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Normal Forms for Symplectic Matrices

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Abstract

We give a self contained and elementary description of normal forms for symplectic matrices, based on geometrical considerations. The normal forms in question are expressed in terms of elementary Jordan matrices and integers with values in $\{-1, 0, 1\}$ related to signatures of quadratic forms naturally associated to the symplectic matrix.

Introduction

Let V be a real vector space of dimension $2n$ with a non degenerate skewsymmetric bilinear form Ω . The symplectic group $\mathrm{Sp}(V, \Omega)$ is the set of linear transformations of V which preserve Ω :

$$\mathrm{Sp}(V, \Omega) = \{ A : V \rightarrow V \mid A \text{ linear and } \Omega(Au, Av) = \Omega(u, v) \text{ for all } u, v \in V \}.$$

A *symplectic basis* of the symplectic vector space (V, Ω) of dimension $2n$ is a basis $\{e_1, \dots, e_{2n}\}$ in which the matrix representing the symplectic form is $\Omega_0 = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$. In a symplectic basis, the matrix A' representing an element $A \in \mathrm{Sp}(V, \Omega)$ belongs to

$$\mathrm{Sp}(2n, \mathbb{R}) = \left\{ A' \in \mathrm{Mat}(2n \times 2n, \mathbb{R}) \mid A'^{\tau} \Omega_0 A' = \Omega_0 \right\}$$

where $(\cdot)^{\tau}$ denotes the transpose of a matrix.

Given an element A in the symplectic group $\mathrm{Sp}(V, \Omega)$, we want to find a symplectic basis of V in which the matrix A' representing A has a distinguished form; to give a *normal form* for matrices in $\mathrm{Sp}(2n, \mathbb{R})$ means to describe a distinguished representative in each conjugacy class. In general, one cannot find a symplectic basis of the complexified vector space for which the matrix representing A has Jordan normal form.

The normal forms considered here are expressed in terms of elementary Jordan matrices and matrices depending on an integer $s \in \{-1, 0, 1\}$. They are closely related to the forms given by Long in [7, 9]; the main difference is that, in those references, some indeterminacy was left in the choice of matrices in each conjugacy class, in particular when the matrix admits 1 as an eigenvalue. We speak in this case of *quasi-normal forms*. Other constructions can be found in [16, 5, 6, 15, 12] but they are either quasi-normal or far from Jordan normal forms. Closely related are the constructions of normal forms for real matrices that are selfadjoint, skewadjoint or unitary with respect to an indefinite inner product where sign characteristics are introduced; they have been studied in many sources; for instance -mainly for selfadjoint and skewadjoint matrices- in the monograph of I. Gohberg, P. Lancaster and L. Rodman [2], and for unitary matrices in the papers [1, 3, 10, 13]. Normal forms for symplectic matrices have been given by C. Mehl in [11] and by V. Sergeichuk in [14]; in those descriptions, the basis producing the normal form is not required to be symplectic.

We construct here normal forms using elementary geometrical methods.

The choice of representatives for normal (or quasi normal) forms of matrices depends on the application one has in view. Quasi normal forms were used by Long to get precise formulas for indices of iterates of Hamiltonian orbits in [8]. The forms obtained here were useful for us to give new characterisations of Conley-Zehnder indices of general paths of symplectic matrices [4]. We have chosen to give a normal form in a symplectic basis. The main interest of our description is the natural interpretation of the signs appearing in the decomposition, and the description of the decomposition for matrices with 1 as an eigenvalue. It also yields an easy natural characterization of the conjugacy class of an element in $\text{Sp}(2n, \mathbb{R})$. We hope it can be useful in other situations.

Assume that V decomposes as a direct sum $V = V_1 \oplus V_2$ where V_1 and V_2 are Ω -orthogonal A -invariant subspaces. Suppose that $\{e_1, \dots, e_{2k}\}$ is a symplectic basis of V_1 in which the matrix representing $A|_{V_1}$ is $A' = \begin{pmatrix} A'_1 & A'_2 \\ A'_3 & A'_4 \end{pmatrix}$. Suppose also that $\{f_1, \dots, f_{2l}\}$ is a symplectic basis of V_2 in which the matrix representing $A|_{V_2}$ is $A'' = \begin{pmatrix} A''_1 & A''_2 \\ A''_3 & A''_4 \end{pmatrix}$. Then $\{e_1, \dots, e_k, f_1, \dots, f_l, e_{k+1}, \dots, e_{2k}, f_{l+1}, \dots, f_{2l}\}$ is a symplectic basis of V and the matrix representing A in this basis is

$$\begin{pmatrix} A'_1 & 0 & A'_2 & 0 \\ 0 & A'_1 & 0 & A'_2 \\ A'_3 & 0 & A'_4 & 0 \\ 0 & A'_3 & 0 & A'_4 \end{pmatrix}.$$

The notation $A' \diamond A''$ is used in Long [8] for this matrix. It is “a direct sum of matrices with obvious identifications”. We call it the *symplectic direct sum* of the matrices A' and A'' .

We \mathbb{C} -linearly extend Ω to the complexified vector space $V^{\mathbb{C}}$ and we \mathbb{C} -linearly extend any $A \in \text{Sp}(V, \Omega)$ to $V^{\mathbb{C}}$. If v_λ denotes an eigenvector of A in

$V^{\mathbb{C}}$ of the eigenvalue λ , then $\Omega(Av_\lambda, Av_\mu) = \Omega(\lambda v_\lambda, \mu v_\mu) = \lambda\mu\Omega(v_\lambda, v_\mu)$, thus $\Omega(v_\lambda, v_\mu) = 0$ unless $\mu = \frac{1}{\lambda}$. Hence the eigenvalues of A arise in ‘‘quadruples’’

$$[\lambda] := \left\{ \lambda, \frac{1}{\lambda}, \bar{\lambda}, \frac{1}{\bar{\lambda}} \right\}. \quad (1)$$

We find a symplectic basis of $V^{\mathbb{C}}$ so that A is a symplectic direct sum of block-upper-triangular matrices of the form

$$\begin{pmatrix} J(\lambda, k)^{-1} & 0 \\ 0 & J(\lambda, k)^\tau \end{pmatrix} \begin{pmatrix} \text{Id} & D(k, s) \\ 0 & \text{Id} \end{pmatrix},$$

or

$$\begin{pmatrix} J(\bar{\lambda}, k)^{-1} & & 0 \\ & J(\lambda, k)^{-1} & \\ 0 & & J(\bar{\lambda}, k)^\tau \\ & & & J(\lambda, k)^\tau \end{pmatrix} \begin{pmatrix} \text{Id} & 0 & 0 & D(k, s) \\ & \text{Id} & D(k, s) & 0 \\ & & \text{Id} & 0 \\ 0 & & & \text{Id} \end{pmatrix},$$

or

$$\begin{pmatrix} J(\bar{\lambda}, k)^{-1} & & 0 \\ & J(\lambda, k+1)^{-1} & \\ & & J(\bar{\lambda}, k)^\tau \\ 0 & & & J(\lambda, k+1)^\tau \end{pmatrix} \begin{pmatrix} \text{Id} & 0 & 0 & S(k, s, \lambda) \\ & \text{Id} & S(k, s, \lambda)^\tau & 0 \\ & & \text{Id} & 0 \\ 0 & & & \text{Id} \end{pmatrix}.$$

Here, $J(\lambda, k)$ is the elementary $k \times k$ Jordan matrix corresponding to an eigenvalue λ , $D(k, s)$ is the diagonal $k \times k$ matrix

$$D(k, s) = \text{diag}(0, \dots, 0, s),$$

and $S(k, s, \lambda)$ is the $k \times (k+1)$ matrix defined by

$$S(k, s, \lambda) := \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & \frac{1}{2}is & \lambda is \end{pmatrix},$$

with s an integer in $\{-1, 0, 1\}$. Each $s \in \{\pm 1\}$ is called a sign and the collection of such signs appearing in the decomposition of a matrix A is called the sign characteristic of A .

More precisely, on the real vector space V , we shall prove:

Theorem 1 (Normal forms for symplectic matrices) *Any symplectic endomorphism A of a finite dimensional symplectic vector space (V, Ω) is the direct sum of its restrictions $A|_{V_{[\lambda]}}$ to the real A -invariant symplectic subspace $V_{[\lambda]}$ whose complexification is the direct sum of the generalized eigenspaces of eigenvalues $\lambda, \frac{1}{\lambda}, \bar{\lambda}$ and $\frac{1}{\bar{\lambda}}$:*

$$V_{[\lambda]}^{\mathbb{C}} := E_\lambda \oplus E_{\frac{1}{\lambda}} \oplus E_{\bar{\lambda}} \oplus E_{\frac{1}{\bar{\lambda}}}.$$

We distinguish three cases : $\lambda \notin S^1$, $\lambda = \pm 1$ and $\lambda \in S^1 \setminus \{\pm 1\}$.

Normal form for $A|_{V_{[\lambda]}}$ for $\lambda \notin S^1$:

Let $\lambda \notin S^1$ be an eigenvalue of A . Let $k := \dim_{\mathbb{C}} \text{Ker}(A - \lambda \text{Id})$ (on $V^{\mathbb{C}}$) and q be the smallest integer so that $(A - \lambda \text{Id})^q$ is identically zero on the generalized eigenspace E_{λ} .

- If λ is a real eigenvalue of A ($\lambda \notin S^1$ so $\lambda \neq \pm 1$), there exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of k matrices of the form

$$\begin{pmatrix} J(\lambda, q_j)^{-1} & 0 \\ 0 & J(\lambda, q_j)^{\tau} \end{pmatrix}$$

with $q = q_1 \geq q_2 \geq \dots \geq q_k$ and $J(\lambda, m)$ is the elementary $m \times m$ Jordan matrix associated to λ

$$J(\lambda, m) = \begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}.$$

This decomposition is unique, when λ has been chosen in $\{\lambda, \lambda^{-1}\}$. It is determined by the chosen λ and by the dimension $\dim(\text{Ker}(A - \lambda \text{Id})^r)$ for each $r > 0$.

- If $\lambda = re^{i\phi} \notin (S^1 \cup \mathbb{R})$ is a complex eigenvalue of A , there exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of k matrices of the form

$$\begin{pmatrix} J_{\mathbb{R}}(re^{-i\phi}, 2q_j)^{-1} & 0 \\ 0 & J_{\mathbb{R}}(re^{-i\phi}, 2q_j)^{\tau} \end{pmatrix}$$

with $q = q_1 \geq q_2 \geq \dots \geq q_k$ and $J_{\mathbb{R}}(re^{i\phi}, k)$ is the $2m \times 2m$ block upper triangular matrix defined by

$$J_{\mathbb{R}}(re^{i\phi}, 2m) := \begin{pmatrix} R(re^{i\phi}) & \text{Id} & & & & \\ & R(re^{i\phi}) & \text{Id} & & & 0 \\ & & R(re^{i\phi}) & \text{Id} & & \\ & & & \ddots & \ddots & \\ 0 & & & & R(re^{i\phi}) & \text{Id} \\ & & & & & R(re^{i\phi}) & \text{Id} \\ & & & & & & R(re^{i\phi}) \end{pmatrix}$$

with $R(re^{i\phi}) = \begin{pmatrix} r \cos \phi & -r \sin \phi \\ r \sin \phi & r \cos \phi \end{pmatrix}$.

This decomposition is unique, when λ has been chosen in $\{\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}\}$. It is determined by the chosen λ and by the dimension $\dim(\text{Ker}(A - \lambda \text{Id})^r)$ for each $r > 0$.

Normal form for $A|_{V_{[\lambda]}}$ for $\lambda = \pm 1$:

Let $\lambda = \pm 1$ be an eigenvalue of A . There exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of matrices of the form

$$\begin{pmatrix} J(\lambda, r_j)^{-1} & C(r_j, s_j, \lambda) \\ 0 & J(\lambda, r_j)^\tau \end{pmatrix}$$

where $C(r_j, s_j, \lambda) := J(\lambda, r_j)^{-1} \text{diag}(0, \dots, 0, s_j)$ with $s_j \in \{0, 1, -1\}$. If $s_j = 0$, then r_j is odd. The dimension of the eigenspace of the eigenvalue λ is given by $2 \text{Card}\{j \mid s_j = 0\} + \text{Card}\{j \mid s_j \neq 0\}$.

The number of s_j equal to $+1$ (resp. -1) arising in blocks of dimension $2k$ (i.e. with corresponding $r_j = k$) is equal to the number of positive (resp. negative) eigenvalues of the symmetric 2-form

$$\begin{aligned} \hat{Q}_{2k}^\lambda : \text{Ker}((A - \lambda \text{Id})^{2k}) \times \text{Ker}((A - \lambda \text{Id})^{2k}) &\rightarrow \mathbb{R} \\ (v, w) &\mapsto \lambda \Omega((A - \lambda \text{Id})^k v, (A - \lambda \text{Id})^{k-1} w). \end{aligned}$$

The decomposition is unique up to a permutation of the blocks and is determined by λ , by the dimension $\dim(\text{Ker}(A - \lambda \text{Id})^r)$ for each $r \geq 1$, and by the rank and the signature of the symmetric bilinear 2-form \hat{Q}_{2k}^λ for each $k \geq 1$.

Normal form for $A|_{V_{[\lambda]}}$ for $\lambda \in S^1 \setminus \{\pm 1\}$:

Let $\lambda \in S^1, \lambda \neq \pm 1$ be an eigenvalue of A . There exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of $4k_j \times 4k_j$ matrices ($k_j \geq 1$) of the form

$$\left(\begin{array}{c|ccc} (J_{\mathbb{R}}(\bar{\lambda}, 2k_j))^{-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & s_j V_{k_j}^1(\phi) \quad s_j V_{k_j}^2(\phi) \\ 0 & 0 & \dots & 0 \\ \hline 0 & & & (J_{\mathbb{R}}(\bar{\lambda}, 2k_j))^\tau \end{array} \right) \quad (2)$$

and $(4k_j + 2) \times (4k_j + 2)$ matrices ($k_j \geq 0$) of the form

$$\left(\begin{array}{c|cc|ccc} (J_{\mathbb{R}}(\bar{\lambda}, 2k_j))^{-1} & s_j U_{k_j}^2(\phi) & \vdots & \vdots & \frac{s_j}{2} V_{k_j}^2(\phi) & \frac{-s_j}{2} V_{k_j}^1(\phi) & U_{k_j}^1(\phi) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cos \phi & 0 & \dots & 0 & 1 & 0 & s_j \sin \phi \\ \hline 0 & 0 & & & & & & 0 \\ \vdots & \vdots & & & & & & \vdots \\ 0 & 0 & & & & & & 0 \\ \hline 0 & -s_j \sin \phi & 0 & \dots & 0 & 0 & -s_j & \cos \phi \end{array} \right) \quad (3)$$

where $J_{\mathbb{R}}(e^{i\phi}, 2k)$ is defined as above, where $(V_{k_j}^1(\phi) V_{k_j}^2(\phi))$ is the $2k_j \times 2$ matrix defined by

$$(V_{k_j}^1(\phi) V_{k_j}^2(\phi)) = \begin{pmatrix} (-1)^{k_j-1} R(e^{ik_j \phi}) \\ \vdots \\ R(e^{i\phi}) \end{pmatrix} \quad (4)$$

with $R(e^{i\phi}) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$, where

$$\left(U_{k_j}^1(\phi) U_{k_j}^2(\phi) \right) = \left(V_{k_j}^1(\phi) V_{k_j}^2(\phi) \right) (R(e^{i\phi})) \quad (5)$$

and where $s_j = \pm 1$. The complex dimension of the eigenspace of the eigenvalue λ in $V^{\mathbb{C}}$ is given by the number of such matrices.

The number of s_j equal to +1 (resp. -1) arising in blocks of dimension $2m$ in the normal decomposition given above is equal to the number of positive (resp. negative) eigenvalues of the Hermitian 2-form \hat{Q}_m^λ defined on $\text{Ker}((A - \lambda \text{Id})^m)$ by:

$$\begin{aligned} \hat{Q}_m^\lambda : \text{Ker}((A - \lambda \text{Id})^m) \times \text{Ker}((A - \lambda \text{Id})^m) &\rightarrow \mathbb{C} \\ (v, w) &\mapsto \frac{1}{\lambda} \Omega((A - \lambda \text{Id})^k v, (A - \bar{\lambda} \text{Id})^{k-1} \bar{w}) && \text{if } m = 2k \\ (v, w) &\mapsto i \Omega((A - \lambda \text{Id})^k v, (A - \bar{\lambda} \text{Id})^k \bar{w}) && \text{if } m = 2k + 1. \end{aligned}$$

This decomposition is unique up to a permutation of the blocks, when λ has been chosen in $\{\lambda, \bar{\lambda}\}$. It is determined by the chosen λ , by the dimension $\dim(\text{Ker}(A - \lambda \text{Id})^r)$ for each $r \geq 1$ and by the rank and the signature of the Hermitian bilinear 2-form \hat{Q}_m^λ for each $m \geq 1$.

The normal form for $A|_{V_{[\lambda]}}$ is given in Theorem 9 for $\lambda \notin S^1$, in Theorem 10 for $\lambda = \pm 1$, and in Theorem 15 for $\lambda \in S^1 \setminus \{\pm 1\}$. The characterisation of the signs is given in Proposition 12 for $\lambda = \pm 1$ and in Proposition 17 for $\lambda \in S^1 \setminus \{\pm 1\}$.

A direct consequence of Theorem 1 is the following characterization of the conjugacy class of a matrix in the symplectic group.

Theorem 2 *The conjugacy class of a matrix $A \in \text{Sp}(2n, \mathbb{R})$ is determined by the following data:*

- the eigenvalues of A which arise in quadruples $[\lambda] = \{\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}\}$;
- the dimension $\dim(\text{Ker}(A - \lambda \text{Id})^r)$ for each $r \geq 1$ for one eigenvalue in each class $[\lambda]$;
- for $\lambda = \pm 1$, the rank and the signature of the symmetric form \hat{Q}_{2k}^λ for each $k \geq 1$ and for an eigenvalue λ in $S^1 \setminus \{\pm 1\}$ chosen in each $[\lambda]$, the rank and the signature of the Hermitian form \hat{Q}_m^λ for each $m \geq 1$, with

$$\begin{aligned} \hat{Q}_m^\lambda : \text{Ker}((A - \lambda \text{Id})^m) \times \text{Ker}((A - \lambda \text{Id})^m) &\rightarrow \mathbb{C} \\ (v, w) &\mapsto \frac{1}{\lambda} \Omega((A - \lambda \text{Id})^k v, (A - \bar{\lambda} \text{Id})^{k-1} \bar{w}) && \text{if } m = 2k \\ (v, w) &\mapsto i \Omega((A - \lambda \text{Id})^k v, (A - \bar{\lambda} \text{Id})^k \bar{w}) && \text{if } m = 2k + 1. \end{aligned}$$

□

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1 Preliminaries

Lemma 3 *Consider $A \in \text{Sp}(V, \Omega)$ and let $0 \neq \lambda \in \mathbb{C}$. Then $\text{Ker}(A - \lambda \text{Id})^j$ in $V^{\mathbb{C}}$ is the symplectic orthogonal complement of $\text{Im}(A - \frac{1}{\lambda} \text{Id})^j$.*

PROOF:

$$\begin{aligned} \Omega((A - \lambda \text{Id})u, Av) &= \Omega(Au, Av) - \lambda \Omega(u, Av) = \Omega(u, v) - \lambda \Omega(u, Av) \\ &= -\lambda \Omega\left(u, \left(A - \frac{1}{\lambda} \text{Id}\right)v\right) \end{aligned}$$

and by induction

$$\Omega((A - \lambda \text{Id})^j u, A^j v) = (-\lambda)^j \Omega\left(u, \left(A - \frac{1}{\lambda} \text{Id}\right)^j v\right). \quad (6)$$

The result follows from the fact that A is invertible. \square

Corollary 4 *If E_λ denotes the generalized eigenspace of eigenvalue λ , i.e $E_\lambda := \{v \in V^{\mathbb{C}} \mid (A - \lambda \text{Id})^j v = 0 \text{ for an integer } j > 0\}$, we have*

$$\Omega(E_\lambda, E_\mu) = 0 \quad \text{when } \lambda\mu \neq 1.$$

Indeed the symplectic orthogonal complement of $E_\lambda = \cup_j \text{Ker}(A - \lambda \text{Id})^j$ is the intersection of the $\text{Im}(A - \frac{1}{\lambda} \text{Id})^j$. By Jordan normal form, this intersection is the sum of the generalized eigenspaces corresponding to the eigenvalues which are not $\frac{1}{\lambda}$.

If $v = u + iu'$ is in $\text{Ker}(A - \lambda \text{Id})^j$ with u and u' in V then $\bar{v} = u - iu'$ is in $\text{Ker}(A - \bar{\lambda} \text{Id})^j$ so that $E_\lambda \oplus E_{\bar{\lambda}}$ is the complexification of a real subspace of V . From this remark and corollary 4 the space

$$W_{[\lambda]} := E_\lambda \oplus E_{\frac{1}{\lambda}} \oplus E_{\bar{\lambda}} \oplus E_{\frac{1}{\bar{\lambda}}} \quad (7)$$

is the complexification of a real and symplectic A -invariant subspace $V_{[\lambda]}$ and

$$V = V_{[\lambda_1]} \oplus V_{[\lambda_2]} \oplus \dots \oplus V_{[\lambda_\kappa]} \quad (8)$$

where we denote by $[\lambda]$ the set $\{\lambda, \bar{\lambda}, \frac{1}{\lambda}, \frac{1}{\bar{\lambda}}\}$ and by $[\lambda_1], \dots, [\lambda_K]$ the distinct such sets exhausting the eigenvalues of A .

We denote by $A_{[\lambda_i]}$ the restriction of A to $V_{[\lambda_i]}$. It is clearly enough to obtain normal forms for each $A_{[\lambda_i]}$ since A will be a symplectic direct sum of those.

We shall construct a symplectic basis of $W_{[\lambda]}$ (and of $V_{[\lambda]}$) adapted to A for a given eigenvalue λ of A . We assume that $(A - \lambda \text{Id})^{p+1} = 0$ and $(A - \lambda \text{Id})^p \neq 0$ on the generalized eigenspace E_λ . Since A is real, this integer p is the same for $\bar{\lambda}$. By lemma 3, $\text{Ker}(A - \lambda \text{Id})^j$ is the symplectic orthogonal complement of $\text{Im}(A - \frac{1}{\lambda} \text{Id})^j$ for all j , thus $\dim \text{Ker}(A - \lambda \text{Id})^j = \dim \text{Ker}(A - \frac{1}{\lambda} \text{Id})^j$; hence the integer p is the same for λ and $\frac{1}{\lambda}$.

We decompose $W_{[\lambda]}$ (and $V_{[\lambda]}$) into a direct sum of A -invariant symplectic subspaces. Given a symplectic subspace Z of $V_{[\lambda]}$ which is A -invariant, its orthogonal complement (with respect to the symplectic 2-form) $V' := Z^{\perp_\Omega}$ is again symplectic and A -invariant. The generalized eigenspace for A on $V'^{\mathbb{C}}$ are $E'_\mu = V'^{\mathbb{C}} \cap E_\mu$, and the smallest integer p' for which $(A - \lambda \text{Id})^{p'+1} = 0$ on E'_λ is such that $p' \leq p$.

Hence, to get the decomposition of $W_{[\lambda]}$ (and $V_{[\lambda]}$) it is enough to build a symplectic subspace of $W_{[\lambda]}$ which is A -invariant and closed under complex conjugation and to proceed inductively. We shall construct such a subspace, containing a well chosen vector $v \in E_\lambda$ so that $(A - \lambda \text{Id})^p v \neq 0$.

We shall distinguish three cases; first $\lambda \notin S^1$ then $\lambda = \pm 1$ and finally $\lambda \in S^1 \setminus \{\pm 1\}$.

We first present a few technical lemmas which will be used for this construction.

1.1 A few technical lemmas

Let (V, Ω) be a real symplectic vector space. Consider $A \in \text{Sp}(V, \Omega)$ and let λ be an eigenvalue of A in $V^{\mathbb{C}}$.

Lemma 5 *For any positive integer j , the bilinear map*

$$\tilde{Q}_j : E_\lambda / \text{Ker}(A - \lambda \text{Id})^j \times E_{\frac{1}{\lambda}} / \text{Ker}(A - \frac{1}{\lambda} \text{Id})^j \rightarrow \mathbb{C}$$

$$([v], [w]) \mapsto \tilde{Q}_j([v], [w]) := \Omega((A - \lambda \text{Id})^j v, w) \quad v \in E_\lambda, w \in E_{\frac{1}{\lambda}} \quad (9)$$

is well defined and non degenerate. In the formula, $[v]$ denotes the class containing v in the appropriate quotient.

PROOF: The fact that \tilde{Q}_j is well defined follows from equation (6); indeed, for any integer j , we have

$$\Omega((A - \lambda \text{Id})^j u, v) = (-\lambda)^j \Omega(A^j u, (A - \frac{1}{\lambda} \text{Id})^j v). \quad (10)$$

The map is non degenerate because $\tilde{Q}_j([v], [w]) = 0$ for all w if and only if $(A - \lambda \text{Id})^j v = 0$ since Ω is a non degenerate pairing between E_λ and $E_{\frac{1}{\lambda}}$, thus

if and only if $[v] = 0$. Similarly, $\tilde{Q}_j([v], [w]) = 0$ for all v if and only if w is Ω -orthogonal to $\text{Im}(A - \lambda \text{Id})^j$, thus if and only if $w \in \text{Ker}(A - \frac{1}{\lambda} \text{Id})^j$ hence $[w] = 0$. \square

Lemma 6 *For any $v, w \in V$, any $\lambda \in \mathbb{C} \setminus \{0\}$ and any integers $i \geq 0$, $j > 0$ we have:*

$$\begin{aligned} \Omega\left((A - \lambda \text{Id})^i v, (A - \frac{1}{\lambda} \text{Id})^j w\right) &= -\frac{1}{\lambda} \Omega\left((A - \lambda \text{Id})^{i+1} v, (A - \frac{1}{\lambda} \text{Id})^j w\right) \quad (11) \\ &\quad - \frac{1}{\lambda^2} \Omega\left((A - \lambda \text{Id})^{i+1} v, (A - \frac{1}{\lambda} \text{Id})^{j-1} w\right). \end{aligned}$$

In particular, if λ is an eigenvalue of A , if $v \in E_\lambda$ is such that $p \geq 0$ is the largest integer for which $(A - \lambda \text{Id})^p v \neq 0$, we have for any integers $k, j \geq 0$:

$$\Omega\left((A - \lambda \text{Id})^{p+k} v, w\right) = (-\lambda^2)^j \Omega\left((A - \lambda \text{Id})^{p+k-j} v, (A - \frac{1}{\lambda} \text{Id})^j w\right) \quad (12)$$

so that

$$\Omega\left((A - \lambda \text{Id})^p v, w\right) = (-\lambda^2)^p \Omega\left(v, (A - \frac{1}{\lambda} \text{Id})^p w\right) \quad (13)$$

and

$$\Omega\left((A - \lambda \text{Id})^k v, (A - \frac{1}{\lambda} \text{Id})^j w\right) = 0 \text{ if } k + j > p. \quad (14)$$

PROOF: We have:

$$\begin{aligned} &\Omega\left((A - \lambda \text{Id})^i v, (A - \frac{1}{\lambda} \text{Id})^j w\right) \\ &= -\frac{1}{\lambda} \Omega\left((A - \lambda \text{Id} - A)(A - \lambda \text{Id})^i v, (A - \frac{1}{\lambda} \text{Id})^j w\right) \\ &= -\frac{1}{\lambda} \Omega\left((A - \lambda \text{Id})^{i+1} v, (A - \frac{1}{\lambda} \text{Id})^j w\right) \\ &\quad + \frac{1}{\lambda} \Omega\left(A(A - \lambda \text{Id})^i v, (A - \frac{1}{\lambda} \text{Id})(A - \frac{1}{\lambda} \text{Id})^{j-1} w\right) \\ &= -\frac{1}{\lambda} \Omega\left((A - \lambda \text{Id})^{i+1} v, (A - \frac{1}{\lambda} \text{Id})^j w\right) \\ &\quad + \frac{1}{\lambda} \Omega\left((A - \lambda \text{Id})^i v, (A - \frac{1}{\lambda} \text{Id})^{j-1} w\right) \\ &\quad - \frac{1}{\lambda^2} \Omega\left(A(A - \lambda \text{Id})^i v, (A - \frac{1}{\lambda} \text{Id})^{j-1} w\right) \end{aligned}$$

and formula (11) follows.

For any integers $k, j \geq 0$ and any v such that $(A - \lambda \text{Id})^p v = 0$, we have, by (6),

$$(-\lambda)^j \Omega\left((A - \lambda \text{Id})^{p+k+1-j} v, (A - \frac{1}{\lambda} \text{Id})^j w\right) = \Omega\left((A - \lambda \text{Id})^{p+k+1} v, A^j w\right) = 0.$$

Hence, applying formula (11) with a decreasing induction on j , we get formula (12). The other formulas follow readily. \square

Definition 7 For $\lambda \in S^1$ an eigenvalue of A and $v \in E_\lambda$ a generalized eigenvector, we define

$$T_{i,j}(v) := \frac{1}{\lambda^i \bar{\lambda}^j} \Omega((A - \lambda \text{Id})^i v, (A - \bar{\lambda})^j \bar{v}). \quad (15)$$

We have, by equation (11) :

$$T_{i,j}(v) = -T_{i+1,j}(v) - T_{i+1,j-1}(v), \quad (16)$$

and also,

$$T_{i,j}(v) = -\overline{T_{j,i}(v)}. \quad (17)$$

Lemma 8 Let $\lambda \in S^1$ be an eigenvalue of A and $v \in E_\lambda$ be a generalised eigenvector such that the largest integer p so that $(A - \lambda \text{Id})^p v \neq 0$ is odd, say, $p = 2k - 1$. Then, in the A -invariant subspace E_λ^v of E_λ generated by v , there exists a vector v' generating the same A -invariant subspace $E_\lambda^{v'} = E_\lambda^v$, so that $(A - \lambda \text{Id})^p v' \neq 0$ and so that

$$T_{i,j}(v') = 0 \quad \text{for all } i, j \leq k - 1.$$

If λ is real (i.e. ± 1), and if v is a real vector (i.e. in V), the vector v' can be chosen to be real as well.

PROOF: Observe that

$$\begin{aligned} T_{k,k-1}(v) &= -T_{k,k}(v) - T_{k-1,k}(v) \quad \text{by (11)} \\ &= -T_{k-1,k}(v) \quad \text{by (14)} \\ &= \overline{T_{k,k-1}(v)} \quad \text{by (17)} \end{aligned}$$

is real and can be put to $d = \pm 1$ by rescaling the vector. We use formulas (11) and (17) and we proceed by decreasing induction on $i + j$ as follows:

- if $T_{k-1,k-1}(v) = \alpha_1$, this α_1 is purely imaginary, we replace v by

$$v' := v - \frac{\alpha_1}{2\lambda d} (A - \lambda \text{Id})v;$$

clearly $E_\lambda^{v'} = E_\lambda^v$ and $T_{i,j}(v') = T_{i,j}(v)$ for $i + j \geq 2k - 1$ but now

$$T_{k-1,k-1}(v') = \alpha_1 - \frac{\alpha_1}{2d} T_{k,k-1}(v) - \frac{\bar{\alpha}_1}{2d} T_{k-1,k}(v) = 0;$$

so we can now assume $T_{k-1,k-1}(v) = 0$; observe that if λ is real and v is in V , then $\alpha_1 = 0$ and $v' = v$;

- if $T_{k-2,k-1}(v) = \alpha_2 = -T_{k-1,k-2}(v)$, this α_2 is real and we replace v by

$$v - \frac{\alpha_2}{2\lambda^2 d} (A - \lambda \text{Id})^2 v;$$

the space E_λ^v does not change and the quantities $T_{i,j}(v)$ do not vary for $i + j \geq 2k - 2$; now

$$T_{k-2,k-1}(v') = \alpha_2 - \frac{\alpha_2}{2d}T_{k,k-1}(v) - \frac{\overline{\alpha_2}}{2d}T_{k-2,k+1}(v) = 0,$$

hence also $T_{k-1,k-2}(v') = 0$; observe that if λ is real and v is in V , then v' is in V .

- we now assume by induction to have a $J > 0$ so that $T_{i,j}(v) = 0$ for all $0 \leq i, j \leq k - 1$ so that $i + j > 2k - 1 - J$;
- if $T_{k-J,k-1}(v) = \alpha_J$, then $T_{k-J,k-1}(v) = (-1)^{J-1}T_{k-1,k-J}(v)$ so that α_J is real when J is even and is imaginary when J is odd; we replace v by

$$v - \frac{\alpha_J}{2\lambda^J d}(A - \lambda \text{Id})^J v;$$

the space E_λ^v does not change and the quantities $T_{i,j}(v)$ do not vary for $i + j \geq 2k - J$; but now

$$\begin{aligned} T_{k-J,k-1}(v') &= \alpha_J - \frac{\alpha_J}{2d}T_{k,k-1}(v) - \frac{\overline{\alpha_J}}{2d}T_{k-J,k+J-1}(v) \\ &= \alpha_J - \frac{\alpha_J}{2} - (-1)^J \frac{\overline{\alpha_J}}{2} = 0. \end{aligned}$$

Hence also $T_{k-J+1,k-2}(v') = 0, \dots, T_{k-1,k-J+1}(v') = 0$; so the induction proceeds. Observe that if λ is real and v is in V then v' is in V . □

We shall use repeatedly that a $n \times n$ block triangular symplectic matrix is of the form

$$A' = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \in \text{Sp}(2n, \mathbb{R}) \Leftrightarrow \begin{cases} B = (D^\tau)^{-1} \\ C = (D^\tau)^{-1}S \text{ with } S \text{ symmetric.} \end{cases} \quad (18)$$

2 Normal forms for $A|_{V_\lambda}$ when $\lambda \notin S^1$.

As before, p denotes the largest integer such that $(A - \lambda \text{Id})^p$ does not vanish identically on the generalized eigenspace E_λ . Let us choose an element $v \in E_\lambda$ and an element $w \in E_\lambda^\perp$ such that

$$\tilde{Q}_p([v], [w]) = \Omega((A - \lambda \text{Id})^p v, w) \neq 0.$$

Let us consider the smallest A -invariant subspace E_λ^v of E_λ containing v ; it is of dimension $p + 1$ and a basis is given by

$$\{a_0 := v, \dots, a_i := (A - \lambda \text{Id})^i v, \dots, a_p := (A - \lambda \text{Id})^p v\}.$$

Observe that $Aa_i = (A - \lambda \text{Id})a_i + \lambda a_i$ so that $Aa_i = \lambda a_i + a_{i+1}$ for $i < p$ and $Aa_p = a_p$.

Similarly, we consider the smallest A -invariant subspace $E_{\frac{1}{\lambda}}^w$ of $E_{\frac{1}{\lambda}}$ containing w ; it is also of dimension $p + 1$ and a basis is given by

$$\left\{ b_0 := w, \dots, b_j := (A - \frac{1}{\lambda} \text{Id})^j w, \dots, b_p := (A - \frac{1}{\lambda} \text{Id})^p w \right\}.$$

One has

$$\Omega(a_i, a_j) = 0 \text{ and } \Omega(b_i, b_j) = 0 \text{ because } \Omega(E_\lambda, E_\mu) = 0 \text{ if } \lambda\mu \neq 1;$$

$$\Omega(a_i, b_j) = 0 \text{ if } i + j > p \text{ by equation (14) ;}$$

$$\Omega(a_i, b_{p-i}) = \left(\frac{-1}{\lambda^2}\right)^{p-i} \Omega\left((A - \lambda \text{Id})^p v, w\right) \text{ by equation (12) and is non zero by the choice of } v, w.$$

The matrix representing Ω in the basis $\{b_p, \dots, b_0, a_0, \dots, a_p\}$ is thus of the form

$$\left(\begin{array}{ccc|ccc} 0 & 0 & \bar{*} & 0 & & \\ & \ddots & & \ddots & & \\ 0 & 0 & 0 & * & \bar{*} & \\ \bar{*} & * & 0 & 0 & 0 & \\ & \ddots & & \ddots & & \\ 0 & \bar{*} & 0 & 0 & 0 & \end{array} \right)$$

with non vanishing $\bar{*}$. Hence Ω is non degenerate on $E_\lambda^v \oplus E_{\frac{1}{\lambda}}^w$ which is thus a symplectic A -invariant subspace.

We now construct a symplectic basis $\{b'_p, \dots, b'_0, a_0, \dots, a_p\}$ of $E_\lambda^v \oplus E_{\frac{1}{\lambda}}^w$, extending $\{a_0, \dots, a_p\}$, using a Gram-Schmidt procedure on the b_i 's. This gives a normal form for A on $E_\lambda^v \oplus E_{\frac{1}{\lambda}}^w$.

If λ is real, we take v, w in the real generalized eigenspaces $E_\lambda^{\mathbb{R}}$ and $E_{\frac{1}{\lambda}}^{\mathbb{R}}$ and we obtain a symplectic basis of the real A -invariant symplectic vector space, $E_\lambda^{\mathbb{R}v} \oplus E_{\frac{1}{\lambda}}^{\mathbb{R}w}$. If λ is not real, one considers the basis of $E_\lambda^{\bar{v}} \oplus E_{\frac{1}{\lambda}}^{\bar{w}}$ defined by the conjugate vectors $\{\bar{b}'_p, \dots, \bar{b}'_0, \bar{a}_0, \dots, \bar{a}_p\}$ and this yields a conjugate normal form on $E_{\bar{\lambda}} \oplus E_{\frac{1}{\bar{\lambda}}}$, hence a normal form on $W_{[\lambda]}$ and this will induce a real normal form on $V_{[\lambda]}$.

We choose v and w such that $\Omega\left((A - \frac{1}{\lambda} \text{Id})^p w, v\right) = 1$. We define inductively on j

$$b'_p := \frac{1}{\Omega(b_p, a_0)} b_p = b_p;$$

$$b'_{p-j} = \frac{1}{\Omega(b_{p-j}, a_j)} \left(b_{p-j} - \sum_{k < j} \Omega(b_{p-j}, a_k) b'_{p-k} \right),$$

so that any b'_j is a linear combination of the b_r with $r \geq j$.

In the symplectic basis $\{b'_p, \dots, b'_0, a_0, \dots, a_p\}$ the matrix representing A is

$$\left(\begin{array}{cc} B & 0 \\ 0 & J(\lambda, p+1)^\tau \end{array} \right)$$

- If $\lambda = re^{i\phi} \notin (S^1 \cup \mathbb{R})$ is a complex eigenvalue of A , there exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of k matrices of the form

$$\begin{pmatrix} J_{\mathbb{R}}(re^{-i\phi}, 2(p_j + 1))^{-1} & 0 \\ 0 & J_{\mathbb{R}}(re^{-i\phi}, 2(p_j + 1))^{\tau} \end{pmatrix}$$

with $p = p_1 \geq p_2 \geq \dots \geq p_k$ and $J_{\mathbb{R}}(re^{i\phi}, k)$ defined by (20). To eliminate the ambiguity in the choice of λ in $[\lambda] = \{\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}\}$ we can choose the eigenvalue λ with a positive imaginary part and a modulus greater than 1. The size of the blocks is determined, knowing the dimension $\dim_{\mathbb{C}}(\text{Ker}(A - \lambda \text{Id})^r)$ for each $r \geq 1$.

This normal form is unique, when a choice of λ in the set $[\lambda]$ is fixed.

3 Normal forms for $A|_{V_{[\lambda]}}$ when $\lambda = \pm 1$.

In this situation $[\lambda] = \{\lambda\}$ and $V_{[\lambda]}$ is the generalized real eigenspace of eigenvalue λ , still denoted –with a slight abuse of notation– E_{λ} . Again, p denotes the largest integer such that $(A - \lambda \text{Id})^p$ does not vanish identically on E_{λ} . We consider $\tilde{Q}_p : E_{\lambda}/\text{Ker}(A - \lambda \text{Id})^p \times E_{\lambda}/\text{Ker}(A - \lambda \text{Id})^p \rightarrow \mathbb{R}$ the non degenerate form defined by $\tilde{Q}_p([v], [w]) = \Omega((A - \lambda \text{Id})^p v, w)$. We see directly from equation (13) that \tilde{Q}_p is symmetric if p is odd and antisymmetric if p is even.

3.1 If $p = 2k - 1$ is odd

we choose $v \in E_{\lambda}$ such that

$$\tilde{Q}([v], [v]) = \Omega((A - \lambda \text{Id})^p v, v) \neq 0$$

and consider the smallest A -invariant subspace E_{λ}^v of E_{λ} containing v ; it is spanned by

$$\{a_p := (A - \lambda \text{Id})^p v, \dots, a_i := (A - \lambda \text{Id})^i v, \dots, a_0 := v\}.$$

We have

$$\Omega(a_i, a_j) = 0 \text{ if } i + j \geq p + 1 (= 2k) \text{ by equation (14);}$$

$$\Omega(a_i, a_{p-i}) \neq 0; \text{ by equation(12) and by the choice of } v.$$

Hence E_{λ}^v is a symplectic subspace because, in the basis defined by the e_i 's, Ω has the triangular form $\begin{pmatrix} 0 & & * \\ & \cdot & \\ * & & * \end{pmatrix}$ and has a non-zero determinant.

We can choose v in $E_\lambda \subset V$ so that $\Omega((A - \lambda \text{Id})^k v, (A - \lambda \text{Id})^{k-1} v) = \lambda s$ with $s = \pm 1$ by rescaling the vector and one may further assume, by lemma 8, that

$$T_{i,j}(v) = \frac{1}{\lambda^i} \frac{1}{\lambda^j} \Omega((A - \lambda \text{Id})^i v, (A - \lambda \text{Id})^j v) = 0 \quad \text{for all } 0 \leq i, j \leq k-1.$$

We now construct a symplectic basis $\{a'_p, \dots, a'_k, a_0, \dots, a_{k-1}\}$ of E_λ^v , extending $\{a_0, \dots, a_{k-1}\}$, by a Gram-Schmidt procedure, having chosen v as above. We define inductively on $0 \leq j \leq k-1$

$$a'_p := \frac{1}{\Omega(a_p, a_0)} a_p;$$

$$a'_{p-j} = \frac{1}{\Omega(a_{p-j}, a_j)} (a_{p-j} - \sum_{k < j} \Omega(a_{p-j}, a_k) a'_{p-k}),$$

so that any a'_j is a linear combination of the a_r 's with $r \geq j$ and in particular $a'_k = \frac{1}{s\lambda} a_k + \sum_{j=1}^{k-1} c_j a_{k+j}$.

In the symplectic basis $\{a'_p, \dots, a'_k, a_0, \dots, a_{k-1}\}$ the matrix representing A is

$$A' = \begin{pmatrix} B & C \\ 0 & J(\lambda, k)^\tau \end{pmatrix}$$

with $J(\lambda, m)$ defined by (19) and with C identically zero except for the last column, and the coefficient $C_k^k = s\lambda$. Since the matrix is symplectic, B is the transpose of the inverse of $J(\lambda, p+1)^\tau$ by (18), so $B = J(\lambda, k)^{-1}$ and $J(\lambda, k)C$ is symmetric with zeroes except in the last column, hence diagonal of the form $\text{diag}(0, \dots, 0, s)$. Thus

$$\begin{pmatrix} J(\lambda, k)^{-1} & J(\lambda, k)^{-1} \text{diag}(0, \dots, 0, s) \\ 0 & J(\lambda, k)^\tau \end{pmatrix},$$

with $s = \pm 1$, is the normal form of A restricted to E_λ^v . Recall that

$$s = \lambda^{-1} \Omega((A - \lambda \text{Id})^k v, (A - \lambda \text{Id})^{k-1} v).$$

3.2 If $p = 2k$ is even

we choose v and w in E_λ such that

$$\tilde{Q}([v], [w]) = \Omega((A - \lambda \text{Id})^p v, w) = \lambda^p = 1$$

and we consider the smallest A -invariant subspace $E_\lambda^v \oplus E_\lambda^w$ of E_λ containing v and w . It is of dimension $4k + 2$. Remark that $\Omega((A - \lambda \text{Id})^p v, v) = 0$. We can choose v so that

$$T_{r,s}(v) = \frac{1}{\lambda^{r+s}} \Omega((A - \lambda \text{Id})^r v, (A - \lambda \text{Id})^s v) = 0 \quad \text{for all } r, s.$$

Indeed, by formula (11) we have $T_{i,j}(v) = -T_{i+1,j}(v) - T_{i+1,j-1}(v)$. Observe that $T_{i,j}(v) = -T_{j,i}(v)$ so that $T_{i,i}(v) = 0$ and $T_{j,i}(v) = -T_{j,i+1}(v) - T_{j-1,i+1}(v)$. We proceed by induction, as in lemma 8 :

- $T_{p,0}(v) = 0$ implies $T_{p-r,r}(v) = 0$ for all $0 \leq r \leq p$ by equation (12).
- We assume by decreasing induction on J , starting from $J = p$, that we have $T_{i,j}(v) = 0$ for all $i + j \geq J$. Then we have $T_{J-1-s,s}(v) = -T_{J-1-s,s+1}(v) - T_{J-2-s,s+1}(v)$; the first term on the righthand side vanishes by the induction hypothesis, so $T_{J-1,0}(v) = (-1)^s T_{J-1-s,s}(v) = (-1)^{J-1} T_{0,J-1}(v) = (-1)^J T_{J-1,0}$.

If $T_{J-1,0}(v) = \alpha \neq 0$, J must be even and we replace v by

$$v' = v + \frac{\alpha}{2\lambda^{p-J+1}}(A - \lambda \text{Id})^{p-J+1}w.$$

Then $v' \in E_\lambda^v \oplus E_\lambda^w$, $E_\lambda^v \oplus E_\lambda^w = E_\lambda^{v'} \oplus E_\lambda^w$, $\Omega((A - \lambda \text{Id})^p v', w) = \lambda^p$ and $T_{i,j}(v') = T_{i,j}(v) = 0$ for all $i + j \geq J$ but now

$$\begin{aligned} T_{J-1,0}(v') &= T_{J-1,0}(v) + \frac{\alpha}{2\lambda^p} \Omega((A - \lambda \text{Id})^p w, v) \\ &\quad + \frac{\alpha}{2\lambda^p} \Omega((A - \lambda \text{Id})^{J-1} v, (A - \lambda \text{Id})^{p-J+1} w) \\ &\quad + \frac{\alpha^2}{4\lambda^p} \Omega((A - \lambda \text{Id})^p w, (A - \lambda \text{Id})^{p-J+1} w) \\ &= \alpha - \frac{\alpha}{2} - \frac{\alpha}{2} = 0 \end{aligned}$$

so that $T_{i,j}(v') = 0$ for all $i + j \geq J - 1$ and the induction proceeds.

We assume from now on that we have chosen v and w in E_λ so that $\Omega((A - \lambda \text{Id})^p v, w) = 1$ and $\Omega((A - \lambda \text{Id})^r v, (A - \frac{1}{\lambda} \text{Id})^s v) = 0$ for all r, s . We can proceed similarly with w so we can thus furthermore assume that $\Omega((A - \lambda \text{Id})^j w, (A - \lambda \text{Id})^k w) = 0$ for all j, k .

A basis of $E_\lambda^v \oplus E_\lambda^w$ is given by

$$\{a_p = (A - \lambda \text{Id})^p v, \dots, a_0 = v, b_0 = w, \dots, b_p = (A - \lambda \text{Id})^p w\}.$$

We have

$$\Omega(a_i, a_j) = 0 \text{ and } \Omega(b_i, b_j) = 0 \text{ by the choice of } v \text{ and } w;$$

$$\Omega(a_i, b_j) = 0 \text{ if } i + j > p \text{ by equation (14) ;}$$

$$\Omega(a_i, b_{p-i}) \neq 0 \text{ by equation (12) and the choice of } v, w.$$

The matrix representing Ω has the form $\left(\begin{array}{ccc|ccc} & & & \bar{*} & & 0 \\ & 0 & & & \ddots & \\ & & & * & & \bar{*} \\ \hline \bar{*} & & & & & \\ & * & & & & \\ & & \ddots & & & 0 \\ 0 & & & \bar{*} & & \end{array} \right)$ hence is non singular and the subspace $E_\lambda^v \oplus E_\lambda^w$ is symplectic. We now construct a symplectic basis $\{a'_p, \dots, a'_0, b_0, \dots, b_p\}$ of $E_\lambda^v \oplus E_\lambda^w$, extending $\{b_0, \dots, b_p\}$, using a Gram-Schmidt procedure on the a_i 's. We define inductively on j

$$a'_p := \frac{1}{\Omega(a_p, b_0)} a_p;$$

$$a'_{p-j} = \frac{1}{\Omega(a_{p-j}, b_j)} (a_{p-j} - \sum_{k < j} \Omega(a_{p-j}, b_k) a'_{p-k}),$$

so that any a'_j is a linear combination of the a'_k with $k \geq j$.

In the symplectic basis $\{a'_p, \dots, a'_0, b_0, \dots, b_p\}$ the matrix representing A is

$$\begin{pmatrix} B & 0 \\ 0 & J(\lambda, p+1)^\tau \end{pmatrix}.$$

Hence, the matrix

$$\begin{pmatrix} J(\lambda, p+1)^{-1} & 0 \\ 0 & J(\lambda, p+1)^\tau \end{pmatrix}$$

is a normal form for A restricted to $E_\lambda^v \oplus E_\lambda^w$. Thus we have:

Theorem 10 (Normal form for $A|_{V_{[\lambda]}}$ for $\lambda = \pm 1$.) *Let $\lambda = \pm 1$ be an eigenvalue of A . There exists a symplectic basis of $V_{[\lambda]}$ in which the matrix representing the restriction of A to $V_{[\lambda]}$ is a symplectic direct sum of matrices of the form*

$$\begin{pmatrix} J(\lambda, r_j)^{-1} & C(r_j, s_j, \lambda) \\ 0 & J(\lambda, r_j)^\tau \end{pmatrix}$$

where $C(r_j, s_j, \lambda) := J(\lambda, r_j)^{-1} \text{diag}(0, \dots, 0, s_j)$ with $s_j \in \{0, 1, -1\}$. If $s_j = 0$, then r_j is odd. The dimension of the eigenspace of eigenvalue 1 is given by $2 \text{Card}\{j \mid s_j = 0\} + \text{Card}\{j \mid s_j \neq 0\}$.

Definition 11 Given $\lambda \in \{\pm 1\}$, we define, for any integer $k \geq 1$, a bilinear form \hat{Q}_{2k}^λ on $\text{Ker}((A - \lambda \text{Id})^{2k})$:

$$\begin{aligned} \hat{Q}_{2k}^\lambda &: \text{Ker}((A - \lambda \text{Id})^{2k}) \times \text{Ker}((A - \lambda \text{Id})^{2k}) \rightarrow \mathbb{R} \\ (v, w) &\mapsto \lambda \Omega((A - \lambda \text{Id})^k v, (A - \lambda \text{Id})^{k-1} w). \end{aligned} \quad (21)$$

It is symmetric.

Proposition 12 *Given $\lambda \in \{\pm 1\}$, the number of positive (resp. negative) eigenvalues of the symmetric 2-form \hat{Q}_{2k}^λ is equal to the number of s_j equal to +1 (resp. -1) arising in blocks of dimension $2k$ (i.e. with corresponding $r_j = k$) in the normal decomposition of A on $V_{[\lambda]}$ given in theorem 10.*

On $V_{[\lambda]}$, we have:

$$\sum_j s_j = \sum_{k=1}^{\dim V} \text{Signature}(\hat{Q}_{2k}^\lambda) \quad (22)$$

PROOF: On the intersection of $\text{Ker}((A - \lambda \text{Id})^{2k})$ with one of the symplectically orthogonal subspaces E_λ^v constructed above for an odd $p \neq 2k - 1$, the form \hat{Q}_{2k}^λ vanishes identically. On the intersection of $\text{Ker}((A - \lambda \text{Id})^{2k})$ with a subspace E_λ^v for a v so that $p = 2k - 1$ and $\Omega((A - \lambda \text{Id})^k v, (A - \lambda \text{Id})^{k-1} v) = \lambda s$

the only non vanishing component is $\hat{Q}_{2k}^\lambda(v, v) = s$.
Indeed, $\text{Ker}((A - \lambda \text{Id})^{2k}) \cap E_\lambda^v$ is spanned by

$$\{(A - \lambda \text{Id})^r v; r \geq 0 \text{ and } r + 2k > p\},$$

and $\Omega((A - \lambda \text{Id})^{k+r} v, (A - \lambda \text{Id})^{k-1+r'} v) = 0$ when $2k + r + r' - 1 > p$ so the only non vanishing cases arise when $r = r' = 0$ and $p = 2k - 1$.

Similarly, the 2 form \hat{Q}_{2k}^λ vanishes on the intersection of $\text{Ker}((A - \lambda \text{Id})^{2k})$ with a subspace $E_\lambda^v \oplus E_\lambda^w$ constructed above for an even p . \square

The numbers s_j appearing in the decomposition of A are thus invariant of the matrix.

Corollary 13 *The normal decomposition described in theorem 10 is determined by the eigenvalue λ , by the dimension $\dim(\text{Ker}(A - \lambda \text{Id})^r)$ for each $r \geq 1$, and by the rank and the signature of the symmetric bilinear 2-forms \hat{Q}_{2k}^λ for each $k \geq 1$. It is unique up to a permutation of the blocks.* \square

4 Normal forms for $A|_{V[\lambda]}$ when $\lambda = e^{i\phi} \in S^1 \setminus \{\pm 1\}$.

We denote again by p the largest integer such that $(A - \lambda \text{Id})^p$ does not vanish identically on E_λ and we consider the non degenerate sesquilinear form

$$\begin{aligned} \hat{Q} : E_\lambda / \text{Ker}(A - \lambda \text{Id})^p \times E_\lambda / \text{Ker}(A - \lambda \text{Id})^p &\rightarrow \mathbb{C} \\ \hat{Q}([v], [w]) &= \overline{\lambda}^p \Omega((A - \lambda \text{Id})^p v, \overline{w}). \end{aligned}$$

Since \hat{Q} is non degenerate, we can choose $v \in E_\lambda$ such that $\hat{Q}([v], [v]) \neq 0$ thus $(A - \lambda \text{Id})^p v \neq 0$ and we consider the smallest A -invariant subspace, stable by complex conjugaison, and containing $v : E_\lambda^v \oplus E_\lambda^{\overline{v}} \subset E_\lambda \oplus E_{\overline{\lambda}}$. A basis is given by

$$\{a_i := (A - \lambda \text{Id})^i v, b_j := (A - \overline{\lambda} \text{Id})^j \overline{v} \quad 0 \leq i, j \leq p\}.$$

We have $a_i = \overline{b_i}$ and

- $\Omega(a_i, a_j) = 0, \Omega(b_i, b_j) = 0$ because $\Omega(E_\lambda, E_\lambda) = 0$;
- $\Omega(a_i, b_k) = 0$ if $i + k \geq p + 1$ by equation (14);
- $\Omega(a_i, b_k) \neq 0$ if $p = i + k$ by equation (12) and by the choice of v .

We conclude that $E_\lambda^v \oplus E_\lambda^{\overline{v}}$ is a symplectic subspace.

4.1 If $p = 2k - 1$ is odd

observe that $T_{k,k-1}(v) := \frac{1}{\lambda} \Omega((A - \lambda \text{Id})^k v, (A - \bar{\lambda} \text{Id})^{k-1} \bar{v}) = s$ is real and can be put to ± 1 by rescaling the vector (we could even put it to 1 exchanging if needed λ and its conjugate). One may further assume, by lemma 8 that

$$T_{i,j}(v) = \frac{1}{\lambda^i} \frac{1}{\bar{\lambda}^j} \Omega((A - \lambda \text{Id})^i v, (A - \bar{\lambda} \text{Id})^j \bar{v}) = 0 \quad \text{for all } 0 \leq i, j \leq k-1.$$

We consider the basis $\{a_{2k-1}, \dots, a_k, b_p, \dots, b_k, b_0, \dots, b_{k-1}, a_0, \dots, a_{k-1}\}$ for such a vector v with $T_{k,k-1}(v) = s = \pm 1$ and $T_{i,j}(v) = 0$ for all $0 \leq i, j \leq k-1$; the matrix representing Ω has the form

$$\left(\begin{array}{ccc|ccc} & & & \bar{*} & 0 & \\ & & & \cdot & \cdot & 0 \\ & & & * & \bar{*} & \\ \hline & 0 & & \bar{*} & 0 & \\ & & & 0 & \cdot & \cdot \\ & & & & * & \bar{*} \\ \hline \bar{*} & * & & & & \\ \cdot & \cdot & & 0 & & \\ \hline 0 & \bar{*} & & & & \\ 0 & & & \bar{*} & * & \\ & & & 0 & \cdot & \cdot \\ & & & & 0 & \bar{*} \end{array} \right)$$

and we transform it by a Gram-Schmidt method into a symplectic basis composed of pairs of conjugate vectors, extending $\{b_0, \dots, b_{k-1}, a_0, \dots, a_{k-1}\}$ on which Ω identically vanishes. We define

$$\begin{aligned} a'_{2k-1} &= \frac{1}{\Omega(a_{2k-1}, b_0)} a_{2k-1}, \\ b'_{2k-1} &= \frac{1}{\Omega(b_{2k-1}, a_0)} b_{2k-1} = \overline{a'_{2k-1}} \end{aligned}$$

and, inductively on increasing j with $1 < j \leq k$

$$\begin{aligned} a'_{2k-j} &= \frac{1}{\Omega(a_{2k-j}, b_{j-1})} \left(a_{2k-j} - \sum_{r=1}^{j-1} \Omega(a_{2k-j}, b_{r-1}) a'_{2k-r} \right), \\ b'_{2k-j} &= \overline{a'_{2k-j}}. \end{aligned}$$

Any a'_{2k-j} is a linear combination of the a_{2k-i} for $1 \leq i \leq j$; reciprocally any a_{2k-j} can be written as a linear combination of the a'_{2k-i} for $1 \leq i \leq j$, and the coefficient of a'_{2k-j} is equal to $\Omega(a_{2k-j}, b_{j-1})$.

The basis $\{a'_{2k-1}, \dots, a'_k, b'_{2k-1}, \dots, b'_k, b_0, \dots, b_{k-1}, a_0, \dots, a_{k-1}\}$ is symplectic, and in that basis, since $A(a_r) = \lambda a_r + a_{r+1}$ and $A(b_r) = \bar{\lambda} b_r + b_{r+1}$ for all

$r < 2k - 2$, the matrix representing A is of the block upper triangular form

$$\begin{pmatrix} * & 0 & 0 & C \\ & * & \overline{C} & 0 \\ & & J(\overline{\lambda}, k)^\tau & 0 \\ 0 & & & J(\lambda, k)^\tau \end{pmatrix}$$

where C is a $k \times k$ matrix such that the only non vanishing terms are on the last column ($C_j^i = 0$ when $j < k$) and $C_k^k = \Omega(a_k, b_{k-1}) = s\lambda$. The fact that the matrix is symplectic implies that $S := J(\overline{\lambda}, k)C$ is hermitean; since $S_j^i = 0$ when $j \neq k$, we have,

$$C = J(\overline{\lambda}, k)^{-1} \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & s \end{pmatrix} = C(k, s, \overline{\lambda})$$

and the matrix of the restriction of A to the subspace $E_\lambda^v \oplus E_\lambda^{\overline{v}}$ has the block triangular normal form

$$\begin{pmatrix} J(\overline{\lambda}, k)^{-1} & 0 & 0 & C(k, s, \overline{\lambda}) \\ & J(\lambda, k)^{-1} & C(k, s, \lambda) & 0 \\ & & J(\overline{\lambda}, k)^\tau & 0 \\ 0 & & & J(\lambda, k)^\tau \end{pmatrix}. \quad (23)$$

Writing $a'_{2k-j} = \frac{1}{\sqrt{2}}(e_{2j-1} - ie_{2j})$, $b'_{2k-j} = \overline{a'_{2k-j}} = \frac{1}{\sqrt{2}}(e_{2j-1} + ie_{2j})$, as well as $a_{j-1} = \frac{1}{\sqrt{2}}(f_{2j-1} - if_{2j})$ and $b_{j-1} = \overline{a_{j-1}} = \frac{1}{\sqrt{2}}(f_{2j-1} + if_{2j})$ for $1 \leq j \leq k$, the vectors e_i, f_j all belong to the real subspace denoted $V_{[\lambda]}^v$ of V whose complexification is $E_\lambda^v \oplus E_\lambda^{\overline{v}}$ and we get a symplectic basis

$$\{e_1, \dots, e_{2k}, f_1, \dots, f_{2k}\}$$

of this real subspace $V_{[\lambda]}^v$. The matrix representing A in this basis is :

$$\begin{pmatrix} (J_{\mathbb{R}}(\overline{\lambda}, 2k))^{-1} & C_{\mathbb{R}}(k, s, \overline{\lambda}) \\ 0 & (J_{\mathbb{R}}(\overline{\lambda}, 2k))^\tau \end{pmatrix} \quad (24)$$

where $J_{\mathbb{R}}(e^{i\phi}, 2k)$ is defined as in (20) and where $C_{\mathbb{R}}(k, s, e^{i\phi})$ is the $(p+1) \times (p+1)$ matrix written in terms of two by two matrices as

$$C_{\mathbb{R}}(k, s, e^{i\phi})^\tau = s \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ (-1)^{k-1} R(e^{ik\phi}) & \dots & -R(e^{i2\phi}) & R(e^{i\phi}) \end{pmatrix} \quad (25)$$

with $R(e^{i\phi}) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$ as before and $s = \pm 1$. This is the normal form of A restricted to $V_{[\lambda]}^v$; recall that

$$s = \lambda^{-1} \Omega((A - \lambda \text{Id})^k v, (A - \overline{\lambda} \text{Id})^{k-1} \overline{v}).$$

4.2 If $p = 2k$ is even

we observe that $\Omega((A - \bar{\lambda}\text{Id})^k \bar{v}, (A - \lambda\text{Id})^k v)$ is purely imaginary and we choose v so that it is $\Omega((A - \bar{\lambda}\text{Id})^k \bar{v}, (A - \lambda\text{Id})^k v) = si$ where $s = \pm 1$ (remark that the sign changes if one permutes λ and $\bar{\lambda}$). We can further choose the vector v so that :

$$\begin{aligned} \Omega((A - \lambda\text{Id})^k v, (A - \bar{\lambda}\text{Id})^{k-1} \bar{v}) &= \frac{1}{2} \lambda si & (26) \\ T_{i,j}(v) := \frac{1}{\lambda^i \bar{\lambda}^j} \Omega((A - \lambda\text{Id})^i v, (A - \bar{\lambda}\text{Id})^j \bar{v}) &= 0 \quad \text{for all } 0 \leq i, j \leq k-1; \end{aligned}$$

Indeed, as before, by (11), we have $T_{i,j}(v) = -T_{i+1,j}(v) - T_{i+1,j-1}(v)$ and $T_{i,j}(v) = -\overline{T_{j,i}(v)}$ and we proceed as in lemma 8 by decreasing induction on $i + j$:

- if $T_{k,k-1}(v) = \alpha_1$, since $T_{k-1,k}(v) = si - T_{k,k-1}(v)$ the imaginary part of α_1 is equal to $\frac{1}{2}si$ and we replace v by $v - \frac{\alpha_1}{2\lambda si}(A - \lambda\text{Id})v$; it generates the same A -invariant subspace and the quantities $T_{i,j}(v)$ do not vary for $i + j \geq 2k$ but now $T_{k,k-1}(v) = \alpha_1 - \frac{\alpha_1}{2si}T_{k+1,k-1}(v) + \frac{\alpha_1}{2si}T_{k,k}(v) = \alpha_1 - \frac{1}{2}\alpha_1 - \frac{1}{2}\bar{\alpha}_1 = \frac{1}{2}si$ since $T_{k,k}(v) = -T_{k+1,k-1}(v) = -si$; so we can now assume $T_{k,k-1}(v) = \frac{1}{2}si$;
- if $T_{k-1,k-1}(v) = \alpha_2$, this α_2 is purely imaginary and we replace v by $v - \frac{\alpha_2}{2\lambda^2 si}(A - \lambda\text{Id})^2 v$; it generates the same A -invariant subspace and the quantities $T_{i,j}(v)$ do not vary for $i + j \geq 2k - 1$; now $T_{k-1,k-1}(v) = \alpha_2 - \frac{\alpha_2}{2si}T_{k+1,k-1}(v) + \frac{\alpha_2}{2si}T_{k-1,k+1}(v) = \alpha_2 - \frac{1}{2}\alpha_2 + \frac{1}{2}\bar{\alpha}_2 = 0$. We may thus assume this property to hold for v .
- if $T_{k-2,k-1}(v) = \alpha_3 = -T_{k-1,k-1}(v) - T_{k-1,k-2}(v) = \overline{T_{k-2,k-1}(v)}$, this α_3 is real and we replace v by $v - \frac{\alpha_3}{2\lambda^3 si}(A - \lambda\text{Id})^3 v$; it generates and the the same A -invariant subspace and the quantities $T_{i,j}(v)$ do not vary for $i + j \geq 2k - 2$; now $T_{k-2,k-1}(v) = \alpha_3 - \frac{\alpha_3}{2si}T_{k+1,k-1}(v) + \frac{\alpha_3}{2si}T_{k-2,k+2}(v) = 0$, since $T_{k+1,k-1}(v) = -T_{k,k}(v) = -T_{k-2,k+2}(v) = si$; hence also $T_{k-1,k-2}(v) = 0$;
- we now assume by induction to have a $J > 1$ so that $T_{i,j}(v) = 0$ for all $0 \leq i, j \leq k - 1$ so that $i + j > 2k - 1 - J$;
- if $T_{k-J,k-1}(v) = \alpha_{J+1}$, then $T_{k-J,k-1}(v) = (-1)^{J-1}T_{k-1,k-J}(v)$ so that α_{J+1} is real when J is even and is imaginary when J is odd; we replace v by $v - \frac{\alpha_{J+1}}{2\lambda^{J+1} si}(A - \lambda\text{Id})^{J+1} v$; it sgenerates the same A -invariant subspace and the quantities $T_{i,j}(v)$ do not vary for $i + j \geq 2k - J$, but now $T_{k-J,k-1}(v) = \alpha_{J+1} - \frac{\alpha_{J+1}}{2si}T_{k+1,k-1}(v) + \frac{\alpha_{J+1}}{2si}T_{k-J,k+J}(v) = \alpha_{J+1} - \frac{\alpha_{J+1}}{2} + (-1)^{J+1}\frac{\alpha_{J+1}}{2} = 0$. Hence also $T_{k-J+1,k-2}(v) = 0, \dots, T_{k-1,k-J+1}(v) = 0$; so the induction step is proven.

on which Ω identically vanishes. We define inductively, for increasing j with $1 \leq j \leq k-1$

$$\begin{aligned}
a'_{2k} &:= \frac{1}{\Omega((A - \lambda \text{Id})^{2k} v, \bar{v})} (A - \lambda \text{Id})^{2k} v = \frac{1}{\Omega(a_{2k}, b_0)} a_{2k} \\
b'_{2k} &:= \frac{1}{\Omega((A - \bar{\lambda} \text{Id})^{2k}, \bar{v}, v)} (A - \bar{\lambda} \text{Id})^{2k} \bar{v} = \frac{1}{\Omega(b_{2k}, a_0)} b_{2k} = \overline{a'_{2k}} \\
a'_{2k-j} &:= \frac{1}{\Omega(a_{2k-j}, b_j)} \left(a_{2k-j} - \sum_{r=0}^{j-1} \Omega(a_{2k-j}, b_r) a'_{2k-r} \right) \\
b'_{2k-j} &:= \frac{1}{\Omega(b_{2k-j}, a_j)} \left(b_{2k-j} - \sum_{r=0}^{j-1} \Omega(b_{2k-j}, a_r) b'_{2k-r} \right) = \overline{a'_{2k-j}} \\
a'_k &:= a_k - \sum_{r=0}^{k-1} \Omega(a_k, b_r) a'_{2k-r} \\
b'_k &:= \frac{1}{\Omega(b_k, a_k)} \left(b_k - \sum_{r=0}^{k-1} \Omega(b_k, a_r) b'_{2k-r} \right) = \frac{1}{iS} \overline{a'_k}.
\end{aligned}$$

Each a'_{2k-j} is a linear combination of the $(A - \lambda \text{Id})^{2k-r} v$ for $0 \leq r \leq j$. The basis

$$\{a'_{2k}, \dots, a'_{k+1}, b'_{2k}, \dots, b'_{k+1}, b'_k; b_0, \dots, b_{k-1}, a_0, \dots, a_{k-1}, a'_k\}$$

is now symplectic. Since $A(a_r) = \lambda a_r + a_{r+1}$ for all $r < 2k$, and $A(a_{2k}) = \lambda a_{2k}$, the matrix representing A in that basis is of the form

$$\begin{pmatrix}
A_1 & 0 & 0 & \begin{pmatrix} c^{2k} & d^{2k} \\ 0 & \vdots & \vdots \\ c^{k+1} & d^{k+1} \end{pmatrix} \\
0 & A_2 & \begin{pmatrix} e^{2k} \\ 0 & \vdots \\ e^{k+1} \\ e^k \end{pmatrix} & 0 \\
0 & 0 & J(\bar{\lambda}, k)^\tau & 0 \\
0 & 0 & 0 & J(\lambda, k+1)^\tau
\end{pmatrix}$$

with $A(b_{k-1}) = \bar{\lambda} b_{k-1} + \sum_{j=0}^k e^{k+j} b'_{k+j}$, $A(a_{k-1}) = \lambda a_{k-1} + a'_k + \sum_{j=1}^k c^{k+j} a'_{k+j}$ and $A(a'_k) = \lambda a'_k + \sum_{j=1}^k d^{k+j} a'_{k+j}$.

Since a matrix $\begin{pmatrix} A' & E \\ 0 & D \end{pmatrix}$ is symplectic if and only if $A' = (D^\tau)^{-1}$ and $D^\tau E$ is symmetric, we have

$$A_1 = J(\bar{\lambda}, k)^{-1} \quad A_2 = J(\lambda, k+1)^{-1}$$

and

$$J(\bar{\lambda}, k) \begin{pmatrix} c^{2k} & d^{2k} \\ 0 & \vdots \\ \vdots & \vdots \\ c^{k+1} & d^{k+1} \end{pmatrix} = \left(J(\lambda, k+1) \begin{pmatrix} e^{2k} \\ \vdots \\ e^{k+1} \\ e^k \end{pmatrix} \right)^\tau.$$

This implies

$$J(\bar{\lambda}, k) \begin{pmatrix} c^{2k} & d^{2k} \\ \vdots & \vdots \\ c^{k+2} & d^{k+2} \\ c^{k+1} & d^{k+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ s_1 & s_2 \end{pmatrix} \quad J(\lambda, k+1) \begin{pmatrix} e^{2k} \\ \vdots \\ e^{k+2} \\ e^{k+1} \\ e^k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ s_1 \\ s_2 \end{pmatrix}$$

so that $s_1 = \bar{\lambda}c^{k+1}$ and $s_2 = \bar{\lambda}d^{k+1}$. Now

$$\begin{aligned} A(a'_k) &= A\left(a_k + \sum_{j \geq 1} F_k^j a_{k+j}\right) = \lambda a'_k + a_{k+1} + \sum_{j \geq 1} F_k^j a_{k+j+1} \\ &= \lambda a'_k + a'_{k+1} \Omega(a_{k+1}, b_{k-1}) + \sum_{j \geq 1} F_k^j a'_{k+j+1} \end{aligned}$$

so that $d^{k+1} = \Omega(a_{k+1}, b_{k-1}) = \lambda^2 i s$ and $s_2 = \lambda i s$. We also have

$$A(a_{k-1}) = \lambda a_{k-1} + a_k = \lambda a_{k-1} + a'_k + \Omega(a_k, b_{k-1}) a'_{k+1} + \sum_{j \geq 2} G^j a'_{k+j}$$

so that $c^{k+1} = \Omega(a_k, b_{k-1}) = \lambda \frac{1}{2} i s$ and $s_1 = \frac{1}{2} i s$.

We have thus shown that the matrix representing A in the chosen basis has the block upper-triangular normal form

$$\begin{pmatrix} J(\bar{\lambda}, k)^{-1} & 0 & 0 & J(\bar{\lambda}, k)^{-1} S \\ & J(\lambda, k+1)^{-1} & J(\lambda, k+1)^{-1} S^\tau & 0 \\ & 0 & J(\bar{\lambda}, k)^\tau & 0 \\ & & & J(\lambda, k+1)^\tau \end{pmatrix} \quad (27)$$

where S is the $k \times (k+1)$ matrix defined by

$$S = S(k, d, \lambda) := \begin{pmatrix} 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ 0 & \dots & 0 & \frac{1}{2} i s & \lambda i s \end{pmatrix}. \quad (28)$$

We write $a'_{2k+1-j} = \frac{1}{\sqrt{2}}(e_{2j-1} - i e_{2j})$, $b'_{2k+1-j} = \overline{a'_{2k+1-j}} = \frac{1}{\sqrt{2}}(e_{2j-1} + i e_{2j})$, as well as $a_{j-1} = \frac{1}{\sqrt{2}}(f_{2j-1} - i f_{2j})$ and $b_{j-1} = \overline{a_{j-1}} = \frac{1}{\sqrt{2}}(f_{2j-1} + i f_{2j})$ for $1 \leq j \leq k$, and $a'_k = \frac{1}{\sqrt{2}}(e_{2k+1} + i d f_{2k+1})$, $b'_k = -i d \overline{a'_k} = \frac{1}{\sqrt{2}}(-f_{2k+1} - i d e_{2k+1})$. The vectors e_i, f_j all belong to the real subspace $V_{[\lambda]}^v$ of V whose complexification is $E_\lambda^v \oplus E_{\bar{\lambda}}^v$ and we get a symplectic basis

$$\{e_1, \dots, e_{2k+1}, f_1, \dots, f_{2k+1}\}$$

matrix defined by

$$\left(V_{k_j}^1(\phi) V_{k_j}^2(\phi) \right) = \begin{pmatrix} (-1)^{k_j-1} R(e^{ik_j\phi}) \\ \vdots \\ R(e^{i\phi}) \end{pmatrix} \quad (31)$$

with $R(e^{i\phi}) = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$, where

$$\left(U_{k_j}^1(\phi) U_{k_j}^2(\phi) \right) = \left(V_{k_j}^1(\phi) V_{k_j}^2(\phi) \right) (R(e^{i\phi})) \quad (32)$$

and where $s_j = \pm 1$. The complex dimension of the eigenspace of eigenvalue λ in $V^{\mathbb{C}}$ is given by the number of such matrices.

Definition 16 Given $\lambda \in S^1 \setminus \{\pm 1\}$, we define, for any integer $m \geq 1$, a Hermitian form \hat{Q}_m^λ on $\text{Ker}((A - \lambda \text{Id})^m)$ by:

$$\begin{aligned} \hat{Q}_m^\lambda : \text{Ker}((A - \lambda \text{Id})^m) \times \text{Ker}((A - \lambda \text{Id})^m) &\rightarrow \mathbb{C} \\ (v, w) &\mapsto \frac{1}{\lambda} \Omega((A - \lambda \text{Id})^k v, (A - \bar{\lambda} \text{Id})^{k-1} \bar{w}) && \text{if } m = 2k \\ (v, w) &\mapsto i \Omega((A - \lambda \text{Id})^k v, (A - \bar{\lambda} \text{Id})^k \bar{w}) && \text{if } m = 2k + 1. \end{aligned}$$

Proposition 17 For $\lambda \in S^1 \setminus \{\pm 1\}$, the number of positive (resp. negative) eigenvalues of the Hermitian 2-form \hat{Q}_m^λ is equal to the number of s_j equal to +1 (resp. -1) arising in blocks of dimension $2m$ in the normal decomposition of A on $V_{[\lambda]}$ given in theorem 15.

PROOF: On the intersection of $\text{Ker}((A - \lambda \text{Id})^m)$ with one of the symplectically orthogonal subspaces $E_\lambda^v \oplus E_\lambda^{\bar{v}}$ constructed above from a v such that $(A - \lambda \text{Id})^p v \neq 0$ and $(A - \lambda \text{Id})^{p+1} v = 0$, the form \hat{Q}_m^λ vanishes identically, except if $p = m - 1$ and the only non vanishing component is $\hat{Q}_m^\lambda(v, v) = s$. Indeed, $\text{Ker}((A - \lambda \text{Id})^m) \cap E_\lambda^v$ is spanned by

$$\{(A - \lambda \text{Id})^r v; r \geq 0 \text{ and } r + m > p\},$$

and $\hat{Q}_m^\lambda((A - \lambda \text{Id})^r v, (A - \lambda \text{Id})^{r'} v) = 0$ when $m + r + r' - 1 > p$ so the only non vanishing cases arise when $r = r' = 0$ and $m = p + 1$ so for $\hat{Q}_m^\lambda(v, v)$. This is equal to $\frac{1}{\lambda} \Omega((A - \lambda \text{Id})^k v, (A - \bar{\lambda} \text{Id})^{k-1} \bar{v}) = \frac{1}{\lambda} \lambda s = s$ if $m = 2k$, and to $i \Omega((A - \lambda \text{Id})^k v, (A - \bar{\lambda} \text{Id})^k \bar{v}) = i(-is) = s$ if $m = 2k + 1$. \square

The numbers s_j appearing in the decomposition are thus invariant of the matrix.

Corollary 18 The normal decomposition described in theorem 15 is unique up to a permutation of the blocks when the eigenvalue λ has been chosen in $\{\lambda, \bar{\lambda}\}$, for instance by specifying that its imaginary part is positive. It is completely determined by this chosen λ , by the dimension $\dim_{\mathbb{C}}(\text{Ker}(A - \lambda \text{Id})^r)$ for each $r \geq 1$ and by the rank and the signature of the Hermitian bilinear 2-forms \hat{Q}_m^λ for each $m \geq 1$. \square

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