



**HAL**  
open science

# Morphisms of pre-Calabi-Yau categories and morphisms of cyclic $A_\infty$ -categories

Marion Boucrot

► **To cite this version:**

Marion Boucrot. Morphisms of pre-Calabi-Yau categories and morphisms of cyclic  $A_\infty$ -categories. 2023. hal-04083732v1

**HAL Id: hal-04083732**

**<https://hal.science/hal-04083732v1>**

Preprint submitted on 27 Apr 2023 (v1), last revised 24 May 2023 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Morphisms of pre-Calabi-Yau categories and morphisms of cyclic $A_\infty$ -categories

Marion Boucrot

---

## Abstract

In this article we prove that there exists a relation between  $d$ -pre-Calabi-morphisms introduced by M. Kontsevich, A. Takeda and Y. Vlassopoulos and cyclic  $A_\infty$ -morphisms, extending a result proved by D. Fernández and E. Herscovich. This leads to a functor between the category of  $d$ -pre-Calabi-Yau structures and the partial category of  $A_\infty$ -categories of the form  $\mathcal{A} \oplus \mathcal{A}^*[d-1]$  with  $\mathcal{A}$  a graded quiver and whose morphisms are the data of an  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$  together with  $A_\infty$ -morphisms  $\mathcal{A}[1] \oplus \mathcal{B}^*[d] \rightarrow \mathcal{A}[1] \oplus \mathcal{A}^*[d]$  and  $\mathcal{A}[1] \oplus \mathcal{B}^*[d] \rightarrow \mathcal{B}[1] \oplus \mathcal{B}^*[d]$ .

**Mathematics subject classification 2020:** 16E45, 18G70, 14A22

**Keywords:**  $A_\infty$ -categories, pre-Calabi-Yau categories

---

## 1 Introduction

Pre-Calabi-Yau algebras were introduced by M. Kontsevich and Y. Vlassopoulos in the last decade. These structures have also appeared under different names, such as  $V_\infty$ -algebras in [10],  $A_\infty$ -algebras with boundary in [7], or weak Calabi-Yau structures in [3] for example. These references show that pre-Calabi-Yau structures play an important role in homological algebra, symplectic geometry, string topology, noncommutative geometry and even in Topological Quantum Field Theory.

In the finite dimensional case, pre-Calabi-Yau algebras are strongly related to  $A_\infty$ -algebras. Actually, for  $d \in \mathbb{Z}$ , a  $d$ -pre-Calabi-Yau structure on a finite dimensional vector space  $A$  is equivalent to a cyclic  $A_\infty$ -structure on  $A \oplus A^*[d-1]$  that restricts to  $A$ . The definition of pre-Calabi-Yau morphisms first appeared in [5] and then in [6], in the properadic setting. A natural question is then about the link between pre-Calabi-Yau morphisms and  $A_\infty$ -morphisms of the corresponding boundary construction.

D. Fernández and E. Herscovich studied this link in [1] at the level of double Poisson dg algebras and a restricted class of pre-Calabi-Yau algebras, when the multiplications  $m_n$  vanish for  $n \geq 4$ . In this paper, we study the relation between  $A_\infty$ -morphisms and pre-Calabi-Yau morphisms in a larger generality. The main result of this paper is the existence of a functor from the category of  $d$ -pre-Calabi-Yau structures  $pCY_d$  to the partial category  $\widehat{A}_{\infty d}$  whose objects are  $A_\infty$ -categories of the form  $\mathcal{A} \oplus \mathcal{A}^*[d-1]$  and whose morphisms are the data of an  $A_\infty$ -structure on

$\mathcal{A} \oplus \mathcal{B}^*[d-1]$  together with a diagram of the form

$$\begin{array}{ccc}
 & \mathcal{A}[1] \oplus \mathcal{B}^*[d] & \\
 \swarrow & & \searrow \\
 \mathcal{A}[1] \oplus \mathcal{A}^*[d] & & \mathcal{B}[1] \oplus \mathcal{B}^*[d]
 \end{array} \tag{1.1}$$

where each of the arrows are  $A_\infty$ -morphisms.

We also show that this functor restricts to a functor between a subcategory of  $pCY_d$  and the partial subcategory of  $\widehat{A}_{\infty d}$  whose objects are those of  $\widehat{A}_{\infty d}$  and whose morphisms are the data of an almost cyclic  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$  together with a diagram of the form (1.1) where the arrows are  $A_\infty$ -morphisms.

Let us briefly present the contents of the article. In Section 2, we fix the notations and conventions we use in this paper and in Section 3, we recall the notions related to  $A_\infty$ -categories. Section 4 is devoted to present the notion of discs and diagrams as well as the notion of pre-Calabi-Yau structures based on the necklace bracket introduced in [5], which is given as the commutator of a necklace product, and their link with  $A_\infty$ -structures in the case of a Hom-finite graded quiver. We incidentally show that the necklace product for a graded quiver  $\mathcal{A}$  is in fact equivalent to the usual Gerstenhaber circle product on  $\mathcal{A} \oplus \mathcal{A}^*[d-1]$  (see Proposition 4.29), which does not seem to have been observed in the literature so far. In Section 5, we recall the definitions of pre-Calabi-Yau morphisms and of the category  $pCY_d$  given in [5].

Section 6 is the core of the article. In Subsection 6.1, we prove that given  $d$ -pre-Calabi-Yau categories  $\mathcal{A}$  and  $\mathcal{B}$  and a strict  $d$ -pre-Calabi-Yau morphism  $\mathcal{A} \rightarrow \mathcal{B}$ , we can produce a cyclic  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$  and a diagram of the form (1.1) whose arrows are cyclic strict  $A_\infty$ -morphisms. We summarize these results in Corollary 6.14. In Subsection 6.2, we prove that given  $d$ -pre-Calabi-Yau categories  $\mathcal{A}$  and  $\mathcal{B}$  and any  $d$ -pre-Calabi-Yau morphism  $\mathcal{A} \rightarrow \mathcal{B}$ , we can produce an  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$  and a diagram of the form (1.1) where the arrows are  $A_\infty$ -morphisms. Moreover, with an additional assumption on the pre-Calabi-Yau morphism  $\mathcal{A} \rightarrow \mathcal{B}$ , the  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$  is almost cyclic. We summarize this in Corollary 6.20 in terms of functors.

**Acknowledgements.** This work is supported by the French National Research Agency in the framework of the ‘‘France 2030’’ program (ANR-15-IDEX-0002) and by the LabEx PERSYVAL-Lab (ANR-11-LABX-0025-01).

## 2 Notations and conventions

In what follows  $\mathbb{k}$  will be a field of characteristic 0 and to simplify we will denote  $\otimes$  for  $\otimes_{\mathbb{k}}$ . We will denote by  $\mathbb{N} = \{0, 1, 2, \dots\}$  the set of natural numbers and we define  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ . For  $i, j \in \mathbb{N}$ , we define the interval of integers  $\llbracket i, j \rrbracket = \{n \in \mathbb{N} \mid i \leq n \leq j\}$ .

Recall that if we have a (cohomologically) graded vector space  $V = \bigoplus_{i \in \mathbb{Z}} V^i$ , we define for  $n \in \mathbb{Z}$  the graded vector space  $V[n]$  given by  $V[n]^i = V^{n+i}$  for  $i \in \mathbb{Z}$  and the map  $s_{V,n} : V \rightarrow V[n]$  whose underlying set theoretic map is the identity. Moreover, if  $f : V \rightarrow W$  is a morphism of graded vector spaces, we define the map  $f[n] : V[n] \rightarrow W[n]$  sending an element  $s_{V,n}(v)$  to  $s_{W,n}(f(v))$  for all  $v \in V$ . We will denote  $s_{V,n}$  simply by  $s_n$  when there is no possible confusion, and  $s_1$  just by  $s$ .

We now recall the Koszul sign rules, that are the ones we use to determine the signs appearing in this paper. If  $V, W$  are graded vector spaces, we have a map  $\tau_{V,W} : V \otimes W \rightarrow W \otimes V$  defined as

$$\tau_{V,W}(v \otimes w) = (-1)^{|w||v|} w \otimes v$$

where  $v \in V$  is a homogeneous element of degree  $|v|$  and  $w \in W$  is a homogeneous element of degree  $|w|$ . More generally, given graded vector spaces  $V_1, \dots, V_n$ , we have a map

$$\tau_{V_1, \dots, V_n} : V_1 \otimes \dots \otimes V_n \rightarrow V_{\sigma^{-1}(1)} \otimes \dots \otimes V_{\sigma^{-1}(n)}$$

defined as

$$\tau_{V_1, \dots, V_n}(v_1 \otimes \dots \otimes v_n) = (-1)^\epsilon (v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(n)})$$

with

$$\epsilon = \sum_{\substack{i>j \\ \sigma^{-1}(i) < \sigma^{-1}(j)}} |v_{\sigma^{-1}(i)}| |v_{\sigma^{-1}(j)}|$$

where  $v_i \in V_i$  is a homogeneous element of degree  $|v_i|$  for  $i \in \llbracket 1, n \rrbracket$ .

Throughout this paper, when we consider an element  $v$  of degree  $|v|$  in a graded vector space  $V$ , we mean a homogeneous element  $v$  of  $V$ . Also, we will denote by  $\text{id}$  the identity map of every space of morphisms, without specifying it. All the products in this paper will be products in the category of graded vector spaces. Given graded vector spaces  $(V_i)_{i \in I}$ , we thus have

$$\prod_{i \in I} V_i = \bigoplus_{n \in \mathbb{Z}} \prod_{i \in I} V_i^n$$

where the second product is the usual product of vector spaces.

Given graded vector spaces  $V, W$  we will denote by  $\text{Hom}_{\mathbb{k}}(V, W)$  the vector space of  $\mathbb{k}$ -linear maps  $f : V \rightarrow W$  and by  $\text{hom}_{\mathbb{k}}^d(V, W)$  the vector space of homogeneous  $\mathbb{k}$ -linear maps  $f : V \rightarrow W$  of degree  $d$ , *i.e.*  $f(v) \in W^{n+d}$  for all  $v \in V^n$ . We assemble them in the graded vector space  $\mathcal{H}om_{\mathbb{k}}(V, W) = \bigoplus_{d \in \mathbb{Z}} \text{hom}_{\mathbb{k}}^d(V, W) \subseteq \text{Hom}_{\mathbb{k}}(V, W)$ . We define the **graded dual** of a graded vector space  $V = \bigoplus_{n \in \mathbb{Z}} V^n$  as the graded vector space  $V^* = \mathcal{H}om_{\mathbb{k}}(V, \mathbb{k})$ . Moreover, given graded vector spaces  $V, V', W, W'$  and homogeneous elements  $f \in \mathcal{H}om_{\mathbb{k}}(V, V')$  and  $g \in \mathcal{H}om_{\mathbb{k}}(W, W')$ , we have that

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w)$$

for homogeneous elements  $v \in V$  and  $w \in W$ . Recall that given graded vector spaces  $V_1, \dots, V_n$  and  $d \in \mathbb{Z}$  we have a homogeneous linear isomorphism of degree 0

$$H_j : \left( \bigotimes_{i=1}^n V_i \right) [d] \rightarrow V_1 \otimes \dots \otimes V_{j-1} \otimes V_j [d] \otimes V_{j+1} \otimes \dots \otimes V_n \quad (2.1)$$

sending an element  $s_d(v_1 \otimes \dots \otimes v_n)$  to  $(-1)^{d(|v_1| + \dots + |v_{j-1}|)} v_1 \otimes \dots \otimes v_{j-1} \otimes s_d v_j \otimes v_{j+1} \otimes \dots \otimes v_n$ . Moreover, given graded vector spaces  $V$  and  $W$  and an integer  $d \in \mathbb{Z}$ , we have homogeneous linear isomorphisms of degree 0

$$\mathcal{H}om_{\mathbb{k}}(V, W) [d] \rightarrow \mathcal{H}om_{\mathbb{k}}(V, W [d]) \quad (2.2)$$

sending  $s_d f \in \mathcal{H}om_{\mathbb{k}}(V, W)[d]$  to the map sending  $v \in V$  to  $s_d(f(v))$  and

$$\mathcal{H}om_{\mathbb{k}}(V, W)[d] \rightarrow \mathcal{H}om_{\mathbb{k}}(V[-d], W) \quad (2.3)$$

sending  $s_d f \in \mathcal{H}om_{\mathbb{k}}(V, W)[d]$  to the map sending  $s_{-d} v \in V[-d]$  to  $(-1)^{d|f|} f(v)$ .

Recall that a graded quiver  $\mathcal{A}$  consists of a set of objects  $\mathcal{O}$  together with graded vector spaces  ${}_y \mathcal{A}_x$  for every  $x, y \in \mathcal{O}$ . A dg quiver  $\mathcal{A}$  is a graded quiver such that  ${}_y \mathcal{A}_x$  is a dg vector space for every  $x, y \in \mathcal{O}$ . Given a quiver  $\mathcal{A}$ , its **enveloping graded quiver** is the graded quiver  $\mathcal{A}^e = \mathcal{A}^{op} \otimes \mathcal{A}$  whose set of objects is  $\mathcal{O} \times \mathcal{O}$  and whose space of morphisms from an object  $(x, y)$  to an object  $(x', y')$  is defined as the graded vector space  $(\mathcal{A}^{op} \otimes \mathcal{A})_{(x, y)(x', y')} = {}_x \mathcal{A}_{x'} \otimes {}_{y'} \mathcal{A}_y$ . A graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$  is said to be **Hom-finite** if  ${}_y \mathcal{A}_x$  is finite dimensional for every  $x, y \in \mathcal{O}$ . Given graded quivers  $\mathcal{A}$  and  $\mathcal{B}$  with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$ , a **morphism of graded quivers**  $(\Phi_0, \Phi) : \mathcal{A} \rightarrow \mathcal{B}$  is the data of a map  $\Phi_0 : \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{B}}$  between the sets of objects together with a collection  $\Phi = ({}_y \Phi_x)_{x, y \in \mathcal{O}_{\mathcal{A}}}$  of morphisms of graded vector spaces  ${}_y \Phi_x : {}_y \mathcal{A}_x \rightarrow {}_{\Phi_0(y)} \mathcal{B}_{\Phi_0(x)}$  for every  $x, y \in \mathcal{O}_{\mathcal{A}}$ . In this paper we will only consider small graded quivers and small categories.

We will denote

$$\bar{\mathcal{O}} = \bigsqcup_{n \in \mathbb{N}^*} \mathcal{O}^n$$

and more generally, we will denote by  $\bar{\bar{\mathcal{O}}}$  the set formed by all finite tuples of elements of  $\bar{\mathcal{O}}$ , i.e.

$$\bar{\bar{\mathcal{O}}} = \bigsqcup_{n \in \mathbb{N}^*} \bar{\mathcal{O}}^n = \bigsqcup_{n > 0} \bigsqcup_{(p_1, \dots, p_n) \in \mathcal{T}_n} \mathcal{O}^{p_1} \times \dots \times \mathcal{O}^{p_n}$$

where  $\mathcal{T}_n = \mathbb{N}^n$  for  $n > 1$  and  $\mathcal{T}_1 = \mathbb{N}^*$ . Given  $\bar{x} = (x_1, \dots, x_n) \in \bar{\mathcal{O}}$  we define its **length** as  $\text{lg}(\bar{x}) = n$ , its **left term** as  $\text{lt}(\bar{x}) = x_1$  and **right term** as  $\text{rt}(\bar{x}) = x_n$ . For  $i \in \llbracket 1, n \rrbracket$ , we define  $\bar{x}_{<i} = (x_1, \dots, x_i)$ ,  $\bar{x}_{\geq i} = (x_i, \dots, x_n)$  and for  $j > i$ ,  $\bar{x}_{\llbracket i, j \rrbracket} = (x_i, x_{i+1}, \dots, x_j)$ . One can similarly define  $\bar{x}_{<i}$  and  $\bar{x}_{>i}$ . Moreover, given  $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\bar{\mathcal{O}}}$  we define its **length** as  $\text{lg}(\bar{\bar{x}}) = n$ , its **left term** as  $\text{lt}(\bar{\bar{x}}) = \bar{x}^1$  and its **right term** as  $\text{rt}(\bar{\bar{x}}) = \bar{x}^n$ . For  $\bar{x} = (x_1, \dots, x_n) \in \bar{\mathcal{O}}$ , we will denote

$$\mathcal{A}^{\otimes \bar{x}} = {}_{x_1} \mathcal{A}_{x_2} \otimes {}_{x_2} \mathcal{A}_{x_3} \otimes \dots \otimes {}_{x_{\text{lg}(\bar{x})-1}} \mathcal{A}_{x_{\text{lg}(\bar{x})}}$$

and we will often denote an element of  $\mathcal{A}^{\otimes \bar{x}}$  as  $a_1, a_2, \dots, a_{\text{lg}(\bar{x})-1}$  instead of  $a_1 \otimes a_2 \otimes \dots \otimes a_{\text{lg}(\bar{x})-1}$  for  $a_i \in {}_{x_i} \mathcal{A}_{x_{i+1}}$ ,  $i \in \llbracket 1, \text{lg}(\bar{x}) - 1 \rrbracket$ . Moreover, given a tuple  $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\bar{\mathcal{O}}}$  we will denote  $\mathcal{A}^{\otimes \bar{\bar{x}}} = \mathcal{A}^{\otimes \bar{x}^1} \otimes \mathcal{A}^{\otimes \bar{x}^2} \otimes \dots \otimes \mathcal{A}^{\otimes \bar{x}^n}$ . Given tuples  $\bar{x} = (x_1, \dots, x_n), \bar{y} = (y_1, \dots, y_m) \in \bar{\mathcal{O}}$ , we define their concatenation as  $\bar{x} \sqcup \bar{y} = (x_1, \dots, x_n, y_1, \dots, y_m)$ . We also define the inverse of a tuple  $\bar{x} = (x_1, \dots, x_n) \in \bar{\mathcal{O}}$  as  $\bar{x}^{-1} = (x_n, x_{n-1}, \dots, x_1)$ . If  $\sigma \in \mathcal{S}_n$  and  $\bar{x} = (x_1, \dots, x_n) \in \bar{\mathcal{O}}^n$ , we define  $\bar{x} \cdot \sigma = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)})$ . Moreover, given  $\sigma \in \mathcal{S}_n$  and  $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\bar{\mathcal{O}}}^n$ , we define  $\bar{\bar{x}} \cdot \sigma = (\bar{x}^{\sigma(1)}, \bar{x}^{\sigma(2)}, \dots, \bar{x}^{\sigma(n)})$ . We denote by  $C_n$  the subgroup of  $\mathcal{S}_n$  generated by the cycle  $\sigma = (12 \dots n)$  which sends  $i \in \llbracket 1, n - 1 \rrbracket$  to  $i + 1$  and  $n$  to 1.

### 3 $A_{\infty}$ -categories

In this section, we recall the notion of (cyclic)  $A_{\infty}$ -categories and (cyclic)  $A_{\infty}$ -morphisms as well as the definition of the natural bilinear form associated to a graded quiver. We also introduce a bilinear form on categories of the form  $\mathcal{A} \oplus \mathcal{B}^*$  where  $\mathcal{A}$  and  $\mathcal{B}$  are graded quivers related by a morphism  $\mathcal{A} \rightarrow \mathcal{B}$ . We refer the reader to [7] for the definitions of modules and bimodules over a category.

**Definition 3.1.** Given a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$ , we define the graded vector space

$$C(\mathcal{A}) = \prod_{p \geq 1} \prod_{\bar{x} \in \mathcal{O}^p} \mathcal{H}om_{\mathbb{k}}(\mathcal{A}[1]^{\otimes \bar{x}}, \text{lt}(\bar{x})\mathcal{A}_{\text{rt}(\bar{x})})$$

Given  $\bar{x} = (x_1, \dots, x_n) \in \mathcal{O}^n$  and a map  $F^{\bar{x}} : \mathcal{A}[1]^{\otimes \bar{x}} \rightarrow \text{lt}(\bar{x})\mathcal{A}_{\text{rt}(\bar{x})}$ , we associate to  $F^{\bar{x}}$  a disc with several incoming arrows and one outgoing arrow (see Figure 3.1).

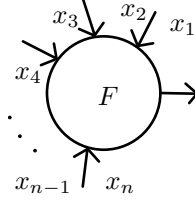


Figure 3.1: A disc representing a map  $F^{\bar{x}} : \mathcal{A}[1]^{\otimes \bar{x}} \rightarrow \text{lt}(\bar{x})\mathcal{A}_{\text{rt}(\bar{x})}$ , where  $\bar{x} = (x_1, \dots, x_n)$

To simplify, we will often omit the objects and assemble the incoming arrows in a big arrow (see Figure 3.2).

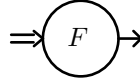


Figure 3.2: The representation of a homogeneous element in  $C(\mathcal{A})$

**Definition 3.2.** The **type** of a disc representing a map  $F^{\bar{x}} : \mathcal{A}[1]^{\otimes \bar{x}} \rightarrow \text{lt}(\bar{x})\mathcal{A}_{\text{rt}(\bar{x})}$  is the tuple  $\bar{x} \in \bar{\mathcal{O}}$ .

**Definition 3.3.** Let  $\mathcal{A}$  be a graded quiver. By the isomorphism (2.2), an element  $sF \in C(\mathcal{A})[1]$  induces maps

$$\mathcal{A}[1]^{\otimes \bar{x}} \rightarrow \text{lt}(\bar{x})\mathcal{A}_{\text{rt}(\bar{x})}[1]$$

sending an element  $(sa_1, \dots, sa_{n-1})$  to  $s(F^{\bar{x}}(sa_1, \dots, sa_{n-1}))$  for  $\bar{x} = (x_1, \dots, x_n) \in \bar{\mathcal{O}}$ ,  $a_i \in x_i\mathcal{A}_{x_{i+1}}$ ,  $i \in [1, n-1]$ . To a homogeneous element  $sF \in C(\mathcal{A})[1]$ , we thus associate disc with a bold outgoing arrow (see Figure 3.3) to indicate that the output of  $sF$  is an element in  $\mathcal{A}[1]$ .

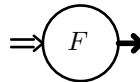
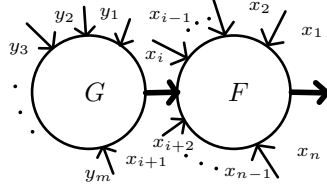


Figure 3.3: The representation of a homogeneous element in  $C(\mathcal{A})[1]$

**Example 3.4.** *The diagram*



is the composition of a disc associated to a map

$$G^{(x_i, y_1, \dots, y_m, x_{i+1})} : \mathcal{A}[1]^{\otimes (x_i, y_1, \dots, y_m, x_{i+1})} \rightarrow {}_{x_i} \mathcal{A}_{x_{i+1}}[1]$$

with a disc associated to a map

$$F^{\bar{x}} : \mathcal{A}[1]^{\otimes \bar{x}} \rightarrow {}_{x_1} \mathcal{A}_{x_n}[1]$$

with  $\bar{x} = (x_1, \dots, x_n)$ . The type of this diagram is  $\bar{x}_{\leq i} \sqcup \bar{y} \sqcup \bar{x}_{> i}$ . This diagram is associated to the map  $F^{\bar{x}} \circ (\text{id}^{\otimes i-1} \otimes G^{(x_i, y_1, \dots, y_m, x_{i+1})} \otimes \text{id}^{\otimes n-i})$ .

**Definition 3.5.** Given a dg bimodule  $M$  over a dg category  $\mathcal{A}$  with set of objects  $\mathcal{O}$ , its **naturalization** is the chain complex  $M_{\text{nat}} = (\bigoplus_{x \in \mathcal{O}} {}_x M_x) / (fg - (-1)^{|g||f|} gf)$  where  $f \in {}_y \mathcal{A}_x$  and  $g \in {}_x M_y$ .

**Definition 3.6.** Given dg category  $\mathcal{A}$  with differential  $d_{\mathcal{A}}$  and product  $\mu$ , we define the dg bimodule  $\text{Bar}(\mathcal{A})$  as  $\text{Bar}(\mathcal{A}) = ({}_{x'} \text{Bar}(\mathcal{A})_x)_{x, x' \in \mathcal{O}}$ , where

$${}_{x'} \text{Bar}(\mathcal{A})_x = \bigoplus_{p \geq 0} \bigoplus_{(x_0, \dots, x_p) \in \mathcal{O}^p} {}_{x'} \mathcal{A}_{x_0} \otimes {}_{x_0} \mathcal{A}_{x_1}[1] \otimes \dots \otimes {}_{x_{p-1}} \mathcal{A}_{x_p}[1] \otimes {}_{x_p} \mathcal{A}_x$$

whose differential restricted to  ${}_{x'} \text{Bar}(\mathcal{A})_x$  is given by  $\sum_{\bar{x} \in \mathcal{O}} d_0^{\bar{x}} + d_1^{\bar{x}}$  where

$$d_0^{\bar{x}}(f_0 \otimes s f_1 \otimes \dots \otimes s f_n \otimes f_{n+1}) = \sum_{i=0}^{n+1} (-1)^{\sum_{j=0}^{i-1} (|f_j| + 1)} f_0 \otimes s f_1 \otimes \dots \otimes s(d_{\mathcal{A}}(f_i)) \otimes \dots \otimes s f_n \otimes f_{n+1}$$

and

$$\begin{aligned} d_1^{\bar{x}}(f_0 \otimes s f_1 \otimes \dots \otimes s f_n \otimes f_{n+1}) &= (-1)^{\epsilon_1} f_0 f_1 \otimes s f_2 \otimes \dots \otimes s f_n \otimes f_{n+1} \\ &\quad + \sum_{i=1}^{n-1} (-1)^{\epsilon_i} f_0 \otimes s f_1 \otimes \dots \otimes s(f_i f_{i+1}) \otimes \dots \otimes s f_n \otimes f_{n+1} \\ &\quad + (-1)^{\epsilon_n} f_0 \otimes s f_1 \otimes \dots \otimes s f_{n-1} \otimes f_n f_{n+1} \end{aligned}$$

for all  $\bar{x} = (x_0, x_1, \dots, x_n)$ ,  $f_0 \in {}_{x'} \mathcal{A}_{x_0}$ ,  $f_{n+1} \in {}_{x_n} \mathcal{A}_x$  and  $f_i \in {}_{x_{i-1}} \mathcal{A}_{x_i}$ ,  $i \in \llbracket 1, n \rrbracket$ , with

$$\epsilon_i = |f_0| + \sum_{j=1}^{i-1} (|f_j| + 1)$$

where we have written  $f_i f_{i+1}$  instead of  $\mu(f_i, f_{i+1})$  for  $i \in \llbracket 0, n \rrbracket$  to denote the composition of  $\mathcal{A}$ .  
The bimodule structure of  $\text{Bar}(\mathcal{A})$  is given by  $\rho = ((x', y')\rho_{(x, y)})_{x, y, x', y' \in \mathcal{O}}$  where

$$(x', y')\rho_{(x, y)} : (x', y')\mathcal{A}_{(x, y)}^e \otimes_y \text{Bar}(\mathcal{A})_x \rightarrow_{y'} \text{Bar}(\mathcal{A})_{x'}$$

sends  $(f \otimes g) \otimes (a \otimes sa_0 \otimes sa_1 \otimes \cdots \otimes sa_n \otimes a')$  to  $(-1)^{|f|(|a|+|b|+\sum_{i=0}^n |sa_i|)} (ga \otimes sa_0 \otimes sa_1 \otimes \cdots \otimes sa_n \otimes a' f)$  for all  $(f \otimes g) \in (x', y')\mathcal{A}_{(x, y)}^e$ ,  $(x_0, \dots, x_{n+1}) \in \tilde{\mathcal{O}}^{n+1}$ ,  $a \in {}_y\mathcal{A}_{x_0}$ ,  $b \in {}_{x_{n+1}}\mathcal{A}_x$  and  $a_i \in {}_{x_i}\mathcal{A}_{x_{i+1}}$  for  $i \in \llbracket 0, n \rrbracket$ . Moreover, in that case, we have a quasi-isomorphism of dg bimodules  $\text{Bar}(\mathcal{A}) \rightarrow \mathcal{A}$  whose restriction to  $\text{Bar}(\mathcal{A})_p$  vanishes for  $p \geq 1$  and whose restriction to  $\text{Bar}(\mathcal{A})_0$  is  $\mu$ .

**Definition 3.7.** Given a dg category  $\mathcal{A}$ , the **dual bimodule** of the dg bimodule  $\text{Bar}(\mathcal{A})$  is given by  $\text{Bar}(\mathcal{A})^\vee = \prod_{x, y \in \mathcal{O}} {}_y \text{Bar}(\mathcal{A})_x^\vee$ , with

$${}_y \text{Bar}(\mathcal{A})_x^\vee = \text{Hom}_{-\mathcal{A}^e}({}_{y'} \text{Bar}(\mathcal{A})_{x'}, {}_{y'} \mathcal{A}_x \otimes_y \mathcal{A}_{x'})$$

where the subscript  $-\mathcal{A}^e$  of the  $\text{Hom}$  indicates that we consider the space of morphisms of right  $\mathcal{A}^e$ -modules. In particular,  ${}_y \text{Bar}(\mathcal{A})_x^\vee$  is a right graded  $\mathcal{A}^e$ -module for each  $x, y \in \mathcal{O}$  for the left  $\mathcal{A}^e$ -structure of  $\mathcal{A}^e$  which is the inner one.

More precisely,  $\text{Bar}(\mathcal{A})$  is a  $\mathcal{A}$ -bimodule with the action given by  $\rho = ((x'', y'')\rho_{(x, y)})_{x, y, x'', y'' \in \mathcal{O}}$  where

$$(x'', y'')\rho_{(x, y)} : (x'', y'')\mathcal{A}_{(x, y)}^e \otimes_y \text{Bar}(\mathcal{A})_x^\vee \rightarrow_{y''} \text{Bar}(\mathcal{A})_{x''}^\vee$$

sends  $(f \otimes g) \otimes {}_{y'} \Phi_x^{x'}$  to the map  ${}_{y''} \Psi_{x''}^{x'} : {}_{y''} \text{Bar}(\mathcal{A})_{x''}^\vee \rightarrow {}_{y''} \mathcal{A}_{x''} \otimes_{y''} \mathcal{A}_{x'}$  defined by

$${}_{y''} \Psi_{x''}^{x'}(a \otimes sa_0 \otimes sa_1 \otimes \cdots \otimes sa_n \otimes a') = (-1)^{(|f|+|g|)|\Phi^{(1)}|} \Phi^{(1)} f \otimes g \Phi^{(2)}$$

where we have written  ${}_{y'} \Phi_x^{x'}(a \otimes sa_0 \otimes sa_1 \otimes \cdots \otimes sa_n \otimes a')$  as a tensor product  $\Phi^{(1)} \otimes \Phi^{(2)}$ , for all  $(f \otimes g) \in (x', y')\mathcal{A}_{(x, y)}^e$ ,  $(x_0, \dots, x_{n+1}) \in \tilde{\mathcal{O}}^{n+1}$ ,  $a \in {}_y\mathcal{A}_{x_0}$ ,  $a' \in {}_{x_{n+1}}\mathcal{A}_x$  and  $a_i \in {}_{x_i}\mathcal{A}_{x_{i+1}}$  for  $i \in \llbracket 0, n \rrbracket$ .

**Remark 3.8.** If  $\mathcal{A}$  is a dg category, we have a map

$$\text{Bar}(\mathcal{A})_{nat}^\vee \rightarrow \prod_{p \geq 1} \prod_{\bar{x} \in \mathcal{O}^p} \text{Hom}_{\mathbb{k}}(\mathcal{A}[1]^{\otimes \bar{x}}, \text{lt}(\bar{x})\mathcal{A}_{\text{rt}(\bar{x})})$$

sending  ${}^z \Phi_x^z \in \text{Hom}_{-\mathcal{A}^e}({}_z \text{Bar}(\mathcal{A})_z, {}_z \mathcal{A}_x \otimes_y \mathcal{A}_z)$  to the collection of  $\mathbb{k}$ -linear maps

$$\Psi : \bigoplus_{p \geq 0} \bigoplus_{(x_0, \dots, x_p) \in \mathcal{O}^p} {}_{x_0} \mathcal{A}_{x_1}[1] \otimes \cdots \otimes {}_{x_{p-1}} \mathcal{A}_{x_p}[1] \rightarrow {}_{x_0} \mathcal{A}_{x_p}$$

given by  $\Psi(sa_0, \dots, sa_{p-1}) = \mu(\tau \circ {}_{x_0} \Phi_{x_p}^{x_0}(1_{x_0}, sa_0, \dots, sa_{p-1}, 1_{x_0}))$  for  $a_i \in {}_{x_i} \mathcal{A}_{x_{i+1}}$  and where  $1_{x_0}$  is the identity of  ${}_{x_0} \mathcal{A}_{x_0}$ . Moreover, this map is an isomorphism of graded vector spaces.

**Definition 3.9.** Given a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$ , the **Gerstenhaber product** of elements  $s\mathbf{F}, s\mathbf{G} \in C(\mathcal{A})[1]$  is defined as the element  $s\mathbf{F} \circ_G s\mathbf{G} \in C(\mathcal{A})[1]$  given by

$$(s\mathbf{F} \circ_G s\mathbf{G})^{\bar{x}} = \sum_{1 \leq i < j \leq \text{lg}(\bar{x})} sF^{\bar{x} \leq i \sqcup \bar{x} \geq j} \circ_{G, i, j} sG^{\bar{x} \llbracket i, j \rrbracket}$$



for  $\bar{x} \in \bar{\mathcal{O}}$  where

$$\begin{aligned} & (sF^{\bar{x} \leq i \sqcup \bar{x} \geq j} \circ_{G, i, j} sG^{\bar{x} \llbracket i, j \rrbracket}) (sa_1, \dots, sa_{n-1}) \\ &= (-1)^{\epsilon_i} sF^{\bar{x} \leq i \sqcup \bar{x} \geq j} (sa_1, \dots, sa_{i-1}, sG^{\bar{x} \llbracket i, j \rrbracket} (sa_i, \dots, sa_{j-1}), sa_j, \dots, sa_{n-1}) \end{aligned}$$

for  $a_i \in {}_{x_i} \mathcal{A}_{x_{i+1}}$ , with  $\epsilon = (|\mathbf{G}| + 1) \sum_{r=1}^{i-1} |sa_r|$ .

The map  $(sF \circ_G sG)^{\bar{x}}$  is by definition the sum of the maps associated to diagrams of type  $\bar{x}$  and of the form



**Definition 3.10.** Given a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$ , the **Gerstenhaber bracket** is the graded Lie bracket  $[-, -]_G$  on  $C(\mathcal{A})[1]$  defined for elements  $sF, sG \in C(\mathcal{A})[1]$  as the element  $[sF, sG]_G \in C(\mathcal{A})[1]$  given by

$$[sF, sG]_G^{\bar{x}} = (sF \circ_G sG)^{\bar{x}} - (-1)^{(|F|+1)(|G|+1)} (sG \circ_G sF)^{\bar{x}}$$

for  $\bar{x} \in \bar{\mathcal{O}}$ .

**Definition 3.11.** An  $A_\infty$ -**structure** on a graded quiver  $\mathcal{A}$  is a homogeneous element  $sm_{\mathcal{A}} \in C(\mathcal{A})[1]$  of degree 1 satisfying the Maurer–Cartan equation  $[sm_{\mathcal{A}}, sm_{\mathcal{A}}]_G = 0$ .

**Remark 3.12.** An  $A_\infty$ -structure on a graded quiver  $\mathcal{A}$  is tantamount to the data of a homogeneous element  $sm_{\mathcal{A}} \in C(\mathcal{A})[1]$  satisfying the following identities

$$\sum_{1 \leq i < j \leq n} sm_{\mathcal{A}}^{\bar{x} \leq i \sqcup \bar{x} \geq j} \circ (\text{id}^{\otimes \bar{x} \leq i} \otimes sm_{\mathcal{A}}^{\bar{x} \llbracket i, j \rrbracket} \otimes \text{id}^{\otimes \bar{x} \geq j}) = 0 \quad (\text{SI})$$

for every  $n \in \mathbb{N}^*$  and  $\bar{x} \in \mathcal{O}^n$ . These are called the **Stasheff identities** and were first introduced in [8] by J. Stasheff.

**Example 3.13.** If  $\mathcal{A}$  is a dg category with differential  $d_{\mathcal{A}}$  and product  $\mu$ , it carries a natural  $A_\infty$ -structure  $sm_{\mathcal{A}} \in C(\mathcal{A})[1]$  with  $m_{\mathcal{A}}^1 = d_{\mathcal{A}}$ ,  $m_{\mathcal{A}}^2 = \mu$  and  $m_{\mathcal{A}}^n = 0$  for  $n \geq 3$ .

**Definition 3.14.** Given a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$  its **graded dual quiver** is the quiver  $\mathcal{A}^*$  whose set of objects is  $\mathcal{O}$  and for  $x, y \in \mathcal{O}$ , the space of morphisms from  $x$  to  $y$  is defined as  ${}_y \mathcal{A}_x^* = ({}_x \mathcal{A}_y)^*$ .

**Definition 3.15.** A **bilinear form of degree  $d$**  on a graded quiver  $\mathcal{A}$  is a collection  $\Gamma = ({}_y \Gamma_x)_{x, y \in \mathcal{O}}$  of homogeneous  $\mathbb{k}$ -linear maps  ${}_y \Gamma_x : {}_y \mathcal{A}_x[1] \otimes {}_x \mathcal{A}_y[1] \rightarrow \mathbb{k}$  of degree  $d + 2$ .

**Definition 3.16.** A bilinear form  $\Gamma$  on a graded quiver  $\mathcal{A}$  is **nondegenerate** if the induced map

$${}_y \mathcal{A}_x[1] \rightarrow ({}_y \mathcal{A}_x[1])^*$$

sending an element  $sa \in {}_y \mathcal{A}_x[1]$  to the map sending  $sb \in {}_x \mathcal{A}_y[1]$  to  ${}_y \Gamma_x(sa, sb)$  is an isomorphism.

**Example 3.17.** Consider two graded quivers  $\mathcal{A}$  and  $\mathcal{B}$  and a morphism  $(\Phi_0, \Phi) : \mathcal{A} \rightarrow \mathcal{B}$  of graded quivers. Define the bilinear form  $\Gamma^\Phi : (\mathcal{A}[1] \oplus \mathcal{B}^*[d])^{\otimes 2} \rightarrow \mathbb{k}$  of degree  $d + 2$  by

$${}_y\Gamma_x^\Phi(tf, sa) = -(-1)^{|sa||tf|} {}_x\Gamma_y^\Phi(sa, tf) = (-1)^{|tf|+1} (f \circ {}_x\Phi_y)(a)$$

for  $f \in {}_{\Phi_0(y)}\mathcal{B}_{\Phi_0(x)}^*$ ,  $a \in {}_x\mathcal{A}_y$ , where  $t$  stands for the shift morphism  $\mathcal{B}^* \rightarrow \mathcal{B}^*[d]$  and

$${}_y\Gamma_x^\Phi(tf, tg) = {}_y\Gamma_x^\Phi(sa, sb) = 0$$

for  $f \in {}_{\Phi_0(y)}\mathcal{B}_{\Phi_0(x)}^*$ ,  $g \in {}_{\Phi_0(x)}\mathcal{B}_{\Phi_0(y)}^*$ ,  $a \in {}_y\mathcal{A}_x$  and  $b \in {}_x\mathcal{A}_y$ . This bilinear form  $\Gamma^\Phi$  will be called the  **$\Phi$ -mixed bilinear form**.

**Example 3.18.** If  $\mathcal{B} = \mathcal{A}$ , the bilinear form  $\Gamma^{\text{id}}$  of the previous example is called the **natural bilinear form** on  $\mathcal{A}$  and will be denoted  $\Gamma^{\mathcal{A}}$ .

**Remark 3.19.** The natural bilinear form on a Hom-finite graded quiver  $\mathcal{A}$  is nondegenerate, whereas the  $\Phi$ -mixed bilinear form is not in general.

**Definition 3.20.** An  $A_\infty$ -structure  $sm_{\mathcal{A}} \in \mathcal{C}(\mathcal{A})[1]$  on a graded quiver  $\mathcal{A}$  is **almost cyclic** with respect to a homogeneous bilinear form  $\Gamma : \mathcal{A}[1]^{\otimes 2} \rightarrow \mathbb{k}$  if the following holds:

$$\begin{aligned} & {}_{x_n}\Gamma_{x_1}(sm_{\mathcal{A}}^{\bar{x}}(sa_1, \dots, sa_{n-1}), sa_n) \\ &= (-1)^{|sa_n|(\sum_{i=1}^{n-1} |sa_i|)} {}_{x_{n-1}}\Gamma_{x_n}(sm_{\mathcal{A}}^{\bar{x} \cdot \sigma^{-1}}(sa_n, sa_1, \dots, sa_{n-2}), sa_{n-1}) \end{aligned} \quad (3.1)$$

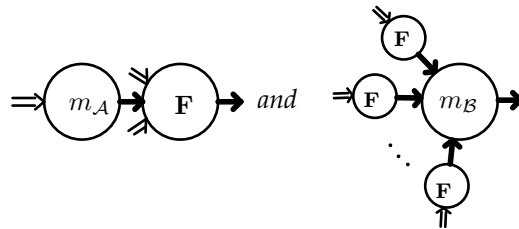
for each  $n \in \mathbb{N}^*$ ,  $\bar{x} = (x_1, \dots, x_n) \in \bar{\mathcal{O}}$ ,  $\sigma = (12 \dots n)$  with  $a_i \in {}_{x_i}\mathcal{A}_{x_{i+1}}$  for  $i \in \llbracket 1, n-1 \rrbracket$  and  $a_n \in {}_{x_n}\mathcal{A}_{x_1}$ . An **almost cyclic**  $A_\infty$ -category is an  $A_\infty$ -category whose  $A_\infty$ -structure is almost cyclic with respect to a fixed homogeneous bilinear form. An almost cyclic  $A_\infty$ -category with respect to a nondegenerate bilinear form is called a **cyclic**  $A_\infty$ -category.

The following definition was first introduced by M. Sugawara in [9].

**Definition 3.21.** An  $A_\infty$ -**morphism** between  $A_\infty$ -categories  $(\mathcal{A}, sm_{\mathcal{A}})$  and  $(\mathcal{B}, sm_{\mathcal{B}})$  with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  is a map  $F_0 : \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{B}}$  together with a collection  $\mathbf{F} = (F^{\bar{x}})_{\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}}$ , where  $F^{\bar{x}} : \mathcal{A}[1]^{\otimes \bar{x}} \rightarrow {}_{F_0(\text{lt}(\bar{x}))}\mathcal{B}_{F_0(\text{rt}(\bar{x}))}[1]$  is a map of degree 0, that satisfies

$$\begin{aligned} & \sum_{1 \leq i < j \leq \text{lg}(\bar{x})} F^{\bar{x} \leq i \sqcup \bar{x} \geq j} \circ (\text{id}^{\otimes \bar{x} \leq i} \otimes sm_{\mathcal{A}}^{\bar{x} \llbracket i, j \rrbracket} \otimes \text{id}^{\otimes \bar{x} \geq j}) \\ &= \sum_{1 \leq i_1 < \dots < i_n \leq \text{lg}(\bar{x})} sm_{\mathcal{B}}^{\bar{y}}(F^{\bar{x} \leq i_1} \otimes F^{\bar{x} \llbracket i_1, i_2 \rrbracket} \otimes \dots \otimes F^{\bar{x} \llbracket i_n, \text{lg}(\bar{x}) \rrbracket}) \end{aligned} \quad (\text{MI})$$

for every  $\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}$  and with  $\bar{y} = (F_0(x_1), F_0(x_{i_1}), \dots, F_0(x_{\text{lg}(\bar{x})}))$ . Note that given  $\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}$  the terms in both sums are sums of maps associated with diagrams of type  $\bar{x}$  that are respectively of the form



The following definition was introduced in [2] by H. Kajiura in the cyclic case.

**Definition 3.22.** An  $A_\infty$ -morphism  $(F_0, \mathbf{F})$  between almost cyclic  $A_\infty$ -categories  $(\mathcal{A}, sm_{\mathcal{A}})$  and  $(\mathcal{B}, sm_{\mathcal{B}})$  with respect to bilinear forms  $\gamma$  and  $\Gamma$  is **cyclic** if

$${}_{F_0(y)}\Gamma_{F_0(x)}(F^1(sa), F^1(sb)) = {}_y\gamma_x(sa, sb)$$

for  $x, y \in \mathcal{O}$ ,  $a \in {}_y\mathcal{A}_x$  and  $b \in {}_x\mathcal{A}_y$  and for  $n \geq 3$

$$\sum_{\substack{\bar{x} \in Z \\ \bar{y} \in Z'}} \text{lt}(\bar{z})\Gamma_{\text{rt}(\bar{z})}(F^{\bar{x}}(sa_1, \dots, sa_i), F^{\bar{y}}(sa_{i+1}, \dots, sa_n)) = 0$$

for  $\bar{z} \in \bar{\mathcal{O}}$ ,  $(sa_1, \dots, sa_i) \in \mathcal{A}[1]^{\otimes \bar{x}}$  and  $(sa_{i+1}, \dots, sa_n) \in \mathcal{A}[1]^{\otimes \bar{y}}$  where

$$Z = \{\bar{x} \in \bar{\mathcal{O}} \mid \text{lt}(\bar{x}) = \text{lt}(\bar{z}), \text{rt}(\bar{x}) = \text{rt}(\bar{z})\} \text{ and } Z' = \{\bar{y} \in \bar{\mathcal{O}} \mid \text{lt}(\bar{y}) = \text{rt}(\bar{z}), \text{rt}(\bar{y}) = \text{lt}(\bar{z})\}$$

## 4 Pre-Calabi-Yau categories

In this section, we present the diagrammatic calculus and recall the definition of  $d$ -pre-Calabi-Yau structures,  $d \in \mathbb{Z}$ , appearing in [5] and [11] as well as their relation with  $A_\infty$ -structures when the graded quiver considered is Hom-finite.

### 4.1 Diagrammatic calculus

In this subsection, we define discs and diagrams and we explain how to evaluate and compose them.

Following [11] we define the following graded vector space.

**Definition 4.1.** Given a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$ , we define the graded vector space

$$\text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d]) = \prod_{n \in \mathbb{N}^*} \text{Multi}^n(\text{Bar}(\mathcal{A})[d]) = \prod_{n \in \mathbb{N}^*} \prod_{\bar{x} \in \bar{\mathcal{O}}^n} \text{Multi}^{\bar{x}}(\text{Bar}(\mathcal{A})[d])$$

where  $\text{Multi}^{\bar{x}}(\text{Bar}(\mathcal{A})[d])$  is the graded vector space consisting of sums of homogeneous  $\mathbb{k}$ -linear maps of the form

$$F^{\bar{x}^1, \dots, \bar{x}^n} : \mathcal{A}[1]^{\otimes \bar{x}^1} \otimes \mathcal{A}[1]^{\otimes \bar{x}^2} \otimes \dots \otimes \mathcal{A}[1]^{\otimes \bar{x}^n} \rightarrow \text{lt}(\bar{x}^1)\mathcal{A}_{\text{rt}(\bar{x}^2)}[-d] \otimes \text{lt}(\bar{x}^2)\mathcal{A}_{\text{rt}(\bar{x}^3)}[-d] \otimes \dots \otimes \text{lt}(\bar{x}^n)\mathcal{A}_{\text{rt}(\bar{x}^1)}[-d]$$

for  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ .

The action of  $\tau = (\tau_n)_{n \in \mathbb{N}^*} \in \prod_{n \in \mathbb{N}^*} C_n$  on an element  $\mathbf{F} = (F^{\bar{x}})_{\bar{x} \in \bar{\mathcal{O}}} \in \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])$  is the element  $\tau \cdot \mathbf{F} \in \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])$  given by

$$(\tau \cdot \mathbf{F})^{\bar{x}} = \tau_{\text{lt}(\bar{x}^1)\mathcal{A}_{\text{rt}(\bar{x}^2)}[-d], \text{lt}(\bar{x}^2)\mathcal{A}_{\text{rt}(\bar{x}^3)}[-d], \dots, \text{lt}(\bar{x}^n)\mathcal{A}_{\text{rt}(\bar{x}^1)}[-d]}^{-1} \circ F^{\bar{x} \cdot \tau} \circ \tau_{\mathcal{A}[1]^{\otimes \bar{x}^1}, \mathcal{A}[1]^{\otimes \bar{x}^2}, \dots, \mathcal{A}[1]^{\otimes \bar{x}^n}}$$

for  $\bar{x} \in \bar{\mathcal{O}}$ . We will denote by  $\text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])^{C_{\text{lg}(\bullet)}}$  the space of elements of  $\text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])$  that are invariant under the action of  $\prod_{n \in \mathbb{N}^*} C_n$ .

**Remark 4.2.** If  $\mathcal{A}$  is a dg category,  $\text{Multi}^n(\text{Bar}(\mathcal{A})[d])$  is the dg vector space

$$\text{Hom}_{(\mathcal{A}^e)^{\otimes n}}(\text{Bar}(\mathcal{A})[d], \text{id}(\mathcal{A}^{\otimes n})_\sigma)$$

where  $\text{id}(\mathcal{A}^{\otimes n})_\sigma$  denotes the  $\mathcal{A}^{\otimes n}$ -bimodule  $\mathcal{A}^{\otimes n}$  whose structure is given by

$${}_{(y_1, \dots, y_n)}(\mathcal{A}^{\otimes n})_{(x_1, \dots, x_n)} = \bigotimes_{i=1}^n {}_{y_i} \mathcal{A}_{x_{i+1}}$$

with the convention that  $x_{n+1} = x_1$ .

The action on morphisms is given by

$$(g_1 \otimes \dots \otimes g_n) \bullet \omega \bullet (f_1 \otimes \dots \otimes f_n) = (-1)^{|f_1|(|f_2| + \dots + |f_n|)} (g_1 \otimes \dots \otimes g_n) \cdot \omega \cdot (f_n \otimes f_2 \otimes \dots \otimes f_{n-1} \otimes f_1)$$

for objects  $x_i, x'_i, y_i, y'_i \in \mathcal{O}$ ,  $i \in \llbracket 1, n \rrbracket$ ,  $\omega \in \bigotimes_{i=1}^n {}_{y_i} \mathcal{A}_{x_{i+1}}$ ,  $f_i \in {}_{x_i} \mathcal{A}_{x'_i}$  and  $g_i \in {}_{y_i} \mathcal{A}_{x'_i}$  where  $\cdot$  denotes the usual bimodule structure of  $\text{Hom}_{(\mathcal{A}^e)^{\otimes n}}(\text{Bar}(\mathcal{A})[d], \mathcal{A}^{\otimes n})$ .

**Definition 4.3.** A **disc**  $D$  is a circle with distinguished set of points which are either incoming or outgoing points. An incoming (resp. outgoing) point will be pictured as an incoming (resp. outgoing) arrow (see Figure 4.1). The **size** of the disc  $D$  is the number of outgoing arrows, and it will be denoted by  $|D|$ .

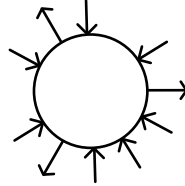


Figure 4.1: A disc of size 3

**Definition 4.4.** Given a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$ , a **decorated disc of size  $n$**  is a disc of size  $n$  together with a clockwise labeling of the outgoing arrows from 1 to  $n$  and a distinguished object of  $\mathcal{O}$  between any couple of consecutive arrows. Given an arrow  $\alpha$  of the disc  $D$ , we will write it  $\alpha = {}_y \alpha_x$  with  $x, y \in \mathcal{O}$  to indicate that  $x$  clockwise precedes the arrow  $\alpha$  in  $D$  and  $\alpha$  clockwise precedes  $y$  in  $D$ . For instance, the outgoing arrow labeled by 3 in Figure 4.2 might be denoted by  ${}_{x_3} \alpha_{x_1^3}$ .

The **type** of the decorated disc is the tuple of the form  $(\bar{x}^1, \dots, \bar{x}^n)$  where  $\bar{x}^i$  is the tuple formed by objects of  $\mathcal{O}$ , read in counterclockwise order, between the outgoing arrows  $i - 1$  and  $i$ , with the convention that the arrow 0 is the arrow  $n$  (see Figure 4.2).

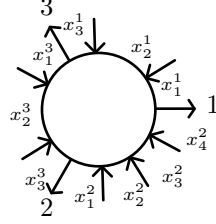


Figure 4.2: A decorated disc of type  $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$  where  $\bar{x}^1 = (x_1^1, x_2^1, x_3^1)$ ,  $\bar{x}^2 = (x_1^2, x_2^2, x_3^2, x_4^2)$  and  $\bar{x}^3 = (x_1^3, x_2^3, x_3^3)$

**Definition 4.5.** Given  $\bar{x} \in \bar{\mathcal{O}}$  we associate to a map  $F^{\bar{x}} \in \text{Multi}^{\bar{x}}(\text{Bar}(\mathcal{A}[d]))$  the unique decorated diagram of type  $\bar{x}$ .

**Definition 4.6.** A **marked** disc is a decorated disc with a bold arrow (see Figure 4.3).

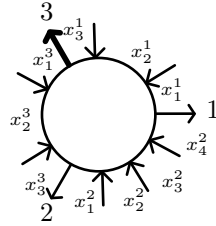


Figure 4.3: A marked disc

**Definition 4.7.** Consider  $s_{d+1}F^{\bar{x}} \in \text{Multi}^{\bar{x}}(\text{Bar}(\mathcal{A})[d])[d+1]$  with  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}^n$ . Given  $(a, b) \in (\{i\} \times \llbracket 1, \text{lg}(\bar{x}^1) + \dots + \text{lg}(\bar{x}^n) \rrbracket) \sqcup (\{o\} \times \llbracket 1, n \rrbracket)$ , by the isomorphisms (2.1), (2.2) and (2.3),  $s_{d+1}F^{\bar{x}}$  induces a morphism of the form

$$\begin{aligned} & \mathcal{A}[1]^{\otimes \bar{x}^1} \otimes \dots \otimes \mathcal{A}[1]^{\otimes \bar{x}^{j-1}} \otimes \mathcal{A}[1]^{\otimes \bar{x}^j \leq b'} \otimes_{x_{b'}^j} \mathcal{A}[-d]_{x_{b'+1}^j} \otimes \mathcal{A}[1]^{\otimes \bar{x}^j > b'} \otimes \mathcal{A}[1]^{\otimes \bar{x}^n} \\ & \rightarrow \text{lt}_{(\bar{x}^1)} \mathcal{A}_{\text{rt}(\bar{x}^2)}[-d] \otimes \dots \otimes \text{lt}_{(\bar{x}^n)} \mathcal{A}_{\text{rt}(\bar{x}^1)}[-d] \end{aligned} \quad (4.1)$$

given by  $(-1)^{(d+1)|F^{\bar{x}}|} F^{\bar{x}} \circ (\text{id}^{\otimes (\text{lg}(\bar{x}^1) + \dots + \text{lg}(\bar{x}^{j-1}) + b' - j)} \otimes_{s_{d-1}} \text{id}^{\otimes (\text{lg}(\bar{x}^j) - b' + \text{lg}(\bar{x}^{j+1}) + \dots + \text{lg}(\bar{x}^n) - n + j)})$  if  $a = i$  and  $b = \text{lg}(\bar{x}^1) + \dots + \text{lg}(\bar{x}^j) + b'$  with  $j \in \llbracket 1, n \rrbracket$ ,  $b' \in \llbracket 1, \text{lg}(\bar{x}^j) - 1 \rrbracket$ , and a morphism of the form

$$\begin{aligned} & \mathcal{A}[1]^{\otimes \bar{x}^1} \otimes \mathcal{A}[1]^{\otimes \bar{x}^2} \otimes \dots \otimes \mathcal{A}[1]^{\otimes \bar{x}^n} \\ & \rightarrow \text{lt}_{(\bar{x}^1)} \mathcal{A}_{\text{rt}(\bar{x}^2)}[-d] \otimes \dots \otimes \text{lt}_{(\bar{x}^{b-1})} \mathcal{A}_{\text{rt}(\bar{x}^b)}[-d] \otimes \text{lt}_{(\bar{x}^b)} \mathcal{A}_{\text{rt}(\bar{x}^{b+1})}[1] \otimes \dots \otimes \text{lt}_{(\bar{x}^n)} \mathcal{A}_{\text{rt}(\bar{x}^1)}[-d] \end{aligned} \quad (4.2)$$

given by  $(\text{id}^{\otimes (b-1)} \otimes_{s_{d+1}} \text{id}^{\otimes (n-b)}) \circ F^{\bar{x}}$  if  $a = o$ . To  $s_{d+1}F^{\bar{x}}$  and  $(a, b)$  as before, we associate the marked diagram which is decorated as the one of  $F^{\bar{x}}$ , where the bold arrow is the  $b$ -th incoming arrow if  $a = i$  and the  $b$ -th outgoing arrow if  $a = o$  where the incoming arrows are numbered in clockwise order, the

first being the incoming arrow following the last outgoing one. Given a disc  $D$  of type  $\bar{x}$ , we will denote by  $\mathcal{E}(D, s_{d+1}F^{\bar{x}})$  the map (4.1) if  $a = i$  and (4.2) if  $a = o$ .

**Example 4.8.** Given the marked disc in Figure 4.3 and an element  $s_{d+1}\mathbf{F} \in \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])[d+1]$ , the evaluation of  $\mathcal{E}(D, s_{d+1}F^{\bar{x}})$  at elements  $(\bar{s}a^1, \bar{s}a^2, \bar{s}a^3)$  is obtained by first compute  $F^{\bar{x}}(\bar{s}a^1, \bar{s}a^2, \bar{s}a^3) = \text{lt}(\bar{x}^1)F_{\text{rt}(\bar{x}^2)}^{-d} \otimes \text{lt}(\bar{x}^2)F_{\text{rt}(\bar{x}^3)}^{-d} \otimes \text{lt}(\bar{x}^3)F_{\text{rt}(\bar{x}^3)}^{-d}$ , and then apply the shift  $s_{d+1}$  to the third tensor factor of the result since the bold arrow is the third outgoing arrow of the disc. The result is then

$$(-1)^\epsilon \text{lt}(\bar{x}^1)F_{\text{rt}(\bar{x}^2)}^{-d} \otimes \text{lt}(\bar{x}^2)F_{\text{rt}(\bar{x}^3)}^{-d} \otimes \text{lt}(\bar{x}^3)F_{\text{rt}(\bar{x}^3)}^1$$

with  $\epsilon = (d+1)(|\text{lt}(\bar{x}^1)F_{\text{rt}(\bar{x}^2)}^{-d}| + |\text{lt}(\bar{x}^2)F_{\text{rt}(\bar{x}^3)}^{-d}|)$ .

To simplify, from now on we will omit the objects when drawing a marked disc as well as the label of the outgoing arrows. By convention, if the bold arrow of the marked disc (of size  $n$ ) is outgoing, it denotes the  $n$ -th outgoing arrow, and if the bold arrow is incoming, then the clockwise preceding outgoing arrow of the marked disc is the one labeled by  $n$ . Moreover, we will draw a big incoming arrow instead of several consecutive incoming arrows (see Figure 4.4 and 4.5).

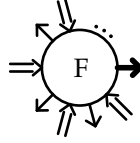


Figure 4.4: A disc representing a map of the form (4.2)

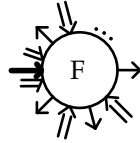


Figure 4.5: A disc representing a map of the form (4.1)

Now, we explain how to compose discs.

**Definition 4.9.** A **diagram** is roughly speaking a finite collection of decorated discs, each of which shares at most an arrow with any other one (see Figure 4.6). More precisely, a diagram  $(D, R)$  is the data of a collection  $D$  of discs  $\{D_1, \dots, D_n\}$ , each of which having sets of arrows  $A_i$  for  $i \in \llbracket 1, n \rrbracket$ , together with a subset  $R \subseteq (\sqcup_{i=1}^n A_i)^2$  satisfying that

(D.1) if  $(\alpha, \beta) \in R \cap (A_i \times A_j)$ , then  $i \neq j$  and  $\alpha$  is an incoming arrow and  $\beta$  is an outgoing arrow;

(D.2) if  $(y'\alpha_{x'}, y\beta_x) \in R$ , then  $x' = y$  and  $x = y'$ ;

where we use the notation introduced in Definition 4.4. We will represent a pair  $(\alpha, \beta) \in R \cap (A_i \times A_j)$  by connecting the outgoing arrow  $\beta$  of  $D_j$  with the incoming arrow  $\alpha$  of  $D_i$  (see Figure 4.6), and we will say that the disc  $D_i$  **shares an arrow** with  $D_j$  or that  $\alpha$  and  $\beta$  are **connected**.

An **incoming (resp. outgoing) arrow** of the diagram  $(D, R)$  is an arrow  $\alpha$  of one of the discs  $D_i$  such that there is no arrow  $\beta$  satisfying that  $(\alpha, \beta) \in R$  (resp.  $(\beta, \alpha) \in R$ ). A diagram  $(D, R)$  has a distinguished object of  $\mathcal{O}$  between any couple of consecutive arrows of  $(D, R)$  given by the decoration of the discs  $D_1, \dots, D_n$ .

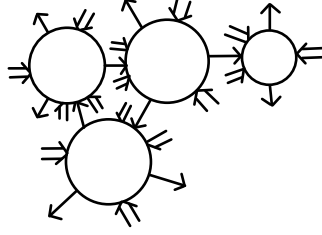


Figure 4.6: A diagram

**Definition 4.10.** We say that a diagram  $(D, R)$  is **admissible** if, roughly speaking, each of the discs in  $D = \{D_1, \dots, D_n\}$  is marked and if  $D$  is connected, with no cycles and if every couple of arrows in  $R$  contains a unique bold arrow. More precisely, a diagram  $(D, R)$  is admissible if the following conditions are satisfied

- (A.1)  $\forall i \in \llbracket 1, n \rrbracket, \exists \alpha_i \in D_i$  such that  $(\alpha_i, \beta_j) \in R$  for some  $j \neq i, \beta_j \in D_j$ ;
- (A.2) for  $(x, y) \in R$ , either  $x$  or  $y$  is a bold arrow, but not both;
- (A.3) there is no family of arrows  $\{x_1, \dots, x_k\}$  such that  $(x_i, x_{i+1}) \in R$  for all  $i \in \llbracket 1, k \rrbracket$  and  $x_k = x_1$ .

Note that there is precisely one disc in  $D = \{D_1, \dots, D_n\}$  whose bold arrow is also an incoming or outgoing arrow of the diagram  $(D, R)$ , i.e. an admissible diagram  $(D, R)$  always has either an incoming or outgoing bold arrow.

The **size** of an admissible diagram  $(D = \{D_1, \dots, D_n\}, R)$  is  $\sum_{i=1}^n |D_i| - n + 1$ , and it will be denoted by  $|D|$ . We will relabel the outgoing arrows of  $(D, R)$  in clockwise direction from 1 to  $|D|$ , such that the outgoing arrow labeled by  $|D|$  is precisely the bold arrow of  $(D, R)$  if the latter arrow is outgoing, and it is the outgoing arrow preceding the bold arrow of  $(D, R)$  in clockwise sense if the bold arrow of  $(D, R)$  is incoming.

**Definition 4.11.** The **type** of an admissible diagram  $(D = \{D_1, \dots, D_n\}, R)$  is the tuple

$$\bar{x} = (\bar{x}^1, \dots, \bar{x}^m) \in \bar{\mathcal{O}}$$

where  $m = |D|$  and  $\bar{x}^i \in \bar{\mathcal{O}}$  is the tuple composed of the objects of  $(D, R)$ , placed in counterclockwise order, that we can read between the outgoing arrows  $i - 1$  and  $i$  of  $(D, R)$ .

**Definition 4.12.** A **source** (resp. **sink**) of an admissible diagram is a disc which shares none of its incoming (resp. outgoing) arrows with another one.

**Remark 4.13.** *An admissible diagram has at least one source and one sink.*

Given an admissible diagram  $(D = \{D_1, \dots, D_n\}, R)$  and a tuple  $(s_{d+1}F_1, \dots, s_{d+1}F_n)$  with  $F_i \in \text{Multi}^{\bar{x}_i}(\text{Bar}(\mathcal{A})[d])$  such that  $\bar{x}_i$  is the type of  $D_i$  for all  $i \in \llbracket 1, n \rrbracket$ , we will define a map  $\mathcal{E}((D, R), s_{d+1}F_1, \dots, s_{d+1}F_n) \in \text{Multi}^{\bar{x}}(\text{Bar}(\mathcal{A})[d])[d+1]$  where  $\bar{x}$  is the type of  $(D, R)$ , as follows. First, we suppose that the bold arrow is on a sink.

**Definition 4.14.** *Let  $(D = \{D_1, \dots, D_n\}, R)$  be a diagram and let  $(s_{d+1}F_1, \dots, s_{d+1}F_n)$  be a tuple of homogeneous elements with  $F_i \in \text{Multi}^{\bar{x}_i}(\text{Bar}(\mathcal{A})[d])$  such that  $\bar{x}_i$  is the type of  $D_i$  for all  $i \in \llbracket 1, n \rrbracket$ . We will define the evaluation of  $\mathcal{E}((D, R), s_{d+1}F_1, \dots, s_{d+1}F_n)$  at an element  $\bar{s}\bar{a}^{\bar{x}}$  of  $\mathcal{A}[1]^{\otimes \bar{x}}$ , where  $\bar{x}$  is the type of  $(D, R)$  by induction on  $n$  as follows. We fix a source  $D_s$  of  $(D, R)$ .*

(E.1) *We place each element  $sa_j^i \in x_j^i \mathcal{A}_{x_{j+1}^i}[1]$  in the incoming arrow  $x_j^i \alpha_{x_{j+1}^i}$  of  $(D, R)$  in counterclockwise order beginning at the bold arrow. This will create a sign, as follows : if an element of degree  $m$  turn around the source representing a map of degree  $m'$ , we add a sign  $(-1)^{mm'}$  and if an element of degree  $\ell$  pass through an element of degree  $\ell'$  to go to its place, we add a sign  $(-1)^{\ell\ell'}$ . Here, turning around the source means passing through all of its inputs.*

(E.2) *We evaluate  $\mathcal{E}(D_s, s_{d+1}F_s)$  at the elements  $sa_j^i \in x_j^i \mathcal{A}_{x_{j+1}^i}[1]$  corresponding to the incoming arrows of  $D_s$ , to obtain an element of the form*

$$(-1)^{(d+1)(|b^1|+\dots+|b^{|D_s|-1}|)} b^1 \otimes \dots \otimes b^{|D_s|-1} \otimes s_{d+1}b^{|D_s|} \in y_1' \mathcal{A}_{y_1}[-d] \otimes \dots \otimes y_{|D_s|}' \mathcal{A}_{y_{|D_s|}}[1]$$

(E.3) *We add a Koszul sign coming from transposing  $b^1 \otimes \dots \otimes b^{|D_s|-1}$  with all of the elements  $sa_j^i \in x_j^i \mathcal{A}_{x_{j+1}^i}[1]$  corresponding to the incoming arrows of  $(D, R)$  between the last outgoing arrow of  $(D, R)$  and the bold arrow of  $D_s$  connected with an arrow of the rest of the diagram in the clockwise order. Recall that the elements  $b^i$  for  $i \in \llbracket 1, |D_s| - 1 \rrbracket$  are labeled with an index from  $\llbracket 1, |D| \rrbracket$  coming from the corresponding outgoing arrow of  $(D, R)$ .*

(E.4) *We consider the diagram  $(D', R')$  given by removing the disc  $D_s$  from  $(D, R)$  and place  $s_{d+1}b^{|D_s|}$  at the incoming arrow of  $(D', R')$  that was previously connected to the bold outgoing arrow of  $D_s$ . By induction we evaluate  $\mathcal{E}((D', R'), s_{d+1}F_1, \dots, s_{d+1}F_s, \dots, s_{d+1}F_n)$  at the elements  $sa_j^i \in x_j^i \mathcal{A}_{x_{j+1}^i}[1]$  corresponding to the incoming arrows of  $D'$ . This evaluation carries a sign*

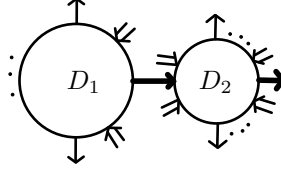
$$(-1)^{b^{|D_s|}(|\bar{s}\bar{a}^u|+|\bar{s}\bar{a}^v|)+|\bar{s}\bar{a}^u||\bar{s}\bar{a}^v|}$$

*where  $|\bar{s}\bar{a}^u|$  and  $|\bar{s}\bar{a}^v|$  are the tuples of objects corresponding to the incoming arrows of  $D'$  that precede and follow the element  $b^{|D_s|}$  without any outgoing arrow between them. Recall that the tensor factors in the evaluation of  $\mathcal{E}((D', R'), s_{d+1}F_1, \dots, s_{d+1}F_s, \dots, s_{d+1}F_n)$  at the elements  $sa_j^i \in x_j^i \mathcal{A}_{x_{j+1}^i}[1]$  stated before are labeled with an index from  $\llbracket 1, |D| \rrbracket$  coming from the corresponding outgoing arrow of  $(D, R)$ .*

(E.5) *We reorder the tensor factors obtained in steps (E.4) and (E.5) according to the labeling of the outgoing arrows of  $(D, R)$ , and add the respective Koszul sign.*

*We illustrate the previous procedure with an admissible diagram  $D$  consisting of two discs, given as follows*





In this case, the evaluation of  $\mathcal{E}(D, s_{d+1}G^{\bar{y}}, s_{d+1}F^{\bar{x}})$  at

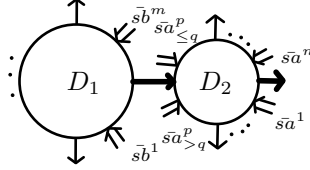
$$(\bar{s}a^1, \dots, \bar{s}a^{p-1}, \bar{s}b^1, \bar{s}a_{>q}^p, \bar{s}b^2, \dots, \bar{s}b^{m-1}, \bar{s}a_{\leq q}^p, \bar{s}b^m, \bar{s}a^{p+1}, \dots, \bar{s}a^n)$$

with  $\bar{s}a^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$  and  $\bar{s}b^i \in \mathcal{A}[1]^{\otimes \bar{y}^i}$ , where  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$  and  $\bar{y} = (\bar{y}^1, \dots, \bar{y}^m)$  are the respective types of  $D_1$  and  $D_2$ , is detailed as follows.

First, we place each tensor factor  $\bar{s}a_j^i$  and  $\bar{s}b_j^i$  of  $\bar{s}a^i$  and  $\bar{s}b^i$  in the corresponding incoming arrow, as explained in (E.1), adding a sign  $(-1)^{(|G^{\bar{y}}|+d+1)(|\bar{s}a^1|+\dots+|\bar{s}a^{p-1}|+|\bar{s}a_{>q}^p|)}$ . Also, we add a sign

$$(-1)^{|\bar{s}b^1||\bar{s}a_{>q}^p|+|\bar{s}b^m||\bar{s}a_{\leq q}^p|}$$

for the permutation of the corresponding elements. We picture this as follows



In step (E.2), we compute  $\mathcal{E}(D_1, s_{d+1}G^{\bar{y}})(\bar{s}b^1, \dots, \bar{s}b^m) = (-1)^{(d+1)(|\epsilon_1|+\dots+|\epsilon_{m-1}|)}\epsilon_1 \otimes \dots \otimes s_{d+1}\epsilon_m$ . After step (E.3) we have gained a total sign  $(-1)^\Delta$  with

$$\Delta = (|G^{\bar{y}}|+d+1)(|\bar{s}a_{\leq q}^p|+|\bar{s}a^{p+1}|+\dots+|\bar{s}a^n|) + (|G^{\bar{y}}|+|\bar{s}a^1|+\dots+|\bar{s}a_{>q}^p|)(|\epsilon_1|+\dots+|\epsilon_{j-i-1}|),$$

multiplying the element  $\epsilon_1 \otimes \dots \otimes \epsilon_{j-i-1} \otimes \mathcal{E}(D_2, s_{d+1}F^{\bar{x}})(\bar{s}a^1 \otimes \dots \otimes \bar{s}a_{>q}^p \otimes s_{d+1}\epsilon_{j-i} \otimes \bar{s}a_{\leq q}^p \otimes \dots \otimes \bar{s}a^n)$ .

In (E.4), we compute

$$\mathcal{E}(D_2, s_{d+1}F^{\bar{x}})(\bar{s}a^1 \otimes \dots \otimes \bar{s}a_{\leq q}^p \otimes s_{d+1}\epsilon_{j-i} \otimes \bar{s}a_{>q}^p \otimes \dots \otimes \bar{s}a^n) = \delta_1 \otimes \dots \otimes \delta_{i-1} \otimes \delta_j \otimes \dots \otimes s_{d+1}\delta_n$$

and add a sign  $(-1)^{\Delta'}$  to the final result, with

$$\Delta' = |\bar{s}a_{>q}^p|(d+1+|\epsilon_{j-i}|+|\bar{s}a_{\leq q}^p|) + |\bar{s}a_{\leq q}^p|(d+1+|\epsilon_{j-i}|) + (d+1)(|\delta_1|+\dots+|\delta_{i-1}|+|\delta_j|+\dots+|\delta_{n-1}|)$$

In (E.5), we reorder the outputs with a sign, giving finally

$$\mathcal{E}(D, s_{d+1}G^{\bar{y}}, s_{d+1}F^{\bar{x}})(\bar{s}a^1, \dots, \bar{s}a^n) = (-1)^{\Delta+\Delta'+\Delta''} \delta_1 \otimes \dots \otimes \delta_{i-1} \otimes \epsilon_1 \otimes \dots \otimes \epsilon_{j-i-1} \otimes \delta_j \otimes \dots \otimes s_{d+1}\delta_n$$

where  $\Delta'' = (|\delta_1|+\dots+|\delta_{i-1}|)(|\epsilon_1|+\dots+|\epsilon_{j-i-1}|)$ .

Now, suppose that the bold arrow is on a source.

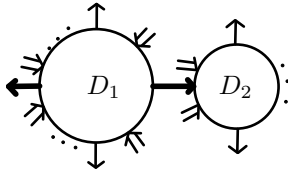
**Definition 4.15.** Let  $(D = \{D_1, \dots, D_n\}, R)$  be a diagram and let  $(s_{d+1}F_1, \dots, s_{d+1}F_n)$  be a tuple of homogeneous elements with  $F_i \in \text{Multi}^{\bar{x}_i}(\text{Bar}(\mathcal{A})[d])$  such that  $\bar{x}_i$  is the type of  $D_i$  for all  $i \in \llbracket 1, n \rrbracket$ . We will define the evaluation  $\mathcal{E}((D, R), s_{d+1}F_1, \dots, s_{d+1}F_n)$  at an element  $\bar{s}a^{\bar{x}}$  of  $\mathcal{A}[1]^{\otimes \bar{x}}$ , where  $\bar{x}$  is the type of  $(D, R)$  by induction on  $n$  as follows. Suppose that the bold arrow is an arrow of the source  $D_s$  of  $(D, R)$  and that the  $p$ -th outgoing arrow  $\alpha$  of  $D_s$  is connected with another disc.

- (F.1) We place each element  $sa_j^i \in x_j^i \mathcal{A}_{x_{j+1}^i}[1]$  in the incoming arrow  $x_j^i \alpha_{x_{j+1}^i}$  of  $(D, R)$  in counterclockwise order beginning at the bold arrow. This will create a sign, as follows : if an element of degree  $m$  turn around a source representing a map of degree  $m'$ , we add a sign  $(-1)^{mm'}$  and if an element of degree  $\ell$  pass through an element of degree  $\ell'$  to go to his place, we add a sign  $(-1)^{\ell\ell'}$ . Again, turning around a source means passing through all of its inputs.
- (F.2) We transpose all the elements that do not correspond to incoming arrows of  $D_s$  with the elements corresponding to incoming arrows of  $D_s$  between  $\alpha$  and the bold arrow. We add the sign coming from this transposition. We evaluate  $\mathcal{E}(D_s, s_{d+1}F_s)$  at the elements  $sa_j^i \in x_j^i \mathcal{A}_{x_{j+1}^i}[1]$  corresponding to the incoming arrows of  $D_s$ , to obtain an element of the form

$$s_{d+1}(b^1 \otimes \dots \otimes b^{|D_s|}) \in (y_1' \mathcal{A}_{y_1}[-d] \otimes \dots \otimes y_{|D_s|}' \mathcal{A}_{y_{|D_s|}}[-d])[d+1]$$

- (F.3) We add a Koszul sign coming from transposing  $b^{p+1} \otimes \dots \otimes b^{|D_s|}$  with all of the elements that do not correspond to an incoming arrow of  $D_s$ . Recall that the elements  $b^i$  for  $i \in \llbracket 1, |D_s| \rrbracket \setminus \{p\}$  are labeled with an index from  $\llbracket 1, |D| \rrbracket$  coming from the corresponding outgoing arrow of  $(D, R)$ .
- (F.4) We then consider the diagram  $(D', R')$  given by removing the disc  $D_s$  from  $(D, R)$ . Its bold arrow is the one previously connected to  $\alpha$ . Note that the element  $b^p$  is associated with this incoming arrow of  $(D', R')$ .
- (F.5) We evaluate  $\mathcal{E}((D', R'), s_{d+1}F_1, \dots, s_{d+1}\check{F}_s, \dots, s_{d+1}F_n)$  at the elements  $sa_j^i \in x_j^i \mathcal{A}_{x_{j+1}^i}[1]$  corresponding to the incoming arrows of  $D'$ . Recall that the tensor factors in this evaluation are labeled with an index from  $\llbracket 1, |D| \rrbracket$  coming from the corresponding outgoing arrow of  $(D, R)$ .

We illustrate the previous procedure with an admissible diagram  $D$  consisting of two discs, given as follows

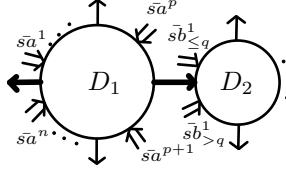


In this case, the evaluation of  $\mathcal{E}(D, s_{d+1}F^{\bar{x}}, s_{d+1}G^{\bar{y}})$  at

$$(\bar{s}a^1, \dots, \bar{s}a^{p-1}, \bar{s}b_{\leq q}^1, \bar{s}a^p, \bar{s}b^2, \dots, \bar{s}b^m, \bar{s}a^{p+1}, \bar{s}b_{> q}^1, \bar{s}a^{p+2}, \dots, \bar{s}a^n)$$

with  $\bar{s}a^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$  and  $\bar{s}b^i \in \mathcal{A}[1]^{\otimes \bar{y}^i}$ , where  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$  and  $\bar{y} = (\bar{y}^1, \dots, \bar{y}^m)$  are the respective types of  $D_1$  and  $D_2$ , is detailed as follows.

First, we place each tensor factor  $sa_j^i$  and  $sb_j^i$  of  $\bar{s}a^i$  and  $\bar{s}b^i$  in the corresponding incoming arrow, as explained in (F.1), adding a sign  $(-1)^{|\bar{s}b_{\leq q}^1||\bar{s}a^p|+|\bar{s}b_{> q}^1||\bar{s}a^{p+1}|}$  for the permutation of the corresponding elements.



In step (F.2), we place all the elements  $\bar{s}b^i$  after the others and multiply by a sign

$$(-1)^{(|\bar{s}b^1|+\dots+|\bar{s}b^m|)(|\bar{s}a^{p+1}|+\dots+|\bar{s}a^n|)}$$

We then compute  $\mathcal{E}(D_1, s_{d+1}F^{\bar{x}})(\bar{s}a^1, \dots, \bar{s}a^n) = (-1)^{(d+1)(|\epsilon_1|+\dots+|\epsilon_{n-1}|)}\epsilon_1 \otimes \dots \otimes s_{d+1}\epsilon_n$ .

After (F.3), we end with

$$\epsilon_1 \otimes \dots \otimes \epsilon_{p-1} \otimes (\mathcal{E}, s_{d+1}G^{\bar{y}})(\epsilon_p \otimes \bar{s}b_{\leq q}^1 \otimes \dots \otimes \bar{s}b^m \otimes \bar{s}b_{> q}^1) \otimes \epsilon_{p+1} \otimes \dots \otimes s_{d+1}\epsilon_n$$

preceded by a sign  $(-1)^{\Delta'}$  with

$$\begin{aligned} \Delta' &= |\bar{s}b_{\leq q}^1||\bar{s}a^p| + |\bar{s}b_{> q}^1||\bar{s}a^{p+1}| + \Delta + (|\bar{s}b_{\leq q}^1| + \dots + |\bar{s}b^m| + |\bar{s}b_{> q}^1|)(|\epsilon_{p+1}| + \dots + |\epsilon_n| + d + 1) \\ &\quad + (|G^{\bar{y}}| + d + 1)(\epsilon_1 + \dots + \epsilon_{p-1}) \end{aligned}$$

In (F.5), we order the elements and compute  $\mathcal{E}(D_2, s_{d+1}G^{\bar{y}})(\bar{s}b_{\leq q}^1 \otimes \epsilon_p \otimes \bar{s}b_{> q}^1, \bar{s}b^2, \dots, \bar{s}b^m)$  and multiply the result by

$$(-1)^{|\bar{s}b_{> q}^1|(|\epsilon_i|+|\bar{s}b^2|+\dots+|\bar{s}b^m|)+(d+1)|G^{\bar{y}}|}$$

Finally, we get

$$\mathcal{E}(D, s_{d+1}F^{\bar{x}}, s_{d+1}G^{\bar{y}})(\bar{s}a^1, \dots, \bar{s}a^n) = (-1)^{\Delta''}\epsilon_1 \otimes \dots \otimes \epsilon_{p-1} \otimes \delta_1 \otimes \dots \otimes \delta_m \otimes \epsilon_{p+1} \otimes \dots \otimes s_{d+1}\epsilon_n$$

where  $\Delta'' = \Delta' + |\bar{s}b_{> q}^1|(|\epsilon_i| + |\bar{s}b^2| + \dots + |\bar{s}b^m|) + (d + 1)|G^{\bar{y}}|$ .

**Definition 4.16.** A *filled diagram* is an admissible diagram  $(D = \{D_1, \dots, D_n\}, R)$  together with elements  $s_{d+1}F_i \in \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])[d + 1]$  for  $i \in \llbracket 1, n \rrbracket$ .

To each filled diagram  $\mathfrak{D} = \{(D = \{D_1, \dots, D_n\}, R), (s_{d+1}F_i)_{i \in \llbracket 1, n \rrbracket}\}$  we associate the element

$$\mathcal{E}(\mathfrak{D}) = \mathcal{E}((D, R), s_{d+1}F^{\bar{x}^1}, \dots, s_{d+1}F^{\bar{x}^n}) \in \text{Multi}^{\bar{x}}(\text{Bar}(\mathcal{A})[d])[d + 1]$$

where  $\bar{x}$  is the type of  $(D, R)$  and  $\bar{x}^i$  is the type of  $D_i$  for  $i \in \llbracket 1, n \rrbracket$ . We depict a filled diagram as a diagram replacing the names of the discs by the corresponding maps (see Figure 4.7).

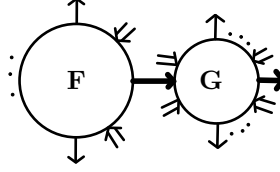
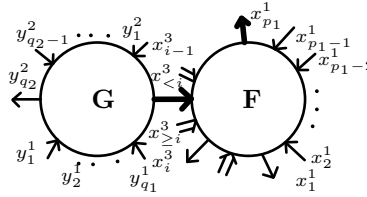


Figure 4.7: A filled diagram

**Example 4.17.** Consider the following filled diagram.



It represents a map

$$s_{d+1}F^{\bar{x}} : \mathcal{A}[1]^{\otimes \bar{x}^1} \otimes \mathcal{A}[1]^{\otimes \bar{x}^2} \otimes \mathcal{A}[1]^{\otimes \bar{x}^3} \rightarrow x_1^1 \mathcal{A}_{x_2^2}[-d] \otimes x_1^2 \mathcal{A}_{x_3^3}[-d] \otimes x_1^3 \mathcal{A}_{x_{p_1}^1}[1]$$

where  $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$  and  $\bar{x}^j = (x_1^j, \dots, x_{\lg(\bar{x}^j)}^j)$  for  $j \in \llbracket 1, 3 \rrbracket$  that takes as an input the last output of a map

$$s_{d+1}G^{\bar{y}^1, \bar{y}^2} : \mathcal{A}[1]^{\otimes \bar{y}^1} \otimes \mathcal{A}[1]^{\otimes \bar{y}^2} \rightarrow y_1^1 \mathcal{A}_{y_m^2}[-d] \otimes x_{i-1}^3 \mathcal{A}_{x_i^3}[1]$$

with  $\bar{y}^1 = (y_1^1, y_2^1, \dots, y_n^1, x_i^3)$ ,  $\bar{y}^2 = (x_{i-1}^3, y_1^2, \dots, y_m^2)$ .

The type of the diagram is  $(\bar{x}^1, \bar{x}^2, \bar{y}^1 \sqcup \bar{x}_{\geq i}^3, \bar{x}_{< i}^3 \sqcup \bar{y}^2)$ , meaning that it represents a map that can be evaluated at  $(\bar{s}a^1, \bar{s}a^2, \bar{s}b^1 \otimes \bar{s}a_{\geq i}^3, \bar{s}a_{< i}^3 \otimes \bar{s}b^2)$  where  $\bar{s}a^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$  and  $\bar{s}b^i \in \mathcal{A}^{\otimes \bar{y}^i}$ . This evaluation goes in 5 steps.

1. We place the elements around the diagram, which creates a sign  $(-1)^\epsilon$  with

$$\epsilon = |s_{d+1}G^{\bar{y}^1, \bar{y}^2}|(|\bar{s}a^1| + |\bar{s}a^2| + |\bar{s}a_{\geq i}^3|) + |\bar{s}a_{\geq i}| |\bar{s}b^1| + |\bar{s}b^2| |\bar{s}a_{> i}|$$

2. We evaluate  $\mathcal{E}(D_1, s_{d+1}G^{\bar{y}^1, \bar{y}^2})$  at  $(\bar{s}b^1, \bar{s}b^2)$ . The result of this evaluation is an element of the shifted tensor product  $(y_1^1 \mathcal{A}_{y_m^2}[-d] \otimes x_{i-1}^3 \mathcal{A}_{x_i^3}[-d])[d+1]$  that we will write  $s_{d+1}(y_1^1 G_{y_m^2}^{-d} \otimes x_{i-1}^3 G_{x_i^3}^{-d})$ . We then use the isomorphism (2.1) to obtain the tensor product

$$(-1)^{\epsilon+(d+1)} |y_1^1 G_{y_m^2}^{-d}| |y_1^1 G_{y_m^2}^{-d} \otimes x_{i-1}^3 G_{x_i^3}^{-d} \in y_1^1 \mathcal{A}_{y_m^2}[-d] \otimes x_{i-1}^3 \mathcal{A}_{x_i^3}[1]$$

3. We put the first output before and get

$$(-1)^{\epsilon+(d+1)|y_1^1 G_{y_m^2}^{-d}|+\epsilon'} y_1^1 G_{y_m^2}^{-d} \otimes \mathcal{E}(D_2, s_{d+1} F^{\bar{x}}(\bar{s}a^1, \bar{s}a^2, \bar{s}a_{\geq i}, \bar{s}a_{< i} \otimes x_{i-1}^3 G_{x_i^3}^1))$$

$$\text{with } \epsilon' = y_1^1 G_{y_m^2}^{-d} (|F| + d + 1 + |\bar{s}a^1| + |\bar{s}a^2| + |\bar{s}a_{\geq i}|)$$

4. We evaluate  $F^{\bar{x}}$  at  $(\bar{s}a^1, \bar{s}a^2, \bar{s}a_{< i} \otimes x_{i-1}^3 G_{x_i^3}^1 \otimes \bar{s}a_{\geq i})$  which add a sign  $(-1)^{|\bar{s}a_{\geq i}|(|\bar{s}a_{< i}|+|x_{i-1}^3 G_{x_i^3}^1|)}$  to the result for the transposition of the corresponding elements.

5. We order the outputs according to the labeling of their corresponding outgoing arrow and finally get

$$(-1)^{\epsilon+(d+1)|y_1^1 G_{y_m^2}^{-d}|+\epsilon'+\epsilon''} x_1^1 F_{x_{p_2}^2}^{-d} \otimes x_2^2 F_{x_{p_3}^3}^{-d} \otimes y_1^1 G_{y_m^2}^{-d} \otimes x_1^3 F_{x_{p_1}^1}^1$$

$$\text{with } \epsilon'' = (-1)^{(|y_1^1 G_{y_m^2}^{-d}|+1)(|x_1^1 F_{x_{p_2}^2}^{-d}|+|x_2^2 F_{x_{p_3}^3}^{-d}|)+(d+1)(|x_1^1 F_{x_{p_2}^2}^{-d}|+\dots+|x_2^2 F_{x_{p_3}^3}^{-d}|+\dots+|y_1^1 G_{y_m^2}^{-d}|)}$$

## 4.2 The necklace graded Lie algebra

In order to recall what a pre-Calabi-Yau structure on  $\mathcal{A}$  is, one first defines a graded Lie algebra, called the **necklace graded Lie algebra** and appearing in [5]. As a graded vector space, this graded Lie algebra is  $\text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])^{C_{1\mathbb{g}(\bullet)}}[d+1]$ . In order to define a graded Lie bracket on this space, we first define a new operation as follows.

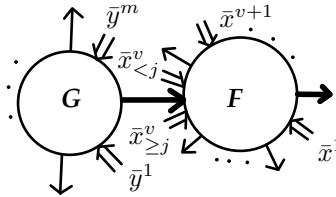
**Definition 4.18.** Consider a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$  as well as tuples of elements of  $\bar{\mathcal{O}}$  given by  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n), \bar{y} = (\bar{y}^1, \dots, \bar{y}^m) \in \bar{\mathcal{O}}$  such that  $\text{rt}(\bar{y}^1) = x_j^v$  and  $\text{lt}(\bar{y}^m) = x_{j-1}^v$  for some  $v \in \llbracket 1, n \rrbracket$  and  $j \in \llbracket 1, \lg(\bar{x}^v) \rrbracket$ . The **inner necklace composition at  $v, j$**  of elements  $s_{d+1} F^{\bar{x}} \in \text{Multi}^{\bar{x}}(\text{Bar}(\mathcal{A})[d])[d+1]$  and  $s_{d+1} G^{\bar{y}} \in \text{Multi}^{\bar{y}}(\text{Bar}(\mathcal{A})[d])[d+1]$  is given by

$$s_{d+1} F^{\bar{x}} \underset{\text{inn}}{\circ}_{\text{nec}, v, j} s_{d+1} G^{\bar{y}} = \mathcal{E}(\mathfrak{D}) \in \text{Multi}^{\bar{x} \sqcup_{v, j, \text{inn}} \bar{y}}(\text{Bar}(\mathcal{A})[d])[d+1]$$

with

$$\bar{x} \sqcup_{v, j, \text{inn}} \bar{y} = (\bar{x}^1, \dots, \bar{x}^{v-1}, \bar{y}^1 \sqcup \bar{x}_{< j}^v, \bar{y}^2, \dots, \bar{y}^{m-1}, \bar{x}_{> j-1}^v \sqcup \bar{y}^m, \bar{x}^{v+1}, \dots, \bar{x}^n)$$

and where  $\mathfrak{D}$  is the filled diagram of type  $\bar{x} \sqcup_{v, j, \text{inn}} \bar{y}$  given by



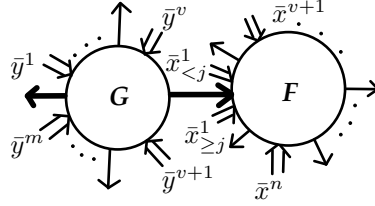
**Definition 4.19.** Consider a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$  as well as tuples of elements of  $\bar{\mathcal{O}}$  given by  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n), \bar{y} = (\bar{y}^1, \dots, \bar{y}^m) \in \bar{\mathcal{O}}$  such that  $\text{lt}(\bar{y}^v) = x_{j-1}^1$  and  $\text{rt}(\bar{y}^{v+1}) = x_j^1$  for some  $v \in \llbracket 1, m \rrbracket$  and  $j \in \llbracket 1, \text{lg}(\bar{x}^1) \rrbracket$ . The **outer necklace composition at  $v, j$**  of elements  $s_{d+1}F^{\bar{x}} \in \text{Multi}^{\bar{x}}(\text{Bar}(\mathcal{A})[d])[d+1]$  and  $s_{d+1}G^{\bar{y}} \in \text{Multi}^{\bar{y}}(\text{Bar}(\mathcal{A})[d])[d+1]$  is given by

$$s_{d+1}F^{\bar{x}} \underset{\substack{\circ \\ \text{nec}, v, j \\ \text{out}}}{\circ} s_{d+1}G^{\bar{y}} = \mathcal{E}(\mathcal{D}) \in \text{Multi}^{\bar{x} \sqcup_{v, j, \text{out}} \bar{y}}(\text{Bar}(\mathcal{A})[d])[d+1]$$

with

$$\bar{x} \sqcup_{v, j, \text{out}} \bar{y} = (\bar{y}^1, \dots, \bar{y}^{v-1}, \bar{x}_{<j-1}^1 \sqcup \bar{y}^v, \bar{x}^2, \dots, \bar{x}^n, \bar{y}^{v+1} \sqcup \bar{x}_{>j}^1, \bar{y}^{v+2}, \dots, \bar{y}^m)$$

and where  $\mathcal{D}$  is the filled diagram of type  $\bar{x} \sqcup_{v, j, \text{out}} \bar{y}$  given by



**Definition 4.20.** Given a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$ , the **necklace product** of elements  $s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G} \in \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])^{C_{\text{lg}(\bullet)}}[d+1]$  is the element

$$s_{d+1}\mathbf{F} \underset{\text{nec}}{\circ} s_{d+1}\mathbf{G} \in \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])[d+1]$$

given by

$$(s_{d+1}\mathbf{F} \underset{\text{nec}}{\circ} s_{d+1}\mathbf{G})^{\bar{z}} = \sum_{(\bar{x}, \bar{y}, v, j) \in \mathcal{I}_{\text{inn}}} s_{d+1}F^{\bar{x}} \underset{\substack{\circ \\ \text{nec}, v, j \\ \text{inn}}}{\circ} s_{d+1}G^{\bar{y}} + \sum_{(\bar{x}, \bar{y}, v, j) \in \mathcal{I}_{\text{out}}} s_{d+1}F^{\bar{x}} \underset{\substack{\circ \\ \text{nec}, v, j \\ \text{out}}}{\circ} s_{d+1}G^{\bar{y}}$$

for all  $\bar{z} \in \bar{\mathcal{O}}$ , where

$$\mathcal{I}_{\text{inn}} = \{(\bar{x}, \bar{y}, v, j) \in \bar{\mathcal{O}} \times \bar{\mathcal{O}} \times \llbracket 1, \text{lg}(\bar{x}) \rrbracket \times \llbracket 1, \text{lg}(\bar{x}^v) \rrbracket \mid \bar{x} \sqcup_{v, j, \text{inn}} \bar{y} = \bar{z}\}$$

$$\mathcal{I}_{\text{out}} = \{(\bar{x}, \bar{y}, v, j) \in \bar{\mathcal{O}} \times \bar{\mathcal{O}} \times \llbracket 1, \text{lg}(\bar{y}) \rrbracket \times \llbracket 1, \text{lg}(\bar{x}^1) \rrbracket \mid \bar{x} \sqcup_{v, j, \text{out}} \bar{y} = \bar{z}\}$$

**Definition 4.21.** Given a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$ , the **necklace bracket** of two elements  $s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G} \in \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])^{C_{\text{lg}(\bullet)}}[d+1]$  is defined as the element

$$[s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G}]_{\text{nec}} \in \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])[d+1]$$

where

$$[s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G}]_{\text{nec}}^{\bar{z}} = (s_{d+1}\mathbf{F} \underset{\text{nec}}{\circ} s_{d+1}\mathbf{G})^{\bar{z}} - (-1)^{(|\mathbf{F}|+d+1)(|\mathbf{G}|+d+1)} (s_{d+1}\mathbf{G} \underset{\text{nec}}{\circ} s_{d+1}\mathbf{F})^{\bar{z}}$$

for every  $\bar{z} \in \bar{\mathcal{O}}$ .

**Definition 4.22.** Given a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$ , we define the graded vector space  $\mathcal{C}_\bullet(\mathcal{A}) = \prod_{\bar{x} \in \bar{\mathcal{O}}} \mathcal{C}_{\bar{x}}(\mathcal{A})$ , where

$$\mathcal{C}_{\bar{x}}(\mathcal{A}) = \text{Hom}_{\mathbb{k}} \left( \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\otimes \bar{x}^{n-i+1}} \otimes_{\text{rt}(\bar{x}^{n-i+1})} \mathcal{A}_{\text{lt}(\bar{x}^{n-i})}^*[d]) \otimes \mathcal{A}[1]^{\otimes \bar{x}^1}, \text{lt}(\bar{x}^n) \mathcal{A}_{\text{rt}(\bar{x}^1)} \right)$$

for  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}$ .

**Lemma 4.23.** Let  $\mathcal{A}$  be a graded quiver with set of objects  $\mathcal{O}$ . Then, we have an injective map

$$\mathbf{j} : \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])[d+1] \rightarrow \mathcal{C}_\bullet(\mathcal{A})[1]$$

given by the family  $\mathbf{j} = (\mathbf{j}_{\bar{x}})_{\bar{x} \in \bar{\mathcal{O}}}$  where  $\mathbf{j}_{\bar{x}} : \text{Multi}^{\bar{x}}(\text{Bar}(\mathcal{A})[d])[d+1] \rightarrow \mathcal{C}_{\bar{x}}(\mathcal{A})[1]$  is defined by  $\mathbf{j}_{\bar{x}}(s_{d+1}\phi) = s\psi_{\bar{x}}$  with

$$\psi_{\bar{x}}(\bar{s}a^n, tf^{n-1}, \bar{s}a^{n-2}, tf^{n-2}, \dots, \bar{s}a^2, tf^2, \bar{s}a^1) = (-1)^\epsilon \left( \bigotimes_{i=1}^{n-1} (f^i \circ s_d) \otimes s_d \right) (\phi(\bar{s}a^1, \bar{s}a^2, \dots, \bar{s}a^n))$$

for  $\phi \in \text{Multi}^{\bar{x}}(\text{Bar}(\mathcal{A})[d])$ ,  $n = \text{lg}(\bar{x})$ ,  $\bar{s}a^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$  for  $i \in \llbracket 1, n \rrbracket$  and  $tf^i \in \text{rt}(\bar{x}^{i+1}) \mathcal{A}_{\text{lt}(\bar{x}^i)}^*[d]$  for  $i \in \llbracket 1, n-1 \rrbracket$  with

$$\epsilon = \sum_{i=1}^{n-1} |tf^i| \sum_{j=i+1}^n |\bar{s}a^j| + (|\phi| + d) \sum_{i=1}^{n-1} |tf^i| + d(n-1) + \sum_{1 \leq i < j \leq n} |\bar{s}a^i| |\bar{s}a^j| + \sum_{1 \leq i < j \leq n-1} |tf^i| |tf^j|$$

We denote by  $\Gamma$  the natural bilinear form on the boundary quiver  $\partial_{d-1}\mathcal{A} = \mathcal{A} \oplus \mathcal{A}^*[d-1]$ , defined in Example 3.18.

**Definition 4.24.** Given a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$ , the **noncounitary cofree conilpotent graded cocategory on  $\mathcal{A}$**  is defined as  $\bar{T}^c(\mathcal{A}[1]) = \bigoplus_{\bar{x} \in \bar{\mathcal{O}}} \mathcal{A}[1]^{\otimes \bar{x}}$  with coproduct given by deconcatenation and the induced grading by the one of  $\mathcal{A}$ .

**Remark 4.25.** We have that

$$\bar{T}^c(\partial_{d-1}\mathcal{A}[1]) = \bigoplus_{\bar{x} \in \bar{\mathcal{O}}} \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\otimes \bar{x}^{n-i+1}} \otimes_{\text{rt}(\bar{x}^{n-i+1})} \mathcal{A}_{\text{lt}(\bar{x}^{n-i})}^*[d]) \otimes \mathcal{A}[1]^{\otimes \bar{x}^1} \quad (4.3)$$

**Definition 4.26.** Given a graded cocategory  $\mathcal{A}$  with coproduct  $\Delta$  given by maps  ${}_y\Delta_x^z : {}_y\mathcal{A}_x \rightarrow {}_y\mathcal{A}_z \otimes {}_z\mathcal{A}_x$  for  $x, y, z \in \mathcal{O}$ , a **homogeneous coderivation of degree  $d$**  on  $\mathcal{A}$  is a collection  ${}_y(B_x)_{x, y \in \mathcal{O}}$  of  $\mathbb{k}$ -linear maps  ${}_yB_x : {}_y\mathcal{A}_x \rightarrow {}_y\mathcal{A}_x$  of degree  $d$  such that

$${}_y\Delta_x^z \circ {}_yB_x = ({}_yB_z \otimes \text{id} + \text{id} \otimes {}_zB_x) \circ {}_y\Delta_x^z$$

for each  $x, y, z \in \mathcal{O}$ .

We will denote by  $\text{Coder}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$  the graded vector space formed by sums of homogeneous coderivations of  $\bar{T}^c(\partial_{d-1}\mathcal{A}[1])$ .

If  $B \in \text{Coder}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$ , we will denote by  $B^{\bar{x}}$  the restriction of  $B$  to the tuple indexed by  $\bar{x} \in \bar{\mathcal{O}}$  in (4.3), by  $B_{\mathcal{A}}^{\bar{x}}$  the composition of  $B^{\bar{x}}$  and the canonical projection  $\bar{T}^c(\partial_{d-1}\mathcal{A}[1]) \rightarrow \mathcal{A}[1]$  and by  $B_{\mathcal{A}^*}^{\bar{x}}$  the composition of  $B^{\bar{x}}$  and the canonical projection  $\bar{T}^c(\partial_{d-1}\mathcal{A}[1]) \rightarrow \mathcal{A}^*[d]$ .

**Definition 4.27.** A coderivation  $B \in \text{Coder}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$  is *cyclic* with respect to the natural bilinear form  $\Gamma$  if the following equations are satisfied

$$\Gamma(B_{\mathcal{A}}^{\bar{x}}(\bar{s}a^n, tf^{n-1}, \dots, \bar{s}a^2, tf^1, \bar{s}a^1), tf^n) = (-1)^\epsilon \Gamma(B_{\mathcal{A}}^{\sigma \cdot \bar{x}}(\bar{s}a^1, tf^n, \bar{s}a^n, tf^{n-1}, \dots, \bar{s}a^2), tf^1) \quad (4.4)$$

and

$$\Gamma(B_{\mathcal{A}^*}^{\bar{x}}(\bar{s}a^n, tf^{n-1}, \dots, \bar{s}a^2, tf^1, \bar{s}a^1), sb) = (-1)^{\epsilon'} \Gamma(B_{\mathcal{A}}^{\bar{x}'}(\bar{s}a^1 \otimes sb \otimes \bar{s}a^n, tf^{n-1}, \dots, \bar{s}a^2), tf^1) \quad (4.5)$$

for  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ ,  $\sigma = (12 \dots n)$  and  $\bar{x}' = (\bar{x}^2, \dots, \bar{x}^1 \sqcup \bar{x}^n)$ , where the arguments are  $\bar{s}a^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ ,  $tf^i \in \text{rt}(\bar{x}^{i+1})\mathcal{A}_{\text{lt}(\bar{x}^i)}^*[d]$  for  $i \in \llbracket 1, n \rrbracket$ ,  $sb \in \text{rt}(\bar{x}^1)\mathcal{A}_{\text{lt}(\bar{x}^n)}[1]$  and

$$\epsilon = (|\bar{s}a^1| + |tf^n|) \left( \sum_{i=2}^n |\bar{s}a^i| + \sum_{i=1}^{n-1} |tf^i| \right), \quad \epsilon' = (|\bar{s}a^1| + |sb|) \left( \sum_{i=2}^n |\bar{s}a^i| + \sum_{i=1}^{n-1} |tf^i| \right)$$

We define  $\text{Coder}_{\text{cyc}}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$  as the graded vector subspace of  $\text{Coder}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$  consisting of sums of homogeneous coderivations that are cyclic with respect to the natural bilinear form  $\Gamma$ . We also define the graded vector subspace of  $\text{Coder}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$  which we denote  $\text{Coder}_{\text{cyc},0}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$  and which is formed by the elements of  $\text{Coder}_{\text{cyc}}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1]))$  sending  $\bar{T}^c(\mathcal{A}[1])$  to itself.

**Proposition 4.28.** The linear map

$$\mathcal{R} : \text{Coder}_{\text{cyc},0}(\bar{T}^c(\partial_{d-1}\mathcal{A}[1])) \rightarrow \mathcal{C}_\bullet(\mathcal{A})[1]$$

of degree 0, sending a coderivation  $B$  to the collection  $(B_{\mathcal{A}}^{\bar{x}})_{\bar{x} \in \bar{\mathcal{O}}}$  is an isomorphism of graded vector spaces.

*Proof.* We will construct the inverse  $\mathcal{R}^{-1}$  of this map. Consider  $B_{\mathcal{A}}^{\bar{x}} \in \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])^{C_{\text{is}}(\bullet)}$ . We construct a collection  $(B_{\mathcal{A}^*}^{\bar{x}})_{\bar{x} \in \bar{\mathcal{O}}}$  of maps

$$B_{\mathcal{A}^*}^{\bar{x}} : \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\otimes \bar{x}^{n-i+1}} \otimes_{\text{rt}(\bar{x}^{n-i+1})} \mathcal{A}_{\text{lt}(\bar{x}^{n-i})}^*[d]) \otimes \mathcal{A}[1]^{\otimes \bar{x}^1} \longrightarrow \text{lt}(\bar{x}^n) \mathcal{A}_{\text{rt}(\bar{x}^1)}^*[d]$$

given by

$$\Gamma(B_{\mathcal{A}^*}^{\bar{x}}(\bar{s}a^n, tf^n, \dots, \bar{s}a^2, tf^1, \bar{s}a^1), sb) = (-1)^{\epsilon'} \Gamma(B_{\mathcal{A}}^{\bar{x}'}(\bar{s}a^1 \otimes sb \otimes \bar{s}a^n, tf^{n-1}, \dots, \bar{s}a^2), tf^1)$$

for each  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}$  and elements  $\bar{s}a^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$  for  $i \in \llbracket 1, n \rrbracket$ ,  $tf^i \in \text{rt}(\bar{x}^{i+1})\mathcal{A}_{\text{lt}(\bar{x}^i)}^*[d]$  for  $i \in \llbracket 1, n-1 \rrbracket$  and  $sb \in \text{rt}(\bar{x}^1)\mathcal{A}_{\text{lt}(\bar{x}^n)}[1]$ , where  $\bar{x}' = (\bar{x}^2, \dots, \bar{x}^1 \sqcup \bar{x}^n)$  and

$$\epsilon' = (|\bar{s}a^1| + |sb|) \left( \sum_{i=1}^{n-1} |\bar{s}a^i| + \sum_{i=2}^{n-1} |tf^i| \right)$$

One can easily check that the map that associates to  $B_{\mathcal{A}}^{\bar{x}}$  the coderivation given by  $(B_{\mathcal{A}}^{\bar{x}}, B_{\mathcal{A}^*}^{\bar{x}})_{\bar{x}}$  is the inverse of  $\mathcal{R}$ .  $\square$

We will denote by  $\mathcal{R}_{\mathcal{A}^*}^{-1}$  (resp.  $\mathcal{R}_{\mathcal{A}}^{-1}$ ) the composition of  $\mathcal{R}^{-1}$  and the canonical projection  $\bar{T}^c(\partial_{d-1}\mathcal{A}[1]) \rightarrow \mathcal{A}^*[d]$  (resp.  $\bar{T}^c(\partial_{d-1}\mathcal{A}[1]) \rightarrow \mathcal{A}[1]$ ).

We have the following relation between the necklace product and the usual Gerstenhaber circle product, which does not seem to have been observed in the literature so far.



**Proposition 4.29.** *Let  $F, G$  be elements in  $\text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])^{C_{\text{lg}(\bullet)}}$ . Then, we have*

$$\mathbf{j}_{v,j,\text{inn}}^{\bar{x}} \sqcup_{\bar{y}} (s_{d+1} F^{\bar{x}} \circ_{\text{inn}}^{\text{nec},v,j} s_{d+1} G^{\bar{y}}) = \mathbf{j}_{\bar{x}}(s_{d+1} F^{\bar{x}}) \circ_{G,p,q} \mathbf{j}_{\bar{y}}(s_{d+1} G^{\bar{y}}) \quad (4.6)$$

for  $\bar{x}, \bar{y} \in \bar{\mathcal{O}}$ ,  $v \in \llbracket 1, \text{lg}(\bar{x}) \rrbracket$  and  $j \in \llbracket 1, \text{lg}(\bar{x}^v) \rrbracket$  and

$$\mathbf{j}_{v,j,\text{out}}^{\bar{x}} \sqcup_{\bar{y}} (s_{d+1} F^{\bar{x}} \circ_{\text{out}}^{\text{nec},v,j} s_{d+1} G^{\bar{y}}) = -(-1)^{(|\mathbf{F}|+d+1)(|\mathbf{G}|+d+1)} \mathbf{j}_{\bar{y}}(s_{d+1} G^{\bar{y}}) \circ_{G,p,q} \mathcal{R}_{\mathcal{A}^*}^{-1}(\mathbf{j}_{\bar{x}}(s_{d+1} F^{\bar{x}})) \quad (4.7)$$

for  $v \in \llbracket 1, \text{lg}(\bar{y}) \rrbracket$  and  $j \in \llbracket 1, \text{lg}(\bar{x}^1) \rrbracket$ , where  $p = \text{lg}(\bar{x}^1) + \dots + \text{lg}(\bar{x}^v) + j + 2$  and  $q = p + \sum_{i=1}^{\text{lg}(\bar{y})} \text{lg}(\bar{y}^i)$ .

*Proof.* We will first show the identity (4.6). Given  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$ ,  $\bar{y} = (\bar{y}^1, \dots, \bar{y}^m) \in \bar{\mathcal{O}}$ , both the compositions

$$s_{d+1} F^{\bar{x}} \circ_{\text{inn}}^{\text{nec},v,j} s_{d+1} G^{\bar{y}}$$

and

$$\mathbf{j}_{\bar{x}}(s_{d+1} F^{\bar{x}}) \circ_{G,p,q} \mathbf{j}_{\bar{y}}(s_{d+1} G^{\bar{y}})$$

are zero if there are no  $u, v \in \llbracket 1, n \rrbracket$  and  $j \in \llbracket 1, \text{lg}(\bar{x}^v) \rrbracket$  such that  $\text{lt}(\bar{y}^u) = x_j^v$  and  $\text{rt}(\bar{y}^{u+1}) = x_{j+1}^v$  with the convention that  $\bar{x}^{n+1} = \bar{x}^1$ . We will thus assume that there exist such  $u, v \in \llbracket 1, n \rrbracket$  and  $j \in \llbracket 1, \text{lg}(\bar{x}^v) \rrbracket$ . We can further suppose that  $u = n$  because of the invariance under the action of  $C_n$ . To simplify the expressions, we will denote the result of applying  $F^{\bar{x}}$  on any argument as a tensor product  $\text{lt}(\bar{x}^1) F_{\text{rt}(\bar{x}^2)}^{-d} \otimes \dots \otimes \text{lt}(\bar{x}^{n-1}) F_{\text{rt}(\bar{x}^n)}^{-d} \otimes \text{lt}(\bar{x}^n) F_{\text{rt}(\bar{x}^1)}^{-d}$ , where we omit those arguments. We have

$$\begin{aligned} & \mathbf{j}_{v,j,\text{inn}}^{\bar{x}} \sqcup_{\bar{y}} (s_{d+1} F^{\bar{x}} \circ_{\text{inn}}^{\text{nec},v,j} s_{d+1} G^{\bar{y}}) \\ & (\bar{s}a^n, t f^{n-1}, \dots, \bar{s}a^{v+1}, t f^v, \bar{s}a_{>j}^v, \bar{s}b^m, t g^{m-1}, \dots, \bar{s}b^2, t g^1, \bar{s}b^1, \bar{s}a_{\leq j}^v, t f^{v-1}, \dots, \bar{s}a^2, t f^1, \bar{s}a^1) \\ & = (-1)^\epsilon ((f^1 \circ s_d) \otimes \dots \otimes (f^v \circ s_d) \otimes (g^1 \circ s_d) \otimes \dots \otimes (g^{m-1} \circ s_d) \otimes (f^{v+1} \circ s_d) \otimes \dots \otimes (f^{n-1} \circ s_d) \otimes \text{id}) \\ & (s_{d+1} F^{\bar{x}} \circ_{\text{inn}}^{\text{nec},v,j} s_{d+1} G^{\bar{y}}) (\bar{s}a^1, \bar{s}a^2, \dots, \bar{s}a^{v-1}, \bar{s}a_{\leq j}^v, \bar{s}b^1, \dots, \bar{s}b^{m-1}, \bar{s}b^m, \bar{s}a_{>j}^v, \dots, \bar{s}a^{n-1}, \bar{s}a^n) \\ & = (-1)^{\epsilon+\epsilon'} \\ & ((f^1 \circ s_d) \otimes \dots \otimes (f^v \circ s_d) \otimes (g^1 \circ s_d) \otimes \dots \otimes (g^{m-1} \circ s_d) \otimes (f^{v+1} \circ s_d) \otimes \dots \otimes (f^{n-1} \circ s_d) \otimes \text{id}) \\ & s_{d+1} (\text{lt}(\bar{x}^1) F_{\text{rt}(\bar{x}^2)}^{-d}, \dots, \text{lt}(\bar{x}^{v-1}) F_{\text{rt}(\bar{x}^v)}^{-d}, \text{lt}(\bar{y}^1) G_{\text{rt}(\bar{y}^2)}^{-d}, \dots, \text{lt}(\bar{y}^{m-1}) G_{\text{rt}(\bar{y}^m)}^{-d}, \text{lt}(\bar{x}^v) F_{\text{rt}(\bar{x}^{v+1})}^{-d}, \dots, \text{lt}(\bar{x}^n) F_{\text{rt}(\bar{x}^1)}^{-d}) \end{aligned}$$

for elements  $\bar{s}a^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ ,  $\bar{s}b^i \in \mathcal{A}[1]^{\otimes \bar{y}^i}$ ,  $t f^i \in \text{rt}(\bar{x}^{i+1}) \mathcal{A}_{\text{lt}(\bar{x}^i)}^*[d]$  and  $t g^i \in \text{rt}(\bar{y}^{i+1}) \mathcal{A}_{\text{lt}(\bar{y}^i)}^*[d]$ , with

$$\begin{aligned} \epsilon & = \sum_{i=v}^{n-1} |t f^i| \sum_{k=i+1}^n |\bar{s}a^k| + \sum_{i=1}^{m-1} |t g^i| \left( \sum_{k=v+1}^n |\bar{s}a^k| + |\bar{s}a_{>j}^v| + \sum_{k=i+1}^m |\bar{s}b^k| \right) \\ & + \sum_{i=1}^{v-1} |t f^i| \left( \sum_{k=v+1}^n |\bar{s}a^k| + |\bar{s}a_{>j}^v| + \sum_{k=1}^m |\bar{s}b^k| + |\bar{s}a_{\leq j}^v| + \sum_{k=i+1}^{v-1} |\bar{s}a^k| \right) + d(n+m) \\ & + (|\mathbf{F}| + |\mathbf{G}| + 1) \left( \sum_{i=1}^{n-1} |t f^i| + \sum_{i=1}^{m-1} |t g^i| \right) + \sum_{i=1}^{n-1} |t f^i| \sum_{i=1}^{m-1} |t g^i| + \sum_{1 \leq i < k \leq n-1} |t f^i| |t f^k| \end{aligned}$$

$$\begin{aligned}
& + \sum_{1 \leq i < k \leq m-1} |tg^i| |tg^k| + \sum_{1 \leq i < k \leq n} |\bar{s}a^i| |\bar{s}a^k| + |\bar{s}a_{>j}^v| |\bar{s}a_{\leq j}^v| + \sum_{1 \leq i < k \leq m} |\bar{s}b^i| |\bar{s}b^k| + \sum_{i=1}^n |\bar{s}a^i| \sum_{i=1}^m |\bar{s}b^i| \\
\epsilon' & = (|\mathbf{G}| + d + 1) \left( \sum_{i=1}^{v-1} |\bar{s}a^i| + |\bar{s}a_{\leq j}^v| \right) \\
& + \sum_{i=1}^{m-1} |\text{lt}(\bar{y}^i) G_{\text{rt}(\bar{y}^{i+1})}^{-d}| (|\bar{s}a_{>j}^v| + \sum_{i=v+1}^n |\bar{s}a^i| + \sum_{i=v}^n |\text{lt}(\bar{x}^i) F_{\text{rt}(\bar{x}^{i+1})}^{-d}| + |\text{lt}(\bar{y}^m) G_{\text{rt}(\bar{y}^1)}^1| + d + 1)
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
& \mathbf{j}_{\bar{x}}(s_{d+1} F^{\bar{x}}) \circ_{G,p,q} \mathbf{j}_{\bar{y}}(s_{d+1} G^{\bar{y}}) \\
& (\bar{s}a^n, tf^{n-1}, \dots, \bar{s}a^{v+1}, tf^v, \bar{s}a_{>j}^v, \bar{s}b^m, tg^{m-1}, \dots, \bar{s}b^2, tg^1, \bar{s}b^1, \bar{s}a_{\leq j}^v, tf^{v-1}, \dots, \bar{s}a^2, tf^1, \bar{s}a^1) \\
& = (-1)^\delta \mathbf{j}_{\bar{x}}(s_{d+1} F^{\bar{x}}) (\bar{s}a^n, tf^{n-1}, \dots, \bar{s}a^{v+1}, tf^v, \bar{s}a_{>j}^v, \left( \bigotimes_{i=1}^{m-1} (g^i \circ s_d) \otimes \text{id} \right) s_{d+1} G^{\bar{y}} (\bar{s}b^1, \dots, \bar{s}b^m), \\
& \bar{s}a_{\leq j}^v, tf^{v-1}, \dots, \bar{s}a^2, tf^1, \bar{s}a^1) \\
& = (-1)^{\delta+\delta'} ((f^1 \circ s_d) \otimes \dots \otimes (f^{v-1} \circ s_d) \otimes (f^v \circ s_d) \otimes \dots \otimes (f^{n-1} \circ s_d) \otimes \text{id}) \\
& s_{d+1} (\text{lt}(\bar{x}^1) F_{\text{rt}(\bar{x}^2)}^{-d}, \dots, \text{lt}(\bar{x}^{v-1}) F_{\text{rt}(\bar{x}^v)}^{-d}, \lambda_G, \text{lt}(\bar{x}^v) F_{\text{rt}(\bar{x}^{v+1})}^{-d}, \dots, \text{lt}(\bar{x}^n) F_{\text{rt}(\bar{x}^1)}^{-d})
\end{aligned}$$

where  $\lambda_G = \bigotimes_{i=1}^{m-1} (g^i \circ s_d) (\text{lt}(\bar{y}^1) G_{\text{rt}(\bar{y}^2)}^{-d}, \dots, \text{lt}(\bar{y}^{m-1}) G_{\text{rt}(\bar{y}^m)}^{-d}) \in \mathbb{k}$  and

$$\begin{aligned}
\delta & = (|\mathbf{G}| + d + 1) \left( \sum_{i=v+1}^n |\bar{s}a^i| + |\bar{s}a_{>j}^v| + \sum_{i=v}^{n-1} |tf^i| \right) + \sum_{i=1}^{m-1} |tg^i| \sum_{k=i+1}^m |\bar{s}b^k| + d(m-1) \\
& + (|\mathbf{G}| + 1) \sum_{i=1}^{m-1} |tg^i| + \sum_{1 \leq i < k \leq m-1} |tg^i| |tg^k| + \sum_{1 \leq i < k \leq m} |\bar{s}b^i| |\bar{s}b^k| \\
\delta' & = \sum_{1 \leq i < k \leq n-1} |tf^i| |tf^k| + \sum_{1 \leq i < k \leq n} |\bar{s}a^i| |\bar{s}a^k| + |\bar{s}a_{\leq j}^v| |\bar{s}a_{>j}^v| + \sum_{i=1}^n |\bar{s}a^i| |\text{lt}(\bar{y}^m) G_{\text{rt}(\bar{y}^1)}^1| \\
& + \sum_{i=1}^{v-1} |tf^i| \left( \sum_{k=i+1}^n |\bar{s}a^k| + |\text{lt}(\bar{y}^m) G_{\text{rt}(\bar{y}^1)}^1| \right) + \sum_{i=v}^{n-1} |tf^i| \sum_{k=i+1}^n |\bar{s}a^k| \\
& + |\lambda_G| \left( \sum_{i=1}^{v-1} |\bar{s}a^i| + |\bar{s}a_{\leq j}^v| + |\text{lt}(\bar{y}^m) G_{\text{rt}(\bar{y}^1)}^1| + \sum_{i=1}^{v-1} |tf^i| + \sum_{i=v}^n |\text{lt}(\bar{x}^i) F_{\text{rt}(\bar{x}^{i+1})}^{-d}| \right) + d(n-1) \\
& + (|\mathbf{F}| + d) \sum_{i=1}^{n-1} |tf^i| + (d+1) \sum_{i=1}^{m-1} |\text{lt}(\bar{y}^i) G_{\text{rt}(\bar{y}^{i+1})}^{-d}|
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& \mathbf{j}_{\bar{x}}(s_{d+1} F^{\bar{x}}) \circ_{G,p,q} \mathbf{j}_{\bar{y}}(s_{d+1} G^{\bar{y}}) \\
& (\bar{s}a^n, tf^{n-1}, \dots, \bar{s}a^{v+1}, tf^v, \bar{s}a_{>j}^v, \bar{s}b^m, tg^{m-1}, \dots, \bar{s}b^2, tg^1, \bar{s}b^1, \bar{s}a_{\leq j}^v, tf^{v-1}, \dots, \bar{s}a^2, tf^1, \bar{s}a^1) \\
& = (-1)^{\delta+\delta'+\delta''}
\end{aligned}$$

$$((f^1 \circ s_d) \otimes \cdots \otimes (f^v \circ s_d) \otimes (g^1 \circ s_d) \otimes \cdots \otimes (g^{m-1} \circ s_d) \otimes (f^{v+1} \circ s_d) \otimes \cdots \otimes (f^{n-1} \circ s_d) \otimes \text{id}) \\ s_{d+1}(\text{lt}(\bar{x}^1)F_{\text{rt}(\bar{x}^2)}^{-d}, \dots, \text{lt}(\bar{x}^{v-1})F_{\text{rt}(\bar{x}^v)}^{-d}, \text{lt}(\bar{y}^1)G_{\text{rt}(\bar{y}^2)}^{-d}, \dots, \text{lt}(\bar{y}^{m-1})G_{\text{rt}(\bar{y}^m)}^{-d}, \text{lt}(\bar{x}^v)F_{\text{rt}(\bar{x}^{v+1})}^{-d} \cdots \text{lt}(\bar{x}^n)F_{\text{rt}(\bar{x}^1)}^{-d})$$

with  $\delta'' = \sum_{i=1}^{m-1} |tg^i| (\sum_{i=v}^{n-1} |tf^i| + \sum_{i=1}^{v-1} \text{lt}(\bar{x}^i)F_{\text{rt}(\bar{x}^{i+1})}^{-d})$ . One can easily check that  $\epsilon + \epsilon' = \delta + \delta' + \delta'' \pmod{2}$ . Then, the first identity is proved.

We now prove the identity (4.7). Given  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n), \bar{y} = (\bar{y}^1, \dots, \bar{y}^m) \in \bar{\mathcal{O}}$ , both the compositions

$$s_{d+1}F_{\text{rt}(\bar{x}^1)}^{-d} \underset{\text{out}}{\circ}_{\text{nec}, v, j} s_{d+1}G_{\text{rt}(\bar{y}^1)}^{-d}$$

and

$$j_{\bar{y}}(s_{d+1}G_{\text{rt}(\bar{y}^1)}^{-d}) \underset{G, p, q}{\circ} \mathcal{R}_{\mathcal{A}^*}^{-1}(j_{\bar{x}}(s_{d+1}F_{\text{rt}(\bar{x}^1)}^{-d}))$$

are zero if there are no  $v \in \llbracket 1, m \rrbracket$  and  $j \in \llbracket 1, \text{lg}(\bar{x}^1) \rrbracket$  such that  $\bar{x}_{j-1}^1 = \text{lt}(\bar{y}^v)$  and  $\bar{x}_j^1 = \text{rt}(\bar{y}^{v+1})$ . We will thus assume that such  $v \in \llbracket 1, m \rrbracket$  and  $j \in \llbracket 1, \text{lg}(\bar{x}^1) \rrbracket$  exist. Then, we have

$$j_{\bar{x}} \underset{v, j, \text{out}}{\sqcup} \bar{y} (s_{d+1}F_{\text{rt}(\bar{x}^1)}^{-d} \underset{\text{out}}{\circ}_{\text{nec}, v, j} s_{d+1}G_{\text{rt}(\bar{y}^1)}^{-d}) \\ (\bar{s}b^m, tg^{m-1}, \dots, \bar{s}b^{v+1}, \bar{s}a_{\geq j}^1, tf^n, \bar{s}a^n, tf^{n-1}, \bar{s}a^{n-1}, \dots, tf^1, \bar{s}a_{< j}^1, \bar{s}b^v, tg^{v-1}, \dots, \bar{s}b^1) \\ = (-1)^\epsilon \\ ((g^1 \circ s_d) \otimes \cdots \otimes (g^{v-1} \circ s_d) \otimes (f^1 \circ s_d) \otimes \cdots \otimes (f^n \circ s_d) \otimes (g^{v+1} \circ s_d) \otimes \cdots \otimes (g^{m-1} \circ s_d) \otimes \text{id}) \\ (s_{d+1}F_{\text{rt}(\bar{x}^1)}^{-d} \underset{\text{out}}{\circ}_{\text{nec}, v, j} s_{d+1}G_{\text{rt}(\bar{y}^1)}^{-d})(\bar{s}b^1, \dots, \bar{s}b^v, \bar{s}a_{< j}^1, \dots, \bar{s}a^n, \bar{s}a_{\geq j}^1, \bar{s}b^{v+1}, \dots, \bar{s}b^m) \\ = (-1)^{\epsilon+\epsilon'} \\ ((g^1 \circ s_d) \otimes \cdots \otimes (g^{v-1} \circ s_d) \otimes (f^1 \circ s_d) \otimes \cdots \otimes (f^n \circ s_d) \otimes (g^{v+1} \circ s_d) \otimes \cdots \otimes (g^{m-1} \circ s_d) \otimes \text{id}) \\ s_{d+1}(\text{lt}(\bar{y}^1)G_{\text{rt}(\bar{y}^2)}^{-d}, \dots, \text{lt}(\bar{y}^{v-1})G_{\text{rt}(\bar{y}^v)}^{-d}, \text{lt}(\bar{x}^1)F_{\text{rt}(\bar{x}^2)}^{-d}, \dots, \text{lt}(\bar{x}^n)F_{\text{rt}(\bar{x}^1)}^{-d}, \text{lt}(\bar{y}^{v+1})G_{\text{rt}(\bar{y}^{v+2})}^{-d}, \dots, \text{lt}(\bar{y}^m)G_{\text{rt}(\bar{y}^1)}^{-d})$$

with

$$\epsilon = (|\mathbf{F}| + |\mathbf{G}| + 1) \left( \sum_{i=1}^{m-1} |tg^i| + \sum_{i=1}^n |tf^i| \right) + d(n+m) + \sum_{i=v+1}^{m-1} |tg^i| \sum_{k=i+1}^m |\bar{s}b^k| + |\bar{s}a_{\geq j}^1| |\bar{s}a_{< j}^1| \\ + \sum_{1 \leq i < k \leq n} |\bar{s}a^i| |\bar{s}a^k| + \sum_{i=1}^n |tf^i| \left( \sum_{k=i+1}^n |\bar{s}a^k| + |\bar{s}a_{\geq j}^1| + \sum_{i=v+1}^m |\bar{s}b^i| \right) + \sum_{i=1}^{v-1} |tg^i| \left( \sum_{i=1}^n |\bar{s}a^i| + \sum_{k=i+1}^m |\bar{s}b^k| \right) \\ + \sum_{1 \leq i < k \leq m} |\bar{s}b^i| |\bar{s}b^k| + \sum_{1 \leq i < k \leq m-1} |tg^i| |tg^k| + \sum_{1 \leq i < k \leq n} |tf^i| |tf^k| + \sum_{i=1}^{m-1} |tg^i| \sum_{i=1}^n |tf^i| + \sum_{i=1}^m |\bar{s}b^i| \sum_{i=1}^n |\bar{s}a^i| \\ \epsilon' = \sum_{i=1}^n |\bar{s}a^i| \left( \sum_{i=v+1}^m |\bar{s}b^i| + \sum_{i=v+1}^m |\text{lt}(\bar{y}^i)G_{\text{rt}(\bar{y}^{i+1})}^{-d}| \right) + (|\mathbf{F}| + d + 1) \sum_{i=1}^{v-1} |\text{lt}(\bar{y}^i)G_{\text{rt}(\bar{y}^{i+1})}^{-d}| \\ + |\bar{s}a_{\geq j}^1| \sum_{i=2}^n |\bar{s}a^i| + |\bar{s}a_{< j}^v| |\text{lt}(\bar{y}^v)G_{\text{rt}(\bar{y}^{v+1})}^{-d}|$$

On the other hand, we have that

$$\begin{aligned} & \mathcal{J}_{\bar{y}}(s_{d+1}G^{\bar{y}}) \circ_{G,p,q} \mathcal{R}_{\mathcal{A}^*}^{-1}(\mathcal{J}_{\bar{x}}(s_{d+1}F^{\bar{x}})) \\ & (\bar{s}b^m, tg^{m-1}, \dots, \bar{s}b^{v+1}, \bar{s}a_{\geq j}^1, tf^n, \bar{s}a^n, tf^{n-1}, \bar{s}a^{n-1}, \dots, \bar{s}a_{< j}^1, \bar{s}b^v, tg^{v-1}, \dots, \bar{s}b^1) \\ & = (-1)^\delta \mathcal{J}_{\bar{y}}(s_{d+1}G^{\bar{y}})(\bar{s}b^m, tg^{m-1}, \dots, \bar{s}b^{v+1}, tg, \bar{s}b^v, tg^{v-1}, \dots, \bar{s}b^1) \end{aligned}$$

where  $tg = \mathcal{R}_{\mathcal{A}^*}^{-1}(\mathcal{J}_{\bar{x}}(s_{d+1}F^{\bar{x}}))(\bar{s}a_{\geq j}^1, tf^n, \bar{s}a^n, tf^{n-1}, \bar{s}a^{n-1}, \dots, \bar{s}a_{< j}^1) \in \mathcal{A}^*[d]$  and

$$\delta = (|\mathbf{F}| + d + 1) \left( \sum_{i=v+1}^m |\bar{s}b^i| + \sum_{i=v+1}^{m-1} |tg^i| \right)$$

Therefore, we have that

$$\begin{aligned} & \mathcal{J}_{\bar{y}}(s_{d+1}G^{\bar{y}}) \circ_{G,p,q} \mathcal{R}_{\mathcal{A}^*}^{-1}(\mathcal{J}_{\bar{x}}(s_{d+1}F^{\bar{x}})) \\ & (\bar{s}b^m, tg^{m-1}, \dots, \bar{s}b^{v+1}, tg^{v+1}, \bar{s}a_{\geq j}^1, \bar{s}a^n, tf^{n-1}, \bar{s}a^{n-1}, \dots, \bar{s}a_{< j}^1, \bar{s}b^v, tg^{v-1}, \dots, \bar{s}b^1) \\ & = (-1)^{\delta+\delta'} ((g^1 \circ s_d) \otimes \dots \otimes (g^{v-1} \circ s_d) \otimes (g \circ s_d) \otimes (g^{v+1} \circ s_d) \otimes \dots \otimes (g^m \circ s_d) \otimes \text{id}) \\ & s_{d+1}(\text{lt}(\bar{y}^1)G_{\text{rt}(\bar{y}^2)}^{-d}, \dots, \text{lt}(\bar{y}^{v-1})G_{\text{rt}(\bar{y}^v)}^{-d}, \text{lt}(\bar{y}^v)G_{\text{rt}(\bar{y}^{v+1})}^{-d}, \dots, \text{lt}(\bar{y}^m)G_{\text{rt}(\bar{y}^1)}^{-d}) \end{aligned}$$

where

$$\begin{aligned} \delta' & = |tg| \sum_{k=v+1}^m |\bar{s}b^k| + \sum_{i=1}^{m-1} |tg^i| \sum_{k=i+1}^m |\bar{s}b^k| + \sum_{1 \leq i < k \leq m} |\bar{s}b^i| |\bar{s}b^k| + \sum_{1 \leq i < k \leq m-1} |tg^i| |tg^k| \\ & + \sum_{i=1}^{m-1} |tg^i| |tg| + (|\mathbf{G}| + 1) \left( \sum_{i=1}^{m-1} |tg^i| + |tg| \right) + d(m-1) \end{aligned}$$

Furthermore, by definition, we have that

$$\begin{aligned} & g(\text{lt}(\bar{y}^v)G_{\text{rt}(\bar{y}^{v+1})}) \\ & = (-1)^{|tg|+1} \Gamma(\mathcal{R}_{\mathcal{A}^*}^{-1}(\mathcal{J}_{\bar{x}}(s_{d+1}F^{\bar{x}}))(\bar{s}a_{\geq j}^1, tf^n, \bar{s}a^n, tf^{n-1}, \bar{s}a^{n-1}, \dots, \bar{s}a_{< j}^1), \text{lt}(\bar{y}^{v-1})G_{\text{rt}(\bar{y}^v)}^1) \\ & = (-1)^{|tg|+1+\Delta} \Gamma(\mathcal{J}_{\bar{x}}(s_{d+1}F^{\bar{x}})(\bar{s}a^n, tf^{n-1}, \bar{s}a^{n-1}, \dots, \bar{s}a_{< j}^1 \otimes \text{lt}(\bar{y}^{v-1})G_{\text{rt}(\bar{y}^v)}^1 \otimes \bar{s}a_{\geq j}^1), tf^n) \\ & = (-1)^{|tg|+1+\Delta+\Delta'} \Gamma(((f^1 \circ s_d) \otimes \dots \otimes (f^{n-1} \circ s_d) \otimes \text{id})(\text{lt}(\bar{x}^1)F_{\text{rt}(\bar{x}^2)}^{-d} \dots \text{lt}(\bar{x}^n)F_{\text{rt}(\bar{x}^1)}^{-d}), tf^n) \\ & = (-1)^{|tg|+1+\Delta+\Delta'+\Delta''} ((f^1 \circ s_d) \otimes \dots \otimes (f^{n-1} \circ s_d) \otimes (f^n \circ s_d))(\text{lt}(\bar{x}^1)F_{\text{rt}(\bar{x}^2)}^{-d} \dots \otimes \text{lt}(\bar{x}^n)F_{\text{rt}(\bar{x}^1)}^{-d}) \end{aligned}$$

where

$$\begin{aligned} \Delta & = (|\bar{s}a_{\geq j}^1| + |tf^n|) \left( \sum_{i=2}^n |\bar{s}a^i| + |\bar{s}a_{< j}^1| + |\text{lt}(\bar{y}^v)G_{\text{rt}(\bar{y}^{v+1})}^1| + \sum_{i=1}^{n-1} |tf^i| \right) \\ \Delta' & = \sum_{1 \leq i < k \leq n} |\bar{s}a^i| |\bar{s}a^k| + |\text{lt}(\bar{y}^v)G_{\text{rt}(\bar{y}^{v+1})}^1| \sum_{i=2}^n |\bar{s}a^i| + \sum_{1 \leq i < k \leq n-1} |tf^i| |tf^k| \\ & + d(n-1) + (|\mathbf{F}| + d) \sum_{i=1}^{n-1} |tf^i| + \sum_{i=1}^{n-1} |tf^i| \sum_{j=i+1}^n |\bar{s}a^j| \\ \Delta'' & = |tf^n| \left( 1 + \sum_{i=1}^n |\text{lt}(\bar{x}^i)F_{\text{rt}(\bar{x}^{i+1})}^{-d}| \right) \end{aligned}$$

Finally, we have that

$$\begin{aligned}
& \mathbf{j}_{\bar{y}}(s_{d+1}G^{\bar{y}}) \circ_{G,p,q} \mathcal{R}_{\mathcal{A}^*}^{-1}(\mathbf{j}_{\bar{x}}(s_{d+1}F^{\bar{x}})) \\
& (\bar{s}^m, tg^{m-1}, \dots, \bar{s}^v, tg^{v-1}, \bar{s}^1_{\geq j}, tf^n, \bar{s}^n, tf^{n-1}, \bar{s}^{n-1}, \dots, \bar{s}^1_{< j}, \bar{s}^v, tg^{v-1}, \dots, \bar{s}^1) \\
& = (-1)^\gamma \\
& ((g^1 \circ s_d) \otimes \dots \otimes (g^{v-1} \circ s_d) \otimes (f^1 \circ s_d) \otimes \dots \otimes (f^n \circ s_d) \otimes (g^{v+1} \circ s_d) \otimes \dots \otimes (g^{m-1} \circ s_d) \otimes \text{id}) \\
& s_{d+1}(\text{lt}(\bar{y}^1)G_{\text{rt}(\bar{y}^2)}^{-d}, \dots, \text{lt}(\bar{y}^{v-1})G_{\text{rt}(\bar{y}^v)}^{-d}, \text{lt}(\bar{x}^1)F_{\text{rt}(\bar{x}^2)}^{-d}, \dots, \text{lt}(\bar{x}^n)F_{\text{rt}(\bar{x}^1)}^{-d}, \text{lt}(\bar{y}^{v+1})G_{\text{rt}(\bar{y}^{v+2})}^{-d}, \dots, \text{lt}(\bar{y}^m)G_{\text{rt}(\bar{y}^1)}^{-d})
\end{aligned}$$

where

$$\begin{aligned}
\gamma &= \delta + \delta' + |tg| \left( \sum_{i=v+1}^{m-1} |tg^i| + d + 1 + \sum_{i=1}^{v-1} |\text{lt}(\bar{y}^i)G_{\text{rt}(\bar{y}^{i+1})}^{-d}| \right) \\
&+ \Delta + \Delta' + \Delta'' + |tg| + 1 + \sum_{i=1}^n |tf^i| \left( \sum_{i=1}^{v-1} |\text{lt}(\bar{y}^i)G_{\text{rt}(\bar{y}^{i+1})}^{-d}| + \sum_{i=v+1}^{m-1} |tg^i| \right)
\end{aligned}$$

It is straightforward to check that  $\epsilon + \epsilon' + \gamma = 1 + (|\mathbf{F}| + d + 1)(|\mathbf{G}| + d + 1) \pmod{2}$ .  $\square$

**Corollary 4.30.** *The necklace bracket  $[-, -]_{\text{nec}}$  introduced in Definition 4.21 gives a graded Lie algebra structure on  $\text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])^{C_{\text{lg}(\bullet)}}[d+1]$ .*

*Proof.* For  $\mathbf{F}, \mathbf{G} \in \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])^{C_{\text{lg}(\bullet)}}$ , we have that

$$\mathbf{j}([s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G}]_{\text{nec}}) = \pi_{\mathcal{A}}([\mathbf{j}(s_{d+1}\mathbf{F}), \mathbf{j}(s_{d+1}\mathbf{G})]_G)$$

where  $\pi_{\mathcal{A}} : \mathcal{A}[1] \oplus \mathcal{A}^*[d] \rightarrow \mathcal{A}[1]$  is the canonical projection. Moreover, using that  $\mathbf{j}$  is injective, we have that  $[-, -]_{\text{nec}}$  is a graded Lie bracket.  $\square$

### 4.3 Pre-Calabi-Yau structures

**Definition 4.31.** *A  $d$ -pre-Calabi-Yau structure on a graded quiver  $\mathcal{A}$  is an element*

$$s_{d+1}M_{\mathcal{A}} \in \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])^{C_{\text{lg}(\bullet)}}[d+1]$$

*of degree 1, solving the Maurer-Cartan equation*

$$[s_{d+1}M_{\mathcal{A}}, s_{d+1}M_{\mathcal{A}}]_{\text{nec}} = 0$$

*Note that, since  $s_{d+1}M_{\mathcal{A}}$  has degree 1, this is tantamount to requiring that  $s_{d+1}M_{\mathcal{A}} \circ_{\text{nec}} s_{d+1}M_{\mathcal{A}} = 0$ .*

We now recall the following result of [5] which states the link between a  $d$ -pre-Calabi-Yau structure on a Hom-finite graded quiver  $\mathcal{A}$  and a cyclic  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{A}^*[d-1]$ .

**Proposition 4.32.** *A  $d$ -pre-Calabi-Yau structure on a Hom-finite graded quiver  $\mathcal{A}$  is equivalent to the data of an  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{A}^*[d-1]$  that restricts to  $\mathcal{A}$  and that satisfies the cyclic condition (3.1).*

*Proof.* Consider an element  $s_{d+1}M_{\mathcal{A}} \in \text{Multi}^\bullet(\text{Bar}(\mathcal{A})[d])^{C_{\text{lg}(\bullet)}}[d+1]$  of degree 1. We then define maps

$$m_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}}^{\bar{x}^1, \dots, \bar{x}^n} : \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\otimes \bar{x}^i} \otimes_{\text{rt}(\bar{x}^{i+1})} \mathcal{A}_{\text{lt}(\bar{x}^i)}^*[d]) \otimes \mathcal{A}[1]^{\otimes \bar{x}^n} \rightarrow \text{lt}(\bar{x}^1) \mathcal{A}_{\text{rt}(\bar{x}^n)} \quad (4.9)$$

and

$$m_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}^*}^{\bar{x}^1, \dots, \bar{x}^n} : \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\otimes \bar{x}^i} \otimes_{\text{rt}(\bar{x}^{i+1})} \mathcal{A}_{\text{lt}(\bar{x}^i)}^*[d]) \otimes \mathcal{A}[1]^{\otimes \bar{x}^n} \rightarrow \text{lt}(\bar{x}^1) \mathcal{A}_{\text{rt}(\bar{x}^n)}^*[d-1] \quad (4.10)$$

uniquely determined by  $sm_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}}^{\bar{x}^1, \dots, \bar{x}^n} = j_{\bar{x}^1-1}(s_{d+1}M_{\mathcal{A}}^{\bar{x}^1, \dots, \bar{x}^1})$  for  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n)$  and such that the cyclicity conditions (4.4) and (4.5) hold. This is well-defined, since the bilinear form  $\Gamma^{\mathcal{A}}$  is nondegenerate. By Proposition 4.29,  $s_{d+1}M_{\mathcal{A}}$  defines a  $d$ -pre-Calabi-Yau structure if and only if  $sm_{\mathcal{A} \oplus \mathcal{A}^*}$  defines a cyclic  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{A}^*[d-1]$ . If  $\mathcal{A}$  is Hom-finite, the bijectivity of  $j$  tells us that the collection of maps (4.9) and (4.10) are in correspondence with maps of the form  $M_{\mathcal{A}}$ , which shows the equivalence.  $\square$

## 5 Pre-Calabi-Yau morphisms

### 5.1 The mixed necklace graded Lie algebra

One can also define a “mixed” necklace bracket, which will be useful in the next section. As we did for the necklace bracket, we first define the following graded vector space.

**Definition 5.1.** *Given graded quivers  $\mathcal{A}$  and  $\mathcal{B}$  with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  and a map  $\Phi : \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{B}}$ , consider the graded vector space*

$$\begin{aligned} & \mathfrak{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d]) \\ &= \prod_{n \in \mathbb{N}} \prod_{\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}^n} \text{Hom}_{\mathbb{k}} \left( \bigotimes_{i=1}^n \mathcal{A}[1]^{\otimes \bar{x}^i}, \bigotimes_{i=1}^{n-1} \mathcal{B}_{\Phi(\text{lt}(\bar{x}^i))} \mathcal{B}_{\Phi(\text{rt}(\bar{x}^{i+1}))}[-d] \otimes (\text{lt}(\bar{x}^n) \mathcal{A}[-d]_{\text{rt}(\bar{x}^1)} \oplus_{\Phi(\text{lt}(\bar{x}^n))} \mathcal{B}^*[-1]_{\Phi(\text{rt}(\bar{x}^1))}) \right) \end{aligned}$$

**Definition 5.2.** *Consider  $s_{d+1}F^{\bar{x}} \in \mathfrak{B}_{\bar{x}}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$  with  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}^n$ . Then  $s_{d+1}F^{\bar{x}}$  induces a morphism of the form*

$$\mathcal{A}[1]^{\otimes \bar{x}^1} \otimes_{\Phi(\text{rt}(\bar{x}^1))} \mathcal{B}_{\Phi(\text{lt}(\bar{x}^n))}[-d] \otimes \mathcal{A}[1]^{\otimes \bar{x}^n} \otimes \cdots \otimes \mathcal{A}[1]^{\otimes \bar{x}^{n-1}} \rightarrow \bigotimes_{i=1}^{n-1} \mathcal{B}_{\Phi(\text{lt}(\bar{x}^i))} \mathcal{B}_{\Phi(\text{rt}(\bar{x}^{i+1}))}[-d] \quad (5.1)$$

sending  $(\bar{s}a^1, s_{-d}b, \bar{s}a^n, \dots, \bar{s}a^{n-1})$  to

$$(-1)^\epsilon F_{\Phi(\text{lt}(\bar{x}^1))}^{-d} \otimes \cdots \otimes F_{\Phi(\text{lt}(\bar{x}^{n-1}))}^{-d} \otimes F_{\Phi(\text{rt}(\bar{x}^n))}^{-d} \otimes F_{\Phi(\text{lt}(\bar{x}^n))}^1 F_{\Phi(\text{rt}(\bar{x}^1))}^1(sb)$$

where we have written

$$\begin{aligned} & (\text{id}^{\otimes (n-1)} \otimes \pi_{\mathcal{B}})(F^{\bar{x}^1, \dots, \bar{x}^n}(\bar{s}a^1, \dots, \bar{s}a^n)) \\ &= F_{\Phi(\text{rt}(\bar{x}^1))}^{-d} \otimes \cdots \otimes F_{\Phi(\text{rt}(\bar{x}^{n-1}))}^{-d} \otimes F_{\Phi(\text{lt}(\bar{x}^n))}^{-d} \otimes F_{\Phi(\text{rt}(\bar{x}^n))}^1 F_{\Phi(\text{lt}(\bar{x}^1))}^1 \end{aligned}$$

where  $\pi_{\mathcal{B}}$  denotes the canonical projection

$$\mathrm{lt}(\bar{x}^n)\mathcal{A}[-d]_{\mathrm{rt}(\bar{x}^1)} \oplus_{\Phi(\mathrm{lt}(\bar{x}^n))} \mathcal{B}^*[-1]_{\Phi(\mathrm{rt}(\bar{x}^1))} \rightarrow \Phi(\mathrm{lt}(\bar{x}^n))\mathcal{B}^*[-1]_{\Phi(\mathrm{rt}(\bar{x}^1))}$$

for  $\bar{s}a^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$  with

$$\epsilon = (d+1)(|\bar{s}a^1| + |F^{\bar{x}}|) + (|sb| + |\bar{s}a^n|) \sum_{i=2}^{n-1} |\bar{s}a^i| + |sb||\bar{s}a^n| + (d+1) \sum_{i=1}^{n-1} |\Phi(\mathrm{lt}(\bar{x}^i))F_{\Phi(\mathrm{rt}(\bar{x}^{i+1}))}^{-d}|$$

and a morphism of the form

$$\mathcal{A}[1]^{\otimes \bar{x}^1} \otimes \mathcal{A}[1]^{\otimes \bar{x}^2} \otimes \cdots \otimes \mathcal{A}[1]^{\otimes \bar{x}^n} \rightarrow \bigotimes_{i=2}^{n-1} \Phi(\mathrm{lt}(\bar{x}^i))\mathcal{B}_{\Phi(\mathrm{rt}(\bar{x}^{i+1}))}[-d] \otimes \mathrm{lt}(\bar{x}^n)\mathcal{A}[1]_{\mathrm{rt}(\bar{x}^1)} \quad (5.2)$$

given by  $(\mathrm{id}^{\otimes (n-1)} \otimes (s_{d+1} \circ \pi_{\mathcal{A}})) \circ F^{\bar{x}^1, \dots, \bar{x}^n}$  where  $\pi_{\mathcal{A}}$  denotes the canonical projection

$$\mathrm{lt}(\bar{x}^n)\mathcal{A}[-d]_{\mathrm{rt}(\bar{x}^1)} \oplus_{\Phi(\mathrm{lt}(\bar{x}^n))} \mathcal{B}^*[-1]_{\Phi(\mathrm{rt}(\bar{x}^1))} \rightarrow \mathrm{lt}(\bar{x}^n)\mathcal{A}[-d]_{\mathrm{rt}(\bar{x}^1)}$$

**Lemma 5.3.** *Given two graded quivers  $\mathcal{A}$  and  $\mathcal{B}$  with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  and a map  $\Phi : \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{B}}$ , we have an isomorphism*

$$B : \mathfrak{B}_{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1] \xrightarrow{\cong} \mathfrak{B}_{\bullet}^{\mathcal{A}}(\mathcal{A}[1], \mathcal{B}[-d]) \oplus \mathfrak{B}_{\bullet}^{\mathcal{B}}(\mathcal{A}[1], \mathcal{B}[-d])$$

where

$$\mathfrak{B}_{\bullet}^{\mathcal{A}}(\mathcal{A}[1], \mathcal{B}[-d]) = \prod_{n \in \mathbb{N}} \prod_{\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}^n} \mathrm{Hom}_{\mathbb{k}} \left( \bigotimes_{i=1}^n \mathcal{A}[1]^{\otimes \bar{x}^i}, \bigotimes_{i=1}^{n-1} \Phi(\mathrm{lt}(\bar{x}^i))\mathcal{B}_{\Phi(\mathrm{rt}(\bar{x}^{i+1}))}[-d] \otimes \mathrm{lt}(\bar{x}^n)\mathcal{A}[1]_{\mathrm{rt}(\bar{x}^1)} \right)$$

and

$$\mathfrak{B}_{\bullet}^{\mathcal{B}}(\mathcal{A}[1], \mathcal{B}[-d]) = \prod_{n \in \mathbb{N}} \prod_{\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}^n} \mathrm{Hom}_{\mathbb{k}} \left( \mathcal{A}[1]^{\otimes \bar{x}^1} \otimes_{\Phi(\mathrm{rt}(\bar{x}^1))} \mathcal{B}[-d]_{\Phi(\mathrm{lt}(\bar{x}^n))} \otimes \mathcal{A}[1]^{\otimes \bar{x}^n} \otimes \cdots \otimes \mathcal{A}[1]^{\otimes \bar{x}^{n-1}}, \bigotimes_{i=1}^{n-1} \Phi(\mathrm{lt}(\bar{x}^i))\mathcal{B}_{\Phi(\mathrm{rt}(\bar{x}^{i+1}))}[-d] \right)$$

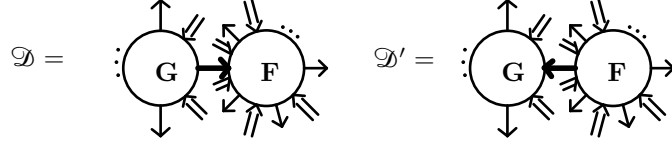
sending an element  $s_{d+1}\mathbf{F} \in \mathfrak{B}_{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$  to the elements defined in (5.2) and (5.1).

**Definition 5.4.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be graded quivers with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  and consider a map  $\Phi : \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{B}}$ . We define the graded quiver  $\mathcal{Q}_{\Phi}$  whose set of objects is  $\mathcal{O}_{\mathcal{A}}$  and whose spaces of morphisms are  ${}_y(\mathcal{Q}_{\Phi})_x = {}_y\mathcal{A}_x \oplus_{\Phi(y)} \mathcal{B}_{\Phi(x)}^*[d-1]$  for  $x, y \in \mathcal{O}_{\mathcal{A}}$ .*

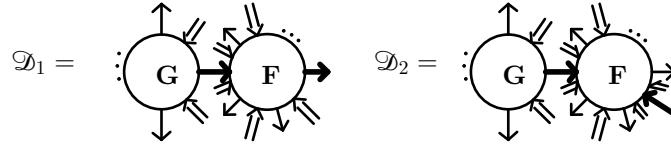
**Definition 5.5.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be graded quivers with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  and consider a map  $\Phi : \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{B}}$ . Given  $s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G} \in \mathfrak{B}_{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$ , we define their  **$\Phi$ -mixed necklace product** as the element  $s_{d+1}\mathbf{F} \circ_{\Phi \text{ nec}} s_{d+1}\mathbf{G} \in \mathfrak{B}_{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$  given by*

$$(s_{d+1}\mathbf{F} \circ_{\Phi \text{ nec}} s_{d+1}\mathbf{G})^{\bar{x}} = \sum \mathcal{E}(\mathcal{D}) + \sum \mathcal{E}(\mathcal{D}') \in \mathfrak{B}_{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$$

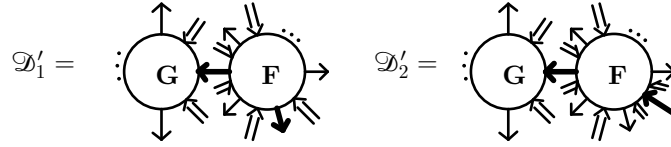
where the sums are over all the filled diagrams  $\mathcal{D}$  and  $\mathcal{D}'$  of type  $\bar{x}$  and of the form



More precisely, each of  $\mathcal{D}$  and  $\mathcal{D}'$  can be pictured as two different filled diagrams by put in bold the arrow corresponding to the last output of  $\mathbf{F}$  which is either an element in  $\mathcal{A}[1]$  or in  $\mathcal{B}^*[d]$ . Thus,  $\mathcal{D}$  is one of the following



and  $\mathcal{D}'$  is one of the following



**Remark 5.6.** The diagrams of Definition 5.5 are filled with  $\mathbf{F}$  and  $\mathbf{G}$  seen as elements of  $\text{Multi}^\bullet(\text{Bar}(\mathcal{Q}'_\Phi)[d])$  where  $\mathcal{Q}'_\Phi$  is the graded quiver whose set of objects is  $\mathcal{O}_\mathcal{A}$  and whose spaces of morphisms are  ${}_y(\mathcal{Q}'_\Phi)_x = {}_y\mathcal{A}_x \oplus_{\Phi(y)} \mathcal{B}_{\Phi(x)}$ .

**Definition 5.7.** Given graded quivers  $\mathcal{A}$  and  $\mathcal{B}$  with respective sets of objects  $\mathcal{O}_\mathcal{A}$  and  $\mathcal{O}_\mathcal{B}$  and a map  $\Phi : \mathcal{O}_\mathcal{A} \rightarrow \mathcal{O}_\mathcal{B}$ , the  $\Phi$ -mixed necklace bracket is the graded Lie bracket which is defined for elements  $\mathbf{F}, \mathbf{G} \in \mathfrak{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])$  by

$$[s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G}]_{\Phi \text{ nec}} = s_{d+1}\mathbf{F} \circ_{\Phi \text{ nec}} s_{d+1}\mathbf{G} - (-1)^{(|\mathbf{F}|+d+1)(|\mathbf{G}|+d+1)} s_{d+1}\mathbf{G} \circ_{\Phi \text{ nec}} s_{d+1}\mathbf{F}$$

**Lemma 5.8.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be graded quivers with respective sets of objects  $\mathcal{O}_\mathcal{A}$  and  $\mathcal{O}_\mathcal{B}$  and consider a map  $\Phi : \mathcal{O}_\mathcal{A} \rightarrow \mathcal{O}_\mathcal{B}$ . Then, we have an injective map

$$j^\Phi : \mathfrak{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])[d+1] \rightarrow C(\mathcal{A} \oplus \mathcal{B}^*[d-1])[1]$$

sending  $s_{d+1}\phi \in \mathfrak{B}_{\bar{x}}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$  to  $s\psi_{\bar{x}}$  where

$$\psi_{\bar{x}} : \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\otimes \bar{x}^{n-i+1}} \otimes_{\Phi(\text{rt}(\bar{x}^{n-i+1}))} \mathcal{B}^*[-d]_{\Phi(\text{lt}(\bar{x}^{n-i}))}) \otimes \mathcal{A}[1]^{\otimes \bar{x}^1} \rightarrow \text{lt}(\bar{x}^n)\mathcal{A}[-d]_{\text{rt}(\bar{x}^1)} \oplus_{\Phi(\text{lt}(\bar{x}^n))} \mathcal{B}^*[-1]_{\Phi(\text{rt}(\bar{x}^1))}$$



is given by

$$\psi_{\bar{x}}(\bar{s}a^n, tf^{n-1}, \bar{s}a^{n-2}, tf^{n-2}, \dots, \bar{s}a^2, tf^2, \bar{s}a^1) = (-1)^\epsilon \left( \bigotimes_{i=1}^n (f^i \circ s_d) \otimes s_d \right) (\phi(\bar{s}a^1, \bar{s}a^2, \dots, \bar{s}a^n))$$

for  $n = \lg(\bar{x})$ ,  $\bar{s}a^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$  and  $tf^i \in \Phi(\text{rt}(\bar{x}^{i+1}))\mathcal{B}_{\Phi(\text{lt}(\bar{x}^i))}^*[d]$  with

$$\epsilon = \sum_{1 \leq i \leq j \leq n-1} |\bar{s}a^i| |tf^j| + (|\phi| + d) \sum_{i=1}^{n-1} |tf^i| + d(n-1) + \sum_{1 \leq i < j \leq n} |\bar{s}a^i| |\bar{s}a^j| + \sum_{1 \leq i < j \leq n-1} |tf^i| |tf^j|$$

**Proposition 5.9.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be graded quivers with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  and consider a map  $\Phi : \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{B}}$ . Then, we have*

$$\dot{j}^{\Phi}([s_{d+1}\mathbf{F}, s_{d+1}\mathbf{G}]_{\Phi \text{ nec}}) = [j^{\Phi}(s_{d+1}\mathbf{F}), j^{\Phi}(s_{d+1}\mathbf{G})]_G$$

for  $\mathbf{F}, \mathbf{G} \in \mathcal{B}_{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])$ .

## 5.2 Pre-Calabi-Yau morphisms (after M. Kontsevich, A. Takeda and Y. Vlasopoulos)

Following the article [5], we recall the definition of the category of  $d$ -pre-Calabi-Yau categories.

**Definition 5.10.** *Given graded quivers  $\mathcal{A}$  and  $\mathcal{B}$  with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  and a map  $F_0 : \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{B}}$ , consider the graded vector space*

$$\begin{aligned} & \text{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d]) \\ &= \prod_{n \in \mathbb{N}} \prod_{\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}^n} \text{Hom}_{\mathbb{k}} \left( \bigotimes_{i=1}^n \mathcal{A}[1]^{\otimes \bar{x}^i}, \bigotimes_{i=1}^{n-1} \mathcal{B}_{F_0(\text{lt}(\bar{x}^i))} \mathcal{B}_{F_0(\text{rt}(\bar{x}^{i+1}))}[-d] \otimes_{F_0(\text{lt}(\bar{x}^n))} \mathcal{B}_{F_0(\text{rt}(\bar{x}^1))}[-d] \right) \end{aligned}$$

The action of  $\tau = (\tau_n)_{n \in \mathbb{N}^*} \in \prod_{n \in \mathbb{N}^*} C_n$  on an element  $\mathbf{F} = (F^{\bar{x}})_{\bar{x} \in \bar{\mathcal{O}}} \in \text{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])$  is the element  $\tau \cdot \mathbf{F} \in \text{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])$  given by

$$(\tau \cdot \mathbf{F})^{\bar{x}} = \tau^{-1}_{F_0(\text{lt}(\bar{x}^1)) \mathcal{B}_{F_0(\text{rt}(\bar{x}^2))}[-d], \dots, F_0(\text{lt}(\bar{x}^n)) \mathcal{B}_{F_0(\text{rt}(\bar{x}^1))}[-d]} \circ F^{\bar{x} \cdot \tau} \circ \tau_{\mathcal{A}[1]^{\otimes \bar{x}^1}, \mathcal{A}[1]^{\otimes \bar{x}^2}, \dots, \mathcal{A}[1]^{\otimes \bar{x}^n}}$$

for  $\bar{x} \in \bar{\mathcal{O}}$ . We will denote by  $\text{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])^{C_{\lg(\bullet)}}$  the space of elements of  $\text{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])$  that are invariant under the action of  $C_{\lg(\bullet)}$ .

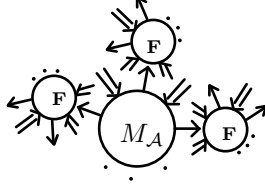
**Definition 5.11.** *Given  $d$ -pre-Calabi-Yau categories  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$  and  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  and an element  $s_{d+1}\mathbf{F} \in \text{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$  of degree 0, the **multinecklace composition** of  $s_{d+1}M_{\mathcal{A}}$  and  $s_{d+1}\mathbf{F}$  is the element*

$$s_{d+1}\mathbf{F} \underset{\text{multinec}}{\circ} s_{d+1}M_{\mathcal{A}} \in \text{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$$

given by

$$(s_{d+1}\mathbf{F} \underset{\text{multinec}}{\circ} s_{d+1}M_{\mathcal{A}})^{\bar{x}} = \sum \mathcal{E}(\mathcal{D})$$

for  $\bar{x} \in \mathcal{O}_{\mathcal{A}}$ , where the sum is over all the filled diagrams  $\mathcal{D}$  of type  $\bar{x}$  of the form



and where we have omitted the bold arrow, meaning that it is any of the outgoing arrows.

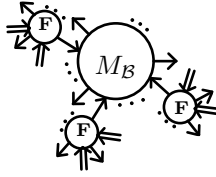
**Definition 5.12.** Given  $d$ -pre-Calabi-Yau categories  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$  and  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  and an element  $s_{d+1}\mathbf{F} \in \text{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$  of degree 0, the **pre composition** of  $s_{d+1}\mathbf{F}$  and  $s_{d+1}M_{\mathcal{B}}$  is the element

$$s_{d+1}M_{\mathcal{B}} \underset{\text{pre}}{\circ} s_{d+1}\mathbf{F} \in \text{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$$

given by

$$(s_{d+1}M_{\mathcal{B}} \underset{\text{pre}}{\circ} s_{d+1}\mathbf{F})^{\bar{x}} = \sum \mathcal{E}(\mathcal{D}')$$

for  $\bar{x} \in \mathcal{O}_{\mathcal{A}}$ , where the sum is over all the filled diagrams  $\mathcal{D}'$  of type  $\bar{x}$  of the form



and where we have omitted the bold arrow, meaning that it is any of the outgoing arrows.

**Definition 5.13.** Given  $d$ -pre-Calabi-Yau categories  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$  and  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  a  $d$ -pre-Calabi-Yau morphism  $(F_0, \mathbf{F}) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \rightarrow (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  is a map  $F_0 : \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{B}}$  together with an element  $s_{d+1}\mathbf{F} \in \text{Multi}^{\bullet}(\mathcal{A}[1], \mathcal{B}[-d])^{C_{1\mathfrak{g}(\bullet)}}[d+1]$  of degree 0 satisfying the following equation

$$(s_{d+1}\mathbf{F} \underset{\text{multinec}}{\circ} M_{\mathcal{A}})^{\bar{x}} = (M_{\mathcal{B}} \underset{\text{pre}}{\circ} s_{d+1}\mathbf{F})^{\bar{x}} \quad (5.3)$$

for all  $\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}$ . Note that the left member and right member of the previous identity belong to

$$\mathcal{H}om_{\mathbb{k}} \left( \bigotimes_{i=1}^n \mathcal{A}[1]^{\otimes \bar{x}^i}, \bigotimes_{i=1}^{n-1} F_0(\text{lt}(\bar{x}^i)) \mathcal{B}_{F_0(\text{rt}(\bar{x}^{i+1}))}[-d] \otimes_{F_0(\text{lt}(\bar{x}^n))} \mathcal{B}_{F_0(\text{rt}(\bar{x}^1))}[1] \right)$$

We now recall how to compose  $d$ -pre-Calabi-Yau morphisms.

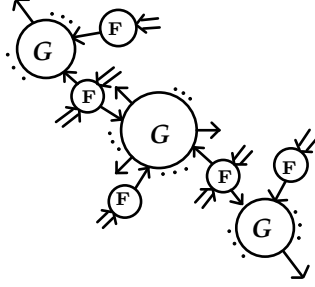
**Definition 5.14.** Let  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ ,  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  and  $(\mathcal{C}, s_{d+1}M_{\mathcal{C}})$  be  $d$ -pre-Calabi-Yau categories with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$ ,  $\mathcal{O}_{\mathcal{B}}$  and  $\mathcal{O}_{\mathcal{C}}$  and let  $(F_0, \mathbf{F}) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \rightarrow (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  and  $(G_0, \mathbf{G}) : (\mathcal{B}, s_{d+1}M_{\mathcal{B}}) \rightarrow (\mathcal{C}, s_{d+1}M_{\mathcal{C}})$  be  $d$ -pre-Calabi-Yau morphisms. The **composition of  $s_{d+1}\mathbf{F}$  and  $s_{d+1}\mathbf{G}$**  is the pair

$$(G_0 \circ F_0, s_{d+1}\mathbf{F} \underset{pCY}{\circ} s_{d+1}\mathbf{G})$$

where

$$s_{d+1}\mathbf{F} \circ_{pCY} s_{d+1}\mathbf{G} \in \text{Multi}^\bullet(\mathcal{A}[1], \mathcal{B}[-d])^{C_{\text{lg}(\bullet)}}[d+1]$$

is of degree 0 and is given by  $(s_{d+1}\mathbf{G} \circ_{pCY} s_{d+1}\mathbf{F})^{\bar{x}} = \sum \mathcal{E}(\mathcal{D})$  where the sum is over all filled diagrams  $\mathcal{D}$  of type  $\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}$  of the form

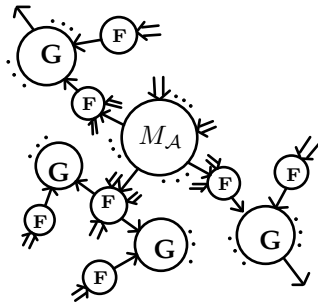


(5.4)

and where we have omitted the bold arrow, meaning that it any of the outgoing arrows.

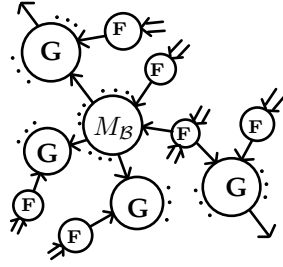
**Proposition 5.15.** For  $d \in \mathbb{Z}$ ,  $d$ -pre-Calabi-Yau categories and  $d$ -pre-Calabi-Yau morphisms together with the composition given in Definition 5.14 define a category, denoted as  $pCY_d$ . Given a graded quiver  $\mathcal{A}$  with set of objects  $\mathcal{O}$ , the identity morphism  $\text{Id} : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \rightarrow (\mathcal{A}, s_{d+1}M_{\mathcal{A}})$  is given by  $\text{Id}^x = \text{id}_{x\mathcal{A}x}$  for  $x \in \mathcal{O}$  and  $\text{Id}^{\bar{x}^1, \dots, \bar{x}^n} = 0$  for  $(\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}^n$  such that  $n \neq 1$  or  $n = 1$  and  $\text{lg}(\bar{x}^1) \neq 1$ .

*Proof.* We only have to check that the composition is associative and that the composition of any two  $d$ -pre-Calabi-Yau morphisms  $\mathbf{F}$  and  $\mathbf{G}$  is a  $d$ -pre-Calabi-Yau morphism. The associativity of the composition is clear. Now, consider two  $d$ -pre-Calabi-Yau morphisms  $\mathbf{F}$  and  $\mathbf{G}$ . Their composition is the sum of diagrams of the form (5.4). Therefore, the multinecklace composition of this composition and the pre-Calabi-Yau structure  $s_{d+1}M_{\mathcal{A}}$  is a sum of diagrams of the form



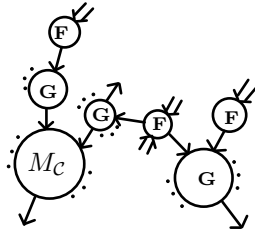
(5.5)

Given  $\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}$ , we have to sum over all diagrams of such type and we have several possibilities for the type of the inner diagram, defined as the subdiagram consisting of the disc filled with  $M_{\mathcal{A}}$  together with all those discs directly connected to the one filled with  $M_{\mathcal{A}}$ . Note that if we fix the type of the outer diagram given as the complement of the inner diagram, the type of the inner one is fixed. Moreover, changing the inner diagram for one of same type does not change the type of the whole diagram. Therefore, taking the sum over all diagrams of type  $\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}$  is the same as taking the sum over all the possible types for the outer diagram and for each of those, taking the sum over all the suitable types for the inner one. This second sum allows us to use that  $\mathbf{F}$  is a pre-Calabi-Yau morphism to replace the inner diagram by one consisting of a disc filled with  $M_{\mathcal{B}}$  whose incoming arrows are connected with outgoing arrows of discs filled with  $\mathbf{F}$ . Then, the sum of all the diagrams of type  $\bar{x}$  of the form (5.5) is equal to the sum of all the diagrams of type  $\bar{x}$  of the form



(5.6)

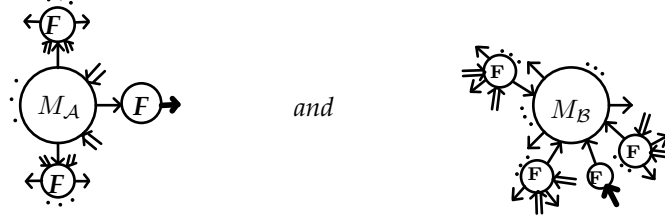
and we now define the inner diagram as filled diagram consisting of the disc filled with  $M_{\mathcal{B}}$  and of all the discs connected to it. The previous remarks on the types of the inner and outer diagrams still hold. Thus, the sum over all possible types for the whole diagram is again the sum over all the possible types for the outer diagram and for each of those, taking the sum over all the suitable types for the inner one.  $\mathbf{G}$  being a pre-Calabi-Yau morphism, one can again use (5.3) and say that the sum of all the diagrams of type  $\bar{x}$  of the form (5.6) is now equal to the sum of all the diagrams of type  $\bar{x}$  of the form



Therefore,  $s_{d+1}G \circ_{pCY} s_{d+1}F$  is a pre-Calabi-Yau morphism. □

The following class of morphisms will be useful in the next subsection.

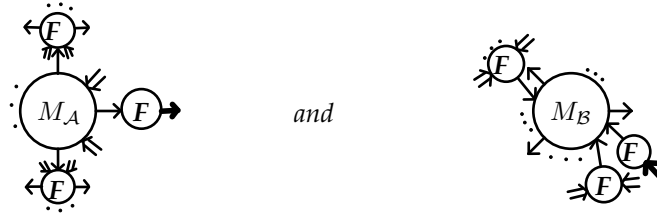
**Definition 5.16.** Given  $d$ -pre-Calabi-Yau categories  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$  and  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  and a  $d$ -pre-Calabi-Yau morphism  $F = (F_0, \mathbf{F}) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \rightarrow (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ , we say that  $F$  is **good** if  $\sum \mathcal{E}(\mathcal{D}) = \sum \mathcal{E}(\mathcal{D}')$  where the sums are over all the filled diagrams  $\mathcal{D}$  and  $\mathcal{D}'$  of type  $\bar{x}$  of the form



respectively.

Note that this condition is not closed under the pre-Calabi-Yau composition. We thus restrict this notion of good morphisms and give the following definition.

**Definition 5.17.** Given  $d$ -pre-Calabi-Yau categories  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$  and  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  and a  $d$ -pre-Calabi-Yau morphism  $F = (F_0, \mathbf{F}) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \rightarrow (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ , we say that  $F$  is **nice** if  $\sum \mathcal{E}(\mathcal{D}) = \sum \mathcal{E}(\mathcal{D}')$  where the sums are over all the filled diagrams  $\mathcal{D}$  and  $\mathcal{D}'$  of type  $\bar{x}$  of the form



respectively.

## 6 The relation between pre-Calabi-Yau morphisms and $A_{\infty}$ -morphisms

Recall that given a Hom-finite graded quiver  $\mathcal{A}$ , we have an equivalence between the data of a  $d$ -pre-Calabi-Yau structure on  $\mathcal{A}$  and a cyclic  $A_{\infty}$ -structure on  $\mathcal{A} \oplus \mathcal{A}^*[d-1]$  that restricts to  $\mathcal{A}$ . In this section, we study the relation between  $d$ -pre-Calabi-Yau morphisms and  $A_{\infty}$ -morphisms.

### 6.1 The case of strict morphisms

In this subsection, we study the relation between strict  $d$ -pre-Calabi-Yau morphisms and  $A_{\infty}$ -morphisms. We first recall the notion of strict  $d$ -pre-Calabi-Yau morphism.

**Definition 6.1.** Let  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ ,  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  be  $d$ -pre-Calabi-Yau categories with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$ . A  $d$ -pre-Calabi-Yau morphism  $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \rightarrow (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  is **strict** if  $\Phi^{\bar{x}}$  vanishes for each  $\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}^n$  with  $n > 1$  and  $\lg(\bar{x}^1) \neq 2$  if  $n = 1$ .

Equivalently, a strict  $d$ -pre-Calabi-Yau morphism between  $d$ -pre-Calabi-Yau categories  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$  and  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  is the data of a map between their sets of objects  $\Phi_0 : \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{B}}$  together with a collection  $\Phi = (\Phi^{x,y} : {}_x\mathcal{A}_y[1] \rightarrow {}_x\mathcal{B}_y[1])_{x,y \in \mathcal{O}_{\mathcal{A}}}$  of maps of degree 0 that satisfies

$$(\Phi^{\text{lt}(\bar{x}^1), \text{rt}(\bar{x}^2)} \otimes \dots \otimes \Phi^{\text{lt}(\bar{x}^n), \text{rt}(\bar{x}^1)}) \circ s_{d+1}M_{\mathcal{A}}^{\bar{x}^1, \dots, \bar{x}^n} = s_{d+1}M_{\mathcal{B}}^{\bar{x}^1, \dots, \bar{x}^n} \circ (\Phi^{\otimes \lg(\bar{x}^1)-1} \otimes \dots \otimes \Phi^{\otimes \lg(\bar{x}^n)-1})$$

for each  $n \in \mathbb{N}^*$ ,  $(\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}_{\mathcal{A}}^n$ .

For simplicity, we will omit the elements when writing the map  $\Phi$ . We will denote

$$\overline{\Phi_0(x)} = (\Phi_0(x_1), \dots, \Phi_0(x_n))$$

for  $\bar{x} = (x_1, \dots, x_n) \in \mathcal{O}_{\mathcal{A}}^n$  and

$$\overline{\overline{\Phi_0(x)}} = (\overline{\Phi_0(\bar{x}^1)}, \dots, \overline{\Phi_0(\bar{x}^n)})$$

for  $\bar{\bar{x}} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}_{\mathcal{A}}^n$ .

**Definition 6.2.** We denote by  $SpCY_d$  the subcategory of  $pCY_d$  whose objects are  $d$ -pre-Calabi-Yau categories and whose morphisms are strict  $d$ -pre-Calabi-Yau morphisms.

Given  $d$ -pre-Calabi-Yau categories  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$  and  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$  and a strict  $d$ -pre-Calabi-Yau morphism  $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \rightarrow (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ , we now construct an  $A_{\infty}$ -structure on  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$ . For  $\bar{x} \in \bar{\mathcal{O}}_{\mathcal{A}}$ , we denote by  $sm_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}}^{\bar{x}}$  the composition of  $j_{\bar{x}^{-1}}(s_{d+1}M_{\mathcal{A}}^{\bar{x}^{-1}})$  and the canonical projection on  $\mathcal{A}[1]$ .

Similarly, we will denote by  $sm_{\mathcal{B} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}^{\overline{\Phi_0(x)}}$  the composition of  $j_{\overline{\Phi_0(x)}^{-1}}(s_{d+1}M_{\mathcal{B}}^{\overline{\Phi_0(x)}^{-1}})$  and the canonical projection on  $\mathcal{B}^*[d]$ .

**Definition 6.3.** We define  $sm_{\mathcal{A} \oplus \mathcal{B}^*} \in C(\mathcal{A} \oplus \mathcal{B}^*[d-1])[1]$  as the unique element such that the composition of  $m_{\mathcal{A} \oplus \mathcal{B}^*}^{\bar{x}^1, \dots, \bar{x}^n}$  with the canonical projection

$$\text{lt}(\bar{x}^1)\mathcal{A}_{\text{rt}(\bar{x}^n)} \oplus \Phi_0(\text{lt}(\bar{x}^1))\mathcal{B}_{\Phi_0(\text{rt}(\bar{x}^n))}^*[d-1] \rightarrow \text{lt}(\bar{x}^1)\mathcal{A}_{\text{rt}(\bar{x}^n)}$$

is given by the map

$$m_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{A}}^{\bar{x}^1, \dots, \bar{x}^n} : \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\otimes \bar{x}^i} \otimes_{\Phi_0(\text{rt}(\bar{x}^{i+1}))} \mathcal{B}^*[d]_{\Phi_0(\text{lt}(\bar{x}^i))}) \otimes \mathcal{A}[1]^{\otimes \bar{x}^n} \rightarrow \text{lt}(\bar{x}^1)\mathcal{A}_{\text{rt}(\bar{x}^n)}$$

defined by

$$m_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{A}}^{\bar{x}^1, \dots, \bar{x}^n} = m_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}}^{\bar{x}^1, \dots, \bar{x}^n} \circ (\text{id}^{\otimes \lg(\bar{x}^1)-1} \otimes \Phi^* \otimes \text{id}^{\otimes \lg(\bar{x}^2)-1} \otimes \dots \otimes \text{id}^{\otimes \lg(\bar{x}^{n-1})-1} \otimes \Phi^* \otimes \text{id}^{\otimes \lg(\bar{x}^n)-1})$$

and the composition of  $m_{\mathcal{A} \oplus \mathcal{B}^*}^{\bar{x}^1, \dots, \bar{x}^n}$  with the canonical projection

$$\text{lt}(\bar{x}^1)\mathcal{A}_{\text{rt}(\bar{x}^n)} \oplus \Phi_0(\text{lt}(\bar{x}^1))\mathcal{B}_{\Phi_0(\text{rt}(\bar{x}^n))}^*[d-1] \rightarrow \Phi_0(\text{lt}(\bar{x}^1))\mathcal{B}_{\Phi_0(\text{rt}(\bar{x}^n))}^*[d-1]$$

is given by the map

$$m_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}^{\bar{x}^1, \dots, \bar{x}^n} : \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\otimes \bar{x}^i} \otimes_{\Phi_0(\text{rt}(\bar{x}^{i+1}))} \mathcal{B}^*[d]_{\Phi_0(\text{lt}(\bar{x}^i))}) \otimes \mathcal{A}[1]^{\otimes \bar{x}^n} \rightarrow_{\Phi_0(\text{lt}(\bar{x}^1))} \mathcal{B}^*_{\Phi_0(\text{rt}(\bar{x}^n))}[d-1]$$

defined by

$$m_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}^{\bar{x}^1, \dots, \bar{x}^n} = m_{\mathcal{B} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}^{\overline{\Phi_0(\bar{x}^1)}, \dots, \overline{\Phi_0(\bar{x}^n)}} \circ (\Phi^{\otimes \text{lg}(\bar{x}^1)-1} \otimes \text{id} \otimes \Phi^{\otimes \text{lg}(\bar{x}^2)-1} \otimes \dots \otimes \text{id} \otimes \Phi^{\otimes \text{lg}(\bar{x}^n)-1})$$

**Proposition 6.4.** *The element  $sm_{\mathcal{A} \oplus \mathcal{B}^*} \in C(\mathcal{A} \oplus \mathcal{B}^*[d-1])[1]$  defines an  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$  that is almost cyclic with respect to the  $\Phi$ -mixed bilinear form  $\Gamma^\Phi$ , defined in Example 3.17.*

*Proof.* Using the Proposition 5.9, we have that the equality  $sm_{\mathcal{A} \oplus \mathcal{B}^*} \circ_G sm_{\mathcal{A} \oplus \mathcal{B}^*} = 0$  is tantamount to  $s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*} \circ_{\Phi_{\text{nec}}} s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*} = 0$  where  $s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*} \in \mathcal{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$  is uniquely determined by

$$M_{\mathcal{A} \oplus \mathcal{B}^*}^{\bar{x}} = (\Phi^{\otimes (n-1)} \otimes \text{id}) \circ s_{d+1}M_{\mathcal{A}}^{\bar{x}} \in \mathcal{B}_{\bar{x}}^{\mathcal{A}}(\mathcal{A}[1], \mathcal{B}[-d])$$

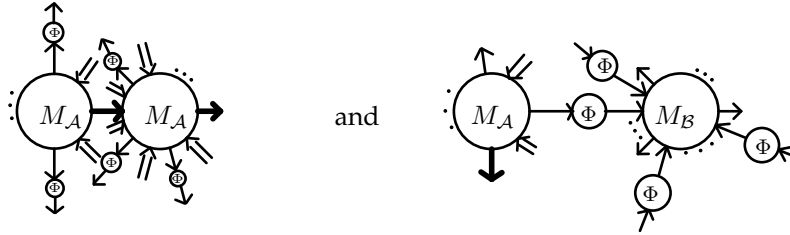
and

$$M_{\mathcal{A} \oplus \mathcal{B}^*}^{\overline{\Phi_0(\bar{x})}} = s_{d+1}M_{\mathcal{B}}^{\overline{\Phi_0(\bar{y})}} \circ (\Phi^{\otimes \text{lg}(\bar{x}^1)-1} \otimes \text{id} \otimes \Phi^{\otimes \text{lg}(\bar{x}^n)-1} \otimes \Phi^{\otimes \text{lg}(\bar{x}^2)-1} \otimes \dots \otimes \Phi^{\otimes \text{lg}(\bar{x}^{n-1})-1}) \in \mathcal{B}_{\bar{x}}^{\mathcal{B}}(\mathcal{A}[1], \mathcal{B}[-d])$$

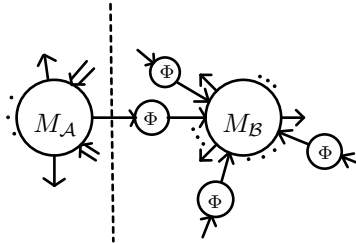
for  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}^n$  and  $\bar{y} = (\bar{x}^1 \sqcup \bar{x}^n, \bar{x}^2, \dots, \bar{x}^{n-1})$ . Moreover,

$$\pi_{\mathcal{A}}(s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*} \circ_{\Phi_{\text{nec}}} s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*})^{\bar{x}} = \sum \mathcal{E}(\mathcal{D}) + \sum \mathcal{E}(\mathcal{D}')$$

where  $\pi_{\mathcal{A}}$  is the canonical projection  $\mathcal{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])[d+1] \rightarrow \mathcal{B}_\bullet^{\mathcal{A}}(\mathcal{A}[1], \mathcal{B}[-d])$  and where the sums are over all the filled diagrams  $\mathcal{D}$  and  $\mathcal{D}'$  of type  $\bar{x}$  that are of the form



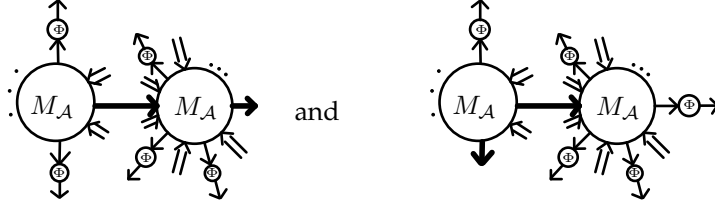
respectively. Now, note that the second diagram can be cut into two as follows.



Using the morphism identity satisfied by  $\Phi$ , the diagram on the right can be replaced by one with a disc filled with  $M_{\mathcal{A}}$  whose outgoing arrows are connected with the unique incoming arrow of discs of size 1 filled with  $\Phi$ . We thus obtain that

$$\pi_{\mathcal{A}}(s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*} \circ_G s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*})^{\bar{x}} = \sum \mathcal{E}(\mathcal{D}) + \sum \mathcal{E}(\mathcal{D}')$$

where the sums are over all the filled diagrams  $\mathcal{D}$  and  $\mathcal{D}'$  of type  $\bar{x}$  that are of the form



respectively. Moreover,  $\sum \mathcal{E}(\mathcal{D}) + \sum \mathcal{E}(\mathcal{D}') = 0$  since  $s_{d+1}M_{\mathcal{A}}$  is a  $d$ -pre-Calabi-Yau structure. Thus, if we show that this structure satisfies the cyclicity condition (3.1),  $sm_{\mathcal{A} \oplus \mathcal{B}^*}$  satisfies the Stasheff identities (SI).

Using the definition of  $\Gamma^{\Phi}$  and  $sm_{\mathcal{B} \oplus \mathcal{B}^*}$  and since  $sm_{\mathcal{B} \oplus \mathcal{B}^*}$  is cyclic with respect to  $\Gamma^{\mathcal{B}}$ , we have that

$$\begin{aligned} & \Gamma^{\Phi}(sm_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}^{\bar{x}^1, \dots, \bar{x}^n}(\bar{s}a^1, tf^1, \dots, \bar{s}a^{n-1}, tf^{n-1}, \bar{s}a^n), sb) \\ &= \Gamma^{\mathcal{B}}(sm_{\mathcal{B} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}^{\bar{\Phi}_0(\bar{x}^1), \dots, \bar{\Phi}_0(\bar{x}^n)}(\Phi^{\otimes \lg(\bar{x}^1)-1}(\bar{s}a^1), tf^1, \dots, tf^{n-1}, \Phi^{\otimes \lg(\bar{x}^n)-1}(\bar{s}a^n)), \Phi(sb)) \\ &= (-1)^{\epsilon} \Gamma^{\mathcal{B}}(sm_{\mathcal{B} \oplus \mathcal{B}^* \rightarrow \mathcal{B}}^{\bar{\Phi}_0(\bar{x}^n) \sqcup \bar{\Phi}_0(\bar{x}^1), \bar{\Phi}_0(\bar{x}^2), \dots, \bar{\Phi}_0(\bar{x}^{n-1})}(\Phi^{\otimes \lg(\bar{x}^n)-1}(\bar{s}a^n) \otimes \Phi(sb) \otimes \Phi^{\otimes \lg(\bar{x}^1)-1}(\bar{s}a^1), tf^1, \dots, \\ & tf^{n-2}, \Phi^{\otimes \lg(\bar{x}^{n-1})-1}(\bar{s}a^{n-1})), tf^{n-1}) \\ &= (-1)^{\epsilon+\delta} \Gamma^{\mathcal{B}}(((f^{n-2} \circ s_d) \otimes \dots \otimes (f^1 \circ s_d) \otimes \text{id}) \\ & (M_{\mathcal{B}}^{\bar{\Phi}_0(\bar{x}^{n-1}), \dots, \bar{\Phi}_0(\bar{x}^n) \sqcup \bar{\Phi}_0(\bar{x}^1)}(\Phi^{\otimes \lg(\bar{x}^{n-1})}(\bar{s}a^{n-1}), \dots, \Phi^{\otimes \lg(\bar{x}^n)}(\bar{s}a^n) \otimes \Phi(sb) \otimes \Phi^{\otimes \lg(\bar{x}^1)}(\bar{s}a^1)), tf^{n-1})) \end{aligned} \quad (6.1)$$

for  $\bar{s}a_i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ ,  $tf^i \in \Phi_0(\text{rt}(\bar{x}^{i+1}))\mathcal{B}^*[d]_{\Phi_0(\text{lt}(\bar{x}^i))}$ , with  $\epsilon = (|\bar{s}a^n| + |sb|)(\sum_{i=1}^{n-1} (|\bar{s}a^i| + |tf^i|))$  and

$$\begin{aligned} \delta &= (|\bar{s}a^1| + |sb| + |\bar{s}a^n|)(\sum_{i=1}^{n-1} |\bar{s}a^i| + \sum_{i=1}^{n-2} |tf^i|) + \sum_{2 \leq i < j \leq n-2} |\bar{s}a^i| |tf^j| + dn \\ &+ \sum_{2 \leq i < j \leq n-1} |\bar{s}a^i| |\bar{s}a^j| + \sum_{1 \leq i < j \leq n-2} |tf^i| |tf^j| \end{aligned}$$

Moreover, using that  $\Phi$  is a  $d$ -pre-Calabi-Yau morphism, we have that the last member of (6.1) is

$$\begin{aligned} & (-1)^{\epsilon+\delta} \Gamma^{\mathcal{A}}\left(\left(\bigotimes_{i=2}^{n-1} (f^{n-i} \circ \tilde{\Phi}) \otimes \text{id}\right) \circ s_{d+1}M_{\mathcal{A}}^{\bar{x}^{n-1}, \dots, \bar{x}^n \sqcup \bar{x}^1}(\bar{s}a^{n-1}, \dots, \bar{s}a^2, \bar{s}a^n \otimes sb \otimes \bar{s}a^1), tf^{n-1} \circ \tilde{\Phi}\right) \\ &= (-1)^{\epsilon} \Gamma^{\mathcal{A}}(sm_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}}^{\bar{x}^n \sqcup \bar{x}^1, \bar{x}^2, \dots, \bar{x}^{n-1}}(\bar{s}a^n \otimes sb \otimes \bar{s}a^1, tf^1 \circ \tilde{\Phi}, \bar{s}a^2, \dots, tf^{n-2} \circ \tilde{\Phi}, \bar{s}a^{n-1}), tf^{n-1} \circ \tilde{\Phi}) \end{aligned} \quad (6.2)$$



where  $\tilde{\Phi}$  denotes the morphism  $\Phi[d-1]$ . Thus, comparing (6.1) and (6.2) we get that

$$\begin{aligned} & \Gamma^{\mathcal{B}}(sm_{\mathcal{B} \oplus \mathcal{B}^* \rightarrow \mathcal{B}}^{\overline{\Phi_0(\bar{x}^n) \sqcup \Phi_0(\bar{x}^1)}, \overline{\Phi_0(\bar{x}^2)}, \dots, \overline{\Phi_0(\bar{x}^{n-1})}}(\Phi^{\otimes \lg(\bar{x}^n)-1}(\bar{s}a^n) \otimes \Phi(sb) \otimes \Phi^{\otimes \lg(\bar{x}^1)-1}(\bar{s}a^1), tf^1, \dots, \\ & tf^{n-2}, \Phi^{\otimes \lg(\bar{x}^{n-1})-1}(\bar{s}a^{n-1})), tf^{n-1}) \\ &= \Gamma^{\mathcal{A}}(sm_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}}^{\bar{x}^n \sqcup \bar{x}^1, \bar{x}^2, \dots, \bar{x}^{n-1}}(\bar{s}a^n \otimes sb \otimes \bar{s}a^1, tf^1 \circ \tilde{\Phi}, \bar{s}a^2, \dots, tf^{n-2} \circ \tilde{\Phi}, \bar{s}a^{n-1}), tf^{n-1} \circ \tilde{\Phi}) \end{aligned} \quad (6.3)$$

Finally, we have that

$$\begin{aligned} & (-1)^\epsilon \Gamma^{\mathcal{A}}(sm_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}}^{\bar{x}^n \sqcup \bar{x}^1, \bar{x}^2, \dots, \bar{x}^{n-1}}(\bar{s}a^n \otimes sb \otimes \bar{s}a^1, tf^1 \circ \tilde{\Phi}, \bar{s}a^2, \dots, tf^{n-2} \circ \tilde{\Phi}, \bar{s}a^{n-1}), tf^{n-1} \circ \tilde{\Phi}) \\ &= (-1)^\epsilon \Gamma^{\Phi}(sm_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{A}}^{\bar{x}^n \sqcup \bar{x}^1, \bar{x}^2, \dots, \bar{x}^{n-1}}(\bar{s}a^n \otimes sb \otimes \bar{s}a^1, tf^1, \bar{s}a^2, \dots, tf^{n-2}, \bar{s}a^{n-1}), tf^{n-1}) \end{aligned}$$

Therefore,  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$  together with  $sm_{\mathcal{A} \oplus \mathcal{B}^*}$  is an  $A_\infty$ -category that is almost cyclic with respect to  $\Gamma^\Phi$ .  $\square$

**Definition 6.5.** Let  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ ,  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  be  $d$ -pre-Calabi-Yau categories with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$ . Consider a strict  $d$ -pre-Calabi-Yau morphism  $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \rightarrow (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ . We define maps of graded vector spaces

$$\varphi_{\mathcal{A}}^{x,y} : {}_x\mathcal{A}[1]_y \oplus_{\Phi_0(x)} \mathcal{B}^*[d]_{\Phi_0(y)} \rightarrow {}_x(\mathcal{A}[1] \oplus \mathcal{A}^*[d])_y$$

and

$$\varphi_{\mathcal{B}}^{x,y} : {}_x\mathcal{A}[1]_y \oplus_{\Phi_0(x)} \mathcal{B}^*[d]_{\Phi_0(y)} \rightarrow_{\Phi_0(x)} (\mathcal{B}[1] \oplus \mathcal{B}^*[d])_{\Phi_0(y)}$$

given by  $\varphi_{\mathcal{A}}^{x,y}(sa) = sa$ ,  $\varphi_{\mathcal{B}}^{x,y}(sa) = \Phi^{x,y}(sa)$ ,  $\varphi_{\mathcal{A}}^{x,y}(tf) = tf \circ \Phi^{y,x}[d]$  and  $\varphi_{\mathcal{B}}^{x,y}(tf) = tf$  for  $x, y \in \mathcal{O}_{\mathcal{A}}$ ,  $sa \in {}_x\mathcal{A}[1]_y$  and  $tf \in {}_{\Phi_0(x)}\mathcal{B}^*[d]_{\Phi_0(y)}$ .

**Proposition 6.6.** Let  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$ ,  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  be  $d$ -pre-Calabi-Yau categories with respective sets of objects  $\mathcal{O}_{\mathcal{A}}$  and  $\mathcal{O}_{\mathcal{B}}$ . Consider a strict  $d$ -pre-Calabi-Yau morphism  $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \rightarrow (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  and the  $A_\infty$ -category  $(\mathcal{A} \oplus \mathcal{B}^*[d-1], sm_{\mathcal{A} \oplus \mathcal{B}^*})$  where  $sm_{\mathcal{A} \oplus \mathcal{B}^*} \in C(\mathcal{A} \oplus \mathcal{B}^*[d-1])[1]$  is given in Definition 6.3. The maps  $\varphi_{\mathcal{A}}$  and  $\varphi_{\mathcal{B}}$  defined in Definition 6.5 are cyclic  $A_\infty$ -morphisms, in the sense of Definition 3.22.

*Proof.* We only check that  $\varphi_{\mathcal{A}}$  is a morphism since the case of  $\varphi_{\mathcal{B}}$  is similar. We only have to verify that

$$\varphi_{\mathcal{A}}^{x,y} \circ sm_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{A}}^{\bar{x}^1, \dots, \bar{x}^n} = sm_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}}^{\bar{x}^1, \dots, \bar{x}^n} \circ (\varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^1)-1} \otimes \varphi_{\mathcal{A}} \otimes \varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^2)-1} \dots \otimes \varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^n)-1}) \quad (6.4)$$

and

$$\varphi_{\mathcal{A}}^{x,y} \circ sm_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}^{\bar{x}^1, \dots, \bar{x}^n} = sm_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}^*}^{\bar{x}^1, \dots, \bar{x}^n} \circ (\varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^1)-1} \otimes \varphi_{\mathcal{A}} \otimes \varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^2)-1} \dots \otimes \varphi_{\mathcal{A}}^{\otimes \lg(\bar{x}^n)-1}) \quad (6.5)$$

for  $x, y \in \mathcal{O}_{\mathcal{A}}$ ,  $(\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}_{\mathcal{A}}^n$  such that  $\text{lt}(\bar{x}^n) = x$ ,  $\text{rt}(\bar{x}^1) = y$ .

First, note that

$$\begin{aligned} & \varphi_{\mathcal{A}}^{x,y}(sm_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{A}}^{\bar{x}^1, \dots, \bar{x}^n}(\bar{s}a^1, tf^1, \bar{s}a^2, \dots, \bar{s}a^{n-1}, tf^{n-1}, \bar{s}a^n)) \\ &= sm_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{A}}^{\bar{x}^1, \dots, \bar{x}^n}(\bar{s}a^1, tf^1, \bar{s}a^2, \dots, \bar{s}a^{n-1}, tf^{n-1}, \bar{s}a^n) \\ &= sm_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}}^{\bar{x}^1, \dots, \bar{x}^n}(\bar{s}a^1, \varphi_{\mathcal{A}}(tf^1), \bar{s}a^2, \dots, \bar{s}a^{n-1}, \varphi_{\mathcal{A}}(tf^{n-1}), \bar{s}a^n) \end{aligned}$$

for  $\bar{s}a^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$  and  $tf^i \in \Phi_0(\text{rt}(\bar{x}^{i+1}))\mathcal{B}^*[d]_{\Phi_0(\text{lt}(\bar{x}^i))}$ , so that (6.4) holds.

Moreover, we also have

$$\begin{aligned} & \varphi_{\mathcal{A}}^{x,y} \circ sm_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}^{\bar{x}^1, \dots, \bar{x}^n}(\bar{s}a^1, tf^1, \bar{s}a^2, \dots, \bar{s}a^{n-1}, tf^{n-1}, \bar{s}a^n) \\ &= sm_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}^{\bar{x}^1, \dots, \bar{x}^n}(\bar{s}a^1, tf^1, \bar{s}a^2, \dots, \bar{s}a^{n-1}, tf^{n-1}, \bar{s}a^n) \circ \Phi^{x,y} \end{aligned}$$

and given  $sb \in {}_x\mathcal{A}_y$ , we have that

$$\begin{aligned} & sm_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}^{\bar{x}^1, \dots, \bar{x}^n}(\bar{s}a^1, tf^1, \bar{s}a^2, \dots, \bar{s}a^{n-1}, tf^{n-1}, \bar{s}a^n)(\Phi^{x,y}(sb)) \\ &= \Gamma^{\Phi}(sm_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}^{\bar{x}^1, \dots, \bar{x}^n}(\bar{s}a^1, tf^1, \bar{s}a^2, \dots, \bar{s}a^{n-1}, tf^{n-1}, \bar{s}a^n), sb) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & sm_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}^*}^{\bar{x}^1, \dots, \bar{x}^n}(\varphi_{\mathcal{A}}^{\otimes \text{lg}(\bar{x}^1)-1}(\bar{s}a^1) \otimes \varphi_{\mathcal{A}}(tf^1) \otimes \varphi_{\mathcal{A}}^{\otimes \text{lg}(\bar{x}^2)-1}(\bar{s}a^2) \dots \otimes \varphi_{\mathcal{A}}^{\otimes \text{lg}(\bar{x}^n)-1}(\bar{s}a^n))(sb) \\ &= \Gamma^{\mathcal{A}}(sm_{\mathcal{A} \oplus \mathcal{A}^* \rightarrow \mathcal{A}^*}^{\bar{x}^1, \dots, \bar{x}^n}(\varphi_{\mathcal{A}}^{\otimes \text{lg}(\bar{x}^1)-1}(\bar{s}a^1) \otimes \varphi_{\mathcal{A}}(tf^1) \otimes \varphi_{\mathcal{A}}^{\otimes \text{lg}(\bar{x}^2)-1}(\bar{s}a^2) \dots \otimes \varphi_{\mathcal{A}}^{\otimes \text{lg}(\bar{x}^n)-1}(\bar{s}a^n)), sb) \end{aligned}$$

Using the identity (6.3), we thus get (6.5).

It remains to show that the morphisms are cyclic. To prove it, we note that

$${}_y\Gamma_x^{\mathcal{A}}(\varphi_{\mathcal{A}}^{y,x}(tf), \varphi_{\mathcal{A}}^{x,y}(sa)) = {}_y\Gamma_x^{\mathcal{A}}(tf \circ \Phi^{x,y}, sa) = f(\Phi^{x,y}(sa)) = {}_y\Gamma_x^{\Phi}(tf, sa)$$

as well as

$${}_y\Gamma_x^{\mathcal{B}}(\varphi_{\mathcal{B}}^{y,x}(tf), \varphi_{\mathcal{B}}^{x,y}(sa)) = {}_y\Gamma_x^{\mathcal{B}}(tf, \Phi^{x,y}(sa)) = f(\Phi^{x,y}(sa)) = {}_y\Gamma_x^{\Phi}(tf, sa)$$

for  $sa \in {}_x\mathcal{A}[1]_y$ ,  $tf \in \Phi_0(y)\mathcal{B}^*[d]_{\Phi_0(x)}$ . The second condition to be a cyclic morphism is obviously satisfied since  $\varphi_{\mathcal{A}}^{\bar{x}}$  and  $\varphi_{\mathcal{B}}^{\bar{x}}$  vanish for  $\bar{x} \in \mathcal{O}^n$  with  $n > 2$ .  $\square$

**Definition 6.7.** Let  $(\mathcal{A} \oplus \mathcal{A}^*[d-1], sm_{\mathcal{A} \oplus \mathcal{A}^*})$ ,  $(\mathcal{B} \oplus \mathcal{B}^*[d-1], sm_{\mathcal{B} \oplus \mathcal{B}^*})$  be  $A_{\infty}$ -categories. A **hat morphism** from  $\mathcal{A} \oplus \mathcal{A}^*[d-1]$  to  $\mathcal{B} \oplus \mathcal{B}^*[d-1]$  is a triple  $(sm_{\mathcal{A} \oplus \mathcal{B}^*}, \varphi_{\mathcal{A}}, \varphi_{\mathcal{B}})$  where  $sm_{\mathcal{A} \oplus \mathcal{B}^*}$  is an  $A_{\infty}$ -structure on  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$  and  $\varphi_{\mathcal{A}}, \varphi_{\mathcal{B}}$  are  $A_{\infty}$ -morphisms

$$\begin{array}{ccc} & \mathcal{A}[1] \oplus \mathcal{B}^*[d] & \\ \varphi_{\mathcal{A}} \swarrow & & \searrow \varphi_{\mathcal{B}} \\ \mathcal{A}[1] \oplus \mathcal{A}^*[d] & & \mathcal{B}[1] \oplus \mathcal{B}^*[d] \end{array} \quad (6.6)$$

**Definition 6.8.** Let  $(\mathcal{A} \oplus \mathcal{A}^*[d-1], sm_{\mathcal{A} \oplus \mathcal{A}^*})$ ,  $(\mathcal{B} \oplus \mathcal{B}^*[d-1], sm_{\mathcal{B} \oplus \mathcal{B}^*})$  and  $(\mathcal{C} \oplus \mathcal{C}^*[d-1], sm_{\mathcal{C} \oplus \mathcal{C}^*})$  be  $A_{\infty}$ -categories. Two hat morphisms  $(sm_{\mathcal{A} \oplus \mathcal{B}^*}, \varphi_{\mathcal{A}}, \varphi_{\mathcal{B}}) : \mathcal{A} \oplus \mathcal{A}^*[d-1] \rightarrow \mathcal{B} \oplus \mathcal{B}^*[d-1]$ ,  $(sm_{\mathcal{B} \oplus \mathcal{C}^*}, \psi_{\mathcal{B}}, \psi_{\mathcal{C}}) : \mathcal{B} \oplus \mathcal{B}^*[d-1] \rightarrow \mathcal{C} \oplus \mathcal{C}^*[d-1]$  are **composable** if there exist a triple  $(sm_{\mathcal{A} \oplus \mathcal{C}^*}, \chi_{\mathcal{A}}, \chi_{\mathcal{C}})$  where  $\chi_{\mathcal{A}} : \mathcal{A} \oplus \mathcal{C}^*[d-1] \rightarrow \mathcal{A} \oplus \mathcal{B}^*[d-1]$  and  $\chi_{\mathcal{C}} : \mathcal{A} \oplus \mathcal{C}^*[d-1] \rightarrow \mathcal{B} \oplus \mathcal{C}^*[d-1]$  are  $A_{\infty}$ -morphisms and such that  $\varphi_{\mathcal{B}} \circ \chi_{\mathcal{A}} = \psi_{\mathcal{B}} \circ \chi_{\mathcal{C}}$ . The **composition** of  $(sm_{\mathcal{A} \oplus \mathcal{B}^*}, \varphi_{\mathcal{A}}, \varphi_{\mathcal{B}})$  and  $(sm_{\mathcal{B} \oplus \mathcal{C}^*}, \psi_{\mathcal{B}}, \psi_{\mathcal{C}})$  is then given by  $(sm_{\mathcal{A} \oplus \mathcal{C}^*}, \varphi_{\mathcal{A}} \circ \chi_{\mathcal{A}}, \psi_{\mathcal{C}} \circ \chi_{\mathcal{C}})$ .

**Definition 6.9.** A **partial category** is an  $A_{\infty}$ -pre-category as defined in [4] where the multiplications  $m_n$  vanish for  $n > 2$ .

**Definition 6.10.** The  $A_\infty$ -*hat category* is the partial category  $\widehat{A_\infty d}$  whose objects are  $A_\infty$ -categories of the form  $\mathcal{A} \oplus \mathcal{A}^*[d-1]$  and whose morphisms are hat morphisms.

**Definition 6.11.** A *functor* between partial categories  $\mathcal{A}$  and  $\mathcal{B}$  with respective sets of objects  $\mathcal{O}_\mathcal{A}$  and  $\mathcal{O}_\mathcal{B}$  is the data of a map  $F_0 : \mathcal{O}_\mathcal{A} \rightarrow \mathcal{O}_\mathcal{B}$  together with a family  $F = ({}_yF_x)_{x,y \in \mathcal{O}_\mathcal{A}}$  sending a morphism  $f : x \rightarrow y$  to a morphism  ${}_yF_x(f) : F_0(x) \rightarrow F_0(y)$  such that if two morphisms  $f : x \rightarrow y$  and  $g : y \rightarrow z$  are composable, then  ${}_yF_x(f)$  and  ${}_zF_y(g)$  are composable and their composition is given by the morphism  ${}_zF_y(g) \circ {}_yF_x(f) = {}_zF_x(g \circ f)$ .

**Definition 6.12.** We define the partial subcategory  $\widehat{cycA_\infty d}$  of  $\widehat{A_\infty d}$  whose objects are cyclic  $A_\infty$ -categories of the form  $\mathcal{A} \oplus \mathcal{A}^*[d-1]$  and whose morphisms  $\mathcal{A} \oplus \mathcal{A}^*[d-1] \rightarrow \mathcal{B} \oplus \mathcal{B}^*[d-1]$  are the data of an almost cyclic  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$  together with a diagram of the form (6.6) where  $\varphi_\mathcal{A}$  and  $\varphi_\mathcal{B}$  are  $A_\infty$ -morphisms.

**Definition 6.13.** We define the partial subcategory  $\widehat{ScycA_\infty d}$  of  $\widehat{cycA_\infty d}$  whose objects are the ones of  $\widehat{cycA_\infty d}$  and whose morphisms are strict cyclic morphisms of  $\widehat{cycA_\infty d}$ .

We now summarize the results of Propositions 6.4 and 6.6.

**Corollary 6.14.** There exists a functor  $SpCY_d \rightarrow \widehat{ScycA_\infty d}$  which sends a  $d$ -pre-Calabi-Yau category  $(\mathcal{A}, s_{d+1}M_\mathcal{A})$  to the  $A_\infty$ -category  $(\mathcal{A} \oplus \mathcal{A}^*[d-1], sm_{\mathcal{A} \oplus \mathcal{A}^*})$  defined in Proposition 4.32 and a  $d$ -pre-Calabi-Yau morphism  $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_\mathcal{A}) \rightarrow (\mathcal{B}, s_{d+1}M_\mathcal{B})$  to the  $A_\infty$ -structure  $sm_{\mathcal{A} \oplus \mathcal{B}^*}$  defined in Definition 6.3 together with cyclic  $A_\infty$ -morphisms

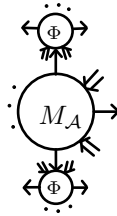
$$\begin{array}{ccc} & \mathcal{A}[1] \oplus \mathcal{B}^*[d] & \\ \varphi_\mathcal{A} \swarrow & & \searrow \varphi_\mathcal{B} \\ \mathcal{A}[1] \oplus \mathcal{A}^*[d] & & \mathcal{B}[1] \oplus \mathcal{B}^*[d] \end{array}$$

given in Definition 6.5.

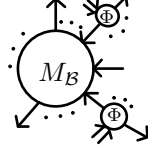
## 6.2 General case

We now present the relation between not necessarily strict  $d$ -pre-Calabi-Yau morphisms and  $A_\infty$ -morphisms. Consider  $d$ -pre-Calabi-Yau categories  $(\mathcal{A}, s_{d+1}M_\mathcal{A})$ ,  $(\mathcal{B}, s_{d+1}M_\mathcal{B})$  as well as a  $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_\mathcal{A}) \rightarrow (\mathcal{B}, s_{d+1}M_\mathcal{B})$  a  $d$ -pre-Calabi-Yau morphism as defined in Definition 5.13. We first construct an  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$ .

**Definition 6.15.** We define  $M_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{A}} \in \mathfrak{B}_\bullet^{\mathcal{A}}(\mathcal{A}[1], \mathcal{B}[-d])$  by  $M_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{A}}^{\bar{x}} = \sum \mathcal{E}(\mathfrak{D})$  where the sum is over all the filled diagrams  $\mathfrak{D}$  of type  $\bar{x}$  and of the form



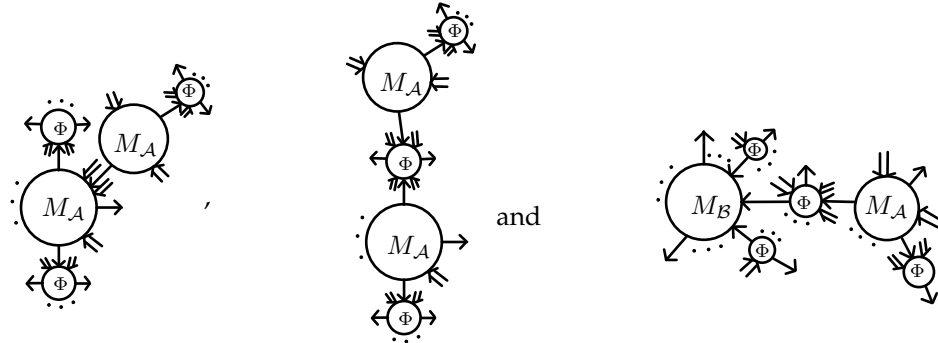
and  $M_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*} \in \mathfrak{B}_\bullet^{\mathcal{B}}(\mathcal{A}[1], \mathcal{B}[-d])$  by  $M_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}^{\bar{x}} = \sum \mathcal{E}(\mathcal{D}')$  where the sum is over all the filled diagrams  $\mathcal{D}'$  of type  $\bar{x}$  and of the form



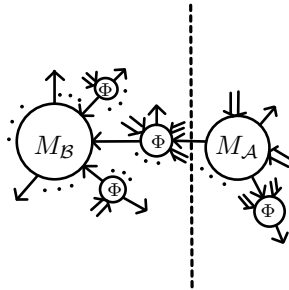
By Lemma 5.3, this defines an element  $s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*} \in \mathfrak{B}_\bullet(\mathcal{A}[1], \mathcal{B}[-d])[d+1]$  and we thus define the element  $sm_{\mathcal{A} \oplus \mathcal{B}^*} \in C(\mathcal{A} \oplus \mathcal{B}^*[d-1])[1]$  as  $sm_{\mathcal{A} \oplus \mathcal{B}^*}^{\bar{x}} = j_{\bar{x}-1}^{\Phi}(s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*}) \in C(\mathcal{A} \oplus \mathcal{B}^*[d-1])[1]$ . We will denote by  $m_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{A}}$  (resp.  $m_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*}$ ) the composition of  $m_{\mathcal{A} \oplus \mathcal{B}^*}$  with the canonical projection on  $\mathcal{A}$  (resp. on  $\mathcal{B}^*[d-1]$ ).

**Proposition 6.16.** *The element  $sm_{\mathcal{A} \oplus \mathcal{B}^*} \in C(\mathcal{A} \oplus \mathcal{B}^*[d-1])[1]$  defines an  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$ . Moreover, if the morphism  $\Phi$  is good,  $sm_{\mathcal{A} \oplus \mathcal{B}^*}$  satisfies the cyclicity condition (3.1).*

*Proof.* Using Proposition 5.9, it suffices to show that  $s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*} \circ_{\Phi \text{ nec}} s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*} = 0$ . We have that  $\pi_{\mathcal{A}}(s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*} \circ_{\Phi \text{ nec}} s_{d+1}M_{\mathcal{A} \oplus \mathcal{B}^*}) = 0$  is tantamount to  $\sum \mathcal{E}(\mathcal{D}) + \sum \mathcal{E}(\mathcal{D}') + \sum \mathcal{E}(\mathcal{D}'') = 0$  where the sums are over all the filled diagrams  $\mathcal{D}, \mathcal{D}'$  and  $\mathcal{D}''$  of type  $\bar{x}$  of the form



respectively. The third diagram can be cut into two as follows

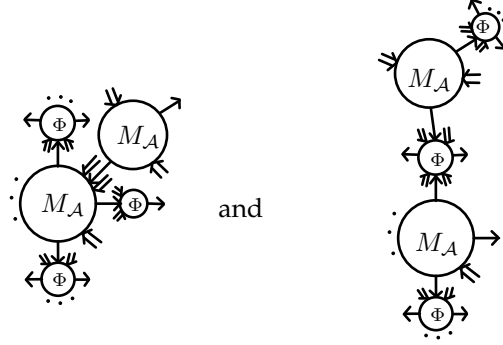


Using that  $\Phi$  is a pre-Calabi-Yau morphism, the left side can be changed into a diagram consisting of a disc filled with  $M_A$  whose outgoing arrows are shared with discs filled with  $\Phi$ . We thus get

that

$$\sum \mathcal{E}(\mathcal{D}'') = \sum \mathcal{E}(\mathcal{D}_1) - \sum \mathcal{E}(\mathcal{D}_2) \quad (6.7)$$

where  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are filled diagrams of the form

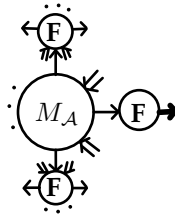


The minus in the identity (6.7) comes from the fact that the discs filled with  $M_A$  change their place, in the sense that the order of the labeling of their first outgoing arrow changes. Since  $s_{d+1}M_A$  is of degree 1, this create a minus sign. Moreover,  $\sum \mathcal{E}(\mathcal{D}') = \sum \mathcal{E}(\mathcal{D}_2)$  so that it remains to show that  $\sum \mathcal{E}(\mathcal{D}) + \sum \mathcal{E}(\mathcal{D}_1) = 0$ . This is the case since  $s_{d+1}M_A$  is a pre-Calabi-Yau structure. Indeed, the sum of these evaluations of diagrams is the composition of

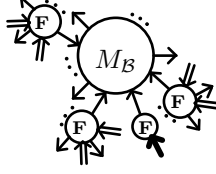
$$s_{d+1}M_A \underset{\text{nec}}{\circ} s_{d+1}M_A$$

with a tensor product composed of maps of the collection  $\Phi$  and of the identity map  $\text{id}$  in the last tensor factor. Therefore, the element  $sm_{\mathcal{A} \oplus \mathcal{B}^*}$  satisfies the Stasheff identities (SI).

It is clear that if the morphism  $\Phi$  is good, then the  $A_\infty$ -structure on  $\mathcal{A} \oplus \mathcal{B}^*[d-1]$  is cyclic. Indeed,  $\Gamma^\Phi \circ (sm_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{A}} \otimes \text{id}_{\mathcal{A}^*}) = \sum \mathcal{E}(\mathcal{D})$  where the sum is over all the filled diagrams  $\mathcal{E}(\mathcal{D})$  of the form



On the other hand,  $\Gamma^\Phi \circ (sm_{\mathcal{A} \oplus \mathcal{B}^* \rightarrow \mathcal{B}^*} \otimes \text{id}_{\mathcal{B}}) = \sum \mathcal{E}(\mathcal{D}')$  where the sum is over all the filled diagrams  $\mathcal{E}(\mathcal{D}')$  of the form



which shows the cyclicity of  $sm_{\mathcal{A} \oplus \mathcal{B}^*}$ .  $\square$

Now, we define two morphisms of  $A_\infty$ -categories  $\varphi_{\mathcal{A}} : \mathcal{A}[1] \oplus \mathcal{B}^*[d] \rightarrow \mathcal{A}[1] \oplus \mathcal{A}^*[d]$  and  $\varphi_{\mathcal{B}} : \mathcal{A}[1] \oplus \mathcal{B}^*[d] \rightarrow \mathcal{B}[1] \oplus \mathcal{B}^*[d]$  using the  $d$ -pre-Calabi-Yau morphism  $\Phi$ .

**Lemma 6.17.** *Consider  $d$ -pre-Calabi-Yau categories  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$  and  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  as well as a  $d$ -pre-Calabi-Yau morphism  $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \rightarrow (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ . Then,  $\Phi$  induces morphisms*

$$\phi_{\mathcal{A}^*}^{\bar{x}} : \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\bar{x}^i} \otimes_{\Phi_0(\text{rt}(\bar{x}^{i+1}))} \mathcal{B}^*[d]_{\Phi_0(\text{lt}(\bar{x}^i))}) \otimes \mathcal{A}[1]^{\bar{x}^n} \rightarrow \text{lt}(\bar{x}^1) \mathcal{A}^*[d]_{\text{rt}(\bar{x}^n)}$$

defined by

$$\begin{aligned} \phi_{\mathcal{A}^*}^{\bar{x}}(\bar{s}a^1, tf^1, \dots, \bar{s}a^{n-1}, tf^{n-1}, \bar{s}a^n)(s_{-d}b) = \\ (-1)^\epsilon ((f^{n-1} \circ s_d) \otimes \dots \otimes (f^1 \circ s_d)) (\Phi^{\bar{x}^{n-1}, \bar{x}^{n-2}, \dots, \bar{x}^n \sqcup \bar{x}^1}(\bar{s}a^{n-1}, \dots, \bar{s}a^n \otimes sb \otimes \bar{s}a^1)) \end{aligned}$$

for  $\bar{s}a^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ ,  $tf^i \in \Phi_0(\text{rt}(\bar{x}^{i+1})) \mathcal{B}^*[d]_{\Phi_0(\text{lt}(\bar{x}^i))}$  and  $sb \in \text{rt}(\bar{x}^1) \mathcal{A}[1]_{\text{lt}(\bar{x}^n)}$  where

$$\begin{aligned} \epsilon = \sum_{i=1}^{n-1} |tf^i| \sum_{j=i+1}^n |\bar{s}a^i| + \sum_{1 \leq i < j \leq n} |\bar{s}a^i| |\bar{s}a^j| \sum_{1 \leq i < j \leq n-1} |tf^i| |tf^j| \\ + d(n-1) + |\bar{s}a^n| (\sum_{i=1}^{n-1} |\bar{s}a^i| + |sb|) + d \sum_{i=1}^{n-1} |tf^i| + (d+1) \sum_{i=1}^n |\bar{s}a^i| \end{aligned}$$

and a morphism

$$\phi_{\mathcal{B}}^{\bar{x}} : \bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\bar{x}^i} \otimes_{\Phi_0(\text{rt}(\bar{x}^{i+1}))} \mathcal{B}^*[d]_{\Phi_0(\text{lt}(\bar{x}^i))}) \otimes \mathcal{A}[1]^{\bar{x}^n} \rightarrow \Phi_0(\text{lt}(\bar{x}^1)) \mathcal{B}[1]_{\Phi_0(\text{rt}(\bar{x}^n))}$$

defined by

$$\begin{aligned} \phi_{\mathcal{B}}^{\bar{x}}(\bar{s}a^1, tf^1, \dots, \bar{s}a^{n-1}, tf^{n-1}, \bar{s}a^n) \\ = (-1)^\delta ((f^{n-1} \circ s_d) \otimes \dots \otimes (f^1 \circ s_d) \otimes \text{id})(s_{d+1} \Phi^{\bar{x}^{n-1}}(\bar{s}a^n, \dots, \bar{s}a^1)) \end{aligned}$$

for  $\bar{s}a^i \in \mathcal{A}[1]^{\otimes \bar{x}^i}$ ,  $tf^i \in \Phi_0(\text{rt}(\bar{x}^{i+1})) \mathcal{B}^*[d]_{\Phi_0(\text{lt}(\bar{x}^i))}$  with

$$\delta = \sum_{i=1}^{n-1} |tf^i| \sum_{j=i+1}^n |\bar{s}a^i| + \sum_{1 \leq i < j \leq n} |\bar{s}a^i| |\bar{s}a^j| \sum_{1 \leq i < j \leq n-1} |tf^i| |tf^j| + d(n-1) + d \sum_{i=1}^{n-1} |tf^i|$$

and for each  $n \in \mathbb{N}^*$ ,  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}_{\mathcal{A}}^n$ .

**Definition 6.18.** Consider  $d$ -pre-Calabi-Yau categories  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$  and  $(\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  as well as a  $d$ -pre-Calabi-Yau morphism  $(\Phi_0, \Phi) : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \rightarrow (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$ . We define maps of graded vector spaces  $\varphi_{\mathcal{A}} : \mathcal{A}[1] \oplus \mathcal{B}^*[d] \rightarrow \mathcal{A}[1] \oplus \mathcal{A}^*[d]$  and  $\varphi_{\mathcal{B}} : \mathcal{A}[1] \oplus \mathcal{B}^*[d] \rightarrow \mathcal{B}[1] \oplus \mathcal{B}^*[d]$  given by  $\varphi_{\mathcal{A}}^{x,y}(sa) = sa$ ,  $\varphi_{\mathcal{B}}^{x,y}(sa) = \Phi^{x,y}(sa)$  for  $sa \in {}_x\mathcal{A}[1]_y$  and  $\varphi_{\mathcal{A}}^{x,y}(tf) = tf \circ \Phi^{y,x}[d]$ ,  $\varphi_{\mathcal{B}}^{x,y}(tf) = tf$ , for  $tf \in {}_{\Phi_0(x)}\mathcal{B}^*[d]_{\Phi_0(y)}$ ,  $x, y \in \mathcal{O}_{\mathcal{A}}$ , as well as

$$\pi_{\mathcal{A}^*[d]} \circ \varphi_{\mathcal{A}}^{\bar{x}} = \phi_{\mathcal{A}^*}^{\bar{x}}, \pi_{\mathcal{A}[1]} \circ \varphi_{\mathcal{A}}^{\bar{x}} = 0$$

and

$$\pi_{\mathcal{B}[1]} \circ \varphi_{\mathcal{B}}^{\bar{x}} = \phi_{\mathcal{B}}^{\bar{x}}, \pi_{\mathcal{B}^*[d]} \circ \varphi_{\mathcal{B}}^{\bar{x}} = 0$$

where  $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n) \in \bar{\mathcal{O}}_{\mathcal{A}}^n$  for  $n > 1$  and  $\text{lg}(\bar{x}^1) > 2$  if  $n = 1$ .

**Proposition 6.19.** The maps  $\varphi_{\mathcal{A}}$  and  $\varphi_{\mathcal{B}}$  are morphisms of  $A_{\infty}$ -categories.

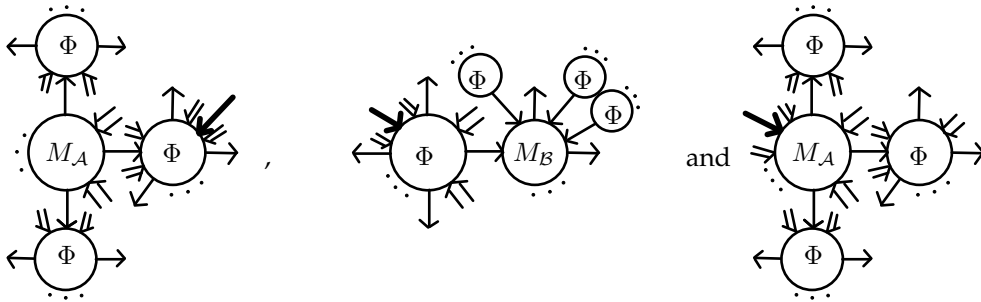
*Proof.* The part of the identity (MI) for  $\varphi_{\mathcal{A}}$  that takes place in

$$\text{Hom}_{\mathbb{k}}\left(\bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\bar{x}^i} \otimes_{\Phi_0(\text{rt}(\bar{x}^{i+1}))} \mathcal{B}^*[d]_{\Phi_0(\text{lt}(\bar{x}^i))}) \otimes \mathcal{A}[1]^{\bar{x}^n}, \text{lt}(\bar{x}^1)\mathcal{A}[1]_{\text{rt}(\bar{x}^n)}\right)$$

is clearly satisfied. Moreover, by definition of the  $A_{\infty}$ -structure  $sm_{\mathcal{A} \oplus \mathcal{B}^*}$ , the part of the identity (MI) that takes place in

$$\text{Hom}_{\mathbb{k}}\left(\bigotimes_{i=1}^{n-1} (\mathcal{A}[1]^{\bar{x}^i} \otimes_{\Phi_0(\text{rt}(\bar{x}^{i+1}))} \mathcal{B}^*[d]_{\Phi_0(\text{lt}(\bar{x}^i))}) \otimes \mathcal{A}[1]^{\bar{x}^n}, \text{lt}(\bar{x}^1)\mathcal{A}^*[d]_{\text{rt}(\bar{x}^n)}\right)$$

is tantamount to  $\sum \mathcal{E}(\mathcal{D}_1) - \sum \mathcal{E}(\mathcal{D}_2) = -\sum \mathcal{E}(\mathcal{D}_3)$  where the sums are over all the filled diagrams  $\mathcal{D}_1$ ,  $\mathcal{D}_2$  and  $\mathcal{D}_3$  of type  $\bar{x}$  of the form



respectively. Using that  $\Phi$  is a pre-Calabi-Yau morphism, we thus have that  $\varphi_{\mathcal{A}}$  is an  $A_{\infty}$ -morphism. The case of  $\varphi_{\mathcal{B}}$  is similar.  $\square$

We now summarize the results of Propositions 6.16 and 6.19.

**Corollary 6.20.** *There is a functor  $pCY_d \rightarrow \widehat{A}_{\infty d}$  sending a  $d$ -pre-Calabi-Yau category  $(\mathcal{A}, s_{d+1}M_{\mathcal{A}})$  to the  $A_{\infty}$ -category  $(\mathcal{A} \oplus \mathcal{A}^*[d-1], sm_{\mathcal{A} \oplus \mathcal{A}^*})$  defined in Proposition 4.32 and a  $d$ -pre-Calabi-Yau morphism  $\Phi : (\mathcal{A}, s_{d+1}M_{\mathcal{A}}) \rightarrow (\mathcal{B}, s_{d+1}M_{\mathcal{B}})$  to the  $A_{\infty}$ -structure  $sm_{\mathcal{A} \oplus \mathcal{B}^*}$  defined in Definition 6.15 together with the  $A_{\infty}$ -morphisms*

$$\begin{array}{ccc}
 & \mathcal{A}[1] \oplus \mathcal{B}^*[d] & \\
 \varphi_{\mathcal{A}} \swarrow & & \searrow \varphi_{\mathcal{B}} \\
 \mathcal{A}[1] \oplus \mathcal{A}^*[d] & & \mathcal{B}[1] \oplus \mathcal{B}^*[d]
 \end{array}$$

defined in Definition 6.18. Moreover, this functor restricts to a functor  $NpCY_d \rightarrow cyc\widehat{A}_{\infty d}$  where  $NpCY_d$  is the subcategory of  $pCY_d$  whose objects are  $d$ -pre-Calabi-Yau categories and whose morphisms are nice  $d$ -pre-Calabi-Yau morphisms defined in Definition 5.17.

## References

- [1] David Fernández and Estanislao Herscovich, *Cyclic  $A_{\infty}$ -algebras and double Poisson algebras*, J. Noncommut. Geom. **15** (2021), no. 1, 241–278. MR4248213 ↑1
- [2] Hiroshige Kajiura, *Noncommutative homotopy algebras associated with open strings*, Rev. Math. Phys. **19** (2007), no. 1, 1–99. MR2293085 ↑10
- [3] Maxim Kontsevich, *Weak Calabi-Yau algebras* (2013), available at <https://math.berkeley.edu/~auroux/miami2013-notes>. Notes taken from the talk at *Conference on Homological Mirror Symmetry*. ↑1
- [4] Maxim Kontsevich and Yan Soibelman, *Homological mirror symmetry and torus fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), 2001, pp. 203–263. MR1882331 ↑41
- [5] Maxim Kontsevich, Alex Takeda, and Yiannis Vlassopoulos, *Pre-Calabi-Yau algebras and topological quantum field theories*, arXiv, 2021. arXiv:2112.14667 [math.AG]. ↑1, 2, 10, 20, 28, 32
- [6] Johan Leray and Bruno Vallette, *Pre-Calabi-Yau algebras and homotopy double poisson gbras*, arXiv, 2022. arXiv:2203.05062 [math.QA]. ↑1
- [7] Paul Seidel, *Fukaya  $A_{\infty}$ -structures associated to Lefschetz fibrations. II*, Algebra, geometry, and physics in the 21st century, 2017, pp. 295–364. MR3727564 ↑1, 4
- [8] James Dillon Stasheff, *Homotopy associativity of  $H$ -spaces. I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–292; *ibid.* **108** (1963), 293–312. MR0158400 ↑8
- [9] Masahiro Sugawara, *On the homotopy-commutativity of groups and loop spaces*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **33** (1960/61), 257–269. MR120645 ↑9
- [10] Thomas Tradler and Mahmoud Zeinalian, *Algebraic string operations*, K-Theory **38** (2007), no. 1, 59–82. MR2353864 ↑1
- [11] Wai-Kit Yeung, *Pre-Calabi-Yau structures and moduli of representations*, arXiv, 2018. arXiv:1802.05398v3 [math.AG]. ↑10

MARION BOUCROT: Univ. Grenoble Alpes, CNRS, IF, 38000 Grenoble, France  
*E-mail adress* : marion.boucrot@univ-grenoble-alpes.fr