Stationary solutions to the Boltzmann equation in the plane

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Abstract
The paper proves existence of stationary solutions to the Boltzmann equation in a bounded set of $\mathbb{R}^2$ for given indata, hard forces and truncation in the collision kernel for small velocities and close to parallel colliding velocities. It does not use any averaging in velocity lemma and uses techniques from the discrete velocity stationary case recently developped in [7]-[8]-[9], where the averaging in velocity lemmas are not valid.

1 Introduction.

Consider the stationary Boltzmann equation in $\Omega \subset \mathbb{R}^2$,

$$v \cdot \nabla_z f(z,v) = Q(f,f), \quad z \in \Omega, \quad v \in \mathbb{R}^2, \tag{1.1}$$

where Ω is a strictly convex domain with C^1 boundary. The nonnegative function f represents the density of a rarefied gas with z the position and v the velocity. The operator Q is the nonlinear Boltzmann collision operator with angular cut-off and a truncation for small velocities and close to parallel colliding velocities,

$$Q(f,f)(z,v) = \int_{\mathbb{R}^2}\int_{\mathcal{S}^1} \chi_\eta(v,v_*,\omega)\bar{B}(v-v_*,\omega)\big(f(z,v')f(z,v'_*) - f(z,v)f(z,v_*)\big)dv_*d\omega. \tag{1.2}$$

$\mathcal{S}^1$ is the unit circle in $\mathbb{R}^2$, $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\omega$, $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\omega$. For $\eta \in]0,1[$ and fixed,

$$\chi_\eta(v, v_*, \omega) = 0 \text{ if } |v| \le \eta \text{ or } |v_*| \le \eta \text{ or } |v'| \le \eta \text{ or } |v'_*| \le \eta,$$
$$\chi_\eta(v, v_*, \omega) = 1, \quad \text{else.} \tag{1.3}$$

Here $\widehat{(u,v)}$ denotes the angle between vectors u and v.

The function $\bar{B}$ is the kernel of the classical nonlinear Boltzmann operator for hard forces,

$$|v - v_*|^\beta b(\omega) \text{ with } \beta \in [0,2[, \quad b \in L^1_+(\mathcal{S}^1), \quad b(\omega) \ge c > 0 \text{ a.e.} \tag{1.4}$$

From now on denote $\chi_\eta \bar{B}$ by B. The inward and outward boundaries in phase space are

$$\partial\Omega^+ = \{(z,v) \in \partial\Omega \times \mathbb{R}^2 \ ; \ v \cdot n(z) > 0\},$$
$$\partial\Omega^- = \{(z,v) \in \partial\Omega \times \mathbb{R}^2 \ ; \ v \cdot n(z) < 0\},$$

where $n(z)$ denotes the inward normal on $\partial\Omega$.
Given a function f_b defined on $\partial\Omega^+$, solutions f to (1.1) are sought with

$$f(z,v) = f_b(z,v), \quad (z,v) \in \partial\Omega^+. \tag{1.5}$$

For any $(z,v) \in \Omega \times \mathbb{R}^2$, denote by

$$s^+(z,v) = \inf\{s > 0; \ (z - sv, v) \in \partial\Omega^+\},$$
$$s^-(z,v) = \inf\{s > 0; \ (z + sv, v) \in \partial\Omega^-\},$$
$$z^+(z,v) = z - s^+(z,v)v, \quad z^-(z,v) = z + s^-(z,v)v. \tag{1.6}$$

Solutions are understood in mild form, i.e.

$$f(z,v) = f_b(z^+(z,v), v) + \int_0^{s^+(z,v)} Q(f,f)(z^+(z,v) + sv, v)ds,$$
$$\text{a.a. } (z,v) \in \Omega \times \mathbb{R}^2. \tag{1.7}$$

The main result of the paper is the following.

Theorem 1.1 *Let f_b be a non negative measurable function such that*

$$\int_{\partial\Omega^+} v \cdot n(z)\big(1 + v^2 + \ln f_b\big) f_b(z,v) d\sigma(z) dv < \infty. \tag{1.8}$$

Then equation (1.1) *has a non negative solution satisfying the boundary condition* (1.5).

For the stationary Boltzmann equation the control of mass and entropy is not straightforward, contray to the evolutionary case. Existence results in the slab, i.e. one-dimensional spatial and three-dimensional velocity frame, together with an invariance with respect to the two remaining space variables of the distribution function were first established for the nonlinear Boltzmann equation. Stationary integrable solutions to the Boltzmann equation in a slab have been proven in [3], [4] and [5], for different boundary conditions, bounds on the entropy production term and a weighted moment of the distribution function giving control of the entropy. For higher space dimension, stationary unsigned solutions close to Maxwellians were constructed in convex domains [14], [15]. In [12], the existence and uniqueness of the stationary solution to the Boltzmann equation close to a uniform Maxwellian have been proven in a bounded domain of $\mathbb{R}^n$, $1 \leq n \leq 3$, for diffuse reflection boundary conditions. Its hydrodynamic limit to a solution to the steady incompressible Navier-Stokes-Fourier system has been performed in [13]. Existence of stationary solutions to the Boltzmann equation in a bounded domain of R^n, $n \geq 1$, and given indata has been proven in [2]. There, scaling arguments from [1] were used. In this paper, we prove existence of solutions to the stationary Boltzmann equation in the plane with the help of the entropy production term and the construction of 'good' characteristics where the distribution function is bounded and 'bad' characteristics of arbitrarily small measure. This is inspired by recent results for discrete velocity models for the Boltzmann equation where averaging lemmas do not hold and new arguments are required. In [7], [8], [9] a weaker property than L^1 compactness of averages in velocity, i.e. the L^1 compactness of the integrated collision frequencies of a sequence of approximations is proven. It strongly depends on the two-dimensional spatial dimension. In this paper we use the tools developped in [7], [8], [9] and do not use any averaging lemma.
Work is in progress to fill a gap in the proof of Lemma 4.1 of [10] that uses these techniques in the discrete velocity evolutionary case.
The construction of a first sequence of approximations with damping and convolutions is performed in Section 2. In Section 3, the damping and convolutions are removed, leading to a more involved sequence of approximations. In Section 3, the phase space is split into 'good' characteristics where the approximations are uniformly bounded and 'bad' characteristics of arbitrarily small measure. In Section 4, the L^1 compactness of the integrated collision frequency sequence is proven. The passage to the limit in the mild form satisfied by the approximations is performed in Section 5.

2 First approximations.

In the paper we denote by c constants that do not depend on approximations. To emphasize their dependence on the indatum f_b, we sometimes denote them by c_b. We use the following approximation scheme. Let $(B_\alpha)_{\alpha\in]0,1[}$, be a family of C^∞ regularizations of B. Let $(\varphi_\alpha)_{\alpha\in]0,1[}$ be mollifiers defined from $\varphi \in C_0^\infty(\mathbb{R}^4)$ such that

$$\varphi(z,v) = 0 \text{ for } |z| \geq 1 \text{ or } |v| \geq 1, \quad \int \varphi(z,v)dzdv = 1,$$

by $\varphi_\alpha(z,v) = \frac{1}{\alpha^4}\varphi(\frac{z}{\alpha}, \frac{v}{\alpha})$. Outside the boundary the function to be convolved with μ_α, is continued in the normal direction by its boundary value. Let $\tilde{\mu}_k$ be a smooth mollifier on $\partial\Omega \times \mathbb{R}^2$ in a ball of radius $\frac{1}{k}$. Denote by

$$f_{bk} = \big(\min\{f_b, k\}\big) * \tilde{\mu}_k\,.$$

Lemma 2.1
For every $(\alpha, k) \in]0,1[\times\mathbb{N}^*$, there is a non negative solution F to

$$\alpha F + v\cdot\nabla_z F = \int B_\alpha\Big(\frac{F}{1+\frac{F}{k}}(z,v')\frac{F*\varphi_\alpha}{1+\frac{F*\varphi_\alpha}{k}}(z,v'_*) - \frac{F}{1+\frac{F}{k}}(z,v)\frac{F*\varphi_\alpha}{1+\frac{F*\varphi_\alpha}{k}}(z,v_*)\Big)dv_*d\omega, \quad (z,v)\in\Omega\times\mathbb{R}^2, \tag{2.1}$$

$$F(z,v) = f_{bk}(z,v), \quad (z,v)\in\partial\Omega^+. \tag{2.2}$$

Proof of Lemma 2.1.
It follows the lines of the proofs in Section 2 in [8] that we refer to for details. Let $(\alpha, k) \in]0,1[\times\mathbb{N}^*$ be given. Let K be the closed and convex subset of $L^1(\Omega\times\mathbb{R}^2)$ defined by

$$K = \{f \in L^1_+(\Omega\times\mathbb{R}^2);\ \int f(z,v)dzdv \leq \frac{1}{\alpha}\int_{\partial\Omega^+} v\cdot n(z) f_b(z,v)d\sigma(z)dv\}.$$

Define the map $\mathcal{T}$ from K into K by $\mathcal{T}(f) = F$, where F is the solution to

$$\alpha F(z,v) + v\cdot\nabla_z F(z,v) = \int_{\mathbb{R}^2\times\mathcal{S}^1} B_\alpha\Big(\frac{F}{1+\frac{F}{k}}(z,v')\frac{f*\varphi_\alpha}{1+\frac{f*\varphi_\alpha}{k}}(z,v'_*) - \frac{F}{1+\frac{F}{k}}(z,v)\frac{f*\varphi_\alpha}{1+\frac{f*\varphi_\alpha}{k}}(z,v_*)\Big)dv_*d\omega, \quad (z,v)\in\Omega\times\mathbb{R}^2, \tag{2.3}$$

$$F(z,v) = f_{bk}(z,v), \quad (z,v)\in\partial\Omega^+. \tag{2.4}$$

$F = \mathcal{T}(f)$ can be obtained as the limit in $L^1_+(\Omega \times \mathbb{R}^2)$ of the sequence $(F^q)_{q\in\mathbb{N}}$ defined by $F^0 = 0$ and

$$\alpha F^{q+1} + v \cdot \nabla_z F^{q+1} = \int_{\mathbb{R}^2\times\mathcal{S}^1} B_\alpha \Big(\frac{F^q}{1+\frac{F^q}{k}}(z,v') \frac{f*\mu_\alpha}{1+\frac{f*\mu_\alpha}{k}}(z,v'_*) - \frac{F^{q+1}}{1+\frac{F^q}{k}}(z,v) \frac{f*\mu_\alpha}{1+\frac{f*\mu_\alpha}{k}}(z,v_*) \Big) dv_* d\omega \,, \tag{2.5}$$

$$F^{q+1}(z,v) = f_{bk}(z,v), \quad (z,v) \in \partial\Omega^+, \qquad q \in \mathbb{N}. \tag{2.6}$$

In exponential form F^{q+1} can be written as

$$\begin{aligned} F^{q+1}(z,v) &= f_{bk}(z^+(z,v),v) e^{-\alpha s^+(z,v)} \\ & e^{-\int_{-s^+(z,v)}^0 \frac{1}{1+\frac{F^q}{k}(z+sv,v)} \int B_\alpha \frac{f*\mu_\alpha}{1+\frac{f*\mu_\alpha}{k}}(z+sv,u_*) du_* d\gamma \, ds} \\ &+ \int_{-s^+(z,v)}^0 \int B_\alpha \frac{F^q}{1+\frac{F^q}{k}}(z+sv,v') \frac{f*\mu_\alpha}{1+\frac{f*\mu_\alpha}{k}}(z+sv,v'_*) e^{\alpha s} \\ & e^{\int_0^s \frac{1}{1+\frac{F^q}{k}(z+rv,v)} \int B_\alpha \frac{f*\mu_\alpha}{1+\frac{f*\mu_\alpha}{k}}(z+rv,u_*) du_* d\gamma \, dr} . \end{aligned} \tag{2.7}$$

The sequence $(F^q)_{q\in\mathbb{N}}$ is monotone. Indeed, $F^0 \le F^1$, by the exponential form of F^1. If $F^q \le F^{q+1}$, then it follows from the exponential forms of F^{q+1} and F^{q+2} that $F^{q+1} \le F^{q+2}$. Moreover,

$$\begin{aligned} \alpha \int F^{q+1}(z,v) dz dv \le & \int_{\partial\Omega^+} v \cdot n(z) f_{bk}(z,v) d\sigma(z) dv \\ & + \int B_\alpha \frac{F^q - F^{q+1}}{1+\frac{F^q}{k}}(z,v) \frac{f*\mu_\alpha}{1+\frac{f*\mu_\alpha}{k}}(z,v_*) dz dv dv_* d\omega, \end{aligned}$$

so that

$$\int F^{q+1}(z,v) dz dv \le \frac{1}{\alpha} \int_{\partial\Omega^+} v \cdot n(z) f_b(z,v) d\sigma(z) dv, \quad q \in \mathbb{N}. \tag{2.8}$$

By the monotone convergence theorem, $(F^q)_{q\in\mathbb{N}}$ converges in $L^1(\Omega \times \mathbb{R}^2)$ to a solution F of (2.3)-(2.4). The solution of (2.3)-(2.4) is unique in the set of non negative functions. Indeed, let G be a non negative solution of (2.3)-(2.4). It follows by induction that

$$F^q \le G, \quad q \in \mathbb{N}. \tag{2.9}$$

Indeed, (2.9) holds for $q = 0$, since $G \geq 0$. Assume (2.9) holds for q. Using the exponential form of F^{q+1} implies $F^{q+1} \leq G$. Consequently,

$$F \leq G. \tag{2.10}$$

Moreover, subtracting the equation satisfied by G from the equation satisfied by F, and integrating the resulting equation on $\Omega \times \mathbb{R}^2$ leads to

$$\alpha \int_{\Omega\times\mathbb{R}^2} (G - F)(z, v)dzdv + \int_{\partial\Omega^-} |v \cdot n(z)|(G - F)(z, v)d\sigma(z)dv = 0. \tag{2.11}$$

It results from (2.10)-(2.11) that $G = F$.
The map $\mathcal{T}$ is continuous in the L^1-norm topology (cf [6] pages 124-5). Namely, let a sequence $(f^q)_{q\in\mathbb{N}}$ in K converge in $L^1(\Omega \times \mathbb{R}^2)$ to $f \in K$. Set $F^q = \mathcal{T}(f^q)$. Because of the uniqueness of the solution to (2.3)-(2.4), it is enough to prove that there is a subsequence of (F^q) converging to $F = \mathcal{T}(f)$. Now there is a subsequence of (f^q), still denoted (f^q), such that decreasingly (resp. increasingly) $(G^q) = (\sup_{r\geq q} f^r)$ (resp. $(g^q) = (\inf_{r\geq q} f^r)$) converges to f in L^1. Let (S^q) (resp. (s^q)) be the sequence of solutions to

$$\begin{aligned}\alpha S^q + v \cdot \nabla_z S^q = \int B_\alpha \Big(&\frac{S^q}{1 + \frac{S^q}{k}}(z, v')\frac{G^q * \mu_\alpha}{1 + \frac{G^q * \mu_\alpha}{k}}(z, v'_*) \\ &- \frac{S^q}{1 + \frac{S^q}{k}}(z, v)\frac{g^q * \mu_\alpha}{1 + \frac{g^q * \mu_\alpha}{k}}(z, v_*)\Big) dv_* d\omega \,, \end{aligned} \tag{2.12}$$

$$S^q(z, v) = f_{bk}(z, v), \quad (z, v) \in \partial\Omega^+, \tag{2.13}$$

$$\begin{aligned}\alpha s^q + v \cdot \nabla_z s^q = \int B_\alpha \Big(&\frac{s^q}{1 + \frac{s^q}{k}}(z, v')\frac{g^q * \mu_\alpha}{1 + \frac{g^q * \mu_\alpha}{k}}(z, v'_*) \\ &- \frac{s^q}{1 + \frac{s^q}{k}}(z, v)\frac{G^q * \mu_\alpha}{1 + \frac{G^q * \mu_\alpha}{k}}(z, v_*)\Big) dv_* d\omega \,, \end{aligned} \tag{2.14}$$

$$s^q(z, v) = f_{bk}(z, v), \quad (z, v) \in \partial\Omega^+ \,. \tag{2.15}$$

(S^q) is a non-increasing sequence, since that holds for the successive iterates defining the sequence. Then (S^q) decreasingly converges in L^1 to some S. Similarly (s^q) increasingly converges in L^1 to some s. The limits S and s satisfy (2.3)-(2.4). It follows by uniqueness that $s = F = S$, hence that (F^q) converges in L^1 to F.
The map $\mathcal{T}$ is compact in the L^1-norm topology. In [8] an averaging lemma was used. Here we replace it by the following argument. Let $(f_l)_{l\in\mathbb{N}}$ be a

bounded sequence of L^1 and $F_l = T(f_l)$. Denote by $\tilde{F}_l = \frac{F_l}{1+\frac{F_l}{k}}$, $\tilde{f}_l = \frac{f_l * \varphi_j}{1+\frac{f_l * \varphi_j}{k}}$, by $B(0,V)$ the open ball in $\mathbb{R}^2$ centered at the origin and of radius $V > 0$, and by $I(\alpha,\beta)$ the interval with end points $(\alpha,\beta) \in \mathbb{R}^2$. $(F_l)_{l\in\mathbb{N}}$ is the sum of

$$\Big(f_{bk}(z^+(z,v),v) + \int_{-s^+(z,v)}^{0} \int B_\alpha \tilde{F}_l(z+sv,v')\tilde{f}_l(z+sv,v'_*)dv_* d\omega ds\Big)_{l\in\mathbb{N}^*}, \tag{2.16}$$

and

$$\Big(-\int_{-s^+(z,v)}^{0} \int B_\alpha \tilde{F}_l(z+sv,v)\tilde{f}_l(z+sv,v_*)dv_* d\omega ds\Big)_{l\in\mathbb{N}^*}. \tag{2.17}$$

In order to prove that (2.16) is compact in L^1, it is sufficient to prove that for any $V > 0$ and $\mu \in]0,1[$,

$$\Big(\int_{-s^+(z,v)}^{0} \int_{B(0,V)\times A_\mu} B_\alpha \tilde{F}_l(z+sv,v')\tilde{f}_l(z+sv,v'_*)dv_* d\omega ds\Big)_{l\in\mathbb{N}^*}, \tag{2.18}$$

is compact in $L^1(\Omega \times B(0,V))$ where

$$A_\mu = \{\omega \in \mathcal{S}^1;\ |\widehat{(v,\omega)}| \geq \mu \text{ and } |\pi - \widehat{(v,\omega)}| \geq \mu\}. \tag{2.19}$$

Indeed, noticing that v' is parallel to v if and only if $\omega = \pm\frac{v}{|v|}$, and expressing ω by its angle with the vector $\frac{v}{|v|}$, the integral over $\Omega \times B(0,V)$ of

$$\int_{-s^+(z,v)}^{0} \int_{B(0,V)\times A_\mu^c} B_j \tilde{F}_l(z+sv,v')\tilde{f}_l(z+sv,v'_*)dv_* d\omega ds dz dv$$

is smaller than $ck^2V^2\mu$. The sequence (2.18) is uniformly bounded in $L^1(\Omega\times B(0,V))$. Let us prove that it is uniformly equiintegrable with respect to the z variable. By the restriction to A_μ in (2.18), any $(v,v') \in \mathbb{R}^2 \times \mathbb{R}^2$ considered when integrating the absolute value of (2.18) over $\Omega \times B(0,V)$ forms a basis in $\mathbb{R}^2$. For any $h \in \mathbb{R}^2$, denote by $(a(h),b(h))$ its coordinates in this basis. Split the difference of (2.18) between (z,v) and $(z+h,v)$ into the three following terms,

$$\int_{I(-s^+(z,v),-s^+(z+h,v))} \int_{A_\mu} B_\alpha \tilde{F}_l(z+h+sv,v')\tilde{f}_l(z+h+sv,v'_*)dv_* d\omega ds,$$

$$\int_{-s^+(z,v)}^0 \Big(\int_{A_\mu} B_\alpha \tilde{F}_l(z+h+sv,v') \tilde{f}_l(z+h+sv,v'_*) dv_* d\omega$$
$$- \int_{A_\mu} B_\alpha \tilde{F}_l(z+b(h)v'+sv,v') \tilde{f}_l(z+b(h)v'+sv,v'_*) dv_* d\omega \Big) ds$$

$$= \int_{I(-s^+(z,v)-a(h),-s^+(z,v))\cup I(0,a(h))} \Big(\int_{A_\mu} B_\alpha \tilde{F}_l(z+h+sv,v') \tilde{f}_l(z+h+sv,v'_*) dv_* d\omega \Big) ds,$$

$$\int_{-s^+(z,v)}^0 \int_{A_\mu} B_\alpha \tilde{F}_l(z+b(h)v'+sv,v') \big(\tilde{f}_l(z+b(h)v'+sv,v'_*)$$
$$- \tilde{f}_l(z+sv,v'_*) \big) dv_* d\omega \Big) ds,$$

that tend to zero $h \to 0$ when integrated over $\Omega \times \{v \in \mathbb{R}^2; |v| < V\}$, and

$$\int_{-s^+(z,v)}^0 \int_{A_\mu} B_\alpha \big(\tilde{F}_l(z+b(h)v'+sv,v') - \tilde{F}_l(z+sv,v') \big)$$
$$\tilde{f}_l(z+sv,v'_*) dv_* d\omega ds. \tag{2.20}$$

Notice that the integrand in the first line of (2.20) is a directional derivative in the direction v'. Consequently, (2.20) is equal to

$$\int_{-s^+(z,v)}^0 \int_{A_\mu} B_\alpha \tilde{F}_l(z+sv,v') \tilde{f}_l(z+sv,v'_*) e^{-\alpha b(h)}$$
$$\Big(\exp(-\int_0^{b(h)} \frac{\int B_\alpha \tilde{f}_l(z+sv+rv',u_*) du_* d\gamma}{(1+\frac{F_l}{k})(z+sv+rv',v')} dr) - 1 \Big) dv_* d\omega ds$$

$$+ \int_{-s^+(z,v)}^0 \int_{A_\mu} B_\alpha \tilde{f}_l(z+sv,v'_*) \Big(\int_0^{b(h)} \int B_\alpha \tilde{F}_l(z+sv+rv',V') \tilde{f}_l(z+sv+rv',V'_*) du_* d\gamma$$
$$e^{\alpha(r-b(h))} \exp\big(- \int_r^{b(h)} \frac{\int B_\alpha \tilde{f}_l(z+sv+tv',u_*) du_* d\gamma}{(1+\frac{F_l}{k})(z+sv+tv',v')} dt \big) dr \Big) dv_* d\omega ds. \tag{2.21}$$

Here, V' (resp V'_*) denotes $v' - (v' - u_*, \gamma)\gamma$ (resp. $u_* + (v' - u_*, \gamma)$). (2.21) tends to zero when $h \to 0$ when integrated over $\Omega \times B(0, V)$ in absolute value since all integrands are uniformly bounded and the domains of integration are of order h. This ends the proof of the uniform equiintegrability of (2.18) w.r.t. the z variable. The proof of its uniform equiintegrability w.r.t. the v variable is analogous. The L^1 compactness of (2.17) can be proven analogously. Hence the sequence $(F_l)_{l\in\mathbb{N}^*}$ is compact in L^1. And so, the Schauder fixed point theorem applies to $\mathcal{T}$, leading to a solution F of (2.1)-(2.2). ∎

3 Removal of the damping and convolutions.

For any $k \in \mathbb{N}^*$, denote by Q_k^+ (resp. Q_k, resp. $\nu_k(F)$, resp. D^k) the approximate gain term (resp. collision operator, resp. collision frequency, resp. entropy production term) defined by

$$Q_k^+(F,F) = \int_{\mathbb{R}^2\times\mathcal{S}^1} B \frac{F(v')}{1+\frac{F(v')}{k}} \frac{F(v'_*)}{1+\frac{F(v'_*)}{k}} dv_* d\omega, \tag{3.1}$$

$$\nu_k(F)(v) = \frac{1}{1+\frac{F(v)}{k}} \int B \frac{F(v_*)}{1+\frac{F(v_*)}{k}} dv_* d\omega, \tag{3.2}$$

$$Q_k(F,F) = Q_k^+(F,F) - F\nu_k(F),$$

$$D^k(z,v) = \int_{\mathbb{R}^2\times\mathcal{S}^1} B\Big(\frac{F^{k\prime}}{1+\frac{F^{k\prime}}{k}} \frac{F_*^{k\prime}}{1+\frac{F_*^{k\prime}}{k}} - \frac{F^k}{1+\frac{F^k}{k}} \frac{F_*^k}{1+\frac{F_*^k}{k}}\Big)$$
$$\ln \frac{F^{k\prime}F_*^{k\prime}(1+\frac{F^k}{k})(1+\frac{F_*^k}{k})}{F^kF_*^k(1+\frac{F^{k\prime}}{k})(1+\frac{F_*^{k\prime}}{k})} dv_* d\omega\,. \tag{3.3}$$

For any $(\alpha, k) \in]0,1[\times\mathbb{N}^*$, denote by $F^{\alpha,k}$ the solution to (2.1)-(2.2) obtained in the previous section. $(F^{\alpha,k})_{\alpha\in]0,1[}$ is weakly compact in $L^1_{loc}(\Omega\times\mathbb{R}^2)$ since it is bounded by a multiple of k^2. Denote by F^k the limit for the weak topology in $L^1_{loc}(\Omega\times\mathbb{R}^2)$ of a converging subsequence when $\alpha \to 0$. In the next lemma we prove that for a subsequence, the convergence is strong in $L^1(\Omega\times\mathbb{R}^2)$.

Lemma 3.1
There is a sequence $(\alpha_q)_{q\in\mathbb{N}}$ tending to zero when $q \to +\infty$, such that when

$q \to +\infty$, $(F^{\alpha_q,k})_{q\in\mathbb{N}}$ *strongly converges in* $L^1(\Omega\times\mathbb{R}^2)$ *to* F^k. *Moreover,* F^k *is a solution to*

$$v\cdot\nabla_z F^k = Q_k(F^k,F^k), \quad (z,v)\in\Omega\times\mathbb{R}^2, \tag{3.4}$$

$$F^k(z,v) = f_{bk}(z,v), \quad (z,v)\in\partial\Omega^+, \tag{3.5}$$

and

$$\int (1+v^2)F^k(z,v)dzdv \le c_b, \tag{3.6}$$

$$\int D^k(z,v)dzdv \le c_b, \quad k\in\mathbb{N}^*. \tag{3.7}$$

$$\int_{\partial\Omega^-} F^k(z,v)\mid v\cdot n(z)\mid d\sigma(z)dv + \int_{\partial\Omega^-,F^k\le k} F^k\ln F^k(z,v)\mid v\cdot n(z)\mid d\sigma(z)dv$$
$$+\ln\frac{k}{2}\int_{\partial\Omega^-,F^k\ge k} F^k\mid v\cdot n(z)\mid d\sigma(z)dv \le c_b. \tag{3.8}$$

Proof of Lemma 3.1

Consider the approximation scheme $(f^{\alpha,\rho})_{\rho\in\mathbb{N}}$ of $F^{\alpha,k}$,

$$f^{\alpha,0} = 0, \tag{3.9}$$

$$\alpha f^{\alpha,\rho+1} + v\cdot\nabla_z f^{\alpha,\rho+1} = \int B_\alpha\Big(\frac{F^{\alpha,k}}{1+\frac{F^{\alpha,k}}{k}}(z,v')\frac{F^{\alpha,k}*\mu_\alpha}{1+\frac{F^{\alpha,k}*\mu_\alpha}{k}}(z,v'_*)$$
$$-\frac{f^{\alpha,\rho+1}}{1+\frac{f^{\alpha,\rho+1}}{k}}(z,v)\frac{f^{\alpha,\rho}*\mu_\alpha}{1+\frac{f^{\alpha,\rho}*\mu_\alpha}{k}}(z,v_*)\Big)dv_*d\omega, \tag{3.10}$$

$$f^{\alpha,\rho+1}(z,v) = f_{bk}(z,v), \quad (z,v)\in\partial\Omega^+, \quad \rho\in\mathbb{N}. \tag{3.11}$$

$f^{\alpha,1}$ is explicitly given in terms of $F^{\alpha,k}$. It follows from the exponential forms of $F^{\alpha,k}$ and $f^{\alpha,1}$ that

$$F^{\alpha,k}\le f^{\alpha,1}, \quad \alpha\in]0,1[.$$

The sequence $(f^{\alpha,\rho})_{\rho\ge2}$ is constructed as follows. Denote by $\mathcal{S}$ the map from $(L^1(\Omega\times\mathbb{R}^2))^2$ mapping (X,Z) into $W=\mathcal{S}(X,Z)\in L^1(\Omega\times\mathbb{R}^2)$ solution to

$$\alpha W + v\cdot\nabla_z W = \int B_\alpha\Big(\frac{F^{\alpha,k}}{1+\frac{F^{\alpha,k}}{k}}(z,v')\frac{F^{\alpha,k}*\mu_\alpha}{1+\frac{F^{\alpha,k}*\mu_\alpha}{k}}(z,v'_*)$$
$$-\frac{W}{1+\frac{X}{k}}(z,v)\frac{Z*\mu_\alpha}{1+\frac{Z*\mu_\alpha}{k}}(z,v_*)\Big)dv_*d\omega\,,$$
$$W(z,v) = f_{bk}(z,v), \quad (z,v)\in\partial\Omega^+.$$

Denote by

$$f^{\alpha,1,0} = \mathcal{S}(0, f^{\alpha,1}), \quad f^{\alpha,1,r} = \mathcal{S}(f^{\alpha,1,r-1}, f^{\alpha,1}),$$
$$F^{\alpha,k,0} = \mathcal{S}(0, F^{\alpha,k}), \quad F^{\alpha,k,r} = \mathcal{S}(F^{\alpha,k,r-1}, F^{\alpha,k}), \quad r \in \mathbb{N}^*.$$

First,

$$f^{\alpha,1,0} \leq F^{\alpha,k,0}.$$

Then the sequence $(f^{\alpha,1,r})_{r\in\mathbb{N}}$ (resp. $(F^{\alpha,k,r})_{r\in\mathbb{N}}$) is increasing with limit $f^{\alpha,2}$ (resp. $F^{\alpha,k}$). It follows from $f^{\alpha,1,r} \leq F^{\alpha,k,r}$, $r \in \mathbb{N}$, that

$$f^{\alpha,2} \leq F^{\alpha,k}. \tag{3.12}$$

Let

$$f^{\alpha,2,0} := \mathcal{S}(0, f^{\alpha,2}), \quad f^{\alpha,2,r} := \mathcal{S}(f^{\alpha,2,r-1}, f^{\alpha,2}), \quad r \in \mathbb{N}^*.$$

It follows from (3.12) that

$$f^{\alpha,2,0} \geq F^{\alpha,k,0}.$$

The sequence $(f^{\alpha,2,r})_{r\in\mathbb{N}}$ is also increasing with limit $f^{\alpha,3}$ and with $f^{\alpha,2,r} \geq F^{\alpha,k,r}$. Hence

$$f^{\alpha,3} \geq F^{\alpha,k}.$$

From here by induction on ρ, it holds that

$$f^{\alpha,2\rho} \leq f^{\alpha,2\rho+2} \leq F^{\alpha,k} \leq f^{\alpha,2\rho+3} \leq f^{\alpha,2\rho+1}, \quad \alpha \in]0,1[, \quad \rho \in \mathbb{N}. \tag{3.13}$$

By induction on r, for each r the sequence $(f^{\alpha,1,r})_{\alpha\in]0,1[}$ is translationally equicontinuous in α. The limit sequence $(f^{\alpha,2})_{\alpha\in]0,1[}$ is also translationally equicontinuous. This is so, since given $\epsilon > 0$, r and then h_0 can be taken so that

$$\int (f^{\alpha,2} - f^{\alpha,1,r})(z,v)dzdv < \epsilon$$
$$\text{and } \int |f^{\alpha,1,r}(z+h, v+\tilde{h}) - f^{\alpha,1,r}(z,v)|dzdv < \epsilon, \ |h| < h_0, \ |\tilde{h}| < h_0.$$

It can analogously be proven that for each $\rho \in \mathbb{N}$, $(f^{\alpha,\rho})_{\alpha\in]0,1[}$ is translationally equicontinuous in α. Let $(\alpha_q)_{q\in\mathbb{N}}$ be a sequence tending to zero. Take a

subsequence in $(\alpha_q)_{q\in\mathbb{N}}$, still denoted by $(\alpha_q)_{q\in\mathbb{N}}$, such that $(f^{\alpha_q,2})_{q\in\mathbb{N}}$ converges in L^1 to some $f^{0,2}$ when $q\to+\infty$.

Continuing by induction gives a sequence $(f^{0,\rho})_{\rho\in\mathbb{N}}$ satisfying

$$f^{0,2\rho}\le f^{0,2\rho+2}\le F^k\le f^{0,2\rho+3}\le f^{0,2\rho+1},\quad \rho\in\mathbb{N}, \tag{3.14}$$

$$\begin{aligned}
&v\cdot\nabla_z f^{0,\rho+1}=G-\int B\frac{f^{0,\rho+1}}{1+\frac{f^{0,\rho+1}}{k}}(z,v)\frac{f^{0,\rho}}{1+\frac{f^{0\rho}}{k}}(z,v_*)dv_*d\omega,\\
&f^{0,\rho+1}(z,v)=f_{bk}(z,v),\quad (z,v)\in\partial\Omega^+.
\end{aligned}$$

Here, G is the weak L^1 limit when $\alpha\to 0$ of the gain term

$$\int B_\alpha\frac{F^{\alpha,k}}{1+\frac{F^{\alpha,k}}{k}}(z,v')\frac{F^{\alpha,k}*\mu_\alpha}{1+\frac{F^{\alpha,k}*\mu_\alpha}{k}}(z,v'_*)dv_*d\omega\ .$$

In particular, $(f^{0,2\rho})_{\rho\in\mathbb{N}}$ (resp. $(f^{0,2\rho+1})_{\rho\in\mathbb{N}}$) non decreasingly (resp. non increasingly) converges in L^1 to some g (resp. h) when $\rho\to+\infty$. The limits satisfy

$$0\le g\le F^k\le h, \tag{3.15}$$

$$v\cdot\nabla_z h=G-\int B\frac{h}{1+\frac{h}{k}}(z,v)\frac{g}{1+\frac{g}{k}}(z,v_*)dv_*d\omega, \tag{3.16}$$

$$v\cdot\nabla_z g=G-\int B\frac{g}{1+\frac{g}{k}}(z,v)\frac{h}{1+\frac{h}{k}}(z,v_*)dv_*d\omega, \tag{3.17}$$

$$(h-g)(z,v)=0,\quad (z,v)\in\partial\Omega^+.$$

Subtracting (3.17) from (3.16) and integrating the resulting equation on $\Omega\times\mathbb{R}^2$ gives that

$$\int_{\partial\Omega^-}\mid v\cdot n(Z)\mid(h-g)(Z,v)d\sigma(Z)dv=0,$$

so that $h-g=0$ also on $\partial\Omega^-$. Hence,

$$\begin{aligned}
&\int_{-s^+(z,v)}^{s^-(z,v)}h(z+sv,v)e^{\int_0^s\int Bh(z+rv,u)dud\gamma dr}\\
&\qquad\Big(\int B(h-g)(z+sv,v_*)dv_*d\omega\Big)ds=0,\quad (z,v)\in\Omega\times\mathbb{R}^2.
\end{aligned} \tag{3.18}$$

(3.15) and (3.18) imply that $g = h$ and is equal to F^k.
$(F^{\alpha_q,k})_{q\in\mathbb{N}}$ converges to F^k in $L^1(\Omega\times\mathbb{R}^2)$ when $q\to+\infty$. Indeed, given $\eta>0$, choose ρ_0 big enough so that

$$\| f^{0,2\rho_0+1} - f^{0,2\rho_0} \|_{L^1} < \eta \quad \text{and} \quad \| f^{0,2\rho_0} - F^k \|_{L^1} < \eta,$$

then q_0 big enough, so that

$$\| f^{\alpha_q,2\rho_0+1} - f^{0,2\rho_0+1} \|_{L^1} \leq \eta \quad \text{and} \quad \| f^{\alpha_q,2\rho_0} - f^{0,2\rho_0} \|_{L^1} \leq \eta, \quad q\geq q_0.$$

Then split $\| F^{\alpha_q,k} - F^k \|_{L^1}$ as follows,

$$\begin{aligned}
&\| F^{\alpha_q,k} - F^k \|_{L^1} \\
&\leq \| F^{\alpha_q,k} - f^{\alpha,2\rho_0} \|_{L^1} + \| f^{\alpha,2\rho_0} - f^{0,2\rho_0} \|_{L^1} + \| f^{0,2\rho_0} - F^k \|_{L^1} \\
&\leq \| f^{\alpha,2\rho_0+1} - f^{\alpha,2\rho_0} \|_{L^1} + 2\eta \quad \text{by} \quad (3.13) \\
&\leq \| f^{\alpha,2\rho_0+1} - f^{0,2\rho_0+1} \|_{L^1} + \| f^{0,2\rho_0+1} - f^{0,2\rho_0} \|_{L^1} + \| f^{0,2\rho_0} - f^{\alpha,2\rho_0} \|_{L^1} + 2\eta \\
&\leq 5\eta, \quad q\geq q_0.
\end{aligned}$$

It remains to prove (3.6)-(3.8). Multiplying (3.4) by $1+v^2$ and integrating over $\Omega\times\mathbb{R}^2$, leads to

$$\begin{aligned}
&\int_{\partial\Omega^-} |v\cdot n(z)|(1+v^2)F^k(z,v)d\sigma(z)dv \\
&\leq \int_{\partial\Omega^+} v\cdot n(z)(1+v^2)f_b(z,v)d\sigma(z)dv. \qquad (3.19)
\end{aligned}$$

Denote by (v_1,v_2) the components of v. Multiply (3.4) by v_1 and integrate it over $\Omega_a\times\mathbb{R}^2$, where Ω_a is the part of Ω with $z_1<a$. Set

$$S_a = \Omega\cap\{z_1=a\} \quad \text{and} \quad \partial\Omega_a = \partial\Omega\cap\bar{\Omega}_a.$$

This gives

$$\int_{S_a\times\mathbb{R}^2} v_1^2 F^k(a,z_2,v)dz_2dv = -\int_{\partial\Omega_a\times\mathbb{R}^2} v_1 v\cdot n(z)F^k(z,v)dzdv. \qquad (3.20)$$

Integrating (3.20) on $[l,L]$, where

$$l=\inf\{a; S_a\neq\emptyset\}, \quad L=\sup\{a; S_a\neq\emptyset\},$$

and using (3.19) leads to

$$\int_{\Omega\times\mathbb{R}^2} v_1^2 F^k(z,v)dzdv \le c_b. \tag{3.21}$$

$(\int v_2^2 F^k(z,v)dzdv)_{k\in\mathbb{N}^*}$ is analogously bounded from above. Thus the boundedness of energy holds. Recalling the small velocity cut-off χ_η, this in turn implies the boundedness of mass. The boundedness of the mass outflow in (3.8) follows from an integration of (3.4) on $\Omega\times\mathbb{R}^2$. Finally, Green's formula for $F^k \ln \frac{F^k}{1+\frac{F^k}{k}}$ implies that for some $c_b > 0$,

$$\begin{aligned}&\int_{\partial\Omega^-} |v\cdot n(z)|\big(F^k \ln F^k - (k+F^k)\ln(1+\frac{F^k}{k})\big)(z,v)d\sigma(z)dv\\ &+\int_{\Omega\times\mathbb{R}^2} D^k(z,v)dzdv \le c_b, \quad k\in\mathbb{N}^*.\end{aligned}$$

Moreover,

$$\begin{aligned}k\int_{\partial\Omega^-} \ln(1+\frac{F^k}{k})(z,v)\mid v\cdot n(z)\mid d\sigma(z)dv &\le \int_{\partial\Omega^-} F^k(z,v)\mid v\cdot n(z)\mid d\sigma(z)dv\\ &\le c_b.\end{aligned}$$

Hence

$$\int_{\partial\Omega^-} F^k \ln\frac{F^k}{1+\frac{F^k}{k}}(z,v)\mid v\cdot n(z)\mid d\sigma(z)dv + \int_{\Omega\times\mathbb{R}^2} D^k(z,v)dzdv \le c_b. \tag{3.22}$$

It holds that

$$\begin{aligned}&\int_{\partial\Omega^-} F^k \ln(1+\frac{F^k}{k})(z,v)\mid v\cdot n(z)\mid d\sigma(z)dv\\ &\le \int_{\partial\Omega^-,F^k\le k} F^k \ln(1+\frac{F^k}{k})(z,v)\mid v\cdot n(z)\mid d\sigma(z)dv\\ &+\int_{\partial\Omega^-,F^k\ge k} F^k \ln(1+\frac{F^k}{k})(z,v)\mid v\cdot n(z)\mid d\sigma(z)dv\\ &\le \ln 2\int_{\partial\Omega^-} F^k(z,v)\mid v\cdot n(z)\mid d\sigma(z)dv + \int_{\partial\Omega^-,F^k\ge k} F^k\ln\frac{2F^k}{k}(z,v)\mid v\cdot n(z)\mid d\sigma(z)dv\\ &\le c_b + \int_{\partial\Omega^-,F^k\ge k} F^k\ln F^k(z,v)\mid v\cdot n(z)\mid d\sigma(z)dv\\ &-\ln\frac{k}{2}\int_{\partial\Omega^-,F^k\ge k} F^k(z,v)\mid v\cdot n(z)\mid d\sigma(z)dv.\end{aligned}$$

Together with (3.22), this implies that

$$\int_{\partial\Omega^-,F^k\leq k} F^k \ln F^k(z,v) \mid v\cdot n(z)\mid d\sigma(z)dv + \ln\frac{k}{2}\int_{\partial\Omega^-,F^k\geq k} F^k \mid v\cdot n(z)\mid d\sigma(z)dv \leq c_b.$$

This ends the proof of Lemma 3.1. ∎

4 Compactness of the integrated collision frequency.

Denote by $Q^+(f,f)$ (resp. $\nu(f)$) the gain term (resp. the collision frequency) of the nonlinear Boltzmann collision operator,

$$\begin{aligned} Q^+(f,f)(z,v) &= \int Bf(z,v')f(z,v'_*)dv_*d\omega, \\ \nu(f)(z,v) &= \int Bf(z,v_*)dv_*d\omega, \end{aligned} \tag{4.1}$$

so that

$$Q(f,f) = Q^+(f,f) - f\nu(f).$$

Lemma 4.1
For any $V>1$, the sequence

$$\Big(\int_{-s^+(z,v)}^{s^-(z,v)} \nu(F^k)(z+sv,v)ds\Big)_{k\in\mathbb{N}^*}$$

is uniformly bounded by cV^2 on $\Omega\times\{v\in\mathbb{R}^2;\,\eta<|v|<V\}$.

Proof of Lemma 4.1.
For any $(Z,v)\in\partial\Omega^+$ with $\eta<|v|<V$, the truncation χ_η in B implies that

$$\begin{aligned} \nu(F^k)(Z+sv,v) &= \int_{|v_*|>\eta} B(v,v_*,\omega)F^k(Z+sv,v_*)dv_*d\omega \\ &\leq c\int \chi_\eta(v,v_*,\omega)(v^2+v_*^2)F^k(Z+sv,v_*)dv_* \\ &\leq c\frac{V^2+\eta^2}{\eta^4}\int v_*^2\sin^2\widehat{(v,v_*)}F^k(Z+sv,v_*)dv_*. \end{aligned} \tag{4.2}$$

Let $\Omega_{Z,v}$ be one of the two subsets of Ω split by the segment $[Z,Z+s^-(Z,v)v]$ and $\partial\Omega_{Z,v}=\partial\Omega\cap\bar\Omega_{Z,v}$. Let $v^\perp$ be one of the vectors orthogonal to v such

that $|v^\perp| = |v|$. Multiply (2.1) written in the variables (z, v_*) by $v_* \cdot \frac{v^\perp}{|v|}$ and integrate the resulting equation over $(z, v_*) \in \Omega_{Z,v} \times \mathbb{R}^2$. This gives

$$\begin{aligned}
&\int_0^{s^-(Z,v)} \int \frac{(v_* \cdot v^\perp)^2}{|v|^2} F^k(Z + sv, v_*) dv_* ds \\
&\leq \int_{\partial\Omega_{Z,v} \times \mathbb{R}^2} \big(v_* \cdot \frac{v^\perp}{|v|}\big)\big(v_* \cdot n(z)\big) F^k(z, v_*) d\sigma(z) dv_* \\
&\leq c_b, \quad (Z, v) \in \partial\Omega^+. \qquad (4.3)
\end{aligned}$$

Together with (4.2) this ends the proof of the lemma. ∎

Lemma 4.2
The sequence $(F^k)_{k\in\mathbb{N}^*}$ is weakly compact in L^1.

Proof of Lemma 4.2.
By (3.6) it is sufficient to prove that for any $V > 1$ the sequence $(F^k_{/\Omega\times\{v\in\mathbb{R}^2;|v|\leq V\}})_{k\in\mathbb{N}^*}$ is weakly compact in $L^1(\Omega \times \{v \in \mathbb{R}^2; \eta \leq |v| \leq V\})$. It follows from the exponential form of $F^k(z, v)$ from the outgoing boundary $\partial\Omega^-$,

$$\begin{aligned}
F^k(z, v) &= F^k(z^-(z, v), v) \exp\Big(\int_0^{s^-(z,v)} \nu(F^k)(z + sv, v) ds\Big) \\
&- \int_0^{s^-(z,v)} Q^+(F^k, F^k)(z + sv, v) \exp\Big(\int_0^s \nu(F^k)(z + rv, v) dr\Big) ds,
\end{aligned}$$

and Lemma 4.1 that

$$F^k(z, v) \leq c e^{cV^2} F^k(z^-(z, v), v), \ z \in \Omega, \ \eta \leq |v| \leq V. \qquad (4.4)$$

By (3.8), $(F^k/_{\partial\Omega^-})_{k\in\mathbb{N}^*}$ is weakly compact in $L^1_{|v\cdot n(Z)|}(\partial\Omega^-)$. This completes the proof of the lemma. ∎

Lemma 4.3
For $k \in \mathbb{N}^*$ and $\epsilon \in]0, 1[$, there is a subset $\Omega^{k,\epsilon}$ of characteristics of

$$\Omega \times \{v \in \mathbb{R}^2; \quad \eta < |v| < \frac{1}{\epsilon}\},$$

with measure smaller than $c\epsilon$, such that

$$F^k(z,v) \leq \frac{c}{\epsilon^3} e^{\frac{c}{\epsilon^2}}, \quad (z,v) \in \Omega \times \{v \in \mathbb{R}^2;\ \eta < |v| < \frac{1}{\epsilon}\} \setminus \Omega^{k,\epsilon}. \tag{4.5}$$

Proof of Lemma 4.3.
Let $\epsilon \in]0,1[$ be given. By the strict convexity of Ω and its C^1 regularity, the set

$$\omega_1^\epsilon := \{(Z,v) \in \partial\Omega \times \mathbb{R}^2;\ \eta < |v| < \frac{1}{\epsilon} \text{ and } -\epsilon^2 < v \cdot n(Z) < 0\} \tag{4.6}$$

is of measure smaller than $c\epsilon$ for some constant $c > 0$ and ϵ small enough. It follows from (3.8) that the measure of the subset of $\partial\Omega^-$ where

$$\eta < |v| < \frac{1}{\epsilon}, \quad v \cdot n(Z) < -\epsilon^2 \text{ and } F^k(Z,v) > \frac{1}{\epsilon^3},$$

is smaller than $c_b\epsilon$. Denote this set by $\omega_2^{k,\epsilon}$. Define $\Omega^{k,\epsilon}$ as

$$\Omega^{k,\epsilon} = \{(Z+sv,v);\quad (Z,v) \in \omega_1^\epsilon \cup \omega_2^{k,\epsilon},\quad s \in [-s^+(Z,v),0]\}. \tag{4.7}$$

Together with (4.4), this ends the proof of the lemma. ∎

For any $(k,\epsilon) \in \mathbb{N}^* \times]0,+\infty[$, denote by $\chi^{k,\epsilon}$ the characteristic function of $(\Omega^{k,\epsilon})^c$.

Lemma 4.4
For any $V > 0$ the sequence $(\int_{-s^+(z,v)}^{s^-(z,v)} \nu(F^k)(z+sv,v)ds)_{k\in\mathbb{N}^*}$ is compact in $L^1(\Omega \times \{v \in \mathbb{R}^2; |v| < V\})$.

Proof of Lemma 4.4.
Let $V > 0$ be given. By (3.6) and Lemma 4.2, it is sufficient to prove that for any $\epsilon > 0$ and $W > 0$, the sequence

$$\Big(\int_{-s^+(z,v)}^{s^-(z,v)} \int_{|v_*|<W} B\chi^{k,\epsilon}F^k(z+sv,v_*)dv_*ds\Big)_{k\in\mathbb{N}^*} \tag{4.8}$$

is compact in $L^1(\Omega \times \{v \in \mathbb{R}^2; |v| < V\})$. By (3.6) this sequence is bounded in L^1. Let us prove that it is uniformly equiintegrable w.r.t. the z variable.

For any $(\alpha,\beta)\in\mathbb{R}^2$, denote by $I(\alpha,\beta)$ the interval with end points α and β. For any $h\in\mathbb{R}^2$, split

$$\int_{-s^+(z+h,v)}^{s^-(z+h,v)}\int B\chi^{k,\epsilon}F^k(z+h+sv,v_*)dv_*ds$$
$$-\int_{-s^+(z,v)}^{s^-(z,v)}\int B\chi^{k,\epsilon}F^k(z+sv,v_*)dv_*ds \tag{4.9}$$

into

$$\int_{I(-s^+(z,v),-s^+(z+h,v))\cup I(s^-(z,v),s^-(z+h,v))}\int B\chi^{k,\epsilon}F^k(z+h+sv,v_*)dv_*ds, \tag{4.10}$$

which absolute value tends to zero when integrated over $\Omega\times\mathbb{R}^2$ and $h\to 0$ by the continuity of (s^+,s^-) on $\bar{\Omega}\times\{v;|v|\le V\}$, and

$$\int_{-s^+(z,v)}^{s^-(z,v)}\int B\chi^{k,\epsilon}\big(F^k(z+h+sv,v_*)-F^k(z+sv,v_*)\big)dv_*ds. \tag{4.11}$$

Almost every $(v,v_*)\in\mathbb{R}^2\times\mathbb{R}^2$ considered when integrating the absolute value of (4.11) over $\Omega\times\{v\in\mathbb{R}^2;|v|<V\}$ forms a basis in $\mathbb{R}^2$. Denote by $(a(h),b(h))$ the coordinates of h in this basis. Split (4.11) into

$$\begin{aligned}&\int_{-s^+(z,v)}^{s^-(z,v)}\int B\chi^{k,\epsilon}\big(F^k(z+h+sv,v_*)-F^k(z+b(h)v_*+sv,v_*)\big)dv_*ds\\&=\int_{-s^+(z,v)}^{s^-(z,v)}\int B\chi^{k,\epsilon}\big(F^k(z+h+sv,v_*)-F^k(z+h+(s-a(h))v,v_*)\big)dv_*ds\\&=\int_{I(-s^+(z,v)-a(h),-s^+(z,v))\cup I(s^-(z,v),s^-(z,v)-a(h))}\int B\chi^{k,\epsilon}F^k(z+h+sv,v_*)dv_*ds,\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}&\int_{-s^+(z,v)}^{s^-(z,v)}\int B\chi^{k,\epsilon}\big(F^k(z+b(h)v_*+sv,v_*)-F^k(z+sv,v_*)\big)dv_*ds\\&=\int_{-s^+(z,v)}^{s^-(z,v)}\int B\chi^{k,\epsilon}\big(f_{bk}(z^+(z+b(h)v_*+sv,v_*),v_*)-f_{bk}(z^+(z+sv,v_*),v_*)\big)dv_*ds\\&+\int_{-s^+(z,v)}^{s^-(z,v)}\int B\int_{I(0,b(h))}\chi^{k,\epsilon}Q_k(F^k,F^k)(z+sv+rv_*,v_*)drdv_*ds.\end{aligned} \tag{4.13}$$

When integrated over $\Omega \times \{v \in \mathbb{R}^2; |v| < V\}$, the limit when $h \to 0$ of the first term of (4.13) is zero by the integrability of f_b. Notice that $(\chi^{k,\epsilon} Q_k(F^k, F^k))_{k\in\mathbb{N}^*}$ is weakly compact in $L^1(\Omega \times \{v \in \mathbb{R}^2; |v| < V\})$, since

$$\chi^{k,\epsilon} Q_k(F^k, F^k) \leq \frac{1}{\ln \Lambda} D^k + \frac{c\Lambda}{\epsilon^3} e^{\frac{c}{\epsilon^2}} \nu(F^k), \quad \Lambda > 1.$$

When integrated over $\Omega \times \{v \in \mathbb{R}^2; |v| < V\}$, the limit when $h \to 0$ of the second term of (4.13) is zero by the weak L^1 compactness of $(\chi^{k,\epsilon} Q_k(F^k, F^k))_{k\in\mathbb{N}^*}$ and

$$\lim_{h\to 0} b(h) = 0, \tag{4.14}$$

uniformly on $\Omega \times \{v \in \mathbb{R}^2; |v| < V\}$.
The uniform equiintegrability w.r.t. the v variable of

$$\Big(\int_{-s^+(z,v)}^{s^-(z,v)} \int_{|v_*|<V} B\chi^{k,\epsilon} F^k(z + sv, v_*) dv_* ds \Big)_{k\in\mathbb{N}^*}$$

follows from similar arguments. ■

5 Passage to the limit in the approximations

For each $\epsilon > 0$, let F^ϵ be the weak L^1 limit of a subsequence of $(\chi^{k,\epsilon} F^k)_{k\in\mathbb{N}^*}$. $(F^\epsilon)_{\epsilon\in]0,1[}$ is non increasing with respect to decreasing ϵ and bounded in L^1. Let f be its strong L^1 limit when $\epsilon \to 0$. Notice that f is also the weak L^1 limit of $(F^k)_{k\in\mathbb{N}}$ when $k \to +\infty$.

For proving that f is a mild solution of (1.1)-(1.5), it is sufficient to prove that for any $\beta > 0$, there is a set X^β of characteristics with complementary set of measure smaller than $c\beta$, such that if χ_β denotes the corresponding characteristic function,

$$\begin{aligned}(\chi_\beta f)(z, v) &= (\chi_\beta f_b)(z^+(z, v), v) \\ &+ \int_{-s^+(z,v)}^{0} (\chi_\beta Q(f, f))(z + sv, v) ds, \quad (z, v) \in \Omega \times \mathbb{R}^2. \end{aligned} \tag{5.1}$$

This in turn is satisfied if for any test function $\varphi \in L^\infty(\bar{\Omega} \times \mathbb{R}^2)$, continuously differentiable along characteristics, with $v \cdot \nabla_z \varphi \in L^\infty(\Omega \times \mathbb{R}^2)$, compact

support and vanishing on $\partial\Omega^-$,

$$\int_{\Omega\times\mathbb{R}^2}\varphi\chi_\beta f(z,v)dzdv=\int_{\Omega\times\mathbb{R}^2}\varphi\chi_\beta f_b(z^+(z,v),v)dzdv$$
$$+\int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^0\big(\chi_\beta f\,v\cdot\nabla_z\varphi+\varphi\chi_\beta Q(f,f)\big)(z+sv,v)ds\,dzdv. \tag{5.2}$$

Let $\epsilon_0>0$ be such that the support of φ is included in $\bar\Omega\times\{v\in\mathbb{R}^2;|v|\le\frac{1}{\epsilon_0}\}$. Define the set X^β as follows. Using the weak $L^1(\Omega\times\mathbb{R}^2)$ compactness of $(F^k)_{k\in\mathbb{N}^*}$ and the weak $L^1(\partial\Omega^-)$ compactness of $(F^k_{/\partial\Omega^-})_{k\in\mathbb{N}^*}$, pass to the limit when $k\to+\infty$ in

$$F^k(z,v)\le ce^{\frac{c}{\epsilon_0^2}}F^k(z^-(z,v),v),\quad \text{a.a. } z\in\Omega,\quad \eta<|v|\le\frac{1}{\epsilon_0},\quad k\in\mathbb{N}^*. \tag{5.3}$$

It implies that

$$f(z,v)\le ce^{\frac{c}{\epsilon_0^2}}f(z^-(z,v),v),\quad \text{a.a. } z\in\Omega,\quad \eta<|v|\le\frac{1}{\epsilon_0}.$$

From here the proof follows the lines of the proof of Lemma 4.3, so that given $\beta>0$, X^β can be defined as a set of characteristics, with complementary set of measure smaller than $c\beta$, such that

$$f(z,v)\le\frac{c}{\beta^3}e^{c\left(\frac{1}{\epsilon_0^2}+\frac{1}{\beta^2}\right)},\quad \text{a.a. } (z,v)\in X^\beta. \tag{5.4}$$

The following lemma is a preliminary step to pass to the limit when $k\to+\infty$ in quadratic terms along the 'good characteristics' $(z+sv,v)$, $(z,v)\notin\Omega^{k,\epsilon}$.

Lemma 5.1
For any test function $\varphi\in L^\infty(\bar\Omega\times\mathbb{R}^2)$, continuously differentiable along characteristics, with $v\cdot\nabla_z\varphi\in L^\infty(\Omega\times\mathbb{R}^2)$, compact support and vanishing on $\partial\Omega^-$,

$$\lim_{k\to+\infty}\int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^0\varphi\chi_\beta\chi^{k,\epsilon}F^k\nu(F^k)(z+sv,v)ds\,dzdv$$
$$=\int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^0\varphi\chi_\beta F^\epsilon\nu(f)(z+sv,v)ds\,dzdv. \tag{5.5}$$

Proof of Lemma 5.1.
Since φ has compact support, one can restrict to the passage to the limit when $k \to +\infty$ and V fixed of

$$\int_{\Omega\times\{v\in\mathbb{R}^2;|v|<V\}} \int_{-s^+(z,v)}^0 \varphi\chi_\beta\chi^{k,\epsilon}F^k\nu(F^k)(z+sv,v)ds\,dzdv. \tag{5.6}$$

By an integration by parts,

$$\begin{aligned}
&\int_{\Omega\times\{v\in\mathbb{R}^2;|v|<V\}} \int_{-s^+(z,v)}^0 \varphi\chi_\beta\chi^{k,\epsilon}F^k\nu(F^k)(z+sv,v)ds\,dzdv\\
&= \int_{\Omega\times\{v\in\mathbb{R}^2;|v|<V\}} \int_{-s^+(z,v)}^0 \varphi\chi_\beta\chi^{k,\epsilon}f_{bk}(z^+(z,v),v)\\
&\qquad\qquad \Big(\int_{-s^+(z,v)}^0 \nu(F^k)(z+sv,v)ds(z+sv,v)ds\Big)\,dzdv\\
&+ \int_{\Omega\times\{v\in\mathbb{R}^2;|v|<V\}} \int_{-s^+(z,v)}^0 \chi_\beta\chi^{k,\epsilon}\big(v\cdot\nabla_z\varphi\,F^k+\varphi Q_k(F^k,F^k)\big)(z+sv,v)\\
&\qquad\qquad \Big(\int_s^0 \nu(F^k)(z+rv,v)dr\Big)\,ds\,dzdv.
\end{aligned} \tag{5.7}$$

The change of variables

$$(z,s)\to(Z,s)=(z+sv,s), \tag{5.8}$$

moves the domain $\Omega\times]-s^+(z,v),0[$ into the domain

$$\{(Z,s)\quad;\quad Z-sv\in\Omega\quad\text{and}\quad s<0\},\quad\text{i.e.}\quad (Z,-s)\in\Omega\times]0,s^-(Z,v)[.$$

Hence,

$$\begin{aligned}
&\int_{\Omega\times\{v\in\mathbb{R}^2;|v|<V\}} \int_{-s^+(z,v)}^0 \varphi\chi_\beta\chi^{k,\epsilon}F^k\nu(F^k)(z+sv,v)ds\,dzdv\\
&= \int_{\Omega\times\{v\in\mathbb{R}^2;|v|<V\}} \int_{-s^+(z,v)}^0 \varphi\chi_\beta\chi^{k,\epsilon}f_{bk}(z^+(z,v),v)\\
&\qquad\qquad \Big(\int_{-s^+(z,v)}^0 \nu(F^k)(z+sv,v)ds(z+sv,v)ds\Big)\,dzdv\\
&+ \int_{\Omega\times\{v\in\mathbb{R}^2;|v|<V\}} \chi_\beta\chi^{k,\epsilon}\big((v\cdot\nabla_z\varphi)F^k+\varphi Q_k(F^k,F^k)\big)(Z,v)\\
&\qquad\qquad \Big(\int_0^{s^-(Z,v)} (s^-(Z,v)-r)\nu(F^k)(Z+rv,v)dr\Big)\,dZdv.
\end{aligned} \tag{5.9}$$

Lemma 5.1 follows from the passage to the limit when $k \to +\infty$ in (5.9). It uses the weak L^1 compactness of $(\chi^{k,\epsilon}F^k)$ and $(\chi^{k,\epsilon}Q_k(F^k,F^k))$, the strong L^1 compactness of $\big(\int_0^{s^-(Z,v)}(s^-(Z,v)-r)\nu(F^k)(Z+rv,v)dr\big)$, and the L^∞ boundedness of $\big(\chi^{k,\epsilon}(Z,v)\int_0^{s^-(Z,v)}(s^-(Z,v)-r)\nu(F^k)(Z+rv,v)dr\big)$. An integration by parts back in the s variable is finally performed, taking into account that

$$\chi^{k,\epsilon}Q_k(F^k,F^k)(z+sv,v)=\frac{d}{ds}\big((\chi^{k,\epsilon}F^k)(z+sv,v)\big).$$

■

Lemma 5.2
f is a solution of (1.1)-(1.5)*, i.e. for test functions φ defined as in Lemma 5.1,*

$$\int_{\Omega\times\mathbb{R}^2}\varphi\chi_\beta f(z,v)dzdv=\int_{\Omega\times\mathbb{R}^2}\varphi\chi_\beta f_b(z^+(z,v),v)dzdv$$
$$+\int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^0\big(\chi_\beta f\,v\cdot\nabla_z\varphi+\varphi\chi_\beta Q(f,f)\big)(z+sv,v)ds\,dzdv. \tag{5.10}$$

Proof of Lemma 5.2.
For $\epsilon\in]0,\epsilon_0[$, write the mild form of $\varphi\chi_\beta\chi^{k,\epsilon}F^k$ and integrate it on $\Omega\times\mathbb{R}^2$. It results

$$\int_{\Omega\times\mathbb{R}^2}\varphi\chi_\beta\chi^{k,\epsilon}F^k(z,v)dzdv=\int_{\Omega\times\mathbb{R}^2}\varphi\chi_\beta\chi^{k,\epsilon}f_{bk}(z^+(z,v),v)dzdv$$
$$+\int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^0\chi_\beta\chi^{k,\epsilon}F^k\,v\cdot\nabla_z\varphi(z+sv,v)ds\,dzdv$$
$$+\int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^0\varphi\chi_\beta\chi^{k,\epsilon}\big(Q^+(F^k,F^k)-F^k\nu(F^k)\big)(z+sv,v)ds\,dzdv. \tag{5.11}$$

By the weak L^1 compactness of $(F^k)_{k\in\mathbb{N}^*}$ and the linearity with respect to $\chi^{k,\epsilon}F^k$ of the first two lines of (5.11), their passage to the limit when $k\to+\infty$ is straightforward. The passage to the limit when $k\to+\infty$ in the last term of (5.11) follows from Lemma 5.1. Finally, using monotonicity

arguments together with the L^∞ boundedness of $\chi_\beta f$ allows to pass to the limit when $\epsilon \to 0$ in

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^2} \varphi\chi_\beta F^\epsilon(z,v)dzdv - \int_{\Omega\times\mathbb{R}^2} \varphi\chi_\beta f_b(z^+(z,v),v)dzdv \\
&- \int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^0 \chi_\beta F^\epsilon\, v\cdot\nabla_z\varphi(z+sv,v)ds\,dzdv \\
&+ \int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^0 \varphi\chi_\beta F^\epsilon \nu(f)(z+sv,v)ds\,dzdv, \qquad (5.12)
\end{aligned}$$

and obtain the limit

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^2} \varphi\chi_\beta f(z,v)dzdv - \int_{\Omega\times\mathbb{R}^2} \varphi\chi_\beta f_b(z^+(z,v),v)dzdv \\
&- \int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^0 \chi_\beta f\, v\cdot\nabla\varphi(z+sv,v)ds\,dzdv \\
&+ \int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^0 \varphi\chi_\beta f\nu(f)(z+sv,v)ds\,dzdv. \qquad (5.13)
\end{aligned}$$

Let us prove that

$$\begin{aligned}
&\lim_{\epsilon\to 0}\lim_{k\to+\infty}\int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^0 \varphi\chi_\beta\chi^{k,\epsilon}Q^+(F^k,F^k)(z+sv,v)ds\,dzdv \\
&= \int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^0 \varphi\chi_\beta Q^+(f,f)(z+sv,v)ds\,dzdv. \qquad (5.14)
\end{aligned}$$

For any (v,v_*,ω), the change of variables

$$(z,s)\to(Z,s)=(z+sv',s), \qquad (5.15)$$

moves the domain $\Omega\times]-s^+(z,v'),0[$ into the domain

$$\{(Z,s) \quad ; \quad Z-sv'\in\Omega \quad \text{and} \quad s<0\}, \quad \text{i.e.} \quad (Z,-s)\in\Omega\times]0,s^-(Z,v')[.$$

Hence,

$$\int_{\Omega\times\mathbb{R}^2}\int_{-s^+(z,v)}^{0}\varphi\chi_\beta\chi^{k,\epsilon}Q^+(F^k,F^k)(z+sv,v)ds\,dzdv$$
$$=\int\Big(\int_\Omega\int_{-s^+(z,v')}^{0}\varphi\chi_\beta\chi^{k,\epsilon}(z+sv',v')B$$
$$\frac{F^k}{1+\frac{F^k}{k}}(z+sv',v)\frac{F^k}{1+\frac{F^k}{k}}(z+sv',v_*)dsdz\Big)\,dvdv_*d\omega$$
$$=\int_{\Omega\times\mathbb{R}^2}\Big(\int\varphi\chi_\beta\chi^{k,\epsilon}s^-(Z,v')B\frac{F^k}{1+\frac{F^k}{k}}(Z,v_*)dv_*d\omega\Big)\frac{F^k}{1+\frac{F^k}{k}}(Z,v)dZdv. \tag{5.16}$$

One can restrict to the study of the limit of

$$\int_{\Omega\times\mathbb{R}^2}\Big(\int_{\{v_*\in\mathbb{R}^2;|\sin(\widehat{v,v_*})|>\mu\}\times\mathcal{S}^1}\varphi\chi_\beta\chi^{k,\epsilon}s^-(Z,v')B\frac{F^k}{1+\frac{F^k}{k}}(Z,v_*)dv_*d\omega\Big)$$
$$\frac{F^k}{1+\frac{F^k}{k}}(Z,v)dZdv,\quad \mu\in]0,1[, \tag{5.17}$$

since

$$\int_{\Omega\times\mathbb{R}^2}\Big(\int_{\{v_*\in\mathbb{R}^2;|\sin(\widehat{v,v_*})|<\mu\}\times\mathcal{S}^1}\varphi\chi_\beta\chi^{k,\epsilon}s^-(Z,v')B\frac{F^k}{1+\frac{F^k}{k}}(Z,v_*)dv_*d\omega\Big)$$
$$\frac{F^k}{1+\frac{F^k}{k}}(Z,v)dZdv$$
$$\le\frac{c}{\ln\Lambda}+\frac{c\Lambda\mu}{\epsilon^3}e^{\frac{c}{\epsilon^2}}\quad\Lambda>1,\quad\mu\in]0,1[. \tag{5.18}$$

The sequence

$$\Big((Z,v)\to F^k(Z,v)\int_{\{v_*\in\mathbb{R}^2;|\sin(\widehat{v,v_*})|>\mu\}\times\mathcal{S}^1}\varphi\chi_\beta\chi^{k,\epsilon}s^-(Z,v')BF^k(Z,v_*)dv_*d\omega\Big)_{k\in\mathbb{N}^*}$$

is weakly compact in L^1. Indeed, using the change of variables $v_*\to v'_*$ for every (v,ω), which holds since $|\sin(\widehat{v,v_*})|>\mu$,

$$F^k(Z,v)\int\varphi\chi_\beta\chi^{k,\epsilon}s^-(Z,v')BF^k(Z,v_*)dv_*d\omega$$
$$\le\frac{c}{\ln\Lambda}D^k(Z,v)+c\frac{\Lambda}{\epsilon^3}e^{\frac{c}{\epsilon^2}}\int F^k(Z,v'_*)dv'_*,\quad\Lambda>1,$$

where $(D^k)_{k\in\mathbb{N}^*}$ is defined in (3.1) and uniformly bounded in L^1 by (3.7). Consequently one can restrict to the passage to the limit when $k\to+\infty$ in

$$\int_{(Z,v)\in\Omega\times\mathbb{R}^2;s^-(Z,v)>\alpha}\Big(\int_{\{v_*\in\mathbb{R}^2;|\sin(\widehat{v,v_*})|>\mu\}\times\mathcal{S}^1}\varphi\chi_\beta\chi^{k,\epsilon}s^-(Z,v')BF^k(Z,v_*)dv_*d\omega\Big)$$
$$\big(\chi^{k,\epsilon_1}F^k\big)(Z,v)dZdv,\quad(\alpha,\epsilon_1)\in]0,1[^2.\quad(5.19)$$

Moreover,

$$\begin{aligned}
&\int_{(Z,v);s^-(Z,v)>\alpha}\Big(\int_{\{v_*\in\mathbb{R}^2;|\sin(\widehat{v,v_*})|<\mu\}\times\mathcal{S}^1}\varphi\chi_\beta\chi^{k,\epsilon}s^-(Z,v')BF^k(Z,v_*)dv_*d\omega\Big)\\
&\qquad\big(\chi^{k,\epsilon_1}F^k\big)(Z,v)dZdv\\
&=\int_{\{v_*\in\mathbb{R}^2;|\sin(\widehat{v,v_*})|<\mu\}\times\mathcal{S}^1}\Big(\int_{Z;s^-(Z,v)>\alpha}\int_0^{s^-(Z,v)}\varphi\chi_\beta\chi^{k,\epsilon}s^-(Z,v')\frac{\chi^{k,\epsilon_1}F^k}{s^-}(Z,v)\\
&\qquad F^k(Z,v_*)dsdZ\Big)Bdvdv_*d\omega\\
&=\int_{\{v_*\in\mathbb{R}^2;|\sin(\widehat{v,v_*})|<\mu\}\times\mathcal{S}^1}\Big(\int_\Omega\int_{\max\{0,\alpha-s^-(z,v)\}}^{s^-(z,v)}\varphi\chi_\beta\chi^{k,\epsilon}s^-(z+sv,v')\frac{\chi^{k,\epsilon_1}F^k}{s^-}(z+sv,v)\\
&\qquad F^k(z+sv,v_*)dsdz\Big)Bdvdv_*d\omega\\
&=\int_{\Omega\times\mathbb{R}^2}\Big(\int_{\max\{0,\alpha-s^-(z,v)\}}^{s^-(z,v)}\frac{\chi^{k,\epsilon_1}F^k}{s^-}(z+sv,v)\\
&\qquad\Big[\int B\varphi\chi_\beta\chi^{k,\epsilon}s^-(z+sv,v')F^k(z+sv,v_*)dv_*\Big]ds\Big)dzdv.
\end{aligned}\quad(5.20)$$

As in the proof of Lemma 5.1,

$$\begin{aligned}
&\lim_{k\to+\infty}\int_{\Omega\times\mathbb{R}^2}\Big(\int_{\max\{0,\alpha-s^-(z,v)\}}^{s^-(z,v)}\frac{\chi^{k,\epsilon_1}F^k}{s^-}(z+sv,v)\\
&\qquad\Big[\int B\varphi\chi_\beta\chi^{k,\epsilon}s^-(z+sv,v')F^k(z+sv,v_*)dv_*\Big]ds\Big)dzdv\\
&=\int_{\Omega\times\mathbb{R}^2}\Big(\int_{\max\{0,\alpha-s^-(z,v)\}}^{s^-(z,v)}\frac{F^{\epsilon_1}}{s^-}(z+sv,v)\\
&\Big[\int_{\{v_*\in\mathbb{R}^2;|\sin(\widehat{v,v_*})|<\mu\}\times\mathcal{S}^1}B\varphi\chi_\beta s^-(z+sv,v')f(z+sv,v_*)dv_*\Big]ds\Big)dzdv\\
&=\int_{(Z,v);s^-(Z,v)>\alpha}\Big(\int_{\{v_*\in\mathbb{R}^2;|\sin(\widehat{v,v_*})|<\mu\}\times\mathcal{S}^1}\varphi\chi_\beta s^-(Z,v')Bf(Z,v_*)dv_*d\omega\Big)\\
&\qquad F^{\epsilon_1}(Z,v)dZdv.
\end{aligned}\quad(5.21)$$

Passing to the limit when $(\alpha, \epsilon_1, \mu) \to (0, 0, 0)$ in (5.21) leads to (5.14). ■

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