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On the minimal number of periodic orbits on some hypersurfaces in \mathbb{R}^{2n}

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Abstract

We study periodic orbits on a nondegenerate dynamically convex starshaped hypersurface in \mathbb{R}^{2n} along the lines of Long and Zhu [LZ02], but using properties of the S^1 -equivariant symplectic homology. We prove that there exist at least n distinct simple periodic orbits on any nondegenerate starshaped hypersurface in \mathbb{R}^{2n} satisfying the condition that the minimal Conley-Zehnder index is at least $n - 1$. The condition is weaker than dynamical convexity.

1 Introduction

We consider a starshaped hypersurface Σ in \mathbb{R}^{2n} endowed with the standard contact form α which is the restriction of the 1-form λ on \mathbb{R}^{2n} defined by

$$\lambda = \frac{1}{2} \sum_{j=1}^n (x^j dy^j - y^j dx^j).$$

The *Reeb vector field* R_α associated to a contact form α is the unique vector field on Σ characterized by: $\iota(R_\alpha)d\alpha = 0$ and $\alpha(R_\alpha) = 1$. Since this vector field does not vanish anywhere, there are no fixed points of its flow, and periodic orbits are the most noticeable objects of its flow.

The existence of a periodic orbit is known from Rabinowitz [Rab79] and a long-standing question is to know the (minimal) number of geometrically distinct periodic orbits of (Σ, α) . This question has been studied in depth in the lowest dimensional case $\Sigma \subset \mathbb{R}^4$, in which the question is nontrivial, in [HWZ98, HWZ03, HT09, CGH12, GHHM13, LL14, GG15]. It turns out that (Σ, α) carries at least two periodic orbits and if there are more than two periodic orbits, infinitely many of them are guaranteed generically. In higher dimensions, nearly all known multiplicity results concern hypersurfaces in \mathbb{R}^{2n} which satisfy some geometric conditions and appear in [EL80, BLMR85, EH87, LZ02, WHL07, Wan13].

This paper is based on the approach due to Long and Zhu [LZ02]. They prove a certain lower bound on the number of simple periodic orbits on a strictly convex hypersurface and in particular this lower bound equals n if the hypersurface is additionally nondegenerate. In their proof, strict

convexity plays a role twice. First they use the fact that the index of periodic orbits behaves very well under iteration in the strictly convex case and this remains true under dynamical convexity. We call (Σ, α) *dynamically convex* if every periodic orbit has Conley-Zehnder index at least $n+1$; this is the case whenever Σ is strictly convex. Secondly, they use a result due to [EH87] to get information about the interval where the indices of periodic orbits of (Σ, α) sit. For this they need the Clarke dual action functional, which exists only when Σ is strictly convex. By using the positive S^1 -equivariant symplectic homology instead, we observe that the idea of [LZ02] works under a weaker assumption and proves a stronger statement.

A simple periodic orbit is called *even* if the Conley-Zehnder indices of all its iterates have the same parity; or, equivalently, if the linearized Poincaré return map has an even number of real eigenvalues in the open interval $] -1, 0[$.

Theorem 1.1 *If a starshaped hypersurface (Σ, α) in \mathbb{R}^{2n} is nondegenerate and dynamically convex, there are at least n even simple periodic orbits. Moreover if there are precisely n simple periodic orbits, all periodic orbits have different indices.*

The dynamical convexity assumption can be slightly weakened, see Theorem 3.1. A diffeomorphism $f : (\Sigma, \alpha) \rightarrow (\Sigma, \alpha)$ is called a *(anti-) strict contactomorphism* if $f^*\alpha = \alpha$ (if $f^*\alpha = -\alpha$). The following corollary directly follows from the fact that a strict contactomorphism preserves the Conley-Zehnder index of periodic orbits.

Corollary 1.2 *Suppose that a nondegenerate starshaped hypersurface (Σ, α) in \mathbb{R}^{2n} is dynamically convex and possesses precisely n simple periodic orbits. If there is a (anti-) strict contactomorphism from (Σ, α) to itself, all periodic orbits are fixed by it.*

An interesting class of (anti-) strict contactomorphisms arises when $(\mathbb{R}^{2n} = \mathbb{C}^n, \lambda)$ admits symmetry and Σ is an invariant hypersurface. For example, let $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $(z_1, \dots, z_n) \mapsto (e^{2q_1\pi i}z_1, \dots, e^{2q_n\pi i}z_n)$, $q_1, \dots, q_n \in \mathbb{N}$ or $(z_1, \dots, z_n) \mapsto (\bar{z}_1, \dots, \bar{z}_n)$ and assume Σ is invariant under f , i.e. $f(\Sigma) = \Sigma$. Then the corollary yields that if there are precisely n periodic orbits, all of them are symmetric. In low dimensional cases for a particular symmetry without nondegeneracy, this result is proved in [Wan09, LLWZ14].

A nondegenerate contact form α is called *perfect* if the associated positive S^1 -equivariant symplectic homology differential vanishes. The following corollary generalizes a result due to Gürel [Gür15]. We note from Theorem 1.1 that if (Σ, α) is dynamically convex and has precisely n periodic orbits, it is perfect by degree reason.

Corollary 1.3 *Suppose that a nondegenerate contact form α on a starshaped hypersurface Σ in \mathbb{R}^{2n} is perfect. Then there are precisely n even simple periodic orbits.*

Now a natural question is whether dynamical convexity is indispensable for multiplicity results. The followings is our partial answer.

Theorem 1.4 *Let (Σ, α) be a nondegenerate starshaped hypersurface in \mathbb{R}^{2n} such that every periodic orbit has Conley-Zehnder index at least $n-1$. Then Σ possesses at least n simple periodic orbits.*

We point out that every periodic geodesic flow of a Finsler n -sphere has at least Conley-Zehnder index $n-1$ under a certain pinching condition. Under this pinching condition and nondegeneracy, Wang [Wan12, Theorem 1.2] proved a conjecture of Anosov on the number of periodic geodesics on Finsler spheres. The proof of Theorem 1.4 can be used to give an alternative rather short proof of this result. This will be discussed in our future paper.

It is easy to show that every nondegenerate starshaped hypersurface in \mathbb{R}^{2n} has two periodic orbits, see for example [Kan13, Gür15]. The following statement shows that if two periodic orbits on (Σ, α) do not satisfy a certain action-index resonance relation, there has to be a third one. This can be thought of as a generalisation of a theorem due to Ekeland and Hofer [EH87, Corollary 1] (or see [Corollary V.3.17][Eke90]).

Proposition 1.5 *Let (Σ, α) be a nondegenerate starshaped hypersurface in \mathbb{R}^{2n} with two simple periodic orbits γ and δ . Then Σ carries another simple periodic orbit unless*

$$\frac{\mathcal{A}(\gamma)}{\widehat{\mu}_{CZ}(\gamma)} = \frac{\mathcal{A}(\delta)}{\widehat{\mu}_{CZ}(\delta)} \quad (1.1)$$

where $\widehat{\mu}_{CZ}$ and \mathcal{A} stand for the mean Conley-Zehnder index and the action respectively.

The rest of the paper is divided in three sections. Section 2.1 is devoted to Long's index iteration formula and gives a proof of a slight generalisation of the common index jump Theorem due to Long and Zhu. In Section 2.2, we recall the properties of positive S^1 -equivariant symplectic homology that we need. Section 3 contains the proofs of Theorem 1.1, Corollary 1.3, and Theorem 1.4. We prove Proposition 1.5 in Section 4.

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2 The main tools

2.1 Index iterations

The Conley-Zehnder index associates an integer to any continuous path ψ defined on the interval $[0, 1]$ with values in the group $\mathrm{Sp}(\mathbb{R}^{2n-2})$ of $2(n-1) \times 2(n-1)$ symplectic matrices, starting from the identity and ending at a matrix which does not admit 1 as an eigenvalue. This index is used, for instance, in the definition of the grading of Floer homology theories. If the path ψ were a loop with values in the unitary group, one could define an integer by looking at the degree of the loop in the circle defined by the (complex) determinant -or an integer power of it. One uses a continuous map ρ from the symplectic group $\mathrm{Sp}(\mathbb{R}^{2n-2})$ into S^1 and an "admissible" extension of ψ to a path $\widetilde{\psi} : [0, 2] \rightarrow \mathrm{Sp}(\mathbb{R}^{2n-2})$ in such a way that $\rho^2 \circ \widetilde{\psi} : [0, 2] \rightarrow S^1$ is a loop. The Conley-Zehnder index of ψ is defined as the degree of this loop

$$\mu_{CZ}(\psi) := \deg(\rho^2 \circ \widetilde{\psi}).$$

Let ϕ^t denotes the flow of the Reeb vector field R_α on (Σ, α) . The linearized flow $T\phi^t$ respects the splitting $T\Sigma = \mathbb{R}R_\alpha \oplus \ker \alpha$, we have $T\phi^t|_{\ker \alpha} : \ker \alpha \rightarrow \ker \alpha$. Throughout the paper we assume that all the periodic orbits (including all iterates) are nondegenerate; this means that 1 is not an eigenvalue of the linearized Poincaré return map $T\phi^T|_{\ker \alpha}(\gamma(0))$ of a periodic orbit $\gamma : [0, T] \rightarrow (\Sigma, \alpha)$ with $\gamma(0) = \gamma(T)$ and $\dot{\gamma}(t) = R_\alpha(\gamma(t))$. The Conley-Zehnder index of a periodic orbit γ is defined by

$$\mu_{CZ}(\gamma) := \mu_{CZ}(\psi_\gamma)$$

where $\psi_\gamma(t) \in \mathrm{Sp}(\mathbb{R}^{2n-2})$, $t \in [0, 1]$ is the linearized flow $T\phi^{Tt}|_{\ker \alpha}(\gamma(0))$ expressed in a symplectic trivialization of $\ker \alpha$ along γ extendable over a capping disk of γ . For a complete presentation

of the Conley-Zehnder index we refer to [CZ84, SZ92, Sal99, Lon02, AD10, Gut15]. The *mean Conley-Zehnder index* of a periodic orbit γ is defined to be

$$\widehat{\mu}_{CZ}(\gamma) := \lim_{k \rightarrow \infty} \frac{\mu_{CZ}(\gamma^k)}{k}.$$

To begin with, we recall Long's index iteration formula in the nondegenerate case and immediate corollaries of this which are used in the proof of the main results. The proof of the following theorem can be found in [Lon02, Section 8.3]. See also [Kan13, Theorem 3.2].

Theorem 2.1 ([Lon02]) *Given a nondegenerate periodic orbit γ , so that all its iterates are nondegenerate, there exist an integer $p \in \mathbb{Z}$, an integer $q \in [0, n-1]$ and q irrational numbers θ_j in $[0, 1]$, such that, for any positive integer $k \in \mathbb{N}$, the Conley-Zehnder index of the k -th iterate of γ is given by*

$$\mu_{CZ}(\gamma^k) = kp + 2 \sum_{j=1}^q [k\theta_j] + q \quad (2.1)$$

where $[r]$ denotes the largest integer which is lower or equal to r , and where q can be $n-1$ only when p is even. In particular,

$$\mu_{CZ}(\gamma) = p + q, \quad \widehat{\mu}_{CZ}(\gamma) = p + 2 \sum_{j=1}^q \theta_j,$$

and

$$|\mu_{CZ}(\gamma^k) - k \widehat{\mu}_{CZ}(\gamma)| < n - 1. \quad (2.2)$$

Moreover if $\mu_{CZ}(\gamma) \geq n - 1 + c$ for some $c \in \mathbb{N} \cup \{0\}$, then $p \geq c$ and $\mu_{CZ}(\gamma^{k+1}) \geq \mu_{CZ}(\gamma^k) + c$. The Conley-Zehnder indices of all even (resp. odd) iterates of a periodic orbit have the same parity.

We remark that there is an alternative way to see (2.2), see [SZ92, Lemma 3.4]. The following so called common index jump theorem due to Long and Zhu [LZ02, Theorem 4.3] is a key tool of the paper. We include the proof of the theorem because their idea in fact proves a slightly generalised statement which will be used later. They used Bott's iteration formula and included the degenerate case, but the nondegenerate case is simple enough for a proof using Long's iteration formula.

Theorem 2.2 ([LZ02]) *Let $\gamma_1, \dots, \gamma_k$ be simple periodic orbits on a given manifold. Assume that all the iterates of the periodic orbits are nondegenerate and that all the mean indices of the periodic orbits are positive; $\widehat{\mu}_{CZ}(\gamma_i) > 0$ for all $i \in [0, k]$. Then, for any given $M \in \mathbb{N}$, there exist infinitely many $N \in \mathbb{N}$ and $(m_1, \dots, m_k) \in \mathbb{N}^k$ such that for any $m \in \{1, \dots, M\}$*

$$\mu_{CZ}(\gamma_i^{2m_i - m}) = 2N - \mu_{CZ}(\gamma_i^m) \quad \text{and} \quad \mu_{CZ}(\gamma_i^{2m_i + m}) = 2N + \mu_{CZ}(\gamma_i^m)$$

and

$$2N - (n - 1) \leq \mu_{CZ}(\gamma_i^{2m_i}) \leq 2N + (n - 1).$$

PROOF: Let v be the vector in $\mathbb{R}^{k + \sum_{i=1}^k q_i}$ defined by

$$v := \left(\frac{1}{\widehat{\mu}_{CZ}(\gamma_1)}, \dots, \frac{1}{\widehat{\mu}_{CZ}(\gamma_k)}, \frac{\theta_{1,1}}{\widehat{\mu}_{CZ}(\gamma_1)}, \dots, \frac{\theta_{1,q_1}}{\widehat{\mu}_{CZ}(\gamma_1)}, \frac{\theta_{2,1}}{\widehat{\mu}_{CZ}(\gamma_2)}, \dots, \frac{\theta_{k,q_k}}{\widehat{\mu}_{CZ}(\gamma_k)} \right).$$

where $\mu_{\text{CZ}}(\gamma_i^k) = kp_i + 2 \sum_{j=1}^{q_i} [k\theta_{i,j}] + q_i$. Consider the closure of the projection on the torus $T^{k+\sum_{i=1}^k q_i} = \mathbb{R}^{k+\sum_{i=1}^k q_i} / \mathbb{Z}^{k+\sum_{i=1}^k q_i}$ of the set $\{kv\}_{k \in \mathbb{N}}$; it is a closed subgroup of the torus $T^{k+\sum_{i=1}^k q_i}$. If we denote by $[a]$ the non integer part of a , $[a] := a - \lfloor a \rfloor$, for any given $\epsilon > 0$, there exist infinitely many $N \in \mathbb{N}$ such that all

$$\left\lfloor \frac{N\theta_{i,j}}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor \text{ and } \left\lfloor \frac{N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor \text{ are in } [0, \epsilon[\text{ or in }]1 - \epsilon, 1[.$$

With N as above, if $\left\lfloor \frac{N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor$ is in $[0, \epsilon[$ define $m_i := \left\lfloor \frac{N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor$ and $\eta_i = 1$. Then

$$[2m_i\theta_{i,j}] = \left\lfloor \left\lfloor \frac{N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor 2\theta_{i,j} \right\rfloor = \left\lfloor \frac{2N\theta_{i,j}}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} - \left\lfloor \frac{N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor 2\theta_{i,j} \right\rfloor \quad (2.3)$$

lies in $[0, 4\epsilon[\cup]1 - 4\epsilon, 1[$. If $\left\lfloor \frac{N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor$ is in $]1 - \epsilon, 1[$, define $m_i := -\left\lfloor \frac{-N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor$ and $\eta_i = -1$.

Then

$$[2m_i\theta_{i,j}] = \left\lfloor -\left\lfloor \frac{-N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor 2\theta_{i,j} \right\rfloor = \left\lfloor \frac{2N\theta_{i,j}}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} + \left\lfloor \frac{-N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor 2\theta_{i,j} \right\rfloor \quad (2.4)$$

lies in $[0, 4\epsilon[\cup]1 - 4\epsilon, 1[$. Observe that $\left\lfloor \frac{-N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor$ is in $[0, \epsilon[$ so that $\left\lfloor \frac{\eta_i N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor$ is always in $[0, \epsilon[$.

For each $i \in \{1, \dots, m\}$, we denote by \mathcal{E}_i the set

$$\mathcal{E}_i := \{j \in \{1, \dots, q_i\} \mid [2m_i\theta_{i,j}] \in [0, 4\epsilon[\}$$

and by \mathcal{E}_i^c its complementary ($\mathcal{E}_i^c := \{1, \dots, q_i\} \setminus \mathcal{E}_i$).

Given a positive integer M we pick the ϵ such that

$$4\epsilon < \min\{\theta_{i,j}, [2\theta_{i,j}], \dots, [M\theta_{i,j}], 1 - \theta_{i,j}, [2(1 - \theta_{i,j})], \dots, [M(1 - \theta_{i,j})], \frac{1}{6q_i}, \frac{1}{6\widehat{\mu}_{\text{CZ}}(\gamma_i)} \mid \forall i, j\}.$$

For any N corresponding as above to this ϵ and with the corresponding m_i , we have, for all $j \in \mathcal{E}_i$, $[2m_i\theta_{i,j}] < 4\epsilon$ and for all $j \in \mathcal{E}_i^c$, $1 - [2m_i\theta_{i,j}] < 4\epsilon$.

Equation (2.1) now yields:

$$\begin{aligned} 2m_i p_i + 2 \sum_{j=1}^{q_i} [2m_i\theta_{i,j}] &= 2m_i \widehat{\mu}_{\text{CZ}}(\gamma_i) + 2 \sum_{j=1}^{q_i} ([2m_i\theta_{i,j}] - 2m_i\theta_{i,j}) \\ &= \eta_i \left\lfloor \frac{\eta_i N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor 2\widehat{\mu}_{\text{CZ}}(\gamma_i) - 2 \sum_{j=1}^{q_i} [2m_i\theta_{i,j}] \\ &= 2N - \eta_i \left\lfloor \frac{\eta_i N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor 2\widehat{\mu}_{\text{CZ}}(\gamma_i) - 2 \sum_{j=1}^{q_i} [2m_i\theta_{i,j}], \end{aligned}$$

and, with our choice of ϵ , N m_i 's and η_i 's, since the $\widehat{\mu}_{\text{CZ}}(\gamma_i)$'s are positive, we have

$$\begin{aligned} \left| 2m_i p_i + 2 \sum_{j=1}^{q_i} [2m_i\theta_{i,j}] - 2N + 2\#\mathcal{E}_i^c \right| &\leq 2 \sum_{j \in \mathcal{E}_i^c} (1 - [2m_i\theta_{i,j}]) \\ &\quad + 2 \sum_{j \in \mathcal{E}_i} ([2m_i\theta_{i,j}]) + \left\lfloor \frac{\eta_i N}{\widehat{\mu}_{\text{CZ}}(\gamma_i)} \right\rfloor 2\widehat{\mu}_{\text{CZ}}(\gamma_i) \\ &< 8q^i \epsilon + 2\widehat{\mu}_{\text{CZ}}(\gamma_i) \epsilon < 1. \end{aligned}$$

Since the difference of two integers is still an integer, this in turn implies

$$2m_i p_i + 2 \sum_{j=1}^{q_i} \lfloor 2m_i \theta_{i,j} \rfloor = 2N - 2\#\mathcal{E}_i^c.$$

Hence

$$\mu_{\text{CZ}}(\gamma_i^{2m_i}) = 2N - 2\#\mathcal{E}_i^c + q_i \in [2N - (n-1), 2N + (n-1)].$$

We also compute

$$\begin{aligned} \mu_{\text{CZ}}(\gamma_i^{2m_i-1}) &= 2N + 2 \sum_{j=1}^{q_i} \left(\lfloor (2m_i - 1)\theta_{i,j} \rfloor - \lfloor 2m_i \theta_{i,j} \rfloor \right) - p_i + q_i - 2\#\mathcal{E}_i^c \\ &= 2N + 2 \sum_{j \in \mathcal{E}_i} (-1) - p_i + q_i - 2\#\mathcal{E}_i^c = 2N - p_i - q_i = 2N - \mu_{\text{CZ}}(\gamma_i) \end{aligned}$$

and

$$\begin{aligned} \mu_{\text{CZ}}(\gamma_i^{2m_i+1}) &= 2N + 2 \sum_{j=1}^{q_i} \left(\lfloor (2m_i + 1)\theta_{i,j} \rfloor - \lfloor 2m_i \theta_{i,j} \rfloor \right) + p_i + q_i - 2\#\mathcal{E}_i^c \\ &= 2N + 2 \sum_{j \in \mathcal{E}_i^c} 1 + p_i + q_i - 2\#\mathcal{E}_i^c = 2N + \mu_{\text{CZ}}(\gamma_i). \end{aligned}$$

More generally for any positive integer $m \leq M$, we have

$$\begin{aligned} \mu_{\text{CZ}}(\gamma_i^{2m_i+m}) &= 2N + 2 \sum_{j=1}^{q_i} \left(\lfloor (2m_i + m)\theta_{i,j} \rfloor - \lfloor 2m_i \theta_{i,j} \rfloor \right) + mp_i + q_i - 2\#\mathcal{E}_i^c \\ &= 2N + mp_i + 2 \sum_{j=1}^{q_i} \left(\lfloor (2m_i + m)\theta_{i,j} \rfloor - \lfloor (2m_i + 1)\theta_{i,j} \rfloor \right) + q_i \\ &= 2N + mp_i + 2 \sum_{j=1}^{q_i} \left(\lfloor \lfloor 2m_i \theta_{i,j} \rfloor + \theta_{i,j} \rfloor + (m-1)\theta_{i,j} \right) + q_i \\ &= 2N + mp_i + 2 \sum_{j=1}^{q_i} \lfloor m\theta_{i,j} \rfloor + q_i = 2N + \mu_{\text{CZ}}(\gamma_i^m) \end{aligned}$$

The second equality follows from

$$\begin{cases} \lfloor 2m_i \theta_{i,j} \rfloor = \lfloor (2m_i + 1)\theta_{i,j} \rfloor & \text{if } j \in \mathcal{E}_i, \\ \lfloor 2m_i \theta_{i,j} \rfloor = \lfloor (2m_i + 1)\theta_{i,j} \rfloor - 1 & \text{if } j \in \mathcal{E}_i^c. \end{cases}$$

In the third equality we used the identity

$$\lfloor a + b \rfloor = \lfloor a \rfloor + \lfloor \{a\} + b \rfloor, \quad a, b \in \mathbb{R}.$$

For the last equality we compute that if $j \in \mathcal{E}_i$,

$$\lfloor \lfloor 2m_i \theta_{i,j} \rfloor + \theta_{i,j} \rfloor + (m-1)\theta_{i,j} = \lfloor m\theta_{i,j} + \lfloor 2m_i \theta_{i,j} \rfloor \rfloor = \lfloor m\theta_{i,j} \rfloor$$

and otherwise, i.e. if $j \in \mathcal{E}_i^c$,

$$\lfloor \lfloor 2m_i \theta_{i,j} \rfloor + \theta_{i,j} \rfloor + (m-1)\theta_{i,j} = \lfloor m\theta_{i,j} - (1 - \lfloor 2m_i \theta_{i,j} \rfloor) \rfloor = \lfloor m\theta_{i,j} \rfloor.$$

The computation for $\mu_{\text{CZ}}(\gamma_i^{2m_i-m})$ is analogous. □

2.2 Positive S^1 -equivariant symplectic homology

Symplectic homology is defined for a compact symplectic manifold with nondegenerate contact type boundary. Very roughly, it is the semi-infinite dimensional Morse homology for the symplectic action functional defined on the contractible component of the free loop space of such symplectic manifolds. In our situation, a nondegenerate starshaped hypersurface Σ in \mathbb{R}^{2n} is a contact type boundary of the compact region bounded by Σ . The version of homology which we will use is so called the positive S^1 -equivariant symplectic homology for $(\Sigma, \alpha) \subset \mathbb{R}^{2n}$ with rational coefficients, denoted by $SH_*^{S^1,+}(\Sigma, \mathbb{R}^{2n}; \mathbb{Q})$. The S^1 -action we are referring to is the reparametrization action on the free loop space and by positive we mean that only periodic orbits of the Reeb vector field are taken into account. Rather than giving a precise definition we recall some important properties of it. For details we refer the reader to [Vit99, Sei08, BO10, BO12, BO13, Gut15]. We can think that the chain complex for $SH_*^{S^1,+}(\Sigma, \mathbb{R}^{2n}; \mathbb{Q})$ is built over *unparametrized* periodic orbits of (Σ, α) with grading given by the Conley-Zehnder index, in light of [BO12], see also [Kan13, Gut15]. The differential is counting gradient flow trajectories of the action functional between periodic orbits modulo the S^1 -action, which solve a certain elliptic PDE. Moreover bad periodic orbits do not contribute to this homology. Recall that a periodic orbit γ is called *good* if the parity of its Conley-Zehnder index is the same as that of the underlying simple orbit and is called *bad* otherwise.

More precisely, for any large real number K , there exists an integer N , such that the S^1 -equivariant symplectic homology $SH_*^{S^1,+}(\Sigma, \mathbb{R}^{2n}; \mathbb{Q})$, truncated at level K for the action, and up to degree N , is the limit of homologies which can be computed via a spectral sequence for which the complex of the first page up to degree N is spanned by the good periodic orbits on the boundary $\partial\Sigma$ of period at most K , graded by their Conley-Zehnder index, and with a differential ∂ , so that the action $\mathcal{A}(\gamma) := \int_\gamma \alpha$ of a periodic orbit decreases along ∂ (see [Gut15]).

The following computation is by now well known.

Theorem 2.3 *Let Σ be nondegenerate starshaped hypersurface in \mathbb{R}^{2n} . Then we have*

$$SH_*^{S^1,+}(\Sigma, \mathbb{R}^{2n}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } * \in n - 1 + 2\mathbb{N}_{\geq 1} \\ 0 & \text{otherwise.} \end{cases}$$

3 Multiplicity of periodic orbits

We denote by \mathcal{P}_{n+1} the set of periodic orbits on (Σ, α) whose Conley-Zehnder indices are congruent to $n + 1$ modulo 2.

Theorem 3.1 *Let (Σ, α) be a nondegenerate starshaped hypersurface in \mathbb{R}^{2n} . Suppose that every simple periodic orbit in \mathcal{P}_{n+1} has Conley-Zehnder index at least $n + 1$. Then (Σ, α) possesses at least n simple periodic orbits, all iterations of which are in \mathcal{P}_{n+1} .*

PROOF: Knowing the positive S^1 -equivariant symplectic homology from Theorem 2.3, which has generators in all degrees which are congruent to $n + 1$ modulo 2, no iterate of a simple periodic orbit not in \mathcal{P}_{n+1} can generate a nonzero homology class since if some iterate of this is in \mathcal{P}_{n+1} , it is a bad periodic orbit. We can assume without loss of generality that there are only a finite number of simple periodic orbits in \mathcal{P}_{n+1} , say $\gamma_1, \dots, \gamma_k$. Periodic orbits with Conley-Zehnder indices at least $n + 1$ have positive mean indices and thus by Theorem 2.2, with $M = 1$, there exists an interval

$$]2N - (n + 1), 2N + (n + 1)[$$

for some $N \in \mathbb{N}$, in which the Conley-Zehnder index of precisely one iterate of each of those orbits sits. Indeed, we have, with the notations of that theorem,

$$\mu_{\text{CZ}}(\gamma_i^{2m_i-1}) = 2N - \mu_{\text{CZ}}(\gamma_i) \leq 2N - (n+1),$$

$$\mu_{\text{CZ}}(\gamma_i^{2m_i+1}) = 2N + \mu_{\text{CZ}}(\gamma_i) \geq 2N + (n+1)$$

and, by Long's iteration formula (Theorem 2.1) $\mu_{\text{CZ}}(\gamma_i^k) < \mu_{\text{CZ}}(\gamma_i^{k+1})$ for all $k \in \mathbb{N}$. In view of Theorem 2.3 again, there must be generators in the n degrees which correspond to the Conley-Zehnder indices in the interval (i.e. indices $2N - (n-1), 2N - (n-3), \dots, 2N + (n-3), 2N + n - 1$). Since they can only correspond to $\gamma_1^{2m_1}, \dots, \gamma_k^{2m_k}$, all of them have to be good and $k \geq n$. \square

This together with the following corollary prove Theorem 1.1.

Corollary 3.2 *If a nondegenerate dynamically convex starshaped hypersurface (Σ, α) in \mathbb{R}^{2n} possesses precisely n simple periodic orbits, then all periodic orbits are in \mathcal{P}_{n+1} and all Conley-Zehnder indices of periodic orbits are different.*

PROOF: The first assertion directly follows from the theorem. If two periodic orbits have the same index $n - 1 + 2k$ with $k \in \mathbb{N}$, there would exist a good periodic orbit with index $n + 2k$ or $n + 2(k+1)$ by Theorem 2.3 and this is not in \mathcal{P}_{n+1} . \square

Corollary 3.3 *Suppose that a nondegenerate contact form α on a starshaped hypersurface Σ in \mathbb{R}^{2n} is perfect. Then there are precisely n even simple periodic orbits.*

PROOF: From Theorem 3.1, we know that there are at least n even simple periodic orbits since perfectness implies dynamical convexity. Indeed if there is a periodic orbit whose Conley-Zehnder index is less than $n + 1$, perfectness is violated since $SH_{n+1}^{S^1, +}(\Sigma, \mathbb{R}^{2n}; \mathbb{Q})$ is the first nonzero homology group, see Theorem 2.3. Now we show that there are at most n even simple periodic orbits, see also [Gür15, Corollary 1.6]. Assume by contradiction that there are more than n even simple periodic orbits. We choose $n + 1$ even simple periodic orbits and then apply Theorem 2.2. Then there are $n + 1$ good periodic orbits with index sitting in $[2N - (n - 1), 2N + (n - 1)]$. By Theorem 2.3, this contradicts the perfectness assumption. \square

This proves Corollary 1.3. Next we provide a proof of Theorem 1.4.

Theorem 3.4 *Let (Σ, α) be a nondegenerate starshaped hypersurface in \mathbb{R}^{2n} such that all periodic orbits have Conley-Zehnder index at least $n - 1$. Then (Σ, α) possesses at least n simple periodic orbits.*

PROOF: We study the complex built with the good periodic orbits and see its compatibility with the positive S^1 -equivariant symplectic homology computation in Theorem 2.3. Due to Theorem 3.1, we may assume that there is a periodic orbit Γ whose Conley-Zehnder index is $n - 1$. Using the same argument as in Theorem 3.1, we know that there exist at least $n - 2$ geometrically distinct simple periodic orbits $\gamma_1, \dots, \gamma_{n-2}$ for which all iterates are in \mathcal{P}_{n+1} .

We also know that $SH_{n-1}^{S^1, +} = 0$; since we have a generator Γ in the chain complex in that degree, there must exist a good periodic orbit δ of index n . Observe that δ cannot be an iterate of Γ or any of the γ_i 's because of the parity of its index. This shows that either we already have n simple periodic orbits and there is nothing more to prove, or Γ is one of the orbits γ_i 's, say $\Gamma = \gamma_1$. We assume by contradiction that $\gamma_1, \dots, \gamma_{n-2}, \delta$ are the only simple periodic orbits. We

can also assume that γ_1 is the only periodic orbit of Conley-Zehnder index $n-1$. Indeed another periodic orbit of index $n-1$ would imply the existence of a second orbit δ of index n ; it would be geometrically distinct from δ since $\mu_{\text{CZ}}(\delta^{m+1}) \geq \mu_{\text{CZ}}(\delta^m) + 1$ and we would have shown the existence of n simple periodic orbits.

Thus we can assume that $\mu_{\text{CZ}}(\gamma_1^2) \geq n+1$ and $\mu_{\text{CZ}}(\gamma_i) \geq n+1$ for all $i \in \{2, \dots, n-2\}$. Hence by Theorem 2.1,

$$\mu_{\text{CZ}}(\delta^s) > \mu_{\text{CZ}}(\delta), \quad \mu_{\text{CZ}}(\gamma_i^s) > \mu_{\text{CZ}}(\gamma_i)$$

for all $i \in \{2, \dots, n-2\}$ and for all integers $s \geq 2$. Since all $\gamma_1, \dots, \gamma_{n-2}, \delta$ have positive mean Conley-Zehnder indices, we can apply Theorem 2.2. Let $(N, m_1, \dots, m_{n-2}, m_\delta) \in \mathbb{N}^n$ be given by Theorem 2.2 for $M = 2$. We have, for all integers $s \geq 2$ and for all $i \in \{2, \dots, n-2\}$:

$$\begin{aligned} \mu_{\text{CZ}}(\gamma_1^{2m_1-s}) &< \mu_{\text{CZ}}(\gamma_1^{2m_1-1}) = 2N - n + 1 \quad \text{and} \quad 2N + n - 1 = \mu_{\text{CZ}}(\gamma_1^{2m_1+1}) < \mu_{\text{CZ}}(\gamma_1^{2m_1+s}). \\ \mu_{\text{CZ}}(\gamma_i^{2m_i-s}) &< \mu_{\text{CZ}}(\gamma_i^{2m_i-1}) \leq 2N - n - 1 \quad \text{and} \quad 2N + n + 1 \leq \mu_{\text{CZ}}(\gamma_i^{2m_i+1}) < \mu_{\text{CZ}}(\gamma_i^{2m_i+s}). \\ \mu_{\text{CZ}}(\delta^{2m_\delta-s}) &< \mu_{\text{CZ}}(\delta^{2m_\delta-1}) = 2N - n \quad \text{and} \quad 2N + n = \mu_{\text{CZ}}(\delta^{2m_\delta+1}) < \mu_{\text{CZ}}(\delta^{2m_\delta+s}). \end{aligned}$$

Hence the only periodic orbits whose Conley-Zehnder indices lie in $[2N-n, 2N+n]$ are $\delta^{2m_\delta-1}$ with index $2N-n$, $\gamma_1^{2m_1-1}$ with index $2N-n+1$, the $n-1$ orbits $\gamma_i^{2m_i}$, $1 \leq i \leq n-1$, whose indices are in $]2N-n+1, 2N+n-1[\cap \{n+1+2\mathbb{N}\}$, $\delta^{2m_\delta+1}$ with index $2N+n$, $\gamma_1^{2m_1+1}$ with index $2N+n-1$, and δ^{2m_δ} with index in $[2N-(n-1), 2N+(n-1)]$. We distinguish two cases, whether δ^{2m_δ} is good or bad.

Case 1 : The even iterates of δ are good. Then the index of δ^{2m_δ} sits in $[2N-n+2, 2N+n-2]$ and the orbit generates a 1-dimensional piece in the complex and also in the homology since $\gamma_1^{2m_1-1}, \gamma_1^{2m_1}, \dots, \gamma_{n-2}^{2m_{n-2}-1}, \gamma_1^{2m_1+1}$ have to generate all homology classes of $SH^{S^1,+}$ degrees in $[2N-n+1, 2N+n-1]$ and therefore δ^{2m_δ} is a cycle and not a boundary. This contradicts the computation of $SH^{S^1,+}$ given in Theorem 2.3.

Case 2 : The even iterates of δ are bad. We claim that $\mu_{\text{CZ}}(\delta^3) \geq n+3$. Indeed by Theorem 2.1, $\mu_{\text{CZ}}(\delta^3) \geq n+2$ and $\mu_{\text{CZ}}(\delta^3) \neq n+2$ since otherwise $\mu_{\text{CZ}}(\delta) = p+q$ with $p=1$ and $q=n-1$ which contradicts the fact that p must be even if $q=n-1$ (cfr Theorem 2.1). This implies in particular that there are no periodic orbits of index $n+2$, therefore there is only one periodic orbit of index $n+1$. By Theorem 2.2, we know that

$$\#\{\mu_{\text{CZ}}^{-1}(2N+n+1)\} = 1 \quad \text{and} \quad \#\{\mu_{\text{CZ}}^{-1}(2N+n-1)\} = 1$$

and they generate the nonzero homology classes of $SH^{S^1,+}$ in each degree. But $\delta^{2m_\delta+1}$ is a good orbit of index $2N+n$, thus generates a homology class which is a contradiction with the computation of $SH^{S^1,+}$ given in Theorem 2.3. \square

4 Third periodic orbit

This section is devoted to the proof of Proposition 1.5. Let (Σ, α) be a nondegenerate starshaped hypersurface in \mathbb{R}^{2n} . In the case $n=2$, if there are precisely two periodic orbits, it is known that there is the action-index resonance relation between them, i.e. (1.1) holds, see [BCE07, Gür15]. Now we consider the cases when $n \geq 3$. From Theorem 2.3, we need at least one simple periodic orbit $\gamma \in \mathcal{P}_{n+1}$ to generate non-zero homology classes. Due to Theorem 1.4, we may assume that $\mu_{\text{CZ}}(\gamma) \leq n-3$. Due to Theorem 2.3 again, we know that there is another simple periodic orbit δ such that δ^ℓ is good with $\mu_{\text{CZ}}(\delta^\ell) \in \{\mu_{\text{CZ}}(\gamma) - 1, \mu_{\text{CZ}}(\gamma) + 1\}$ for some $\ell \in \mathbb{N}$. Note that

if some iterates of δ are in P_{n+1} , they are bad. We assume for a contradiction that γ and δ are the only simple periodic orbits. Note that both periodic orbits have positive mean indices since otherwise we need an additional periodic orbit to meet the homology computation in Theorem 2.3, in view of (2.2). We also may assume that

$$\{\mu_{\text{CZ}}(\gamma^k) \mid k \in \mathbb{N}\} = \min\{\mu_{\text{CZ}}(\gamma^k) \mid k \in \mathbb{N}\} - 2 + 2\mathbb{N} \quad (4.1)$$

since otherwise there exist infinitely many $q \in n - 1 + 2\mathbb{N}$ such that $q \notin \{\mu_{\text{CZ}}(\gamma^k) \mid k \in \mathbb{N}\}$ due to Theorem 2.2 which immediately guarantees an additional periodic orbit.

4.1 First case: $\frac{\mathcal{A}(\gamma)}{\widehat{\mu}_{\text{CZ}}(\gamma)} > \frac{\mathcal{A}(\delta)}{\widehat{\mu}_{\text{CZ}}(\delta)}$

We denote by \mathcal{G} the set of good periodic orbits in (Σ, α) . Since bad periodic orbits do not have any contribution to the homology $SH^{S^1,+}$, we consider in this section the Conley-Zehnder index only defined on good periodic orbits:

$$\mu_{\text{CZ}} : \mathcal{G} \rightarrow \mathbb{Z}.$$

Observe from (2.2) that $\mu_{\text{CZ}}(\gamma^k) = r$ implies $k\widehat{\mu}_{\text{CZ}}(\gamma) \in]r - (n-1), r + (n-1)[$ [and $\mu_{\text{CZ}}(\delta^\ell) = r \pm 1$ implies $\ell\widehat{\mu}_{\text{CZ}}(\delta) \in]r - 1 - (n-1), r + 1 + (n-1)[$. Hence

$$(r+1)\frac{\mathcal{A}(\gamma)}{\widehat{\mu}_{\text{CZ}}(\gamma)} > \mathcal{A}(\gamma^k) = k\mathcal{A}(\gamma) > (r-1)\frac{\mathcal{A}(\gamma)}{\widehat{\mu}_{\text{CZ}}(\gamma)}$$

and

$$(r-2)\frac{\mathcal{A}(\delta)}{\widehat{\mu}_{\text{CZ}}(\delta)} < \mathcal{A}(\delta^\ell) = \ell\mathcal{A}(\delta) < (r+2)\frac{\mathcal{A}(\delta)}{\widehat{\mu}_{\text{CZ}}(\delta)}$$

so that $\mathcal{A}(\gamma^k) > \mathcal{A}(\delta^\ell)$ when $\frac{r-1}{r+2}\frac{\mathcal{A}(\gamma)}{\widehat{\mu}_{\text{CZ}}(\gamma)} > \frac{\mathcal{A}(\delta)}{\widehat{\mu}_{\text{CZ}}(\delta)}$. Hence, if $\frac{\mathcal{A}(\gamma)}{\widehat{\mu}_{\text{CZ}}(\gamma)} > \frac{\mathcal{A}(\delta)}{\widehat{\mu}_{\text{CZ}}(\delta)}$, there is $\kappa_0 \in \mathbb{N}$ such that for any $\kappa \geq \kappa_0$, if $\mu_{\text{CZ}}(\gamma^k) = 2\kappa + n + 1$ and $\mu_{\text{CZ}}(\delta^\ell) \in \{\mu_{\text{CZ}}(\gamma^k) - 1, \mu_{\text{CZ}}(\gamma^k) + 1\}$ for some $k, \ell \in \mathbb{N}$, then

$$\mathcal{A}(\gamma^k) > \mathcal{A}(\delta^\ell). \quad (4.2)$$

Therefore since $SH_*^{S^1,+}$ is \mathbb{Q} for $* \in 2\mathbb{N} + n - 1$ and 0 for $* \in 2\mathbb{Z} + n$, all high good iterates of δ must be killed by good iterates of γ due to (4.2) which in turn implies

$$\#\mu_{\text{CZ}}^{-1}(2\kappa + n) + 1 = \#\mu_{\text{CZ}}^{-1}(2\kappa + n + 1), \quad \kappa \geq \kappa_0 \quad (4.3)$$

and

$$\partial_{2\kappa+n+2} : SC_{2\kappa+n+2}^{S^1,+} \xrightarrow{0} SC_{2\kappa+n+1}^{S^1,+}, \quad \kappa \geq \kappa_0. \quad (4.4)$$

Here $SC^{S^1,+}$ is the chain complex spanned by the (unparametrized) good periodic orbits of period at most $K \gg 0$ and ∂ is the differential. Here we use the fact that the action \mathcal{A} decreases along ∂ . Recall that we have assumed that $\widehat{\mu}_{\text{CZ}}(\gamma) > 0$ and $\widehat{\mu}_{\text{CZ}}(\delta) > 0$ since otherwise the proposition immediately follows. From this we are able to choose $M \in \mathbb{N}$ sufficiently large such that for any $k \geq M$,

$$\mu_{\text{CZ}}(\gamma^k) > 2\kappa_0 + n + 3 + 2(n-1). \quad (4.5)$$

According to Theorem 2.2, we can find $(N, m_\gamma, m_\delta) \in \mathbb{N}^3$ with $N \geq \kappa_0 + n$ satisfying

$$\mu_{\text{CZ}}(\gamma^{2m_\gamma - m}) = 2N - \mu_{\text{CZ}}(\gamma^m), \quad \mu_{\text{CZ}}(\gamma^{2m_\gamma + m}) = 2N + \mu_{\text{CZ}}(\gamma^m), \quad 1 \leq m \leq M \quad (4.6)$$

and

$$\mu_{\text{CZ}}(\delta^{2m_\delta - m}) = 2N - \mu_{\text{CZ}}(\delta^m), \quad \mu_{\text{CZ}}(\delta^{2m_\delta + m}) = 2N + \mu_{\text{CZ}}(\delta^m), \quad 1 \leq m \leq M. \quad (4.7)$$

Since $\mu_{\text{CZ}}(\gamma^{k+i}) - \mu_{\text{CZ}}(\gamma^k) > -2(n-1)$ for any $k, i \in \mathbb{N}$ from (2.2), (4.5) and (4.6) yield that for any $m \geq M$,

$$\mu_{\text{CZ}}(\gamma^{2m_\gamma - m}) \leq 2N - n + 1, \quad \mu_{\text{CZ}}(\gamma^{2m_\gamma + m}) \geq 2N + 2\kappa_0 + n + 3. \quad (4.8)$$

Here one can deduce a better estimate for $\mu_{\text{CZ}}(\gamma^{2m_\gamma - m})$ but our estimate is enough for the proof. The same holds for δ : for any $m \geq M$,

$$\mu_{\text{CZ}}(\delta^{2m_\delta - m}) \leq 2N - n + 1, \quad \mu_{\text{CZ}}(\delta^{2m_\delta + m}) \geq 2N + 2\kappa_0 + n + 3. \quad (4.9)$$

From Theorem 2.2, we also know

$$\mu_{\text{CZ}}(\gamma^{2m_\gamma}) \leq 2N + n - 1, \quad \mu_{\text{CZ}}(\delta^{2m_\delta}) \leq 2N + n - 1. \quad (4.10)$$

Moreover, the fact that both $\widehat{\mu}_{\text{CZ}}(\gamma)$ and $\widehat{\mu}_{\text{CZ}}(\delta)$ are positive together with (2.2) imply that for all $k \in \mathbb{N}$, $\mu_{\text{CZ}}(\gamma^k)$ and $\mu_{\text{CZ}}(\delta^k)$ are bigger than $-n+1$ and therefore

$$\mu_{\text{CZ}}(\gamma^{2m_\gamma - m}) < 2N + n - 1, \quad \mu_{\text{CZ}}(\delta^{2m_\delta - m}) < 2N + n - 1. \quad (4.11)$$

for all $1 \leq m \leq M$ due to (4.6) and (4.7). From (4.8), (4.9), (4.10), and (4.11), we deduce that if

$$\mu_{\text{CZ}}(\gamma^k), \mu_{\text{CZ}}(\delta^\ell) \in [2N + n, 2N + 2\kappa_0 + n + 2],$$

then $k, \ell \in \mathbb{N}$ are of the form

$$k = 2m_\gamma + m, \quad \ell = 2m_\delta + m \quad \text{for some } 1 \leq m \leq M.$$

Therefore (4.6) and (4.7) show that

$$\#\mu_{\text{CZ}}^{-1}(q) = \#\mu_{\text{CZ}}^{-1}(2N + q), \quad n \leq q \leq 2\kappa_0 + n + 2.$$

From this and (4.3), we deduce

$$\#\mu_{\text{CZ}}^{-1}(n - 2 + q) + 1 = \#\mu_{\text{CZ}}^{-1}(n - 1 + q), \quad q \in 2\mathbb{N}. \quad (4.12)$$

This implies that the differential $\partial_n : SC_n^{S^1, +} \rightarrow SC_{n-1}^{S^1, +}$ vanishes. Indeed if this were not true, the differential ∂_* would be nonzero for all $*$ in $2\mathbb{N} + n$ to obtain the homology results of Theorem 2.3, in view of (4.12). This would contradict (4.4). This implies that

$$(SC_*^{S^1, +}, \partial_*)_{* \in I}, \quad I = \mathbb{Z} \cap [-n + 3, n - 1]$$

is a chain complex with zero homology in view of Theorem 2.3 again. We claim that this is impossible by showing that

$$\sum_{q \in 2\mathbb{Z} + n + 1 \cap I} \#\mu_{\text{CZ}}^{-1}(q) > \sum_{q \in 2\mathbb{Z} + n \cap I} \#\mu_{\text{CZ}}^{-1}(q).$$

To show the claim we observe again from (4.6) and (4.7) that

$$\#(\mu_{\text{CZ}}^{-1}(-q) \cup \mu_{\text{CZ}}^{-1}(q)) = \#\mu_{\text{CZ}}^{-1}(2N + q), \quad 0 \leq q \leq n - 1$$

except in the case where γ^{2m_γ} or δ^{2m_δ} is good and has index $2N+q$. Indeed if $\mu_{\text{CZ}}(\gamma^k) = 2N+q$ where $k = 2m_\gamma + k_0$ and $k_0 \neq 0$, $\mu_{\text{CZ}}(\gamma^{|k_0|})$ is either q or $-q$ by (4.6). Conversely, if $\mu_{\text{CZ}}(\gamma^{k_0})$ is q (or $-q$) for some $k_0 \in \mathbb{N}$, then $\mu_{\text{CZ}}(\gamma^{2m_\gamma+k_0})$ (or $\mu_{\text{CZ}}(\gamma^{2m_\gamma-k_0})$) is $2N+q$. The same holds for δ . Therefore we have

$$e_\gamma + \sum_{q \in 2\mathbb{Z}+n+1 \cap I} \#\mu_{\text{CZ}}^{-1}(q) = \sum_{q \in 2\mathbb{N}+n+1 \cap [0, n-1]} \#\mu_{\text{CZ}}^{-1}(2N+q) \quad (4.13)$$

where $e_\gamma = 1$ if γ^{2m_γ} is good and otherwise $e_\gamma = 0$. We set $e_\delta \in \{0, 1\}$ in the same way and have

$$e_\delta + \sum_{q \in 2\mathbb{Z}+n \cap I} \#\mu_{\text{CZ}}^{-1}(q) = \sum_{q \in 2\mathbb{N}+n \cap [0, n-2]} \#\mu_{\text{CZ}}^{-1}(2N+q) \quad (4.14)$$

Since we have assumed that $\mu_{\text{CZ}}(\gamma) \leq n-3$, $\#\mu^{-1}(2\mathbb{Z}+n+1 \cap I) \geq 2$ due to (4.1). Therefore using (4.12), (4.13), and (4.14) we deduce

$$\sum_{q \in 2\mathbb{Z}+n+1 \cap I} \#\mu_{\text{CZ}}^{-1}(q) \geq \sum_{q \in 2\mathbb{Z}+n \cap I} \#\mu_{\text{CZ}}^{-1}(q) + 2 + e_\delta - e_\gamma > \sum_{q \in 2\mathbb{Z}+n \cap I} \#\mu_{\text{CZ}}^{-1}(q).$$

This proves the claim and hence the first case.

4.2 Second case: $\frac{\mathcal{A}(\gamma)}{\bar{\mu}_{\text{CZ}}(\gamma)} < \frac{\mathcal{A}(\delta)}{\bar{\mu}_{\text{CZ}}(\delta)}$

We derive a contradiction in a similar manner to the first case. In the same way as before, the condition $\frac{\mathcal{A}(\gamma)}{\bar{\mu}_{\text{CZ}}(\gamma)} < \frac{\mathcal{A}(\delta)}{\bar{\mu}_{\text{CZ}}(\delta)}$ implies that there is $\kappa_0 \in \mathbb{N}$ such that $\kappa \geq \kappa_0$, if $\mu_{\text{CZ}}(\gamma^k) \geq 2\kappa+n+1$ and $\mu_{\text{CZ}}(\delta^\ell) \in \{\mu_{\text{CZ}}(\gamma^k) - 1, \mu_{\text{CZ}}(\gamma^k) + 1\}$ for some $k, \ell \in \mathbb{N}$,

$$\mathcal{A}(\gamma^m) < \mathcal{A}(\delta^\ell).$$

As before, this implies that

$$\#\mu_{\text{CZ}}^{-1}(2\kappa+n+1) + 1 = \#\mu_{\text{CZ}}^{-1}(2\kappa+n+2), \quad \kappa \geq \kappa_0$$

and

$$\partial_{2\kappa+n+1} : SC_{2\kappa+n+1}^{S^1,+} \xrightarrow{0} SC_{2\kappa+n}^{S^1,+}, \quad \kappa \geq \kappa_0.$$

We choose $(N, m_\gamma, m_\delta) \in \mathbb{N}^3$ to satisfy (4.5) and (4.6). In a similar way as in the first case, we deduce that $\partial_{n+1} \equiv 0$ and therefore the chain complex

$$(SC_*^{S^1,+}, \partial_*)_{* \in I}, \quad I = \mathbb{Z} \cap [-n+3, n]$$

has vanishing homology. However this is impossible. Indeed since we have assumed that $\mu_{\text{CZ}}(\gamma) \leq n-3$, $\mu_{\text{CZ}}(\delta) \leq n-2$ and $\#\mu^{-1}(2\mathbb{Z}+n \cap I) \geq 2$ due to (4.1). Therefore using (4.3) we deduce

$$\sum_{q \in 2\mathbb{Z}+n \cap I} \#\mu_{\text{CZ}}^{-1}(q) \geq \sum_{q \in 2\mathbb{Z}+n+1 \cap I} \#\mu_{\text{CZ}}^{-1}(q) + 2 + e_\gamma - e_\delta > \sum_{q \in 2\mathbb{Z}+n+1 \cap I} \#\mu_{\text{CZ}}^{-1}(q).$$

This proves the second case and hence finishes the proof of Proposition 1.5.

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