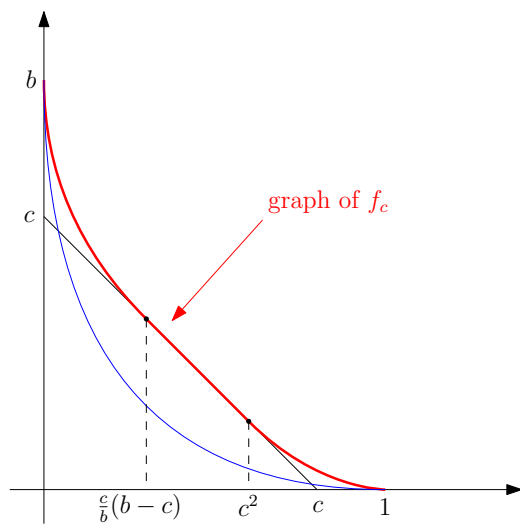
$$g_c(\mu_1) = \begin{cases} \sqrt{b} - \sqrt{\frac{b-c}{c}}\mu_1 & \text{if } 0 \leq \mu_1 \leq \sqrt{\frac{c}{b}}(b-c) \\ \sqrt{c - \mu_1^2} & \text{if } \sqrt{\frac{c}{b}}(b-c) \leq \mu_1 \leq c \\ \sqrt{\frac{c}{1-c}}(1 - \mu_1) & \text{if } c \leq \mu_1 \leq 1 \end{cases}$$

By a change of variables $w_i = \mu_i^2$, we know that the boundary $\overline{\partial_+\tilde{\Omega}}$ (minus the components on w_1 -axis and w_2 -axis) is a function $w_2 = f_c(w_1) := g_c(\sqrt{w_1})^2$ given by

FIGURE 5. Graph of g_c in red.

$$f_c(w_1) = \begin{cases} (\sqrt{b} - \sqrt{\frac{b-c}{c}w_1})^2 & \text{if } 0 \leq w_1 \leq \frac{c}{b}(b-c) \\ c - w_1 & \text{if } \frac{c}{b}(b-c) \leq w_1 \leq c^2, \\ \frac{c}{1-c}(1 - \sqrt{w_1})^2 & \text{if } c^2 \leq w_1 \leq 1 \end{cases}$$

also see Figure 6.

FIGURE 6. Graph of f_c in red.

Denote by X_{f_c} the monotone toric domain such that f_c is the boundary of its moment map minus the w_1 and w_2 -axis. Then we have

$$\frac{c_{\text{Gr}}(X_\Omega)}{\text{Vol}(X_\Omega)} \leq \frac{c_{\text{Gr}}(X_{f_c})}{\text{Vol}(X_{f_c})}.$$

Meanwhile, by integrating along the graph $f_c(w_1)$, we get that

$$\text{Vol}(X_{f_c}) = \frac{c^2}{2} + \frac{(b-c)^2 c}{6b} + \frac{c(1-c)^2}{6},$$

therefore,

$$\frac{c_{\text{Gr}}(X_{f_c})}{\text{Vol}(X_{f_c})} = \frac{c}{\text{Vol}(X_{f_c})} = \frac{6}{3c + \frac{(b-c)^2}{b} + (1-c)^2}.$$

Moreover, since X_Ω is geometrically convex with the w_1 -intercept and w_2 -intercept being $a(\Omega) = 1$ and $b(\Omega) = b$, respectively, we have $\frac{b}{1+b} \leq c_{\text{Gr}}(X_\Omega) \leq 1$, since the Gromov width of the domain with boundary $w_2 = b(1 - \sqrt{w_1})^2$ is $\frac{b}{1+b}$. Therefore, we get

$$\max_{c \in [\frac{b}{1+b}, 1]} \left\{ \frac{c_{\text{Gr}}(X_{f_c})}{\text{Vol}(X_{f_c})} \right\} = \frac{6}{1+b}$$

where the maximum is obtained for $c = \frac{b}{1+b}$ i.e. for the domain whose boundary is given by $f(w_1) = b(1 - \sqrt{w_1})^2$. Hence,

$$\text{ru}(X_\Omega)^2 \cdot \text{sys}(X_\Omega) = \frac{(1+b)^2 c_{\text{Gr}}(X_\Omega)^2}{4\text{Vol}(X_\Omega)^2} \leq 9.$$

Thus we completed the proof of the second conclusion. \square

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