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# Lyapunov functions for linear damped wave equations in one-dimensional space with dynamic boundary conditions.

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## Abstract

We establish the exponential decay of the solutions of the damped wave equations in one-dimensional space where the damping coefficient is a nowhere-vanishing function of space. The considered PDE is associated with several dynamic boundary conditions, also referred to as Wentzell/Ventzel boundary conditions in the literature. The analysis is based on the determination of appropriate Lyapunov functions and some further analysis. This result is associated with a regulation problem inspired by a real experiment with a proportional-integral control. Some numerical simulations and additional results on closed wave equations are also provided.

*Key words:* Infinite dimensional system, regulation, output feedback control.

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The wave equation is one of the classical partial differential equations. The actual reason is that the wave equation is the continuous pendant of Newton's second law of motion, i.e., where momentum is equal to the sum of the forces. As a consequence, it is also linked with the Euler-Lagrange framework, and therefore with the principle of least action. For stationary systems, the energy is conserved and the action (or Lagrangian) is stationary. Other physical phenomena are therefore associated with the wave equation, such that electromagnetic law, and quantum phenomena with the Klein-Gordon equation.

In the control community, the wave equation has been mainly used for the modelization, estimation, and control of mechanical vibration and deformation phenomena. The regulation and control problem applied on the one dimensional wave equation with dynamic boundary condition has attracted the attention of many researchers in the control community: crane regulation [8], [7], [11], and [5], hanging cable immersed in water [4], drilling torsional vibrations [25], [31], [1], [32], piezoelectric control [18], and flexible structure [13]. There are

nowadays two main classes of issues : on the one hand, longitudinal variation with for example overhead crane and underwater cable, and on the other hand torsional variation with drilling string dynamics. The difference is on the control objective: one aims at controlling the position in the first case, and instead the velocity in the second case.

The behavior of the wave equation is strongly related to its boundary conditions. In the case of classical boundary condition (i.e., Dirichlet, Neumann, Robin) that issue is well-understood in the linear case and without high-order terms. Particular terms at one boundary can compensate for anti-damping terms at other boundaries and even in the domain, for example, see [28], [27] and [23]. Moreover, there are cases where even if the energy of the one-dimensional linear wave equation decreases along trajectories, it still does not decay exponentially [17, Section 4].

The wave equation under consideration is subject to two dynamic boundary conditions. This model results from an identification problem associated with a laboratory experiment [24].

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## 1 Problem statement.

The considered system is defined for  $t \geq 0$  and for  $x \in (0, 1)$ , by

$$\begin{cases} v_{tt}(t, x) = (a(x)v_x)_x(t, x) - q(x)v_t(t, x) + f(x), & (1a) \\ v_{tt}(t, 1) = -\beta_1 v_x(t, 1) - q_1 v_t(t, 1) + U(t) + f_1, & (1b) \\ v_{tt}(t, 0) = \mu_1 v_x(t, 0) - \gamma_1 v_t(t, 0) + f_2, & (1c) \\ v(0, \cdot) = v_0, \quad v_t(0, \cdot) = v_1. & (1d) \end{cases}$$

We assume that

- ( $h_1$ ) The function  $a : [0, 1] \rightarrow \mathbb{R}_+^*$  is in  $W^{1,\infty}(0, 1)$  and that there exist  $\underline{a}, \bar{a} > 0$  such that  $\underline{a} \leq a(\cdot) \leq \bar{a}$  a.e. on  $[0, 1]$ . This function is associated with the mass and elasticity of the wave and it is also linked with the velocity.
- ( $h_2$ ) The function  $q : [0, 1] \rightarrow \mathbb{R}_+^*$ , describing the in-domain damping is in  $L^\infty(0, 1)$  and satisfies  $\underline{q} \leq q(\cdot) \leq \bar{q}$  a.e. on  $[0, 1]$  for some  $\underline{q}, \bar{q} > 0$ .
- ( $h_3$ ) The constants  $\beta_1, \gamma_1, \mu_1$  are positive real numbers, and  $q_1$  is real.
- ( $h_4$ )  $U(t)$  is the control input. The source terms  $f(\cdot)$  is in  $L^\infty(0, 1)$ , and the real constants  $f_1, f_2$  are unknown and therefore they are not used in the computation of the control law  $U(t)$ .

The regularity of  $a$  follows by classical arguments. In the sequel, we need  $a(\cdot)u_x(t, \cdot)$  to be in  $H^1(0, 1)$ . To be more precise everything will be the same as in the constant parameters case if  $a(\cdot)u_x(t, \cdot)$  and  $u_x(t, \cdot)$  have the same regularity. To get strong solutions for (1), one needs to have that  $u \in H^2(0, 1)$ . Next, it can be easily shown that if  $a \in W^{1,\infty}$  and  $u \in H^2(0, 1)$  then  $a(\cdot)u_x(t, \cdot)$  is in  $H^1(0, 1)$ . Note that this is just a sufficient condition for regularity and we refer the reader to [30, Chapter 21] for more details. In the sequel, we also need  $qu_t(t, \cdot)^2$  need to be integrable, this means  $q \in L^\infty(0, 1)$ . For  $f$  we actually only need it to be integrable, it holds nonetheless  $L^\infty(0, 1) \subset L^1(0, 1)$ .

The objective of the paper is to regulate  $v_t(\cdot, t)$  to  $v_1^{\text{ref}}$ , by means of a proportional-integral (PI) control law using the measurement of the velocity collocated with the actuation,  $v_t(1, t)$ , in other words, the control  $U(t)$  should take the form

$$U(t) := -k(v_t(1, t) - v_1^{\text{ref}}) - \alpha_2 \int_0^t (v_t(1, t) - v_1^{\text{ref}}) dt. \quad (2)$$

This can be equivalently written as

$$\begin{cases} U(t) = -k(v_t(1, t) - v_1^{\text{ref}}) - \alpha_2 \eta_v(t), & (3a) \\ \dot{\eta}_v(t) = v_t(1, t) - v_1^{\text{ref}}, \quad \eta_v(0) = 0. & (3b) \end{cases}$$

In the literature boundary conditions of the type (1b)-(1c) can be recasted as Wentzell's boundary conditions

[12]. It involves a modification of the usual state space which in our case requires the addition of two finite-dimensional state variables, in a similar way as in [26], [7], [11] and [5]. When the wave equation is more than a one-dimensional, the reader is referred to [12] and references therein.

The closest approach associated with the present paper is [31] where the velocity regulation with a PI is considered. However, the controlled boundary condition considered in [31] is not a second order dynamic one, and thus is different from the one considered in this paper. Nevertheless, the boundary condition considered in [31] implies the exponential stability even with small viscous anti-damping at the boundary opposite to the actuation. In the case under consideration, only viscous damping at the opposite boundary is considered, and exponential stability is achieved. In [5] the wave equation is subject to two dynamic boundary conditions but they focus on the position stabilization, and no viscous terms are considered.

PI controllers have been successfully and recently used in order to regulate linear and non-linear PDE, see [6], [16]. An identification procedure has been presented in [24] for the system (1) without source terms on experimental data. This means that the considered problem can be associated with an experimental setup. A first study has been made on this system in [21] using classical form a Lyapunov functional but it failed to prove the exponential stability. Only asymptotic stability was established, by using the LaSalle invariance principle.

This paper provides a new term in the Lyapunov functional and an associated methodology, for the present particular setup. In Section 4, the proof is compared with the existing result, and we study some associated boundary conditions. The latter stage on the paper deals with numerical simulations. The numerical scheme is not derived from the usual approximation of space and time derivatives. We used the fact that the wave equation can be derived from the Lagrangian and the least action principle to approximate the system space energy by a finite dimensional continuous time Euler-Lagrange equation. The finite dimensional continuous time system is then numerically solved by using symplectic integrators. This suggested numerical scheme is new up to the authors' knowledge and provides an interesting alternative compares to more standard discretization schemes.

**Notations:** If  $I$  is an interval of real numbers,  $L^2(I; \mathbb{R})$  denotes (the class of equivalence of) square-integrable functions from  $I$  to  $\mathbb{R}$ . Moreover  $L^2([0, 1]; \mathbb{R})$  is abusively denoted  $L^2(0, 1)$ . Furthermore  $H^n$  denotes the Sobolev space  $W^{n,2}$ , i.e.,

$$u \in H^1 \Leftrightarrow u \in L^2, \quad u' \in L^2, \quad (4)$$

in which  $u'$  denotes the derivative of  $u$ .

## 2 Main result

Consider the following change of variable for all  $(x, t) \in [0, 1] \times [0, \infty)$

$$\begin{aligned} u(t, x) &:= v(t, x) - tv_1^{\text{ref}} \\ &\quad + \int_0^x \frac{1}{a(s)} \int_0^s [-v_1^{\text{ref}}q(\chi) + f(\chi)]d\chi ds \\ &\quad + \frac{a(0)}{\mu_1} [-\gamma_1 v_1^{\text{ref}} + f_2] \int_0^x \frac{1}{a(s)} ds, \end{aligned} \quad (5)$$

$$\begin{aligned} \eta_2(t) &:= \eta_v(t) - \frac{\beta_1}{\alpha_2 a(1)} \int_0^1 [-v_1^{\text{ref}}q(s) + f(s)]ds \\ &\quad - \frac{\beta_1 a(0)}{\alpha_2 \mu_1 a(1)} [-\gamma_1 v_1^{\text{ref}} + f_2] + \frac{q_1 v_1^{\text{ref}} - f_1}{\alpha_2}. \end{aligned} \quad (6)$$

Direct computations yield that the variable  $u(x, t)$  is the solution of the following system:

$$\begin{cases} u_{tt} - (a(x)u_x)_x = -q(x)u_t, & (t, x) \in \mathbb{R}^+ \times (0, 1), & (7a) \\ u_t(t, 1) = \eta_1(t) & (7b) \\ u_t(t, 0) = \xi_1(t) & (7c) \\ \dot{\eta}_1(t) = -\alpha_1 \eta_1(t) - \alpha_2 \eta_2(t) - \beta_1 u_x(t, 1), & (7d) \\ \dot{\eta}_2(t) = \eta_1(t), & (7e) \\ \dot{\xi}_1(t) = -\gamma_1 \xi_1(t) + \mu_1 u_x(t, 0), & (7f) \\ u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1 \quad \text{on } (0, 1), & (7g) \\ \eta_1(0) = \eta_0, \quad \eta_2(0) = \eta_{2,0} \quad \xi_1(0) = \xi_0. & (7h) \end{cases}$$

where  $\alpha_1 = k_p + q$ , and  $k_p$  is chosen such that  $\alpha_1$  is positive.

Consider the following Hilbert spaces

$$X_w := H^1((0, 1); \mathbb{R}) \times L^2((0, 1); \mathbb{R}) \times \mathbb{R}^3, \quad (8)$$

$$X_s := H^2((0, 1); \mathbb{R}) \times H^1((0, 1); \mathbb{R}) \times \mathbb{R}^3, \quad (9)$$

The wave equation is associated with the following abstract problem

$$\begin{cases} \dot{\mathcal{X}}(t) + \mathcal{A}\mathcal{X}(t) = 0, & (10a) \\ \mathcal{X}(0) = \mathcal{X}_0 \in \text{Dom}(\mathcal{A}) \subset X_s \subset X_w, & (10b) \end{cases}$$

in which

$$\forall z \in \text{Dom}(\mathcal{A}), \quad \mathcal{A}z := \begin{bmatrix} -z_2 \\ -(az_1')' + qz_2 \\ \alpha_1 z_3 + \alpha_2 z_4 + \beta_1 z_1'(1) \\ -z_3 \\ \gamma_1 z_5 - \mu_1 z_1'(0) \end{bmatrix}, \quad (11)$$

and

$$\text{Dom}(\mathcal{A}) := \{z \in X_s; z_2(1) = z_3, z_2(0) = z_5\}. \quad (12)$$

Our first result goes as follows.

**Theorem 1** *The abstract problem is well-posed and for  $\mathcal{X}_0 \in \text{Dom}(\mathcal{A})$ , the solution of (7) belongs to*

$$\mathcal{X} \in C^1([0, \infty); X_w) \cap C([0, \infty); \text{Dom}(\mathcal{A})). \quad (13)$$

$X_w$  is the state space of weak solutions and the Hilbert space considered. Note that  $\text{Dom}(\mathcal{A})$  is dense in  $X_w$ .

The proof is based on finding a transformation such that the abstract problem is associated with a linear maximal monotone operator. Then the conclusion is drawn by using Hille-Yosida theorem. Details are provided in the appendix.

The state is

$$\begin{aligned} \mathcal{X}(t) &:= [u(t, \cdot), u_t(t, \cdot), \\ &\quad \eta_1(t), \eta_2(t), \xi_1(t)] \in \text{Dom}(\mathcal{A}) \subset X_s. \end{aligned} \quad (14)$$

We define the *energy*  $E_u$  of a solution of (7) as

$$E_u(t) := \frac{1}{2} \int_0^1 (u_t(t, x)^2 + a(x)u_x(t, x)^2) dx, \quad t \geq 0. \quad (15)$$

Note that this energy is invariant by translations with constants, i.e.,  $E_u = E_v$  if  $u - v$  is a constant function. Moreover, the absolutely continuous function  $u(\cdot, 1) - \eta_2(\cdot)$  is constant along a trajectory of (7) and equal to  $u_*$  where

$$u_* := u_0(1) - \eta_2(0). \quad (16)$$

Our objective is to establish the exponential stability of the trajectory with respect to the following attractor

$$S := \{z \in X_w, z_1(\cdot) = d, d \in \mathbb{R}, z_2(\cdot) = 0, z_3 = 0, z_4 = 0, z_5 = 0\}. \quad (17)$$

This attractor is the kernel of the following functional

$$\begin{aligned} \Gamma(\mathcal{X}(t)) &:= \int_0^1 [u_t(t, x)^2 + u_x(t, x)^2] dx \\ &\quad + \eta_1(t)^2 + \eta_2(t)^2 + \xi_1(t)^2, \end{aligned} \quad (18)$$

indeed it holds

$$\Gamma(z) = 0 \Leftrightarrow z \in S. \quad (19)$$

We establish the following result.

**Theorem 2** Consider the 1D wave equation (7) with the assumptions  $(h_1)$ ,  $(h_2)$ ,  $(h_3)$ ,  $(h_4)$ , and with  $\alpha_2, \alpha_1 > 0$ . Then, there exist a positive constant  $\rho$ , and a positive constant  $M$  such that, for every weak solution  $\mathcal{X}$ , it holds,

$$\Gamma(\mathcal{X}(t)) \leq M\Gamma(\mathcal{X}(0))e^{-\rho t}. \quad (20)$$

and the system is exponentially stable towards the attractor  $S$ .

In addition it holds that  $\max_{x \in [0,1]} |u(t, x) - u_*|$  tends exponentially to zero as  $t$  tends to infinity, with a decay rate larger than or equal to  $\rho$ .

### 3 Proof of Theorem 2

This proof follows a standard strategy: the result is first established for strong solutions by the determination of Lyapunov functions verifying an appropriate differential inequality, and then it is extended to weak solutions by a classical density argument. Hence, in the sequel, solutions of (7) are all assumed to be strong.

We start with the time derivative of  $E_u$  along a strong solution. It holds for  $t \geq 0$

$$\begin{aligned} \dot{E}_u &= - \int_0^1 q u_t^2 dx + a(1)\eta_1(t)u_x(t, 1) \\ &\quad - a(0)\xi_1(t)u_x(t, 0). \end{aligned} \quad (21)$$

One also has, for  $t \geq 0$ , after using (7d)

$$\begin{aligned} a(1)\eta_1(t)u_x(t, 1) &= -\frac{a(1)}{\beta_1}\eta_1(t)\left(\dot{\eta}_1(t) + \alpha_1\eta_1(t) \right. \\ &\quad \left. + \alpha_2\eta_2(t)\right) \\ &= -\frac{d}{dt}\left(\frac{a(1)}{2\beta_1}(\eta_1^2(t) + \alpha_2\eta_2^2(t))\right) \\ &\quad - \frac{a(1)\alpha_1}{\beta_1}\eta_1^2(t). \end{aligned} \quad (22)$$

Similarly, one also has, for  $t \geq 0$ , after using (7f)

$$\begin{aligned} -a(0)\xi_1(t)u_x(t, 0) &= -\frac{a(0)}{\mu_1}\xi_1(t)\left(\dot{\xi}_1(t) + \gamma_1\xi_1(t)\right) \\ &= -\frac{d}{dt}\left(\frac{a(0)}{2\mu_1}\xi_1^2(t)\right) - \frac{a(0)\gamma_1}{\mu_1}\xi_1^2(t). \end{aligned} \quad (23)$$

Define for  $t \geq 0$

$$F(\mathcal{X}(t)) := E_u(t) + \frac{a(1)}{2\beta_1}\eta_1^2(t) + \frac{a(0)}{2\mu_1}\xi_1^2(t). \quad (24)$$

Then, by gathering (21), (22) and (23), one deduces that, for  $t \geq 0$ ,

$$\begin{aligned} \frac{d}{dt}\left(F + \frac{a(1)\alpha_2}{2\beta_1}\eta_2^2(t)\right) &= - \int_0^1 q u_t^2 dx \\ &\quad - \frac{a(1)\alpha_1}{\beta_1}\eta_1^2(t) - \frac{a(0)\gamma_1}{\mu_1}\xi_1^2(t). \end{aligned} \quad (25)$$

We next consider an extra term which will be added in the candidate Lyapunov function in the sequel. From (16) it holds that

$$\eta_2(t) = u(t, 1) - u_*, \quad t \geq 0. \quad (26)$$

Set

$$\xi_2(t) := u(t, 0) - u_*, \quad t \geq 0. \quad (27)$$

One has, for  $t \geq 0$ , that

$$\begin{aligned} \frac{d}{dt}\left(\int_0^1 (u - u_*)u_t dx\right) &= \int_0^1 u_t^2 + \int_0^1 (u - u_*)u_{tt} \\ &= \int_0^1 u_t^2 + \int_0^1 (u - u_*)(au_x)_x - \int_0^1 q(u - u_*)u_t \\ &= \int_0^1 u_t^2 - \int_0^1 au_x^2 - \frac{d}{dt}\left(\int_0^1 \frac{q}{2}(u - u_*)^2 dx\right) \\ &\quad + a(1)\eta_2 u_x(t, 1) - a(0)\xi_2 u_x(t, 0). \end{aligned} \quad (28)$$

Using (7d) and (7f), one deduces after computations similar to those performed to get (22) and (23), that, for  $t \geq 0$ ,

$$\begin{aligned} \eta_2 u_x(t, 1) &= \frac{-\alpha_2\eta_2^2(t) + \eta_1^2(t)}{\beta_1} \\ &\quad - \frac{d}{dt}\left(\frac{\frac{\alpha_1}{2}\eta_2^2(t) + \eta_1(t)\eta_2(t)}{\beta_1}\right), \end{aligned} \quad (29)$$

$$-\xi_2 u_x(t, 0) = \frac{\xi_1^2(t)}{\mu_1} - \frac{d}{dt}\left(\frac{\frac{\gamma_1}{2}\xi_2^2(t) + \xi_2(t)\xi_1(t)}{\mu_1}\right). \quad (30)$$

We next define for  $t \geq 0$

$$\begin{aligned} W(\mathcal{X}(t)) &= \int_0^1 (u - u_*)u_t dx + \int_0^1 \frac{q}{2}(u - u_*)^2 dx \\ &\quad + \frac{a(1)}{\beta_1}\left(\frac{\alpha_1}{2}\eta_2^2(t) + \eta_2(t)\eta_1(t)\right) \\ &\quad + \frac{a(0)}{\mu_1}\left(\frac{\gamma_1}{2}\xi_2^2(t) + \xi_2(t)\xi_1(t)\right). \end{aligned} \quad (31)$$

Gathering (28), (29) and (30), it holds for  $t \geq 0$

$$\begin{aligned} \dot{W} &= - \int_0^1 au_x^2 - \frac{a(1)\alpha_2}{\beta_1}\eta_2^2(t) + \int_0^1 u_t^2 \\ &\quad + \frac{a(1)}{\beta_1}\eta_1^2(t) + \frac{a(0)}{\mu_1}\xi_1^2(t). \end{aligned} \quad (32)$$

We finally define the candidate Lyapunov function  $V$  used for proving Theorem 2, which is positive definite for some constant  $\ell$  such that  $\sqrt{2q} > \ell > 0$  by

$$V(\mathcal{X}(t)) = F(\mathcal{X}(t)) + \frac{a(1)\alpha_2}{2\beta_1}\eta_2^2(t) + \ell W(\mathcal{X}(t)), \quad \geq 0. \quad (33)$$

Putting together (24) and (31), it holds for  $t \geq 0$ ,

$$\begin{aligned} V(\mathcal{X}(t)) &= E_u(t) \\ &+ \ell \int_0^1 \left( (u - u_*)u_t + \frac{q}{2}(u - u_*)^2 \right) dx \\ &+ \frac{a(1)}{2\beta_1} \left( \eta_1^2 + \ell(2\eta_2\eta_1 + \alpha_2\eta_2^2) \right) \\ &+ \frac{a(0)}{2\mu_1} \left( \xi_1^2 + \ell(2\xi_2\xi_1 + \gamma_1\xi_2^2) \right). \end{aligned} \quad (34)$$

and similarly, putting together (25) and (32), it holds for  $t \geq 0$ ,

$$\begin{aligned} \dot{V} &= - \int_0^1 \left( \frac{q}{2} - \ell \right) u_t^2 dx - \ell \int_0^1 a u_x^2 dx \\ &- \frac{a(1)}{\beta_1} \left( (\alpha_1 - \ell)\eta_1^2 + \alpha_2\ell\eta_2^2 \right) \\ &- \frac{a(0)(\gamma_1 - \ell)}{\mu_1} \xi_1^2. \end{aligned} \quad (35)$$

**Proposition 3** *With the notations above, and  $\Gamma$  defined in (18), there exist  $\ell > 0$  and two positive constants  $c, C, \rho > 0$  such that for every strong solution  $\mathcal{X}(t)$  of (7), one gets, for  $t \geq 0$ ,*

$$c\Gamma(\mathcal{X}(t)) \leq V(\mathcal{X}(t)) \leq C\Gamma(\mathcal{X}(t)), \quad (36)$$

$$\dot{V}(\mathcal{X}(t)) \leq -C\rho\Gamma(\mathcal{X}(t)). \quad (37)$$

**PROOF.** The proof of the proposition is based on the observation that for every  $t \geq 0$  and  $x \in [0, 1]$  it holds

$$\begin{aligned} |u(t, x) - u_*|^2 &\leq 2|u(t, x) - u(t, 1)|^2 + 2\eta_2^2(t) \\ &\leq 2 \int_0^1 u_x^2(t, x) dx + 2\eta_2^2(t) \\ &\leq \frac{4}{a} E_u(t) + 2\eta_2^2(t), \end{aligned} \quad (38)$$

As an immediate consequence, one gets that, for  $t \geq 0$ ,

$$\int_0^1 (u - u_*)^2 dx \leq \frac{4}{a} E_u(t) + 2\eta_2^2(t), \quad (39)$$

$$\xi_2^2(t) \leq \frac{4}{a} E_u(t) + 2\eta_2^2(t). \quad (40)$$

The proof of (36) relies now on the combination of (34), (39) and (40), several completions of squares and

Cauchy-Schwarz inequality. As for the argument of (37), it is obtained similarly by using (35), (39) and (40),

**Remark 4** *Note that one must use finite upper bounds for  $a$  and  $q$ . Indeed using Holder's inequality*

$$\int_0^1 a(x)u_x(x, t)^2 dx \leq \|a\|_{L^\infty} \|u_x\|_{L^2}. \quad (41)$$

*Similar facts hold for  $q(x)$ .*

Relying on Proposition 3, we complete the proof of Theorem 2.

From (36) and (37), it follows that  $\dot{V} \leq -\rho V$  hence yielding exponential decrease of  $V$  at the rate  $\rho$  and the similar conclusion holds for  $\Gamma$ , thanks to (37). All items of Theorem 2 are proved after using (38).

#### 4 Discussion on the proof of the main theorem and associated results.

There exist cases where the linear one-dimensional wave does not decay exponentially. For example, the solution  $u$  of the system

$$\begin{cases} u_{tt} = u_{xx}, & (t, x) \in \mathbb{R}^+ \times (0, 1), \end{cases} \quad (42a)$$

$$\begin{cases} u = 0, & (t, 0), \end{cases} \quad (42b)$$

$$\begin{cases} u_{tt} = -u_x - u_t, & (t, 1), \end{cases} \quad (42c)$$

does not decrease exponentially [17, Section 4]. However, the energy of the following two systems

$$\begin{cases} u_{tt} = u_{xx}, & (t, x) \in \mathbb{R}^+ \times (0, 1), \end{cases} \quad (43a)$$

$$\begin{cases} u_x = u_t, & (t, 0), \end{cases} \quad (43b)$$

$$\begin{cases} u_x = -u_t, & (t, 1), \end{cases} \quad (43c)$$

and

$$\begin{cases} u_{tt} = u_{xx}, & (t, x) \in \mathbb{R}^+ \times (0, 1), \end{cases} \quad (44a)$$

$$\begin{cases} u_x = u_t, & (t, 0), \end{cases} \quad (44b)$$

$$\begin{cases} u_{tt} = -u_x - u_t, & (t, 1), \end{cases} \quad (44c)$$

are exponentially decreasing [22]. Typically, for both previous cases, the exponential decrease and stability can be obtained via Energy/Lyapunov approach using cross terms in the following form.

$$\int_0^1 (1 - 2x)u_t u_x dx, \quad (45)$$

which can make in-domain term in  $u_x^2$  and  $u_t^2$  appear for the Energy/Lyapunov functional derivative. However, this term implies boundary terms in the following form.

$$[u_x^2 + u_t^2]_0^1, \quad (46)$$

and one is stuck since one cannot make negatively bound  $u_x$  both in 0 and 1. This is properly shown in [21] with more general form of  $u_x u_t$  cross terms. Note that this is not the case when we also consider the term in position on the domain as we can use cross terms like

$$\int_0^1 u u_t, \quad (47)$$

In fact, this term can be used when we have a position term in the wave equation [20, Chapter 9]. This term is close to the one we suggest

$$\int_0^1 (u - u_*) u_t, \quad (48)$$

This mostly corresponds to the beforehand knowledge of the limit value of  $u$  for the system. This can be made because the integrator part of the system capture the distance between the state and the attractor. Note that in our case this term can be added because  $q$  is strictly positive, see (28).

In the following we investigate and establish results on associated problems.

**Proposition 5** Consider the following 1D wave equation

$$\begin{cases} u_{tt} - (a(x)u_x)_x = -q(x)u_t, & (t, x) \in \mathbb{R}^+ \times (0, 1), & (49a) \\ u_t(t, 1) = \eta_1(t), & & (49b) \\ \dot{\eta}_1(t) = -\alpha_1 \eta_1(t) - \beta_1 u_x(t, 1), & & (49c) \\ u(t, 0) = 0, \quad t \geq 0, & & (49d) \\ u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1, & \text{on } (0, 1), & (49e) \\ \eta_1(0) = \eta_0. & & (49f) \end{cases}$$

where  $a(\cdot)$ ,  $q(\cdot)$  are respecting  $(h_1)$ - $(h_2)$ , and with  $\alpha_1$  and  $\beta_1$  are strictly positive. The state of this system is

$$\mathcal{X}_2(t) = [u(t, \cdot), u_t(t, \cdot), \eta_1(t)] \in \text{Dom}(\mathcal{A}_2), \quad (50)$$

where  $\mathcal{A}_2$  is the unbounded operator associated with (49). The domain is defined as

$$\text{Dom}(\mathcal{A}_2) = \{z \in X_{2,s}, z_1(0) = 0, z_2(1) = z_3\}, \quad (51)$$

where  $X_{2,s}$  is the space of strong solutions, and  $X_{2,w}$  is the space of weak solutions defined as

$$X_{2,s} = H^2 \times H^1 \times \mathbb{R}, \quad (52)$$

$$X_{2,w} = H^1 \times L^2 \times \mathbb{R}. \quad (53)$$

Finally, consider

$$\Gamma_2(\mathcal{X}_2(t)) = \int_0^1 (u_t(t, x)^2 + u_x(t, x)^2) dx + \eta_1(t)^2. \quad (54)$$

Then, there exist a positive constant  $\rho$  and a positive constant  $M$  such that for every weak solution  $\mathcal{X}_2$ , it holds

$$\Gamma_2(\mathcal{X}_2(t)) \leq M \Gamma_2(\mathcal{X}_2(0)) e^{-\rho t}. \quad (55)$$

And the system is exponentially stable towards the origin of  $X_{2,w}$ .

In additions, it holds that  $\max_{x \in [0,1]} |u(t, x)|$  tend exponentially to zero as  $t$  tends to infinity, with a decay rate larger than or equal to  $\rho$ .

**Proposition 6** Consider the following 1D wave equation,

$$\begin{cases} u_{tt} - (a(x)u_x)_x = -q(x)u_t, & (t, x) \in \mathbb{R}^+ \times (0, 1), & (56a) \\ u_t(t, 1) = \eta_1(t), & & (56b) \\ \dot{\eta}_1(t) = -\alpha_1 \eta_1(t) - \alpha_2 \eta_2(t) - \beta_1 u_x(t, 1), & & (56c) \\ \dot{\eta}_2(t) = \eta_1(t), & & (56d) \\ u(t, 0) = 0, \quad t \geq 0, & & (56e) \\ u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1, & \text{on } (0, 1), & (56f) \\ \eta_1(0) = \eta_0, \quad \eta_2(0) = \eta_{2,0}. & & (56g) \end{cases}$$

where  $a(\cdot)$ ,  $q(\cdot)$  are respecting  $(h_1)$ - $(h_2)$ , and with  $\alpha_1$ ,  $\alpha_2$  and  $\beta_1$  are strictly positive.

For  $x \in [0, 1]$ , define

$$v(x) := C_2 \int_0^x \frac{ds}{a(s)}, \quad (57)$$

$$C_2 := \frac{a(1)\alpha_2}{a(1)\alpha_2 \int_0^1 \frac{ds}{a(s)} + \beta_1} u_*(1). \quad (58)$$

The state of this system is

$$\mathcal{X}_3(t) = [u(t, \cdot), u_t(t, \cdot), \eta_1(t), \eta_2(t)] \in \text{Dom}(\mathcal{A}_3), \quad (59)$$

where  $\mathcal{A}_3$  is the unbounded operator associated with (56). The domain is defined as

$$\text{Dom}(\mathcal{A}_3) = \{z \in X_{3,s}, z_1(0) = 0, z_2(1) = z_3\}, \quad (60)$$

where  $X_{3,s}$  is the space of strong solutions, and  $X_{3,w}$  is the space of weak solutions defined as

$$X_{3,s} = H^2 \times H^1 \times \mathbb{R}^2, \quad (61)$$

$$X_{3,w} = H^1 \times L^2 \times \mathbb{R}^2. \quad (62)$$

Finally, consider

$$\begin{aligned} \Gamma_3(\mathcal{X}_2(t)) = & \int_0^1 ((u(t, x) - v(x))^2 + u_t(t, x)^2 \\ & + u_x(t, x)^2) dx \\ & + \eta_1(t)^2 + (\eta_2(t) - \frac{\beta_1 C_2}{a(1)\alpha_2})^2. \end{aligned} \quad (63)$$

Then, there exists a positive constant  $\rho$  and a positive constant  $M$  such that for every weak solution  $\mathcal{X}_2$ , it holds

$$\Gamma_3(\mathcal{X}_3(t)) \leq M\Gamma_3(\mathcal{X}_3(0))e^{-\rho t}. \quad (64)$$

And the system is exponentially stable towards the attractor defined as  $\ker(\Gamma_3(\cdot))$ .

In additions, it holds that  $\max_{x \in [0,1]} |u(t, x) - v(x)|$  tend exponentially to zero as  $t$  tends to infinity, with a decay rate larger than or equal to  $\rho$ .

**Proposition 7** Consider the following 1D wave equation,

$$\begin{cases} u_{tt} - (a(x)u_x)_x = -q(x)u_t, & (t, x) \in \mathbb{R}^+ \times (0, 1), & (65a) \\ u_t(t, 1) = \eta_1(t), & & (65b) \\ u_t(t, 0) = \xi_1(t), & & (65c) \\ \dot{\eta}_1(t) = -\alpha_1\eta_1(t) - \beta_1u_x(t, 1), & & (65d) \\ \dot{\xi}_1(t) = -\gamma_1\xi_1(t) + \mu_1u_x(t, 0), & & (65e) \\ u(0, \cdot) = u_0, \quad \partial_t u(0, \cdot) = u_1 \quad \text{on } (0, 1), & & (65f) \\ \eta_1(0) = \eta_0, \quad \eta_2(0) = \eta_{2,0}, \quad \xi_1(0) = \xi_0. & & (65g) \end{cases}$$

where  $a(\cdot)$ ,  $q(\cdot)$  are respecting  $(h_1)$ - $(h_2)$ , and with  $\alpha_1, \beta_1, \gamma_1$  and  $\mu_1$  are positive. The state of this system is

$$\mathcal{X}_4(t) = [u(t, \cdot), u_t(t, \cdot), \eta_1(t), \xi_1(t)] \in \text{Dom}(\mathcal{A}_4), \quad (66)$$

where  $\mathcal{A}_4$  is the unbounded operator associated with (65). The domain is defined as

$$\text{Dom}(\mathcal{A}_4) = \{z \in X_{3,s}; \\ z_2(0) = z_4, z_2(1) = z_3\}, \quad (67)$$

where  $X_{3,s}$  is the space of strong solutions, and  $X_{3,w}$  is the space of weak solutions, both defined in (61)-(62) Finally, consider

$$\Gamma_4(\mathcal{X}_4(t)) = \int_0^1 (u_t(t, x)^2 + u_x(t, x)^2) dx \\ + \eta_1(t)^2 + \xi_1(t)^2. \quad (68)$$

Then, there exists a positive constant  $\rho$  and a positive constant  $M$  such that, for every weak solution  $\mathcal{X}_4$ , it holds

$$\Gamma_4(\mathcal{X}_4(t)) \leq M\Gamma_4(\mathcal{X}_4(0))e^{-\rho t}. \quad (69)$$

And the system is exponentially stable towards the attractor  $S_4$  defined by

$$S_4 = \{z \in X_{3,w}, z_1(\cdot) = d, d \in \mathbb{R}, z_2(\cdot) = 0, \\ z_3 = 0, z_4 = 0\}. \quad (70)$$

which is the kernel of  $\Gamma_4(\cdot)$ .

In addition, there exists  $u_*$  so that  $\max_{x \in [0,1]} |u(t, x) - u_*|$  tends exponentially to zero as  $t$  tends to infinity.

**PROOF.**

We start by proving Proposition 5. As before, the argument is based on an appropriate Lyapunov function  $\bar{V}_u = \bar{F}_u + \ell \bar{W}_u$  where  $\ell$  is a positive constant to be chosen and

$$\bar{F}_u(t) := \frac{1}{2} \int_0^1 (u_t^2 + au_x^2) dx + \frac{a(1)\alpha_1}{2\beta_1} \eta_1^2, \quad (71)$$

$$\bar{W}_u(t) := \int_0^1 uu_t dx + \frac{1}{2} \int_0^1 qu^2 dx + \frac{a(1)}{\beta_1} \left( \frac{\alpha_1}{2} \eta_1^2 \right. \\ \left. + \eta_1 u(t, 1) \right). \quad (72)$$

One gets

$$\dot{\bar{V}}_u = - \int_0^1 (q - \ell)u_t^2 dx - \ell \int_0^1 au_x^2 dx \\ - \frac{a(1)}{\beta_1} (\alpha_1 - \ell) \eta_1^2. \quad (73)$$

The conclusion follows by taking  $\ell > 0$  small enough and noting that, thanks to the Dirichlet boundary condition (56e), for every  $t \geq 0$  and  $x \in [0, 1]$

$$|u(t, x)|^2 = |u(t, x) - u(t, 0)|^2 \\ \leq \int_0^1 u_x^2(t, x) dx \leq 2E_u(t) \leq 2\bar{F}_u(t). \quad (74)$$

One proceeds by establishing an analog to Proposition 3 where  $\Gamma$  and  $V$  are replaced by  $\Gamma_2$  and  $\bar{V}_u$  in order first to obtain that  $\dot{\bar{V}}_u \leq -\rho \bar{V}_u$  for some positive constant  $\rho$  independent of the state and finally to conclude as in the final part of the argument of Theorem 2.

We next turn to the proof of Proposition 6. Using the notations of the proposition, we set

$$w(t, x) := u(t, x) - v(x), \quad t \geq 0, x \in [0, 1], \\ \bar{\eta}_2(t) := \eta_2(t) - \frac{\beta_1 C_2}{a(1)\alpha_2}, \quad t \geq 0, x \in [0, 1]. \quad (75)$$

It is a matter of elementary computations to check that  $w$  is the solution of (65) with the Dirichlet boundary condition at  $x = 0$  (since  $v(0) = 0$ ) and the boundary condition given by

$$\dot{\eta}_1(t) = -\alpha_1\eta_1(t) - \alpha_2\bar{\eta}_2(t) - \beta_1w_x(t, 1), \quad (76)$$

$$\dot{\eta}_2(t) = \eta_1(t). \quad (77)$$



Moreover it holds that  $w_*(1) = 0$ , i.e.,  $w(t, 1) = \bar{\eta}_2(t)$  for all  $t \geq 0$ . Indeed

$$\begin{aligned} w_*(1) &= w(t, 1) - \bar{\eta}_2(t) \\ &= u_*(1) - v(1) - \frac{\beta_1 C_2}{a(1)\alpha_2} \\ &= u_*(1) - C_2 \left( \int_0^1 \frac{ds}{a(s)} + \frac{\beta_1 C_2}{a(1)\alpha_2} \right) = 0. \end{aligned} \quad (78)$$

We have essentially reduced the problem to only deal with solutions of (65) with the Dirichlet boundary condition at  $x = 0$ , with the additional constraint that  $w_*(1) = 0$ . In that case, we consider the candidate Lyapunov function  $\tilde{V}_w = \tilde{F}_w + \ell \tilde{W}_w$  where  $\ell$  is a positive constant to be chosen and

$$\begin{aligned} \tilde{F}_w(t) &:= \frac{1}{2} \int_0^1 (w_t^2 + aw_x^2) dx \\ &\quad + \frac{a(1)}{2\beta_1} (\alpha_1 \eta_1^2 + \alpha_2 \bar{\eta}_2^2), \end{aligned} \quad (79)$$

$$\begin{aligned} \tilde{W}_w(t) &:= \int_0^1 ww_t dx + \frac{1}{2} \int_0^1 qw^2 dx \\ &\quad + \frac{a(1)}{\beta_1} \left( \frac{\alpha_1}{2} \bar{\eta}_2^2 + \eta_1 \bar{\eta}_2 \right). \end{aligned} \quad (80)$$

One gets

$$\begin{aligned} \dot{\tilde{V}}_w &= - \int_0^1 (q - \ell) w_t^2 dx - \ell \int_0^1 aw_x^2 dx \\ &\quad - \frac{a(1)}{\beta_1} (\alpha_1 - \ell) \eta_1^2 - \ell \frac{a(1)\alpha_2}{\beta_1} \bar{\eta}_2^2, \end{aligned} \quad (81)$$

where we have repeatedly used the equality  $w(t, 1) = \bar{\eta}_2(t)$ . By following what has been done previously, the conclusion follows.

We finally prove Proposition 7. As before the argument is based on an appropriate Lyapunov function  $\bar{V}_u$  defined later. We first consider  $F$  given in (24) and note that for  $t \geq 0$  it holds

$$\dot{F} = - \int_0^1 qw_t^2 dx - \frac{a(1)\alpha_1}{\beta_1} \eta_1^2(t) - \frac{a(0)\gamma_1}{\mu_1} \xi_1^2(t). \quad (82)$$

We next compute along solutions of (65) the following

time derivative

$$\begin{aligned} \frac{d}{dt} \left( \int_0^1 (u(t, x) - u(t, 1)) u_t(t, x) dx \right) &= \\ &+ \int_0^1 u_t^2 dx - \eta_1 \int_0^1 u_t dx \\ &- \frac{d}{dt} \int_0^1 \frac{q}{2} (u(t, x) - u(t, 1))^2 dx \\ &+ (u(t, 1) - u(t, 0)) a(0) u_x(t, 0) \\ &- \int_0^1 au_x^2 dx. \end{aligned} \quad (83)$$

In the above equation, we use (65e) to get rid of  $u_x(t, 0)$  and, to obtain for  $t \geq 0$  that

$$\begin{aligned} &(u(t, 1) - u(t, 0)) u_x(t, 0) = \\ &+ (u(t, 1) - u(t, 0)) \frac{\dot{\xi}_1 + \gamma_1 \xi_1}{\mu_1} \\ &= \frac{d}{dt} \left( (u(t, 1) - u(t, 0)) \frac{\xi_1}{\mu_1} \right) \\ &- (\eta_1 - \xi_1) \frac{\xi_1}{\mu_1} + \frac{\gamma_1 \xi_1}{\mu_1} (u(t, 1) - u(t, 0)). \end{aligned} \quad (84)$$

Setting for  $t \geq 0$

$$\begin{aligned} G_u(t) &:= \int_0^1 (u(t, x) - u(t, 1)) u_t(t, x) dx \\ &- \frac{a(0)\xi_1}{\mu_1} (u(t, 1) - u(t, 0)), \end{aligned} \quad (85)$$

we deduce from the above that along with solutions of (65) that

$$\begin{aligned} \dot{G}_u &= \int_0^1 u_t^2 dx - \eta_1 \int_0^1 u_t dx - \int_0^1 au_x^2 dx \\ &- \frac{a(0)\xi_1}{\mu_1} (\eta_1 - \xi_1) + \frac{a(0)\gamma_1 \xi_1}{\mu_1} (u(t, 1) - u(t, 0)) \\ &- \int_0^1 q (u(t, x) - u(t, 1)) u_t dx. \end{aligned} \quad (86)$$

We finally recall that there exists a positive constant  $C_a$  (independent of the solutions of (65)) such that, for  $t \geq 0$ ,

$$\begin{aligned} &\int_0^1 |u_t| dx + \max_{x \in [0, 1]} |u(t, x) - u(t, 1)| \\ &\leq \int_0^1 (|u_t| + |u_x|) dx \\ &\leq C_a E_u^{1/2}(t). \end{aligned} \quad (87)$$

We now choose  $\bar{V}_u = F + \ell G_u$  for  $\ell > 0$  small enough. Using repeatedly Cauchy-Schwarz inequality and (87)

in (82) and (86), one gets for  $\ell$  small enough that (36) and (37) hold true, from which one deduces Item(i) of Proposition 7.

Finally, to get Item(ii) of Proposition 7, first notice that  $u(t, 1)$  admits a limit  $u_*$  as  $t$  tends to infinity since, for every  $t, t' > 0$  it holds  $u(t, 1) - u(t', 1) = \int_{t'}^t \eta_1$  and  $\eta_1$  decreases to zero exponentially. The conclusion follows now by using (87).

**Remark 8** *In the proofs of all our results, one could use the function  $G_u$  (especially the integral term) to obtain the exponential decrease of  $E_u$  and some of the components of the Wentzell's boundary conditions. However, this does not allow one to determine the limit  $u_*$  for the solution  $u$  in terms of initial conditions. In particular, we are not able to characterize  $u_*$  in Proposition 7.*

Note also that

$$u(t, x) - u(t, 1) = - \int_x^1 u_x(t, s) ds. \quad (88)$$

This can be related with the mean of  $u_x$  and therefore we have extended our Lyapunov function with a space moving evaluation of the mean of the force/torque. Indeed  $u_x$  is associated with the torque or the force in mechanical setup.

## 5 Numerical schemes and simulations.

There exist several ways to compute numerical approximation of the solution of evolution problems associated with partial differential equation, [29]. In the case under consideration, spectral methods lead to an estimation of the base function at each time step due to the dynamics boundary condition. This requires an important computing power. As we have only one dimension in space finite-element methods reduce to finite difference methods with (possibly unequal) spacial step. Finite different methods can be delicate to design in order to ensure at the same time numerical stability and good approximation. Note that there also exist specific schemes based on Riemann invariants [2]. These last schemes have good numerical property, but their extension to dynamic boundary conditions is not obvious.

In this paper, we suggest a new approach, which provides numerical scheme stability and therefore achieves structural stability. It is based on the discretization of the Lagrangian associated with the wave equation. This approach leads to a special finite difference scheme. As previously said the wave equation in its stationary form can be associated with a Lagrangian. For the case under

consideration, this Lagrangian is given by

$$L(u(t, \cdot)) = \int_0^1 \frac{1}{2} (u_t(t, x)^2 - a(x)u_x(t, x)^2) dx + \frac{1}{2} \left( \frac{a(1)}{\beta_1} \eta_1(t)^2 + \frac{a(0)}{\mu_1} \xi_1(t)^2 \right), \quad (89)$$

together with the conditions

$$u_t(1, t) = \eta_1(t), \quad u_t(0, t) = \xi_1(t). \quad (90)$$

Following the strategy in [15] and the least action principle, the dynamics of the system is associated with a stationary action. The action for any time interval is given as

$$I(u) = \int_{t_i}^{t_f} L(u(t, \cdot)) dt. \quad (91)$$

A stationary action means that the first variation is equal to zero

$$\delta I(u, \delta u) = 0, \quad (92)$$

where the first variation is defined as

$$\delta I(u, \delta u) = \delta I(u + \delta u) - \delta I(u) + O(\|\delta u\|^2). \quad (93)$$

Computation gives the following stationary system

$$\begin{cases} u_{tt} - (au_x)_x = 0, & (x, t) \text{ in } \mathbb{R}^+ \times (0, 1), \end{cases} \quad (94a)$$

$$\begin{cases} u_t(t, 1) = \eta_1(t), \end{cases} \quad (94b)$$

$$\begin{cases} u_t(t, 0) = \xi_1(t), \end{cases} \quad (94c)$$

$$\begin{cases} \dot{\eta}_1(t) = -\beta_1 u_x(t, 1), \end{cases} \quad (94d)$$

$$\begin{cases} \dot{\xi}_1(t) = \mu_1 u_x(t, 0). \end{cases} \quad (94e)$$

Consider now a discrete version of (89)

$$L_d(u(t)[\cdot]) = \frac{1}{2} \sum_{i=1}^{N-1} [\dot{u}(t)[i]]^2 - \frac{a[i-1]}{12} \left( \frac{u(t)[i] - u(t)[i-1]}{Dx[i]} \right)^2 - \frac{a[i]}{3} \left( \frac{u(t)[i+1] - u(t)[i-1]}{Dx[i] + Dx[i+1]} \right)^2 - \frac{a[i+1]}{12} \left( \frac{u(t)[i+1] - u(t)[i]}{Dx[i+1]} \right)^2 Dx[i] + \frac{1}{2} \frac{a[N]}{\beta_1} \dot{u}(t)[N]^2 + \frac{1}{2} \frac{a[0]}{\mu_1} \dot{u}(t)[0]^2. \quad (95)$$

The integral part in  $u_x^2$  has approximated using Simpson's 1/3 rule. The derivation of the Euler-Largange

equation can then be done by a symbolic numerical computation. This gives an autonomous stationary linear finite dimensional system:

$$E\ddot{u}_d(t) = Au_d(t), \quad (96)$$

with  $\sigma(A) \in \mathbf{i}\mathbb{R}$ . It holds

$$E = \text{diag} \left[ \frac{1}{\beta_1} \ dx_1 \ \dots \ dx_{N-1} \ \frac{1}{\mu_1} \right]. \quad (97)$$

Then we add dissipation with a positive symmetric matrix  $R$ , source term (disturbance and action) and observation,

$$\begin{cases} E\ddot{u}_d(t) = Au_d(t) - R\dot{u}_d(t) + BU(t) + f, & (98a) \\ y(t) = C\dot{u}_d(t). & (98b) \end{cases}$$

The control  $U(t)$  is computed through

$$\begin{cases} \dot{\eta}(t) = y(t) - y_{\text{ref}}, & (99a) \\ U(t) = -k_i\eta - k_p(y(t) - y_{\text{ref}}). & (99b) \end{cases}$$

As the main idea of this discretization scheme is to have a good approximation of the energy, we suggest going on with this idea using symplectic integrator scheme, see [10] and references within. These methods, like the Crank-Nicolson method have the property preserve the energy as time evolves. It is known that for a system which has an eigenvalue in  $\mathbf{i}\mathbb{R}$  explicit schemes are unstable, and implicit schemes are exponentially stable see [10]. As our system has structurally the zero eigenvalue, symplectic numerical discretization schemes tend to give better behaviors approximation.

The idea of a symplectic scheme is to combine an implicit scheme together with an explicit one. This leads to

$$u_d[k+1] = u_d[k] + \Delta t \dot{u}_d[k+1], \quad (100)$$

$$\begin{aligned} E\dot{u}_d[k+1] &= E\dot{u}_d[k] + \Delta t Au_d[k] - \Delta t R\dot{u}_d[k+1] \\ &\quad + \Delta t BU[k] + \Delta t f. \end{aligned} \quad (101)$$

The second line is implicit, but  $R$  in our case is a diagonal matrix and so the associated inverse matrix is easily computed

$$\begin{aligned} \dot{u}_d[k+1] &= (1 + \Delta t E^{-1}R)^{-1} (\dot{u}_d[k] + \Delta t E^{-1}Au_d[k] \\ &\quad + \Delta t E^{-1}BU[k] + \Delta t E^{-1}f). \end{aligned} \quad (102)$$

There are several key points to note in this last equation. First, the term  $(1 + E^{-1}\Delta t R)^{-1}$  correspond to a contraction map in the case where  $R$  is positive, and therefore is associated with dissipation terms. Second, in the case where  $R$  represent anti-dissipation term, there exist discretized steps  $\frac{\Delta t}{Dx}$  where the numerical scheme is

undefined. Third, where  $R = 0$ , these equations are two-step explicit ones. The value selected for the numerical simulation for the wave is summarized in Table 1. The control law parameters and the initial condition are described in Table 2

Symbol	set ①	set ②	set ③
$N$	199	199	199
$a(\cdot)$	1	1	1
$q(\cdot)$	0	0.001	0.005
$f(\cdot)$	0	0	0
$\beta_1$	20	20	20
$\mu_1$	20	20	20
$q_1$	0	0.001	0.005
$\gamma_1$	0	0.001	0.005
$f_1$	0	0	0
$f_2$	0	0	0

Table 1  
Parameter values for the simulation.

Symbol	set ①	set ②	set ③
$k_p$	0	10	1000
$\alpha_2$	0	100	10
$v_1^{\text{ref}}$	.	.5	.5
$u_d[\cdot]$	0	0	0
$\dot{u}_d[0 : N-1]$	0	0	0
$\dot{u}_d[N]$	1	1	1

Table 2  
Control law parameter values and initial condition for the simulation.

The first round of simulations aims at showing the behavior of the wave in some really simple case where the input is neutralized see Figure 1-3. In Figure 1 we can see that the initial condition at  $x = 1$  propagate to the  $x = 0$  boundary and so on. We can see that when time increases, the oscillation resulting from the initial condition degenerates in smaller oscillation which starts to appear and become more frequent. In Figure 2, we have a small damping, and so the amplitude of the wave decreases, but the form of the oscillation remains the same. In Figure 3, since the damping is larger, the decline of the amplitude is more significant.

The purpose of the second round of simulations is to illustrate how the system together with the control law behave. This concerns Figure 4-6. In Figure 4, the wave

has the same parameters as Figure 1, but with the control law the oscillation resulting from the initial condition, is slowly extinguished and the wave is regulated towards the reference. The same observation can be made with Figure 5 and Figure 6.

The last round of simulations concerns the case when a large damping is given by the control law, see Figure 7-9.

We notice from these figures that adding large damping to the boundary does not necessarily result in the wave equation behaving better, as we see some kind of driven oscillation appears. This is more properly captured with the difference between Figure 10 and Figure 12, which depict only the boundary velocities associated with the configuration in Figure 4 and Figure 12. The control law associated with Figure 4 and Figure 10 is given in Figure 11. The control law associated with Figure 7 and Figure 12 is given in Figure 13.

Reconsider Figure 12 with the context of drilling control vibration. With a "bad" design of the PI control law, we can obtain a flat behavior at the  $x = 1$  boundary, but a more violent behavior at the  $x = 0$  compared with Figure 10.

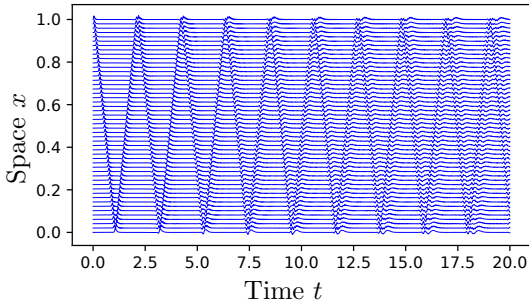


Fig. 1. Wave velocity trajectory with parameters (1) and gain (a).

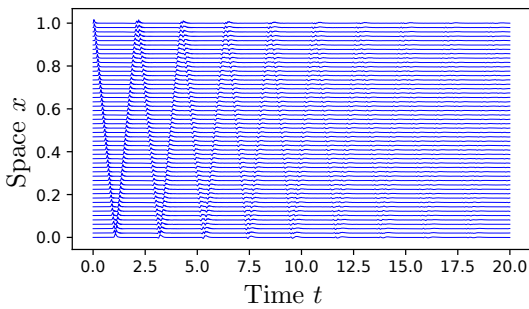


Fig. 2. Wave velocity trajectory with parameters (2) and gain (a).

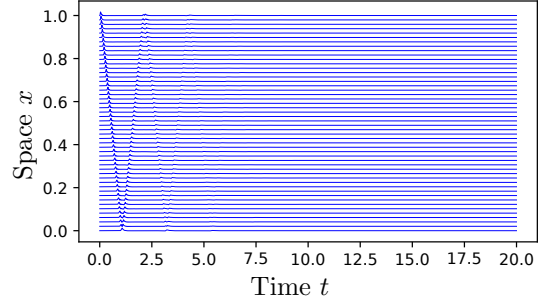


Fig. 3. Wave velocity trajectory with parameters (3) and gain (a).

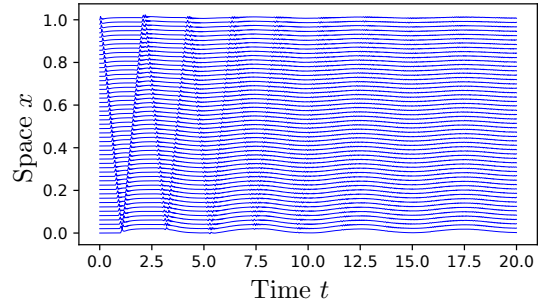


Fig. 4. Wave velocity trajectory with parameters (1) and gain (b).

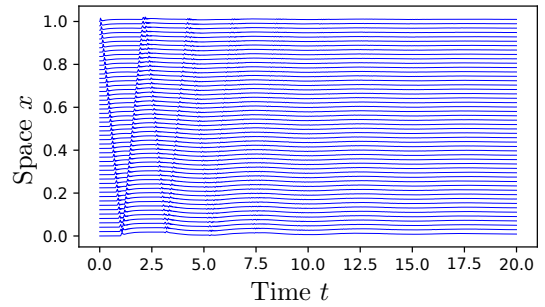


Fig. 5. Wave velocity trajectory with parameters (2) and gain (b).

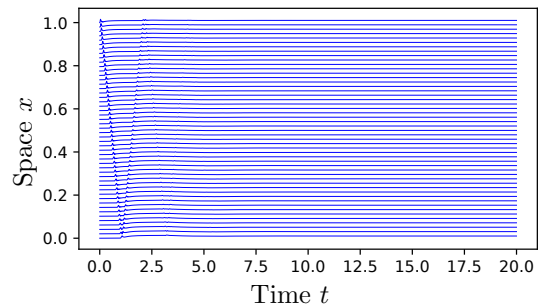


Fig. 6. Wave velocity trajectory with parameters (3) and gain (b).

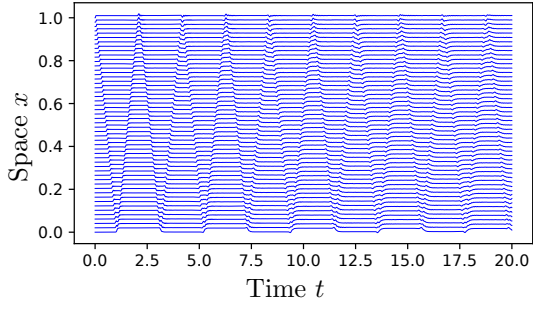


Fig. 7. Wave velocity trajectory with parameters ① and gain ③.

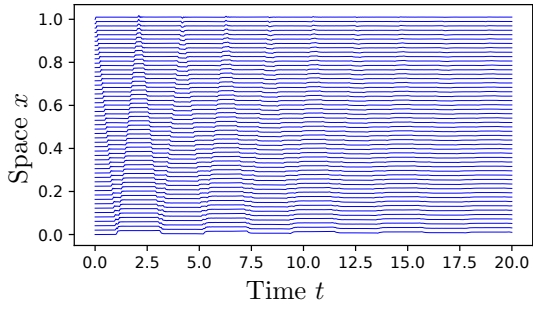


Fig. 8. Wave velocity trajectory with parameters ② and gain ③.

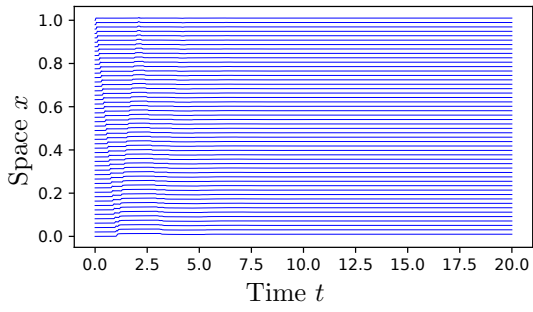


Fig. 9. Wave velocity trajectory with parameters ③ and gain ③.

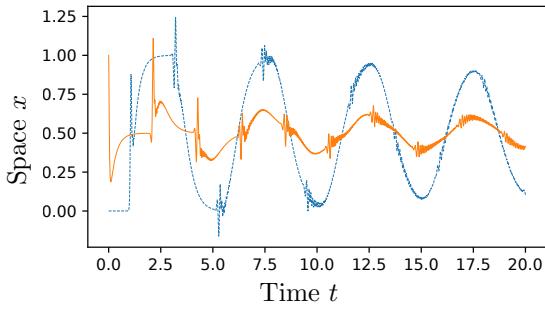


Fig. 10. Wave boundary velocity trajectories with parameters ① and gain ⑥.

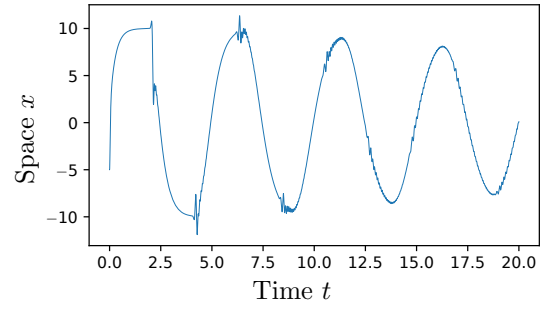


Fig. 11. Control law values with parameters ① and gain ⑥.

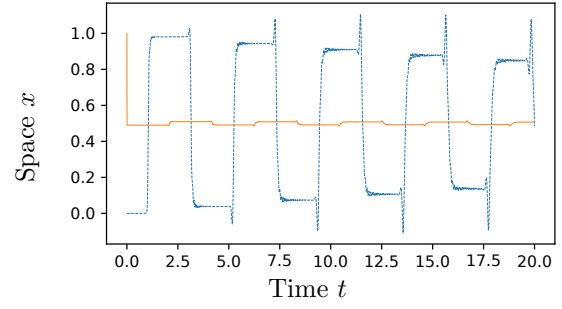


Fig. 12. Wave boundary velocity trajectories with parameters ① and gain ③.

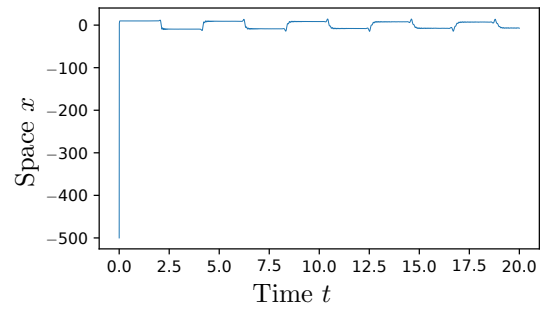


Fig. 13. Control law values with parameters ① and gain ③.

## 6 Conclusion

We present a new term for Lyapunov analysis which is associated with the moving window mean value. This term allows us to establish an exponential decay rate for several wave equations with Wentzell boundary conditions. Among these wave equations one is associated with a controlled system itself associated with an identified experimental model. Future work will be to use some of the exponential decay system as the target system for infinite-dimensional backstepping control design. There is also a great interest considering the boundary opposite to the control with a non-linear friction term, typically LuGre friction term.

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## A Proof of Theorem 1

The proof follows the same lines as the ones exposed in [24].

The idea of the proof is to decompose the operator  $\mathcal{A}$  defined in (11) into a maximal monotone part and a remaining part. We should be able to cancel the remaining part with a bijective change of variable. Finally, we conclude using the following theorem.

### Theorem 9 (Hille-Yosida [3, Theorem 7.4 on p. 185])

Let  $\mathcal{A}$  be a maximal operator on the Hilbert space  $H$  then for every  $X_0 \in \mathcal{D}(\mathcal{A})$  there exists a unique solution  $X$  to the following abstract problem

$$\begin{cases} \frac{dX}{dt}(t) + \mathcal{A}X(t) = 0, & (\text{A.1a}) \\ X(0) = X_0. & (\text{A.1b}) \end{cases}$$

with

$$X \in C^1([0, \infty); H) \cap C([0, \infty); \mathcal{D}(\mathcal{A})). \quad (\text{A.2})$$

Now consider the following operator

$$\forall z \in \mathcal{D}(G), \quad Gz = \begin{bmatrix} -z_2 \\ -(az'_1)' + z_2 + z_1 \\ \beta_1 z'_1(1) \\ 0 \\ -\mu_1 z'_1(0) \end{bmatrix}, \quad (\text{A.3})$$

and the following matrix

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -q + 1 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_1 & -\alpha_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\gamma_1 \end{bmatrix}. \quad (\text{A.4})$$

The domain of  $G$  is equal to the domain of  $\mathcal{A}$ . One gets

$$\mathcal{A} = G + B. \quad (\text{A.5})$$

$G$  is a monotone part, this is established in the following lemma and  $B$  is a bounded operator.

**Lemma 10** *The unbounded linear operator  $G$  defined in (A.3) is a maximal monotone operator on  $X_w$  defined in (9).*

**PROOF.** Considering the following scalar product on  $X_w$

$$\langle z, q \rangle = \int_0^1 (z_1 q_1 + z_2 q_2 + az'_1 q'_1) dx + \frac{a(1)}{\beta_1} z_3 q_3 + z_4 q_4 + \frac{a(0)}{\mu_1} z_5 q_5, \quad (\text{A.6})$$

$$\begin{aligned} \langle z, Gz \rangle &= \int_0^1 [-z_1 z_2 + z_2 (-(az'_1)' + z_2 + z_1) \\ &\quad - a(x) z'_1 z'_2] dx + a(1) z_3 z'_1(1) \\ &\quad - a(0) z_5 z'_1(0), \end{aligned} \quad (\text{A.7})$$

using integration by parts and the fact that  $z \in \mathcal{D}(\mathcal{A})$ , one obtains

$$\langle z, Gz \rangle = \int_0^1 z_2^2 dx \geq 0. \quad (\text{A.8})$$

Thus the operator  $G$  is monotone (see [3, Chapter 7 on Page 181]) on the Hilbert  $X_w$ . In addition, if we establish that

$$R(I + G) = X_w, \quad (\text{A.9})$$

then the operator  $G$  is maximal monotone (see [3, Chapter 7 on Page 181]),  $R$  stands for the range of the operator. Let  $y \in X_w$ , we have to solve

$$z \in \mathcal{D}(\mathcal{A}), \quad z + Gz = y, \quad (\text{A.10})$$

which means that

$$z_1 - z_2 = y_1, \quad (\text{A.11})$$

$$z_2 - (az'_1)' + z_2 + z_1 = y_2, \quad (\text{A.12})$$

$$z_3 + \beta z'_1(1) = y_3, \quad (\text{A.13})$$

$$z_4 + 0 = y_4, \quad (\text{A.14})$$

$$z_5 - \mu_1 z'_1(0) = y_5, \quad (\text{A.15})$$

using the fact that  $z \in \mathcal{D}(\mathcal{A})$  one gets

$$3z_1 - (az'_1)' = 2y_1 + y_2, \quad (\text{A.16})$$

$$\beta_1 z'_1(1) + z_1(1) = (y_3 + y_1(1)), \quad (\text{A.17})$$

$$-\mu_1 z'_1(0) + z_1(0) = (y_5 + y_1(0)). \quad (\text{A.18})$$

This is a classical stationary problem (e.g., see [3]) with Robin's boundaries conditions, using standard result (as done in [3, Example 6, On Page 226] ) one gets that as  $2y_1 + y_2 \in L^2(0, 1)$ , (A.16)-(A.18) has a unique solution  $z_1 \in H^2(0, 1)$ . Now one can check that the element  $z = (z_1, z_2, z_3, z_4, z_5)$  with

$$\begin{cases} z_1 \text{ is the solution to (A.16)-(A.18),} & \text{(A.19a)} \\ z_2 = z_1 - y_1, & \text{(A.19b)} \\ z_3 = y_3 - a(1)z_1'(1), & \text{(A.19c)} \\ z_4 = y_4z_4 = y_5 + a(0)z_1'(0), & \text{(A.19d)} \end{cases}$$

satisfies (A.11)-(A.15). Moreover using (A.16)-(A.18) on (A.19) one gets that  $z$  satisfying (A.19) is in  $\mathcal{D}(\mathcal{A})$ .

Now, we are ready to state the proof of the well posedness of (10). Note that the fact that  $G$  is maximal monotone implies that  $\mathcal{D}(\mathcal{A})$  is dense in  $X_w$  (i.e.,  $\overline{\mathcal{D}(\mathcal{A})} = X_w$ ).

Using the bijective change of variable

$$z_e(t) = z(t)e^{Bt}, \quad \text{(A.20)}$$

$z$  is the solution to (10) is equivalent to,  $z_e \in \mathcal{D}(\mathcal{A})$  is the solution to

$$\begin{cases} \frac{d}{dt}z_e(t) + Gz_e(t) = 0, & \text{(A.21a)} \\ z_e(0) = z_0, & \text{(A.21b)} \end{cases}$$

where  $B$  is defined in (A.4) and  $G$  is defined in (A.3).

From Lemma 10, using Theorem 9 on (A.21), and the change of variable (A.20), one establishes (i). Using argument of density of  $\mathcal{D}(\mathcal{A})$  in  $X_w$ , and  $C_0$ -semigroup theory one obtains the regularity of weak solutions.

Note that we refer the reader to [14], [19] for the notion mild solutions. Moreover part of the proof are inspired from [5] and [9] which in turn originates from [26].