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# A Conditional Logic-Based Argumentation Framework

Philippe besnard, Éric Grégoire, and Badran Raddaoui

<sup>1</sup> IRT-CNRS, Université de Toulouse  
118 route de Narbonne, F-31065 Toulouse Cedex, France  
besnard@irit.fr

<sup>2</sup> Université Lille-Nord de France, Artois  
CRIL-CNRS UMR 8188  
F-62307 Lens Cedex, France  
{gregoire, raddaoui}@cril.fr

**Abstract.** The goal of this paper is twofold. First, a logic-based argumentation framework is introduced in the context of conditional logic, as conditional logic is often regarded as an appealing setting for knowledge representation and reasoning. Second, a concept of conditional contrariety is defined that covers usual inconsistency-based conflicts and puts in light a specific form of conflicts that often occurs in real-life: when an agent asserts an *If then* rule, it can be argued that additional conditions are actually needed to derive the conclusion.

**Keywords:** Conditional Logic, Logical Argumentation Theory, Conditional Contrariety.

## 1 Introduction

Argumentation has long been a major topic in A.I. (see e.g., [1, 2] and for more recent accounts e.g., [3]) that has concerned a large variety of application domains for more than a decade, like e.g., law [4, 5], medicine [6], negotiation [7], decision making [8] and multiagent systems [9, 10]. Two main families of computational models for argumentation have been proposed in the literature: namely, the abstract and the logic-based argumentation frameworks. Following the seminal work of [11], the first family is based on graph-oriented representations and focuses mainly on the interaction between arguments without taking the possible internal structure of the involved arguments into account. On the contrary, the logic-based approaches (e.g., [12–19]) exploit the logical internal structure of arguments and adopt inconsistency as a pivotal paradigm: any pair of conflicting arguments must be contradictory. Consequently, no conflicting arguments can be found together inside a same consistent set of formulas.

However, many natural real-life arguments and counter-arguments do not necessarily appear mutually inconsistent in usual knowledge representation modes. For example, consider the assertion *If there is a match tonight then John will go to the stadium* encoded through a (material) implicative formula in standard

logic as  $MatchTonight \rightarrow JohnGoesToStadium$ , so that in case  $MatchTonight$  is true,  $JohnGoesToStadium$  can be deduced. Now, the following objection can be raised against that argument through the sentence *If there is a match tonight and if John has got enough money then John will go to the stadium*, which requires an additional condition for John to go the stadium if there is a match tonight. The latter sentence is a deductive consequence of the first one and taking both of them does not yield an inconsistent set. This paper aims to extend the logic-based approaches by encompassing this specific form of contrariety.

It is possible to represent the above example as a case of inconsistency-based conflict by using e.g. a modal logic of necessity and possibility, and more information: when the first sentence is augmented so that it excludes the possibility that John does not go to the stadium tonight if there is a match, whereas the other sentence allows this possibility to happen. Such an alternative representation requires all this additional or implicit information to be asserted and represented in some way. On the contrary, we provide a representation framework where the motivating example can be modeled in way close to natural language implications and without resorting to logical inconsistency; moreover, the framework allows a form of contrariety to be recognized between the implicative formulas.

To this end, we resort to conditional logic, which allows an additional specific implicative connective to be used, in addition to standard-logic material implication. Conditional logic is actually rooted in the formalization of hypothetical or counterfactual reasonings of the form *If  $\alpha$  were true then  $\beta$  would be true* and attempts to avoid some pitfalls of material implication to represent patterns of conditional or hypothetical reasonings. Actually, the conditional implication connective is often regarded as a very suitable connective to encode many implicative reasoning patterns from real-life; accordingly, conditional logic has long been investigated in many A.I. areas [20] like belief revision [21], data base and knowledge update [22] natural language semantics for handling hypothetical and counterfactual sentences [23], non-monotonic and prototypical reasonings [24, 25], causal inference [26] and logic programming [27], just to mention some seminal works.

The goal of this paper is thus twofold. First, we revisit frameworks *à la* [15, 28] to lay down the main foundations of a logic-based argumentation framework based on conditional logic. Second, we introduce a concept of conditional contrariety that encompasses both the conflicts through inconsistency and a generalization of the conflict illustrated in the motivating example. One intended benefit is that when an agent represents *If then* rules using the conditional connective, the framework allows one to argue against this rule by stating that additional conditions are required in order for the conclusion of the rule to hold. Accordingly, in this framework, the conditional implication connective is intended to be used to represent hypothetical reasonings and other implications that can be questioned within an argumentation process.

The paper is organized as follows. In the next section, the user is provided with basic elements of conditional logic MP. In section 3, conditional contrariety is motivated and introduced. Section 4 revisits the main foundational concepts

of Besnard and Hunter's framework so that conditional contrariety is covered. The last section discusses some promising paths for further research. For the clarity of presentation, all proofs are given in an appendix.

## 2 Conditional logics

Conditional logics are rooted in the formalization of counterfactual or hypothetical reasonings of the form "If  $\alpha$  were true then  $\beta$  would be true", for which an additional connective, called *conditional connective* and denoted  $\Rightarrow$ , is generally introduced. Roughly, a conditional formula  $\alpha \Rightarrow \beta$  is valid when  $\beta$  is true in the possible worlds where  $\alpha$  is true. Clearly, this diverges from the (material) implicative standard-logic formula  $\alpha \rightarrow \beta$ , since this latter one is equivalent to  $\neg\alpha \vee \beta$  which is also satisfied when  $\alpha$  is false. In the paper, we consider the well-known conditional logic MP, which can be extended to yield many of the other popular conditional logics (see e.g., [29]).

MP is an extension of the language and inference system of classical Boolean logic. It is a language of formulas, denoted  $\mathcal{L}_c$ . We use  $\alpha, \beta, \gamma, \delta, \dots$  to denote formulas of  $\mathcal{L}_c$  and  $\Delta, \Phi, \Psi, \Theta, \dots$  to denote sets of formulas of  $\mathcal{L}_c$ . Formulas are built in the usual way from the standard connectives  $\neg, \wedge, \vee, \rightarrow$  and  $\leftrightarrow$ : they accommodate the conditional connective  $\Rightarrow$  through the additional formation rule: if  $\alpha$  and  $\beta$  are formulas, so is  $\alpha \Rightarrow \beta$ .  $\top$  and  $\perp$  represent truth and falsity, respectively. A concept of *extended literal* proves useful: an extended literal is of the form  $\alpha$  or  $\neg\alpha$  such that  $\alpha$  is either an atom or a formula with the conditional connective as the main connective.

The inferential apparatus of MP consists of the following axioms schemas and inference rules [29], enriching standard Boolean logic to yield an inference relation denoted  $\vdash_c$ .

$$\mathbf{RCEA.} \frac{\vdash_c \alpha \leftrightarrow \beta}{\vdash_c (\alpha \Rightarrow \gamma) \leftrightarrow (\beta \Rightarrow \gamma)}$$

$$\mathbf{RCEC.} \frac{\vdash_c \alpha \leftrightarrow \beta}{\vdash_c (\gamma \Rightarrow \alpha) \leftrightarrow (\gamma \Rightarrow \beta)}$$

$$\mathbf{CC.} \vdash_c ((\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma)) \rightarrow (\alpha \Rightarrow (\beta \wedge \gamma))$$

$$\mathbf{CM.} \vdash_c (\alpha \Rightarrow (\beta \wedge \gamma)) \rightarrow ((\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma))$$

$$\mathbf{CN.} \vdash_c (\alpha \Rightarrow \top)$$

$$\mathbf{MP.} \vdash_c (\alpha \Rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$$

In the paper, an expression of the form  $\alpha \equiv \beta$  will be a shortcut for  $\alpha \vdash_c \beta$  and  $\beta \vdash_c \alpha$ . We make use of the *disjunctive form* of conditional formulas, defined as follows.

**Definition 1.** *The disjunctive form of a formula  $\alpha$  of  $\mathcal{L}_c$ , denoted  $DF(\alpha)$ , is the first (according to the lexicographic order) formula of the form  $\alpha_1 \vee \dots \vee \alpha_n$  that is logically equivalent with  $\alpha$  under  $\vdash_c$  and such that each  $\alpha_i$  is a conjunction of extended literals.*

### 3 Conditional contrariety

*Conditional contrariety* (in short, contrariety) is the cornerstone concept in this paper. It is intended to encompass both logical inconsistency in MP and a form of contrariety involving a pair of conditional implicative formulas where the first one would entail the other one in standard logic if the material implication were used. Let us introduce the concept progressively and refer to items of the next formal definition through their numbering like e.g., (I), (II.a) or (II.2.).

Let  $\alpha$  and  $\beta$  be two formulas of  $\mathcal{L}_c$  s.t.  $DF(\alpha) = \alpha_1 \vee \dots \vee \alpha_n$  and  $DF(\beta) = \beta_1 \vee \dots \vee \beta_m$ .

$\alpha$  *contraries*  $\beta$ , denoted  $\alpha \bowtie \beta$ , in any of the following situations.

(I.) First,  $\alpha$  and  $\beta$  are mutually inconsistent in MP. Note that this also covers the standard-logic occurrences of inconsistency. Formally, whenever  $\{\alpha, \beta\} \vdash_c \perp$  we have  $\alpha \bowtie \beta$ . Taking into account the DF of  $\alpha$  and  $\beta$ , this amounts to  $\forall \alpha_i, \forall \beta_j \{\alpha_i, \beta_j\} \vdash_c \perp$ . Let us note that if  $\beta$  is itself inconsistent then any formula  $\alpha$  contraries  $\beta$ , in particular  $\beta$  contraries  $\beta$ .

(II.) Second, we need to address the case that requires  $\alpha$  of the form  $\phi \wedge \epsilon \Rightarrow \psi$  to contrary  $\beta$  of the form  $\phi \Rightarrow \psi$ , just as in the motivating example from the introduction. Actually, we can be more general and consider the cases where a similar situation occurs with respect to a more general class of pairs of “contrarying” formulas of the form  $\gamma = \gamma_1 \Rightarrow \gamma_2$  and  $\delta = \delta_1 \Rightarrow \delta_2$ , provided that  $\delta$  and  $\gamma$  would be “derivable” in some sense from  $\alpha$  and  $\beta$ . The class of pairs of formulas is defined through specific inferential links between their elements  $\gamma_1$ ,  $\gamma_2$ ,  $\delta_1$  and  $\delta_2$ . First,  $\gamma$  and  $\delta$  must be two conditionals about the same conclusion: hence,  $\gamma_2 \equiv \delta_2$  (II.d). Generalizing the motivating example, we require the antecedent of the first conditional to entail the antecedent of the second one (but not conversely), formally  $\gamma_1 \vdash_c \delta_1$  (II.a) and  $\delta_1 \not\vdash_c \gamma_1$  (II.b). Another condition is required to prevent valid formulas from being in a contrariety position:  $\gamma_1 \not\vdash_c \gamma_2$  (II.c). Indeed, consider e.g. the two following Boolean tautologies:  $a \wedge b \rightarrow \top$  and  $b \rightarrow \top$ . Applying the *RCEA* and *RCEC* rules plus the *CM* axiom schema yields  $(a \wedge b \Rightarrow a \wedge b) \rightarrow (a \wedge b \Rightarrow \top)$  and  $(b \Rightarrow b) \rightarrow (b \Rightarrow \top)$ , respectively. This entails  $a \wedge b \Rightarrow \top$  and  $b \Rightarrow \top$ , i.e., two valid formulas that we would thus inappropriately consider as being in a contrariety position.

Then, we need to make clear the inferential links between the pair of formulas  $\alpha$  and  $\beta$  for which we explore a contrariety situation, and the above  $\gamma$  and  $\delta$  formulas that are themselves in contrariety (II.1 and II.2). First, a contrariety situation occurs when  $\beta$  conditionally entails the last two formulas, i.e.  $\beta \vdash_c \gamma \wedge \delta$  (or equivalently, taking the DF of  $\beta$ , this occurs when taking  $\beta_j$  as premisses,  $\forall \beta_j$ ) (II.2). The motivation is as follows. Remember that whenever  $\beta$  was inconsistent, any  $\alpha$  contraries  $\beta$ . Likewise, if  $\beta$  allows by itself the derivation of both  $\gamma$  and  $\delta$  that are in a contrariety position, then  $\beta$  is in some way self-contraryed, and any  $\alpha$  is contrarying  $\beta$ .

Finally,  $\alpha \vdash_c \gamma$  while  $\{\alpha, \beta\} \vdash_c \gamma \wedge \delta$  naturally covers the last  $\alpha \bowtie \beta$  case. In the definition, this condition is expressed taking the DF of both  $\alpha$  and  $\beta$  into account (II.1).

It is important to stress that the contrariety concept that is defined is deductively-based in all of the following senses. First, inconsistency is reached through deduction. Second, the inferential relations between elements of the pair  $(\alpha, \beta)$  with elements of  $(\gamma, \delta)$  are also of a deductive nature. Finally, the condition  $\gamma_1 \vdash_c \delta_1$  is also deductive. Accordingly, we will see that contrariety is not symmetric in the general case.

**Definition 2.** Let  $\alpha$  and  $\beta$  be two formulas of  $\mathcal{L}_c$  s.t.  $DF(\alpha) = \alpha_1 \vee \dots \vee \alpha_n$  and  $DF(\beta) = \beta_1 \vee \dots \vee \beta_m$ .

$\alpha$  contraries  $\beta$ , denoted  $\alpha \bowtie \beta$ ,  
iff  $\forall \alpha_i, \forall \beta_j$

- (I.)  $\{\alpha_i, \beta_j\} \vdash_c \perp$ , or
  - (II.) There exist  $\gamma_1, \gamma_2, \delta_1$  and  $\delta_2$  in  $\mathcal{L}_c$  s.t.
    - (II.a)  $\gamma_1 \vdash_c \delta_1$  and
    - (II.b)  $\delta_1 \not\vdash_c \gamma_1$  and
    - (II.c)  $\gamma_1 \not\vdash_c \gamma_2$  and
    - (II.d)  $\gamma_2 \equiv \delta_2$
- where
- (II.1)  $\{\alpha_i, \beta_j\} \vdash_c (\gamma_1 \Rightarrow \gamma_2) \wedge (\delta_1 \Rightarrow \delta_2)$   
s.t.  $\alpha_i \vdash_c \gamma_1 \Rightarrow \gamma_2$ , or
  - (II.2)  $\beta_j \vdash_c (\gamma_1 \Rightarrow \gamma_2) \wedge (\delta_1 \Rightarrow \delta_2)$ .

*Example 1.*  $a \wedge b \Rightarrow c$  contraries  $a \Rightarrow c$ .  $a \wedge (a \wedge c \Rightarrow f)$  contraries both formulas  $a \rightarrow (a \Rightarrow f)$  and  $\neg a \vee c \Rightarrow f$ . Also,  $\neg a \wedge b$  and  $a \wedge (a \wedge c \Rightarrow b \vee \neg d)$  contrary each other.

The concept of *contrarying a formula* is naturally extended into a concept of *contrarying a set of formulas*.

**Definition 3.** Let  $\Phi$  and  $\alpha$  be a subset and a formula of  $\mathcal{L}_c$ , respectively.  $\alpha$  contraries  $\Phi$ , denoted  $\alpha \bowtie \Phi$ , iff there exists  $\beta$  in  $\mathcal{L}_c$  s.t.  $\Phi \vdash_c \beta$  and  $\alpha \bowtie \beta$ .

*Example 2.* Let  $\Phi = \{a \Rightarrow b, a \vee d \Rightarrow b \wedge c, a \Rightarrow c\}$ . Let  $\alpha = a \Rightarrow b \wedge c$ . Note that  $\Phi \vdash_c \alpha$ . However,  $\alpha \not\bowtie \Phi$  because  $\alpha \bowtie a \vee d \Rightarrow b \wedge c$ .

Obviously,  $\bowtie$  is neither symmetric, nor antisymmetric, nor antireflexive. However, it is *monotonic* and *syntax-independent*.

**Proposition 1.** Let  $\Phi, \Psi$  and  $\alpha$  be two subsets and a formula of  $\mathcal{L}_c$ , respectively. If  $\alpha \bowtie \Phi$  then  $\alpha \bowtie \Phi \cup \Psi$ .

**Proposition 2.** Let  $\Phi, \alpha$  and  $\beta$  be a subset and two formulas of  $\mathcal{L}_c$ , respectively. If  $\alpha \equiv \beta$  then  $\alpha \bowtie \Phi$  iff  $\beta \bowtie \Phi$ .

## 4 A conditional-logic argumentation framework

Having defined the pivotal concept of contrariety, we are now ready to revisit [15, 28]'s framework and lay down the foundations of a conditional-logic argumentation framework based on contrariety. Accordingly, we revisit and extend the following concepts, successively: arguments, conflicts, rebuttals, defeaters and argumentation trees.

In the following, we assume a subset  $\Delta$  of  $\mathcal{L}_c$  that can be inconsistent. All concepts will be implicitly defined relatively to  $\Delta$ . Membership of formulas and inclusion of sets of formulas to  $\mathcal{L}_c$  will also be implicit from now on.

### 4.1 Arguments

After Besnard-Hunter, an argument is made of a set formulas together with a conclusion that can be derived from the set. The usual non-contradiction condition expressed by  $\Phi \not\perp$  is naturally extended and replaced by a non-contrariety requirement (second item).

**Definition 4.** *An argument  $A$  is a pair  $\langle \Phi, \alpha \rangle$  s.t.:*

1.  $\Phi \subseteq \Delta$
2.  $\forall \beta$  s.t.  $\Phi \vdash_c \beta$ ,  $\beta \not\bowtie \Phi$
3.  $\Phi \vdash_c \alpha$
4.  $\forall \Phi' \subset \Phi$ ,  $\Phi' \not\vdash_c \alpha$

$A$  is said to be an *argument* for  $\alpha$ . The set  $\Phi$  and the formula  $\alpha$  are the *support* and the *conclusion* of  $A$ , respectively.

*Example 3.* Let  $\Delta = \{(a \Rightarrow \neg d) \wedge \neg b, a \Rightarrow c, \neg a\}$ . In view of  $\Delta$ , some arguments are:

$$\begin{aligned} & \langle \{-a\}, \neg(a \wedge b) \rangle, \\ & \langle \{a \Rightarrow c\}, a \Rightarrow c \rangle, \\ & \langle \{(a \Rightarrow \neg d) \wedge \neg b, a \Rightarrow c\}, a \Rightarrow \neg d \wedge c \rangle. \end{aligned}$$

Note that  $CC$  is used to obtain the conclusion of the last argument.

The following result shows that the revisited concept of argument still preserves coherence, such a coherence concept being in some sense extended to  $\bowtie$ .

**Proposition 3.** *If  $\langle \Phi, \alpha \rangle$  is an argument then  $\alpha \not\bowtie \Phi$  and  $\neg \alpha \bowtie \Phi$ .*

A notion of *quasi-identical* arguments is now introduced as follows. It is intended to capture situations where two arguments can be said to make the same point on the same grounds.

**Definition 5.** *Two arguments  $\langle \Phi, \alpha \rangle$  and  $\langle \Psi, \beta \rangle$  are quasi-identical iff  $\Phi = \Psi$  and  $\alpha \equiv \beta$ .*



Not surprisingly, provided one argument, its quasi-identical ones form an infinite set.

**Proposition 4.** *Let  $\langle \Phi, \alpha \rangle$  be an argument. There is an infinite set of arguments of the form  $\langle \Psi, \beta \rangle$  s.t.  $\langle \Phi, \alpha \rangle$  and  $\langle \Psi, \beta \rangle$  are quasi-identical.*

Arguments are not necessarily independent. The definition of *more conservative arguments* captures a notion of subsumption between arguments, translating situations where an argument is in some sense contained within another one.

**Definition 6.** *An argument  $\langle \Phi, \alpha \rangle$  is more conservative than an argument  $\langle \Psi, \beta \rangle$  iff  $\Phi \subseteq \Psi$  and  $\beta \vdash_c \alpha$ .*

*Example 4.* The argument  $\langle \{a\}, a \vee b \rangle$  is more conservative than  $\langle \{-a \vee b, a\}, b \rangle$ . Also,  $\langle \{(a \Rightarrow b) \wedge c, c \rightarrow d\}, (a \Rightarrow b) \wedge d \rangle$  is more conservative than  $\langle \{(a \Rightarrow b) \wedge c, c \rightarrow d\}, (a \Rightarrow b) \wedge c \wedge d \rangle$ .

**Proposition 5.** *If  $\langle \Phi, \alpha \rangle$  is more conservative than  $\langle \Psi, \beta \rangle$ , then  $\beta \not\bowtie \Phi$ ,  $\alpha \not\bowtie \Psi$  and  $\neg\alpha \not\bowtie \Psi$ .*

In this last result, it is worth noting that  $\neg\beta \not\bowtie \Phi$  does not hold in full generality. A counter-example consists of the two arguments  $\langle \{a\}, a \vee b \rangle$  and  $\langle \{a, b\}, a \wedge b \rangle$ ;  $\langle \{a\}, a \vee b \rangle$  is more conservative than  $\langle \{a, b\}, a \wedge b \rangle$  but  $\neg(a \wedge b) \not\bowtie a$ .

Actually, the concept of being more conservative induces the concept of quasi-identical arguments, and conversely.

**Proposition 6.** *Two arguments  $\langle \Phi, \alpha \rangle$  and  $\langle \Psi, \beta \rangle$  are quasi-identical iff each one is more conservative than the other.*

*Example 5.*  $\langle \{a \Rightarrow b, a \Rightarrow c\}, (a \Rightarrow b) \wedge (a \Rightarrow c) \rangle$  and  $\langle \{a \Rightarrow b, a \Rightarrow c\}, a \Rightarrow b \wedge c \rangle$  are quasi-identical as each one is more conservative than the other. In fact, the proof of equivalence between  $(a \Rightarrow b) \wedge (a \Rightarrow c)$  and  $a \Rightarrow b \wedge c$  is obtained by applying the *CC* and *CM* axioms schemas.

The notions of quasi-identicality and of being more conservative will be used in the next Subsection to avoid some redundancy when counter-arguments need to be listed.

## 4.2 Conflicts between arguments

We now revisit conflicts-related concepts in light of conditional contrariety. Let us start with *rebuttals*.

**Definition 7.** *A rebuttal for an argument  $\langle \Phi, \alpha \rangle$  is an argument  $\langle \Psi, \beta \rangle$  s.t.  $\beta \not\bowtie \alpha$ .*

*Example 6.* A rebuttal for  $\langle \{\neg a \vee b, \neg b\}, \neg a \wedge \neg b \rangle$  is  $\langle \{a\}, \neg\neg a \rangle$ . Also,  $\langle \{a \wedge e \Rightarrow b, a \wedge e \Rightarrow \neg c\}, \neg f \vee (a \wedge e \Rightarrow b \wedge \neg c) \rangle$  is a rebuttal for  $\langle \{a \vee d \Rightarrow b \wedge \neg c, f\}, (a \vee d \Rightarrow b \wedge \neg c) \wedge f \rangle$ .

Note that *CC* is used to obtain the conclusion  $\neg f \vee (a \wedge e \Rightarrow b \wedge \neg c)$ .

In classical-logic-based argumentation [15], if  $\langle \Phi, \alpha \rangle$  is a rebuttal for  $\langle \Psi, \beta \rangle$  then  $\langle \Psi, \beta \rangle$  is also a rebuttal for  $\langle \Phi, \alpha \rangle$ . Consequently, the notion of rebuttal is symmetric. However, this property does not hold with respect to the contrariety paradigm. Let us return to Example 6, the argument  $\langle \{a \vee d \Rightarrow b \wedge \neg c, f\}, (a \vee d \Rightarrow b \wedge \neg c) \wedge f \rangle$  is not a rebuttal for  $\langle \{a \wedge e \Rightarrow b, a \wedge e \Rightarrow \neg c\}, \neg f \vee (a \wedge e \Rightarrow b \wedge \neg c) \rangle$ . The notion of rebuttal defined here is thus asymmetric.

Another concept is captured by *defeaters*, which are arguments whose conclusion contraries the support of their targeted argument.

**Definition 8.** A *defeater* for an argument  $\langle \Phi, \alpha \rangle$  is an argument  $\langle \Psi, \beta \rangle$  s.t.  $\beta \boxtimes \Phi$ .

*Example 7.* Some defeaters for  $\langle \{a \vee \neg d \Rightarrow b \wedge c, f \vee \neg b, b\}, f \wedge (a \vee \neg d \Rightarrow b \wedge c) \rangle$  are listed below:

$\langle \{\neg b\}, \neg b \rangle,$   
 $\langle \{\neg b\}, \neg(\neg b \rightarrow b) \rangle,$   
 $\langle \{\neg b, \neg a \rightarrow b\}, \neg b \wedge a \rangle,$   
 $\langle \{e \wedge \neg d \Rightarrow b \wedge c\}, e \wedge \neg d \Rightarrow b \wedge c \rangle,$   
 $\langle \{a \Rightarrow b, a \Rightarrow c\}, a \Rightarrow b \wedge c \rangle,$   
 $\langle \{a \Rightarrow b, a \Rightarrow c\}, \neg \neg(a \Rightarrow b \wedge c) \rangle,$   
 $\langle \{a \Rightarrow b, a \Rightarrow c\}, (a \Rightarrow b) \wedge (a \Rightarrow c) \rangle.$

**Proposition 7.** If  $\langle \Psi, \beta \rangle$  is a rebuttal for  $\langle \Phi, \alpha \rangle$  then  $\langle \Psi, \beta \rangle$  is a defeater for  $\langle \Phi, \alpha \rangle$ .

*Example 8.*  $\langle \{((a \wedge e) \Rightarrow b \wedge c) \wedge \neg d\}, ((a \wedge e) \Rightarrow b \wedge c) \wedge \neg d \rangle$  is a rebuttal for  $\langle \{a \Rightarrow b, a \Rightarrow c\}, (a \Rightarrow b \wedge c) \vee d \rangle$ .

Here  $\{a \Rightarrow b, a \Rightarrow c\} \vdash_c (a \Rightarrow b \wedge c) \vee d$ , then  $\langle \{((a \wedge e) \Rightarrow b \wedge c) \wedge \neg d\}, ((a \wedge e) \Rightarrow b \wedge c) \wedge \neg d \rangle \boxtimes \{a \Rightarrow b, a \Rightarrow c\}$ . Consequently,  $\langle \{((a \wedge e) \Rightarrow b \wedge c) \wedge \neg d\}, ((a \wedge e) \Rightarrow b \wedge c) \wedge \neg d \rangle$  is a defeater for  $\langle \{a \Rightarrow b, a \Rightarrow c\}, (a \Rightarrow b \wedge c) \vee d \rangle$ .

An interesting special kind of defeaters are *challenges*. As the next proposition shows, challenges capture some situations where the defeat relation is asymmetric.

**Definition 9.** Let  $\langle \Phi, \alpha \rangle$  and  $\langle \Psi, \beta \rangle$  be two arguments.  $\langle \Phi, \alpha \rangle$  is a *challenge* to  $\langle \Psi, \beta \rangle$  iff  $\alpha \boxtimes \Psi$  and  $\forall \gamma$  s.t.  $\Psi \vdash_c \gamma, \gamma \not\boxtimes \Phi$ .

*Example 9.* The argument  $\langle \{a \wedge e \Rightarrow b, a \wedge e \Rightarrow c\}, a \wedge e \Rightarrow b \wedge c \rangle$  is a challenge to the argument  $\langle \{a \vee d \Rightarrow b \wedge c\}, a \vee d \Rightarrow b \wedge c \rangle$ .

**Proposition 8.** If  $\langle \Phi, \alpha \rangle$  is a challenge to  $\langle \Psi, \beta \rangle$  then  $\langle \Phi, \alpha \rangle$  is a defeater for  $\langle \Psi, \beta \rangle$  and  $\langle \Psi, \beta \rangle$  is not a defeater for  $\langle \Phi, \alpha \rangle$ .

As intended, defeaters can exist even though there is no inconsistency involved. The next result shows that the support of a challenge is consistent with the support of the argument that it attacks.

**Proposition 9.** If  $\langle \Phi, \alpha \rangle$  is a challenge to  $\langle \Psi, \beta \rangle$  then  $\Phi \cup \Psi \not\vdash_c \perp$ .

**Definition 10.** An argument  $\langle \Psi, \beta \rangle$  is a maximally conservative defeater for  $\langle \Phi, \alpha \rangle$  iff  $\langle \Psi, \beta \rangle$  is a defeater for  $\langle \Phi, \alpha \rangle$  such that no defeaters for  $\langle \Phi, \alpha \rangle$  are strictly more conservative than  $\langle \Psi, \beta \rangle$ .

We assume that there exists an enumeration which we call *canonical enumeration* of all maximally conservative defeaters for  $\langle \Phi, \alpha \rangle$ .

*Example 10.* Let us return to Example 7. Both of the following  $\langle \{a \Rightarrow b, a \Rightarrow c\}, a \Rightarrow b \wedge c \rangle$ ,  $\langle \{e \wedge \neg d \Rightarrow b \wedge c\}, e \wedge \neg d \Rightarrow b \wedge c \rangle$ ,  $\langle \{\neg b\}, \neg b \rangle$ ,  $\langle \{a \Rightarrow b, a \Rightarrow c\}, \neg \neg(a \Rightarrow b \wedge c) \rangle$ ,  $\langle \{\neg b\}, \neg(\neg b \rightarrow b) \rangle$ , and  $\langle \{a \Rightarrow b, a \Rightarrow c\}, (a \Rightarrow b) \wedge (a \Rightarrow c) \rangle$  are maximally conservative defeaters for the argument  $\langle \{a \vee \neg d \Rightarrow b \wedge c, f \vee \neg b, b\}, f \wedge (a \vee \neg d \Rightarrow b \wedge c) \rangle$ .

Note that, like arguments, maximally conservative defeaters are in an infinite number, as shown by the following results.

**Proposition 10.** Let  $\langle \Psi, \beta \rangle$  be a maximally conservative defeater for  $\langle \Phi, \alpha \rangle$ .  $\langle \Psi, \gamma \rangle$  is a maximally conservative defeater for  $\langle \Phi, \alpha \rangle$  iff  $\langle \Psi, \beta \rangle$  and  $\langle \Psi, \gamma \rangle$  are quasi-identical.

**Corollary 1.** Let  $\langle \Psi, \beta \rangle$  be a maximally conservative defeater for  $\langle \Phi, \alpha \rangle$ . There is an infinite set of maximally conservative defeaters for  $\langle \Phi, \alpha \rangle$  of the form  $\langle \Theta, \gamma \rangle$  such that  $\langle \Psi, \beta \rangle$  and  $\langle \Theta, \gamma \rangle$  are quasi-identical.

Now, it is possible to avoid some amount of redundancy among counter-arguments by ignoring the unnecessary variants of maximally conservative defeaters. To this end, we define a concept of *pertinent defeaters* as follows.

**Definition 11.** Let  $\langle \Psi_1, \beta_1 \rangle, \dots, \langle \Psi_n, \beta_n \rangle, \dots$  be the canonical enumeration of all maximally conservative defeaters for  $\langle \Phi, \alpha \rangle$ .  $\langle \Psi_i, \beta_i \rangle$  is a pertinent defeater for  $\langle \Phi, \alpha \rangle$  iff for every  $j < i$ ,  $\langle \Psi_i, \beta_i \rangle$  and  $\langle \Psi_j, \beta_j \rangle$  are not quasi-identical.

Thus, a pertinent defeater can be interpreted as the representative of a set of counter-arguments.

*Example 11.* Let us return to Example 10. Suppose that  $\langle \{a \Rightarrow b, a \Rightarrow c\}, a \Rightarrow b \wedge c \rangle$ ,  $\langle \{e \wedge \neg d \Rightarrow b \wedge c\}, e \wedge \neg d \Rightarrow b \wedge c \rangle$ ,  $\langle \{\neg b\}, \neg b \rangle$ ,  $\langle \{a \Rightarrow b, a \Rightarrow c\}, \neg \neg(a \Rightarrow b \wedge c) \rangle$ ,  $\langle \{\neg b\}, \neg(\neg b \rightarrow b) \rangle$ ,  $\langle \{a \Rightarrow b, a \Rightarrow c\}, (a \Rightarrow b) \wedge (a \Rightarrow c) \rangle, \dots$  is the canonical enumeration of the maximally conservative defeaters. Both of the following  $\langle \{a \Rightarrow b, a \Rightarrow c\}, a \Rightarrow b \wedge c \rangle$ ,  $\langle \{e \wedge \neg d \Rightarrow b \wedge c\}, e \wedge \neg d \Rightarrow b \wedge c \rangle$ ,  $\langle \{\neg b\}, \neg b \rangle$  are pertinent defeaters for the argument  $\langle \{a \vee \neg d \Rightarrow b \wedge c, f \vee \neg b, b\}, f \wedge (a \vee \neg d \Rightarrow b \wedge c) \rangle$ .

Clearly, an argument may have more than one pertinent defeater. The next result shows how the pertinent defeaters for the same argument differ from one another.

**Proposition 11.** Any two different pertinent defeaters for the same argument have distinct supports.

**Proposition 12.** An argument cannot be its own pertinent defeater.

### 4.3 Argumentation trees

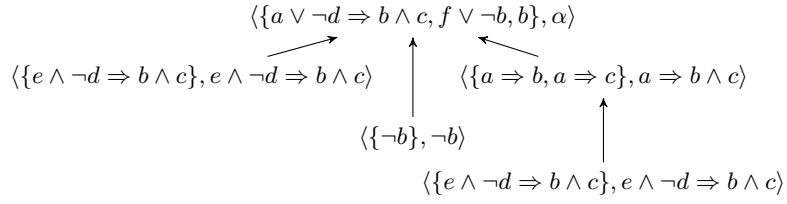
A last basic bricks of logic-based argumentation theories that we revisit is the notion *argumentation trees* and its related topics. From a set  $\Delta$  of formulas, several possibly interconnected arguments can co-exist that should be assembled to get a full understanding about the pros and cons conducting a conclusion to be accepted or rejected. Argumentation trees are intended to collect and organize those arguments.

**Definition 12.** *An argumentation tree for  $\alpha$  is a tree  $T$  whose nodes are arguments s.t.:*

1. *The root of  $T$  is an argument for  $\alpha$*
2. *For every node  $\langle \Psi, \beta \rangle$  whose ancestor nodes are  $\langle \Psi_1, \beta_1 \rangle, \dots, \langle \Psi_n, \beta_n \rangle$ , there exists  $\gamma \in \Psi$  s.t.  $\gamma \notin \Psi_i$  for  $i = 1..n$ ,*
3. *Each child node is a pertinent defeater of its parent node.*

An argumentation tree aims to exhaustively (but implicitly) capture the way counter-arguments can take place as a dispute develops. Condition 2 requires that each counter-argument involves extra information thereby precluding cycles.

*Example 12.* Let us return to Example 7. Let  $\alpha = f \wedge (a \vee \neg d \Rightarrow b \wedge c)$ .



**Fig. 1.** Argumentation tree for  $f \wedge (a \vee \neg d \Rightarrow b \wedge c)$ .

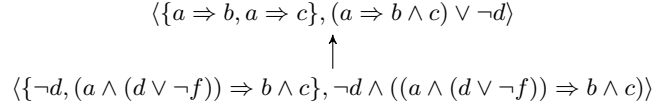
**Proposition 13.** *An argumentation tree for a formula  $\alpha$  is finite.*

**Proposition 14.** *For any  $\alpha$  s.t.  $\Delta \vdash_c \alpha$ , there is only a finite number of argumentation trees for  $\alpha$ .*

Clearly, the last two properties are important in practice. They show that an argumentation tree can indeed be an effective way of representing an argumentation process.

In standard-logic argumentation [15], if  $\Delta$  is consistent then all argumentation trees have exactly one node. This is not the case in contrariety-based argumentation: from a consistent knowledge base, argumentation trees that do not collapse into a single node exist.

Example 13 illustrates that attacks between arguments need not be inconsistency-based but can indeed be rooted in contrariety in conditional logic.



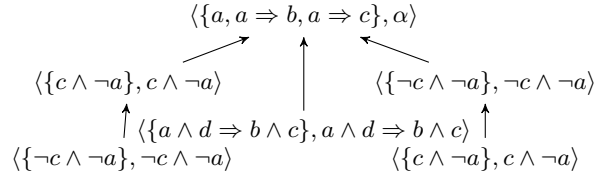
**Fig. 2.** Argumentation tree for  $(a \Rightarrow b \wedge c) \vee \neg d$ .

*Example 13.* Let  $\Delta = \{a \Rightarrow b, \neg d, (a \wedge (d \vee \neg f)) \Rightarrow b \wedge c, a \Rightarrow c\}$ . Note that  $\Delta$  is consistent. Let  $\alpha = (a \Rightarrow b \wedge c) \vee \neg d$ .

As several different argumentation trees for a given formula  $\alpha$  can co-exist, the following *full argumentation tree* concept aims to represent them in a global manner by considering all pertinent defeaters and all possible attacks.

**Definition 13.** Let  $T$  be an argumentation tree for  $\alpha$ .  $T$  is a *full argumentation tree* for  $\alpha$  if the children of any node  $A$  consists of all pertinent defeaters of  $A$ .

*Example 14.* Let  $\Delta = \{a \Rightarrow b, a, a \wedge d \Rightarrow b \wedge c, c \wedge \neg a, a \Rightarrow c, \neg c \wedge \neg a\}$ . Let  $\alpha = a \wedge (a \Rightarrow b \wedge c)$ .



**Fig. 3.** Full argumentation tree for  $a \wedge (a \Rightarrow b \wedge c)$ .

## 5 Perspectives and conclusion

Conditional logic is a widespread tool in A.I. This paper is an attempt to lay down the basic bricks of logic-based argumentation in conditional logic. Interestingly, it has allowed us to put in light and encompass a specific form of conflict that often occurs in real-life argumentation: i.e., claims that additional conditions are required for the conclusion of a rule to hold. In this respect and to some extend, this paper targets some patterns of reasoning similar to those in [30, 31], where preemption operators are investigated in the framework of standard logic: preemption operators allow a logically weaker piece of information to replace a stronger one. However, the problem that we have addressed in this paper is different: the focus has been on confronting arguments, with a specific attention

to comparing conditional formulas making use of the conditional connective, allowing in some sense weaker formulas of that kind to be compared to stronger ones.

In the future, we plan to investigate how to compare and rationalize argumentation trees in conditional logic, consider audience and impact-related issues on arguments and build various algorithmic tools for handling arguments and reasoning about them.

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## Appendix. Proofs.

**Proof of Proposition 1.** Let  $\alpha \boxtimes \Phi$ . By Definition 3 there exists  $\beta$  such that  $\Phi \vdash_c \beta$  and  $\alpha \boxtimes \beta$ . Let  $\Psi$  be a set of formulas. Then,  $\Phi \cup \Psi \vdash_c \beta$  and it happens that  $\alpha \boxtimes \Phi \cup \Psi$ . ■

**Proof of Proposition 2.** Let  $\alpha \equiv \beta$  where  $DF(\alpha) = \alpha_1 \vee \dots \vee \alpha_n$  and  $DF(\beta) = \beta_1 \vee \dots \vee \beta_m$ . Assume that  $\alpha \boxtimes \Phi$  then there exists  $\gamma$  (where  $DF(\gamma) = \gamma_1 \vee \dots \vee \gamma_k$ ) such that  $\Phi \vdash_c \gamma$  and  $\alpha \boxtimes \gamma$ . By Definition 2,  $\forall \alpha_i, \gamma_j$  (I)  $\{\alpha_i, \gamma_j\} \vdash_c \perp$  or (II) there exist  $\delta_1, \delta_2, \sigma_1, \sigma_2$  s.t.  $\delta_1 \vdash_c \sigma_1$ ,  $\sigma_1 \not\vdash_c \delta_1$ ,  $\delta_1 \not\vdash_c \delta_2$  and  $\delta_2 \equiv \sigma_2$ , where (1)  $\{\alpha_i, \gamma_j\} \vdash_c (\delta_1 \Rightarrow \delta_2) \wedge (\sigma_1 \Rightarrow \sigma_2)$  s.t.  $\alpha_i \vdash_c \delta_1 \Rightarrow \delta_2$ , or (2)  $\gamma_j \vdash_c (\delta_1 \Rightarrow \delta_2) \wedge (\sigma_1 \Rightarrow \sigma_2)$ . Let us first consider the case (I).  $\{\alpha_i, \gamma_j\} \vdash_c \perp$ . As  $\alpha \equiv \beta$ , we have that  $\alpha_1 \vee \dots \vee \alpha_n \equiv \beta_1 \vee \dots \vee \beta_m$ . Then, for all  $\alpha_i \in \{\alpha_1, \dots, \alpha_n\}$ , there exists  $\beta_l \in \{\beta_1, \dots, \beta_m\}$  such that  $\alpha_i \equiv \beta_l$ . Thus,  $\{\beta_l, \gamma_j\} \vdash_c \perp$  (3). Turning now to the case (II). By (1)  $\{\alpha_i, \gamma_j\} \vdash_c (\delta_1 \Rightarrow \delta_2) \wedge (\sigma_1 \Rightarrow \sigma_2)$  s.t.  $\alpha_i \vdash_c \delta_1 \Rightarrow \delta_2$ .

Since for all  $\alpha_i \in \{\alpha_1, \dots, \alpha_n\}$ , there exists  $\beta_l \in \{\beta_1, \dots, \beta_m\}$  such that  $\alpha_i \equiv \beta_l$ , it follows that  $\{\beta_l, \gamma_j\} \vdash_c (\delta_1 \Rightarrow \delta_2) \wedge (\sigma_1 \Rightarrow \sigma_2)$  s.t.  $\beta_l \vdash_c \delta_1 \Rightarrow \delta_2$  (4).

As consequently, by the results (3), (4), and (2)  $\beta \bowtie \gamma$  holds. Lastly, that is  $\Phi \vdash_c \gamma$  and  $\beta \bowtie \gamma$  yields  $\beta \bowtie \Phi$ . ■

**Proof of Proposition 3.** By definition of an argument,  $\forall \beta$  s.t.  $\Phi \vdash_c \beta$ ,  $\beta \not\bowtie \Phi$  and  $\Phi \vdash_c \alpha$ . Then,  $\alpha \not\bowtie \Phi$ .

As  $\langle \Phi, \alpha \rangle$  is an argument, then  $\Phi \vdash_c \alpha$ . Thus,  $\Phi \cup \{\neg \alpha\} \vdash_c \perp$ . Hence, there exists  $\beta$  such that  $\Phi \vdash_c \beta$  and  $\{\neg \alpha, \beta\} \vdash_c \perp$ . Assume that  $DF(\neg \alpha) = \alpha_1 \vee \dots \vee \alpha_n$  and  $DF(\beta) = \beta_1 \vee \dots \vee \beta_m$ . It then follows that  $\forall \alpha_i, \forall \beta_j, \{\alpha_i, \beta_j\} \vdash_c \perp$ . As a consequence,  $\neg \alpha \bowtie \beta$ . Accordingly, by Definition 3  $\neg \alpha \bowtie \Phi$  holds. ■

**Proof of Proposition 4.** Let  $\langle \Phi, \alpha \rangle$  be an argument, then  $\Phi \vdash_c \alpha$ . Also, there exists an infinite number of formulas  $\beta$  s.t.  $\alpha \equiv \beta$ . Since  $\Phi$  is minimal to infer  $\alpha$ , it follows that  $\Phi$  is also minimal to infer  $\beta$ . Then,  $\langle \Phi, \beta \rangle$  is an argument. By Definition 5,  $\langle \Phi, \alpha \rangle$  and  $\langle \Phi, \beta \rangle$  are quasi-identical. That is, there exists an infinite number of arguments quasi-identical to  $\langle \Phi, \alpha \rangle$ . ■

**Proof of Proposition 5.** Let  $\langle \Psi, \beta \rangle$  be an argument, then  $\beta \not\bowtie \Psi$ . That is,  $\forall \gamma$  s.t.  $\Psi \vdash_c \gamma$ ,  $\beta \not\bowtie \gamma$ . As  $\langle \Phi, \alpha \rangle$  is more conservative than  $\langle \Psi, \beta \rangle$ ,  $\Phi \subseteq \Psi$ . Then,  $\forall \alpha$  s.t.  $\Phi \vdash_c \alpha$ ,  $\Psi \vdash_c \alpha$ . As a consequence,  $\forall \alpha$  s.t.  $\Phi \vdash_c \alpha$ ,  $\beta \not\bowtie \alpha$ . Thus,  $\beta \not\bowtie \Phi$ .

$\langle \Phi, \alpha \rangle$  is an argument, then  $\Phi \vdash_c \alpha$ . As  $\Phi \subseteq \Psi$ ,  $\Psi \vdash_c \alpha$ . Since  $\langle \Psi, \beta \rangle$  is an argument, then  $\forall \gamma$  s.t.  $\Psi \vdash_c \gamma$ ,  $\gamma \not\bowtie \Psi$ . Thus,  $\alpha \not\bowtie \Psi$ .

As  $\langle \Phi, \alpha \rangle$  is an argument, then by Proposition 3  $\neg \alpha \bowtie \Phi$ . Then, there exists  $\gamma$  such that  $\Phi \vdash_c \gamma$  and  $\neg \alpha \bowtie \gamma$ . Since  $\langle \Phi, \alpha \rangle$  is more conservative than  $\langle \Psi, \beta \rangle$ ,  $\Phi \subseteq \Psi$ . Thus,  $\Psi \vdash_c \gamma$ . Hence, by Definition 3,  $\neg \alpha \bowtie \Psi$ . ■

**Proof of Proposition 6.** We prove the nontrivial part. Consider two arguments  $\langle \Phi, \alpha \rangle$  and  $\langle \Psi, \beta \rangle$  s.t. each one is more conservative than the other. Of course,  $\Phi \subseteq \Psi$  and  $\Psi \subseteq \Phi$ , then  $\Phi = \Psi$ . Therefore,  $\alpha \vdash_c \beta$  and  $\beta \vdash_c \alpha$ , then  $\alpha$  being logically equivalent with  $\beta$ . Thus,  $\langle \Phi, \alpha \rangle$  and  $\langle \Psi, \beta \rangle$  are quasi-identical. ■

**Proof of Proposition 7.** Let  $\langle \Psi, \beta \rangle$  be a rebuttal for  $\langle \Phi, \alpha \rangle$ . That is,  $\beta \bowtie \alpha$ . As  $\langle \Phi, \alpha \rangle$  is an argument, we have  $\Phi \vdash_c \alpha$ . Accordingly,  $\beta \bowtie \Phi$ . Thus,  $\langle \Psi, \beta \rangle$  is a defeater for  $\langle \Phi, \alpha \rangle$ . ■

**Proof of Proposition 8.**  $\langle \Phi, \alpha \rangle$  is a challenge to  $\langle \Psi, \beta \rangle$ . By Definition 9, (1)  $\alpha \bowtie \Psi$  and (2)  $\forall \gamma$  s.t.  $\Psi \vdash_c \gamma$ ,  $\gamma \not\bowtie \Phi$ . An immediate consequence from (1) is  $\langle \Phi, \alpha \rangle$  a defeater for  $\langle \Psi, \beta \rangle$ . In view of (2),  $\beta \not\bowtie \Phi$  because  $\Psi \vdash_c \beta$  (by the definition of an argument). Thus,  $\langle \Psi, \beta \rangle$  can not be a defeater for  $\langle \Phi, \alpha \rangle$ . ■

**Proof of Proposition 9.** Let  $\langle \Phi, \alpha \rangle$  be a challenge to  $\langle \Psi, \beta \rangle$ . By Definition 9,  $\forall \gamma$  s.t.  $\Psi \vdash_c \gamma$ ,  $\gamma \not\bowtie \Phi$ . Then,  $\forall \delta$  s.t.  $\Phi \vdash_c \delta$ ,  $\gamma \not\bowtie \delta$ . It then follows that there exist  $\gamma_i, \delta_j$  where  $DF(\gamma) = \gamma_1 \vee \dots \vee \gamma_n$  and  $DF(\delta) = \delta_1 \vee \dots \vee \delta_m$  s.t.  $\{\gamma_i, \delta_j\} \not\vdash_c \perp$ . Then,  $\{\gamma, \delta\} \not\vdash_c \perp$ . Thus,  $\forall \gamma, \delta$  s.t.  $\Psi \vdash_c \gamma$  and  $\Phi \vdash_c \delta$ ,  $\{\gamma, \delta\} \not\vdash_c \perp$ . Consequently,  $\{\Phi, \Psi\} \not\vdash_c \perp$ . ■



**Proof of Proposition 10.** ( $\rightarrow$ ) Let  $\langle \Psi, \beta \rangle$  and  $\langle \Psi, \gamma \rangle$  be maximally conservative defeaters for  $\langle \Phi, \alpha \rangle$ . Then,  $\beta \bowtie \Phi$  and  $\gamma \bowtie \Phi$ . By Definition 3, there exist  $\delta, \sigma$  s.t.  $\Phi \vdash_c \delta$  and  $\Phi \vdash_c \sigma$ , where (1)  $\beta \bowtie \delta$  and (2)  $\gamma \bowtie \sigma$ . Assume that  $DF(\beta) = \beta_1 \vee \dots \vee \beta_n$ ,  $DF(\gamma) = \gamma_1 \vee \dots \vee \gamma_m$ ,  $DF(\delta) = \delta_1 \vee \dots \vee \delta_s$  and  $DF(\sigma) = \sigma_1 \vee \dots \vee \sigma_v$ . Now,  $DF(\beta \vee \gamma) = \omega_1 \vee \dots \vee \omega_x$  where  $\omega_z = \beta_i$  or  $\omega_z = \gamma_k$  and  $DF(\delta \wedge \sigma) = (\delta_1 \wedge \sigma_1) \vee \dots \vee (\delta_s \wedge \sigma_v)$ . Let us first consider the first case. By (1),  $\{\beta_i, \delta_j\} \vdash_c \perp$ . Therefore, by (2)  $\{\gamma_k, \sigma_t\} \vdash_c \perp$ . Thus,  $\{\omega_z, \delta_j \wedge \sigma_t\} \vdash_c \perp$  (3) because  $\{\beta_j, \delta_j \wedge \sigma_t\} \vdash_c \perp$  and  $\{\gamma_k, \delta_j \wedge \sigma_t\} \vdash_c \perp$ . Turning to the second case. By (1), there exist  $\mu_1, \mu_2, \rho_1, \rho_2$  s.t.  $\mu_1 \vdash_c \rho_1$ ,  $\rho_1 \not\vdash_c \mu_1$ ,  $\mu_1 \not\vdash_c \mu_2$  and  $\mu_2 \equiv \rho_2$ , where  $\{\beta_i, \delta_j\} \vdash_c (\mu_1 \Rightarrow \mu_2) \wedge (\rho_1 \Rightarrow \rho_2)$  s.t.  $\beta_i \vdash_c \mu_1 \Rightarrow \mu_2$  (4)<sup>3</sup>. Also, by (2) there exist  $\mu_3, \mu_4, \rho_3, \rho_4$  s.t.  $\mu_3 \vdash_c \rho_3$ ,  $\rho_3 \not\vdash_c \mu_3$ ,  $\mu_3 \not\vdash_c \mu_4$  and  $\mu_4 \equiv \rho_4$ , where  $\{\gamma_k, \sigma_t\} \vdash_c (\mu_3 \Rightarrow \mu_4) \wedge (\rho_3 \Rightarrow \rho_4)$  s.t.  $\gamma_k \vdash_c \mu_3 \Rightarrow \mu_4$  (5)<sup>4</sup>. From (4),  $\{\beta_i, \delta_j \wedge \sigma_t\} \vdash_c (\mu_1 \Rightarrow \mu_2) \wedge (\rho_1 \Rightarrow \rho_2)$  s.t.  $\beta_i \vdash_c \mu_1 \Rightarrow \mu_2$ . Also from (5),  $\{\gamma_k, \sigma_t \wedge \delta_j\} \vdash_c (\mu_3 \Rightarrow \mu_4) \wedge (\rho_3 \Rightarrow \rho_4)$  s.t.  $\gamma_k \vdash_c \mu_3 \Rightarrow \mu_4$ . Hence, there exist  $\mu_5, \mu_6, \rho_5, \rho_6$  s.t.  $\mu_5 \vdash_c \rho_5$ ,  $\rho_5 \not\vdash_c \mu_5$ ,  $\mu_5 \not\vdash_c \mu_6$  and  $\mu_6 \equiv \rho_6$ , where  $\{\omega_z, \delta_j \wedge \sigma_t\} \vdash_c (\mu_5 \Rightarrow \mu_6) \wedge (\rho_5 \Rightarrow \rho_6)$  s.t.  $\omega_z \vdash_c \mu_5 \Rightarrow \mu_6$  (6)<sup>5</sup>; (note here that  $\mu_5 \Rightarrow \mu_6$  is  $\mu_1 \Rightarrow \mu_2$  or  $\mu_3 \Rightarrow \mu_4$  and  $\rho_5 \Rightarrow \rho_6$  is  $\rho_1 \Rightarrow \rho_2$  or  $\rho_3 \Rightarrow \rho_4$ ). That is, by (3) and (6)  $\forall \omega_z, \forall \delta_j, \forall \sigma_t \{\omega_z, \delta_j \wedge \sigma_t\} \vdash_c \perp$  or there exist  $\mu_5, \mu_6, \rho_5, \rho_6$  s.t.  $\mu_5 \vdash_c \rho_5$ ,  $\rho_5 \not\vdash_c \mu_5$ ,  $\mu_5 \not\vdash_c \mu_6$  and  $\mu_6 \equiv \rho_6$ , where  $\{\omega_z, \delta_j \wedge \sigma_t\} \vdash_c (\mu_5 \Rightarrow \mu_6) \wedge (\rho_5 \Rightarrow \rho_6)$  s.t.  $\omega_z \vdash_c \mu_5 \Rightarrow \mu_6$ . Then,  $\beta \vee \gamma \bowtie \delta \wedge \sigma$ . As,  $\Phi \vdash_c \delta \wedge \sigma$ , then  $\beta \vee \gamma \bowtie \Phi$ . Now, as  $\Psi \vdash_c \beta$  and  $\Psi \vdash_c \gamma$  there exists some minimal  $\Psi' \subseteq \Psi$  s.t.  $\Psi' \vdash_c \beta \vee \gamma$ . Moreover,  $\Psi' \not\vdash_c \perp$  because  $\Psi \not\vdash_c \perp$ . Hence,  $\langle \Psi', \beta \vee \gamma \rangle$  is an argument. As we have proven  $\beta \vee \gamma \bowtie \Phi$ ,  $\langle \Psi', \beta \vee \gamma \rangle$  is a defeater for  $\langle \Phi, \alpha \rangle$ . Thus,  $\langle \Psi', \beta \vee \gamma \rangle$  is more conservative than  $\langle \Psi, \beta \rangle$  and  $\langle \Psi, \gamma \rangle$  because  $\Psi' \subseteq \Psi$ ,  $\beta \vdash_c \beta \vee \gamma$  and  $\gamma \vdash_c \beta \vee \gamma$ . Since  $\langle \Psi, \beta \rangle$  and  $\langle \Psi, \gamma \rangle$  are maximally conservative defeaters for  $\langle \Phi, \alpha \rangle$ , we obtain  $\beta \vee \gamma \vdash_c \beta$  and  $\beta \vee \gamma \vdash_c \gamma$ . As consequently,  $\beta \equiv \gamma$  and then  $\langle \Psi, \beta \rangle$  and  $\langle \Psi, \gamma \rangle$  are quasi-identical.

( $\leftarrow$ ) Let  $\langle \Psi, \beta \rangle$  be a maximally conservative defeater for  $\langle \Phi, \alpha \rangle$ . Then,  $\beta \bowtie \Phi$ . Assume that  $\langle \Psi, \gamma \rangle$  is an argument s.t.  $\langle \Psi, \gamma \rangle$  and  $\langle \Psi, \beta \rangle$  are quasi-identical. Hence,  $\gamma \equiv \beta$ . Then, by Proposition 2  $\gamma \bowtie \Phi$ . It follows that  $\langle \Psi, \gamma \rangle$  is a defeater for  $\langle \Phi, \alpha \rangle$ . Since,  $\Psi \subseteq \Psi$  and  $\beta \vdash_c \gamma$ , then  $\langle \Theta, \gamma \rangle$  is a maximally conservative defeater for  $\langle \Phi, \alpha \rangle$ . ■

**Proof of Corollary 1.** Let  $\langle \Psi, \beta \rangle$  be a maximally conservative defeater for  $\langle \Phi, \alpha \rangle$ . By Proposition 10, every argument  $\langle \Theta, \gamma \rangle$ , where  $\langle \Theta, \gamma \rangle$  and  $\langle \Psi, \beta \rangle$  are quasi-identical, is maximally conservative defeater for  $\langle \Phi, \alpha \rangle$ . Then,  $\beta \equiv \gamma$ . Due to the existence of an infinite number of formulas  $\gamma$  s.t.  $\gamma \equiv \beta$ , there exists an infinite number of maximally conservative defeaters for  $\langle \Phi, \alpha \rangle$  which are quasi-identical to  $\langle \Psi, \beta \rangle$ . ■

<sup>3</sup> Here, we do not consider the case  $\delta_j \vdash_c (\mu_1 \Rightarrow \mu_2) \wedge (\rho_1 \Rightarrow \rho_2)$  because  $\delta \not\bowtie \delta$  by  $\Phi \vdash_c \delta$ .

<sup>4</sup> Here, we do not consider the case  $\sigma_t \vdash_c (\mu_3 \Rightarrow \mu_4) \wedge (\rho_3 \Rightarrow \rho_4)$  because  $\sigma \not\bowtie \sigma$  by  $\Phi \vdash_c \sigma$ .

<sup>5</sup> Here, we do not consider the case  $\delta_j \wedge \sigma_t \vdash_c (\mu_5 \Rightarrow \mu_6) \wedge (\rho_5 \Rightarrow \rho_6)$  because  $\delta \wedge \sigma \not\bowtie \delta \wedge \sigma$ .

**Proof of Proposition 11.** Suppose there exist two pertinent defeaters for  $\langle \Phi, \alpha \rangle$  which have the same support. So  $\langle \Psi, \beta \rangle$  and  $\langle \Psi, \gamma \rangle$  are pertinent defeaters for  $\langle \Phi, \alpha \rangle$ . Since,  $\Psi \vdash_c \beta$  and  $\Psi \vdash_c \gamma$ , then  $\Psi \vdash_c \beta \vee \gamma$ . Thus, there exists some minimal subset  $\Psi' \subseteq \Psi$  such that  $\langle \Psi', \beta \vee \gamma \rangle$  is an argument. As proven in proof of Proposition 10, if  $\beta \bowtie \Phi$  and  $\gamma \bowtie \Phi$ , then  $\beta \vee \gamma \bowtie \Phi$ . Thus,  $\langle \Psi', \beta \vee \gamma \rangle$  is a defeater for  $\langle \Phi, \alpha \rangle$ . Therefore,  $\langle \Psi', \beta \vee \gamma \rangle$  is more conservative than  $\langle \Psi, \beta \rangle$  and  $\langle \Psi, \gamma \rangle$ . It then follows that  $\langle \Psi', \beta \vee \gamma \rangle$  is maximally conservative defeater for  $\langle \Phi, \alpha \rangle$  and this contradicts the assumption that  $\langle \Psi, \beta \rangle$  and  $\langle \Psi, \gamma \rangle$  are maximally conservative defeater for  $\langle \Phi, \alpha \rangle$ . ■

**Proof of Proposition 12.** Let  $\langle \Phi, \alpha \rangle$  be an argument. Suppose that  $\langle \Phi, \alpha \rangle$  is a defeater for itself. Then,  $\alpha \bowtie \Phi$ . Hence, there exists  $\beta$  s.t.  $\Phi \vdash_c \beta$  and  $\alpha \bowtie \beta$ . This contradicts the condition 2 of Definition 4 because  $\Phi \vdash_c \alpha$ . Accordingly,  $\langle \Phi, \alpha \rangle$  cannot be a pertinent defeater for itself. ■

**Proof of Proposition 13.** Since  $\Delta$  is finite, there is only a finite number of pertinent defeaters for each argument. Then, by condition 3 of Definition 12, the number of branches of the argumentation tree is finite. Also, by condition 2 of Definition 12, no branch in an argumentation tree can then be infinite. Thus, each argumentation tree is finite. ■

**Proof of Proposition 14.** Since  $\Delta$  is finite, there is only a finite number of pertinent defeaters for each argument. Then, there exists a finite number of nodes that can be used to construct an argumentation tree for  $\alpha$ . ■