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Maximal operators on hyperbolic triangles

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Abstract

We characterize the boundedness properties on the spaces $L^p(\mathbb{H}^2)$ of the maximal operator $M_{\mathcal{B}}$ where \mathcal{B} is an arbitrary family of hyperbolic triangles stable by isometries.

1 Introduction

The theory of maximal operators have a long history in analysis and those objects are still intensively studied today: in a measured metric space (X, μ, d) , it is common to consider the *centered Hardy-Littlewood* maximal operator M^c defined as

$$M^c f(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f|$$

or its non *non-centered* version M defined as

$$Mf(x) := \sup_{x \in B \in \mathcal{Q}} \frac{1}{\mu(B)} \int_B |f|.$$

Here, we have denoted by \mathcal{Q} the family containing all the balls of the space X .

In the Euclidean space \mathbb{R}^n endowed with the Lebesgue measure, the operator M and M^c are pointwise comparable (*i.e.* for any function f and $x \in \mathbb{R}^n$, we have $M^c f(x) \simeq_n Mf(x)$) and both operators have weak-type $(1, 1)$.

Theorem 1 (Maximal Theorem in \mathbb{R}^n). *For any f and $t > 0$, one has the following estimate*

$$|\{Mf > t\}|_n \lesssim_n \int_{\mathbb{R}^n} \frac{|f|}{t}.$$

It is well known that the proof of Theorem 1 relies on the classic Vitali's covering Theorem. Also by interpolation arguments with $p = \infty$, we immediately obtain that the operators M and M^c are bounded from L^p to L^p for any $p > 1$. Such an estimate coupled with classic arguments of measure theory yield the following Theorem of differentiation.

Theorem 2 (Lebesgue's Theorem of differentiation). *For any locally integrable function f , we have almost everywhere the following identity*

$$f(x) = \lim_{r \rightarrow 0} A_r f(x).$$

Here, $A_r f(x)$ stands for the average of f on a ball (or a cube) centered at the point x or radius r .

If the centered or non centered version maximal operator behave in the same fashion in the Euclidean space the story is quite different on non-compact symmetric spaces. Precisely, let \mathbb{G} be a non-compact connected semisimple Lie group with finite center, \mathbb{K} a maximal compact subgroup and consider the non-compact symmetric space

$$\mathbb{X} = \mathbb{G}/\mathbb{K}$$

equipped with its natural \mathbb{G} -invariant measure μ and distance d . The case of the centered Hardy-Littlewood operator M^c in this setting has been studied in [8] by Stromberg.

Theorem 3 (Stromberg). *On (\mathbb{X}, μ, d) , the centered Hardy-Littlewood operator M^c has weak-type $(1, 1)$.*

In contrast with the Euclidean space, one cannot use covering argument on non-compact symmetric space since the growth of the volume of the balls are exponential. To deal with this difficulty, Stromberg uses a local-global strategy to decompose the operator M^c and to detail its optimal boundedness property. On the other hand, the non-centered operator M has been studied by Ionescu in [6].

Theorem 4 (Ionescu). *On (\mathbb{X}, μ, d) , the non-centered Hardy-Littlewood operator M is bounded from L^p to L^p in the sharp range of exponents $p \in (2, \infty]$.*

The proof of Ionescu relies on the relation between the boundedness property of maximal operator and the covering property of the geometric family that defines it. Precisely, in a measured metric space (X, μ, d) , given an arbitrary family \mathcal{B} of measurable sets of finite measure (the *geometric data*) one can define the maximal operator $M_{\mathcal{B}}$ associated as

$$M_{\mathcal{B}} f(x) := \sup_{x \in R \in \mathcal{B}} \frac{1}{\mu(R)} \int_R |f|.$$

Here, we have implicitly supposed that we have

$$X = \bigcup_{R \in \mathcal{B}} R.$$

In [4], Cordoba and Fefferman say that such a family \mathcal{B} satisfies the covering property V_q for $q \in (1, \infty)$ if given any finite family $\{R_i : i \in I\}$ included in \mathcal{B} , there exists a subfamily $J \subset I$ such that we have

$$\mu \left(\bigcup_{i \in I} R_i \right) \lesssim \mu \left(\bigcup_{j \in J} R_j \right)$$

and also

$$\left\| \sum_{j \in J} \mathbb{1}_{R_j} \right\|_{L^p(\mu)} \lesssim \mu \left(\bigcup_{j \in J} R_j \right)^{\frac{1}{q}}.$$

They proved then the following.

Theorem 5 (Cordoba and R. Fefferman). *The family \mathcal{B} satisfies the covering property V_q if and only if the associated maximal operator $M_{\mathcal{B}}$ has weak-type (p, p) i.e. if for any f and $t > 0$, one has*

$$\mu(\{M_{\mathcal{B}}f > t\})^{\frac{1}{p}} \lesssim \frac{\|f\|_{L^p(\mu)}}{t}.$$

Here we have supposed that $\frac{1}{p} + \frac{1}{q} = 1$.

Hence, in regards of Theorems 3 and 4, one can see that the *geometry* deeply influence the behavior of M^c or M . Let us consider a slightly different problematic: in a fixed measured metric space (X, μ, d) and with this general definition of a maximal operator $M_{\mathcal{B}}$, one can wonder what happens when the family \mathcal{B} is composed of more complex geometric objects than balls? With appropriate geometric restrictions on the family \mathcal{B} , this problematic can unravel deep phenomena of the underlying space X and the interaction of the elements of the family \mathcal{B} .

Let us give an example in the Euclidean plane \mathbb{R}^2 : given an arbitrary set of directions $\Omega \subset \mathbb{S}^1$, consider the *directional* family \mathcal{R}_{Ω} composed of every rectangles whose longest side makes an angle $\omega \in \Omega$ with the Ox -axis and denote by M_{Ω} the maximal operator associated to this family. Those *directional* maximal operators have been intensively studied in the past decades (see for examples [1], [3] and [7]) and it was only after a long serie of different works that Bateman classified in [2] the behavior of directional maximal operators on the L^p spaces for $1 < p < \infty$. We give some examples before stating Bateman's Theorem:

- if the set of directions Ω is finite, then the maximal operator M_{Ω} is bounded on L^p for any $p > 1$.
- if we consider the set of directions $\Omega_{\text{lac}} = \{\frac{\pi}{2^k} : k \geq 1\}$, then the maximal operator $M_{\Omega_{\text{lac}}}$ is bounded on L^p for any $p > 1$, see [7]. The proof relies on Fourier analysis techniques
- if we consider the set of directions $\Omega_{\text{lin}} = \{\frac{\pi}{k} : k \geq 1\}$, then the maximal operator $M_{\Omega_{\text{lin}}}$ is not bounded on L^p for any $p < \infty$. The proof relies on the construction of a so-called *Perron's tree* which is a concrete realization of a *Keakeya blow* in the plane.
- if we consider a classic ternary Cantor set Ω_{cantor} embedded into \mathbb{S}^1 , then the maximal operator $M_{\Omega_{\text{cantor}}}$ is not bounded on L^p for any $p < \infty$, see [3]. As for Ω_{lin} , the proof relies on the realization of a *Keakeya blow* in the plane but *via* a sophisticated random construction.

Here, we have used the term *Keakeya blow* to designate the following *geometric obstruction* of the Euclidean space (we denote by \mathcal{R} the family containing every plane rectangle in \mathbb{R}^2).

Theorem 6 (Keakeya blow with \mathcal{R}). *Given any large constant $A \gg 1$, there exists a finite family of rectangles*

$$\{R_i : i \in I\} \subset \mathcal{R}$$

such that we have

$$\left| \bigcup_{i \in I} TR_i \right| \geq A \left| \bigcup_{i \in I} R_i \right|.$$

Here, we have denoted by TR the rectangle R translated along its longest side by its own length

This Theorem has several applications in analysis and is related to the traditional Keakeya problem. For example, C. Fefferman used a more refined version of this property in order to disprove the famous Ball multiplier Theorem: see [5] which also contains a proof of Theorem 6. For an arbitrary set of directions $\Omega \subset \mathbb{S}^1$, let us state Bateman's Theorem which indicates that the correct notion to consider is *finiteness lacunarity*. We invite the reader to look at Bateman's paper [2] for a precise definition of this notion.

Theorem 7 (Bateman). *We have the following alternative:*

- if the set of directions Ω is finitely lacunary then M_Ω is bounded on L^p for any $p > 1$.
- if the set of directions Ω is not finitely lacunary then it is possible to make a Keakeya blow with the family \mathcal{R}_Ω and in particular, M_Ω is not bounded on L^p for any $p < \infty$.

In this text, we investigate this problematic in the hyperbolic plane \mathbb{H}^2 endowed with its natural measure μ and where \mathcal{B} is a family included in \mathcal{T} - the family containing all the geodesic triangles of \mathbb{H}^2 - and where in addition \mathcal{B} is stable by isometries of \mathbb{H}^2 . Since we are able to completely determine the behavior of the operator $M_{\mathcal{B}}$, it seems interesting to study less regular geometric families and the following example seems pertinent. We fix a point $\omega \in \partial\mathbb{H}^2$ and we consider the family of curves \mathcal{G}_ω which contains every geodesics that goes to ω at infinite and also the family \mathcal{H}_ω which contains every horocycles tangent to ω : given two geodesics $d \neq d' \in \mathcal{G}_\omega$ and two horocycles $h \neq h' \in \mathcal{H}_\omega$, we consider the open bounded set $P(d, d', h, h')$ whose border is delimited by the four curves d, d', h and h' and then we define the family

$$\mathcal{P}_\omega = \{P(d, d', h, h') : d \neq d' \in \mathcal{G}_\omega, h \neq h' \in \mathcal{H}_\omega\}.$$

What can be said about the maximal operator

$$M_{\mathcal{P}_\omega} : L^\infty \rightarrow L^\infty$$

associated to the family \mathcal{P}_ω ? Can we expect boundedness property analogue to the strong maximal operator ? Thereafter, given an arbitrary family $\Omega \subset \partial\mathbb{H}^2$, one can consider the family

$$\mathcal{P}_\Omega = \bigcup_{\omega \in \Omega} \mathcal{P}_\omega.$$

What can be said about the operator $M_{\mathcal{P}_\Omega}$ associated to this family ? Does its behavior depend on Ω in the same fashion that in the Euclidean case ? We will return to those questions later.

2 Results

As said earlier, we are interested by families composed of geodesic triangles which are invariant under the action of isometries. In the following, we denote by G the group of isometries of \mathbb{H}^2 and by \mathcal{T} the family containing all geodesic triangles in \mathbb{H}^2 . A family \mathcal{B} included in \mathcal{T} and invariant by G is parameterized by the angles exhibited by the triangles it contains: if we denote by α_T the set of angles of the triangle T and that we let $A_{\mathcal{B}} := \{\alpha_T : T \in \mathcal{B}\}$ then, because \mathcal{B} is invariant under G , we precisely have

$$\mathcal{B} = \{T \in \mathcal{T} : \alpha_T \in A_{\mathcal{B}}\}.$$

We consider the simplex S of admissible angles which is defined as

$$S := \left\{ \alpha \in (\mathbb{R}_+)^3 : s(\alpha) \leq \pi \right\}$$

where $s(\alpha) := \alpha_1 + \alpha_2 + \alpha_3$. We are going to decompose the simplex S in three disjoint parts S_1, S_2 and S_3 according to the *eccentricity* function p defined as

$$p(\alpha) := \alpha_1 \alpha_2 \alpha_3.$$

Precisely we define:

- $S_0 := \{\alpha \in S : s(\alpha) < \pi, p(\alpha) > 0\}$
- $S_1 := \{\alpha \in S : s(\alpha) = \pi, p(\alpha) > 0\}$,
- $S_2 := \{\alpha \in S : s(\alpha) < \pi, p(\alpha) = 0\}$,
- and $S_3 := \{\alpha \in S : s(\alpha) = \pi, p(\alpha) = 0\}$

The functions s and p have simple geometric interpretations: on one hand for any triangle T , one has

$$\mu(T) = \pi - s(\alpha_T)$$

and on the other hand, $p(\alpha_T)$ represents the eccentricity of the triangle T , its *thickness*. We can now state our main result.

Theorem 8. *Let \mathcal{B} be a family of hyperbolic triangles stable by isometries and recall its set of angle is defined as $A := \{\alpha_T : T \in \mathcal{B}\}$. We have the following alternatives:*

- (I) *if $\bar{A} \subset S_0$, then the operator $M_{\mathcal{B}}$ is bounded from L^1 to L^1 i.e. there exists a constant $C_{\mathcal{B}} < \infty$ such that for any function f we have*

$$\|M_{\mathcal{B}}f\| \leq C_{\mathcal{B}}\|f\|_{L^1(\mu)}.$$

- (II) *if $\bar{A} \subset S_1$ and $\bar{A} \cap S_1 \neq \emptyset$, then the operator $M_{\mathcal{B}}$ has weak-type $(1,1)$ i.e. there exists a constant $C_{\mathcal{B}} < \infty$ such that for any function f and $t > 0$, the following estimate holds*

$$\mu(\{M_{\mathcal{B}}f > t\}) \leq C_{\mathcal{B}} \frac{\|f\|_{L^1(\mu)}}{t}.$$

- (III) *if $\bar{A} \cap S_2 \neq \emptyset$, then for any non zero function f , there exists a positive constant $c > 0$ depending on f and \mathcal{B} such that for any $y \in \mathbb{H}^2$, one has*

$$M_{\mathcal{B}}f(y) > c.$$

- (IV) *if $\bar{A} \cap S_3 \neq \emptyset$, then for any large constant $C \gg 1$, there exists a bounded set E in \mathbb{H}^2 satisfying*

$$\mu\left(\left\{M_{\mathcal{B}}\mathbb{1}_E \geq \frac{1}{4}\right\}\right) \geq C\mu(E).$$

The proof of Theorem 8 will be done in different parts and relies on different ideas: in contrast with the work of Stromberg [8], we exclusively use geometric techniques. To begin with, if $\bar{A} \subset S_0$, then the operator $M_{\mathcal{B}}$ is controlled by

$$T : f(x) \rightarrow \frac{1}{B(x,r)} \int_{B(x,r)} |f|$$

where the radius r is independent of x . Such an operator T is bounded from L^1 to L^1 and since $M_{\mathcal{B}} \leq T$, the conclusion comes easily. If $\bar{A} \subset S_1$ and $\bar{A} \cap S_1 \neq \emptyset$, this means that the triangles in the family \mathcal{B} cannot become arbitrarily thin. Hence, one can compare each of them to ball whose radius is uniformly bounded and this will allow us to prove the desired weak-type estimate thanks to a covering argument *à la* Vitali.

In the case where $\bar{A} \cap S_2 \neq \emptyset$, the family \mathcal{B} contains triangles that tend to become partly ideal. We prove that given any pair of point (x_0, y) one can always catch the point y and a small ball centered at x_0 with partly ideal triangles and this will allow us to prove that the maximal function $M_{\mathcal{B}}f$ of any non zero function is uniformly bounded by below. This effect is specific to the hyperbolic plane \mathbb{H}^2 and cannot be observed in the Euclidean space with classic convex sets.

Finally, if $\bar{A} \cap S_3 \neq \emptyset$ then the family \mathcal{B} contains arbitrarily thin triangles that tend to become Euclidean. We will be able to construct the desired set E by considering local families of triangles: we will prove that their images by the exponential map contain enough Euclidean triangles in an arbitrary tangent space $T_{x_0} \mathbb{H}^2$ which will allow us to exploit a Kakeya-type set of the Euclidean plane.

3 Thick triangles (I)

We suppose that we have $\bar{A} \subset S_0$. For any $x \in \mathbb{H}^2$, consider the family $\mathcal{B}(x)$ defined as

$$\mathcal{B}(x) := \{T \in \mathcal{B} : x \in T\}.$$

In other words, $\mathcal{B}(x)$ is composed of every triangles T in \mathcal{B} which contains the point x . We claim the following.

Claim 1. *If $\bar{A} \subset S_0$ then there exists $r = r(\mathcal{B}) > 0$ such that for any $T \in \mathcal{B}(x)$, one has $T \subset B(x, r)$ and also $\mu(T) \simeq_{\mathcal{B}} \mu(B(x, r))$.*

With this proposition at hands, the conclusion comes easily: for any f , we have

$$\int_{\mathbb{H}^2} M_{\mathcal{B}} f(x) d\mu(x) \lesssim_{\mathcal{B}} \int_{\mathbb{H}^2 \times \mathbb{H}^2} |f|(y) \mathbb{1}_{d(x,y) \leq r} d\mu(y) d\mu(x).$$

Indeed for any $x \in \mathbb{H}^2$, there exists $T \in \mathcal{B}(x)$ such that

$$M_{\mathcal{B}} f(x) \leq \frac{2}{\mu(T)} \int_T |f| \lesssim_{\mathcal{B}} \int_{B(x,r)} |f|$$

since we have $T \subset B(x, r)$ and that both sets have comparable volume. Using Fubini we obtain as expected

$$\int_{\mathbb{H}^2} M_{\mathcal{B}} f(x) d\mu(x) \lesssim_{\mathcal{B}} \|f\|_{L^1(\mu)}.$$

4 Thick triangles (II)

We suppose that the family \mathcal{B} satisfies $\bar{A} \subset S_1$ and $\bar{A} \cap S_1 \neq \emptyset$. This condition means that the eccentricity of a triangle in \mathcal{B} is uniformly bounded by below but we do not have control on its volume now. The following Claim holds.

Claim 2. *For any triangle $T \in \mathcal{B}$, there exists a ball $B(T)$ containing T and satisfying*

$$\mu(B(T)) \simeq_{\mathcal{B}} \mu(T).$$

In particular, there exists a constant $R < \infty$ only depending on \mathcal{B} such that

$$\sup_{T \in \mathcal{B}} \text{diam}(B(T)) < R.$$

We recall now the Vitali covering Theorem which is valid in metric space.

Theorem 9 (Vitali covering Theorem). *Given any family \mathcal{F} of balls whose radius is uniformly bounded by a constant $R < \infty$, there exists a sub-family $\mathcal{F}' \subset \mathcal{F}$ satisfying the following properties: on one hand the balls of \mathcal{F}' are pairwise disjoint and on the other hand, one has*

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B' \in \mathcal{F}'} 5B'.$$

Here $5B$ stands for the ball $B(x, 5r)$ if $B = B(x, r)$.

With Claim 2 and Theorem 9 at hands, we can prove that $M_{\mathcal{B}}$ has weak-type $(1, 1)$. We fix f a function on \mathbb{H}^2 and $t > 0$: by definition the level set $\{M_{\mathcal{B}}f > t\}$ is the union of the triangles T belonging to \mathcal{B} satisfying

$$\int_T |f| > \mu(T)t.$$

We denote by \mathcal{F} this family of triangles and we apply Claim 2 for each T in \mathcal{F} and then we apply Theorem 9 to each balls $B(T)$. This gives us a subfamily \mathcal{F}' of \mathcal{F} satisfying the following inclusion

$$\{M_{\mathcal{B}}f > t\} \subset \bigcup_{T' \in \mathcal{F}'} 5B(T')$$

and moreover, the balls $B(T')$ for $T' \in \mathcal{F}'$ are pairwise disjoint. Since all the balls considered have a radius uniformly bounded, we have

$$\mu(5B(T')) \simeq_{\mathcal{B}} \mu(B(T'))$$

and so the following estimate holds

$$\mu\left(\bigcup_{T' \in \mathcal{F}'} 5B(T')\right) \leq \sum_{T' \in \mathcal{F}'} \mu(5B(T')) \lesssim_{\mathcal{B}} \sum_{T' \in \mathcal{F}'} \mu(B(T')).$$

Now we also have $\mu(B(T')) \simeq_{\mathcal{B}} \mu(T')$ and so we obtain

$$\sum_{T' \in \mathcal{F}'} \mu(B(T')) \lesssim_{\mathcal{B}} \sum_{T' \in \mathcal{F}'} \frac{1}{t} \int_{B(T')} |f|.$$

Since the balls B' are pairwise disjoint we obtain as expected

$$\mu(\{M_{\mathcal{B}}f > t\}) \lesssim_{\mathcal{B}} \frac{1}{t} \int |f|.$$

5 Partly ideal triangles

We suppose now that the family \mathcal{B} satisfies $\bar{A} \cap S_2 \neq \emptyset$. We fix an element $\alpha \in \bar{A} \cap S_2$ and without loss of generality we suppose that it is of the form $\alpha = (0, \alpha_2, \alpha_3)$. We prove the following Proposition.

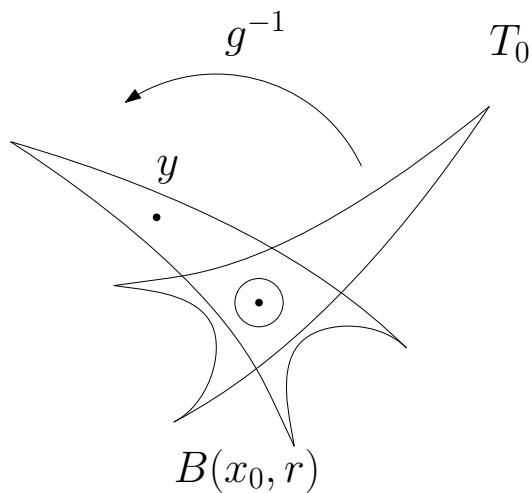


Figure 1: An illustration of the proof of Theorem 1.

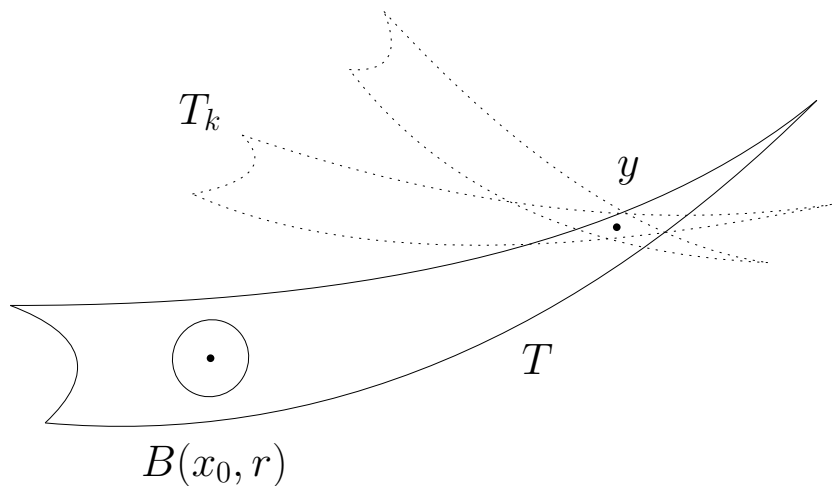


Figure 2: The existence of the sequence $\{T_k : k \geq 1\}$ comes from the fact that α belongs to \bar{A} .

Proposition 1. *For any $x_0 \in \mathbb{H}^2$, there exists $r > 0$ such that for any $y \in \mathbb{H}^2$, there is a triangle T satisfying $\alpha_T = (0, \alpha_2, \alpha_3)$, $y \in T$ and $B(x_0, r) \subset T$.*

Proof. Let $x_0 \in \mathbb{H}^2$ and consider a triangle T_0 of angles $(0, \alpha_2, \alpha_3)$. As the action of G on \mathbb{H}^2 is transitive, we may assume that T_0 contains x_0 in its interior: fix then $r > 0$ such that T_0 contains the ball $B(x_0, r)$.

Given any point $y \in \mathbb{H}^2$, its orbit under the stabilizer of x_0 is exactly the circle $C(x_0, d(x_0, y))$. As T_0 has at least one angle equal to 0 by hypothesis, it

is noncompact and intersect nontrivially $C(x_0, d(x_0, y))$. Hence there is $g \in G$ such that $g(x_0) = x_0$ and $g(y) \in T_0$ i.e. $y \in T := g^{-1}T_0$ and we have

$$B(x_0, r) \subset T.$$

□

Proposition 1 yields the conclusion as follow. Consider a non zero function f on \mathbb{H}^2 : there exists a ball $B(x_0, r)$ such that $\int_{B(x_0, r)} |f| > 0$ and by a standard compactness argument, we can choose r to be arbitrarily small. We fix any point y in \mathbb{H}^2 and we apply Theorem 1 which gives us a triangle T satisfying $\alpha_T = (0, \alpha_2, \alpha_3)$, $y \in T$ and $B(x_0, r) \subset T$. To conclude, it suffices to observe that since $\alpha \in \bar{A} \cap S_2$, there exists a sequence of triangles $\{T_k : k \geq 1\} \subset \mathcal{B}$ such that each of them contain y and satisfying

$$\int_{T_k} |f| \rightarrow \int_T |f|.$$

In particular we have

$$M_{\mathcal{B}}f(y) \geq \frac{1}{\mu(T)} \int_T |f| \geq \frac{1}{\pi} \int_{B(x_0, r)} |f|.$$

This yields the desired conclusion since the integral $\int_{B(x_0, r)} |f|$ is independant of the point y (it only depends on f and \mathcal{B}).

6 Thin and small triangles

We suppose now that the family \mathcal{B} satisfies $\bar{A} \cap S_3 \neq \emptyset$. We are going to prove that for any constant $C \gg 1$, there exists a bounded set E in \mathbb{H}^2 satisfying

$$\mu \left(\left\{ M_{\mathcal{B}} \mathbb{1}_E \geq \frac{1}{4} \right\} \right) \geq C \mu(E).$$

In particular, this property implies that the operator $M_{\mathcal{B}}$ is not bounded on the space $L^p(\mathbb{H}^2)$ for any $p < \infty$. To construct the set E , the idea is to make a Kakeya blow with the triangles of the family \mathcal{B} : this is possible precisely because the family \mathcal{B} contains triangles which are arbitrarily thin and almost Euclidean. To make this argument rigorous, we need to work in a tangent space $T_{x_0} \mathbb{H}^2$ and to precisely describe the image of small triangles in a neighborhood of x_0 by the exponential map.

We introduce the following notation: given two triangles T and T' (Euclidean or hyperbolic), we note

$$T \sim T'$$

if there is an isometry g such that $T = g(T')$. In the following, we fix any point $x_0 \in \mathbb{H}^2$ and let \exp_{x_0} be the exponential map between \mathbb{H}^2 and $T_{x_0} \mathbb{H}^2$. We also

suppose that we work in a neighborhood $B(x_0, r)$ of x_0 so small that we have the following estimate for any open set $U \subset B(x_0, r)$ and with $h = \frac{1}{1000}$

$$(1 - h) |\exp_{x_0}(U)| \leq \mu(U) \leq (1 + h) |\exp_{x_0}(U)|.$$

We consider *local* families of triangles: given any $r > 0$ and a triangle T , we consider the following family of triangles

$$\mathcal{L}_r(T) := \{T' \in \mathcal{T} : T' \sim T, T' \subset B(x_0, r)\}.$$

Because we want to be able to *move* our triangle T in the ball $B(x_0, r)$, we assume that we have

$$\frac{r}{10} > \text{diam}(T).$$

The following proposition states that in the Euclidean plane, a maximal operator defined on a local family of thin triangles can exploit a Kakeya-type set.

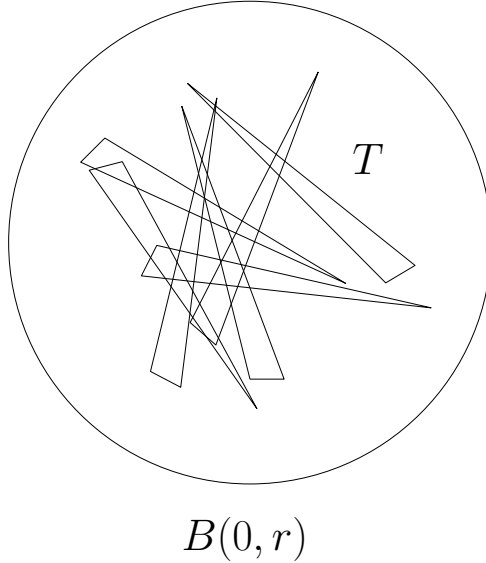


Figure 3: Representation of a local family $\mathcal{L}_r(T)$ in the Euclidean plane.

Proposition 2. *For any large constant $C \gg 1$, there exists $0 < \epsilon \ll 1$ such that if T_i is an Euclidean triangle thin enough (i.e. satisfying $p(\alpha_{T_i}) < \epsilon$) then any maximal operator $M_{\mathcal{L}_r(T_i)}$ defined on a non empty local family*

$$\mathcal{L}_r(T_i) := \{T' \in \mathcal{T} : T' \sim T, T' \subset B(0, r)\}$$

admits a Kakeya-type set F included in $B(x_0, r)$ satisfying

$$\left| \left\{ M_{\mathcal{L}_r(T_i)} \mathbb{1}_F \geq \frac{1}{4} \right\} \right| \geq C |F|.$$

Proposition 2 is a well-known fact in the literature and we omit its proof. To conclude the proof of Theorem 8, we need the following proposition which simply states that given a local family generated by a small triangle T in \mathbb{H}^2 , its image under the exponential maps is compatible with a local family generated by a triangle T_i which is close to T according to their sets of angles.

Proposition 3. *Fix an arbitrary precision $\delta > 0$: there exists $0 < r, \epsilon \ll 1$ (which can be taken arbitrarily small) such that for any set of angles $\alpha_0 \in S$ satisfying $|\pi - s(\alpha_0)| \leq \epsilon$, there exists a set of Euclidean angles α_1 satisfying $\|\alpha_0/\alpha_1 - 1\|_1 \leq \delta$ and such that for any triangle T included in $B(x_0, r)$ and satisfying*

$$\alpha_T = \alpha_0,$$

there exists an Euclidean triangle $\varphi(T)$ which depends on T satisfying

$$\varphi(T) \subset \exp_{x_0}(T) \tag{1}$$

$$|\varphi(T)| \geq \frac{1}{10}|T| \tag{2}$$

$$\alpha_{\varphi(T)} = \alpha_1. \tag{3}$$

Finally, there exists $0 < r' \ll 1$ such that the following inclusion holds

$$\emptyset \neq \mathcal{L}_{r'}(\varphi(T)) \subset \varphi(\mathcal{L}_r(T)).$$

Proof. As it is well known, the differential at 0 of the exponential map is the identity. Hence given $\epsilon > 0$, there is $r > 0$ such that the restriction of the exponential map to $B(0, r)$ is $(1 + \epsilon)$ bilipschitz and quasiconformal. In particular, \exp_{x_0} will be uniformly close to an isometry, so the preimage of a hyperbolic triangle T in $B(x_0, r)$ with angles α_0 will contain a Euclidean triangle with angles α_1 satisfying $\|\alpha_1/\alpha_0 - 1\|_1 \leq \epsilon$. Taking ϵ small enough will give exactly the claimed properties. □

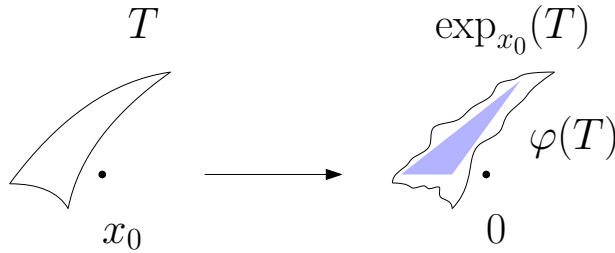


Figure 4: Given a triangle T close to x_0 , we claim that its image $\exp_{x_0}(T)$ contains a triangle $\varphi(T)$ which is not too small compared to T and which is also thin enough.

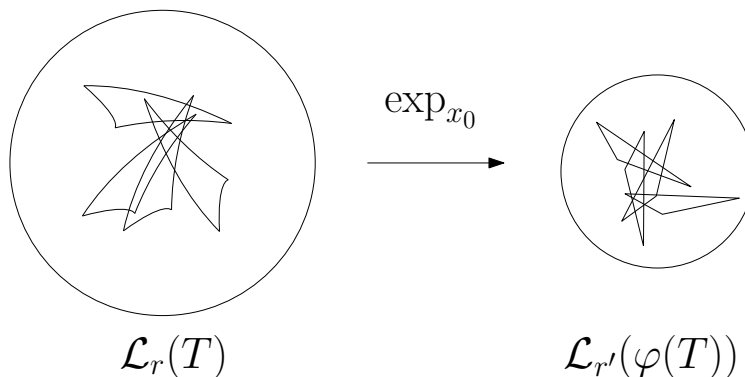


Figure 5: We also claim that the image of the local family $\mathcal{L}_r(T)$ under the exponential will contain a local family $\mathcal{L}_{r'}(\varphi(T))$: this second family is now contained in the Euclidean plane and so we can use Proposition 2.

With Propositions 2 and 3 at hands, we can conclude the proof of Theorem 8. Fix a large constant $C \gg 1$ and let $\epsilon \ll 1$ as in Proposition 2. Because we suppose that $\bar{A} \cap S_3 \neq \emptyset$, we can choose an set of angles $\alpha_0 \in A$ such that if α_1 is the set of Euclidean angles as described in Proposition 3, then

$$p(\alpha_1) < \epsilon.$$

In this situation, the local family $\mathcal{L}_{r'}(\varphi(T))$ given by Proposition 3 satisfies the condition of Proposition 2: hence there exists a Kakeya-type set F satisfying

$$\left| \left\{ M_{\mathcal{L}_{r'}(\varphi(T))} \mathbb{1}_F \geq \frac{1}{4} \right\} \right| \geq C |F|.$$

Define now the set E in \mathbb{H}^2 as the pre-image of F by the exponential map

$$E := \exp_{x_0}^{-1}(F)$$

and let us see why we can exploit the set E with the maximal operator $M_{\mathcal{B}}$. Fix $\varphi(T')$ in $\mathcal{L}_{r'}(\varphi(T))$ such that

$$|\varphi(T') \cap F| \geq \frac{1}{4} |\varphi(T')|$$

and observe that in \mathbb{H}^2 we have

$$\mu(T' \cap E) \simeq |\exp_{x_0}(T' \cap E)| \geq |\varphi(T') \cap F| \geq \frac{1}{4} |\varphi(T')| \simeq \frac{1}{40} \mu(T').$$

Finally, the following estimate follows by inclusion

$$\mu \left(\left\{ M_{\mathcal{B}} \mathbb{1}_E \geq \frac{1}{40} \right\} \right) \geq \mu \left(\bigcup_{T'} T' \right) \simeq \left| \bigcup_{T'} \varphi(T') \right| \geq C |E| \simeq C \mu(F).$$

This concludes the last part of the proof of Theorem 8

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