Morphisms of pre-Calabi-Yau categories and morphisms of cyclic A_∞-categories

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Abstract

In this article we prove that there exists a relation between d-pre-Calabi-morphisms introduced by M. Kontsevich, A. Takeda and Y. Vlassopoulos and cyclic A_∞-morphisms, extending a result proved by D. Fernández and E. Herscovich. This leads to a functor between the category of d-pre-Calabi-Yau structures and the partial category of A_∞-categories of the form $\mathcal{A}\oplus\mathcal{A}^*[d-1]$ with $\mathcal{A}$ a graded quiver and whose morphisms are the data of an A_∞-structure on $\mathcal{A}\oplus\mathcal{B}^*[d-1]$ together with A_∞-morphisms $\mathcal{A}[1]\oplus\mathcal{B}^*[d]\to\mathcal{A}[1]\oplus\mathcal{A}^*[d]$ and $\mathcal{A}[1]\oplus\mathcal{B}^*[d]\to\mathcal{B}[1]\oplus\mathcal{B}^*[d]$.

Mathematics subject classification 2020: 16E45, 18G70, 14A22
Keywords: A_∞-categories, pre-Calabi-Yau categories

1 Introduction

Pre-Calabi-Yau algebras were introduced by M. Kontsevich and Y. Vlassopoulos in the last decade. These structures have also appeared under different names, such as V_∞-algebras in [10], A_∞-algebras with boundary in [7], or weak Calabi-Yau structures in [3] for example. These references show that pre-Calabi-Yau structures play an important role in homological algebra, symplectic geometry, string topology, noncommutative geometry and even in Topological Quantum Field Theory.

In the finite dimensional case, pre-Calabi-Yau algebras are strongly related to A_∞-algebras. Actually, for $d\in\mathbb{Z}$, a d-pre-Calabi-Yau structure on a finite dimensional vector space A is equivalent to a cyclic A_∞-structure on $A\oplus A^*[d-1]$ that restricts to A. The definition of pre-Calabi-Yau morphisms first appeared in [5] and then in [6], in the properadic setting. A natural question is then about the link between pre-Calabi-Yau morphisms and A_∞-morphisms of the corresponding boundary construction.

D. Fernández and E. Herscovich studied this link in [1] at the level of double Poisson dg algebras and a restricted class of pre-Calabi-Yau algebras, when the multiplications m_n vanish for $n\geq 4$. In this paper, we study the relation between A_∞-morphisms and pre-Calabi-Yau morphisms in a larger generality. The main result of this paper is the existence of a functor from the category of d-pre-Calabi-Yau structures pCY_d to the partial category $\widehat{A_{\infty}}_d$ whose objects are A_∞-categories of the form $\mathcal{A}\oplus\mathcal{A}^*[d-1]$ and whose morphisms are the data of an A_∞-structure on

$\mathcal{A} \oplus \mathcal{B}^*[d-1]$ together with a diagram of the form

$$\begin{tikzcd}
& \mathcal{A}[1] \oplus \mathcal{B}^*[d] \arrow[dl] \arrow[dr] & \\
\mathcal{A}[1] \oplus \mathcal{A}^*[d] & & \mathcal{B}[1] \oplus \mathcal{B}^*[d]
\end{tikzcd} \tag{1.1}$$

where each of the arrows are A_∞-morphisms.

We also show that this functor restricts to a functor between a subcategory of pCY_d and the partial subcategory of $\widehat{A_{\infty}}_d$ whose objects are those of $\widehat{A_{\infty}}_d$ and whose morphisms are the data of an almost cyclic A_∞-structure on $\mathcal{A} \oplus \mathcal{B}^*[d-1]$ together with a diagram of the form (1.1) where the arrows are A_∞-morphisms.

Let us briefly present the contents of the article. In Section 2, we fix the notations and conventions we use in this paper and in Section 3, we recall the notions related to A_∞-categories. Section 4 is devoted to present the notion of discs and diagrams as well as the notion of pre-Calabi-Yau structures based on the necklace bracket introduced in [5], which is given as the commutator of a necklace product, and their link with A_∞-structures in the case of a Hom-finite graded quiver. We incidentally show that the necklace product for a graded quiver $\mathcal{A}$ is in fact equivalent to the usual Gerstenhaber circle product on $\mathcal{A} \oplus \mathcal{A}^*[d-1]$ (see Proposition 4.29), which does not seem to have been observed in the literature so far. In Section 5, we recall the definitions of pre-Calabi-Yau morphisms and of the category pCY_d given in [5].

Section 6 is the core of the article. In Subsection 6.1, we prove that given d-pre-Calabi-Yau categories $\mathcal{A}$ and $\mathcal{B}$ and a strict d-pre-Calabi-Yau morphism $\mathcal{A} \to \mathcal{B}$, we can produce a cyclic A_∞-structure on $\mathcal{A} \oplus \mathcal{B}^*[d-1]$ and a diagram of the form (1.1) whose arrows are cyclic strict A_∞-morphisms. We summarize these results in Corollary 6.14. In Subsection 6.2, we prove that given d-pre-Calabi-Yau categories $\mathcal{A}$ and $\mathcal{B}$ and any d-pre-Calabi-Yau morphism $\mathcal{A} \to \mathcal{B}$, we can produce an A_∞-structure on $\mathcal{A} \oplus \mathcal{B}^*[d-1]$ and a diagram of the form (1.1) where the arrows are A_∞-morphisms. Moreover, with an additional assumption on the pre-Calabi-Yau morphism $\mathcal{A} \to \mathcal{B}$, the A_∞-structure on $\mathcal{A} \oplus \mathcal{B}^*[d-1]$ is almost cyclic. We summarize this in Corollary 6.20 in terms of functors.

Acknowledgements. This work is supported by the French National Research Agency in the framework of the "France 2030" program (ANR-15-IDEX-0002) and by the LabEx PERSYVAL-Lab (ANR-11-LABX-0025-01).

2 Notations and conventions

In what follows $\Bbbk$ will be a field of characteristic 0 and to simplify we will denote $\otimes$ for $\otimes_{\Bbbk}$. We will denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of natural numbers and we define $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For $i, j \in \mathbb{N}$, we define the interval of integers $[\![i, j]\!] = \{n \in \mathbb{N} | i \le n \le j\}$.

Recall that if we have a (cohomologically) graded vector space $V = \oplus_{i \in \mathbb{Z}} V^i$, we define for $n \in \mathbb{Z}$ the graded vector space $V[n]$ given by $V[n]^i = V^{n+i}$ for $i \in \mathbb{Z}$ and the map $s_{V,n} : V \to V[n]$ whose underlying set theoretic map is the identity. Moreover, if $f : V \to W$ is a morphism of graded vector spaces, we define the map $f[n] : V[n] \to W[n]$ sending an element $s_{V,n}(v)$ to $s_{W,n}(f(v))$ for all $v \in V$. We will denote $s_{V,n}$ simply by s_n when there is no possible confusion, and s_1 just by s.

We now recall the Koszul sign rules, that are the ones we use to determine the signs appearing in this paper. If V, W are graded vector spaces, we have a map $\tau_{V,W} : V \otimes W \to W \otimes V$ defined as

$$\tau_{V,W}(v \otimes w) = (-1)^{|w||v|} w \otimes v$$

where $v \in V$ is a homogeneous element of degree $|v|$ and $w \in W$ is a homogeneous element of degree $|w|$. More generally, given graded vector spaces $V_1, \ldots, V_n$, we have a map

$$\tau_{V_1,\ldots,V_n} : V_1 \otimes \cdots \otimes V_n \to V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(n)}$$

defined as

$$\tau_{V_1,\ldots,V_n}(v_1 \otimes \cdots \otimes v_n) = (-1)^{\epsilon}(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)})$$

with

$$\epsilon = \sum_{\substack{i>j \\ \sigma^{-1}(i)<\sigma^{-1}(j)}} |v_{\sigma^{-1}(i)}||v_{\sigma^{-1}(j)}|$$

where $v_i \in V_i$ is a homogeneous element of degree $|v_i|$ for $i \in [\![1, n]\!]$.

Throughout this paper, when we consider an element v of degree $|v|$ in a graded vector space V, we mean a homogeneous element v of V. Also, we will denote by id the identity map of every space of morphisms, without specifying it. All the products in this paper will be products in the category of graded vector spaces. Given graded vector spaces $(V_i)_{i\in I}$, we thus have

$$\prod_{i\in I} V_i = \bigoplus_{n\in\mathbb{Z}} \prod_{i\in I} V_i^n$$

where the second product is the usual product of vector spaces.

Given graded vector spaces V, W we will denote by $\mathrm{Hom}_{\Bbbk}(V, W)$ the vector space of $\Bbbk$-linear maps $f : V \to W$ and by $\mathrm{hom}^d_{\Bbbk}(V, W)$ the vector space of homogeneous $\Bbbk$-linear maps $f : V \to W$ of degree d, *i.e.* $f(v) \in W^{n+d}$ for all $v \in V^n$. We assemble them in the graded vector space $\mathcal{H}om_{\Bbbk}(V, W) = \bigoplus_{d\in\mathbb{Z}} \mathrm{hom}^d_{\Bbbk}(V, W) \subseteq \mathrm{Hom}_{\Bbbk}(V, W)$. We define the **graded dual** of a graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V^n$ as the graded vector space $V^* = \mathcal{H}om_{\Bbbk}(V, \Bbbk)$. Moreover, given graded vector spaces V, V', W, W' and homogeneous elements $f \in \mathcal{H}om_{\Bbbk}(V, V')$ and $g \in \mathcal{H}om_{\Bbbk}(W, W')$, we have that

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$$

for homogeneous elements $v \in V$ and $w \in W$. Recall that given graded vector spaces $V_1, \ldots, V_n$ and $d \in \mathbb{Z}$ we have a homogeneous linear isomorphism of degree 0

$$H_j : (\bigotimes_{i=1}^{n} V_i)[d] \to V_1 \otimes \cdots \otimes V_{j-1} \otimes V_j[d] \otimes V_{j+1} \otimes \cdots \otimes V_n \tag{2.1}$$

sending an element $s_d(v_1 \otimes \cdots \otimes v_n)$ to $(-1)^{d(|v_1|+\cdots+|v_{j-1}|)} v_1 \otimes \cdots \otimes v_{j-1} \otimes s_d v_j \otimes v_{j+1} \otimes \cdots \otimes v_n$. Moreover, given graded vector spaces V and W and an integer $d \in \mathbb{Z}$, we have homogeneous linear isomorphisms of degree 0

$$\mathcal{H}om_{\Bbbk}(V, W)[d] \to \mathcal{H}om_{\Bbbk}(V, W[d]) \tag{2.2}$$

sending $s_d f \in \mathcal{H}om_{\Bbbk}(V,W)[d]$ to the map sending $v \in V$ to $s_d(f(v))$ and

$$\mathcal{H}om_{\Bbbk}(V,W)[d] \to \mathcal{H}om_{\Bbbk}(V[-d],W) \tag{2.3}$$

sending $s_d f \in \mathcal{H}om_{\Bbbk}(V,W)[d]$ to the map sending $s_{-d}v \in V[-d]$ to $(-1)^{d|f|}f(v)$.

Recall that a graded quiver $\mathcal{A}$ consists of a set of objects $\mathcal{O}$ together with graded vector spaces ${}_y\mathcal{A}_x$ for every $x, y \in \mathcal{O}$. A dg quiver $\mathcal{A}$ is a graded quiver such that ${}_y\mathcal{A}_x$ is a dg vector space for every $x, y \in \mathcal{O}$. Given a quiver $\mathcal{A}$, its **enveloping graded quiver** is the graded quiver $\mathcal{A}^e = \mathcal{A}^{op} \otimes \mathcal{A}$ whose set of objects is $\mathcal{O} \times \mathcal{O}$ and whose space of morphisms from an object (x, y) to an object (x', y') is defined as the graded vector space ${}_{(x',y')}(\mathcal{A}^{op} \otimes \mathcal{A})_{(x,y)} = {}_x\mathcal{A}_{x'} \otimes {}_{y'}\mathcal{A}_y$. A graded quiver $\mathcal{A}$ with set of objects $\mathcal{O}$ is said to be **Hom-finite** if ${}_y\mathcal{A}_x$ is finite dimensional for every $x, y \in \mathcal{O}$. Given graded quivers $\mathcal{A}$ and $\mathcal{B}$ with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$, a **morphism of graded quivers** $(\Phi_0, \Phi) : \mathcal{A} \to \mathcal{B}$ is the data of a map $\Phi_0 : \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$ between the sets of objects together with a collection $\Phi = ({}_y\Phi_x)_{x,y\in\mathcal{O}_{\mathcal{A}}}$ of morphisms of graded vector spaces ${}_y\Phi_x : {}_y\mathcal{A}_x \to {}_{\Phi_0(y)}\mathcal{B}_{\Phi_0(x)}$ for every $x, y \in \mathcal{O}_{\mathcal{A}}$. In this paper we will only consider small graded quivers and small categories.

We will denote

$$\bar{\mathcal{O}} = \bigsqcup_{n\in\mathbb{N}^*} \mathcal{O}^n$$

and more generally, we will denote by $\bar{\bar{\mathcal{O}}}$ the set formed by all finite tuples of elements of $\bar{\mathcal{O}}$, *i.e.*

$$\bar{\bar{\mathcal{O}}} = \bigsqcup_{n\in\mathbb{N}^*} \bar{\mathcal{O}}^n = \bigsqcup_{n>0} \bigsqcup_{(p_1,\dots,p_n)\in\mathcal{T}_n} \mathcal{O}^{p_1} \times \cdots \times \mathcal{O}^{p_n}$$

where $\mathcal{T}_n = \mathbb{N}^n$ for $n > 1$ and $\mathcal{T}_1 = \mathbb{N}^*$. Given $\bar{x} = (x_1, \dots, x_n) \in \bar{\mathcal{O}}$ we define its **length** as $\mathrm{lg}(\bar{x}) = n$, its **left term** as $\mathrm{lt}(\bar{x}) = x_1$ and **right term** as $\mathrm{rt}(\bar{x}) = x_n$. For $i \in [\![1, n]\!]$, we define $\bar{x}_{\le i} = (x_1, \dots, x_i)$, $\bar{x}_{\ge i} = (x_i, \dots, x_n)$ and for $j > i$, $\bar{x}_{[\![i,j]\!]} = (x_i, x_{i+1}, \dots, x_j)$. One can similarly define $\bar{x}_{<i}$ and $\bar{x}_{>i}$. Moreover, given $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\bar{\mathcal{O}}}$ we define its **length** as $\mathrm{lg}(\bar{\bar{x}}) = n$, its **left term** as $\mathrm{lt}(\bar{x}) = \bar{x}^1$ and its **right term** as $\mathrm{rt}(\bar{x}) = \bar{x}^n$. For $\bar{x} = (x_1, \dots, x_n) \in \bar{\mathcal{O}}$, we will denote

$$\mathcal{A}^{\otimes\bar{x}} = {}_{x_1}\mathcal{A}_{x_2} \otimes {}_{x_2}\mathcal{A}_{x_3} \otimes \cdots \otimes {}_{x_{\mathrm{lg}(\bar{x})-1}}\mathcal{A}_{x_{\mathrm{lg}(\bar{x})}}$$

and we will often denote an element of $\mathcal{A}^{\otimes\bar{x}}$ as $a_1, a_2, \dots, a_{\mathrm{lg}(\bar{x})-1}$ instead of $a_1 \otimes a_2 \otimes \cdots \otimes a_{\mathrm{lg}(\bar{x})-1}$ for $a_i \in {}_{x_i}\mathcal{A}_{x_{i+1}}$, $i \in [\![1, \mathrm{lg}(\bar{x}) - 1]\!]$. Moreover, given a tuple $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\bar{\mathcal{O}}}$ we will denote $\mathcal{A}^{\otimes\bar{\bar{x}}} = \mathcal{A}^{\otimes\bar{x}^1} \otimes \mathcal{A}^{\otimes\bar{x}^2} \otimes \cdots \otimes \mathcal{A}^{\otimes\bar{x}^n}$. Given tuples $\bar{x} = (x_1, \dots, x_n), \bar{y} = (y_1, \dots, y_m) \in \bar{\mathcal{O}}$, we define their concatenation as $\bar{x} \sqcup \bar{y} = (x_1, \dots, x_n, y_1, \dots, y_m)$. We also define the inverse of a tuple $\bar{x} = (x_1, \dots, x_n) \in \bar{\mathcal{O}}$ as $\bar{x}^{-1} = (x_n, x_{n-1}, \dots, x_1)$. If $\sigma \in \mathcal{S}_n$ and $\bar{x} = (x_1, \dots, x_n) \in \mathcal{O}^n$, we define $\bar{x} \cdot \sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$. Moreover, given $\sigma \in \mathcal{S}_n$ and $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}^n$, we define $\bar{\bar{x}} \cdot \sigma = (\bar{x}^{\sigma(1)}, \bar{x}^{\sigma(2)}, \dots, \bar{x}^{\sigma(n)})$. We denote by C_n the subgroup of $\mathcal{S}_n$ generated by the cycle $\sigma = (12 \dots n)$ which sends $i \in [\![1, n - 1]\!]$ to $i + 1$ and n to 1.

3 A_∞-categories

In this section, we recall the notion of (cyclic) A_∞-categories and (cyclic) A_∞-morphisms as well as the definition of the natural bilinear form associated to a graded quiver. We also introduce a bilinear form on categories of the form $\mathcal{A} \oplus \mathcal{B}^*$ where $\mathcal{A}$ and $\mathcal{B}$ are graded quivers related by a morphism $\mathcal{A} \to \mathcal{B}$. We refer the reader to [7] for the definitions of modules and bimodules over a category.

Definition 3.1. *Given a graded quiver $\mathcal{A}$ with set of objects $\mathcal{O}$, we define the graded vector space*

$$C(\mathcal{A}) = \prod_{p \geq 1} \prod_{\bar{x} \in \mathcal{O}^p} \mathcal{H}om_{\Bbbk}(\mathcal{A}[1]^{\otimes \bar{x}}, {}_{\mathrm{lt}(\bar{x})}\mathcal{A}_{\mathrm{rt}(\bar{x})})$$

Given $\bar{x} = (x_1, \ldots, x_n) \in \mathcal{O}^n$ and a map $F^{\bar{x}} : \mathcal{A}[1]^{\otimes \bar{x}} \to {}_{\mathrm{lt}(\bar{x})}\mathcal{A}_{\mathrm{rt}(\bar{x})}$, we associate to $F^{\bar{x}}$ a disc with several incoming arrows and one outgoing arrow (see Figure 3.1).

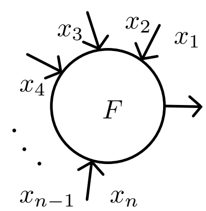


Figure 3.1: A disc representing a map $F^{\bar{x}} : \mathcal{A}[1]^{\otimes \bar{x}} \to {}_{\mathrm{lt}(\bar{x})}\mathcal{A}_{\mathrm{rt}(\bar{x})}$, where $\bar{x} = (x_1, \ldots, x_n)$

To simplify, we will often omit the objects and assemble the incoming arrows in a big arrow (see Figure 3.2).

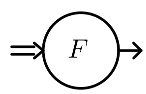


Figure 3.2: The representation of a homogeneous element in $C(\mathcal{A})$

Definition 3.2. *The* ***type*** *of a disc representing a map $F^{\bar{x}} : \mathcal{A}[1]^{\otimes \bar{x}} \to {}_{\mathrm{lt}(\bar{x})}\mathcal{A}_{\mathrm{rt}(\bar{x})}$ is the tuple $\bar{x} \in \bar{\mathcal{O}}$.*

Definition 3.3. *Let $\mathcal{A}$ be a graded quiver. By the isomorphism (2.2), an element $sF \in \mathcal{C}(\mathcal{A})[1]$ induces maps*

$$\mathcal{A}[1]^{\otimes \bar{x}} \to {}_{\mathrm{lt}(\bar{x})}\mathcal{A}_{\mathrm{rt}(\bar{x})}[1]$$

sending an element $(sa_1, \ldots, sa_{n-1})$ to $s(F^{\bar{x}}(sa_1, \ldots, sa_{n-1}))$ for $\bar{x} = (x_1, \ldots, x_n) \in \bar{\mathcal{O}}$, $a_i \in {}_{x_i}\mathcal{A}_{x_{i+1}}$, $i \in [\![1, n-1]\!]$. To a homogeneous element $sF \in \mathcal{C}(\mathcal{A})[1]$, we thus associate disc with a bold outgoing arrow (see Figure 3.3) to indicate that the output of sF is an element in $\mathcal{A}[1]$.

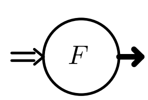
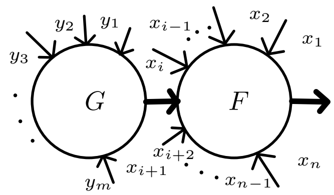


Figure 3.3: The representation of a homogeneous element in $C(\mathcal{A})[1]$

Example 3.4. *The diagram*



is the composition of a disc associated to a map

$$G^{(x_i,y_1,\ldots,y_m,x_{i+1})} : \mathcal{A}[1]^{\otimes(x_i,y_1,\ldots,y_m,x_{i+1})} \to {}_{x_i}\mathcal{A}_{x_{i+1}}[1]$$

with a disc associated to a map

$$F^{\bar{x}} : \mathcal{A}[1]^{\otimes\bar{x}} \to {}_{x_1}\mathcal{A}_{x_n}[1]$$

with $\bar{x} = (x_1,\ldots,x_n)$. *The type of this diagram is* $\bar{x}_{\leq i} \sqcup \bar{y} \sqcup \bar{x}_{>i}$. *This diagram is associated to the map* $F^{\bar{x}} \circ (\mathrm{id}^{\otimes i-1} \otimes G^{(x_i,y_1,\ldots,y_m,x_{i+1})} \otimes \mathrm{id}^{\otimes n-i})$.

Definition 3.5. *Given a dg bimodule* M *over a dg category* $\mathcal{A}$ *with set of objects* $\mathcal{O}$, *its* ***naturalization*** *is the chain complex* $M_{nat} = (\bigoplus_{x\in\mathcal{O}} {}_xM_x)/(fg - (-1)^{|g||f|}gf)$ *where* $f \in {}_y\mathcal{A}_x$ *and* $g \in {}_xM_y$.

Definition 3.6. *Given dg category* $\mathcal{A}$ *with differential* $d_{\mathcal{A}}$ *and product* μ, *we define the dg bimodule* $\mathrm{Bar}(\mathcal{A})$ *as* $\mathrm{Bar}(\mathcal{A}) = ({}_{x'}\mathrm{Bar}(\mathcal{A})_x)_{x,x'\in\mathcal{O}}$, *where*

$${}_{x'}\mathrm{Bar}(\mathcal{A})_x = \bigoplus_{p\geq 0} \bigoplus_{(x_0,\ldots,x_p)\in\mathcal{O}^p} {}_{x'}\mathcal{A}_{x_0} \otimes {}_{x_0}\mathcal{A}_{x_1}[1] \otimes \ldots \otimes {}_{x_{p-1}}\mathcal{A}_{x_p}[1] \otimes {}_{x_p}\mathcal{A}_x$$

whose differential restricted to ${}_{x'}\mathrm{Bar}(\mathcal{A})_x$ *is given by* $\sum_{\bar{x}\in\bar{\mathcal{O}}} d_0^{\bar{x}} + d_1^{\bar{x}}$ *where*

$$d_0^{\bar{x}}(f_0 \otimes sf_1 \otimes \ldots \otimes sf_n \otimes f_{n+1}) = \sum_{i=0}^{n+1} (-1)^{\sum_{j=0}^{i-1}(|f_j|+1)} f_0 \otimes sf_1 \otimes \ldots \otimes s(d_{\mathcal{A}}(f_i)) \otimes \ldots \otimes sf_n \otimes f_{n+1}$$

and

$$\begin{aligned} d_1^{\bar{x}}(f_0 \otimes sf_1 \otimes \ldots \otimes sf_n \otimes f_{n+1}) = {} & (-1)^{\epsilon_1} f_0f_1 \otimes sf_2 \otimes \ldots \otimes sf_n \otimes f_{n+1} \\ & + \sum_{i=1}^{n-1} (-1)^{\epsilon_i} f_0 \otimes sf_1 \otimes \ldots \otimes s(f_if_{i+1}) \otimes \ldots \otimes sf_n \otimes f_{n+1} \\ & + (-1)^{\epsilon_n} f_0 \otimes sf_1 \otimes \ldots \otimes sf_{n-1} \otimes f_nf_{n+1} \end{aligned}$$

for all $\bar{x} = (x_0,x_1,\ldots,x_n)$, $f_0 \in {}_{x'}\mathcal{A}_{x_0}$, $f_{n+1} \in {}_{x_n}\mathcal{A}_x$ *and* $f_i \in {}_{x_{i-1}}\mathcal{A}_{x_i}$, $i \in [\![1,n]\!]$, *with*

$$\epsilon_i = |f_0| + \sum_{j=1}^{i-1}(|f_j|+1)$$

where we have written $f_i f_{i+1}$ instead of $\mu(f_i, f_{i+1})$ for $i \in [\![0, n]\!]$ to denote the composition of $\mathcal{A}$.

The bimodule structure of $\mathrm{Bar}(\mathcal{A})$ is given by $\rho = ({}_{(x',y')}\rho_{(x,y)})_{x,y,x',y' \in \mathcal{O}}$ where

$$ {}_{(x',y')}\rho_{(x,y)} : {}_{(x',y')}\mathcal{A}^e_{(x,y)} \otimes {}_y \mathrm{Bar}(\mathcal{A})_x \to {}_{y'} \mathrm{Bar}(\mathcal{A})_{x'} $$

sends $(f \otimes g) \otimes (a \otimes sa_0 \otimes sa_1 \otimes \cdots \otimes sa_n \otimes a')$ to $(-1)^{|f|(|a|+|b|+\sum\limits_{i=0}^{n}|sa_i|)}(ga \otimes sa_0 \otimes sa_1 \otimes \cdots \otimes sa_n \otimes a'f)$ for all $(f \otimes g) \in {}_{(x',y')}\mathcal{A}^e_{(x,y)}$, $(x_0, \ldots, x_{n+1}) \in \bar{\mathcal{O}}^{n+1}$, $a \in {}_y\mathcal{A}_{x_0}$, $b \in {}_{x_{n+1}}\mathcal{A}_x$ and $a_i \in {}_{x_i}\mathcal{A}_{x_{i+1}}$ for $i \in [\![0, n]\!]$. Moreover, in that case, we have a quasi-isomorphism of dg bimodules $\mathrm{Bar}(\mathcal{A}) \to \mathcal{A}$ whose restriction to $\mathrm{Bar}(\mathcal{A})_p$ vanishes for $p \geq 1$ and whose restriction to $\mathrm{Bar}(\mathcal{A})_0$ is μ.

Definition 3.7. *Given a dg category $\mathcal{A}$, the* ***dual bimodule*** *of the dg bimodule $\mathrm{Bar}(\mathcal{A})$ is given by $\mathrm{Bar}(\mathcal{A})^\vee = \prod\limits_{x,y \in \mathcal{O}} {}_y \mathrm{Bar}(\mathcal{A})^\vee_x$, with*

$$ {}_y \mathrm{Bar}(\mathcal{A})^\vee_x = \mathcal{H}om_{-\mathcal{A}^e}({}_{y'} \mathrm{Bar}(\mathcal{A})_{x'}, {}_{y'}\mathcal{A}_x \otimes {}_y\mathcal{A}_{x'}) $$

where the subscript $-\mathcal{A}^e$ of the $\mathcal{H}om$ indicates that we consider the space of morphisms of right $\mathcal{A}^e$-modules. In particular, ${}_y \mathrm{Bar}^\vee_x$ is a right graded $\mathcal{A}^e$-module for each $x, y \in \mathcal{O}$ for the left $\mathcal{A}^e$-structure of $\mathcal{A}^e$ which is the inner one.

More precisely, $\mathrm{Bar}(\mathcal{A})$ is a $\mathcal{A}$-bimodule with the action given by $\rho = ({}_{(x",y")}\rho_{(x,y)})_{x,y,x",y" \in \mathcal{O}}$ where

$$ {}_{(x",y")}\rho_{(x,y)} : {}_{(x",y")}\mathcal{A}^e_{(x,y)} \otimes {}_y \mathrm{Bar}(\mathcal{A})^\vee_x \to {}_{y"} \mathrm{Bar}(\mathcal{A})^\vee_{x"} $$

sends $(f \otimes g) \otimes {}^{y'}_{y}\Phi^{x'}_{x}$ to the map ${}^{y'}_{y"}\Psi^{x'}_{x"} : {}_{y"} \mathrm{Bar}(\mathcal{A})_{x"} \to {}_{y'}\mathcal{A}_{x"} \otimes {}_{y"}\mathcal{A}_{x'}$ defined by

$$ {}^{y'}_{y"}\Psi^{x'}_{x"}(a \otimes sa_0 \otimes sa_1 \otimes \cdots \otimes sa_n \otimes a') = (-1)^{(|f|+|g|)|\Phi^{(1)}|}\Phi^{(1)} f \otimes g\Phi^{(2)} $$

where we have written ${}^{y'}_{y}\Phi^{x'}_{x}(a \otimes sa_0 \otimes sa_1 \otimes \cdots \otimes sa_n \otimes a')$ as a tensor product $\Phi^{(1)} \otimes \Phi^{(2)}$, for all $(f \otimes g) \in {}_{(x',y')}\mathcal{A}^e_{(x,y)}$, $(x_0, \ldots, x_{n+1}) \in \bar{\mathcal{O}}^{n+1}$, $a \in {}_y\mathcal{A}_{x_0}$, $a' \in {}_{x_{n+1}}\mathcal{A}_x$ and $a_i \in {}_{x_i}\mathcal{A}_{x_{i+1}}$ for $i \in [\![0, n]\!]$.

Remark 3.8. *If $\mathcal{A}$ is a dg category, we have a map*

$$ \mathrm{Bar}(\mathcal{A})^\vee_{nat} \to \prod_{p \geq 1} \prod_{\bar{x} \in \mathcal{O}^p} \mathcal{H}om_{\Bbbk}(\mathcal{A}[1]^{\otimes \bar{x}}, {}_{\mathrm{lt}(\bar{x})}\mathcal{A}_{\mathrm{rt}(\bar{x})}) $$

sending ${}^{z}_{y}\Phi^{z}_{x} \in \mathcal{H}om_{-\mathcal{A}^e}({}_z \mathrm{Bar}(\mathcal{A})_z, {}_z\mathcal{A}_x \otimes {}_y\mathcal{A}_z)$ to the collection of $\Bbbk$-linear maps

$$ \Psi : \bigoplus_{p \geq 0} \bigoplus_{(x_0, \ldots, x_p) \in \mathcal{O}^p} {}_{x_0}\mathcal{A}_{x_1}[1] \otimes \ldots \otimes {}_{x_{p-1}}\mathcal{A}_{x_p}[1] \to {}_{x_0}\mathcal{A}_{x_p} $$

given by $\Psi(sa_0, \ldots, sa_{p-1}) = \mu(\tau \circ {}^{x_0}_{x_0}\Phi^{x_0}_{x_p}(1_{x_0}, sa_0, \ldots, sa_{p-1}, 1_{x_0}))$ for $a_i \in {}_{x_i}\mathcal{A}_{x_{i+1}}$ and where 1_{x_0} is the identity of ${}_{x_0}\mathcal{A}_{x_0}$. Moreover, this map is an isomorphism of graded vector spaces.

Definition 3.9. *Given a graded quiver $\mathcal{A}$ with set of objects $\mathcal{O}$, the* ***Gerstenhaber product*** *of elements $s\mathbf{F}, s\mathbf{G} \in C(\mathcal{A})[1]$ is defined as the element $s\mathbf{F} \underset{G}{\circ} s\mathbf{G} \in C(\mathcal{A})[1]$ given by*

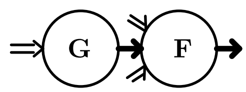
$$ (s\mathbf{F} \underset{G}{\circ} s\mathbf{G})^{\bar{x}} = \sum_{1 \leq i < j \leq \lg(\bar{x})} sF^{\bar{x}_{\leq i} \sqcup \bar{x}_{\geq j}} \underset{G,i,j}{\circ} sG^{\bar{x}_{[\![i,j]\!]}} $$

for $\bar{x} \in \bar{\mathcal{O}}$ where

$$\begin{aligned}&\left(sF^{\bar{x}_{\leq i}\sqcup\bar{x}_{\geq j}} \underset{G,i,j}{\circ} sG^{\bar{x}_{[\![i,j]\!]}}\right)(sa_1,\dots,sa_{n-1})\\ &= (-1)^{\epsilon_i} sF^{\bar{x}_{\leq i}\sqcup\bar{x}_{\geq j}}(sa_1,\dots,sa_{i-1},sG^{\bar{x}_{[\![i,j]\!]}}(sa_i,\dots,sa_{j-1}),sa_j,\dots,sa_{n-1})\end{aligned}$$

for $a_i \in {}_{x_i}\mathcal{A}_{x_{i+1}}$, with $\epsilon = (|\mathbf{G}|+1)\sum\limits_{r=1}^{i-1}|sa_r|$.

The map $(s\mathbf{F} \underset{G}{\circ} s\mathbf{G})^{\bar{x}}$ is by definition the sum of the maps associated to diagrams of type $\bar{x}$ and of the form



Definition 3.10. *Given a graded quiver $\mathcal{A}$ with set of objects $\mathcal{O}$, the* ***Gerstenhaber bracket*** *is the graded Lie bracket $[-,-]_G$ on $C(\mathcal{A})[1]$ defined for elements $s\boldsymbol{F}, s\boldsymbol{G} \in C(\mathcal{A})[1]$ as the element $[s\boldsymbol{F}, s\boldsymbol{G}]_G \in C(\mathcal{A})[1]$ given by*

$$[s\boldsymbol{F}, s\boldsymbol{G}]_G^{\bar{x}} = (s\boldsymbol{F} \underset{G}{\circ} s\boldsymbol{G})^{\bar{x}} - (-1)^{(|\boldsymbol{F}|+1)(|\boldsymbol{G}|+1)}(s\boldsymbol{G} \underset{G}{\circ} s\boldsymbol{F})^{\bar{x}}$$

for $\bar{x} \in \bar{\mathcal{O}}$.

Definition 3.11. *An A_∞**-structure** on a graded quiver $\mathcal{A}$ is a homogeneous element $sm_{\mathcal{A}} \in C(\mathcal{A})[1]$ of degree 1 satisfying the Maurer-Cartan equation $[sm_{\mathcal{A}}, sm_{\mathcal{A}}]_G = 0$.*

Remark 3.12. *An A_∞-structure on a graded quiver $\mathcal{A}$ is tantamount to the data of a homogeneous element $sm_{\mathcal{A}} \in C(\mathcal{A})[1]$ satisfying the following identities*

$$\sum_{1\leq i<j\leq n} sm_{\mathcal{A}}^{\bar{x}_{\leq i}\sqcup\bar{x}_{\geq j}} \circ (\mathrm{id}^{\otimes\bar{x}_{\leq i}} \otimes sm_{\mathcal{A}}^{\bar{x}_{[\![i,j]\!]}} \otimes \mathrm{id}^{\otimes\bar{x}_{\geq j}}) = 0 \tag{SI}$$

for every $n \in \mathbb{N}^*$ and $\bar{x} \in \mathcal{O}^n$. These are called the ***Stasheff identities*** *and were first introduced in* [8] *by J. Stasheff.*

Example 3.13. *If $\mathcal{A}$ is a dg category with differential $d_{\mathcal{A}}$ and product μ, it carries a natural A_∞-structure $sm_{\mathcal{A}} \in \mathcal{C}(\mathcal{A})[1]$ with $m^1_{\mathcal{A}} = d_{\mathcal{A}}$, $m^2_{\mathcal{A}} = \mu$ and $m^n_{\mathcal{A}} = 0$ for $n \geq 3$.*

Definition 3.14. *Given a graded quiver $\mathcal{A}$ with set of objects $\mathcal{O}$ its* ***graded dual quiver*** *is the quiver $\mathcal{A}^*$ whose set of objects is $\mathcal{O}$ and for $x, y \in \mathcal{O}$, the space of morphisms from x to y is defined as ${}_y\mathcal{A}^*_x = ({}_x\mathcal{A}_y)^*$.*

Definition 3.15. *A* ***bilinear form of degree d*** *on a graded quiver $\mathcal{A}$ is a collection $\Gamma = ({}_y\Gamma_x)_{x,y\in\mathcal{O}}$ of homogeneous $\Bbbk$-linear maps ${}_y\Gamma_x : {}_y\mathcal{A}_x[1] \otimes {}_x\mathcal{A}_y[1] \to \Bbbk$ of degree $d+2$.*

Definition 3.16. *A bilinear form Γ on a graded quiver $\mathcal{A}$ is* ***nondegenerate*** *if the induced map*

$${}_y\mathcal{A}_x[1] \to ({}_y\mathcal{A}_x[1])^*$$

sending an element $sa \in {}_y\mathcal{A}_x[1]$ to the map sending $sb \in {}_x\mathcal{A}_y[1]$ to ${}_y\Gamma_x(sa, sb)$ is an isomorphism.

Example 3.17. *Consider two graded quivers $\mathcal{A}$ and $\mathcal{B}$ and a morphism $(\Phi_0, \Phi) : \mathcal{A} \to \mathcal{B}$ of graded quivers. Define the bilinear form $\Gamma^{\Phi} : (\mathcal{A}[1] \oplus \mathcal{B}^*[d])^{\otimes 2} \to \Bbbk$ of degree $d+2$ by*

$$ {}_{y}\Gamma^{\Phi}_{x}(tf, sa) = -(-1)^{|sa||tf|}\, {}_{x}\Gamma^{\Phi}_{y}(sa, tf) = (-1)^{|tf|+1}(f \circ {}_{x}\Phi_{y})(a) $$

for $f \in {}_{\Phi_0(y)}\mathcal{B}^*_{\Phi_0(x)}$, $a \in {}_{x}\mathcal{A}_{y}$, where t stands for the shift morphism $\mathcal{B}^* \to \mathcal{B}^*[d]$ and

$$ {}_{y}\Gamma^{\Phi}_{x}(tf, tg) = {}_{y}\Gamma^{\Phi}_{x}(sa, sb) = 0 $$

for $f \in {}_{\Phi_0(y)}\mathcal{B}^*_{\Phi_0(x)}$, $g \in {}_{\Phi_0(x)}\mathcal{B}^*_{\Phi_0(y)}$, $a \in {}_{y}\mathcal{A}_{x}$ and $b \in {}_{x}\mathcal{A}_{y}$. This bilinear form Γ^{Φ} will be called the ***Φ-mixed bilinear form****.*

Example 3.18. *If $\mathcal{B} = \mathcal{A}$, the bilinear form Γ^{id} of the previous example is called the* ***natural bilinear form*** *on $\mathcal{A}$ and will be denoted $\Gamma^{\mathcal{A}}$.*

Remark 3.19. *The natural bilinear form on a* Hom*-finite graded quiver $\mathcal{A}$ is nondegenerate, whereas the Φ-mixed bilinear form is not in general.*

Definition 3.20. *An A_∞-structure $sm_{\mathcal{A}} \in \mathcal{C}(\mathcal{A})[1]$ on a graded quiver $\mathcal{A}$ is* ***almost cyclic*** *with respect to a homogeneous bilinear form $\Gamma : \mathcal{A}[1]^{\otimes 2} \to \Bbbk$ if the following holds:*

$$ \begin{aligned} &{}_{x_n}\Gamma_{x_1}(sm^{\bar{x}}_{\mathcal{A}}(sa_1, \dots, sa_{n-1}), sa_n) \\ &= (-1)^{|sa_n|(\sum_{i=1}^{n-1}|sa_i|)}\, {}_{x_{n-1}}\Gamma_{x_n}(sm^{\bar{x}\cdot\sigma^{-1}}_{\mathcal{A}}(sa_n, sa_1, \dots, sa_{n-2}), sa_{n-1}) \end{aligned} \tag{3.1} $$

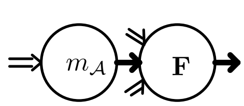
for each $n \in \mathbb{N}^*$, $\bar{x} = (x_1, \dots, x_n) \in \bar{\mathcal{O}}$, $\sigma = (12\dots n)$ with $a_i \in {}_{x_i}\mathcal{A}_{x_{i+1}}$ for $i \in [\![1, n-1]\!]$ and $a_n \in {}_{x_n}\mathcal{A}_{x_1}$. An ***almost cyclic*** *A_∞-category is an A_∞-category whose A_∞-structure is almost cyclic with respect to a fixed homogeneous bilinear form. An almost cyclic A_∞-category with respect to a nondegenerate bilinear form is called a* ***cyclic*** *A_∞-category.*

The following definition was first introduced by M. Sugawara in [9].

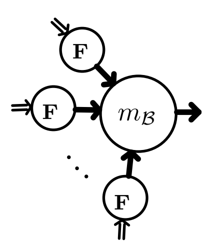
Definition 3.21. *An A_∞**-morphism** between A_∞-categories $(\mathcal{A}, sm_{\mathcal{A}})$ and $(\mathcal{B}, sm_{\mathcal{B}})$ with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$ is a map $F_0 : \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$ together with a collection $\mathbf{F} = (F^{\bar{x}})_{\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}}$, where $F^{\bar{x}} : \mathcal{A}[1]^{\otimes \bar{x}} \to {}_{F_0(\mathrm{lt}(\bar{x}))}\mathcal{B}_{F_0(\mathrm{rt}(\bar{x}))}[1]$ is a map of degree 0, that satisfies*

$$ \begin{aligned} &\sum_{1 \le i < j \le \lg(\bar{x})} F^{\bar{x}_{\le i} \sqcup \bar{x}_{\ge j}} \circ (\mathrm{id}^{\otimes \bar{x}_{\le i}} \otimes sm^{\bar{x}_{[\![i,j]\!]}}_{\mathcal{A}} \otimes \mathrm{id}^{\otimes \bar{x}_{\ge j}}) \\ &\qquad = \sum_{1 \le i_1 < \dots < i_n \le \lg(\bar{x})} sm^{\bar{y}}_{\mathcal{B}}(F^{\bar{x}_{\le i_1}} \otimes F^{\bar{x}_{[\![i_1,i_2]\!]}} \otimes \dots \otimes F^{\bar{x}_{[\![i_n,\lg(\bar{x})]\!]}}) \end{aligned} \tag{MI} $$

for every $\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}$ and with $\bar{y} = (F_0(x_1), F_0(x_{i_1}), \dots, F_0(x_{\lg(\bar{x})}))$. Note that given $\bar{x} \in \mathcal{O}_{\mathcal{A}}$ the terms in both sums are sums of maps associated with diagrams of type $\bar{x}$ that are respectively of the form



and



The following definition was introduced in [2] by H. Kajiura in the cyclic case.

Definition 3.22. *An A_∞-morphism $(F_0, \mathbf{F})$ between almost cyclic A_∞-categories $(\mathcal{A}, sm_{\mathcal{A}})$ and $(\mathcal{B}, sm_{\mathcal{B}})$ with respect to bilinear forms γ and Γ is **cyclic** if*

$$_{F_0(y)}\Gamma_{F_0(x)}(F^1(sa), F^1(sb)) = {}_y\gamma_x(sa, sb)$$

for $x, y \in \mathcal{O}$, $a \in {}_y\mathcal{A}_x$ and $b \in {}_x\mathcal{A}_y$ and for $n \geq 3$

$$\sum_{\substack{\bar{x}\in Z\\ \bar{y}\in Z'}} {}_{\mathrm{lt}(\bar{z})}\Gamma_{\mathrm{rt}(\bar{z})}(F^{\bar{x}}(sa_1, \dots, sa_i), F^{\bar{y}}(sa_{i+1}, \dots, sa_n)) = 0$$

for $\bar{z} \in \bar{\mathcal{O}}$, $(sa_1, \dots, sa_i) \in \mathcal{A}[1]^{\otimes \bar{x}}$ and $(sa_{i+1}, \dots, sa_n) \in \mathcal{A}[1]^{\otimes \bar{y}}$ where

$$Z = \{\bar{x} \in \bar{\mathcal{O}} \mid \mathrm{lt}(\bar{x}) = \mathrm{lt}(\bar{z}), \mathrm{rt}(\bar{x}) = \mathrm{rt}(\bar{z})\} \text{ and } Z' = \{\bar{y} \in \bar{\mathcal{O}} \mid \mathrm{lt}(\bar{y}) = \mathrm{rt}(\bar{z}), \mathrm{rt}(\bar{y}) = \mathrm{lt}(\bar{z})\}$$

4 Pre-Calabi-Yau categories

In this section, we present the diagrammatic calculus and recall the definition of d-pre-Calabi-Yau structures, $d \in \mathbb{Z}$, appearing in [5] and [11] as well as their relation with A_∞-structures when the graded quiver considered is Hom-finite.

4.1 Diagrammatic calculus

In this subsection, we define discs and diagrams and we explain how to evaluate and compose them.

Following [11] we define the following graded vector space.

Definition 4.1. *Given a graded quiver $\mathcal{A}$ with set of objects $\mathcal{O}$, we define the graded vector space*

$$\mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{A})[d]) = \prod_{n\in\mathbb{N}^*} \mathrm{Multi}^n(\mathrm{Bar}(\mathcal{A})[d]) = \prod_{n\in\mathbb{N}^*} \prod_{\bar{\bar{x}}\in\bar{\mathcal{O}}^n} \mathrm{Multi}^{\bar{\bar{x}}}(\mathrm{Bar}(\mathcal{A})[d])$$

where $\mathrm{Multi}^{\bar{\bar{x}}}(\mathrm{Bar}(\mathcal{A})[d])$ is the graded vector space consisting of sums of homogeneous $\Bbbk$-linear maps of the form

$$F^{\bar{x}^1,\dots,\bar{x}^n} : \mathcal{A}[1]^{\otimes\bar{x}^1} \otimes \mathcal{A}[1]^{\otimes\bar{x}^2} \otimes \cdots \otimes \mathcal{A}[1]^{\otimes\bar{x}^n} \to {}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}_{\mathrm{rt}(\bar{x}^2)}[-d] \otimes {}_{\mathrm{lt}(\bar{x}^2)}\mathcal{A}_{\mathrm{rt}(\bar{x}^3)}[-d] \otimes \cdots \otimes {}_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}_{\mathrm{rt}(\bar{x}^1)}[-d]$$

for $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n)$.

The action of $\tau = (\tau_n)_{n\in\mathbb{N}^*} \in \prod_{n\in\mathbb{N}^*} C_n$ on an element $\mathbf{F} = (F^{\bar{\bar{x}}})_{\bar{\bar{x}}\in\bar{\bar{\mathcal{O}}}} \in \mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{A})[d])$ is the element $\tau \cdot \mathbf{F} \in \mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{A})[d])$ given by

$$(\tau \cdot \mathbf{F})^{\bar{\bar{x}}} = \tau^{-1}_{{}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}_{\mathrm{rt}(\bar{x}^2)}[-d], {}_{\mathrm{lt}(\bar{x}^2)}\mathcal{A}_{\mathrm{rt}(\bar{x}^3)}[-d], \dots, {}_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}_{\mathrm{rt}(\bar{x}^1)}[-d]} \circ F^{\bar{\bar{x}}\cdot\tau} \circ \tau_{\mathcal{A}[1]^{\otimes\bar{x}^1}, \mathcal{A}[1]^{\otimes\bar{x}^2}, \dots, \mathcal{A}[1]^{\otimes\bar{x}^n}}$$

for $\bar{\bar{x}} \in \bar{\bar{\mathcal{O}}}$. We will denote by $\mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{A})[d])^{C_{\mathrm{lg}(\bullet)}}$ the space of elements of $\mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{A})[d])$ that are invariant under the action of $\prod_{n\in\mathbb{N}^*} C_n$.

Remark 4.2. *If $\mathcal{A}$ is a dg category,* $\mathrm{Multi}^n(\mathrm{Bar}(\mathcal{A})[d])$ *is the dg vector space*

$$\mathcal{H}om_{(\mathcal{A}^e)^{\otimes n}}(\mathrm{Bar}(\mathcal{A})[d], {}_{\mathrm{id}}(\mathcal{A}^{\otimes n})_\sigma)$$

where ${}_{\mathrm{id}}(\mathcal{A}^{\otimes n})_\sigma$ *denotes the* $\mathcal{A}^{\otimes n}$*-bimodule* $\mathcal{A}^{\otimes n}$ *whose structure is given by*

$$_{(y_1,\dots,y_n)}(\mathcal{A}^{\otimes n})_{(x_1,\dots,x_n)} = \bigotimes_{i=1}^{n} {}_{y_i}\mathcal{A}_{x_{i+1}}$$

with the convention that $x_{n+1} = x_1$.

The action on morphisms is given by

$$(g_1 \otimes \cdots \otimes g_n) \bullet \omega \bullet (f_1 \otimes \cdots \otimes f_n) = (-1)^{|f_1|(|f_2|+\cdots+|f_n|)}(g_1 \otimes \cdots \otimes g_n) \cdot \omega \cdot (f_n \otimes f_2 \otimes \cdots \otimes f_{n-1} \otimes f_1)$$

for objects $x_i, x_i', y_i, y_i' \in \mathcal{O}$*,* $i \in [\![1, n]\!]$*,* $\omega \in \bigotimes_{i=1}^{n} {}_{y_i}\mathcal{A}_{x_{i+1}}$*,* $f_i \in {}_{x_i}\mathcal{A}_{x_i'}$ *and* $g_i \in {}_{y_i}\mathcal{A}_{x_i'}$ *where* $\cdot$ *denotes the usual bimodule structure of* $\mathcal{H}om_{(\mathcal{A}^e)^{\otimes n}}(\mathrm{Bar}(\mathcal{A})[d], \mathcal{A}^{\otimes n})$.

Definition 4.3. *A* ***disc*** D *is a circle with distinguished set of points which are either incoming or outgoing points. An incoming (resp. outgoing) point will be pictured as an incoming (resp. outgoing) arrow (see Figure 4.1). The* ***size*** *of the disc* D *is the number of outgoing arrows, and it will be denoted by* $|D|$.

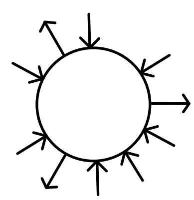


Figure 4.1: A disc of size 3

Definition 4.4. *Given a graded quiver* $\mathcal{A}$ *with set of objects* $\mathcal{O}$*, a* ***decorated disc of size*** n *is a disc of size* n *together with a clockwise labeling of the outgoing arrows from* 1 *to* n *and a distinguished object of* $\mathcal{O}$ *between any couple of consecutive arrows. Given an arrow* α *of the disc* D*, we will write it* $\alpha = {}_y\alpha_x$ *with* $x, y \in \mathcal{O}$ *to indicate that* x *clockwise precedes the arrow* α *in* D *and* α *clockwise precedes* y *in* D*. For instance, the outgoing arrow labeled by* 3 *in Figure 4.2 might be denoted by* ${}_{x_3^1}\alpha_{x_1^3}$.

The ***type*** *of the decorated disc is the tuple of the form* $(\bar{x}^1, \dots, \bar{x}^n)$ *where* $\bar{x}^i$ *is the tuple formed by objects of* $\mathcal{O}$*, read in counterclockwise order, between the outgoing arrows* $i-1$ *and* i*, with the convention that the arrow* 0 *is the arrow* n *(see Figure 4.2).*

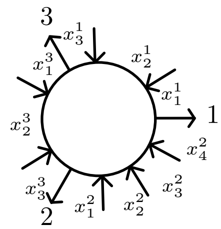


Figure 4.2: A decorated disc of type $\bar{\bar{x}} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$ where $\bar{x}^1 = (x^1_1, x^1_2, x^1_3)$, $\bar{x}^2 = (x^2_1, x^2_2, x^2_3, x^2_4)$ and $\bar{x}^3 = (x^3_1, x^3_2, x^3_3)$

Definition 4.5. *Given $\bar{\bar{x}} \in \bar{\bar{\mathcal{O}}}$ we associate to a map $F^{\bar{\bar{x}}} \in \mathrm{Multi}^{\bar{\bar{x}}}(\mathrm{Bar}(\mathcal{A}[d]))$ the unique decorated diagram of type $\bar{\bar{x}}$.*

Definition 4.6. *A **marked** disc is a decorated disc with a bold arrow (see Figure 4.3).*

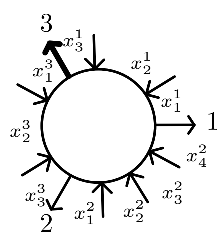


Figure 4.3: A marked disc

Definition 4.7. *Consider $s_{d+1}F^{\bar{\bar{x}}} \in \mathrm{Multi}^{\bar{\bar{x}}}(\mathrm{Bar}(\mathcal{A})[d])[d+1]$ with $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}^n$. Given $(a,b) \in (\{i\} \times [\![1, \mathrm{lg}(\bar{x}^1) + \cdots + \mathrm{lg}(\bar{x}^n)]\!]) \sqcup (\{o\} \times [\![1,n]\!])$, by the isomorphisms (2.1), (2.2) and (2.3), $s_{d+1}F^{\bar{\bar{x}}}$ induces a morphism of the form*

$$\begin{aligned}&\mathcal{A}[1]^{\otimes \bar{x}^1} \otimes \cdots \otimes \mathcal{A}[1]^{\otimes \bar{x}^{j-1}} \otimes \mathcal{A}[1]^{\otimes \bar{x}^j_{\leq b'}} \otimes {}_{x^j_{b'}}\mathcal{A}[-d]_{x^j_{b'+1}} \otimes \mathcal{A}[1]^{\otimes \bar{x}^j_{>b'}} \otimes \mathcal{A}[1]^{\otimes \bar{x}^n}\\ &\to {}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}_{\mathrm{rt}(\bar{x}^2)}[-d] \otimes \cdots \otimes {}_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}_{\mathrm{rt}(\bar{x}^1)}[-d]\end{aligned} \tag{4.1}$$

given by $(-1)^{(d+1)|F^{\bar{\bar{x}}}|}F^{\bar{\bar{x}}} \circ (\mathrm{id}^{\otimes(\mathrm{lg}(\bar{x}^1)+\cdots+\mathrm{lg}(\bar{x}^{j-1})+b'-j)} \otimes s_{-d-1} \otimes \mathrm{id}^{\otimes(\mathrm{lg}(\bar{x}^j)-b'+\mathrm{lg}(\bar{x}^{j+1})+\cdots+\mathrm{lg}(\bar{x}^n)-n+j)})$ if $a = i$ and $b = \mathrm{lg}(\bar{x}^1) + \cdots + \mathrm{lg}(\bar{x}^j) + b'$ with $j \in [\![1,n]\!]$, $b' \in [\![1, \mathrm{lg}(\bar{x}^j) - 1]\!]$, and a morphism of the form

$$\begin{aligned}&\mathcal{A}[1]^{\otimes \bar{x}^1} \otimes \mathcal{A}[1]^{\otimes \bar{x}^2} \otimes \cdots \otimes \mathcal{A}[1]^{\otimes \bar{x}^n}\\ &\to {}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}_{\mathrm{rt}(\bar{x}^2)}[-d] \otimes \cdots \otimes {}_{\mathrm{lt}(\bar{x}^{b-1})}\mathcal{A}_{\mathrm{rt}(\bar{x}^b)}[-d] \otimes {}_{\mathrm{lt}(\bar{x}^b)}\mathcal{A}_{\mathrm{rt}(\bar{x}^{b+1})}[1] \otimes \cdots \otimes {}_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}_{\mathrm{rt}(\bar{x}^1)}[-d]\end{aligned} \tag{4.2}$$

given by $(\mathrm{id}^{\otimes(b-1)} \otimes s_{d+1} \otimes \mathrm{id}^{\otimes(n-b)}) \circ F^{\bar{\bar{x}}}$ if $a = o$. To $s_{d+1}F^{\bar{\bar{x}}}$ and (a,b) as before, we associate the marked diagram which is decorated as the one of $F^{\bar{\bar{x}}}$, where the bold arrow is the b-th incoming arrow if $a = i$ and the b-th outgoing arrow if $a = o$ where the incoming arrows are numbered in clockwise order, the *first being the incoming arrow following the last outgoing one. Given a disc D of type $\bar{\bar{x}}$, we will denote by $\mathcal{E}(D, s_{d+1}F^{\bar{\bar{x}}})$ the map (4.1) if $a = i$ and (4.2) if $a = o$.*

Example 4.8. *Given the marked disc in Figure 4.3 and an element $s_{d+1}\mathbf{F} \in \mathrm{Multi}^{\bullet}(\mathrm{Bar}(\mathcal{A})[d])[d+1]$, the evaluation of $\mathcal{E}(D, s_{d+1}F^{\bar{\bar{x}}})$ at elements $(s\bar{a}^1, s\bar{a}^2, s\bar{a}^3)$ is obtained by first compute $F^{\bar{\bar{x}}}(s\bar{a}^1, s\bar{a}^2, s\bar{a}^3) = {}_{\mathrm{lt}(\bar{x}^1)}F^{-d}_{\mathrm{rt}(\bar{x}^2)} \otimes {}_{\mathrm{lt}(\bar{x}^2)}F^{-d}_{\mathrm{rt}(\bar{x}^3)} \otimes {}_{\mathrm{lt}(\bar{x}^3)}F^{-d}_{\mathrm{rt}(\bar{x}^3)}$, and then apply the shift s_{d+1} to the third tensor factor of the result since the bold arrow is the third outgoing arrow of the disc. The result is then*

$$(-1)^{\epsilon}{}_{\mathrm{lt}(\bar{x}^1)}F^{-d}_{\mathrm{rt}(\bar{x}^2)} \otimes {}_{\mathrm{lt}(\bar{x}^2)}F^{-d}_{\mathrm{rt}(\bar{x}^3)} \otimes {}_{\mathrm{lt}(\bar{x}^3)}F^{1}_{\mathrm{rt}(\bar{x}^3)}$$

with $\epsilon = (d+1)(|{}_{\mathrm{lt}(\bar{x}^1)}F^{-d}_{\mathrm{rt}(\bar{x}^2)}| + |{}_{\mathrm{lt}(\bar{x}^2)}F^{-d}_{\mathrm{rt}(\bar{x}^3)}|)$.

To simplify, from now on we will omit the objects when drawing a marked disc as well as the label of the outgoing arrows. By convention, if the bold arrow of the marked disc (of size n) is outgoing, it denotes the n-th outgoing arrow, and if the bold arrow is incoming, then the clockwise preceding outgoing arrow of the marked disc is the one labeled by n. Moreover, we will draw a big incoming arrow instead of several consecutive incoming arrows (see Figure 4.4 and 4.5).

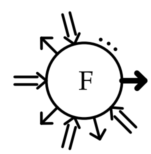


Figure 4.4: A disc representing a map of the form (4.2)

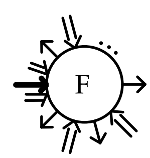


Figure 4.5: A disc representing a map of the form (4.1)

Now, we explain how to compose discs.

Definition 4.9. *A **diagram** is roughly speaking a finite collection of decorated discs, each of which shares at most an arrow with any other one (see Figure 4.6). More precisely, a diagram (D, R) is the data of a collection D of discs $\{D_1, \dots, D_n\}$, each of which having sets of arrows A_i for $i \in [\![1, n]\!]$, together with a subset $R \subseteq (\sqcup_{i=1}^n A_i)^2$ satisfying that*

(D.1) if $(\alpha, \beta) \in R \cap (A_i \times A_j)$, then $i \neq j$ and α is an incoming arrow and β is an outgoing arrow;

(D.2) if $({}_{y'}\alpha_{x'}, {}_{y}\beta_{x}) \in R$, then $x' = y$ and $x = y'$;

*where we use the notation introduced in Definition 4.4. We will represent a pair $(\alpha, \beta) \in R \cap (A_i \times A_j)$ by connecting the outgoing arrow β of D_j with the incoming arrow α of D_i (see Figure 4.6), and we will say that the disc D_i **shares an arrow** with D_j or that α and β are **connected**.*

*An **incoming (resp. outgoing) arrow** of the diagram (D, R) is an arrow α of one of the discs D_i such that there is no arrow β satisfying that $(\alpha, \beta) \in R$ (resp. $(\beta, \alpha) \in R$). A diagram (D, R) has a distinguished object of $\mathcal{O}$ between any couple of consecutive arrows of (D, R) given by the decoration of the discs $D_1, \ldots, D_n$.*

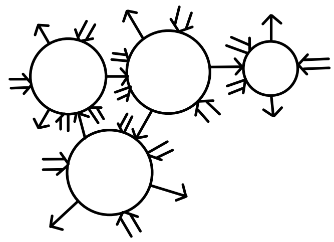


Figure 4.6: A diagram

Definition 4.10. *We say that a diagram (D, R) is **admissible** if, roughly speaking, each of the discs in $D = \{D_1, \ldots, D_n\}$ is marked and if D is connected, with no cycles and if every couple of arrows in R contains a unique bold arrow. More precisely, a diagram (D, R) is admissible if the following conditions are satisfied*

(A.1) $\forall i \in [\![1, n]\!], \exists\, \alpha_i \in D_i$ *such that* $(\alpha_i, \beta_j) \in R$ *for some* $j \neq i$, $\beta_j \in D_j$;

(A.2) for $(x, y) \in R$, *either x or y is a bold arrow, but not both;*

(A.3) there is no family of arrows $\{x_1, \ldots, x_k\}$ *such that* $(x_i, x_{i+1}) \in R$ *for all* $i \in [\![1, k]\!]$ *and* $x_k = x_1$.

Note that there is precisely one disc in $D = \{D_1, \ldots, D_n\}$ whose bold arrow is also an incoming or outgoing arrow of the diagram (D, R), i.e. an admissible diagram (D, R) always has either an incoming or outgoing bold arrow.

*The **size** of an admissible diagram $(D = \{D_1, \ldots, D_n\}, R)$ is $\sum_{i=1}^{n} |D_i| - n + 1$, and it will be denoted by $|D|$. We will relabel the outgoing arrows of (D, R) in clockwise direction from 1 to $|D|$, such that the outgoing arrow labeled by $|D|$ is precisely the bold arrow of (D, R) if the latter arrow is outgoing, and it is the outgoing arrow preceding the bold arrow of (D, R) in clockwise sense if the bold arrow of (D, R) is incoming.*

Definition 4.11. *The **type** of an admissible diagram $(D = \{D_1, \ldots, D_n\}, R)$ is the tuple*

$$\bar{\bar{x}} = (\bar{x}^1, \ldots, \bar{x}^m) \in \bar{\bar{\mathcal{O}}}$$

where $m = |D|$ and $\bar{x}^i \in \bar{\mathcal{O}}$ is the tuple composed of the objects of (D, R), placed in counterclockwise order, that we can read between the outgoing arrows $i - 1$ and i of (D, R).

Definition 4.12. *A **source** (resp. **sink**) of an admissible diagram is a disc which shares none of its incoming (resp. outgoing) arrows with another one.*

Remark 4.13. *An admissible diagram has at least one source and one sink.*

Given an admissible diagram $(D = \{D_1, \dots, D_n\}, R)$ and a tuple $(s_{d+1}F_1, \dots, s_{d+1}F_n)$ with $F_i \in \mathrm{Multi}^{\bar{\bar{x}}_i}(\mathrm{Bar}(\mathcal{A})[d])$ such that $\bar{\bar{x}}_i$ is the type of D_i for all $i \in [\![1, n]\!]$, we will define a map $\mathcal{E}((D, R), s_{d+1}F_1, \dots, s_{d+1}F_n) \in \mathrm{Multi}^{\bar{\bar{x}}}(\mathrm{Bar}(\mathcal{A})[d])[d+1]$ where $\bar{\bar{x}}$ is the type of (D, R), as follows. First, we suppose that the bold arrow is on a sink.

Definition 4.14. *Let $(D = \{D_1, \dots, D_n\}, R)$ be a diagram and let $(s_{d+1}F_1, \dots, s_{d+1}F_n)$ be a tuple of homogeneous elements with $F_i \in \mathrm{Multi}^{\bar{\bar{x}}_i}(\mathrm{Bar}(\mathcal{A})[d])$ such that $\bar{\bar{x}}_i$ is the type of D_i for all $i \in [\![1, n]\!]$. We will define the evaluation of $\mathcal{E}((D, R), s_{d+1}F_1, \dots, s_{d+1}F_n)$ at an element $\bar{sa}^{\bar{\bar{x}}}$ of $\mathcal{A}[1]^{\otimes \bar{\bar{x}}}$, where $\bar{\bar{x}}$ is the type of (D, R) by induction on n as follows. We fix a source D_s of (D, R).*

(E.1) We place each element $sa^i_j \in {}_{x^i_j}\mathcal{A}_{x^i_{j+1}}[1]$ in the incoming arrow ${}_{x^i_j}\alpha_{x^i_{j+1}}$ of (D, R) in counterclockwise order beginning at the bold arrow. This will create a sign, as follows : if an element of degree m turn around the source representing a map of degree m', we add a sign $(-1)^{mm'}$ and if an element of degree ℓ pass through an element of degree ℓ' to go to its place, we add a sign $(-1)^{\ell\ell'}$. Here, turning around the source means passing through all of its inputs.

(E.2) We evaluate $\mathcal{E}(D_s, s_{d+1}F_s)$ at the elements $sa^i_j \in {}_{x^i_j}\mathcal{A}_{x^i_{j+1}}[1]$ corresponding to the incoming arrows of D_s, to obtain an element of the form

$$(-1)^{(d+1)(|b^1|+\cdots+|b^{|D_s|-1}|)} b^1 \otimes \cdots \otimes b^{|D_s|-1} \otimes s_{d+1} b^{|D_s|} \in {}_{y'_1}\mathcal{A}_{y_1}[-d] \otimes \cdots \otimes {}_{y'_{|D_s|}}\mathcal{A}_{y_{|D_s|}}[1]$$

(E.3) We add a Koszul sign coming from transposing $b^1 \otimes \cdots \otimes b^{|D_s|-1}$ with all of the elements $sa^i_j \in {}_{x^i_j}\mathcal{A}_{x^i_{j+1}}[1]$ corresponding to the incoming arrows of (D, R) between the last outgoing arrow of (D, R) and the bold arrow of D_s connected with an arrow of the rest of the diagram in the clockwise order. Recall that the elements b^i for $i \in [\![1, |D_s| - 1]\!]$ are labeled with an index from $[\![1, |D|]\!]$ coming from the corresponding outgoing arrow of (D, R).

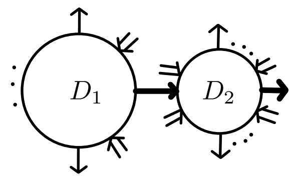
(E.4) We consider the diagram (D', R') given by removing the disc D_s from (D, R) and place $s_{d+1}b^{|D_s|}$ at the incoming arrow of (D', R') that was previously connected to the bold outgoing arrow of D_s. By induction we evaluate $\mathcal{E}((D', R'), s_{d+1}F_1, \dots, s_{d+1}\check{F}_s, \dots, s_{d+1}F_n)$ at the elements $sa^i_j \in {}_{x^i_j}\mathcal{A}_{x^i_{j+1}}[1]$ corresponding to the incoming arrows of D'. This evaluation carries a sign

$$(-1)^{b^{|D_s|(|\bar{sa}^u|+|\bar{sa}^v|)+|\bar{sa}^u||\bar{sa}^v|}}$$

where $|\bar{sa}^u|$ and $|\bar{sa}^v|$ are the tuples of objects corresponding to the incoming arrows of D' that precede and follow the element $b^{|D_s|}$ without any outgoing arrow between them. Recall that the tensor factors in the evaluation of $\mathcal{E}((D', R'), s_{d+1}F_1, \dots, s_{d+1}\check{F}_s, \dots, s_{d+1}F_n)$ at the elements $sa^i_j \in {}_{x^i_j}\mathcal{A}_{x^i_{j+1}}[1]$ stated before are labeled with an index from $[\![1, |D|]\!]$ coming from the corresponding outgoing arrow of (D, R).

(E.5) We reorder the tensor factors obtained in steps (E.4) and (E.5) according to the labeling of the outgoing arrows of (D, R), and add the respective Koszul sign.

We illustrate the previous procedure with an admissible diagram D consisting of two discs, given as follows



In this case, the evaluation of $\mathcal{E}(D, s_{d+1}G^{\bar{\bar{y}}}, s_{d+1}F^{\bar{\bar{x}}})$ *at*

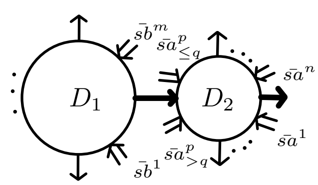
$$(\bar{sa}^1, \dots, \bar{sa}^{p-1}, \bar{sb}^1, \bar{sa}^p_{>q}, \bar{sb}^2 \dots, \bar{sb}^{m-1}, \bar{sa}^p_{\leq q}, \bar{sb}^m, \bar{sa}^{p+1} \dots, \bar{sa}^n)$$

with $\bar{sa}^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ *and* $\bar{sb}^i \in \mathcal{A}[1]^{\otimes \bar{y}^i}$, *where* $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n)$ *and* $\bar{\bar{y}} = (\bar{y}^1, \dots, \bar{y}^m)$ *are the respective types of* D_1 *and* D_2, *is detailed as follows.*

First, we place each tensor factor sa^i_j *and* sb^i_j *of* $\bar{sa}^i$ *and* $\bar{sb}^i$ *in the corresponding incoming arrow, as explained in (E.1), adding a sign* $(-1)^{(|G^{\bar{\bar{y}}}|+d+1)(|\bar{sa}^1|+\cdots+|\bar{sa}^{p-1}|+|\bar{sa}^p_{>q}|)}$. *Also, we add a sign*

$$(-1)^{|\bar{sb}^1||\bar{sa}^p_{>q}|+|\bar{sb}^m||\bar{sa}^p_{\leq q}|}$$

for the permutation of the corresponding elements. We picture this as follows



In step (E.2), we compute $\mathcal{E}(D_1, s_{d+1}G^{\bar{\bar{y}}})(\bar{sb}^1, \dots, \bar{sb}^m) = (-1)^{(d+1)(|\epsilon_1|+\cdots+|\epsilon_{m-1}|)}\epsilon_1 \otimes \cdots \otimes s_{d+1}\epsilon_m$.

After step (E.3) we have gained a total sign $(-1)^{\Delta}$ *with*

$$\Delta = (|G^{\bar{\bar{y}}}|+d+1)(|\bar{sa}^p_{\leq q}|+|\bar{sa}^{p+1}|+\cdots+|\bar{sa}^n|)+(|G^{\bar{\bar{y}}}|+|\bar{sa}^1|+\cdots+|\bar{sa}^p_{>q}|)(|\epsilon_1|+\cdots+|\epsilon_{j-i-1}|),$$

multiplying the element $\epsilon_1 \otimes \cdots \otimes \epsilon_{j-i-1} \otimes \mathcal{E}(D_2, s_{d+1}F^{\bar{\bar{x}}})(\bar{sa}^1 \otimes \cdots \otimes \bar{sa}^p_{>q} \otimes s_{d+1}\epsilon_{j-i} \otimes \bar{sa}^p_{\leq q} \otimes \cdots \otimes \bar{sa}^n)$.

In (E.4), we compute

$$\mathcal{E}(D_2, s_{d+1}F^{\bar{\bar{x}}})(\bar{sa}^1 \otimes \cdots \otimes \bar{sa}^p_{\leq q} \otimes s_{d+1}\epsilon_{j-i} \otimes \bar{sa}^p_{>q} \otimes \cdots \otimes \bar{sa}^n) = \delta_1 \otimes \cdots \otimes \delta_{i-1} \otimes \delta_j \otimes \cdots \otimes s_{d+1}\delta_n$$

and add a sign $(-1)^{\Delta'}$ *to the final result, with*

$$\Delta' = |\bar{sa}^p_{>q}|(d+1+|\epsilon_{j-i}|+|\bar{sa}^p_{\leq q}|)+|\bar{sa}^p_{\leq q}|(d+1+|\epsilon_{j-i}|)+(d+1)(|\delta_1|+\cdots+|\delta_{i-1}|+|\delta_j|+\cdots+|\delta_{n-1}|)$$

In (E.5), we reorder the outputs with a sign, giving finally

$$\mathcal{E}(D, s_{d+1}G^{\bar{\bar{y}}}, s_{d+1}F^{\bar{\bar{x}}})(\bar{sa}^1, \dots, \bar{sa}^n) = (-1)^{\Delta+\Delta'+\Delta''}\delta_1 \otimes \cdots \otimes \delta_{i-1} \otimes \epsilon_1 \otimes \cdots \otimes \epsilon_{j-i-1} \otimes \delta_j \otimes \cdots \otimes s_{d+1}\delta_n$$

where $\Delta'' = (|\delta_1| + \dots |\delta_{i-1}|)(|\epsilon_1| + \cdots + |\epsilon_{j-i-1}|)$.

Now, supose that the bold arrow is on a source.

Definition 4.15. *Let $(D = \{D_1, \dots, D_n\}, R)$ be a diagram and let $(s_{d+1}F_1, \dots, s_{d+1}F_n)$ be a tuple of homogeneous elements with $F_i \in \mathrm{Multi}^{\bar{\bar{x}}_i}(\mathrm{Bar}(\mathcal{A})[d])$ such that $\bar{\bar{x}}_i$ is the type of D_i for all $i \in [\![1, n]\!]$. We will define the evaluation $\mathcal{E}((D, R), s_{d+1}F_1, \dots, s_{d+1}F_n)$ at an element $s\bar{a}^{\bar{\bar{x}}}$ of $\mathcal{A}[1]^{\otimes \bar{\bar{x}}}$, where $\bar{\bar{x}}$ is the type of (D, R) by induction on n as follows. Suppose that the bold arrow is an arrow of the source D_s of (D, R) and that the p-th outgoing arrow α of D_s is connected with another disc.*

(F.1) We place each element $sa^i_j \in {}_{x^i_j}\mathcal{A}_{x^i_{j+1}}[1]$ in the incoming arrow ${}_{x^i_j}\alpha_{x^i_{j+1}}$ of (D, R) in counterclockwise order beginning at the bold arrow. This will create a sign, as follows : if an element of degree m turn around a source representing a map of degree m', we add a sign $(-1)^{mm'}$ and if an element of degree ℓ pass through an element of degree ℓ' to go to his place, we add a sign $(-1)^{\ell\ell'}$. Again, turning around a source means passing through all of its inputs.

(F.2) We transpose all the elements that do not correspond to incoming arrows of D_s with the elements corresponding to incoming arrows of D_s between α and the bold arrow. We add the sign coming from this transposition. We evaluate $\mathcal{E}(D_s, s_{d+1}F_s)$ at the elements $sa^i_j \in {}_{x^i_j}\mathcal{A}_{x^i_{j+1}}[1]$ corresponding to the incoming arrows of D_s, to obtain an element of the form

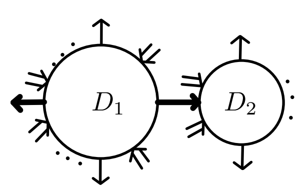
$$s_{d+1}(b^1 \otimes \cdots \otimes b^{|D_s|}) \in ({}_{y'_1}\mathcal{A}_{y_1}[-d] \otimes \cdots \otimes {}_{y'_{|D_s|}}\mathcal{A}_{y_{|D_s|}}[-d])[d+1]$$

(F.3) We add a Koszul sign coming from transposing $b^{p+1} \otimes \cdots \otimes b^{|D_s|}$ with all of the elements that do not correspond to an incoming arrow of D_s. Recall that the elements b^i for $i \in [\![1, |D_s|]\!] \setminus \{p\}$ are labeled with an index from $[\![1, |D|]\!]$ coming from the corresponding outgoing arrow of (D, R).

(F.4) We then consider the diagram (D', R') given by removing the disc D_s from (D, R). Its bold arrow is the one previously connected to α. Note that the element b^p is associated with this incoming arrow of (D', R').

(F.5) We evaluate $\mathcal{E}((D', R'), s_{d+1}F_1, \dots, s_{d+1}\check{F}_s, \dots, s_{d+1}F_n)$ at the elements $sa^i_j \in {}_{x^i_j}\mathcal{A}_{x^i_{j+1}}[1]$ corresponding to the incoming arrows of D'. Recall that the tensor factors in this evaluation are labeled with an index from $[\![1, |D|]\!]$ coming from the corresponding outgoing arrow of (D, R).

We illustrate the previous procedure with an admissible diagram D consisting of two discs, given as follows

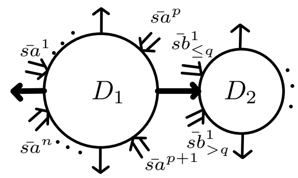


In this case, the evaluation of $\mathcal{E}(D, s_{d+1}F^{\bar{\bar{x}}}, s_{d+1}G^{\bar{\bar{y}}})$ at

$$(s\bar{a}^1, \dots, s\bar{a}^{p-1}, s\bar{b}^1_{\leq q}, s\bar{a}^p, s\bar{b}^2 \dots, s\bar{b}^m, s\bar{a}^{p+1}, s\bar{b}^1_{>q}, s\bar{a}^{p+2} \dots, s\bar{a}^n)$$

with $\bar{sa}^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ and $\bar{sb}^i \in \mathcal{A}[1]^{\otimes \bar{y}^i}$, where $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n)$ and $\bar{\bar{y}} = (\bar{y}^1, \dots, \bar{y}^m)$ are the respective types of D_1 and D_2, is detailed as follows.

First, we place each tensor factor sa^i_j and sb^i_j of $\bar{sa}^i$ and $\bar{sb}^i$ in the corresponding incoming arrow, as explained in (F.1), adding a sign $(-1)^{|\bar{sb}^1_{\leq q}||\bar{sa}^p|+|\bar{sb}^1_{>q}||\bar{sa}^{p+1}|}$ for the permutation of the corresponding elements.



In step (F.2), we place all the elements $\bar{sb}^i$ after the others and multiply by a sign

$$(-1)^{(|\bar{sb}^1|+\cdots+|\bar{sb}^m|)(|\bar{sa}^{p+1}|+\cdots+|\bar{sa}^n|)}$$

We then compute $\mathcal{E}(D_1, s_{d+1}F^{\bar{\bar{x}}})(\bar{sa}^1, \dots, \bar{sa}^n) = (-1)^{(d+1)(|\epsilon_1|+\cdots+|\epsilon_{n-1}|)}\epsilon_1 \otimes \cdots \otimes s_{d+1}\epsilon_n$.

After (F.3), we end with

$$\epsilon_1 \otimes \cdots \otimes \epsilon_{p-1} \otimes (\mathcal{E}, s_{d+1}G^{\bar{\bar{y}}})(\epsilon_p \otimes \bar{sb}^1_{\leq q} \otimes \cdots \otimes \bar{sb}^m \otimes \bar{sb}^1_{>q}) \otimes \epsilon_{p+1} \otimes \cdots \otimes s_{d+1}\epsilon_n$$

preceded by a sign $(-1)^{\Delta'}$ with

$$\begin{aligned}\Delta' = {} & |\bar{sb}^1_{\leq q}||\bar{sa}^p| + |\bar{sb}^1_{>q}||\bar{sa}^{p+1}| + \Delta + (|\bar{sb}^1_{\leq q}| + \cdots + |\bar{sb}^m| + |\bar{sb}^1_{>q})(|\epsilon_{p+1}| + \cdots + |\epsilon_n| + d + 1) \\ & + (|G^{\bar{\bar{y}}}| + d + 1)(\epsilon_1 + \cdots + \epsilon_{p-1})\end{aligned}$$

In (F.5), we order the elements and compute $\mathcal{E}(D_2, s_{d+1}G^{\bar{\bar{y}}})(\bar{sb}^1_{\leq q} \otimes \epsilon_p \otimes \bar{sb}^1_{>q}, \bar{sb}^2, \dots, \bar{sb}^m)$ and multiply the result by

$$(-1)^{|\bar{sb}^1_{>q}|(|\epsilon_i|+|\bar{sb}^2|+\cdots+|\bar{sb}^m|)+(d+1)|G^{\bar{\bar{y}}}|}$$

Finally, we get

$$\mathcal{E}(D, s_{d+1}F^{\bar{\bar{x}}}, s_{d+1}G^{\bar{\bar{y}}})(\bar{sa}^1, \dots, \bar{sa}^n) = (-1)^{\Delta''}\epsilon_1 \otimes \cdots \otimes \epsilon_{p-1} \otimes \delta_1 \otimes \cdots \otimes \delta_m \otimes \epsilon_{p+1} \otimes \cdots \otimes s_{d+1}\epsilon_n$$

where $\Delta'' = \Delta' + |\bar{sb}^1_{>q}|(|\epsilon_i| + |\bar{sb}^2| + \cdots + |\bar{sb}^m|) + (d+1)|G^{\bar{\bar{y}}}|$.

Definition 4.16. *A **filled diagram** is an admissible diagram $(D = \{D_1, \dots, D_n\}, R)$ together with elements $s_{d+1}F_i \in \mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{A})[d])[d+1]$ for $i \in [\![1, n]\!]$.*

To each filled diagram $\mathscr{D} = \{(D = \{D_1, \dots, D_n\}, R), (s_{d+1}F_i)_{i \in [\![1,n]\!]}\}$ we associate the element

$$\mathcal{E}(\mathscr{D}) = \mathcal{E}((D, R), s_{d+1}F^{\bar{\bar{x}}_1}, \dots, s_{d+1}F^{\bar{\bar{x}}_n}) \in \mathrm{Multi}^{\bar{\bar{x}}}(\mathrm{Bar}(\mathcal{A})[d])[d+1]$$

where $\bar{\bar{x}}$ is the type of (D, R) and $\bar{\bar{x}}_i$ is the type of D_i for $i \in [\![1, n]\!]$. We depict a filled diagram as a diagram replacing the names of the discs by the corresponding maps (see Figure 4.7).

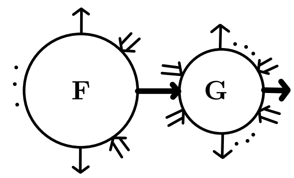
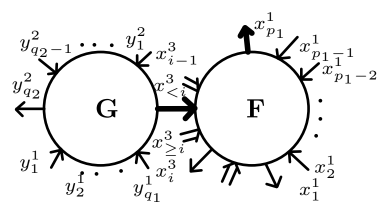


Figure 4.7: A filled diagram

Example 4.17. *Consider the following filled diagram.*



It represents a map

$$s_{d+1}F^{\bar{\bar{x}}} : \mathcal{A}[1]^{\otimes \bar{x}^1} \otimes \mathcal{A}[1]^{\otimes \bar{x}^2} \otimes \mathcal{A}[1]^{\otimes \bar{x}^3} \to {}_{x_1^1}\mathcal{A}_{x_{p_2}^2}[-d] \otimes {}_{x_1^2}\mathcal{A}_{x_{p_3}^3}[-d] \otimes {}_{x_1^3}\mathcal{A}_{x_{p_1}^1}[1]$$

where $\bar{\bar{x}} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$ *and* $\bar{x}^j = (x_1^j, \ldots, x^j_{\lg(\bar{x}^i)})$ *for* $j \in [\![1,3]\!]$ *that takes as an input the last output of a map*

$$s_{d+1}G^{(\bar{y}^1, \bar{y}^2)} : \mathcal{A}[1]^{\otimes \bar{y}^1} \otimes \mathcal{A}[1]^{\otimes \bar{y}^2} \to {}_{y_1^1}\mathcal{A}_{y_m^2}[-d] \otimes {}_{x_{i-1}^3}\mathcal{A}_{x_i^3}[1]$$

with $\bar{y}^1 = (y_1^1, y_2^1, \ldots, y_n^1, x_i^3)$, $\bar{y}^2 = (x_{i-1}^3, y_1^2, \ldots, y_m^2)$.

The type of the diagram is $(\bar{x}^1, \bar{x}^2, \bar{y}^1 \sqcup \bar{x}^3_{\geq i}, \bar{x}^3_{<i} \sqcup \bar{y}^2)$*, meaning that it represents a map that can be evaluated at* $(s\bar{a}^1, s\bar{a}^2, s\bar{b}^1 \otimes s\bar{a}^3_{\geq i}, s\bar{a}^3_{<i} \otimes s\bar{b}^2)$ *where* $s\bar{a}^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ *and* $s\bar{b}^i \in \mathcal{A}^{\otimes \bar{y}^i}$*. This evaluation goes in 5 steps.*

1. *We place the elements around the diagram, which creates a sign* $(-1)^\epsilon$ *with*

$$\epsilon = |s_{d+1}G^{(\bar{y}^1, \bar{y}^2)}|(|s\bar{a}^1| + |s\bar{a}^2| + |s\bar{a}^3_{\geq i}|) + |s\bar{a}_{\geq i}||s\bar{b}^1| + |s\bar{b}^2||s\bar{a}_{>i}|$$

2. *We evaluate* $\mathcal{E}(D_1, s_{d+1}G^{(\bar{y}^1, \bar{y}^2)})$ *at* $(s\bar{b}^1, s\bar{b}^2)$*. The result of this evaluation is an element of the shifted tensor product* $({}_{y_1^1}\mathcal{A}_{y_m^2}[-d] \otimes {}_{x_{i-1}^3}\mathcal{A}_{x_i^3}[-d])[d+1]$ *that we will write* $s_{d+1}({}_{y_1^1}G^{-d}_{y_m^2} \otimes {}_{x_{i-1}^3}G^{-d}_{x_i^3})$*. We then use the isomorphism (2.1) to obtain the tensor product*

$$(-1)^{\epsilon + (d+1)|{}_{y_1^1}G^{-d}_{y_m^2}|}\, {}_{y_1^1}G^{-d}_{y_m^2} \otimes {}_{x_{i-1}^3}G^1_{x_i^3} \in {}_{y_1^1}\mathcal{A}_{y_m^2}[-d] \otimes {}_{x_{i-1}^3}\mathcal{A}_{x_i^3}[1]$$

3. *We put the first output before and get*

$$(-1)^{\epsilon+(d+1)|{}_{y_1^1}G^{-d}_{y_m^2}|+\epsilon'}\;{}_{y_1^1}G^{-d}_{y_m^2} \otimes \mathcal{E}(D_2, s_{d+1}F^{\bar{\bar{x}}}(\bar{sa}^1, \bar{sa}^2, \bar{sa}_{\geq i}, \bar{sa}_{<i} \otimes {}_{x^3_{i-1}}G^1_{x^3_i}))$$

   with $\epsilon' = {}_{y_1^1}G^{-d}_{y_m^2}|(|F| + d + 1 + |\bar{sa}^1| + |\bar{sa}^2| + |\bar{sa}_{\geq i}|)$

4. *We evaluate* $F^{\bar{\bar{x}}}$ *at* $(\bar{sa}^1, \bar{sa}^2, \bar{sa}_{<i} \otimes {}_{x^3_{i-1}}G^1_{x^3_i} \otimes \bar{sa}_{\geq i})$ *which add a sign* $(-1)^{|\bar{sa}_{\geq i}|(|\bar{sa}_{<i}|+|{}_{x^3_{i-1}}G^1_{x^3_i}|)}$ *to the result for the transposition of the corresponding elements.*

5. *We order the outputs according to the labeling of their corresponding outgoing arrow and finally get*

$$(-1)^{\epsilon+(d+1)|{}_{y_1^1}G^{-d}_{y_m^2}|+\epsilon'+\epsilon''}\;{}_{x_1^1}F^{-d}_{x^2_{p_2}} \otimes {}_{x_2^1}F^{-d}_{x^3_{p_3}} \otimes {}_{y_1^1}G^{-d}_{y_m^2} \otimes {}_{x_1^3}F^1_{x^1_{p_1}}$$

   with $\epsilon'' = (-1)^{(|{}_{y_1^1}G^{-d}_{y_m^2}|+1)(|{}_{x_1^1}F^{-d}_{x^2_{p_2}}|+|{}_{x_1^2}F^{-d}_{x^3_{p_3}}|)+(d+1)(|{}_{x_1^1}F^{-d}_{x^2_{p_2}}|+\dots+|{}_{x_2^1}F^{-d}_{x^3_{p_3}}|+\dots+|{}_{y_1^1}G^{-d}_{y_m^2}|)}$.

4.2 The necklace graded Lie algebra

In order to recall what a pre-Calabi-Yau structure on $\mathcal{A}$ is, one first defines a graded Lie algebra, called the **necklace graded Lie algebra** and appearing in [5]. As a graded vector space, this graded Lie algebra is $\mathrm{Multi}^{\bullet}(\mathrm{Bar}(\mathcal{A})[d])^{C_{\mathrm{lg}(\bullet)}}[d+1]$. In order to define a graded Lie bracket on this space, we first define a new operation as follows.

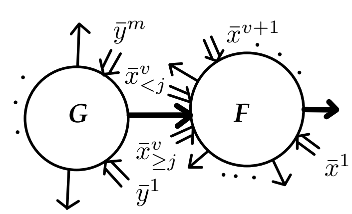
Definition 4.18. *Consider a graded quiver $\mathcal{A}$ with set of objects $\mathcal{O}$ as well as tuples of elements of $\bar{\mathcal{O}}$ given by $\bar{\bar{x}} = (\bar{x}^1, ..., \bar{x}^n), \bar{\bar{y}} = (\bar{y}^1, ..., \bar{y}^m) \in \bar{\bar{\mathcal{O}}}$ such that $\mathrm{rt}(\bar{y}^1) = x^v_j$ and $\mathrm{lt}(\bar{y}^m) = x^v_{j-1}$ for some $v \in [\![1, n]\!]$ and $j \in [\![1, \mathrm{lg}(\bar{x}^v)]\!]$. The **inner necklace composition at v,j** of elements $s_{d+1}F^{\bar{\bar{x}}} \in \mathrm{Multi}^{\bar{\bar{x}}}(\mathrm{Bar}(\mathcal{A})[d])[d+1]$ and $s_{d+1}G^{\bar{\bar{y}}} \in \mathrm{Multi}^{\bar{\bar{y}}}(\mathrm{Bar}(\mathcal{A})[d])[d+1]$ is given by*

$$s_{d+1}F^{\bar{\bar{x}}} \underset{\substack{\mathrm{nec},v,j\\ \mathrm{inn}}}{\circ} s_{d+1}G^{\bar{\bar{y}}} = \mathcal{E}(\mathscr{D}) \in \mathrm{Multi}^{\bar{\bar{x}} \underset{v,j,\mathrm{inn}}{\sqcup} \bar{\bar{y}}}(\mathrm{Bar}(\mathcal{A})[d])[d+1]$$

with

$$\bar{\bar{x}} \underset{v,j,\mathrm{inn}}{\sqcup} \bar{\bar{y}} = (\bar{x}^1, \dots, \bar{x}^{v-1}, \bar{y}^1 \sqcup \bar{x}^v_{<j}, \bar{y}^2, \dots, \bar{y}^{m-1}, \bar{x}^v_{>j-1} \sqcup \bar{y}^m, \bar{x}^{v+1}, \dots, \bar{x}^n)$$

and where $\mathscr{D}$ is the filled diagram of type $\bar{\bar{x}} \underset{v,j,\mathrm{inn}}{\sqcup} \bar{\bar{y}}$ given by



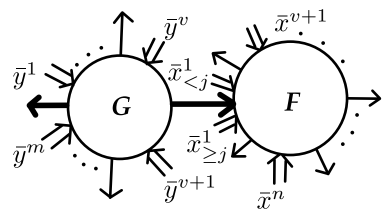
Definition 4.19. *Consider a graded quiver $\mathcal{A}$ with set of objects $\mathcal{O}$ as well as tuples of elements of $\bar{\mathcal{O}}$ given by $\bar{\bar{x}} = (\bar{x}^1, ..., \bar{x}^n), \bar{\bar{y}} = (\bar{y}^1, ..., \bar{y}^m) \in \bar{\bar{\mathcal{O}}}$ such that $\mathrm{lt}(\bar{y}^v) = x^1_{j-1}$ and $\mathrm{rt}(\bar{y}^{v+1}) = x^1_j$ for some $v \in [\![1, m]\!]$ and $j \in [\![1, \mathrm{lg}(\bar{x}^1)]\!]$. The* ***outer necklace composition at*** v,j *of elements $s_{d+1}F^{\bar{\bar{x}}} \in \mathrm{Multi}^{\bar{\bar{x}}}(\mathrm{Bar}(\mathcal{A})[d])[d+1]$ and $s_{d+1}G^{\bar{\bar{y}}} \in \mathrm{Multi}^{\bar{\bar{y}}}(\mathrm{Bar}(\mathcal{A})[d])[d+1]$ is given by*

$$s_{d+1}F^{\bar{\bar{x}}} \underset{\substack{\mathrm{nec},v,j \\ \mathrm{out}}}{\circ} s_{d+1}G^{\bar{\bar{y}}} = \mathcal{E}(\mathscr{D}) \in \mathrm{Multi}^{\bar{\bar{x}} \underset{v,j,\mathrm{out}}{\sqcup} \bar{\bar{y}}}(\mathrm{Bar}(\mathcal{A})[d])[d+1]$$

with

$$\bar{\bar{x}} \underset{v,j,\mathrm{out}}{\sqcup} \bar{\bar{y}} = (\bar{y}^1, \dots, \bar{y}^{v-1}, \bar{x}^1_{<j-1} \sqcup \bar{y}^v, \bar{x}^2, \dots, \bar{x}^n, \bar{y}^{v+1} \sqcup \bar{x}^1_{>j}, \bar{y}^{v+2}, \dots, \bar{y}^m)$$

and where $\mathscr{D}$ is the filled diagram of type $\bar{\bar{x}} \underset{v,j,\mathrm{out}}{\sqcup} \bar{\bar{y}}$ given by



Definition 4.20. *Given a graded quiver $\mathcal{A}$ with set of objects $\mathcal{O}$, the* ***necklace product*** *of elements $s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G} \in \mathrm{Multi}^{\bullet}(\mathrm{Bar}(\mathcal{A})[d])^{C_{\mathrm{lg}(\bullet)}}[d+1]$ is the element*

$$s_{d+1}\mathbf{F} \underset{\mathrm{nec}}{\circ} s_{d+1}\mathbf{G} \in \mathrm{Multi}^{\bullet}(\mathrm{Bar}(\mathcal{A})[d])[d+1]$$

given by

$$(s_{d+1}\mathbf{F} \underset{\mathrm{nec}}{\circ} s_{d+1}\mathbf{G})^{\bar{\bar{z}}} = \sum_{(\bar{\bar{x}}, \bar{\bar{y}}, v, j) \in \mathcal{I}_{\mathrm{inn}}} s_{d+1}F^{\bar{\bar{x}}} \underset{\substack{\mathrm{nec},v,j \\ \mathrm{inn}}}{\circ} s_{d+1}G^{\bar{\bar{y}}} + \sum_{(\bar{\bar{x}}, \bar{\bar{y}}, v, j) \in \mathcal{I}_{\mathrm{out}}} s_{d+1}F^{\bar{\bar{x}}} \underset{\substack{\mathrm{nec},v,j \\ \mathrm{out}}}{\circ} s_{d+1}G^{\bar{\bar{y}}}$$

for all $\bar{\bar{z}} \in \bar{\bar{\mathcal{O}}}$, where

$$\mathcal{I}_{\mathrm{inn}} = \{(\bar{\bar{x}}, \bar{\bar{y}}, v, j) \in \bar{\bar{\mathcal{O}}} \times \bar{\bar{\mathcal{O}}} \times [\![1, \mathrm{lg}(\bar{\bar{x}})]\!] \times [\![1, \mathrm{lg}(\bar{x}^v)]\!] | \bar{\bar{x}} \underset{v,j,\mathrm{inn}}{\sqcup} \bar{\bar{y}} = \bar{\bar{z}}\}$$

$$\mathcal{I}_{\mathrm{out}} = \{(\bar{\bar{x}}, \bar{\bar{y}}, v, j) \in \bar{\bar{\mathcal{O}}} \times \bar{\bar{\mathcal{O}}} \times [\![1, \mathrm{lg}(\bar{\bar{y}})]\!] \times [\![1, \mathrm{lg}(\bar{x}^1)]\!] | \bar{\bar{x}} \underset{v,j,\mathrm{out}}{\sqcup} \bar{\bar{y}} = \bar{\bar{z}}\}$$

Definition 4.21. *Given a graded quiver $\mathcal{A}$ with set of objects $\mathcal{O}$, the* ***necklace bracket*** *of two elements $s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G} \in \mathrm{Multi}^{\bullet}(\mathrm{Bar}(\mathcal{A})[d])^{C_{\mathrm{lg}(\bullet)}}[d+1]$ is defined as the element*

$$[s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G}]_{\mathrm{nec}} \in \mathrm{Multi}^{\bullet}(\mathrm{Bar}(\mathcal{A})[d])[d+1]$$

where

$$[s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G}]^{\bar{\bar{z}}}_{\mathrm{nec}} = (s_{d+1}\mathbf{F} \underset{\mathrm{nec}}{\circ} s_{d+1}\mathbf{G})^{\bar{\bar{z}}} - (-1)^{(|\mathbf{F}|+d+1)(|\mathbf{G}|+d+1)}(s_{d+1}\mathbf{G} \underset{\mathrm{nec}}{\circ} s_{d+1}\mathbf{F})^{\bar{\bar{z}}}$$

for every $\bar{\bar{z}} \in \bar{\bar{\mathcal{O}}}$.

Definition 4.22. *Given a graded quiver $\mathcal{A}$ with set of objects $\mathcal{O}$, we define the graded vector space $\mathscr{C}_\bullet(\mathcal{A}) = \prod_{\bar{\bar{x}}\in\bar{\bar{\mathcal{O}}}} \mathscr{C}_{\bar{\bar{x}}}(\mathcal{A})$, where*

$$\mathscr{C}_{\bar{\bar{x}}}(\mathcal{A}) = \mathcal{H}om_{\Bbbk}\Big(\bigotimes_{i=1}^{n-1}(\mathcal{A}[1]^{\otimes \bar{x}^{n-i+1}} \otimes {}_{\mathrm{rt}(\bar{x}^{n-i+1})}\mathcal{A}^*_{\mathrm{lt}(\bar{x}^{n-i})}[d]) \otimes \mathcal{A}[1]^{\otimes \bar{x}^1}, {}_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}_{\mathrm{rt}(\bar{x}^1)}\Big)$$

for $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\bar{\mathcal{O}}}$.

Lemma 4.23. *Let $\mathcal{A}$ be a graded quiver with set of objects $\mathcal{O}$. Then, we have an injective map*

$$\jmath : \mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{A})[d])[d+1] \to \mathscr{C}_\bullet(\mathcal{A})[1]$$

given by the family $\jmath = (\jmath_{\bar{\bar{x}}})_{\bar{\bar{x}}\in\bar{\bar{\mathcal{O}}}}$ where $\jmath_{\bar{\bar{x}}} : \mathrm{Multi}^{\bar{\bar{x}}}(\mathrm{Bar}(\mathcal{A})[d])[d+1] \to \mathscr{C}_{\bar{\bar{x}}}(\mathcal{A})[1]$ is defined by $\jmath_{\bar{\bar{x}}}(s_{d+1}\phi) = s\psi_{\bar{\bar{x}}}$ with

$$\psi_{\bar{\bar{x}}}(s\bar{a}^n, tf^{n-1}, s\bar{a}^{n-2}, tf^{n-2}, ..., s\bar{a}^2, tf^2, s\bar{a}^1) = (-1)^\epsilon\big(\bigotimes_{i=1}^{n-1}(f^i \circ s_d) \otimes s_d\big)\big(\phi(s\bar{a}^1, s\bar{a}^2, ..., s\bar{a}^n)\big)$$

for $\phi \in \mathrm{Multi}^{\bar{\bar{x}}}(\mathrm{Bar}(\mathcal{A})[d])$, $n = \lg(\bar{\bar{x}})$, $s\bar{a}^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ for $i \in [\![1,n]\!]$ and $tf^i \in {}_{\mathrm{rt}(\bar{x}^{i+1})}\mathcal{A}^*_{\mathrm{lt}(\bar{x}^i)}[d]$ for $i \in [\![1,n-1]\!]$ with

$$\epsilon = \sum_{i=1}^{n-1}|tf^i|\sum_{j=i+1}^{n}|s\bar{a}^j| + (|\phi|+d)\sum_{i=1}^{n-1}|tf^i| + d(n-1) + \sum_{1\leq i<j\leq n}|s\bar{a}^i||s\bar{a}^j| + \sum_{1\leq i<j\leq n-1}|tf^i||tf^j|$$

We denote by Γ the natural bilinear form on the boundary quiver $\partial_{d-1}\mathcal{A} = \mathcal{A} \oplus \mathcal{A}^*[d-1]$, defined in Example 3.18.

Definition 4.24. *Given a graded quiver $\mathcal{A}$ with set of objects $\mathcal{O}$, the* ***noncounitary cofree conilpotent graded cocategory on*** *$\mathcal{A}$ is defined as $\bar{T}^c(\mathcal{A}[1]) = \bigoplus_{\bar{x}\in\bar{\mathcal{O}}} \mathcal{A}[1]^{\otimes \bar{x}}$ with coproduct given by deconcatenation and the induced grading by the one of $\mathcal{A}$.*

Remark 4.25. *We have that*

$$\bar{T}^c(\partial_{d-1}\mathcal{A}[1]) = \bigoplus_{\bar{\bar{x}}\in\bar{\bar{\mathcal{O}}}}\bigotimes_{i=1}^{n-1}(\mathcal{A}[1]^{\otimes \bar{x}^{n-i+1}} \otimes {}_{\mathrm{rt}(\bar{x}^{n-i+1})}\mathcal{A}^*_{\mathrm{lt}(\bar{x}^{n-i})}[d]) \otimes \mathcal{A}[1]^{\otimes \bar{x}^1} \tag{4.3}$$

Definition 4.26. *Given a graded cocategory $\mathcal{A}$ with coproduct Δ given by maps ${}_y\Delta_x^z : {}_y\mathcal{A}_x \to {}_y\mathcal{A}_z \otimes {}_z\mathcal{A}_x$ for $x, y, z \in \mathcal{O}$, a* ***homogeneous coderivation of degree d*** *on $\mathcal{A}$ is a collection ${}_y(B_x)_{x,y\in\mathcal{O}}$ of $\Bbbk$-linear maps ${}_yB_x : {}_y\mathcal{A}_x \to {}_y\mathcal{A}_x$ of degree d such that*

$${}_y\Delta_x^z \circ {}_yB_x = ({}_yB_z \otimes \mathrm{id} + \mathrm{id} \otimes {}_zB_x) \circ {}_y\Delta_x^z$$

for each $x, y, z \in \mathcal{O}$.

We will denote by $\mathrm{Coder}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$ the graded vector space formed by sums of homogeneous coderivations of $\bar{T}^c(\partial_{d-1}\mathcal{A}[1])$.

If $B \in \mathrm{Coder}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$, we will denote by $B^{\bar{\bar{x}}}$ the restriction of B to the tuple indexed by $\bar{\bar{x}} \in \bar{\bar{\mathcal{O}}}$ in (4.3), by $B^{\bar{\bar{x}}}_{\mathcal{A}}$ the composition of $B^{\bar{\bar{x}}}$ and the canonical projection $\bar{T}^c(\partial_{d-1}\mathcal{A}[1]) \to \mathcal{A}[1]$ and by $B^{\bar{\bar{x}}}_{\mathcal{A}^*}$ the composition of $B^{\bar{\bar{x}}}$ and the canonical projection $\bar{T}^c(\partial_{d-1}\mathcal{A}[1]) \to \mathcal{A}^*[d]$.

Definition 4.27. *A coderivation $B \in \mathrm{Coder}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$ is* ***cyclic*** *with respect to the natural bilinear form Γ if the following equations are satisfied*

$$\Gamma(B_{\mathcal{A}}^{\bar{\bar{x}}}(\bar{sa}^n, tf^{n-1}, ..., \bar{sa}^2, tf^1, \bar{sa}^1), tf^n) = (-1)^{\epsilon}\Gamma(B_{\mathcal{A}}^{\sigma\cdot\bar{\bar{x}}}(\bar{sa}^1, tf^n, \bar{sa}^n, tf^{n-1}, ..., \bar{sa}^2), tf^1) \tag{4.4}$$

and

$$\Gamma(B_{\mathcal{A}^*}^{\bar{\bar{x}}}(\bar{sa}^n, tf^{n-1}, ..., \bar{sa}^2, tf^1, \bar{sa}^1), sb) = (-1)^{\epsilon'}\Gamma(B_{\mathcal{A}}^{\bar{\bar{x}}'}(\bar{sa}^1 \otimes sb \otimes \bar{sa}^n, tf^{n-1}, ..., \bar{sa}^2), tf^1) \tag{4.5}$$

for $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n)$, $\sigma = (12\dots n)$ and $\bar{\bar{x}}' = (\bar{x}^2, \dots, \bar{x}^1 \sqcup \bar{x}^n)$, where the arguments are $\bar{sa}^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$, $tf^i \in {}_{\mathrm{rt}(\bar{x}^{i+1})}\mathcal{A}^*_{\mathrm{lt}(\bar{x}^i)}[d]$ for $i \in [\![1,n]\!]$, $sb \in {}_{\mathrm{rt}(\bar{x}^1)}\mathcal{A}_{\mathrm{lt}(\bar{x}^n)}[1]$ and

$$\epsilon = (|\bar{sa}^1| + |tf^n|)(\sum_{i=2}^{n} |\bar{sa}^i| + \sum_{i=1}^{n-1} |tf^i|)\ ,\ \epsilon' = (|\bar{sa}^1| + |sb|)(\sum_{i=2}^{n} |\bar{sa}^i| + \sum_{i=1}^{n-1} |tf^i|)$$

We define $\mathrm{Coder}_{cyc}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$ as the graded vector subspace of $\mathrm{Coder}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$ consisting of sums of homogeneous coderivations that are cyclic with respect to the natural bilinear form Γ. We also define the graded vector subspace of $\mathrm{Coder}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$ which we denote $\mathrm{Coder}_{cyc,0}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$ and which is formed by the elements of $\mathrm{Coder}_{cyc}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$ sending $\bar{T}^c(\mathcal{A}[1])$ to itself.

Proposition 4.28. *The linear map*

$$\mathcal{R} : \mathrm{Coder}_{cyc,0}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1])) \to \mathscr{C}_\bullet(\mathcal{A})[1]$$

of degree 0, sending a coderivation B to the collection $(B_{\mathcal{A}}^{\bar{\bar{x}}})_{\bar{\bar{x}}\in\bar{\bar{\mathcal{O}}}}$ is an isomorphism of graded vector spaces.

Proof. We will construct the inverse $\mathcal{R}^{-1}$ of this map. Consider $B_{\mathcal{A}}^{\bar{\bar{x}}} \in \mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{A})[d])^{C_{\mathrm{lg}(\bullet)}}$. We construct a collection $(B_{\mathcal{A}^*}^{\bar{\bar{x}}})_{\bar{\bar{x}}\in\bar{\bar{\mathcal{O}}}}$ of maps

$$B_{\mathcal{A}^*}^{\bar{\bar{x}}} : \bigotimes_{i=1}^{n-1}(\mathcal{A}[1]^{\otimes \bar{x}^{n-i+1}} \otimes {}_{\mathrm{rt}(\bar{x}^{n-i+1})}\mathcal{A}^*_{\mathrm{lt}(\bar{x}^{n-i})}[d]) \otimes \mathcal{A}[1]^{\otimes \bar{x}^1} \longrightarrow {}_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}^*_{\mathrm{rt}(\bar{x}^1)}[d]$$

given by

$$\Gamma(B_{\mathcal{A}^*}^{\bar{\bar{x}}}(\bar{sa}^n, tf^n, ..., \bar{sa}^2, tf^1, \bar{sa}^1), sb) = (-1)^{\epsilon'}\Gamma(B_{\mathcal{A}}^{\bar{\bar{x}}'}(\bar{sa}^1 \otimes sb \otimes \bar{sa}^n, tf^{n-1}, ..., \bar{sa}^2), tf^1)$$

for each $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\bar{\mathcal{O}}}$ and elements $\bar{sa}^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ for $i \in [\![1,n]\!]$, $tf^i \in {}_{\mathrm{rt}(\bar{x}^{i+1})}\mathcal{A}^*_{\mathrm{lt}(\bar{x}^i)}[d]$ for $i \in [\![1,n-1]\!]$ and $sb \in {}_{\mathrm{rt}(\bar{x}^1)}\mathcal{A}_{\mathrm{lt}(\bar{x}^n)}[1]$, where $\bar{\bar{x}}' = (\bar{x}^2, \dots, \bar{x}^1 \sqcup \bar{x}^n)$ and

$$\epsilon' = (|\bar{sa}^1| + |sb|)(\sum_{i=1}^{n-1} |\bar{sa}^i| + \sum_{i=2}^{n-1} |tf^i|)$$

One can easily check that the map that associates to $B_{\mathcal{A}}^{\bar{\bar{x}}}$ the coderivation given by $(B_{\mathcal{A}}^{\bar{\bar{x}}}, B_{\mathcal{A}^*}^{\bar{\bar{x}}})_{\bar{\bar{x}}}$ is the inverse of $\mathcal{R}$. □

We will denote by $\mathcal{R}_{\mathcal{A}^*}^{-1}$ (resp. $\mathcal{R}_{\mathcal{A}}^{-1}$ the composition of $\mathcal{R}^{-1}$ and the canonical projection $\bar{T}^c(\partial_{d-1}\mathcal{A}[1]) \to \mathcal{A}^*[d]$ (resp. $\bar{T}^c(\partial_{d-1}\mathcal{A}[1]) \to \mathcal{A}[1]$).

We have the following relation between the necklace product and the usual Gerstenhaber circle product, which does not seem to have been observed in the literature so far.

Proposition 4.29. *Let $\mathbf{F}$, $\mathbf{G}$ be elements in* $\mathrm{Multi}^{\bullet}(\mathrm{Bar}(\mathcal{A})[d])^{C_{\mathrm{lg}(\bullet)}}$. *Then, we have*

$$j_{\bar{\bar{x}}\underset{v,j,\mathrm{inn}}{\sqcup}\bar{\bar{y}}}(s_{d+1}F^{\bar{\bar{x}}}\underset{\substack{\mathrm{nec},v,j\\ \mathrm{inn}}}{\circ}s_{d+1}G^{\bar{\bar{y}}})=j_{\bar{\bar{x}}}(s_{d+1}F^{\bar{\bar{x}}})\underset{G,p,q}{\circ}j_{\bar{\bar{y}}}(s_{d+1}G^{\bar{\bar{y}}}) \tag{4.6}$$

for $\bar{\bar{x}},\bar{\bar{y}}\in\bar{\bar{\mathcal{O}}}$, $v\in[\![1,\mathrm{lg}(\bar{\bar{x}})]\!]$ and $j\in[\![1,\mathrm{lg}(\bar{x}^v)]\!]$ and

$$j_{\bar{\bar{x}}\underset{v,j,\mathrm{out}}{\sqcup}\bar{\bar{y}}}(s_{d+1}F^{\bar{\bar{x}}}\underset{\substack{\mathrm{nec},v,j\\ \mathrm{out}}}{\circ}s_{d+1}G^{\bar{\bar{y}}})=-(-1)^{(|\mathbf{F}|+d+1)(|\mathbf{G}|+d+1)}j_{\bar{\bar{y}}}(s_{d+1}G^{\bar{\bar{y}}})\underset{G,p,q}{\circ}\mathcal{R}^{-1}_{\mathcal{A}^*}(j_{\bar{\bar{x}}}(s_{d+1}F^{\bar{\bar{x}}})) \tag{4.7}$$

for $v\in[\![1,\mathrm{lg}(\bar{\bar{y}})]\!]$ and $j\in[\![1,\mathrm{lg}(\bar{x}^1)]\!]$, where $p=\mathrm{lg}(\bar{x}^1)+...+\mathrm{lg}(\bar{x}^v)+j+2$ and $q=p+\sum\limits_{i=1}^{\mathrm{lg}(\bar{\bar{y}})}\mathrm{lg}(\bar{y}^i)$.

Proof. We will first show the identity (4.6). Given $\bar{\bar{x}}=(\bar{x}^1,\dots,\bar{x}^n),\bar{\bar{y}}=(\bar{y}^1,\dots,\bar{y}^m)\in\bar{\bar{\mathcal{O}}}$, both the compositions

$$s_{d+1}F^{\bar{\bar{x}}}\underset{\substack{\mathrm{nec},v,j\\ \mathrm{inn}}}{\circ}s_{d+1}G^{\bar{\bar{y}}}$$

and

$$j_{\bar{\bar{x}}}(s_{d+1}F^{\bar{\bar{x}}})\underset{G,p,q}{\circ}j_{\bar{\bar{y}}}(s_{d+1}G^{\bar{\bar{y}}})$$

are zero if there are no $u,v\in[\![1,n]\!]$ and $j\in[\![1,\mathrm{lg}(\bar{x}^v)]\!]$ such that $\mathrm{lt}(\bar{y}^u)=x^v_j$ and $\mathrm{rt}(\bar{y}^{u+1})=x^v_{j+1}$ with the convention that $\bar{x}^{n+1}=\bar{x}^1$. We will thus assume that there exist such $u,v\in[\![1,n]\!]$ and $j\in[\![1,\mathrm{lg}(\bar{x}^v)]\!]$. We can further suppose that $u=n$ because of the invariance under the action of C_n. To simplify the expressions, we will denote the result of applying $F^{\bar{\bar{x}}}$ on any argument as a tensor product ${}_{\mathrm{lt}(\bar{x}^1)}F^{-d}_{\mathrm{rt}(\bar{x}^2)}\otimes\cdots\otimes{}_{\mathrm{lt}(\bar{x}^{n-1})}F^{-d}_{\mathrm{rt}(\bar{x}^n)}\otimes{}_{\mathrm{lt}(\bar{x}^n)}F^{-d}_{\mathrm{rt}(\bar{x}^1)}$, where we omit those arguments. We have

$$\begin{aligned}
&j_{\bar{\bar{x}}\underset{v,j,\mathrm{inn}}{\sqcup}\bar{\bar{y}}}(s_{d+1}F^{\bar{\bar{x}}}\underset{\substack{\mathrm{nec},v,j\\ \mathrm{inn}}}{\circ}s_{d+1}G^{\bar{\bar{y}}})\\
&(\bar{sa}^n,tf^{n-1},\dots,\bar{sa}^{v+1},tf^v,\bar{sa}^v_{>j},\bar{sb}^m,tg^{m-1},\dots,\bar{sb}^2,tg^1,\bar{sb}^1,\bar{sa}^v_{\leq j},tf^{v-1},\dots,\bar{sa}^2,tf^1,\bar{sa}^1)\\
&=(-1)^{\epsilon}((f^1\circ s_d)\otimes\cdots\otimes(f^v\circ s_d)\otimes(g^1\circ s_d)\otimes\cdots\otimes(g^{m-1}\circ s_d)\otimes(f^{v+1}\circ s_d)\otimes\cdots\otimes(f^{n-1}\circ s_d)\otimes\mathrm{id})\\
&\quad(s_{d+1}F^{\bar{\bar{x}}}\underset{\substack{\mathrm{nec},v,j\\ \mathrm{inn}}}{\circ}s_{d+1}G^{\bar{\bar{y}}})(\bar{sa}^1,\bar{sa}^2,\dots,\bar{sa}^{v-1},\bar{sa}^v_{\leq j},\bar{sb}^1,\dots,\bar{sb}^{m-1},\bar{sb}^m,\bar{sa}^v_{>j},\dots,\bar{sa}^{n-1},\bar{sa}^n)\\
&=(-1)^{\epsilon+\epsilon'}\\
&\quad((f^1\circ s_d)\otimes\cdots\otimes(f^v\circ s_d)\otimes(g^1\circ s_d)\otimes\cdots\otimes(g^{m-1}\circ s_d)\otimes(f^{v+1}\circ s_d)\otimes\cdots\otimes(f^{n-1}\circ s_d)\otimes\mathrm{id})\\
&\quad s_{d+1}({}_{\mathrm{lt}(\bar{x}^1)}F^{-d}_{\mathrm{rt}(\bar{x}^2)},\cdots{}_{\mathrm{lt}(\bar{x}^{v-1})}F^{-d}_{\mathrm{rt}(\bar{x}^v)},{}_{\mathrm{lt}(\bar{y}^1)}G^{-d}_{\mathrm{rt}(\bar{y}^2)},\dots,{}_{\mathrm{lt}(\bar{y}^{m-1})}G^{-d}_{\mathrm{rt}(\bar{y}^m)},{}_{\mathrm{lt}(\bar{x}^v)}F^{-d}_{\mathrm{rt}(\bar{x}^{v+1})}\cdots{}_{\mathrm{lt}(\bar{x}^n)}F^{-d}_{\mathrm{rt}(\bar{x}^1)})
\end{aligned}$$

for elements $\bar{sa}^i\in\mathcal{A}[1]^{\otimes\bar{x}^i}$, $\bar{sb}^i\in\mathcal{A}[1]^{\otimes\bar{y}^i}$, $tf^i\in{}_{\mathrm{rt}(\bar{x}^{i+1})}\mathcal{A}^*_{\mathrm{lt}(\bar{x}^i)}[d]$ and $tg^i\in{}_{\mathrm{rt}(\bar{y}^{i+1})}\mathcal{A}^*_{\mathrm{lt}(\bar{y}^i)}[d]$, with

$$\begin{aligned}
\epsilon=&\sum_{i=v}^{n-1}|tf^i|\sum_{k=i+1}^{n}|\bar{sa}^k|+\sum_{i=1}^{m-1}|tg^i|(\sum_{k=v+1}^{n}|\bar{sa}^k|+|\bar{sa}^v_{>j}|+\sum_{k=i+1}^{m}|\bar{sb}^k|)\\
&+\sum_{i=1}^{v-1}|tf^i|(\sum_{k=v+1}^{n}|\bar{sa}^k|+|\bar{sa}^v_{>j}|+\sum_{k=1}^{m}|\bar{sb}^k|+|\bar{sa}^v_{\leq j}|+\sum_{k=i+1}^{v-1}|\bar{sa}^k|)+d(n+m)\\
&+(|\mathbf{F}|+|\mathbf{G}|+1)(\sum_{i=1}^{n-1}|tf^i|+\sum_{i=1}^{m-1}|tg^i|)+\sum_{i=1}^{n-1}|tf^i|\sum_{i=1}^{m-1}|tg^i|+\sum_{1\leq i<k\leq n-1}|tf^i||tf^k|
\end{aligned}$$

$$+\sum_{1\le i<k\le m-1}|tg^i||tg^k|+\sum_{1\le i<k\le n}|\bar{sa}^i||\bar{sa}^k|+|\bar{sa}^v_{>j}||\bar{sa}^v_{\le j}|+\sum_{1\le i<k\le m}|\bar{sb}^i||\bar{sb}^k|+\sum_{i=1}^{n}|\bar{sa}^i|\sum_{i=1}^{m}|\bar{sb}^i|$$

$$\epsilon'=(|\mathbf{G}|+d+1)(\sum_{i=1}^{v-1}|\bar{sa}^i|+|\bar{sa}^v_{\le j}|)$$

$$+\sum_{i=1}^{m-1}|{}_{\mathrm{lt}(\bar{y}^i)}G^{-d}_{\mathrm{rt}(\bar{y}^{i+1})}|(|\bar{sa}^v_{>j}|+\sum_{i=v+1}^{n}|\bar{sa}^i|+\sum_{i=v}^{n}|{}_{\mathrm{lt}(\bar{x}^i)}F^{-d}_{\mathrm{rt}(\bar{x}^{i+1})}|+|{}_{\mathrm{lt}(\bar{y}^m)}G^{1}_{\mathrm{rt}(\bar{y}^1)}|+d+1)$$

On the other hand, we have that

$$\jmath_{\bar{\bar{x}}}(s_{d+1}F^{\bar{\bar{x}}})\underset{G,p,q}{\circ}\jmath_{\bar{\bar{y}}}(s_{d+1}G^{\bar{\bar{y}}})$$
$$(\bar{sa}^n,tf^{n-1},\dots,\bar{sa}^{v+1},tf^v,\bar{sa}^v_{>j},\bar{sb}^m,tg^{m-1},\dots,\bar{sb}^2,tg^1,\bar{sb}^1,\bar{sa}^v_{\le j},tf^{v-1},\dots,\bar{sa}^2,tf^1,\bar{sa}^1)$$
$$=(-1)^{\delta}\jmath_{\bar{\bar{x}}}(s_{d+1}F^{\bar{\bar{x}}})(\bar{sa}^n,tf^{n-1},\dots,\bar{sa}^{v+1},tf^v,\bar{sa}^v_{>j},(\bigotimes_{i=1}^{m-1}(g^i\circ s_d)\otimes\mathrm{id})s_{d+1}G^{\bar{\bar{y}}}(\bar{sb}^1,\dots,\bar{sb}^m),$$
$$\bar{sa}^v_{\le j},tf^{v-1},\dots,\bar{sa}^2,tf^1,\bar{sa}^1)$$
$$=(-1)^{\delta+\delta'}((f^1\circ s_d)\otimes\cdots\otimes(f^{v-1}\circ s_d)\otimes(f^v\circ s_d)\otimes\cdots\otimes(f^{n-1}\circ s_d)\otimes\mathrm{id})$$
$$s_{d+1}({}_{\mathrm{lt}(\bar{x}^1)}F^{-d}_{\mathrm{rt}(\bar{x}^2)},\dots{}_{\mathrm{lt}(\bar{x}^{v-1})}F^{-d}_{\mathrm{rt}(\bar{x}^v)},\lambda_G,{}_{\mathrm{lt}(\bar{x}^v)}F^{-d}_{\mathrm{rt}(\bar{x}^{v+1})}\cdots{}_{\mathrm{lt}(\bar{x}^n)}F^{-d}_{\mathrm{rt}(\bar{x}^1)})$$

where $\lambda_G=\bigotimes_{i=1}^{m-1}(g^i\circ s_d))({}_{\mathrm{lt}(\bar{y}^1)}G^{-d}_{\mathrm{rt}(\bar{y}^2)},\dots,{}_{\mathrm{lt}(\bar{y}^{m-1})}G^{-d}_{\mathrm{rt}(\bar{y}^m)})\in\Bbbk$ and

$$\delta=(|\mathbf{G}|+d+1)(\sum_{i=v+1}^{n}|\bar{sa}^i|+|\bar{sa}^v_{>j}|+\sum_{i=v}^{n-1}|tf^i|)+\sum_{i=1}^{m-1}|tg^i|\sum_{k=i+1}^{m}|\bar{sb}^k|+d(m-1)$$
$$+(|\mathbf{G}|+1)\sum_{i=1}^{m-1}|tg^i|+\sum_{1\le i<k\le m-1}|tg^i||tg^k|+\sum_{1\le i<k\le m}|\bar{sb}^i||\bar{sb}^k|$$
$$\delta'=\sum_{1\le i<k\le n-1}|tf^i||tf^k|+\sum_{1\le i<k\le n}|\bar{sa}^i||\bar{sa}^k|+|\bar{sa}^v_{\le j}||\bar{sa}^v_{>j}|+\sum_{i=1}^{n}|\bar{sa}^i||{}_{\mathrm{lt}(\bar{y}^m)}G^{1}_{\mathrm{rt}(\bar{y}^1)}|$$
$$+\sum_{i=1}^{v-1}|tf^i|(\sum_{k=i+1}^{n}|\bar{sa}^k|+|{}_{\mathrm{lt}(\bar{y}^m)}G^{1}_{\mathrm{rt}(\bar{y}^1)}|)+\sum_{i=v}^{n-1}|tf^i|\sum_{k=i+1}^{n}|\bar{sa}^k|$$
$$+|\lambda_G|(\sum_{i=1}^{v-1}|\bar{sa}^i|+|\bar{sa}^v_{\le j}|+|{}_{\mathrm{lt}(\bar{y}^m)}G^{1}_{\mathrm{rt}(\bar{y}^1)}|+\sum_{i=1}^{v-1}|tf^i|+\sum_{i=v}^{n}{}_{\mathrm{lt}(\bar{x}^i)}F^{-d}_{\mathrm{rt}(\bar{x}^{i+1})}|)+d(n-1)$$
$$+(|\mathbf{F}|+d)\sum_{i=1}^{n-1}|tf^i|+(d+1)\sum_{i=1}^{m-1}|{}_{\mathrm{lt}(\bar{y}^i)}G^{-d}_{\mathrm{rt}(\bar{y}^{i+1})}|$$

Therefore, we have

$$\jmath_{\bar{\bar{x}}}(s_{d+1}F^{\bar{\bar{x}}})\underset{G,p,q}{\circ}\jmath_{\bar{\bar{y}}}(s_{d+1}G^{\bar{\bar{y}}})$$
$$(\bar{sa}^n,tf^{n-1},\dots,\bar{sa}^{v+1},tf^v,\bar{sa}^v_{>j},\bar{sb}^m,tg^{m-1},\dots,\bar{sb}^2,tg^1,\bar{sb}^1,\bar{sa}^v_{\le j},tf^{v-1},\dots,\bar{sa}^2,tf^1,\bar{sa}^1)$$
$$=(-1)^{\delta+\delta'+\delta''}$$

$$((f^1 \circ s_d) \otimes \cdots \otimes (f^v \circ s_d) \otimes (g^1 \circ s_d) \otimes \cdots \otimes (g^{m-1} \circ s_d) \otimes (f^{v+1} \circ s_d) \otimes \cdots \otimes (f^{n-1} \circ s_d) \otimes \mathrm{id})$$
$$s_{d+1}({}_{\mathrm{lt}(\bar{x}^1)}F^{-d}_{\mathrm{rt}(\bar{x}^2)}, \dots {}_{\mathrm{lt}(\bar{x}^{v-1})}F^{-d}_{\mathrm{rt}(\bar{x}^v)}, {}_{\mathrm{lt}(\bar{y}^1)}G^{-d}_{\mathrm{rt}(\bar{y}^2)}, ..., {}_{\mathrm{lt}(\bar{y}^{m-1})}G^{-d}_{\mathrm{rt}(\bar{y}^m)}, {}_{\mathrm{lt}(\bar{x}^v)}F^{-d}_{\mathrm{rt}(\bar{x}^{v+1})} \dots {}_{\mathrm{lt}(\bar{x}^n)}F^{-d}_{\mathrm{rt}(\bar{x}^1)})$$

with $\delta'' = \sum_{i=1}^{m-1} |tg^i|(\sum_{i=v}^{n-1} |tf^i| + \sum_{i=1}^{v-1} {}_{\mathrm{lt}(\bar{x}^i)}F^{-d}_{\mathrm{rt}(\bar{x}^{i+1})})$. One can easily check that $\epsilon + \epsilon' = \delta + \delta' + \delta''$ $\bmod 2$. Then, the first identity is proved.

We now prove the identity (4.7). Given $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n), \bar{\bar{y}} = (\bar{y}^1, \dots, \bar{y}^m) \in \bar{\bar{\mathcal{O}}}$, both the compositions

$$s_{d+1}F^{\bar{\bar{x}}} \underset{\substack{\mathrm{nec},v,j\\ \mathrm{out}}}{\circ} s_{d+1}G^{\bar{\bar{y}}}$$

and

$$\jmath_{\bar{\bar{y}}}(s_{d+1}G^{\bar{\bar{y}}}) \underset{G,p,q}{\circ} \mathcal{R}^{-1}_{\mathcal{A}^*}(\jmath_{\bar{\bar{x}}}(s_{d+1}F^{\bar{\bar{x}}}))$$

are zero if there are no $v \in [\![1, m]\!]$ and $j \in [\![1, \mathrm{lg}(\bar{x}^1)]\!]$ such that $\bar{x}^1_{j-1} = \mathrm{lt}(\bar{y}^v)$ and $\bar{x}^1_j = \mathrm{rt}(\bar{y}^{v+1})$. We will thus assume that such $v \in [\![1, m]\!]$ and $j \in [\![1, \mathrm{lg}(\bar{x}^1)]\!]$ exist. Then, we have

$$\jmath_{\bar{\bar{x}} \underset{v,j,\mathrm{out}}{\sqcup} \bar{\bar{y}}}(s_{d+1}F^{\bar{\bar{x}}} \underset{\substack{\mathrm{nec},v,j\\ \mathrm{out}}}{\circ} s_{d+1}G^{\bar{\bar{y}}})$$
$$(s\bar{b}^m, tg^{m-1}, \dots, s\bar{b}^{v+1}, s\bar{a}^1_{\geq j}, tf^n, s\bar{a}^n, tf^{n-1}, s\bar{a}^{n-1}, \dots, tf^1, s\bar{a}^1_{<j}, s\bar{b}^v, tg^{v-1}, \dots, s\bar{b}^1)$$
$$= (-1)^{\epsilon}$$
$$((g^1 \circ s_d) \otimes \cdots \otimes (g^{v-1} \circ s_d) \otimes (f^1 \circ s_d) \otimes \cdots \otimes (f^n \circ s_d) \otimes (g^{v+1} \circ s_d) \otimes \cdots \otimes (g^{m-1} \circ s_d) \otimes \mathrm{id})$$
$$(s_{d+1}F^{\bar{\bar{x}}} \underset{\substack{\mathrm{nec},v,j\\ \mathrm{out}}}{\circ} s_{d+1}G^{\bar{\bar{y}}})(s\bar{b}^1, \dots, s\bar{b}^v, s\bar{a}^1_{<j}, \dots, s\bar{a}^n, s\bar{a}^1_{\geq j}, s\bar{b}^{v+1}, \dots, s\bar{b}^m)$$
$$= (-1)^{\epsilon+\epsilon'}$$
$$((g^1 \circ s_d) \otimes \cdots \otimes (g^{v-1} \circ s_d) \otimes (f^1 \circ s_d) \otimes \cdots \otimes (f^n \circ s_d) \otimes (g^{v+1} \circ s_d) \otimes \cdots \otimes (g^{m-1} \circ s_d) \otimes \mathrm{id})$$
$$s_{d+1}({}_{\mathrm{lt}(\bar{y}^1)}G^{-d}_{\mathrm{rt}(\bar{y}^2)}, \dots, {}_{\mathrm{lt}(\bar{y}^{v-1})}G^{-d}_{\mathrm{rt}(\bar{y}^v)}, {}_{\mathrm{lt}(\bar{x}^1)}F^{-d}_{\mathrm{rt}(\bar{x}^2)}, \dots, {}_{\mathrm{lt}(\bar{x}^n)}F^{-d}_{\mathrm{rt}(\bar{x}^1)}, {}_{\mathrm{lt}(\bar{y}^{v+1})}G^{-d}_{\mathrm{rt}(\bar{y}^{v+2})}, \dots, {}_{\mathrm{lt}(\bar{y}^m)}G^{-d}_{\mathrm{rt}(\bar{y}^1)})$$

with

$$\begin{aligned}
\epsilon =& (|\mathbf{F}| + |\mathbf{G}| + 1)(\sum_{i=1}^{m-1} |tg^i| + \sum_{i=1}^{n} |tf^i|) + d(n+m) + \sum_{i=v+1}^{m-1} |tg^i| \sum_{k=i+1}^{m} |s\bar{b}^k| + |s\bar{a}^1_{\geq j}||s\bar{a}^1_{<j}| \\
&+ \sum_{1 \leq i < k \leq n} |s\bar{a}^i||s\bar{a}^k| + \sum_{i=1}^{n} |tf^i|(\sum_{k=i+1}^{n} |s\bar{a}^k| + |s\bar{a}^1_{\geq j}| + \sum_{i=v+1}^{m} |s\bar{b}^i|) + \sum_{i=1}^{v-1} |tg^i|(\sum_{i=1}^{n} |s\bar{a}^i| + \sum_{k=i+1}^{m} |s\bar{b}^k|) \\
&+ \sum_{1 \leq i < k \leq m} |s\bar{b}^i||s\bar{b}^k| + \sum_{1 \leq i < k \leq m-1} |tg^i||tg^k| + \sum_{1 \leq i < k \leq n} |tf^i||tf^k| + \sum_{i=1}^{m-1} |tg^i| \sum_{i=1}^{n} |tf^i| + \sum_{i=1}^{m} |s\bar{b}^i| \sum_{i=1}^{n} |s\bar{a}^i| \\
\epsilon' =& \sum_{i=1}^{n} |s\bar{a}^i|(\sum_{i=v+1}^{m} |s\bar{b}^i| + \sum_{i=v+1}^{m} |{}_{\mathrm{lt}(\bar{y}^i)}G^{-d}_{\mathrm{rt}(\bar{y}^{i+1})}|) + (|\mathbf{F}| + d + 1) \sum_{i=1}^{v-1} |{}_{\mathrm{lt}(\bar{y}^i)}G^{-d}_{\mathrm{rt}(\bar{y}^{i+1})}| \\
&+ |s\bar{a}^1_{\geq j}| \sum_{i=2}^{n} |s\bar{a}^i| + |s\bar{a}^v_{<j}||{}_{\mathrm{lt}(\bar{y}^v)}G^1_{\mathrm{rt}(\bar{y}^{v+1})}|
\end{aligned}$$

On the other hand, we have that

$$\jmath_{\bar{y}}(s_{d+1}G^{\bar{\bar{y}}}) \underset{G,p,q}{\circ} \mathscr{R}_{\mathcal{A}^*}^{-1}(\jmath_{\bar{\bar{x}}}(s_{d+1}F^{\bar{\bar{x}}}))$$
$$(\bar{sb}^m, tg^{m-1}, \ldots, \bar{sb}^{v+1}, \bar{sa}^1_{\geq j}, tf^n, \bar{sa}^n, tf^{n-1}, \bar{sa}^{n-1}, \ldots, \bar{sa}^1_{<j}, \bar{sb}^v, tg^{v-1}, \ldots, \bar{sb}^1)$$
$$= (-1)^{\delta} \jmath_{\bar{y}}(s_{d+1}G^{\bar{\bar{y}}})(\bar{sb}^m, tg^{m-1}, \ldots, \bar{sb}^{v+1}, tg, \bar{sb}^v, tg^{v-1}, \ldots, \bar{sb}^1)$$

where $tg = \mathscr{R}_{\mathcal{A}^*}^{-1}(\jmath_{\bar{\bar{x}}}(s_{d+1}F^{\bar{\bar{x}}}))(\bar{sa}^1_{\geq j}, tf^n, \bar{sa}^n, tf^{n-1}, \bar{sa}^{n-1}, \ldots, \bar{sa}^1_{<j}) \in \mathcal{A}^*[d]$ and

$$\delta = (|\mathbf{F}| + d + 1)(\sum_{i=v+1}^{m} |\bar{sb}^i| + \sum_{i=v+1}^{m-1} |tg^i|)$$

Therefore, we have that

$$\jmath_{\bar{y}}(s_{d+1}G^{\bar{\bar{y}}}) \underset{G,p,q}{\circ} \mathscr{R}_{\mathcal{A}^*}^{-1}(\jmath_{\bar{\bar{x}}}(s_{d+1}F^{\bar{\bar{x}}}))$$
$$(\bar{sb}^m, tg^{m-1}, \ldots, \bar{sb}^{v+1}, tg^{v+1}, \bar{sa}^1_{\geq j}, \bar{sa}^n, tf^{n-1}, \bar{sa}^{n-1}, \ldots, \bar{sa}^1_{<j}, \bar{sb}^v, tg^{v-1}, \ldots, \bar{sb}^1)$$
$$= (-1)^{\delta+\delta'}((g^1 \circ s_d) \otimes \cdots \otimes (g^{v-1} \circ s_d) \otimes (g \circ s_d) \otimes (g^{v+1} \circ s_d) \otimes \cdots \otimes (g^m \circ s_d) \otimes \mathrm{id})$$
$$s_{d+1}({}_{\mathrm{lt}(\bar{y}^1)}G^{-d}_{\mathrm{rt}(\bar{y}^2)}, \ldots, {}_{\mathrm{lt}(\bar{y}^{v-1})}G^{-d}_{\mathrm{rt}(\bar{y}^v)}, {}_{\mathrm{lt}(\bar{y}^v)}G^{-d}_{\mathrm{rt}(\bar{y}^{v+1})}, \ldots, {}_{\mathrm{lt}(\bar{y}^m)}G^{-d}_{\mathrm{rt}(\bar{y}^1)})$$

where

$$\delta' = |tg| \sum_{k=v+1}^{m} |\bar{sb}^k| + \sum_{i=1}^{m-1} |tg^i| \sum_{k=i+1}^{m} |\bar{sb}^k| + \sum_{1 \leq i < k \leq m} |\bar{sb}^i||\bar{sb}^k| + \sum_{1 \leq i < k \leq m-1} |tg^i||tg^k|$$
$$+ \sum_{i=1}^{m-1} |tg^i||tg| + (|\mathbf{G}| + 1)(\sum_{i=1}^{m-1} |tg^i| + |tg|) + d(m-1)$$

Furthermore, by definition, we have that

$$g({}_{\mathrm{lt}(\bar{y}^v)}G_{\mathrm{rt}(\bar{y}^{v+1})})$$
$$= (-1)^{|tg|+1}\Gamma(\mathscr{R}_{\mathcal{A}^*}^{-1}(\jmath_{\bar{\bar{x}}}(s_{d+1}F^{\bar{\bar{x}}}))(\bar{sa}^1_{\geq j}, tf^n, \bar{sa}^n, tf^{n-1}, \bar{sa}^{n-1}, \ldots, \bar{sa}^1_{<j}), {}_{\mathrm{lt}(\bar{y}^{v-1})}G^1_{\mathrm{rt}(\bar{y}^v)})$$
$$= (-1)^{|tg|+1+\Delta}\Gamma(\jmath_{\bar{\bar{x}}}(s_{d+1}F_{\bar{\bar{x}}})(\bar{sa}^n, tf^{n-1}, \bar{sa}^{n-1}, \ldots, \bar{sa}^1_{<j} \otimes {}_{\mathrm{lt}(\bar{y}^{v-1})}G^1_{\mathrm{rt}(\bar{y}^v)} \otimes \bar{sa}^1_{\geq j}), tf^n)$$
$$= (-1)^{|tg|+1+\Delta+\Delta'}\Gamma(((f^1 \circ s_d) \otimes \cdots \otimes (f^{n-1} \circ s_d) \otimes \mathrm{id})({}_{\mathrm{lt}(\bar{x}^1)}F^{-d}_{\mathrm{rt}(\bar{x}^2)} \cdots {}_{\mathrm{lt}(\bar{x}^n)}F^{-d}_{\mathrm{rt}(\bar{x}^1)}), tf^n)$$
$$= (-1)^{|tg|+1+\Delta+\Delta'+\Delta''}((f^1 \circ s_d) \otimes \cdots \otimes (f^{n-1} \circ s_d) \otimes (f^n \circ s_d))({}_{\mathrm{lt}(\bar{x}^1)}F^{-d}_{\mathrm{rt}(\bar{x}^2)} \cdots \otimes {}_{\mathrm{lt}(\bar{x}^n)}F^{-d}_{\mathrm{rt}(\bar{x}^1)})$$

where

$$\Delta = (|\bar{sa}^1_{\geq j}| + |tf^n|)(\sum_{i=2}^{n} |\bar{sa}^i| + |\bar{sa}^1_{<j}| + |{}_{\mathrm{lt}(\bar{y}^v)}G^1_{\mathrm{rt}(\bar{y}^{v+1})}| + \sum_{i=1}^{n-1} |tf^i|)$$

$$\Delta' = \sum_{1 \leq i < k \leq n} |\bar{sa}^i||\bar{sa}^k| + |{}_{\mathrm{lt}(\bar{y}^v)}G^1_{\mathrm{rt}(\bar{y}^{v+1})}| \sum_{i=2}^{n} |\bar{sa}^i| + \sum_{1 \leq i < k \leq n-1} |tf^i||tf^k|$$
$$+ d(n-1) + (|\mathbf{F}| + d) \sum_{i=1}^{n-1} |tf^i| + \sum_{i=1}^{n-1} |tf^i| \sum_{j=i+1}^{n} |\bar{sa}^i|$$

$$\Delta'' = |tf^n|(1 + \sum_{i=1}^{n} |{}_{\mathrm{lt}(\bar{x}^i)}F^{-d}_{\mathrm{rt}(\bar{x}^{i+1})}|)$$

Finally, we have that

$$\jmath_{\bar{\bar{y}}}(s_{d+1}G^{\bar{\bar{y}}}) \underset{G,p,q}{\circ} \mathcal{R}_{\mathcal{A}^*}^{-1}(\jmath_{\bar{\bar{x}}}(s_{d+1}F^{\bar{\bar{x}}}))$$
$$(\bar{sb}^m, tg^{m-1}, \dots, \bar{sb}^{v+1}, tg^{v+1}, \bar{sa}^1_{\geq j}, tf^n, \bar{sa}^n, tf^{n-1}, \bar{sa}^{n-1}, \dots, \bar{sa}^1_{<j}, \bar{sb}^v, tg^{v-1}, \dots, \bar{sb}^1)$$
$$= (-1)^\gamma$$
$$((g^1 \circ s_d) \otimes \cdots \otimes (g^{v-1} \circ s_d) \otimes (f^1 \circ s_d) \otimes \cdots \otimes (f^n \circ s_d) \otimes (g^{v+1} \circ s_d) \otimes \cdots \otimes (g^{m-1} \circ s_d) \otimes \mathrm{id})$$
$$s_{d+1}({}_{\mathrm{lt}(\bar{y}^1)}G^{-d}_{\mathrm{rt}(\bar{y}^2)}, \dots, {}_{\mathrm{lt}(\bar{y}^{v-1})}G^{-d}_{\mathrm{rt}(\bar{y}^v)}, {}_{\mathrm{lt}(\bar{x}^1)}F^{-d}_{\mathrm{rt}(\bar{x}^2)}, \dots, {}_{\mathrm{lt}(\bar{x}^n)}F^{-d}_{\mathrm{rt}(\bar{x}^1)}, {}_{\mathrm{lt}(\bar{y}^{v+1})}G^{-d}_{\mathrm{rt}(\bar{y}^{v+2})}, \dots, {}_{\mathrm{lt}(\bar{y}^m)}G^{-d}_{\mathrm{rt}(\bar{y}^1)})$$

where

$$\begin{aligned}\gamma = \delta + \delta' + |tg|(\sum_{i=v+1}^{m-1} |tg^i| + d + 1 + \sum_{i=1}^{v-1} |{}_{\mathrm{lt}(\bar{y}^i)}G^{-d}_{\mathrm{rt}(\bar{y}^{i+1})}|) \\ + \Delta + \Delta' + \Delta" + |tg| + 1 + \sum_{i=1}^{n} |tf^i|(\sum_{i=1}^{v-1} |{}_{\mathrm{lt}(\bar{y}^i)}G^{-d}_{\mathrm{rt}(\bar{y}^{i+1})}| + \sum_{i=v+1}^{m-1} |tg^i|)\end{aligned}$$

It is straightforward to check that $\epsilon + \epsilon' + \gamma = 1 + (|\mathbf{F}| + d + 1)(|\mathbf{G}| + d + 1) \bmod 2$. □

Corollary 4.30. *The necklace bracket $[-,-]_{\mathrm{nec}}$ introduced in Definition 4.21 gives a graded Lie algebra structure on $\mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{A})[d])^{C_{\mathrm{lg}(\bullet)}}[d+1]$.*

Proof. For $\mathbf{F}, \mathbf{G} \in \mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{A})[d])^{C_{\mathrm{lg}(\bullet)}}$, we have that

$$\jmath([s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G}]_{\mathrm{nec}}) = \pi_{\mathcal{A}}([\jmath(s_{d+1}\mathbf{F}), \jmath(s_{d+1}\mathbf{G})]_G)$$

where $\pi_{\mathcal{A}} : \mathcal{A}[1] \oplus \mathcal{A}^*[d] \to \mathcal{A}[1]$ is the canonical projection. Moreover, using that $\jmath$ is injective, we have that $[-,-]_{\mathrm{nec}}$ is a graded Lie bracket. □

4.3 Pre-Calabi-Yau structures

Definition 4.31. *A* ***d-pre-Calabi-Yau structure*** *on a graded quiver $\mathcal{A}$ is an element*

$$s_{d+1}M_{\mathcal{A}} \in \mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{A})[d])^{C_{\mathrm{lg}(\bullet)}}[d+1]$$

of degree 1, solving the Maurer-Cartan equation

$$[s_{d+1}M_{\mathcal{A}}, s_{d+1}M_{\mathcal{A}}]_{\mathrm{nec}} = 0$$

Note that, since $s_{d+1}M_{\mathcal{A}}$ has degree 1, this is tantamount to requiring that $s_{d+1}M_{\mathcal{A}} \underset{\mathrm{nec}}{\circ} s_{d+1}M_{\mathcal{A}} = 0$.

We now recall the following result of [5] which states the link beteween a d-pre-Calabi-Yau structure on a Hom-finite graded quiver $\mathcal{A}$ and a cyclic A_∞-structure on $\mathcal{A} \oplus \mathcal{A}^*[d-1]$.

Proposition 4.32. *A d-pre-Calabi-Yau structure on a* Hom*-finite graded quiver $\mathcal{A}$ is equivalent to the data of an A_∞-structure on $\mathcal{A} \oplus \mathcal{A}^*[d-1]$ that restricts to $\mathcal{A}$ and that satisfies the cyclic condition (3.1).*

Proof. Consider an element $s_{d+1}M_{\mathcal{A}} \in \mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{A})[d])^{C_{\mathrm{lg}(\bullet)}}[d+1]$ of degree 1. We then define maps

$$m^{\bar{x}^1,\dots,\bar{x}^n}_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}} : \bigotimes_{i=1}^{n-1}(\mathcal{A}[1]^{\otimes\bar{x}^i} \otimes {}_{\mathrm{rt}(\bar{x}^{i+1})}\mathcal{A}^*_{\mathrm{lt}(\bar{x}^i)}[d]) \otimes \mathcal{A}[1]^{\otimes\bar{x}^n} \to {}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}_{\mathrm{rt}(\bar{x}^n)} \tag{4.9}$$

and

$$m^{\bar{x}^1,\dots,\bar{x}^n}_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}^*} : \bigotimes_{i=1}^{n-1}(\mathcal{A}[1]^{\otimes\bar{x}^i} \otimes {}_{\mathrm{rt}(\bar{x}^{i+1})}\mathcal{A}^*_{\mathrm{lt}(\bar{x}^i)}[d]) \otimes \mathcal{A}[1]^{\otimes\bar{x}^n} \to {}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}^*_{\mathrm{rt}(\bar{x}^n)}[d-1] \tag{4.10}$$

uniquely determined by $sm^{\bar{x}^1,\dots,\bar{x}^n}_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}} = \jmath_{\bar{\bar{x}}^{-1}}(s_{d+1}M^{\bar{x}^n,\dots,\bar{x}^1}_{\mathcal{A}})$ for $\bar{\bar{x}} = (\bar{x}^1,\dots,\bar{x}^n)$ and such that the cyclicity conditions (4.4) and (4.5) hold. This is well-defined, since the bilinear form $\Gamma^{\mathcal{A}}$ is nondegenerate. By Proposition 4.29, $s_{d+1}M_{\mathcal{A}}$ defines a d-pre-Calabi-Yau structure if and only if $sm_{\mathcal{A}\oplus\mathcal{A}^*}$ defines a cyclic A_∞-structure on $\mathcal{A}\oplus\mathcal{A}^*[d-1]$. If $\mathcal{A}$ is Hom-finite, the bijectivity of $\jmath$ tells us that the collection of maps (4.9) and (4.10) are in correspondence with maps of the form $M_{\mathcal{A}}$, which shows the equivalence. □

5 Pre-Calabi-Yau morphisms

5.1 The mixed necklace graded Lie algebra

One can also define a "mixed" necklace bracket, which will be useful in the next section. As we did for the necklace bracket, we first define the following graded vector space.

Definition 5.1. *Given graded quivers $\mathcal{A}$ and $\mathcal{B}$ with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$ and a map $\Phi : \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$, consider the graded vector space*

$$\begin{aligned}&\mathscr{B}_\bullet(\mathcal{A}[1],\mathcal{B}[-d])\\ &= \prod_{n\in\mathbb{N}}\prod_{\bar{\bar{x}}\in\bar{\mathcal{O}}^n_{\mathcal{A}}} \mathcal{H}om_{\Bbbk}\Big(\bigotimes_{i=1}^{n}\mathcal{A}[1]^{\otimes\bar{x}^i}, \bigotimes_{i=1}^{n-1} {}_{\Phi(\mathrm{lt}(\bar{x}^i))}\mathcal{B}_{\Phi(\mathrm{rt}(\bar{x}^{i+1}))}[-d] \otimes \big({}_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}[-d]_{\mathrm{rt}(\bar{x}^1)} \oplus {}_{\Phi(\mathrm{lt}(\bar{x}^n))}\mathcal{B}^*[-1]_{\Phi(\mathrm{rt}(\bar{x}^1))}\big)\Big)\end{aligned}$$

Definition 5.2. *Consider $s_{d+1}F^{\bar{\bar{x}}} \in \mathscr{B}_{\bar{\bar{x}}}(\mathcal{A}[1],\mathcal{B}[-d])[d+1]$ with $\bar{\bar{x}} = (\bar{x}^1,\dots,\bar{x}^n) \in \bar{\mathcal{O}}^n$. Then $s_{d+1}F^{\bar{\bar{x}}}$ induces a morphism of the form*

$$\mathcal{A}[1]^{\otimes\bar{x}^1} \otimes {}_{\Phi(\mathrm{rt}(\bar{x}^1))}\mathcal{B}_{\Phi(\mathrm{lt}(\bar{x}^n))}[-d] \otimes \mathcal{A}[1]^{\otimes\bar{x}^n} \otimes\cdots\otimes \mathcal{A}[1]^{\otimes\bar{x}^{n-1}} \to \bigotimes_{i=1}^{n-1} {}_{\Phi(\mathrm{lt}(\bar{x}^i))}\mathcal{B}_{\Phi(\mathrm{rt}(\bar{x}^{i+1}))}[-d] \tag{5.1}$$

sending $(\bar{sa}^1, s_{-d}b, \bar{sa}^n,\dots,\bar{sa}^{n-1})$ to

$$(-1)^\epsilon {}_{\Phi(\mathrm{lt}(\bar{x}^1))}F^{-d}_{\Phi(\mathrm{rt}(\bar{x}^2))} \otimes\cdots\otimes {}_{\Phi(\mathrm{lt}(\bar{x}^{n-1}))}F^{-d}_{\Phi(\mathrm{rt}(\bar{x}^n))} \otimes {}_{\Phi(\mathrm{lt}(\bar{x}^n))}F^{1}_{\Phi(\mathrm{rt}(\bar{x}^1))}(sb)$$

where we have written

$$\begin{aligned}&(\mathrm{id}^{\otimes(n-1)}\otimes\pi_{\mathcal{B}})\big(F^{\bar{x}^1,\dots,\bar{x}^n}(\bar{sa}^1,\dots,\bar{sa}^n)\big)\\ &\qquad= {}_{\Phi(\mathrm{rt}(\bar{x}^1))}F^{-d}_{\Phi(\mathrm{lt}(\bar{x}^2))} \otimes\cdots\otimes {}_{\Phi(\mathrm{rt}(\bar{x}^{n-1}))}F^{-d}_{\Phi(\mathrm{lt}(\bar{x}^n))} \otimes {}_{\Phi(\mathrm{rt}(\bar{x}^n))}F^{1}_{\Phi(\mathrm{lt}(\bar{x}^1))}\end{aligned}$$

where $\pi_{\mathcal{B}}$ *denotes the canonical projection*

$$_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}[-d]_{\mathrm{rt}(\bar{x}^1)} \oplus {}_{\Phi(\mathrm{lt}(\bar{x}^n))}\mathcal{B}^*[-1]_{\Phi(\mathrm{rt}(\bar{x}^1))} \to {}_{\Phi(\mathrm{lt}(\bar{x}^n))}\mathcal{B}^*[-1]_{\Phi(\mathrm{rt}(\bar{x}^1))}$$

for $\bar{sa}^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ *with*

$$\epsilon = (d+1)(|\bar{sa}^1| + |F^{\bar{\bar{x}}}|) + (|sb| + |\bar{sa}^n|)\sum_{i=2}^{n-1}|\bar{sa}^i| + |sb||\bar{sa}^n| + (d+1)\sum_{i=1}^{n-1}|{}_{\Phi(\mathrm{lt}(\bar{x}^i))}F^{-d}_{\Phi(\mathrm{rt}(\bar{x}^{i+1}))}|$$

and a morphism of the form

$$\mathcal{A}[1]^{\otimes \bar{x}^1} \otimes \mathcal{A}[1]^{\otimes \bar{x}^2} \otimes \cdots \otimes \mathcal{A}[1]^{\otimes \bar{x}^n} \to \bigotimes_{i=2}^{n-1} {}_{\Phi(\mathrm{lt}(\bar{x}^i))}\mathcal{B}_{\Phi(\mathrm{rt}(\bar{x}^{i+1}))}[-d] \otimes {}_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}[1]_{\mathrm{rt}(\bar{x}^1)} \tag{5.2}$$

given by $\left(\mathrm{id}^{\otimes(n-1)} \otimes (s_{d+1} \circ \pi_{\mathcal{A}})\right) \circ F^{\bar{x}^1,\ldots,\bar{x}^n}$ *where* $\pi_{\mathcal{A}}$ *denotes the canonical projection*

$$_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}[-d]_{\mathrm{rt}(\bar{x}^1)} \oplus {}_{\Phi(\mathrm{lt}(\bar{x}^n))}\mathcal{B}^*[-1]_{\Phi(\mathrm{rt}(\bar{x}^1))} \to {}_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}[-d]_{\mathrm{rt}(\bar{x}^1)}$$

Lemma 5.3. *Given two graded quivers* $\mathcal{A}$ *and* $\mathcal{B}$ *with respective sets of objects* $\mathcal{O}_{\mathcal{A}}$ *and* $\mathcal{O}_{\mathcal{B}}$ *and a map* $\Phi : \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$*, we have an isomorphism*

$$B : \mathscr{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])[d+1] \xrightarrow{\simeq} \mathscr{B}^{\mathcal{A}}_\bullet(\mathcal{A}[1], \mathcal{B}[-d]) \oplus \mathscr{B}^{\mathcal{B}}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])$$

where

$$\mathscr{B}^{\mathcal{A}}_\bullet(\mathcal{A}[1], \mathcal{B}[-d]) = \prod_{n \in \mathbb{N}} \prod_{\bar{\bar{x}} \in \bar{\mathcal{O}}^n_{\mathcal{A}}} \mathcal{H}om_{\Bbbk}\left(\bigotimes_{i=1}^{n} \mathcal{A}[1]^{\otimes \bar{x}^i}, \bigotimes_{i=1}^{n-1} {}_{\Phi(\mathrm{lt}(\bar{x}^i))}\mathcal{B}_{\Phi(\mathrm{rt}(\bar{x}^{i+1}))}[-d] \otimes {}_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}[1]_{\mathrm{rt}(\bar{x}^1)}\right)$$

and

$$\mathscr{B}^{\mathcal{B}}_\bullet(\mathcal{A}[1], \mathcal{B}[-d]) = \prod_{n \in \mathbb{N}} \prod_{\bar{\bar{x}} \in \bar{\mathcal{O}}^n_{\mathcal{A}}} \mathcal{H}om_{\Bbbk}\Bigg(\mathcal{A}[1]^{\otimes \bar{x}^1} \otimes {}_{\Phi(\mathrm{rt}(\bar{x}^1))}\mathcal{B}[-d]_{\Phi(\mathrm{lt}(\bar{x}^n))} \otimes \mathcal{A}[1]^{\otimes \bar{x}^n} \otimes \cdots \otimes \mathcal{A}[1]^{\otimes \bar{x}^{n-1}},$$
$$\bigotimes_{i=1}^{n-1} {}_{\Phi(\mathrm{lt}(\bar{x}^i))}\mathcal{B}_{\Phi(\mathrm{rt}(\bar{x}^{i+1}))}[-d]\Bigg)$$

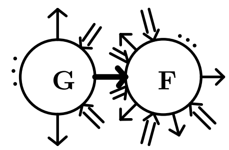
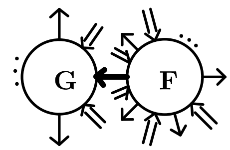
sending an element $s_{d+1}\mathbf{F} \in \mathscr{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$ *to the elements defined in* (5.2) *and* (5.1).

Definition 5.4. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be graded quivers with respective sets of objects* $\mathcal{O}_{\mathcal{A}}$ *and* $\mathcal{O}_{\mathcal{B}}$ *and consider a map* $\Phi : \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$*. We define the graded quiver* $\mathcal{Q}_\Phi$ *whose set of objects is* $\mathcal{O}_{\mathcal{A}}$ *and whose spaces of morphisms are* ${}_y(\mathcal{Q}_\Phi)_x = {}_y\mathcal{A}_x \oplus {}_{\Phi(y)}\mathcal{B}^*_{\Phi(x)}[d-1]$ *for* $x, y \in \mathcal{O}_{\mathcal{A}}$.

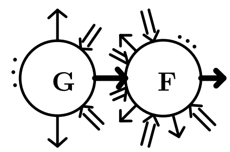
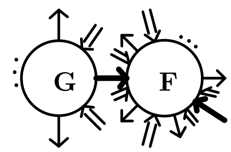
Definition 5.5. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be graded quivers with respective sets of objects* $\mathcal{O}_{\mathcal{A}}$ *and* $\mathcal{O}_{\mathcal{B}}$ *and consider a map* $\Phi : \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$*. Given* $s_{d+1}\boldsymbol{F}, s_{d+1}\boldsymbol{G} \in \mathscr{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$*, we define their* ***$\Phi$-mixed necklace product*** *as the element* $s_{d+1}\boldsymbol{F} \underset{\Phi\,\mathrm{nec}}{\circ} s_{d+1}\boldsymbol{G} \in \mathscr{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$ *given by*

$$(s_{d+1}\boldsymbol{F} \underset{\Phi\,\mathrm{nec}}{\circ} s_{d+1}\boldsymbol{G})^{\bar{\bar{x}}} = \sum \mathcal{E}(\mathscr{D}) + \sum \mathcal{E}(\mathscr{D}') \in \mathscr{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$$

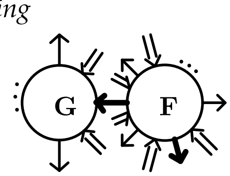
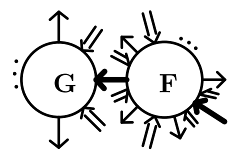
where the sums are over all the filled diagrams $\mathscr{D}$ *and* $\mathscr{D}'$ *of type* $\bar{\bar{x}}$ *and of the form*

$\mathscr{D} =$  $\mathscr{D}' =$ 

More precisely, each of $\mathscr{D}$ *and* $\mathscr{D}'$ *can be pictured as two different filled diagrams by put in bold the arrow corresponding to the last output of* ***F*** *which is either an element in* $\mathcal{A}[1]$ *or in* $\mathcal{B}^*[d]$*. Thus,* $\mathscr{D}$ *is one of the following*

$\mathscr{D}_1 =$  $\mathscr{D}_2 =$ 

and $\mathscr{D}'$ *is one of the following*

$\mathscr{D}'_1 =$  $\mathscr{D}'_2 =$ 

Remark 5.6. *The diagrams of Definition 5.5 are filled with* ***F*** *and* ***G*** *seen as elements of* $\mathrm{Multi}^\bullet(\mathrm{Bar}(\mathcal{Q}'_\Phi)[d])$ *where* $\mathcal{Q}'_\Phi$ *is the graded quiver whose set of objects is* $\mathcal{O}_\mathcal{A}$ *and whose spaces of morphisms are* ${}_y(\mathcal{Q}'_\Phi)_x = {}_y\mathcal{A}_x \oplus {}_{\Phi(y)}\mathcal{B}_{\Phi(x)}$.

Definition 5.7. *Given graded quivers* $\mathcal{A}$ *and* $\mathcal{B}$ *with respective sets of objects* $\mathcal{O}_\mathcal{A}$ *and* $\mathcal{O}_\mathcal{B}$ *and a map* $\Phi : \mathcal{O}_\mathcal{A} \to \mathcal{O}_\mathcal{B}$*, the* ***$\Phi$-mixed necklace bracket*** *is the graded Lie bracket which is defined for elements* $\mathbf{F}, \mathbf{G} \in \mathscr{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])$ *by*

$$[s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G}]_{\Phi\,\mathrm{nec}} = s_{d+1}\mathbf{F} \underset{\Phi\,\mathrm{nec}}{\circ} s_{d+1}\mathbf{G} - (-1)^{(|\mathbf{F}|+d+1)(|\mathbf{G}|+d+1)} s_{d+1}\mathbf{G} \underset{\Phi\,\mathrm{nec}}{\circ} s_{d+1}\mathbf{F}$$

Lemma 5.8. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be graded quivers with respective sets of objects* $\mathcal{O}_\mathcal{A}$ *and* $\mathcal{O}_\mathcal{B}$ *and consider a map* $\Phi : \mathcal{O}_\mathcal{A} \to \mathcal{O}_\mathcal{B}$*. Then, we have an injective map*

$$\jmath^\Phi : \mathscr{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])[d+1] \to C(\mathcal{A} \oplus \mathcal{B}^*[d-1])[1]$$

sending $s_{d+1}\phi \in \mathscr{B}_{\bar{\bar{x}}}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$ *to* $s\psi_{\bar{\bar{x}}}$ *where*

$$\psi_{\bar{\bar{x}}} : \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\otimes \bar{x}^{n-i+1}} \otimes_{\Phi(\mathrm{rt}(\bar{x}^{n-i+1}))} \mathcal{B}^*[-d]_{\Phi(\mathrm{lt}(\bar{x}^{n-i}))}) \otimes \mathcal{A}[1]^{\otimes \bar{x}^1} \to {}_{\mathrm{lt}(\bar{x}^n)}\mathcal{A}[-d]_{\mathrm{rt}(\bar{x}^1)} \oplus {}_{\Phi(\mathrm{lt}(\bar{x}^n))}\mathcal{B}^*[-1]_{\Phi(\mathrm{rt}(\bar{x}^1))}$$

is given by

$$\psi_{\bar{\bar{x}}}(\bar{sa}^n, tf^{n-1}, \bar{sa}^{n-2}, tf^{n-2}, ..., \bar{sa}^2, tf^2, \bar{sa}^1) = (-1)^{\epsilon}\big(\bigotimes_{i=1}^{n}(f^i \circ s_d) \otimes s_d\big)\big(\phi(\bar{sa}^1, \bar{sa}^2, ..., \bar{sa}^n)\big)$$

for $n = \mathrm{lg}(\bar{\bar{x}})$*,* $\bar{sa}^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ *and* $tf^i \in {}_{\Phi(\mathrm{rt}(\bar{x}^{i+1}))}\mathcal{B}^*_{\Phi(\mathrm{lt}(\bar{x}^i))}[d]$ *with*

$$\epsilon = \sum_{1\leq i\leq j\leq n-1} |\bar{sa}^i||tf^j| + (|\phi| + d)\sum_{i=1}^{n-1}|tf^i| + d(n-1) + \sum_{1\leq i<j\leq n}|\bar{sa}^i||\bar{sa}^j| + \sum_{1\leq i<j\leq n-1}|tf^i||tf^j|$$

Proposition 5.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be graded quivers with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$ and consider a map $\Phi : \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$. Then, we have*

$$\jmath^{\Phi}([s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G}]_{\Phi\text{ nec}}) = [\jmath^{\Phi}(s_{d+1}\mathbf{F}), \jmath^{\Phi}(s_{d+1}\mathbf{G})]_G$$

for $\mathbf{F}, \mathbf{G} \in \mathscr{B}_{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])$*.*

5.2 Pre-Calabi-Yau morphisms (after M. Kontsevich, A. Takeda and Y. Vlassopoulos)

Following the article [5], we recall the definition of the category of d-pre-Calabi-Yau categories.

Definition 5.10. *Given graded quivers $\mathcal{A}$ and $\mathcal{B}$ with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$ and a map $F_0 : \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$, consider the graded vector space*

$$\begin{aligned}&\mathrm{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])\\ &= \prod_{n\in\mathbb{N}}\prod_{\bar{\bar{x}}\in\bar{\mathcal{O}}^n_{\mathcal{A}}} \mathcal{H}om_{\Bbbk}\Big(\bigotimes_{i=1}^{n}\mathcal{A}[1]^{\otimes\bar{x}^i}, \bigotimes_{i=1}^{n-1}{}_{F_0(\mathrm{lt}(\bar{x}^i))}\mathcal{B}_{F_0(\mathrm{rt}(\bar{x}^{i+1}))}[-d] \otimes {}_{F_0(\mathrm{lt}(\bar{x}^n))}\mathcal{B}[-d]_{F_0(\mathrm{rt}(\bar{x}^1))}\Big)\end{aligned}$$

The action of $\tau = (\tau_n)_{n\in\mathbb{N}^*} \in \prod_{n\in\mathbb{N}^*} C_n$ on an element $\mathbf{F} = (F^{\bar{\bar{x}}})_{\bar{\bar{x}}\in\bar{\bar{\mathcal{O}}}} \in \mathrm{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])$ is the element $\tau\cdot\mathbf{F} \in \mathrm{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])$ given by

$$(\tau\cdot\mathbf{F})^{\bar{\bar{x}}} = \tau^{-1}_{{}_{F_0(\mathrm{lt}(\bar{x}^1))}\mathcal{B}_{F_0(\mathrm{rt}(\bar{x}^2))}[-d],\dots,{}_{F_0(\mathrm{lt}(\bar{x}^n))}\mathcal{B}_{F_0(\mathrm{rt}(\bar{x}^1))}[-d]} \circ F^{\bar{\bar{x}}\cdot\tau} \circ \tau_{\mathcal{A}[1]^{\otimes\bar{x}^1},\mathcal{A}[1]^{\otimes\bar{x}^2},\dots,\mathcal{A}[1]^{\otimes\bar{x}^n}}$$

for $\bar{\bar{x}} \in \bar{\bar{\mathcal{O}}}$. We will denote by $\mathrm{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])^{C_{\mathrm{lg}(\bullet)}}$ the space of elements of $\mathrm{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])$ that are invariant under the action of $C_{\mathrm{lg}(\bullet)}$.

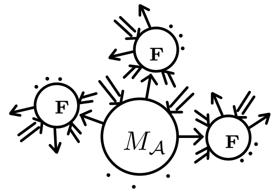
Definition 5.11. *Given d-pre-Calabi-Yau categories $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ and $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$ and an element $s_{d+1}\mathbf{F} \in \mathrm{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$ of degree 0, the* ***multinecklace composition of $s_{d+1}M_{\mathcal{A}}$ and $s_{d+1}\mathbf{F}$*** *is the element*

$$s_{d+1}\mathbf{F} \underset{\text{multinec}}{\circ} s_{d+1}M_{\mathcal{A}} \in \mathrm{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$$

given by

$$(s_{d+1}\mathbf{F} \underset{\text{multinec}}{\circ} s_{d+1}M_{\mathcal{A}})^{\bar{\bar{x}}} = \sum \mathcal{E}(\mathscr{D})$$

for $\bar{\bar{x}} \in \mathcal{O}_{\mathcal{A}}$, where the sum is over all the filled diagrams $\mathscr{D}$ of type $\bar{\bar{x}}$ of the form



and where we have omitted the bold arrow, meaning that it is any of the outgoing arrows.

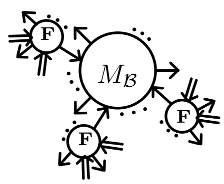
Definition 5.12. *Given d-pre-Calabi-Yau categories $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ and $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$ and an element $s_{d+1}\mathbf{F} \in \mathrm{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$ of degree 0, the* ***pre composition of $s_{d+1}\mathbf{F}$ and $s_{d+1}M_{\mathcal{B}}$*** *is the element*

$$s_{d+1}M_{\mathcal{B}} \underset{\mathrm{pre}}{\circ} s_{d+1}\mathbf{F} \in \mathrm{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$$

given by

$$(s_{d+1}M_{\mathcal{B}} \underset{\mathrm{pre}}{\circ} s_{d+1}\mathbf{F})^{\bar{\bar{x}}} = \sum \mathcal{E}(\mathcal{D}')$$

for $\bar{\bar{x}} \in \mathcal{O}_{\mathcal{A}}$, where the sum is over all the filled diagrams $\mathscr{D}'$ of type $\bar{\bar{x}}$ of the form



and where we have omitted the bold arrow, meaning that it is any of the outgoing arrows.

Definition 5.13. *Given d-pre-Calabi-Yau categories $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ and $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$ a* ***d-pre-Calabi-Yau morphism*** *$(F_0, \boldsymbol{F}) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \to (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ is a map $F_0 : \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$ together with an element $s_{d+1}\boldsymbol{F} \in \mathrm{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])^{C_{\mathrm{lg}(\bullet)}}[d+1]$ of degree 0 satisfying the following equation*

$$(s_{d+1}\mathbf{F} \underset{\mathrm{multinec}}{\circ} M_{\mathcal{A}})^{\bar{\bar{x}}} = (M_{\mathcal{B}} \underset{\mathrm{pre}}{\circ} s_{d+1}\mathbf{F})^{\bar{\bar{x}}} \tag{5.3}$$

for all $\bar{\bar{x}} \in \bar{\bar{\mathcal{O}}}_{\mathcal{A}}$. Note that the left member and right member of the previous identity belong to

$$\mathcal{H}om_{\Bbbk}\left(\bigotimes_{i=1}^{n} \mathcal{A}[1]^{\otimes \bar{x}^i}, \bigotimes_{i=1}^{n-1} {}_{F_0(\mathrm{lt}(\bar{x}^i))}\mathcal{B}_{F_0(\mathrm{rt}(\bar{x}^{i+1}))}[-d] \otimes {}_{F_0(\mathrm{lt}(\bar{x}^n))}\mathcal{B}_{F_0(\mathrm{rt}(\bar{x}^1))}[1]\right)$$

We now recall how to compose d-pre-Calabi-Yau morphisms.

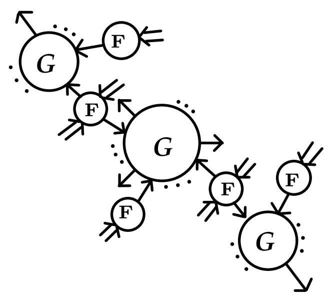
Definition 5.14. *Let $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$, $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ and $(\mathcal{C}, s_{d+1}M_{\mathcal{C}})$ be d-pre-Calabi-Yau categories with respective sets of objects $\mathcal{O}_{\mathcal{A}}$, $\mathcal{O}_{\mathcal{B}}$ and $\mathcal{O}_{\mathcal{C}}$ and let $(F_0, \mathbf{F}) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \to (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ and $(G_0, \mathbf{G}) : (\mathcal{B}, s_{d+1}M_{\mathcal{B}}) \to (\mathcal{C}, s_{d+1}M_{\mathcal{C}})$ be d-pre-Calabi-Yau morphisms. The* ***composition of $s_{d+1}\mathbf{F}$ and $s_{d+1}\mathbf{G}$*** *is the pair*

$$(G_0 \circ F_0, s_{d+1}\mathbf{F} \underset{pCY}{\circ} s_{d+1}\mathbf{G})$$

where

$$s_{d+1}\mathbf{F} \underset{pCY}{\circ} s_{d+1}\mathbf{G} \in \mathrm{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])^{C_{\lg(\bullet)}}[d+1]$$

is of degree 0 *and is given by* $(s_{d+1}G \underset{pCY}{\circ} s_{d+1}F)^{\bar{\bar{x}}} = \sum \mathcal{E}(\mathscr{D})$ *where the sum is over all filled diagrams* $\mathscr{D}$ *of type* $\bar{\bar{x}} \in \bar{\bar{\mathcal{O}}}_{\mathcal{A}}$ *of the form*

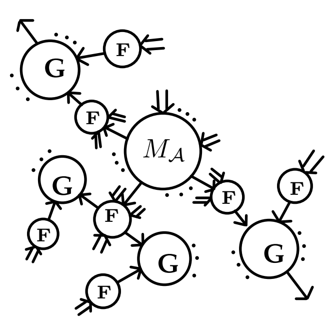


(5.4)

and where we have omitted the bold arrow, meaning that it any of the outgoing arrows.

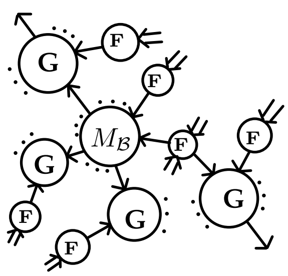
Proposition 5.15. *For* $d \in \mathbb{Z}$, d*-pre-Calabi-Yau categories and* d*-pre-Calabi-Yau morphisms together with the composition given in Definition 5.14 define a category, denoted as* $\mathrm{pCY_d}$*. Given a graded quiver* $\mathcal{A}$ *with set of objects* $\mathcal{O}$*, the identity morphism* $\mathrm{Id} : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \to (\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ *is given by* $\mathrm{Id}^x = \mathrm{id}_{{}_x\mathcal{A}_x}$ *for* $x \in \mathcal{O}$ *and* $\mathrm{Id}^{\bar{x}^1,\dots,\bar{x}^n} = 0$ *for* $(\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}^n$ *such that* $n \neq 1$ *or* $n = 1$ *and* $\lg(\bar{x}^1) \neq 1$.

Proof. We only have to check that the composition is associative and that the composition of any two d-pre-Calabi-Yau morphisms **F** and **G** is a d-pre-Calabi-Yau morphism. The associativity of the composition is clear. Now, consider two d-pre-Calabi-Yau morphisms **F** and **G**. Their composition is the sum of diagrams of the form (5.4). Therefore, the multinecklace composition of this composition and the pre-Calabi-Yau structure $s_{d+1}M_{\mathcal{A}}$ is a sum of diagrams of the form



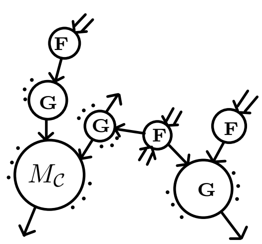
(5.5)

Given $\bar{\bar{x}} \in \bar{\bar{\mathcal{O}}}_{\mathcal{A}}$, we have to sum over all diagrams of such type and we have several possibilities for the type of the inner diagram, defined as the subdiagram consisting of the disc filled with $M_{\mathcal{A}}$ together with all those discs directly connected to the one filled with $M_{\mathcal{A}}$. Note that if we fix the type of the outer diagram given as the complement of the inner diagram, the type of the inner one is fixed. Moreover, changing the inner diagram for one of same type does not change the type of the whole diagram. Therefore, taking the sum over all diagrams of type $\bar{\bar{x}} \in \bar{\bar{\mathcal{O}}}_{\mathcal{A}}$ is the same as taking the sum over all the possible types for the outer diagram and for each of those, taking the sum over all the suitable types for the inner one. This second sum allows us to use that **F** is a pre-Calabi-Yau morphism to replace the inner diagram by one consisting of a discs filled with $M_{\mathcal{B}}$ whose incoming arrows are connected with outgoing arrows of discs filled with **F**. Then, the sum of all the diagrams of type $\bar{\bar{x}}$ of the form (5.5) is equal to the sum of all the diagrams of type $\bar{\bar{x}}$ of the form



(5.6)

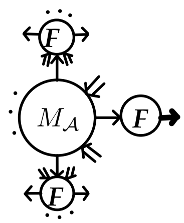
and we now define the inner diagram as filled diagram consisting of the disc filled with $M_{\mathcal{B}}$ and of all the discs connected to it. The previous remarks on the types of the inner and outer diagrams still hold. Thus, the sum over all possible types for the whole diagram is again the sum over all the possible types for the outer diagram and for each of those, taking the sum over all the suitable types for the inner one. **G** being a pre-Calabi-Yau morphism, one can again use (5.3) and say that the sum of all the diagrams of type $\bar{\bar{x}}$ of the form (5.6) is now equal to the sum of all the diagrams of type $\bar{\bar{x}}$ of the form



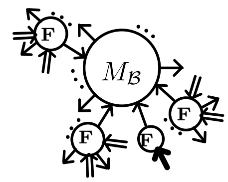
Therefore, $s_{d+1}G \underset{pCY}{\circ} s_{d+1}F$ is a pre-Calabi-Yau morphism. $\square$

The following class of morphisms will be useful in the next subsection.

Definition 5.16. *Given d-pre-Calabi-Yau categories $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ and $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{A}}$ and a d-pre-Calabi-Yau morphism $F = (F_0, \mathbf{F}) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \to (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$, we say that F is **good** if $\sum \mathcal{E}(\mathscr{D}) = \sum \mathcal{E}(\mathscr{D}')$ where the sums are over all the filled diagrams $\mathscr{D}$ and $\mathscr{D}'$ of type $\bar{\bar{x}}$ of the form*



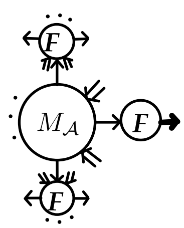
and



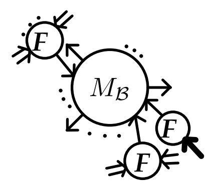
respectively.

Note that this condition is not closed under the pre-Calabi-Yau composition. We thus restrict this notion of good morphisms and give the following definition.

Definition 5.17. *Given d-pre-Calabi-Yau categories $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ and $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{A}}$ and a d-pre-Calabi-Yau morphism $F = (F_0, \mathbf{F}) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \to (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$, we say that F is **nice** if $\sum \mathcal{E}(\mathscr{D}) = \sum \mathcal{E}(\mathscr{D}')$ where the sums are over all the filled diagrams $\mathscr{D}$ and $\mathscr{D}'$ of type $\bar{\bar{x}}$ of the form*



and



respectively.

6 The relation between pre-Calabi-Yau morphisms and A_∞-morphisms

Recall that given a Hom-finite graded quiver $\mathcal{A}$, we have an equivalence between the data of a d-pre-Calabi-Yau structure on $\mathcal{A}$ and a cyclic A_∞-structure on $\mathcal{A} \oplus \mathcal{A}^*[d-1]$ that restricts to $\mathcal{A}$. In this section, we study the relation between d-pre-Calabi-Yau morphisms and A_∞-morphisms.

6.1 The case of strict morphisms

In this subsection, we study the relation between strict d-pre-Calabi-Yau morphisms and A_∞-morphisms. We first recall the notion of strict d-pre-Calabi-Yau morphism.

Definition 6.1. *Let $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$, $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ be d-pre-Calabi-Yau categories with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$. A d-pre-Calabi-Yau morphism $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \to (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ is* ***strict*** *if $\Phi^{\bar{\bar{x}}}$ vanishes for each $\bar{\bar{x}} \in \bar{\mathcal{O}}^n_{\mathcal{A}}$ with $n > 1$ and $\lg(\bar{x}^1) \neq 2$ if $n = 1$.*

Equivalently, a strict d-pre-Calabi-Yau morphism between d-pre-Calabi-Yau categories $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ and $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ is the data of a map between their sets of objects $\Phi_0 : \mathcal{O}_{\mathcal{A}} \to \mathcal{O}_{\mathcal{B}}$ together with a collection $\Phi = (\Phi^{x,y} : {}_x\mathcal{A}_y[1] \to {}_x\mathcal{B}_y[1])_{x,y\in\mathcal{O}_{\mathcal{A}}}$ of maps of degree 0 that satisfies

$$(\Phi^{\mathrm{lt}(\bar{x}^1),\mathrm{rt}(\bar{x}^2)} \otimes \cdots \otimes \Phi^{\mathrm{lt}(\bar{x}^n),\mathrm{rt}(\bar{x}^1)}) \circ s_{d+1}M_{\mathcal{A}}^{\bar{x}^1,\dots,\bar{x}^n} = s_{d+1}M_{\mathcal{B}}^{\bar{x}^1,\dots,\bar{x}^n} \circ (\Phi^{\otimes \lg(\bar{x}^1)-1} \otimes \cdots \otimes \Phi^{\otimes \lg(\bar{x}^n)-1})$$

for each $n \in \mathbb{N}^*$, $(\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}^n_{\mathcal{A}}$.

For simplicity, we will omit the elements when writing the map Φ. We will denote

$$\overline{\Phi_0(x)} = (\Phi_0(x_1), \dots, \Phi_0(x_n))$$

for $\bar{x} = (x_1, \dots, x_n) \in \mathcal{O}^n_{\mathcal{A}}$ and

$$\overline{\overline{\Phi_0(x)}} = (\overline{\Phi_0(\bar{x}^1)}, \dots, \overline{\Phi_0(\bar{x}^n)})$$

for $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}^n_{\mathcal{A}}$.

Definition 6.2. *We denote by $SpCY_d$ the subcategory of pCY_d whose objects are d-pre-Calabi-Yau categories and whose morphisms are strict d-pre-Calabi-Yau morphisms.*

Given d-pre-Calabi-Yau categories $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ and $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$ and a strict d-pre-Calabi-Yau morphism $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \to (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$, we now construct an A_∞-structure on $\mathcal{A} \oplus \mathcal{B}^*[d-1]$. For $\bar{\bar{x}} \in \bar{\bar{\mathcal{O}}}_{\mathcal{A}}$, we denote by $sm^{\bar{\bar{x}}}_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}}$ the composition of $j_{\bar{\bar{x}}^{-1}}(s_{d+1}M_{\mathcal{A}}^{\bar{\bar{x}}^{-1}})$ and the canonical projection on $\mathcal{A}[1]$.

Similarly, we will denote by $sm^{\overline{\overline{\Phi_0(x)}}}_{\mathcal{B}\oplus\mathcal{B}^*\to\mathcal{B}^*}$ the composition of $j_{\overline{\overline{\Phi_0(x)}}^{-1}}(s_{d+1}M_{\mathcal{B}}^{\overline{\overline{\Phi_0(x)}}^{-1}})$ and the canonical projection on $\mathcal{B}^*[d]$.

Definition 6.3. *We define $sm_{\mathcal{A}\oplus\mathcal{B}^*} \in C(\mathcal{A}\oplus\mathcal{B}^*[d-1])[1]$ as the unique element such that the composition of $m^{\bar{x}^1,\dots,\bar{x}^n}_{\mathcal{A}\oplus\mathcal{B}^*}$ with the canonical projection*

$${}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}_{\mathrm{rt}(\bar{x}^n)} \oplus {}_{\Phi_0(\mathrm{lt}(\bar{x}^1))}\mathcal{B}^*_{\Phi_0(\mathrm{rt}(\bar{x}^n))}[d-1] \to {}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}_{\mathrm{rt}(\bar{x}^n)}$$

is given by the map

$$m^{\bar{x}^1,\dots,\bar{x}^n}_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{A}} : \bigotimes_{i=1}^{n-1}(\mathcal{A}[1]^{\otimes\bar{x}^i} \otimes {}_{\Phi_0(\mathrm{rt}(\bar{x}^{i+1}))}\mathcal{B}^*[d]_{\Phi_0(\mathrm{lt}(\bar{x}^i))}) \otimes \mathcal{A}[1]^{\otimes\bar{x}^n} \to {}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}_{\mathrm{rt}(\bar{x}^n)}$$

defined by

$$m^{\bar{x}^1,\dots,\bar{x}^n}_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{A}} = m^{\bar{x}^1,\dots,\bar{x}^n}_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}} \circ (\mathrm{id}^{\otimes\lg(\bar{x}^1)-1} \otimes \Phi^* \otimes \mathrm{id}^{\otimes\lg(\bar{x}^2)-1} \otimes \cdots \otimes \mathrm{id}^{\otimes\lg(\bar{x}^{n-1})-1} \otimes \Phi^* \otimes \mathrm{id}^{\otimes\lg(\bar{x}^n)-1})$$

and the composition of $m^{\bar{x}^1,\dots,\bar{x}^n}_{\mathcal{A}\oplus\mathcal{B}^*}$ with the canonical projection

$${}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}_{\mathrm{rt}(\bar{x}^n)} \oplus {}_{\Phi_0(\mathrm{lt}(\bar{x}^1))}\mathcal{B}^*_{\Phi_0(\mathrm{rt}(\bar{x}^n))}[d-1] \to {}_{\Phi_0(\mathrm{lt}(\bar{x}^1))}\mathcal{B}^*_{\Phi_0(\mathrm{rt}(\bar{x}^n))}[d-1]$$

is given by the map

$$m_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{B}^*}^{\bar{x}^1,\ldots,\bar{x}^n} : \bigotimes_{i=1}^{n-1}(\mathcal{A}[1]^{\otimes\bar{x}^i} \otimes {}_{\Phi_0(\mathrm{rt}(\bar{x}^{i+1}))}\mathcal{B}^*[d]_{\Phi_0(\mathrm{lt}(\bar{x}^i))}) \otimes \mathcal{A}[1]^{\otimes\bar{x}^n} \to {}_{\Phi_0(\mathrm{lt}(\bar{x}^1))}\mathcal{B}^*_{\Phi_0(\mathrm{rt}(\bar{x}^n))}[d-1]$$

defined by

$$m_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{B}^*}^{\bar{x}^1,\ldots,\bar{x}^n} = m_{\mathcal{B}\oplus\mathcal{B}^*\to\mathcal{B}^*}^{\overline{\Phi_0(\bar{x}^1)},\ldots,\overline{\Phi_0(\bar{x}^n)}} \circ (\Phi^{\otimes\,\mathrm{lg}(\bar{x}^1)-1} \otimes \mathrm{id} \otimes \Phi^{\otimes\,\mathrm{lg}(\bar{x}^2)-1} \otimes \cdots \otimes \mathrm{id} \otimes \Phi^{\otimes\,\mathrm{lg}(\bar{x}^n)-1})$$

Proposition 6.4. *The element $sm_{\mathcal{A}\oplus\mathcal{B}^*} \in C(\mathcal{A}\oplus\mathcal{B}^*[d-1])[1]$ defines an A_∞-structure on $\mathcal{A}\oplus\mathcal{B}^*[d-1]$ that is almost cyclic with respect to the Φ-mixed bilinear form Γ^Φ, defined in Example 3.17.*

Proof. Using the Proposition 5.9, we have that the equality $sm_{\mathcal{A}\oplus\mathcal{B}^*} \underset{G}{\circ} sm_{\mathcal{A}\oplus\mathcal{B}^*} = 0$ is tantamount to $s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*} \underset{\Phi\ \mathrm{nec}}{\circ} s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*} = 0$ where $s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*} \in \mathscr{B}_\bullet(\mathcal{A}[1],\mathcal{B}[-d])[d+1]$ is uniquely determined by

$$M_{\mathcal{A}\oplus\mathcal{B}^*}^{\bar{\bar{x}}} = (\Phi^{\otimes(n-1)} \otimes \mathrm{id}) \circ s_{d+1}M_{\mathcal{A}}^{\bar{\bar{x}}} \in \mathscr{B}_{\bar{\bar{x}}}^{\mathcal{A}}(\mathcal{A}[1],\mathcal{B}[-d])$$

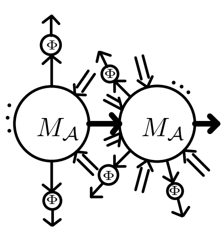
and

$$M_{\mathcal{A}\oplus\mathcal{B}^*}^{\overline{\overline{\Phi_0(x)}}} = s_{d+1}M_{\mathcal{B}}^{\overline{\overline{\Phi_0(y)}}} \circ (\Phi^{\otimes\,\mathrm{lg}(\bar{x}^1)-1} \otimes \mathrm{id} \otimes \Phi^{\otimes\,\mathrm{lg}(\bar{x}^n)-1} \otimes \Phi^{\otimes\,\mathrm{lg}(\bar{x}^2)-1} \otimes \cdots \otimes \Phi^{\otimes\,\mathrm{lg}(\bar{x}^{n-1})-1}) \in \mathscr{B}_{\bar{\bar{x}}}^{\mathcal{B}}(\mathcal{A}[1],\mathcal{B}[-d])$$

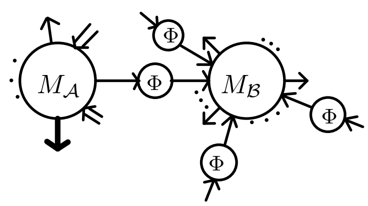
for $\bar{\bar{x}} = (\bar{x}^1,\ldots,\bar{x}^n) \in \bar{\mathcal{O}}^n$ and $\bar{\bar{y}} = (\bar{x}^1 \sqcup \bar{x}^n, \bar{x}^2, \ldots, \bar{x}^{n-1})$. Moreover,

$$\pi_{\mathcal{A}}(s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*} \underset{\Phi\ \mathrm{nec}}{\circ} s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*})^{\bar{\bar{x}}} = \sum \mathcal{E}(\mathscr{D}) + \sum \mathcal{E}(\mathscr{D}')$$

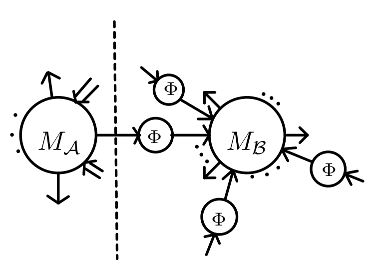
where $\pi_{\mathcal{A}}$ is the canonical projection $\mathscr{B}_\bullet(\mathcal{A}[1],\mathcal{B}[-d])[d+1] \to \mathscr{B}_\bullet^{\mathcal{A}}(\mathcal{A}[1],\mathcal{B}[-d])$ and where the sums are over all the filled diagrams $\mathscr{D}$ and $\mathscr{D}'$ of type $\bar{\bar{x}}$ that are of the form



and



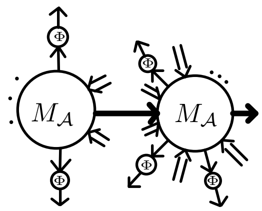
respectively. Now, note that the second diagram can be cut into two as follows.



Using the morphism identity satisfied by Φ, the diagram on the right can be replaced by one with a disc filled with $M_{\mathcal{A}}$ whose outgoing arrows are connected with the unique incoming arrow of discs of size 1 filled with Φ. We thus obtain that

$$\pi_{\mathcal{A}}(s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*} \underset{G}{\circ} s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*})^{\bar{\bar{x}}} = \sum \mathcal{E}(\mathscr{D}) + \sum \mathcal{E}(\mathscr{D}')$$

where the sums are over all the filled diagrams $\mathscr{D}$ and $\mathscr{D}'$ of type $\bar{\bar{x}}$ that are of the form



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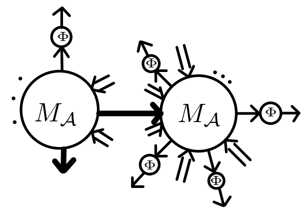


respectively. Moreover, $\sum \mathcal{E}(\mathscr{D}) + \sum \mathcal{E}(\mathscr{D}') = 0$ since $s_{d+1}M_{\mathcal{A}}$ is a d-pre-Calabi-Yau structure. Thus, if we show that this structure satisfies the cyclicity condition (3.1), $sm_{\mathcal{A}\oplus\mathcal{B}^*}$ satisfies the Stasheff identities (SI).

Using the definition of Γ^{Φ} and $sm_{\mathcal{B}\oplus\mathcal{B}^*}$ and since $sm_{\mathcal{B}\oplus\mathcal{B}^*}$ is cyclic with respect to $\Gamma^{\mathcal{B}}$, we have that

$$\begin{aligned}
&\Gamma^{\Phi}(sm^{\bar{x}^1,\dots,\bar{x}^n}_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{B}^*}(s\bar{a}^1, tf^1, \dots, s\bar{a}^{n-1}, tf^{n-1}, s\bar{a}^n), sb) \\
&= \Gamma^{\mathcal{B}}(sm^{\overline{\Phi_0(\bar{x}^1)},\dots,\overline{\Phi_0(\bar{x}^n)}}_{\mathcal{B}\oplus\mathcal{B}^*\to\mathcal{B}^*}(\Phi^{\otimes \lg(\bar{x}^1)-1}(s\bar{a}^1), tf^1, \dots, tf^{n-1}, \Phi^{\otimes \lg(\bar{x}^n)-1}(s\bar{a}^n)), \Phi(sb)) \\
&= (-1)^{\epsilon}\Gamma^{\mathcal{B}}(sm^{\overline{\Phi_0(\bar{x}^n)}\sqcup\overline{\Phi_0(\bar{x}^1)},\overline{\Phi_0(\bar{x}^2)},\dots,\overline{\Phi_0(\bar{x}^{n-1})}}_{\mathcal{B}\oplus\mathcal{B}^*\to\mathcal{B}}\big(\Phi^{\otimes \lg(\bar{x}^n)-1}(s\bar{a}^n) \otimes \Phi(sb) \otimes \Phi^{\otimes \lg(\bar{x}^1)-1}(s\bar{a}^1), tf^1, \dots, \\
&tf^{n-2}, \Phi^{\otimes \lg(\bar{x}^{n-1})-1}(s\bar{a}^{n-1})\big), tf^{n-1}) \\
&= (-1)^{\epsilon+\delta}\Gamma^{\mathcal{B}}\big(((f^{n-2} \circ s_d) \otimes \cdots \otimes (f^1 \circ s_d) \otimes \mathrm{id}) \\
&(M^{\overline{\Phi_0(\bar{x}^{n-1})},\dots\overline{\Phi_0(\bar{x}^n)}\sqcup\overline{\Phi_0(\bar{x}^1)}}_{\mathcal{B}}(\Phi^{\otimes \lg(\bar{x}^{n-1})}(s\bar{a}^{n-1}), \dots, \Phi^{\otimes \lg(\bar{x}^n)}(s\bar{a}^n) \otimes \Phi(sb) \otimes \Phi^{\otimes \lg(\bar{x}^1)}(s\bar{a}^1)), tf^{n-1}\big)
\end{aligned} \tag{6.1}$$

for $s\bar{a}_i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$, $tf^i \in {}_{\Phi_0(\mathrm{rt}(\bar{x}^{i+1}))}\mathcal{B}^*[d]_{\Phi_0(\mathrm{lt}(\bar{x}^i))}$, with $\epsilon = (|s\bar{a}^n| + |sb|)(\sum_{i=1}^{n-1}(|s\bar{a}^i| + |tf^i|)$ and

$$\begin{aligned}
\delta = (|s\bar{a}^1| + |sb| + |s\bar{a}^n|)(\sum_{i=1}^{n-1}|s\bar{a}^i| + \sum_{i=1}^{n-2}|tf^i|) + \sum_{2\leq i\leq j\leq n-2}|s\bar{a}^i||tf^j| + dn \\
+ \sum_{2\leq i<j\leq n-1}|s\bar{a}^i||s\bar{a}^j| + \sum_{1\leq i<j\leq n-2}|tf^i||tf^j|
\end{aligned}$$

Moreover, using that Φ is a d-pre-Calabi-Yau morphism, we have that the last member of (6.1) is

$$\begin{aligned}
&(-1)^{\epsilon+\delta}\Gamma^{\mathcal{A}}\big((\bigotimes_{i=2}^{n-1}(f^{n-i} \circ \tilde{\Phi}) \otimes \mathrm{id}) \circ s_{d+1}M^{\bar{x}^{n-1},\dots,\bar{x}^n\sqcup\bar{x}^1}_{\mathcal{A}}(s\bar{a}^{n-1}, \dots, s\bar{a}^2, s\bar{a}^n \otimes sb \otimes s\bar{a}^1), tf^{n-1} \circ \tilde{\Phi}\big) \\
&= (-1)^{\epsilon}\Gamma^{\mathcal{A}}\big(sm^{\bar{x}^n\sqcup\bar{x}^1,\bar{x}^2,\dots,\bar{x}^{n-1}}_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}}(s\bar{a}^n \otimes sb \otimes s\bar{a}^1, tf^1 \circ \tilde{\Phi}, s\bar{a}^2, \dots, tf^{n-2} \circ \tilde{\Phi}, s\bar{a}^{n-1}), tf^{n-1} \circ \tilde{\Phi}\big)
\end{aligned} \tag{6.2}$$

where $\tilde{\Phi}$ denotes the morphism $\Phi[d-1]$. Thus, comparing (6.1) and (6.2) we get that

$$\begin{aligned}
&\Gamma^{\mathcal{B}}(sm_{\mathcal{B}\oplus\mathcal{B}^*\to\mathcal{B}}^{\overline{\Phi_0(\bar{x}^n)}\sqcup\overline{\Phi_0(\bar{x}^1)},\overline{\Phi_0(\bar{x}^2)},\dots,\overline{\Phi_0(\bar{x}^{n-1})}}\big(\Phi^{\otimes \lg(\bar{x}^n)-1}(\bar{sa}^n)\otimes\Phi(sb)\otimes\Phi^{\otimes \lg(\bar{x}^1)-1}(\bar{sa}^1),tf^1,\dots,\\
&tf^{n-2},\Phi^{\otimes \lg(\bar{x}^{n-1})-1}(\bar{sa}^{n-1})\big),tf^{n-1}) \qquad (6.3)\\
&=\Gamma^{\mathcal{A}}(sm_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}}^{\bar{x}^n\sqcup\bar{x}^1,\bar{x}^2,\dots,\bar{x}^{n-1}}(\bar{sa}^n\otimes sb\otimes\bar{sa}^1,tf^1\circ\tilde{\Phi},\bar{sa}^2,\dots,tf^{n-2}\circ\tilde{\Phi},\bar{sa}^{n-1}),tf^{n-1}\circ\tilde{\Phi})
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
&(-1)^{\epsilon}\Gamma^{\mathcal{A}}(sm_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}}^{\bar{x}^n\sqcup\bar{x}^1,\bar{x}^2,\dots,\bar{x}^{n-1}}(\bar{sa}^n\otimes sb\otimes\bar{sa}^1,tf^1\circ\tilde{\Phi},\bar{sa}^2,\dots,tf^{n-2}\circ\tilde{\Phi},\bar{sa}^{n-1}),tf^{n-1}\circ\tilde{\Phi})\\
&=(-1)^{\epsilon}\Gamma^{\Phi}(sm_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{A}}^{\bar{x}^n\sqcup\bar{x}^1,\bar{x}^2,\dots,\bar{x}^{n-1}}(\bar{sa}^n\otimes sb\otimes\bar{sa}^1,tf^1,\bar{sa}^2,\dots,tf^{n-2},\bar{sa}^{n-1}),tf^{n-1})
\end{aligned}$$

Therefore, $\mathcal{A}\oplus\mathcal{B}^*[d-1]$ together with $sm_{\mathcal{A}\oplus\mathcal{B}^*}$ is an A_∞-category that is almost cyclic with respect to Γ^{Φ}. $\square$

Definition 6.5. *Let $(\mathcal{A},s_{d+1}M_{\mathcal{A}})$, $(\mathcal{B},s_{d+1}M_{\mathcal{B}})$ be d-pre-Calabi-Yau categories with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$. Consider a strict d-pre-Calabi-Yau morphism $(\Phi_0,\Phi):(\mathcal{A},s_{d+1}M_{\mathcal{A}})\to(\mathcal{B},s_{d+1}M_{\mathcal{B}})$. We define maps of graded vector spaces*

$$\varphi_{\mathcal{A}}^{x,y}:{}_x\mathcal{A}[1]_y\oplus{}_{\Phi_0(x)}\mathcal{B}^*[d]_{\Phi_0(y)}\to{}_x(\mathcal{A}[1]\oplus\mathcal{A}^*[d])_y$$

and

$$\varphi_{\mathcal{B}}^{x,y}:{}_x\mathcal{A}[1]_y\oplus{}_{\Phi_0(x)}\mathcal{B}^*[d]_{\Phi_0(y)}\to{}_{\Phi_0(x)}(\mathcal{B}[1]\oplus\mathcal{B}^*[d])_{\Phi_0(y)}$$

given by $\varphi_{\mathcal{A}}^{x,y}(sa)=sa$, $\varphi_{\mathcal{B}}^{x,y}(sa)=\Phi^{x,y}(sa)$, $\varphi_{\mathcal{A}}^{x,y}(tf)=tf\circ\Phi^{y,x}[d]$ and $\varphi_{\mathcal{B}}^{x,y}(tf)=tf$ for $x,y\in\mathcal{O}_{\mathcal{A}}$, $sa\in{}_x\mathcal{A}[1]_y$ and $tf\in{}_{\Phi_0(x)}\mathcal{B}^*[d]_{\Phi_0(y)}$.

Proposition 6.6. *Let $(\mathcal{A},s_{d+1}M_{\mathcal{A}})$, $(\mathcal{B},s_{d+1}M_{\mathcal{B}})$ be d-pre-Calabi-Yau categories with respective sets of objects $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{O}_{\mathcal{B}}$. Consider a strict d-pre-Calabi-Yau morphism $(\Phi_0,\Phi):(\mathcal{A},s_{d+1}M_{\mathcal{A}})\to(\mathcal{B},s_{d+1}M_{\mathcal{B}})$ and the A_∞-category $(\mathcal{A}\oplus\mathcal{B}^*[d-1],sm_{\mathcal{A}\oplus\mathcal{B}^*})$ where $sm_{\mathcal{A}\oplus\mathcal{B}^*}\in C(\mathcal{A}\oplus\mathcal{B}^*[d-1])[1]$ is given in Definition 6.3. The maps $\varphi_{\mathcal{A}}$ and $\varphi_{\mathcal{B}}$ defined in Definition 6.5 are cyclic A_∞-morphisms, in the sense of Definition 3.22.*

Proof. We only check that $\varphi_{\mathcal{A}}$ is a morphism since the case of $\varphi_{\mathcal{B}}$ is similar. We only have to verify that

$$\varphi_{\mathcal{A}}^{x,y}\circ sm_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{A}}^{\bar{x}^1,\dots,\bar{x}^n}=sm_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}}^{\bar{x}^1,\dots,\bar{x}^n}\circ(\varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^1)-1}\otimes\varphi_{\mathcal{A}}\otimes\varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^2)-1}\dots\otimes\varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^n)-1}) \qquad (6.4)$$

and

$$\varphi_{\mathcal{A}}^{x,y}\circ sm_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{B}^*}^{\bar{x}^1,\dots,\bar{x}^n}=sm_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}^*}^{\bar{x}^1,\dots,\bar{x}^n}\circ(\varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^1)-1}\otimes\varphi_{\mathcal{A}}\otimes\varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^2)-1}\dots\otimes\varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^n)-1}) \qquad (6.5)$$

for $x,y\in\mathcal{O}_{\mathcal{A}}$, $(\bar{x}^1,\dots,\bar{x}^n)\in\bar{\mathcal{O}}_{\mathcal{A}}^n$ such that $\mathrm{lt}(\bar{x}^n)=x$, $\mathrm{rt}(\bar{x}^1)=y$.

First, note that

$$\begin{aligned}
\varphi_{\mathcal{A}}^{x,y}(sm_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{A}}^{\bar{x}^1,\dots,\bar{x}^n}&(\bar{sa}^1,tf^1,\bar{sa}^2,\dots,\bar{sa}^{n-1},tf^{n-1},\bar{sa}^n))\\
&=sm_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{A}}^{\bar{x}^1,\dots,\bar{x}^n}(\bar{sa}^1,tf^1,\bar{sa}^2,\dots,\bar{sa}^{n-1},tf^{n-1},\bar{sa}^n)\\
&=sm_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}}^{\bar{x}^1,\dots,\bar{x}^n}(\bar{sa}^1,\varphi_{\mathcal{A}}(tf^1),\bar{sa}^2,\dots,\bar{sa}^{n-1},\varphi_{\mathcal{A}}(tf^{n-1}),\bar{sa}^n)
\end{aligned}$$

for $\bar{sa}^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ and $tf^i \in {}_{\Phi_0(\mathrm{rt}(\bar{x}^{i+1}))}\mathcal{B}^*[d]_{\Phi_0(\mathrm{lt}(\bar{x}^i))}$, so that (6.4) holds.

Moreover, we also have

$$\begin{aligned}&\varphi_{\mathcal{A}}^{x,y} \circ sm_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{B}^*}^{\bar{x}^1,\dots,\bar{x}^n}(\bar{sa}^1, tf^1, \bar{sa}^2, \dots, \bar{sa}^{n-1}, tf^{n-1}, \bar{sa}^n) \\ &= sm_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{B}^*}^{\bar{x}^1,\dots,\bar{x}^n}(\bar{sa}^1, tf^1, \bar{sa}^2, \dots, \bar{sa}^{n-1}, tf^{n-1}, \bar{sa}^n) \circ \Phi^{x,y}\end{aligned}$$

and given $sb \in {}_x\mathcal{A}_y$, we have that

$$\begin{aligned}&sm_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{B}^*}^{\bar{x}^1,\dots,\bar{x}^n}(\bar{sa}^1, tf^1, \bar{sa}^2, \dots, \bar{sa}^{n-1}, tf^{n-1}, \bar{sa}^n)(\Phi^{x,y}(sb)) \\ &= \Gamma^{\Phi}(sm_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{B}^*}^{\bar{x}^1,\dots,\bar{x}^n}(\bar{sa}^1, tf^1, \bar{sa}^2, \dots, \bar{sa}^{n-1}, tf^{n-1}, \bar{sa}^n), sb)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&sm_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}^*}^{\bar{x}^1,\dots,\bar{x}^n}(\varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^1)-1}(\bar{sa}^1) \otimes \varphi_{\mathcal{A}}(tf^1) \otimes \varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^2)-1}(\bar{sa}^2) \cdots \otimes \varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^n)-1}(\bar{sa}^n))(sb) \\ &= \Gamma^{\mathcal{A}}(sm_{\mathcal{A}\oplus\mathcal{A}^*\to\mathcal{A}^*}^{\bar{x}^1,\dots,\bar{x}^n}(\varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^1)-1}(\bar{sa}^1) \otimes \varphi_{\mathcal{A}}(tf^1) \otimes \varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^2)-1}(\bar{sa}^2) \cdots \otimes \varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^n)-1}(\bar{sa}^n)), sb)\end{aligned}$$

Using the identity (6.3), we thus get (6.5).

It remains to show that the morphisms are cyclic. To prove it, we note that

$${}_y\Gamma_x^{\mathcal{A}}(\varphi_{\mathcal{A}}^{y,x}(tf), \varphi_{\mathcal{A}}^{x,y}(sa)) = {}_y\Gamma_x^{\mathcal{A}}(t(f \circ \Phi^{x,y}), sa) = f(\Phi^{x,y}(sa)) = {}_y\Gamma_x^{\Phi}(tf, sa)$$

as well as

$${}_y\Gamma_x^{\mathcal{B}}(\varphi_{\mathcal{B}}^{y,x}(tf), \varphi_{\mathcal{B}}^{x,y}(sa)) = {}_y\Gamma_x^{\mathcal{B}}(tf, \Phi^{x,y}(sa)) = f(\Phi^{x,y}(sa)) = {}_y\Gamma_x^{\Phi}(tf, sa)$$

for $sa \in {}_x\mathcal{A}[1]_y$, $tf \in {}_{\Phi_0(y)}\mathcal{B}^*[d]_{\Phi_0(x)}$. The second condition to be a cyclic morphism is obviously satisfied since $\varphi_{\mathcal{A}}^{\bar{x}}$ and $\varphi_{\mathcal{B}}^{\bar{x}}$ vanish for $\bar{x} \in \mathcal{O}^n$ with $n > 2$. $\square$

Definition 6.7. *Let $(\mathcal{A} \oplus \mathcal{A}^*[d-1], sm_{\mathcal{A}\oplus\mathcal{A}^*})$, $(\mathcal{B} \oplus \mathcal{B}^*[d-1], sm_{\mathcal{B}\oplus\mathcal{B}^*})$ be A_∞-categories. A* ***hat morphism*** *from $\mathcal{A} \oplus \mathcal{A}^*[d-1]$ to $\mathcal{B} \oplus \mathcal{B}^*[d-1]$ is a triple $(sm_{\mathcal{A}\oplus\mathcal{B}^*}, \varphi_{\mathcal{A}}, \varphi_{\mathcal{B}})$ where $sm_{\mathcal{A}\oplus\mathcal{B}^*}$ is an A_∞-structure on $\mathcal{A} \oplus \mathcal{B}^*[d-1]$ and $\varphi_{\mathcal{A}}$, $\varphi_{\mathcal{B}}$ are A_∞-morphisms*

$$\begin{tikzcd} & \mathcal{A}[1] \oplus \mathcal{B}^*[d] \arrow[dl, "\varphi_{\mathcal{A}}"'] \arrow[dr, "\varphi_{\mathcal{B}}"] & \\ \mathcal{A}[1] \oplus \mathcal{A}^*[d] & & \mathcal{B}[1] \oplus \mathcal{B}^*[d] \end{tikzcd} \tag{6.6}$$

Definition 6.8. *Let $(\mathcal{A}\oplus\mathcal{A}^*[d-1], sm_{\mathcal{A}\oplus\mathcal{A}^*})$, $(\mathcal{B}\oplus\mathcal{B}^*[d-1], sm_{\mathcal{B}\oplus\mathcal{B}^*})$ and $(\mathcal{C}\oplus\mathcal{C}^*[d-1], sm_{\mathcal{C}\oplus\mathcal{C}^*})$ be A_∞-categories. Two hat morphisms $(sm_{\mathcal{A}\oplus\mathcal{B}^*}, \varphi_{\mathcal{A}}, \varphi_{\mathcal{B}}) : \mathcal{A} \oplus \mathcal{A}^*[d-1] \to \mathcal{B} \oplus \mathcal{B}^*[d-1]$, $(sm_{\mathcal{B}\oplus\mathcal{C}^*}, \psi_{\mathcal{B}}, \psi_{\mathcal{C}}) : \mathcal{B} \oplus \mathcal{B}^*[d-1] \to \mathcal{C} \oplus \mathcal{C}^*[d-1]$ are* ***composable*** *if there exist a triple $(sm_{\mathcal{A}\oplus\mathcal{C}^*}, \chi_{\mathcal{A}}, \chi_{\mathcal{C}})$ where $\chi_{\mathcal{A}} : \mathcal{A} \oplus \mathcal{C}^*[d-1] \to \mathcal{A} \oplus \mathcal{B}^*[d-1]$ and $\chi_{\mathcal{C}} : \mathcal{A} \oplus \mathcal{C}^*[d-1] \to \mathcal{B} \oplus \mathcal{C}^*[d-1]$ are A_∞-morphisms and such that $\varphi_{\mathcal{B}} \circ \chi_{\mathcal{A}} = \psi_{\mathcal{B}} \circ \chi_{\mathcal{C}}$. The* ***composition*** *of $(sm_{\mathcal{A}\oplus\mathcal{B}^*}, \varphi_{\mathcal{A}}, \varphi_{\mathcal{B}})$ and $(sm_{\mathcal{B}\oplus\mathcal{C}^*}, \psi_{\mathcal{B}}, \psi_{\mathcal{C}})$ is then given by $(sm_{\mathcal{A}\oplus\mathcal{C}^*}, \varphi_{\mathcal{A}} \circ \chi_{\mathcal{A}}, \psi_{\mathcal{C}} \circ \chi_{\mathcal{C}})$.*

Definition 6.9. *A* ***partial category*** *is an A_∞-pre-category as defined in [4] where the multiplications m_n vanish for $n > 2$.*

Definition 6.10. *The* A_∞***-hat category*** *is the partial category* $\widehat{A_{\infty}}_d$ *whose objects are* A_∞*-categories of the form* $\mathcal{A} \oplus \mathcal{A}^*[d-1]$ *and whose morphisms are hat morphisms.*

Definition 6.11. *A* ***functor*** *between partial categories* $\mathcal{A}$ *and* $\mathcal{B}$ *with respective sets of objects* $\mathcal{O}_\mathcal{A}$ *and* $\mathcal{O}_\mathcal{B}$ *is the data of a map* $F_0 : \mathcal{O}_\mathcal{A} \to \mathcal{O}_\mathcal{B}$ *together with a family* $F = ({}_yF_x)_{x,y \in \mathcal{O}_\mathcal{A}}$ *sending a morphism* $f : x \to y$ *to a morphism* ${}_yF_x(f) : F_0(x) \to F_0(y)$ *such that if two morphisms* $f : x \to y$ *and* $g : y \to z$ *are composable, then* ${}_yF_x(f)$ *and* ${}_zF_y(g)$ *are composable and their composition is given by the morphism* ${}_zF_y(g) \circ {}_yF_x(f) = {}_zF_x(g \circ f)$.

Definition 6.12. *We define the partial subcategory* $cyc\widehat{A_{\infty}}_d$ *of* $\widehat{A_{\infty}}_d$ *whose objects are cyclic* A_∞*-categories of the form* $\mathcal{A} \oplus \mathcal{A}^*[d-1]$ *and whose morphisms* $\mathcal{A} \oplus \mathcal{A}^*[d-1] \to \mathcal{B} \oplus \mathcal{B}^*[d-1]$ *are the data of an almost cyclic* A_∞*-structure on* $\mathcal{A} \oplus \mathcal{B}^*[d-1]$ *together with a diagram of the form* (6.6) *where* $\varphi_\mathcal{A}$ *and* $\varphi_\mathcal{B}$ *are* A_∞*-morphisms.*

Definition 6.13. *We define the partial subcategory* $Scyc\widehat{A_{\infty}}_d$ *of* $cyc\widehat{A_{\infty}}_d$ *whose objects are the ones of* $cyc\widehat{A_{\infty}}_d$ *and whose morphisms are strict cyclic morphisms of* $cyc\widehat{A_{\infty}}_d$.

We now summarize the results of Propositions 6.4 and 6.6.

Corollary 6.14. *There exists a functor* $SpCY_d \to Scyc\widehat{A_{\infty}}_d$ *which sends a* d*-pre-Calabi-Yau category* $(\mathcal{A}, s_{d+1}M_\mathcal{A})$ *to the* A_∞*-category* $(\mathcal{A} \oplus \mathcal{A}^*[d-1], sm_{\mathcal{A}\oplus\mathcal{A}^*})$ *defined in Proposition 4.32 and a* d*-pre-Calabi-Yau morphism* $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_\mathcal{A}) \to (\mathcal{B}, s_{d+1}M_\mathcal{B})$ *to the* A_∞*-structure* $sm_{\mathcal{A}\oplus\mathcal{B}^*}$ *defined in Definition 6.3 together with cyclic* A_∞*-morphisms*

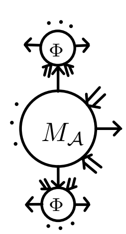
$$\begin{tikzcd}
 & \mathcal{A}[1] \oplus \mathcal{B}^*[d] \arrow[dl, "\varphi_\mathcal{A}"'] \arrow[dr, "\varphi_\mathcal{B}"] & \\
\mathcal{A}[1] \oplus \mathcal{A}^*[d] & & \mathcal{B}[1] \oplus \mathcal{B}^*[d]
\end{tikzcd}$$

given in Definition 6.5.

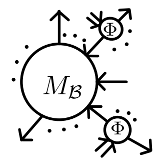
6.2 General case

We now present the relation between not necessarily strict d-pre-Calabi-Yau morphisms and A_∞-morphisms. Consider d-pre-Calabi-Yau categories $(\mathcal{A}, s_{d+1}M_\mathcal{A})$, $(\mathcal{B}, s_{d+1}M_\mathcal{B})$ as well as a $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_\mathcal{A}) \to (\mathcal{B}, s_{d+1}M_\mathcal{B})$ a d-pre-Calabi-Yau morphism as defined in Definition 5.13. We first construct an A_∞-structure on $\mathcal{A} \oplus \mathcal{B}^*[d-1]$.

Definition 6.15. *We define* $M_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{A}} \in \mathscr{B}^{\mathcal{A}}_{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])$ *by* $M^{\bar{\bar{x}}}_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{A}} = \sum \mathcal{E}(\mathscr{D})$ *where the sum is over all the filled diagrams* $\mathscr{D}$ *of type* $\bar{\bar{x}}$ *and of the form*



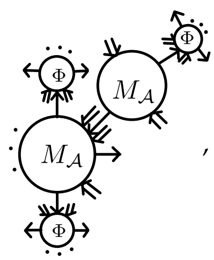
and $M_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{B}^*} \in \mathscr{B}^{\mathcal{B}}_{\bullet}(\mathcal{A}[1],\mathcal{B}[-d])$ by $M^{\bar{\bar{x}}}_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{B}^*} = \sum\mathcal{E}(\mathscr{D}')$ where the sum is over all the filled diagrams $\mathscr{D}'$ of type $\bar{\bar{x}}$ and of the form



By Lemma 5.3, this defines an element $s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*} \in \mathscr{B}_{\bullet}(\mathcal{A}[1],\mathcal{B}[-d])[d+1]$ and we thus define the element $sm_{\mathcal{A}\oplus\mathcal{B}^*} \in C(\mathcal{A}\oplus\mathcal{B}^*[d-1])[1]$ as $sm^{\bar{\bar{x}}}_{\mathcal{A}\oplus\mathcal{B}^*} = \jmath^{\Phi}_{\bar{\bar{x}}^{-1}}(s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*}) \in C(\mathcal{A}\oplus\mathcal{B}^*[d-1])[1]$. We will denote by $m_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{A}}$ (resp. $m_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{B}^*}$) the composition of $m_{\mathcal{A}\oplus\mathcal{B}^*}$ with the canonical projection on $\mathcal{A}$ (resp. on $\mathcal{B}^*[d-1]$).

Proposition 6.16. *The element $sm_{\mathcal{A}\oplus\mathcal{B}^*} \in C(\mathcal{A}\oplus\mathcal{B}^*[d-1])[1]$ defines an A_∞-structure on $\mathcal{A}\oplus\mathcal{B}^*[d-1]$. Moreover, if the morphism Φ is good, $sm_{\mathcal{A}\oplus\mathcal{B}^*}$ satisfies the cyclicity condition* (3.1).

Proof. Using Proposition 5.9, it suffices to show that $s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*} \underset{\Phi\text{ nec}}{\circ} s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*} = 0$. We have that $\pi_{\mathcal{A}}(s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*} \underset{\Phi\text{ nec}}{\circ} s_{d+1}M_{\mathcal{A}\oplus\mathcal{B}^*}) = 0$ is tantamount to $\sum\mathcal{E}(\mathscr{D}) + \sum\mathcal{E}(\mathscr{D}') + \sum\mathcal{E}(\mathscr{D}'') = 0$ where the sums are over all the filled diagrams $\mathscr{D}$, $\mathscr{D}'$ and $\mathscr{D}''$ of type $\bar{\bar{x}}$ of the form



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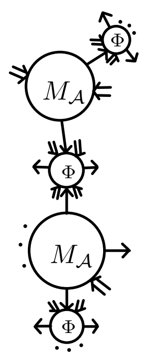
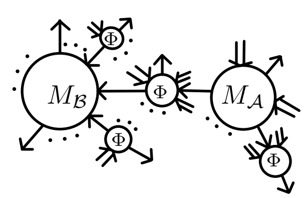
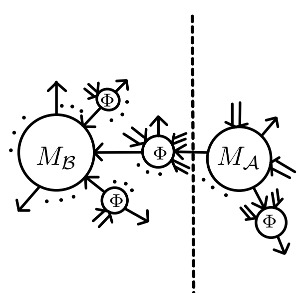


and



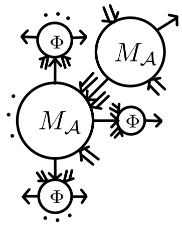
respectively. The third diagram can be cut into two as follows



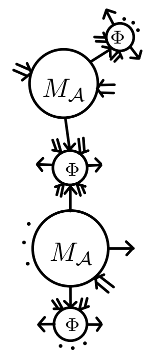
Using that Φ is a pre-Calabi-Yau morphism, the left side can be changed into a diagram consisting of a disc filled with $M_{\mathcal{A}}$ whose outgoing arrows are shared with discs filled with Φ. We thus get that

$$\sum \mathcal{E}(\mathscr{D}'') = \sum \mathcal{E}(\mathscr{D}_1) - \sum \mathcal{E}(\mathscr{D}_2) \tag{6.7}$$

where $\mathscr{D}_1$ and $\mathscr{D}_2$ are filled diagrams of the form



and

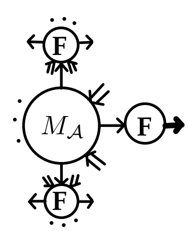


The minus in the identity (6.7) comes from the fact that the discs filled with $M_{\mathcal{A}}$ change their place, in the sense that the order of the labeling of their first outgoing arrow changes. Since $s_{d+1}M_{\mathcal{A}}$ is of degree 1, this create a minus sign. Moreover, $\sum \mathcal{E}(\mathscr{D}') = \sum \mathcal{E}(\mathscr{D}_2)$ so that it remains to show that $\sum \mathcal{E}(\mathscr{D}) + \sum \mathcal{E}(\mathscr{D}_1) = 0$. This is the case since $s_{d+1}M_{\mathcal{A}}$ is a pre-Calabi-Yau structure. Indeed, the sum of these evaluations of diagrams is the composition of

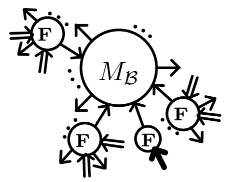
$$s_{d+1}M_{\mathcal{A}} \underset{\text{nec}}{\circ} s_{d+1}M_{\mathcal{A}}$$

with a tensor product composed of maps of the collection Φ and of the identity map id in the last tensor factor. Therefore, the element $sm_{\mathcal{A}\oplus\mathcal{B}^*}$ satisfies the Stasheff identities (SI).

It is clear that if the morphism Φ is good, then the A_∞-structure on $\mathcal{A} \oplus \mathcal{B}^*[d-1]$ is cyclic. Indeed, $\Gamma^\Phi \circ (sm_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}^*}) = \sum \mathcal{E}(\mathscr{D})$ where the sum is over all the filled diagrams $\mathcal{E}(\mathscr{D})$ of the form



On the other hand, $\Gamma^\Phi \circ (sm_{\mathcal{A}\oplus\mathcal{B}^*\to\mathcal{B}^*} \otimes \mathrm{id}_{\mathcal{B}}) = \sum \mathcal{E}(\mathscr{D}')$ where the sum is over all the filled diagrams $\mathcal{E}(\mathscr{D}')$ of the form



which shows the cyclicity of $sm_{\mathcal{A}\oplus\mathcal{B}^*}$. $\square$

Now, we define two morphisms of A_∞-categories $\varphi_{\mathcal{A}} : \mathcal{A}[1] \oplus \mathcal{B}^*[d] \to \mathcal{A}[1] \oplus \mathcal{A}^*[d]$ and $\varphi_{\mathcal{B}} : \mathcal{A}[1] \oplus \mathcal{B}^*[d] \to \mathcal{B}[1] \oplus \mathcal{B}^*[d]$ using the d-pre-Calabi-Yau morphism Φ.

Lemma 6.17. *Consider d-pre-Calabi-Yau categories $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ and $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ as well as a d-pre-Calabi-Yau morphism $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \to (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$. Then, Φ induces morphisms*

$$\phi^{\bar{\bar{x}}}_{\mathcal{A}^*} : \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\bar{x}^i} \otimes {}_{\Phi_0(\mathrm{rt}(\bar{x}^{i+1}))}\mathcal{B}^*[d]_{\Phi_0(\mathrm{lt}(\bar{x}^i))}) \otimes \mathcal{A}[1]^{\bar{x}^n} \to {}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}^*[d]_{\mathrm{rt}(\bar{x}^n)}$$

defined by

$$\begin{aligned}\phi^{\bar{\bar{x}}}_{\mathcal{A}^*}&(s\bar{a}^1, tf^1, \dots, s\bar{a}^{n-1}, tf^{n-1}, s\bar{a}^n)(s_{-d}b) = \\ &(-1)^{\epsilon}((f^{n-1} \circ s_d) \otimes \dots \otimes (f^1 \circ s_d))(\Phi^{\bar{x}^{n-1}, \bar{x}^{n-2}, \dots, \bar{x}^n \sqcup \bar{x}^1}(s\bar{a}^{n-1}, \dots, s\bar{a}^n \otimes sb \otimes s\bar{a}^1))\end{aligned}$$

for $s\bar{a}^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$, $tf^i \in {}_{\Phi_0(\mathrm{rt}(\bar{x}^{i+1}))}\mathcal{B}^*[d]_{\Phi_0(\mathrm{lt}(\bar{x}^i))}$ and $sb \in {}_{\mathrm{rt}(\bar{x}^1)}\mathcal{A}[1]_{\mathrm{lt}(\bar{x}^n)}$ where

$$\begin{aligned}\epsilon = &\sum_{i=1}^{n-1} |tf^i| \sum_{j=i+1}^{n} |s\bar{a}^i| + \sum_{1\leq i<j\leq n} |s\bar{a}^i||s\bar{a}^j| \sum_{1\leq i<j\leq n-1} |tf^i||tf^j| \\ &+ d(n-1) + |s\bar{a}^n|(\sum_{i=1}^{n-1} |s\bar{a}^i| + |sb|) + d\sum_{i=1}^{n-1} |tf^i| + (d+1)\sum_{i=1}^{n} |s\bar{a}^i|\end{aligned}$$

and a morphism

$$\phi^{\bar{\bar{x}}}_{\mathcal{B}} : \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\bar{x}^i} \otimes {}_{\Phi_0(\mathrm{rt}(\bar{x}^{i+1}))}\mathcal{B}^*[d]_{\Phi_0(\mathrm{lt}(\bar{x}^i))}) \otimes \mathcal{A}[1]^{\bar{x}^n} \to {}_{\Phi_0(\mathrm{lt}(\bar{x}^1))}\mathcal{B}[1]_{\Phi_0(\mathrm{rt}(\bar{x}^n))}$$

defined by

$$\begin{aligned}\phi^{\bar{\bar{x}}}_{\mathcal{B}}&(s\bar{a}^1, tf^1, \dots, s\bar{a}^{n-1}, tf^{n-1}, s\bar{a}^n) \\ &= (-1)^{\delta}((f^{n-1} \circ s_d) \otimes \dots \otimes (f^1 \circ s_d) \otimes \mathrm{id})(s_{d+1}\Phi^{\bar{\bar{x}}^{-1}}(s\bar{a}^n, \dots, s\bar{a}^1))\end{aligned}$$

for $s\bar{a}^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$, $tf^i \in {}_{\Phi_0(\mathrm{rt}(\bar{x}^{i+1}))}\mathcal{B}^*[d]_{\Phi_0(\mathrm{lt}(\bar{x}^i))}$ with

$$\delta = \sum_{i=1}^{n-1} |tf^i| \sum_{j=i+1}^{n} |s\bar{a}^i| + \sum_{1\leq i<j\leq n} |s\bar{a}^i||s\bar{a}^j| \sum_{1\leq i<j\leq n-1} |tf^i||tf^j| + d(n-1) + d\sum_{i=1}^{n-1} |tf^i|$$

and for each $n \in \mathbb{N}^*$, $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}^n_{\mathcal{A}}$.

Definition 6.18. *Consider d-pre-Calabi-Yau categories $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ and $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ as well as a d-pre-Calabi-Yau morphism $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \to (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$. We define maps of graded vector spaces $\varphi_{\mathcal{A}} : \mathcal{A}[1] \oplus \mathcal{B}^*[d] \to \mathcal{A}[1] \oplus \mathcal{A}^*[d]$ and $\varphi_{\mathcal{B}} : \mathcal{A}[1] \oplus \mathcal{B}^*[d] \to \mathcal{B}[1] \oplus \mathcal{B}^*[d]$ given by $\varphi_{\mathcal{A}}^{x,y}(sa) = sa$, $\varphi_{\mathcal{B}}^{x,y}(sa) = \Phi^{x,y}(sa)$ for $sa \in {}_x\mathcal{A}[1]_y$ and $\varphi_{\mathcal{A}}^{x,y}(tf) = tf \circ \Phi^{y,x}[d]$, $\varphi_{\mathcal{B}}^{x,y}(tf) = tf$, for $tf \in {}_{\Phi_0(x)}\mathcal{B}^*[d]_{\Phi_0(y)}$, $x, y \in \mathcal{O}_{\mathcal{A}}$, as well as*

$$\pi_{\mathcal{A}^*[d]} \circ \varphi_{\mathcal{A}}^{\bar{\bar{x}}} = \phi_{\mathcal{A}^*}^{\bar{\bar{x}}}, \pi_{\mathcal{A}[1]} \circ \varphi_{\mathcal{A}}^{\bar{\bar{x}}} = 0$$

and

$$\pi_{\mathcal{B}[1]} \circ \varphi_{\mathcal{B}}^{\bar{\bar{x}}} = \phi_{\mathcal{B}}^{\bar{\bar{x}}}, \pi_{\mathcal{B}^*[d]} \circ \varphi_{\mathcal{B}}^{\bar{\bar{x}}} = 0$$

where $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}_{\mathcal{A}}^n$ for $n > 1$ and $\lg(\bar{x}^1) > 2$ if $n = 1$.

Proposition 6.19. *The maps $\varphi_{\mathcal{A}}$ and $\varphi_{\mathcal{B}}$ are morphisms of A_∞-categories.*

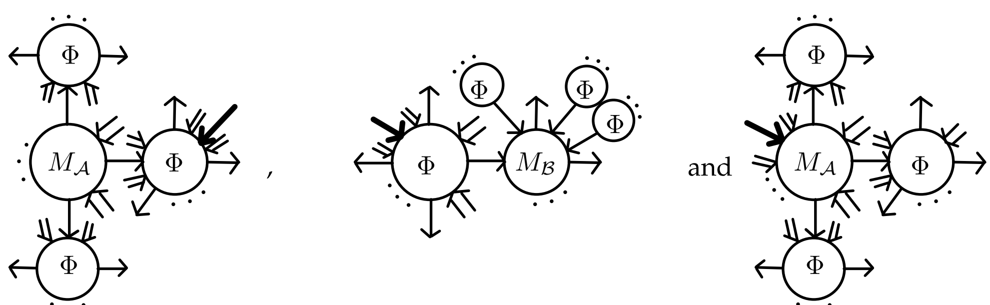
Proof. The part of the identity (MI) for $\varphi_{\mathcal{A}}$ that takes place in

$$\mathcal{H}om_{\Bbbk}\Big(\bigotimes_{i=1}^{n-1}(\mathcal{A}[1]^{\bar{x}^i} \otimes {}_{\Phi_0(\mathrm{rt}(\bar{x}^{i+1}))}\mathcal{B}^*[d]_{\Phi_0(\mathrm{lt}(\bar{x}^i))}) \otimes \mathcal{A}[1]^{\bar{x}^n}, {}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}[1]_{\mathrm{rt}(\bar{x}^n)}\Big)$$

is clearly satisfied. Moreover, by definition of the A_∞-structure $sm_{\mathcal{A}\oplus\mathcal{B}^*}$, the part of the identity (MI) that takes place in

$$\mathcal{H}om_{\Bbbk}\Big(\bigotimes_{i=1}^{n-1}(\mathcal{A}[1]^{\bar{x}^i} \otimes {}_{\Phi_0(\mathrm{rt}(\bar{x}^{i+1}))}\mathcal{B}^*[d]_{\Phi_0(\mathrm{lt}(\bar{x}^i))}) \otimes \mathcal{A}[1]^{\bar{x}^n}, {}_{\mathrm{lt}(\bar{x}^1)}\mathcal{A}^*[d]_{\mathrm{rt}(\bar{x}^n)}\Big)$$

is tantamount to $\sum \mathcal{E}(\mathscr{D}_1) - \sum \mathcal{E}(\mathscr{D}_2) = -\sum \mathcal{E}(\mathscr{D}_3)$ where the sums are over all the filled diagrams $\mathscr{D}_1, \mathscr{D}_2$ and $\mathscr{D}_3$ of type $\bar{\bar{x}}$ of the form



respectively. Using that Φ is a pre-Calabi-Yau morphism, we thus have that $\varphi_{\mathcal{A}}$ is an A_∞-morphism. The case of $\varphi_{\mathcal{B}}$ is similar. □

We now summarize the results of Propositions 6.16 and 6.19.

Corollary 6.20. *There is a functor $pCY_d \to \widehat{A_{\infty}}_d$ sending a d-pre-Calabi-Yau category $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ to the A_∞-category $(\mathcal{A} \oplus \mathcal{A}^*[d-1], sm_{\mathcal{A}\oplus\mathcal{A}^*})$ defined in Proposition 4.32 and a d-pre-Calabi-Yau morphism $\Phi : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \to (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ to the A_∞-structure $sm_{\mathcal{A}\oplus\mathcal{B}^*}$ defined in Definition 6.15 together with the A_∞-morphisms*

$$\begin{tikzcd} & \mathcal{A}[1] \oplus \mathcal{B}^*[d] \arrow[dl, "\varphi_{\mathcal{A}}"'] \arrow[dr, "\varphi_{\mathcal{B}}"] & \\ \mathcal{A}[1] \oplus \mathcal{A}^*[d] & & \mathcal{B}[1] \oplus \mathcal{B}^*[d] \end{tikzcd}$$

defined in Definition 6.18. Moreover, this functor restricts to a functor $NpCY_d \to cyc\widehat{A_{\infty}}_d$ where $NpCY_d$ is the subcategory of pCY_d whose objects are d-pre-Calabi-Yau categories and whose morphisms are nice d-pre-Calabi-Yau morphisms defined in Definition 5.17.

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