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A Global Stochastic Optimization Particle Filter Algorithm

Mathieu Gerber⁽¹⁾, Randal Douc⁽²⁾

(1) School of Mathematics, University of Bristol, UK

(2) SAMOVAR, Telecom SudParis, Institut Polytechnique de Paris, France

We introduce a new online algorithm for expected log-likelihood maximization in situations where the objective function is multi-modal and/or has saddle points, that we term G-PFSO. The key element underpinning G-PFSO is a probability distribution which (a) is shown to concentrate on the target parameter value as the sample size increases and (b) can be efficiently estimated by means of a standard particle filter algorithm. This distribution depends on a learning rate, where the faster the learning rate the quicker it concentrates on the desired element of the search space, but the less likely G-PFSO is to escape from a local optimum of the objective function. In order to achieve a fast convergence rate with a slow learning rate, G-PFSO exploits the acceleration property of averaging, well-known in the stochastic gradient literature. Considering several challenging estimation problems, the numerical experiments show that, with high probability, G-PFSO successfully finds the highest mode of the objective function and converges to its global maximizer at the optimal rate. While the focus of this work is expected log-likelihood maximization, the proposed methodology and its theory apply more generally for optimizing a function defined through an expectation.

1. Introduction

1.1. Set-up and problem formulation

Let $(Y_t)_{t \geq 1}$ be a sequence of i.i.d. random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a measurable space $(\mathcal{Y}, \mathcal{Y})$. Below we will often refer to $(Y_t)_{t \geq 1}$ as the observations and to t as the time at which Y_t is observed. We let $\{f_\theta, \theta \in \Theta \subseteq \mathbb{R}^d\}$ be a collection of probability density functions on \mathcal{Y} with respect to some σ -finite measure $\eta(dy)$ and we assume that $\theta_\star = \operatorname{argmax}_{\theta \in \Theta} \mathbb{E}\{\log f_\theta(Y_1)\}$ is well-defined, with \mathbb{E} the expectation operator associated to $(\Omega, \mathcal{F}, \mathbb{P})$. In this work we consider the problem of estimating θ_\star in an online fashion, where by online we mean that

the memory and computational requirement to process Y_t is finite and bounded uniformly in t . Developing methods with a computational complexity scaling at most linearly with the number of data points is particularly useful for parameter inference in large datasets (National Research Council, 2013, Chap. 2). We stress that we do not assume the model $\{f_\theta, \theta \in \Theta\}$ to be well-specified, i.e. that $Y_1 \sim f_{\theta_\star}(y)\eta(dy)$. Consequently, as explained in Section 1.3, the optimization problem addressed in this paper is relevant to performing various learning tasks and not only to drawing inference in parametric models from i.i.d. observations.

Stochastic gradient algorithms are popular tools to learn the parameter value θ_\star on the fly and at the optimal $t^{-1/2}$ rate (Toulis and Airolidi, 2015). However, when the objective function $\theta \mapsto \mathbb{E}\{\log f_\theta(Y_1)\}$ is multi-modal and/or has saddle points these algorithms are only guaranteed to converge to one of its stationary points (Tadić, 2015). Gelfand and Mitter (1991) have shown that, for such objective functions, we can ensure the convergence of a standard stochastic gradient algorithm towards θ_\star by adding some extra noise at each iteration. Unfortunately, this approach results in an algorithm converging towards θ_\star at an extremely slow rate $r_t \rightarrow 0$ such that $(\log t)^{-1/2} = o(r_t)$ (Pelletier, 1998; Yin, 1999). By contrast, the algorithm recently introduced in Gerber and Heine (2021) is proven to converge to the global optimum of the mapping $\theta \mapsto \mathbb{E}\{\log f_\theta(Y_1)\}$ at a nearly optimal rate. However, as its computational cost increases exponentially fast with the dimension of the parameter space Θ , this algorithm can only be used in practice to tackle small or moderate dimensional optimization problems.

1.2. Contribution of the paper

We introduce in this work a new global stochastic optimization method to learn the parameter value θ_\star in an online manner, that we term *Global Particle Filter Stochastic Optimization* (G-PFSO). Unlike the algorithm proposed in Gelfand and Mitter (1991), we observe empirically that G-PFSO converges to θ_\star at the optimal $t^{-1/2}$ rate and, unlike the method developed in Gerber and Heine (2021), G-PFSO is implementable for any dimension of Θ . It should however be clear that, in practice, we cannot expect G-PFSO to perform well on any optimization problems since, depending on the landscape of the objective function and on the dimension of the search space, finding a small neighbourhood of θ_\star with a reasonable computational cost may simply be an intractable task (Loshchilov and Hutter, 2016).

In G-PFSO the parameter θ_\star is learnt through a sequence $(\tilde{\pi}_t)_{t \geq 1}$ of probability distributions which is proven to converge to the Dirac mass $\delta_{\{\theta_\star\}}$ as $t \rightarrow \infty$. Each distribution $\tilde{\pi}_t$ depends only on $\{Y_s\}_{s=1}^t$ and the sequence $(\tilde{\pi}_t)_{t \geq 1}$ can be easily estimated in an online fashion by means of a standard particle filter algorithm (Chopin and Papaspiliopoulos, 2020, Chap. 10). For every $t \geq 1$ the resulting particle filter approximation $\tilde{\pi}_t^N$ of $\tilde{\pi}_t$, which has a finite support of size $N \in \mathbb{N} = \{1, 2, \dots\}$, is then used to compute an estimate of θ_\star .

The sequence $(\tilde{\pi}_t)_{t \geq 1}$ depends on a learning rate $h_t \rightarrow 0$, with the slower $h_t \rightarrow 0$ the greater the ability of $\tilde{\pi}_t^N$ to escape from a local optimum of the objective function, but the slower the rate at which this distribution can concentrate on θ_\star . Polyak–Ruppert av-

eraging is a well-known method for accelerating stochastic gradient methods (Polyak and Juditsky, 1992) and, following this idea, the G-PFSO estimator of θ_* is $\bar{\theta}_t^N = t^{-1} \sum_{s=1}^t \tilde{\theta}_s^N$, with $\tilde{\theta}_s^N = \int_{\Theta} \theta \tilde{\pi}_s^N(d\theta)$. In all the experiments presented below we observe that, for a fixed N and a slow learning rate $h_t = t^{-1/2}$, the estimator $\bar{\theta}_t^N$ converges at the optimal $t^{-1/2}$ rate.

The theoretical analysis of the convergence behaviour of $\bar{\theta}_t^N$ as $t \rightarrow \infty$, for a fixed $N \in \mathbb{N}$, is a hard task. In this paper we focus on a key preliminary step towards making such an analysis possible, namely the study of the sequence $(\tilde{\pi}_t)_{t \geq 1}$. More precisely, we derive a consistency result for this sequence that holds under weak conditions on the statistical model and on the rate at which $h_t \rightarrow 0$, dealing carefully with the case where the parameter space Θ is unbounded.

An R package implementing the proposed global stochastic optimization algorithm is available on GitHub at github.com/mathieugerber/PFOptim.

1.3. Scope of the theoretical analysis

Our theoretical analysis of $(\tilde{\pi}_t)_{t \geq 1}$ does not assume that the statistical model $\{f_\theta, \theta \in \Theta\}$ is well-specified, making it valid when G-PFSO is used to compute the maximum likelihood estimator for a large class of statistical models. To clarify this point consider n observations $\{(z_i, x_i)\}_{i=1}^n$ in $\mathbb{R}^{d_z} \times \mathbb{R}^{d_x}$ assumed to be such that, for all i , the conditional distribution of Z_i given $X_i = x$ belongs to the set $\{f_\theta(\cdot | x), \theta \in \Theta\}$, and let $f_\theta(z, x) = f_\theta(z | x)f_X(x)$ for some arbitrary probability density function f_X on \mathbb{R}^{d_x} . Then, letting $Y_1 = (Z_1^{(n)}, X_1^{(n)})$ be a random variable distributed according to the empirical distribution of the observations $\{(z_i, x_i)\}_{i=1}^n$, we have $\theta_* = \operatorname{argmax}_{\theta \in \Theta} \mathbb{E}\{\log f_\theta(Y_1)\} = \operatorname{argmax}_{\theta \in \Theta} \sum_{i=1}^n \log f_\theta(z_i | x_i) = \hat{\theta}_{\text{mle}, n}$. Notice that this optimization problem does not depend on f_X and, as one may expect, using G-PFSO to compute $\hat{\theta}_{\text{mle}, n}$ does not require one to specify this probability density function; see Algorithm 1.

A second important consequence of not assuming the model $\{f_\theta, \theta \in \Theta\}$ to be well-specified is that our main result is applicable to estimation problems that are not limited to parameter inference in parametric models. In particular, in order to compute $\theta_* = \operatorname{argmin}_{\theta \in \Theta} \mathbb{E}\{\varphi(\theta, Y_1)\}$ for a measurable function $\varphi : \Theta \times \mathcal{Y} \rightarrow [0, \infty)$ such that $\int_{\mathcal{Y}} \exp\{-\varphi(\theta, y)\} \eta(dy) = 1$ for every $\theta \in \Theta$ and for some σ -finite measure $\eta(dy)$, G-PFSO can be used, and its theoretical guarantees apply, with the probability density function $f_\theta(y) = \exp\{-\varphi(\theta, y)\}$. For instance, a classical machine learning task is to train a function γ_θ to predict a response variable Z from a vector x of features, which amounts to computing $\theta_* = \operatorname{argmin}_{\theta \in \Theta} \mathbb{E}[L\{\gamma_\theta(X), Z\}]$ for some loss function $L(\hat{z}, z)$. For this problem, and with $y = (z, x)$, the condition $\int_{\mathcal{Y}} \exp\{-\varphi(\theta, y)\} \eta(dy) = 1$ holds for $\varphi(\theta, y) = L\{\gamma_\theta(x), z\}$ when L is the quadratic loss, i.e. $L(\hat{z}, z) = \|\hat{z} - z\|^2$, the absolute error loss, i.e. $L(\hat{z}, z) = |\hat{z} - z|$ assuming $z \in \mathbb{R}$, or the cross-entropy loss, i.e. $L(\hat{z}, z) = -z \log(\hat{z}) - (1 - z) \log(1 - \hat{z})$ assuming $z \in \{0, 1\}$ and $\hat{z} \in (0, 1)$.

We stress that neither the definition of $\tilde{\pi}_t$, see Section 2.1, nor the proof of our convergence result for the sequence $(\tilde{\pi}_t)_{t \geq 1}$ requires that $\int_{\mathcal{Y}} f_\theta(y) \eta(dy) = 1$. Consequently, our main theorem provides conditions on a measurable function $\varphi : \Theta \times \mathcal{Y} \rightarrow [0, \infty)$

which ensure that, for $f_\theta(y) = \exp\{-\varphi(\theta, y)\}$, the distribution $\tilde{\pi}_t$ concentrates on $\theta_\star = \operatorname{argmin}_{\theta \in \Theta} \mathbb{E}\{\varphi(\theta, Y_1)\}$ as $t \rightarrow \infty$. In particular, our theoretical analysis applies when $\varphi(\theta, y) = \tilde{\varphi}(\theta)$ for some function $\tilde{\varphi} : \Theta \rightarrow \mathbb{R}$, that is, when G-PFSO is used to address a standard optimization task where the objective function can be easily evaluated point-wise. We however stress that, our proof technique coming from the literature on Bayesian asymptotics, our assumptions on $\{f_\theta, \theta \in \Theta\}$ are particularly standard when this set of functions is a collection of probability density functions.

1.4. Additional notation and outline of the paper

We let $\|\cdot\|_\infty$ be the maximum norm on \mathbb{R}^d , $B_\epsilon(x)$ be the open ball of size $\epsilon > 0$ around $x \in \mathbb{R}^d$ w.r.t. $\|\cdot\|$, the Euclidean norm on \mathbb{R}^d , $V_\epsilon = \Theta \setminus B_\epsilon(\theta_\star)$, $t_{d,\nu}(m, \Sigma)$ be the d -dimensional Student's t -distribution with $\nu > 0$ degrees of freedom, location vector m and scale matrix Σ , and $\mathcal{N}_d(m, \Sigma)$ be the d -dimensional Gaussian distribution with mean m and covariance matrix Σ . If A is a Borelian set of \mathbb{R}^d we denote by $\mathcal{B}(A)$ the Borel σ -algebra on A , by $\mathcal{P}(A)$ the set of probability measures on $(A, \mathcal{B}(A))$ and by $\mathcal{P}_L(A)$ the set of probability measures on $(A, \mathcal{B}(A))$ that are absolutely continuous w.r.t. $d\theta$, the Lebesgue measure on \mathbb{R}^d . Moreover, for a sequence $(\mu_t)_{t \geq 1}$ of probability measures on \mathbb{R}^d , the notation $\mu_t \Rightarrow \mu$ means that the sequence $(\mu_t)_{t \geq 1}$ converges weakly to the probability measure $\mu \in \mathcal{P}(\mathbb{R}^d)$. We assume throughout the paper that $\Theta \in \mathcal{B}(\mathbb{R}^d)$ and, for any probability measure μ on Θ and any sequence $(\mu_t)_{t \geq 1}$ of probability measures on Θ , implicitly indexed by random variables, we say that $\mu_t \Rightarrow \mu$ in \mathbb{P} -probability if the sequence of random variables $(\mathcal{D}_{\text{pro}}(\mu_t, \mu))_{t \geq 1}$ converges in \mathbb{P} -probability to 0, where \mathcal{D}_{pro} denotes the Prohorov distance between probability measures on \mathbb{R}^d .

The rest of the paper is organized as follows. The G-PFSO approach is introduced in Section 2, our convergence result for the sequence $(\tilde{\pi}_t)_{t \geq 1}$ is given in Section 3 and Section 4 proposes some numerical experiments. All the proofs are gathered in Appendix A.

2. Global particle filter stochastic optimization

2.1. The sequence $(\tilde{\pi}_t)_{t \geq 1}$

Let $\nu \in (0, \infty)$, Σ be a $d \times d$ covariance matrix, $h_t \rightarrow 0$ be a learning rate, that is $(h_t)_{t \geq 1}$ is a sequence in $(0, \infty)$ such that $h_t \rightarrow 0$, and let $(t_p)_{p \geq 0}$ be a strictly increasing sequence in \mathbb{N} verifying $t_{p+1} - t_p \rightarrow \infty$ as $p \rightarrow \infty$. Then, for $\tilde{\pi}_0 \in \mathcal{P}_L(\Theta)$ and letting

$$\widetilde{M}_t(\theta', d\theta) = \begin{cases} \mathcal{N}_d(\theta', h_t^2 \Sigma), & t \notin (t_p)_{p \geq 0}, \\ t_{d,\nu}(\theta', h_t^2 \Sigma), & t \in (t_p)_{p \geq 0} \end{cases}, \quad \theta' \in \mathbb{R}^d, \quad t \geq 1, \quad (1)$$

the sequence $(\tilde{\pi}_t)_{t \geq 1}$ is defined by

$$\tilde{\pi}_t(d\theta) = \frac{f_\theta(Y_t) \left\{ \int_{\mathbb{R}^d} \tilde{\pi}_{t-1}(d\theta') \widetilde{M}_{t-1}(\theta', d\theta) \right\} |_{\Theta}}{\int_{\mathbb{R}^d} f_\theta(Y_t) \left\{ \int_{\mathbb{R}^d} \tilde{\pi}_{t-1}(d\theta') \widetilde{M}_{t-1}(\theta', d\theta) \right\} |_{\Theta}} \in \mathcal{P}_L(\Theta), \quad t \geq 1 \quad (2)$$

with the convention $\int_{\mathbb{R}^d} \tilde{\pi}_{t-1}(\mathrm{d}\theta') \widetilde{M}_{t-1}(\theta', \mathrm{d}\theta) = \tilde{\pi}_0(\mathrm{d}\theta)$ when $t = 1$ and where $\mu|_{\Theta}$ denotes the restriction of $\mu \in \mathcal{P}(\mathbb{R}^d)$ on Θ . In (2) we implicitly assume that $\int_{\Theta} \mathrm{d}\theta > 0$.

To motivate the above definition of $(\widetilde{M}_t)_{t \geq 1}$ let us focus on the behaviour of the subsequence $(\tilde{\pi}_{t_p})_{p \geq 0}$. To this aim, for every $t \geq 1$ we let $\tilde{\Psi}_t : \mathcal{P}(\Theta) \rightarrow \mathcal{P}(\Theta)$ and $\Psi_t : \mathcal{P}(\Theta) \rightarrow \mathcal{P}(\Theta)$ be the random mappings such that, for all $\pi \in \mathcal{P}(\Theta)$,

$$\tilde{\Psi}_t(\pi)(\mathrm{d}\theta) \propto f_{\theta}(Y_t) \left\{ \int_{\mathbb{R}^d} \pi(\mathrm{d}\theta') \widetilde{M}_{t-1}(\theta', \mathrm{d}\theta) \right\} |_{\Theta}, \quad \Psi_t(\pi)(\mathrm{d}\theta) \propto f_{\theta}(Y_t) \pi(\mathrm{d}\theta).$$

In this notation, $\tilde{\pi}_{t_{p+1}} = \tilde{\Psi}_{t_{p+1}} \circ \dots \circ \tilde{\Psi}_{t_p+2}(\tilde{\pi}_{t_p+1})$ and, for the sake of argument, let $\pi'_{t_{p+1}} = \Psi_{t_{p+1}} \circ \dots \circ \Psi_{t_p+2}(\tilde{\pi}_{t_p+1})$ be the Bayesian posterior distribution associated with the observations $\{Y_t\}_{t=t_p+2}^{t_{p+1}}$ and the prior distribution $\tilde{\pi}_{t_p+1}$. Let U be a small neighbourhood of θ_* . Then, results on Bayesian asymptotics (Kleijn and van der Vaart, 2012) ensure that if $t_{p+1} - t_p$ is large enough then the mass of $\pi'_{t_{p+1}}$ on U will be large with high probability, where the sample size $t_{p+1} - t_p$ required for this to be true depends on the mass of the ‘prior distribution’ $\tilde{\pi}_{t_p+1}$ around θ_* . In particular, if this mass is small then $t_{p+1} - t_p$ needs to be large, i.e. a large sample size is necessary to compensate for a poor prior distribution. Informally speaking, by letting \widetilde{M}_t be a Gaussian kernel—that is a kernel with thin tails—for all $t \notin (t_p)_{p \geq 0}$, and by letting $h_t \rightarrow 0$ fast enough, we can ensure that the mappings $\{\tilde{\Psi}_t\}_{t=t_p+2}^{t_{p+1}}$ are sufficiently close to the Bayes updates $\{\Psi_t\}_{t=t_p+2}^{t_{p+1}}$ to enable $\tilde{\pi}_t$ to concentrate on θ_* between time $t = t_p + 2$ and time $t = t_{p+1}$. On the other hand, by taking \widetilde{M}_{t_p} to be a Student’s t -kernel—that is a kernel with heavy tails—we can compute a lower bound for the mass of $\tilde{\pi}_{t_{p+1}} = \tilde{\Psi}_{t_{p+1}}(\tilde{\pi}_{t_p})$ around θ_* that holds uniformly in $\tilde{\pi}_{t_p}$ and which does not converge to zero too quickly as $p \rightarrow \infty$. Together with a suitable definition of $t_{p+1} - t_p$ this lower bound allows us to obtain a precise control of $\tilde{\pi}_{t_{p+1}}(U)$ which, in particular, makes it possible to show that $\tilde{\pi}_{t_{p+1}}(U) \rightarrow 1$ as $p \rightarrow \infty$.

Our main result, Theorem 1 in Section 3, provides conditions on the learning rate $h_t \rightarrow 0$ and on $(t_p)_{p \geq 0}$ which ensure the convergence of $\tilde{\pi}_t$ towards $\delta_{\{\theta_*\}}$ in \mathbb{P} -probability, under standard assumptions on the statistical model. For instance, Corollary 1 in Section 3 gives sufficient conditions on $\{f_{\theta}, \theta \in \Theta\}$ to guarantee that $\tilde{\pi}_t \Rightarrow \delta_{\{\theta_*\}}$ in \mathbb{P} -probability when $h_t = t^{-\alpha}$ and when, for some $\varrho \in (0, \alpha \wedge 1)$, the sequence $(t_p)_{p \geq 0}$ is defined by

$$t_0 \in \mathbb{N} \quad t_p = t_{p-1} + \lceil t_{p-1}^{\varrho} \log(t_{p-1}) \rceil, \quad p \geq 1.$$

It is important to mention at this stage that these conditions on $\{f_{\theta}, \theta \in \Theta\}$ do not depend on α when Θ is a bounded set. When Θ is unbounded the smaller α is the stronger the assumptions on the statistical model imposed by our main result. However, as shown in Section 3, Proposition 1, even when Θ is unbounded we may have $\tilde{\pi}_t \Rightarrow \delta_{\{\theta_*\}}$ in \mathbb{P} -probability for any learning rate of the form $h_t = t^{-\alpha}$. To summarize, whenever Θ is bounded, and for some models defined on an unbounded parameter space, we show that $\tilde{\pi}_t$ converges towards $\delta_{\{\theta_*\}}$ even when $h_t \rightarrow 0$ at an arbitrarily slow polynomial rate. As explained in Section 2.4, defining $\tilde{\pi}_t$ with a slow learning rate $h_t \rightarrow 0$ is of particular importance when the function $\theta \mapsto \mathbb{E}\{\log f_{\theta}(Y_1)\}$ has several modes.

Algorithm 1. Global PF Stochastic Optimization

(Operations with index n must be performed for all $n \in \{1, \dots, N\}$.)

Input: $N \in \mathbb{N}$, $c_{\text{ESS}} \in (0, 1]$ and a resampling algorithm $\mathcal{R}(\cdot, \cdot)$

- 1: Let $\theta_1^n \sim \tilde{\pi}_0(d\theta_0)$ and set $w_1^n = f_{\theta_1^n}(Y_1)$ and $W_1^n = w_1^n / \sum_{m=1}^N w_1^m$
 - 2: Let $\tilde{\theta}_1^N = \sum_{n=1}^N W_1^n \theta_1^n$ and $\bar{\theta}_1^N = \tilde{\theta}_1^N$
 - 3: **for** $t \geq 2$ **do**
 - 4: Set $\text{ESS}_{t-1} = 1 / \sum_{m=1}^N (W_{t-1}^m)^2$
 - 5: **if** $\text{ESS}_{t-1} \leq N c_{\text{ESS}}$ **then**
 - 6: Let $\{\hat{\theta}_{t-1}^1, \dots, \hat{\theta}_{t-1}^N\} \sim \mathcal{R}(\{\theta_{t-1}^n, W_{t-1}^n\}_{n=1}^N, N)$ and set $w_{t-1}^n = 1$
 - 7: **else**
 - 8: Let $\hat{\theta}_{t-1}^n = \theta_{t-1}^n$
 - 9: **end if**
 - 10: **if** $(t-1) \in (t_p)_{p \geq 0}$ **then**
 - 11: Set $\theta_t^n = \hat{\theta}_{t-1}^n + h_{t-1} \epsilon_{t-1}^n$ where $\epsilon_{t-1}^n \sim t_{d,\nu}(0, \Sigma)$
 - 12: **else**
 - 13: Set $\theta_t^n = \hat{\theta}_{t-1}^n + h_{t-1} \epsilon_{t-1}^n$ where $\epsilon_{t-1}^n \sim \mathcal{N}_d(0, \Sigma)$
 - 14: **end if**
 - 15: Set $w_t^n = w_{t-1}^n f_{\theta_t^n}(Y_t)$ if $\theta_t^n \in \Theta$ and $w_t^n = 0$ otherwise, and $W_t^n = w_t^n / \sum_{m=1}^N w_t^m$
 - 16: Let $\tilde{\theta}_t^N = \sum_{n=1}^N W_t^n \theta_t^n$ and $\bar{\theta}_t^N = t^{-1}((t-1)\bar{\theta}_{t-1}^N + \tilde{\theta}_t^N)$
 - 17: **end for**
-

2.2. The G-PFSO algorithm

The G-PFSO algorithm is presented in Algorithm 1, which reduces to a simple particle filter algorithm for approximating the sequence $(\tilde{\pi}_t)_{t \geq 1}$ in an online fashion. For every $t \geq 1$ the particle filter estimate $\tilde{\pi}_t^N$ of $\tilde{\pi}_t$ is $\tilde{\pi}_t^N = \sum_{n=1}^N W_t^n \delta_{\theta_t^n}$ while the G-PFSO estimator $\bar{\theta}_t^N$ of θ_* is computed on the last line of the algorithm, where $\tilde{\theta}_t^N = \int_{\Theta} \theta \tilde{\pi}_t^N(d\theta)$ is a particle filter approximation of $\tilde{\theta}_t = \int_{\Theta} \theta \tilde{\pi}_t(d\theta)$. Noting that $\bar{\theta}_t^N = t^{-1} \sum_{s=1}^t \tilde{\theta}_s^N$, it follows that $\bar{\theta}_t^N$ is the Polyak–Ruppert averaging of $\{\tilde{\theta}_s^N\}_{s=1}^t$. Averaging is a well-known acceleration technique in the literature on stochastic gradient algorithms which is illustrated in the next subsection.

In Algorithm 1 the resampling algorithm $\mathcal{R}(\cdot)$ is such that $\mathcal{R}(\{x^n, p^n\}_{n=1}^N)$ is a probability distribution on the set $\{x^1, \dots, x^N\}^N$, where $(p^1, \dots, p^N) \in [0, 1]^N$, $\sum_{n=1}^N p^n = 1$ and $x^n \in \mathbb{R}^d$ for all $n = 1, \dots, N$. We refer the reader to Chap. 9 of Chopin and Papaspiliopoulos (2020) for a detailed discussion of resampling methods, and to Chap. 10 of this reference for explanations concerning the role of the parameter $c_{\text{ESS}} \in (0, 1]$ appearing in Algorithm 1.

$Z_t \setminus \alpha$	0.1	0.3	0.5	0.7	1
$ \tilde{\theta}_t - \theta_\star $	0.05	0.15	0.24	0.36	0.45
$ \bar{\theta}_t - \theta_\star $	0.48	0.48	0.48	0.48	0.50

Table 1.: Example of Section 2.3. Ordinary least square estimate of β_2 in the model $\log(Z_t) = \beta_1 - \beta_2 \log(t) + \epsilon_t$, where Z_t is as defined in the table, $t \in \{10^5, 10^5 + 1, \dots, 10^7\}$ and where $\tilde{\pi}_t$ is as defined in (3) with $h_t = t^{-\alpha}$, $(\tilde{\theta}_0, \tilde{\sigma}_0^2) = (0, 25)$ and with $Y_1 \sim \mathcal{N}_1(0, 1)$ (so that $\theta_\star = 0$).

2.3. Accelerating property of Polyak–Ruppert averaging: an illustrative example

For every $\theta \in \Theta = \mathbb{R}$ we let $f_\theta(y)\eta(dy) = \mathcal{N}_1(\theta, 1)$ and, for some $\tilde{\theta}_0 \in \mathbb{R}$ and $\tilde{\sigma}_0^2 \in (0, \infty)$, we let $\tilde{\pi}_0 = \mathcal{N}_1(\tilde{\theta}_0, \tilde{\sigma}_0^2)$. To make the computation of $(\tilde{\pi}_t)_{t \geq 1}$ tractable we relax the condition that the sequence $(\tilde{M}_t)_{t \geq 1}$ contains infinitely many Student’s t -distributions by letting $\tilde{M}_t(\theta', d\theta) = \mathcal{N}_1(\theta', h_t^2)$ for all $t \geq 1$. Then, with $g(x) = x/(1+x)$ for all $x \in (0, \infty)$, we have

$$\tilde{\pi}_t = \mathcal{N}_1(\tilde{\theta}_t, \tilde{\sigma}_t^2), \quad \tilde{\sigma}_t^2 = g(\tilde{\sigma}_{t-1}^2 + h_{t-1}^2), \quad \tilde{\theta}_t = \tilde{\theta}_{t-1} + \tilde{\sigma}_t^2(Y_t - \tilde{\theta}_{t-1}), \quad t \geq 1. \quad (3)$$

In Table 1 we study the convergence rate of $\tilde{\theta}_t$ and of $\bar{\theta}_t = t^{-1} \sum_{s=1}^t \tilde{\theta}_s$ when $h_t = t^{-\alpha}$. The results reported in this table, obtained for all $\alpha \in \{0.1, 0.3, 0.5, 0.7, 1\}$, suggest that $\tilde{\theta}_t$ converges to θ_\star at rate $t^{-\alpha/2}$ while $\bar{\theta}_t$ converges to this target parameter value at the optimal $t^{-1/2}$ rate.

Noting that the update (3) for $\tilde{\theta}_t$ reduces to that of a standard stochastic gradient algorithm with step sizes $(\tilde{\sigma}_t^2)_{t \geq 1}$, the $t^{-1/2}$ convergence rate for $\bar{\theta}_t$ observed in this example proceeds from well-known results on the acceleration property of averaging for this class of algorithms (Polyak and Juditsky, 1992). However, the numerical experiments presented in Section 4 suggest that, within G-PFSO, the acceleration property of averaging holds beyond the simple example considered above, although the total insensitivity of the convergence rate of $\bar{\theta}_t$ to α observed in Table 1 does not appear to hold in general.

2.4. Choosing the learning rate h_t

As illustrated in Section 2.3, the faster $h_t \rightarrow 0$ the closer to the optimal $t^{-1/2}$ rate is the rate at which $\tilde{\pi}_t$ learns θ_\star . However, it is clear from Algorithm 1 and from the definition of $(\tilde{M}_t)_{t \geq 1}$ that, for a fixed value of N , the smaller the value of h_t the higher the probability of having the support of $\tilde{\pi}_{t+1}^N$ close to that of $\tilde{\pi}_t^N$. Consequently, the ability of G-PFSO to explore Θ deteriorates as the learning rate $h_t \rightarrow 0$ become faster. In particular, when the initial particles $\{\theta_1^n\}_{n=1}^N$ are far from θ_\star and h_t decreases quickly over time, $\tilde{\pi}_t^N$ may fail to reach a small neighbourhood of θ_\star , and thus to approximate $\tilde{\pi}_t$ well, even for large values of t .

With this trade-off involved when choosing h_t , between statistical efficiency, i.e. enabling $\tilde{\pi}_t$ to concentrate quickly on θ_* , and computational efficiency, i.e. making $\tilde{\pi}_t^N$ close to $\tilde{\pi}_t$ for a reasonable value of N , our recommendation is to take $h_t = t^{-1/2}$ as the default choice for the learning rate. This recommendation is based on the fact that, for $h_t = t^{-\alpha}$, in all of our numerical experiments the estimator $\hat{\theta}_t^N$ appears to converge to θ_* at the optimal $t^{-1/2}$ rate when $\alpha = 0.5$, while this is not the case for a smaller value of this parameter; see the example of Section 4.2. It is worth noting that for $h_t = t^{-1/2}$ we have $\sum_{t=1}^{\infty} h_t^2 = \infty$, meaning that if in Algorithm 1 we write θ_t^n as $\theta_t^n = \hat{\theta}_{t-1}^n + \epsilon_t^n$, with $\epsilon_t^n \sim \tilde{M}_t(0, d\theta_t)$, then $\lim_{t \rightarrow \infty} \|\text{Var}(\sum_{s=1}^t \epsilon_s^n)\| = \infty$. In words, at any given time t the sum of all the future noise terms that will be used to propagate a given particle has an infinite variance, a property that may help G-PFSO to escape from a local mode of the objective function even after having processed a large number of observations; see Sections 4.3-4.4 for examples where this phenomenon happens.

2.5. Discussion

If in (1) we define $(\tilde{M}_t)_{t \geq 1}$ using Gaussian and Student's t -distributions only, our theoretical analysis of $\tilde{\pi}_t$ applies more generally for Markov kernels $(\tilde{M}_t)_{t \geq 1}$ whose tails verify certain conditions, as discussed in Appendix B.

In Appendix B we also establish that, for $h_t = t^{-\alpha}$, taking $\nu \leq 1/\alpha$ guarantees that with \mathbb{P} -probability one there is no value of t beyond which all the particles generated by Algorithm 1 remain stuck in a local mode of the mapping $\theta \mapsto \mathbb{E}\{\log f_{\theta}(Y_1)\}$. From a theoretical point of view this is an important result, because this property of G-PFSO is necessary to enable $\hat{\theta}_t^N$ to converge in \mathbb{P} -probability towards θ_* for a fixed $N \in \mathbb{N}$. It is however important to stress that the capacity of the particles to escape from a local mode of the objective function deteriorates over time, and thus that even for $\nu \leq 1/\alpha$ all the particles may be stuck in a local mode for a very long time. Finally, we remark that if we let $h_t = t^{-1/2}$, as recommended in the previous subsection, then the condition $\nu \leq 1/\alpha$ imposes to use, in (1), Student's t -distributions having an infinite variance.

2.6. Related approaches

The idea of using particle filter algorithms, or more generally sequential Monte Carlo methods, for optimizing a function $\varphi : \Theta \rightarrow \mathbb{R}$ that can be easily evaluated pointwise has been considered e.g. in Zhou et al. (2008), Liu et al. (2016) and in Giraud and Del Moral (2013, 2017). By contrast, it is only recently that optimizing a function defined through an expectation by means of a particle filter has been proposed, notably in Akyildiz et al. (2020) and in Liu (2020). When used to estimate θ_* , the approaches introduced in these two references amount to approximating in an online fashion the Bayesian posterior distributions $\{\pi_t\}_{t=1}^T$ and then to using the resulting estimate of π_T to learn the target parameter value.

More precisely, the parallel sequential Monte Carlo optimizer of Akyildiz et al. (2020) relies on the estimate $\pi_{\text{jit},t}^N = \sum_{n=1}^N W_t^n \delta_{\{\theta_t^n\}}$ of π_t , where $\{(W_t^n, \theta_t^n)\}_{n=1}^N$ can be computed using Algorithm 1 by replacing, for all $t \geq 1$, the Markov kernel \tilde{M}_t by a jittering kernel

M_N as introduced in Crisan and Miguez (2018). The distribution $\pi_{\text{Jit},t}^N$ is shown to converge to π_t as $N \rightarrow \infty$. However, our numerical experiments reveal an important limitation of using $\pi_{\text{Jit},t}^N$ to estimate θ_* , namely that for a fixed value of N there exists some finite time t_N after which processing Y_t does not allow $\pi_{\text{Jit},t}^N$ to provide any new information about the target parameter value; see Section 4.2 for an illustration. This issue arises because the Markov kernel M_N on which $\pi_{\text{Jit},t}^N$ relies is time homogenous, which prevents the support of this distribution from concentrating on a particular element of Θ as t increases.

The estimate $\pi_{K,t}^N$ of π_t used in the kernel smoothing particle filter based stochastic optimization (KS-PFSO) algorithm of Liu (2020) is computed as the jittering approximation $\pi_{\text{Jit},t}^N$, the only difference being that in KS-PFSO, for all $t \geq 1$, the Markov kernel \widetilde{M}_t appearing in Algorithm 1 is replaced by the kernel $M_{K,t}$ defined by

$$M_{K,t}(\theta', d\theta_t) = \mathcal{N}_d\left(\sqrt{1-\iota^2}\theta' + (1-\sqrt{1-\iota^2})\theta_{K,t-1}^N, \iota^2 V_{K,t-1}^N\right), \theta' \in \mathbb{R}^d \quad (4)$$

for some $\iota > 0$ and where $\theta_{K,t-1}^N$ and $V_{K,t-1}^N$ respectively denotes the expectation and the covariance matrix of θ under $\pi_{K,t-1}^N$. Remark that, under this kernel and conditionally to $\{(W_{t-1}^n, \theta_{t-1}^n)\}_{n=1}^N$, each particle θ_t^n generated by Algorithm 1 has the same expectation and covariance matrix than a random draw from $\pi_{K,t-1}^N$. This kernel was introduced by Liu and West (2001) in the context of online Bayesian state and parameter learning in state-space models, where it is used to rejuvenate the particle system without inflating the tails of the current approximation of the posterior distribution for the model parameter. Under standard regularity conditions π_t concentrates on θ_* at rate $t^{-1/2}$ (Kleijn and van der Vaart, 2012) and it is therefore expected that $\pi_{K,t}^N$ concentrates at this rate on a particular element of Θ . The numerical experiments presented in Section 4.3 show that this quick concentration of $\pi_{K,t}^N$ over time limits considerably the ability of KS-PFSO to escape from a local optimum of the objective function, making this algorithm a local rather than a global optimization method. It is worth mentioning that, unlike G-PFSO and the sequential Monte Carlo optimizer proposed by Akyildiz et al. (2020), KS-PFSO is not introduced by Liu (2020) as a global optimization method but as a stochastic optimization algorithm which bypasses the need to specify a learning rate. We also remark that $\pi_{K,t}^N$ is actually the approximation of π_t computed by the one-pass SMC sampler of Balakrishnan and Madigan (2006) and is not supported by any theoretical results.

It is interesting to note that the only difference between the algorithm used in G-PFSO to estimate $(\tilde{\pi}_t)_{t \geq 1}$ and those used by Akyildiz et al. (2020) and by Liu (2020) to approximate the posterior distributions $(\pi_t)_{t \geq 1}$ is the choice of the Markov kernels that are employed to rejuvenate the particle system. In G-PFSO the variance of θ_{t+1} under $\widetilde{M}(\theta_t, d\theta_{t+1})$ is of size $\mathcal{O}(h_t^2)$ and thus converges to zero as $t \rightarrow \infty$. Our numerical experiments suggest that this property of \widetilde{M}_t enables $\tilde{\pi}_t^N$ to concentrate on a single point of the parameter space as t increases, unlike the approximation $\pi_{\text{Jit},t}^N$ of π_t . In addition, and as mentioned in Section 2.4, by choosing $h_t \rightarrow 0$ such that $\tilde{\pi}_t$ concentrates on θ_* at a sub-optimal rate, that is at a rate slower than $t^{-1/2}$, we can improve the ability of

G-PFSO to escape from a local optimum of the objective function $\theta \mapsto \mathbb{E}\{\log f_\theta(Y_1)\}$. For this reason, G-PFSO is more suitable than KS-PFSO for multi-modal optimization tasks.

3. Theoretical analysis of the sequence $(\tilde{\pi}_t)_{t \geq 1}$

3.1. Assumptions on the statistical model

To simplify the notation we use below the shorthand $\mathbb{E}(g)$ for $\mathbb{E}\{g(Y_1)\}$ and, for every $\theta \in \mathbb{R}^d$, we let $\tilde{f}_\theta : \mathbb{R}^d \rightarrow \mathbb{R}$ be such that $\tilde{f}_\theta \equiv 0$ if $\theta \notin \Theta$ and such that $f_\theta = \tilde{f}_\theta$ if $\theta \in \Theta$.

The following two assumptions impose some regularity on the random mapping $\theta \mapsto \log f_\theta(Y_1)$ around θ_\star .

Assumption A1. *There exist a constant $\delta_\star > 0$ and a measurable function $m_\star : \mathcal{Y} \rightarrow \mathbb{R}$ such that $\mathbb{E}[m_\star^2] < \infty$ and such that \mathbb{P} -a.s. we have, for all $\theta_1, \theta_2 \in B_{\delta_\star}(\theta_\star) \cap \Theta$,*

$$|\log\{f_{\theta_1}(Y_1)/f_{\theta_2}(Y_1)\}| \leq m_\star(Y_1)\|\theta_1 - \theta_2\|.$$

Assumption A2. *There exist constants $\delta_\star > 0$ and $C_\star < \infty$ such that, for all $\theta \in B_{\delta_\star}(\theta_\star) \cap \Theta$,*

$$\mathbb{E}\{\log(f_{\theta_\star}/f_\theta)\} \leq C_\star\|\theta - \theta_\star\|^2.$$

The next assumption notably implies that θ_\star is identifiable.

Assumption A3. *For every compact set $W \in \mathcal{B}(\Theta)$ such that $\theta_\star \in W$ and every $\epsilon \in (0, \infty)$ there exists a sequence of measurable functions $(\psi'_t)_{t \geq 1}$, with $\psi'_t : \mathcal{Y}^t \rightarrow \{0, 1\}$, such that*

$$\mathbb{E}\{\psi'_t(Y_{1:t})\} \rightarrow 0, \quad \sup_{\theta \in V_\epsilon \cap W} \mathbb{E}\left[\{1 - \psi'_t(Y_{1:t})\} \prod_{s=1}^t (f_\theta/f_{\theta_\star})(Y_s)\right] \rightarrow 0.$$

Assumptions A1-A3 are standard, see e.g. Kleijn and van der Vaart (2012). It is easily checked that Assumption A2 holds when the mapping $\theta \mapsto \mathbb{E}(\log f_\theta)$ admits a second-order Taylor expansion in a neighbourhood of θ_\star . By Kleijn and van der Vaart (2012, Theorem 3.2), Assumption A3 holds for instance when, for every compact set $W \in \mathcal{B}(\Theta)$ containing θ_\star and every $\theta' \in W$, the mapping $W \ni \theta \mapsto \mathbb{E}(f_\theta f_{\theta'}^{-s} f_{\theta_\star}^{s-1})$ is continuous at θ' for every s in a left neighbourhood of 1, and $\mathbb{E}(f_{\theta'}/f_{\theta_\star}) < \infty$. Remark that if the model is well-specified then $\mathbb{E}(f_\theta/f_{\theta_\star}) = 1 < \infty$ for all $\theta \in \Theta$. If the model is miss-specified and the distribution of Y_1 admits a density f_\star w.r.t. $\eta(dy)$, the condition $\mathbb{E}(f_\theta/f_{\theta_\star}) < \infty$ requires the tails of f_{θ_\star} to be not too thin compared to those of f_\star . For instance, if $f_\theta(y)\eta(dy) = t_{1,\nu}(\mu, \sigma^2)$, with $\theta = (\mu, \sigma^2, \nu)$, and $\lim_{|y| \rightarrow \infty} |y|^{\nu_\star+1} f_\star(y) < \infty$ then $\mathbb{E}(f_\theta/f_{\theta_\star}) < \infty$ for all $\theta \in \mathbb{R} \times (0, \infty)^2$.

Remark 1. *Assumption A3 is stronger than needed for our main result to hold, this latter requiring only that Assumption A3 holds for the specific compact set $W = \tilde{A}_\star \cap \Theta$, where \tilde{A}_\star is as defined in Assumption A4 below.*

Assumption A4. For some set $A_\star \in \mathcal{B}(\mathbb{R}^d)$,

1. One of the following conditions holds:
 - a) $\mathbb{E}(\sup_{\theta \notin A_\star} \log \tilde{f}_\theta) < \mathbb{E}(\log f_{\theta_\star})$,
 - b) $\sup_{\theta \notin A_\star} \mathbb{E}(\tilde{f}_\theta / f_{\theta_\star}) < 1$,
 - c) $\log\{\sup_{\theta \notin A_\star} \mathbb{E}(\tilde{f}_\theta)\} < \mathbb{E}(\log f_{\theta_\star})$.
2. There exists a set $\tilde{A}_\star \in \mathcal{B}(\mathbb{R}^d)$, containing a neighbourhood of A_\star , such that $\tilde{A}_\star \cap \Theta$ is compact, the mapping $\theta \mapsto f_\theta(Y_1)$ is \mathbb{P} -a.s. continuous on $\tilde{A}_\star \cap \Theta$ and, for some $\tilde{\delta} > 0$,

$$\mathbb{E} \left[\sup_{(\theta, v) \in (\tilde{A}_\star \cap \Theta) \times B_{\tilde{\delta}}(0)} \{\log(\tilde{f}_{\theta+v} / f_\theta)\}^2 \right] < \infty.$$

Remark 2. If Θ is compact then Assumption A4 holds as soon as $\theta \mapsto f_\theta(Y_1)$ is \mathbb{P} -a.s. continuous on Θ and, for some $\tilde{\delta} > 0$, $\mathbb{E}[\sup_{(\theta, \theta') \in \Theta^2: \|\theta' - \theta\| < \tilde{\delta}} \{\log(f_{\theta'} / f_\theta)\}^2] < \infty$.

Assumption A5. One of the following three conditions hold:

1. $\mathbb{E}(\sup_{\theta \in \Theta} \log f_\theta) < \infty$,
2. $\sup_{\theta \in \Theta} \mathbb{E}(f_\theta / f_{\theta_\star}) < \infty$,
3. $\sup_{\theta \in \Theta} \mathbb{E}(f_\theta) < \infty$.

Remark 3. Assumption A5.2 always holds when the model is well-specified.

The last assumption, Assumption A6 below, is used to obtain a convergence result for $\tilde{\pi}_t$ that holds when Θ is unbounded.

Assumption A6. One of the following three conditions hold for some $k_\star \in \{1/2\} \cup \mathbb{N}$:

1. There exists a constant $C_1 \in (0, \infty)$ such that

$$\sup_{C \geq C_1} \mathbb{E} \left[\left| \sup_{\theta \in V_C} \log(\tilde{f}_\theta / f_{\theta_\star}) - \mathbb{E} \left\{ \sup_{\theta \in V_C} \log(\tilde{f}_\theta / f_{\theta_\star}) \right\} \right|^{2k_\star} \right] < \infty$$

and $\limsup_{C \rightarrow \infty} \zeta(C)(\log C)^{-1} < 0$ with $\zeta(C) = \mathbb{E}\{\sup_{\theta \in V_C} \log(\tilde{f}_\theta / f_{\theta_\star})\}$,

2. $\limsup_{C \rightarrow \infty} \zeta(C)(\log C)^{-1} < 0$ with $\zeta(C) = \log \{\sup_{\theta \in V_C} \mathbb{E}(\tilde{f}_\theta / f_{\theta_\star})\}$,
3. $\mathbb{E}(|\log f_{\theta_\star}|^{2k_\star}) < \infty$ and $\limsup_{C \rightarrow \infty} \zeta(C)(\log C)^{-1} < 0$ with $\zeta(C) = \log \{\sup_{\theta \in V_C} \mathbb{E}(\tilde{f}_\theta)\}$.

Remark 4. If Θ is a bounded set then $V_C \cap \Theta = \{\theta \in \Theta : \|\theta - \theta_\star\| \geq C\}$ is empty for sufficiently large C and Assumption A6.2 is satisfied. Since Assumption A6.2 does not depend on k_\star it follows that if Θ is bounded then Assumption A6 holds for every $k_\star \in \{1/2\} \cup \mathbb{N}$.

Remark 5. Assumption A6 implies the existence of a set $A_\star \in \mathcal{B}(\mathbb{R}^d)$ such that the first part of Assumption A4 holds.

Assumptions A4-A6 are non-standard but are reasonable, as illustrated with the next result.

Proposition 1. *Let $Y = \mathbb{R}$ and $\eta(dy)$ be the Lebesgue measure on \mathbb{R} .*

1. *Let $\Theta = \mathbb{R} \times [\underline{\sigma}^2, \infty)$ for some $\underline{\sigma}^2 \in (0, \infty)$ and, for every $\theta = (\mu, \sigma^2) \in \Theta$, let $f_\theta(y)\eta(dy) = \mathcal{N}_1(\mu, \sigma^2)$. Then, Assumptions A4-A5 hold if $\mathbb{E}(Y_1^4) < \infty$. If in addition we have $\mathbb{E}(e^{c|Y_1|}) < \infty$ for some $c > 0$ then Assumption A6 holds for all $k_\star \in \mathbb{N}$.*
2. *Let $\Theta = \mathbb{R} \times [\underline{c}, \infty)^2$ for some $\underline{c} \in (0, \infty)$ and, for every $\theta = (\mu, \sigma^2, \nu) \in \Theta$, let $f_\theta(y)\eta(dy) = t_{1,\nu}(\mu, \sigma^2)$. Then, Assumptions A4-A5 hold if $\mathbb{E}\{\log(1 + cY_1^2)\} < \infty$ for all $c \in (0, \infty)$. If in addition we have $\mathbb{E}(|Y_1|^c) < \infty$ for some $c > 0$ then Assumption A6 holds for all $k_\star \in \mathbb{N}$.*

3.2. Main result

The following theorem provides conditions on the sequences $(h_t)_{t \geq 1}$ and $(t_p)_{p \geq 0}$ which guarantee that, under Assumptions A1-A6 on $\{f_\theta, \theta \in \Theta\}$, we have $\tilde{\pi}_t \Rightarrow \delta_{\{\theta_\star\}}$ in \mathbb{P} -probability.

Theorem 1. *Assume Assumptions A1-A6, let $k_\star \in \{1/2\} \cup \mathbb{N}$ and $\zeta(C)$, $C \in (0, \infty)$, be as in Assumption A6, and let $(h_t)_{t \geq 1}$ and $(t_p)_{p \geq 0}$ be such that*

1. $\log(h_{t_p})/(t_p - t_{p-1}) \rightarrow 0$ and $(t_p - t_{p-1}) \sum_{s=t_{p-1}+1}^{t_p-1} h_s^2 \rightarrow 0$,
2. $h_{t_p} < h_{t_{p-1}}$ for all $p \geq 0$ and $\liminf_{p \rightarrow \infty} (h_{t_p} t_p^\alpha + t_p^{-\alpha}/h_{t_p}) > 0$ for some $\alpha \in (0, \infty)$,
3. $\limsup_{p \rightarrow \infty} (t_{p+1} - t_p)/(t_p - t_{p-1}) < \infty$,
4. $|\zeta(h_{t_p}^{-\beta_\star})|^{-2k_\star} \sum_{i=1}^p (t_i - t_{i-1})^{-k_\star} \mathbf{1}_{\mathbb{N}(k_\star)} \rightarrow 0$ for some constant $\beta_\star \in (0, \infty)$.

Then, $\tilde{\pi}_t \Rightarrow \delta_{\{\theta_\star\}}$ in \mathbb{P} -probability.

Remark 6. *With a similar reasoning as in Remark 4, if Θ is a bounded set then for sufficiently large C the set $V_C \cap \Theta$ is empty and we have $|\zeta(C)| = \infty$. Hence, Condition 4 of Theorem 1 always holds when the set Θ is bounded.*

3.3. Application of Theorem 1

The following proposition can be used to explicitly define a sequence $(t_p)_{p \geq 0}$ that verifies the conditions of Theorem 1 when $h_t = t^{-\alpha}$ for some $\alpha > 0$.

Proposition 2. *For some constants $C \in [1, \infty)$, $\alpha \in (0, \infty)$, $\varrho_\alpha \in (0, \alpha \wedge 1)$, $c \in (0, 1)$ and $t_0 \in \mathbb{N}$ let $h_t = t^{-\alpha}$ for all $t \geq 1$ and let $(t_p)_{p \geq 0}$ be such that, for all $p \geq 1$,*

$$t_p = t_{p-1} + \lceil C_{p-1} \log(t_{p-1}) \vee C \rceil, \quad p \geq 1 \quad (5)$$

for some $C_{p-1} \in [ct_{p-1}^{\varrho_\alpha}, t_{p-1}^{\varrho_\alpha}/c]$. Then, the sequences $(h_t)_{t \geq 1}$ and $(t_p)_{p \geq 0}$ verify Conditions 1-3 of Theorem 1. Moreover, these two sequences also verify Condition 4 of Theorem 1 if Assumption A6 holds for a $k_\star > (1 + \varrho_\alpha)/\varrho_\alpha$.

The second part of the proposition notably implies that if Assumption A6 holds for all $k_\star \in \mathbb{N}$, as is the case in the examples of Propositions 1 or when Θ is a bounded set, see Remark 4, then for every $\alpha \in (0, \infty)$ the learning rate $h_t = t^{-\alpha}$ and the sequence $(t_p)_{p \geq 0}$ defined in (5) verify Conditions 1-4 of Theorem 1.

The conclusions of Theorem 1 and of Proposition 2 are summarized in the following corollary.

Corollary 1. *Let $\alpha \in (0, \infty)$, $h_t = t^{-\alpha}$ for all $t \geq 1$ and $(t_p)_{p \geq 0}$ be as in Proposition 2 for some $\varrho_\alpha \in (0, \alpha \wedge 1)$. Then, under Assumptions A1–A5, and provided that either Θ is a bounded set or that Assumption A6 holds for some $k_\star > (1 + \varrho_\alpha)/\varrho_\alpha$, we have $\tilde{\pi}_t \Rightarrow \delta_{\{\theta_\star\}}$ in \mathbb{P} -probability.*

4. Numerical experiments

4.1. Implementation of G-PFSO and reference measure $\eta(dy)$

Throughout this section we let $h_t = t^{-\alpha}$ for some $\alpha \in \{0.3, 0.5, 0.8\}$ and, following the result of Corollary 1, we let $(t_p)_{p \geq 0}$ be defined by

$$t_p = t_{p-1} + \lceil A t_{p-1}^\varrho \log(t_{p-1}) \vee B \rceil, \quad p \geq 1 \quad (6)$$

with $A = B = 1$, $t_0 = 5$ and $\varrho = 0.1$.

Since our convergence result for $\tilde{\pi}_t$ imposes a strong constraint on how often Student's t -distributions can be present in the sequence $(\tilde{M}_t)_{t \geq 1}$ it seems judicious to assess the ability of G-PFSO to reach a small neighbourhood of θ_\star without relying on these fat tail distributions. For this reason, unless otherwise mentioned, we let $\nu = 50$ so that each $t_{d,\nu}(\theta', h_t^2 \Sigma)$ distribution appearing in (1) is very close to the $\mathcal{N}_d(\theta', h_t^2 \Sigma)$ distribution in the sense of the Kullback-Leibler divergence (Villa and Rubio, 2018). However, as illustrated in Section 4.3, increasing the tails and the frequency of the Student's t -distributions can improve the performance of G-PFSO.

All the results presented in this section are obtained with $\Sigma = c_\Sigma I_d$ for some constant $c_\Sigma \in (0, \infty)$ and by letting, in Algorithm 1, $\mathcal{R}(\cdot)$ be the SSP resampling algorithm (Gerber et al., 2019) and, somewhat arbitrarily, $c_{\text{ESS}} = 0.7$. Finally, the examples considered below are all such that $\eta(dy)$ is the Lebesgue measure on \mathbb{R}^k for some $k \in \mathbb{N}$.

4.2. A censored quantile regression model

The main objective of this example is to study the convergence rate, as $t \rightarrow \infty$, of the G-PFSO estimator $\tilde{\theta}_t^N$ in a non-trivial statistical model. To this aim we consider a censored quantile regression model with only $d = 5$ parameters that will be learnt by processing sequentially a set of $T = 10^7$ i.i.d. observations. The model and the simulation set-up are precisely described in Appendix C, and we let $\theta_\star^{(\tau)}$ be the target parameter value when the censored quantile regression model is used to estimate the conditional quantile of order $\tau \in (0, 1)$ of the response variable. Below, results are presented for $\tau = 0.5$ and for $\tau = 0.99$.

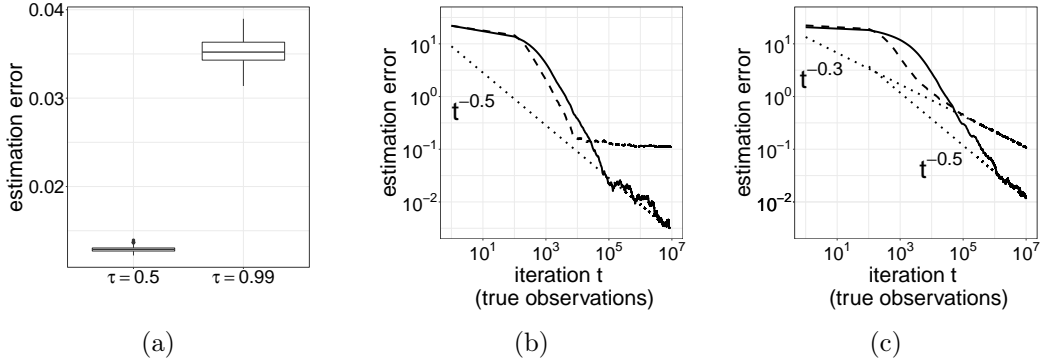


Figure 1.: Example of Section 4.2. Plot (a) summarizes the values of $\|\theta_{\text{ada},T} - \theta_{\star}^{(\tau)}\|_{\infty}$ obtained with Adagrad for 10^3 starting values randomly sampled from the $\mathcal{N}_d(\theta_{\star}^{(0.5)} + 10, 2I_d)$ distribution. In Plot (b) the solid line is for $\bar{\theta}_t^N$ and the dashed line for $\bar{\theta}_{\text{Jit},t}^N$. In Plot (c) the solid line is for $\alpha = 0.5$ and the dashed line for $\alpha = 0.3$. In Plots (b)-(c) the dotted lines show the function $f(t) = ct^{-\gamma}$ for some $c > 0$ and with γ as given in the plots.

In Figure 1a we summarize the estimation error obtained when $\theta_{\star}^{(\tau)}$ is estimated using Adagrad, an adaptive stochastic gradient algorithm introduced by Duchi et al. (2011), randomly initialized far from the target parameter value. The results presented in this figure suggest that for $\tau \in \{0.5, 0.99\}$ the corresponding objective function is uni-modal, at least in a large neighbourhood of $\theta_{\star}^{(\tau)}$. Therefore, we can study the convergence behaviour of $\bar{\theta}_t^N$ without the concern of being trapped in a local optimum.

For this example G-PFSO is implemented with $N = 1000$, $\tilde{\pi}_0(d\theta) = \mathcal{N}_d(\theta_{\star}^{(0.5)} + 10, 2I_d)$ and $c_{\Sigma} = 1$. In Figures 1b-1c we report the evolution as t increases of the average value of $\|\bar{\theta}_t^N - \theta_{\star}^{(\tau)}\|$ obtained in 20 runs of Algorithm 1, where $(\tau, \alpha) = (0.5, 0.5)$ in Figure 1b and where $(\tau, \alpha) \in \{(0.99, 0.3), (0.99, 0.5)\}$ in Figure 1c. The results in these two figures suggest that for $h_t = t^{-1/2}$ the estimator $\bar{\theta}_t^N$ converges at the optimal $t^{-1/2}$ rate to the target parameter value both when $\tau = 0.5$ and when $\tau = 0.99$. However, for the very slow learning rate $h_t = t^{-0.3}$, and for $\tau = 0.99$, the G-PFSO estimator appears to converge towards $\theta_{\star}^{(\tau)}$ at the slow $t^{-0.3}$ rate. Consequently, these results indicate that if $h_t \rightarrow 0$ too slowly then the convergence rate of $\bar{\theta}_t^N$ may be sub-optimal. We finally use the jittering estimate $\pi_{\text{Jit},t}^N$ of π_t to estimate $\theta_{\star}^{(\tau)}$ when $\tau = 0.5$, using the jittering kernel $M_N(\theta', d\theta) = (1 - N^{-1/2})\delta_{\{\theta'\}} + N^{-1/2}\mathcal{N}_d(\theta', I_d)$. Since this kernel is homogenous it is clear that for a fixed number of particles, N , the estimator $\theta_{\text{Jit},t}^N = \int_{\Theta} \theta \pi_{\text{Jit},t}^N(d\theta)$ cannot converge to $\theta_{\star}^{(\tau)}$ as $t \rightarrow \infty$. To study if Polyak–Ruppert averaging can resolve this issue we let $\bar{\theta}_{\text{Jit},t}^N = t^{-1} \sum_{s=1}^t \theta_{\text{Jit},s}^N$ and report in Figure 1b the evolution as t increases of $\|\bar{\theta}_{\text{Jit},t}^N - \theta_{\star}^{(\tau)}\|$, averaged over 20 runs of the algorithm. We remark that after $T_N \approx 10^5$ observations the average estimation error $\|\bar{\theta}_{\text{Jit},t}^N - \theta_{\star}^{(\tau)}\|$ stabilizes around 0.10. As

discussed above, for this example the objective function has apparently no local maxima, and therefore these simulation results suggest that Polyak–Ruppert averaging does not prevent the inference based on $\pi_{\text{Jit},t}^N$ to stop improving after some finite time.

4.3. A toy multi-modal example

In this second example we consider a sequence $(Y_t = (Z_t, X_t))_{t \geq 1}$ of i.i.d. random variables taking value in $\mathbb{R} \times \mathbb{R}^d$, with $d = 20$. Then, inspired by an example in Hunter and Lange (2000), we let $\Theta \subset \mathbb{R}^d$ and $\mu(\theta, x) = \sum_{i=1}^d (e^{-x_i \theta_i^2} + x_i \theta_{d-i+1})$ for all $(\theta, x) \in \Theta \times \mathbb{R}^d$, and our goal is to estimate $\theta_\star = \operatorname{argmin}_{\theta \in \Theta} \mathbb{E}[|Z_1 - \mu(\theta, X_1)|]$. To cast this estimation problem into the set-up of this paper, for every $(\theta, x) \in \Theta \times \mathbb{R}^d$ we let $f_\theta(\cdot | x)$ be the density of the Laplace distribution with scale parameter $b > 0$ and location parameter $\mu(\theta, x)$, so that $\theta_\star = \operatorname{argmax}_{\theta \in \Theta} \mathbb{E}[\log f_\theta(Z_1 | X_1)]$. We let $b = 0.5$ and simulate $T = 10^6$ i.i.d. observations using $Z_1 | X_1 \sim \mathcal{N}_1(\mu(\theta_\star, X_1), 4)$, $X_1 \sim \operatorname{Unif}([-1, 1]^d)$ and $\theta_\star = (-1, \dots, -1)$. As in the previous subsection the observations are processed sequentially while, to avoid numerical problems, we let Θ be the open ball of size 20 around θ_\star w.r.t. the maximum norm. For this example we let $N = 2000$, $\tilde{\pi}_0(d\theta) = \operatorname{Unif}(\Theta)$ and $\alpha = 0.5$.

We first use the KS-PFSO estimator $\theta_{K,t}^N = \int_{\Theta} \theta \pi_{K,t}^N(d\theta)$ to estimate θ_\star where, using equation (10) in Balakrishnan and Madigan (2006), we let $\iota \approx 0.68$ in (4). In Figure 2a we summarize the values of $\|\theta_{K,T'}^N - \theta_\star\|_\infty$ obtained in 100 runs of the algorithm, with $T' = 10^5$. We observe that the estimation error obtained with KS-PFSO is always larger than 1.66, suggesting the existence of some local optima to which this algorithm converges. The existence of local optima can also be observed in Figure 2b which shows the evolution as t increases of the 14-th component of $\tilde{\theta}_t^N$ obtained in a single run of G-PFSO with $c_\Sigma = 10$. Notice that the results presented in Figure 2a for KS-PFSO illustrate the fact that this algorithm is a local optimization method, for reasons explained in Section 2.6.

The second boxplot in Figure 2a summarizes the values of $\|\bar{\theta}_{T'}^N - \theta_\star\|_\infty$ obtained in 100 runs of G-PFSO with $c_\Sigma = 10$ and where, as for KS-PFSO, $T' = 10^5$. For each run of the algorithm the estimation error is smaller than 0.16, showing that $\bar{\theta}_{T'}^N$ successfully finds the global optimum of the function $\theta \mapsto \mathbb{E}[\log f_\theta(Z_1 | X_1)]$ with very high probability. To assess the sensitivity of $\bar{\theta}_{T'}^N$ to the parameter c_Σ in the 3rd and 4th boxplot of Figure 2a we repeat the experiment with $c_\Sigma = 3$ and with $c_\Sigma = 1$. We observe that decreasing c_Σ from 10 to 3 improves the performance of the estimator $\bar{\theta}_{T'}^N$, for the following reason. On the one hand, decreasing the value of c_Σ reduces for all $t \geq 1$ the variance of the distributions $\{\tilde{M}_t(\hat{\theta}_{t-1}^n, d\theta)\}_{n=1}^N$ used in Algorithm 1 to generate $\{\theta_t^n\}_{n=1}^N$, which enables $\tilde{\pi}_t^N$ to be more concentrated around θ_\star and consequently to reduce the estimation error. On the other hand, in this example for $c_\Sigma = 3$ the variance of these Markov kernels is large enough to ensure that a small neighbourhood of θ_\star is quickly reached by G-PFSO. We however remark in Figure 2a that this is no longer the case when $c_\Sigma = 1$ since, for this value of c_Σ , the estimation error of $\bar{\theta}_{T'}^N$ is frequently large and similar to that obtained with KS-PFSO. For a given choice of c_Σ the exploration of Θ can be improved

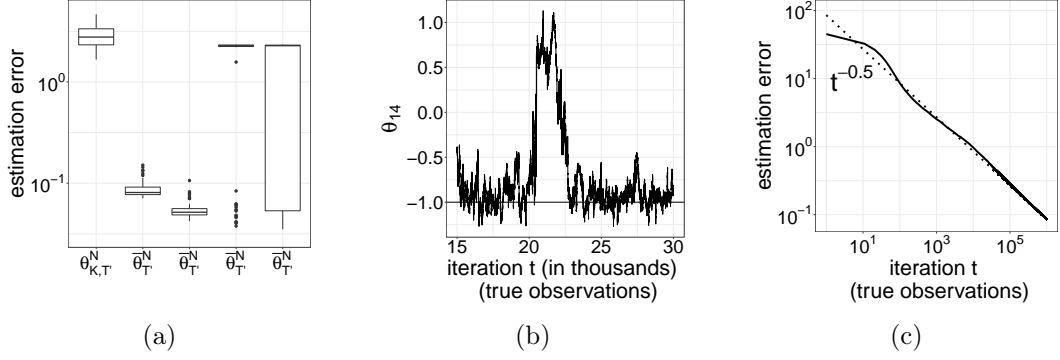


Figure 2.: Example of Section 4.3. In Plot (a), for the estimator $\bar{\theta}_{T'}^N$ the first boxplot is for $(\nu, c_\Sigma) = (50, 10)$, the second boxplot is for $(\nu, c_\Sigma) = (50, 3)$, the third boxplot for $(\nu, c_\Sigma) = (50, 1)$ and the last boxplot is for $(\nu, c_\Sigma) = (1.5, 1)$. In Plot (b) the horizontal line represents the true parameter value and in Plot (c) the dotted line is as in Figure 1.

by reducing ν , the number of degrees of freedom of the Student's t -distributions used in (1). This point is illustrated with the last boxplot in Figure 2a, which shows that for $c_\Sigma = 1$ decreasing ν from 50 to 1.5 increases the probability of $\bar{\theta}_{T'}^N$ having a small estimation error. Notably, in this figure reducing ν from 50 to 1.5 doubles, from 18 to 36, the number of runs of G-PFSO for which $\|\bar{\theta}_{T'}^N - \theta_\star\|_\infty < 0.2$.

Finally, in Figure 2c we show the evolution of $\|\bar{\theta}_t^N - \theta_\star\|$ while processing the available $T = 10^6$ data points, averaged over 10 runs of Algorithm 1 with $c_\Sigma = 10$. The results reported in this plot suggest that, as in the previous example, for $h_t = t^{-1/2}$ the estimator $\bar{\theta}_t^N$ converges to θ_\star at the optimal $t^{-1/2}$ rate.

4.4. A smooth adaptive Gaussian mixture model

Let $(Y_t = (Z_t, X_t))_{t \geq 1}$ be a sequence of random variables taking values in $\mathbb{R} \times \mathbb{R}^{d_x}$ for some $d_x \geq 1$. Then, the smooth adaptive Gaussian mixture model (Villani et al., 2009) with $K \geq 2$ components assumes that, for every $t \geq 1$ and with $d = d_x(3K - 1)$, the conditional distribution of Z_t given $X_t = x$ belongs to the set $\{f_\theta(\cdot | x), \theta \in \Theta \subseteq \mathbb{R}^d\}$ where

$$f_\theta(z | x) = \sum_{k=1}^K w_k(x, \beta^w) \varphi_1 \left\{ z; x^\top \beta_{(k)}^\mu, \exp(-x^\top \beta_{(k)}^\sigma) \right\}, \quad (\theta, z) \in \Theta \times \mathbb{R} \quad (7)$$

with $\varphi_1(\cdot; \mu, \sigma)$ the probability density function of the $\mathcal{N}_1(\mu, \sigma^2)$ distribution, $w_K(x, \beta^w) = 1 - \sum_{k=1}^{K-1} w_k(x, \beta^w)$ and

$$w_k(x, \beta^w) = \frac{\exp(-x^\top \beta_{(k)}^w)}{1 + \sum_{k'=1}^{K-1} \exp(-x^\top \beta_{(k')}^w)}, \quad k = 1, \dots, K-1. \quad (8)$$

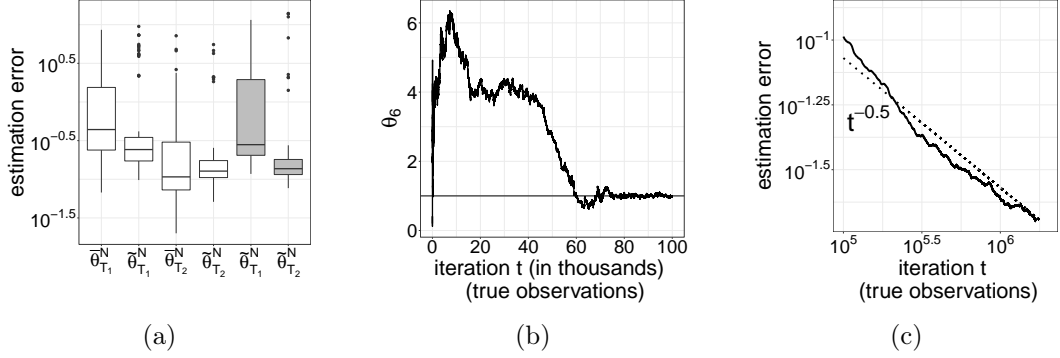


Figure 3.: Example of Section 4.4. In Plot (a), $T_1 = 20\,000$ and $T_2 = 10^5$, the white boxplots are for $\nu = 50$ and the grey boxplots for $\nu = 2$. In Plot (b) the horizontal line represents the true parameter value and in Plot (c) the dotted line is as in Figures 1-2.

In (7)-(8) we have $\beta_{(k)}^i \in \mathbb{R}^{d_x}$ for all k and all $i \in \{\mu, \sigma, w\}$, while $\beta^w = (\beta_{(1)}^w, \dots, \beta_{(K-1)}^w)$ and, letting $\beta^i = (\beta_{(1)}^i, \dots, \beta_{(K)}^i)$ for every $i \in \{\mu, \sigma\}$, $\theta = (\beta^w, \beta^\mu, \beta^\sigma)$.

For this example we let $K = 2$ and $d_x = 4$, resulting in a model with $d = 20$ parameters that will be learnt by processing sequentially a set of $T = 2 \times 10^6$ i.i.d. observations. The simulation set-up is described in Appendix C. Without loss of generality we let $\Theta = \{\theta \in \mathbb{R}^d : \theta_1 \geq 0\}$ and, since our chosen value for θ_* is such that $\theta_{*,1} \neq 0$, it follows that θ_* is the unique global maximizer of the function $\Theta \ni \theta \mapsto \mathbb{E}[\log f_\theta(Z_1 | X_1)]$. Finally, we let $N = 5\,000$, $c_\Sigma = 1$, $\alpha = 0.5$ and $\tilde{\pi}_0(d\theta) = \text{Exp}(1) \otimes \mathcal{N}_{d-1}(0, I_{d-1})$.

We report in Figure 3a a summary of the values obtained for $\|\tilde{\theta}_{T'}^N - \theta_*\|_\infty$ and for $\|\tilde{\theta}_{10^5}^N - \theta_*\|_\infty$ in 100 runs of G-PFSO, with $T' \in \{20\,000, 10^5\}$. Although the initial distribution $\tilde{\pi}_0$ has most of its mass on a ball of size 1 around θ_* , we remark that the estimation error can in some cases be large for the two G-PFSO estimators. We indeed observe in our experiments that, for this example, the value of $\|\tilde{\theta}_t^N - \theta_*\|_\infty$ often increases sharply in the first few iterations of Algorithm 1 and may remain large for a very long time period. This phenomenon is illustrated in Figure 3b where, for one of the 100 runs of G-PFSO, we present the evolution as t increases of $\tilde{\theta}_{t,6}^N$, the 6-th component of $\tilde{\theta}_t^N$. A close look at the results presented in Figure 3a reveals that the estimation error $\|\tilde{\theta}_{10^5}^N - \theta_*\|_\infty$ is smaller than 0.26 for 91 runs of Algorithm 1 and larger than 1.8 for the 9 remaining runs, suggesting the existence of some local optima in which G-PFSO remains trapped after 10^5 observations. The existence of some local optima seems confirmed by the evolution of $\tilde{\theta}_{t,6}^N$ reported in Figure 3b, a figure which also illustrates the ability of G-PFSO to escape from a local optimum after a large number of iterations. This ability of G-PFSO to escape from a local optimum explains why in the 18 runs of Algorithm 1 for which we have $\|\tilde{\theta}_{T'}^N - \theta_*\|_\infty > 2$ after having processed $T' = 20\,000$ observations—the estimation error being smaller than 0.42 in the 82 other runs of G-PFSO—increasing the number of data points to $T' = 10^5$ allows to have $\|\tilde{\theta}_{T'}^N - \theta_*\|_\infty < 0.23$ for 9 of them. Next,

and importantly, the results in Figure 3a show that even for a large sample size t the estimator $\tilde{\theta}_t^N$ can outperform $\bar{\theta}_t^N$ when many iterations are needed by G-PFSO to reach a small neighbourhood of θ_* , as it is often the case in this example. Indeed, recalling that $\bar{\theta}_t^N = t^{-1}\{(t-1)\bar{\theta}_{t-1}^N + \hat{\theta}_t^N\}$, it follows that if $\|\hat{\theta}_t^N - \theta_*\|_\infty$ starts decreasing only after some time $\underline{t} \gg 1$ then, because $\tilde{\theta}_t^N$ is multiplied by a factor $1/t$ in the definition of $\bar{\theta}_t^N$, the number of iterations needed for $\|\bar{\theta}_t^N - \theta_*\|_\infty$ to reach a value close to zero is much larger than for $\|\hat{\theta}_t^N - \theta_*\|_\infty$. In practice this problem can be avoided by defining $\bar{\theta}_t^N$ as $\bar{\theta}_t^N = (t - \underline{t})^{-1} \sum_{s=\underline{t}+1}^t \tilde{\theta}_s^N$ for $t > \underline{t}$, where $\underline{t} \in \mathbb{N}$ is the time needed for the sequence $(\tilde{\theta}_t^N)_{t \geq 1}$ to stabilize around a given point in the parameter space. In the last two boxplots of Figure 3a we repeat the same experiment with $\nu = 2$. Unlike what we observed in Section 4.3, for this example it is not desirable to use, in (1), Student's t -distributions with an infinite variance. In particular, the results in Figure 3a suggest that reducing ν from 50 to 2 decreases the probability of reaching a small neighbourhood of θ_* with $T' = 20\,000$ iterations and increases the probability of having a large estimation error with $T' \in \{20\,000, 10^5\}$ iterations. By improving the ability of the particle system to move across the parameter space, reducing ν enables G-PFSO to find the global mode of the objective function earlier but also increases the probability that the estimator $\hat{\theta}_t^N$ is pushed far away from θ_* at time $t \in (t_p)_{p \geq 0}$. Consequently, depending on the estimation problem at hand, decreasing ν may improve—as in Section 4.3—or deteriorate—as in this example—the performance of G-PFSO.

Finally, we study the convergence behaviour of $\|\bar{\theta}_t^N - \theta_*\|$ as t increases. The results in Figure 3c, which show the time evolution of $\|\bar{\theta}_t^N - \theta_*\|$ averaged over 5 runs of Algorithm 1, suggest again that $\bar{\theta}_t^N$ converges to θ_* at the optimal $t^{-1/2}$ rate when $h_t = t^{-1/2}$.

4.5. The bivariate g-and-k distribution

In this last example we use a real dataset to illustrate the use of G-PFSO for approximating the maximum likelihood estimator. To this aim we consider the problem of parameter inference in the bivariate extension of the g-and-k distribution proposed by Drovandi and Pettitt (2011). For $\theta = (a_1, a_2, b_1, b_2, g_1, g_2, k_1, k_2, \rho) \in \Theta \subset \mathbb{R}^9$ we let f_θ be the probability density function of the bivariate g-and-k distribution with location parameters (a_1, a_2) , scale parameters (b_1, b_2) , skewness parameters (g_1, g_2) , kurtosis parameters (k_1, k_2) and correlation parameter ρ . Following the results in Prangle (2017), to ensure that f_θ is a well-defined probability density function we let $\Theta = \mathbb{R}^2 \times (0, \infty)^2 \times \Theta_{\text{gk}} \times (-1, 1)$ where

$$\Theta_{\text{gk}} = \{(g_1, g_2, k_1, k_2) \in \mathbb{R}^4 : |g_i| < 5.5, k_i > -0.045 - 0.01g_i^2, i = 1, 2\}.$$

As in Drovandi and Pettitt (2011), we let $\{\tilde{y}_{i,1}\}_{i=1}^n$ and $\{\tilde{y}_{i,2}\}_{i=1}^n$ be respectively the exchange rate daily log returns from GBP to AUD and from GBP to EURO, multiplied by 100. We consider data from 4 January 2000 to 1 January 2021 inclusive which, after having removed the dates for which only one of the two exchange rates is available, results in a sample of size $n = 7\,254$. Letting $\tilde{y}_i = (\tilde{y}_{i,1}, \tilde{y}_{i,2})$ for all i , the corresponding log-likelihood function is defined by $l_n(\theta) = \sum_{i=1}^n \log f_\theta(\tilde{y}_i)$ and, following the discussion in Section

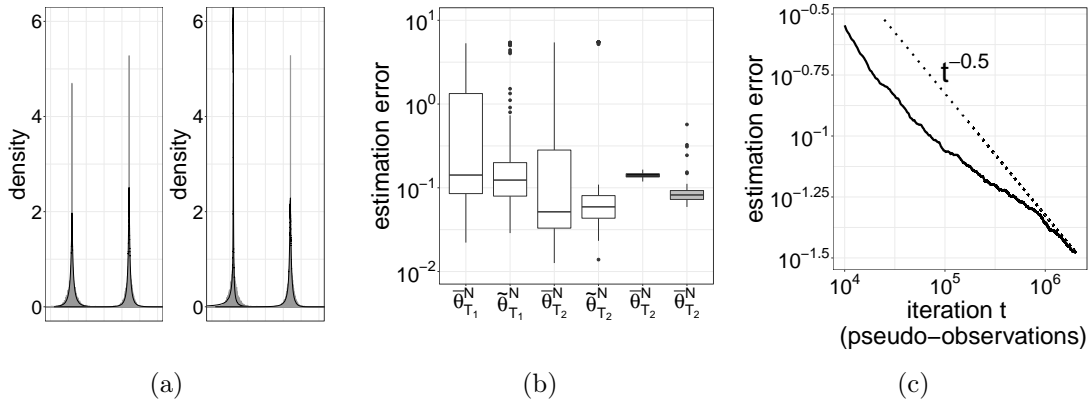


Figure 4.: Example of Section 4.5. Plot (a) compares the marginal distributions of $f_\theta(y)\eta(dy)$ and the empirical marginal distributions for $\theta = \hat{\theta}_{\text{mle},n}$ (left) and for $\theta = \theta^{(-5.5,0)}$ (right). In Plot (b) $T_1 = 10\,000$, $T_2 = 50\,000$ while $(\alpha, c_\Sigma) = (0.8, 10)$ (white boxplots), $(\alpha, c_\Sigma) = (0.5, 10)$ (fifth boxplot) and $(\alpha, c_\Sigma) = (0.5, 1)$ (sixth boxplot). In Plot (c) the dotted line is as in Figures 1-3.

1.4, below we compute the maximum likelihood estimator $\hat{\theta}_{\text{mle},n} = \operatorname{argmax}_{\theta \in \Theta} l_n(\theta)$ by running Algorithm 1 on a set of T pseudo-observations $\{y_t\}_{t=1}^T$ sampled i.i.d. from the empirical distribution of the observations $\{\tilde{y}_i\}_{i=1}^n$. For each run of Algorithm 1 a new set of pseudo-observations $\{y_t\}_{t=1}^T$ is sampled and, for this example, we let $N = 500$ and $\tilde{\pi}_0 \in \mathcal{P}(\Theta)$ be such that if $\theta \sim \tilde{\pi}_0(d\theta)$ then, for $i = 1, 2$, $a_i \sim \mathcal{N}_1(0, 1)$, $b_i \sim \operatorname{Exp}(1)$, $g_i \sim \operatorname{Unif}(-5.5, 5.5)$ and $k_i + 0.045 + 0.01g_i^2 \sim \operatorname{Exp}(1)$, where all the random variables are independent of each other, with the exception of k_1 that depends on g_1 and of k_2 that depends on g_2 .

We start by considering 100 runs of Algorithm 1 with $T = 50\,000$, $c_\Sigma = 10$ and with a fast learning rate $h_t = t^{-0.8}$, so that $\alpha = 0.8$. In a first step we optimize $l_n(\theta)$ using a quasi-Newton algorithm initialized at the G-PFSO estimate of $\hat{\theta}_{\text{mle},n}$ that gives the largest log-likelihood value, and the resulting parameter value is treated in what follows as the true value of $\hat{\theta}_{\text{mle},n}$. From Figure 4a we observe that the g-and-k distribution $f_{\hat{\theta}_{\text{mle},n}}(y)\eta(dy)$ fits the data well, at least as far as the marginal distributions are concerned.

In Figure 4b we summarize the 100 values obtained for $\|\tilde{\theta}_{T'}^N - \hat{\theta}_{\text{mle},n}\|_\infty$ and for $\|\tilde{\theta}_{T'}^N - \hat{\theta}_{\text{mle},n}\|_\infty$, with $T' \in \{10^4, T\}$. We remark that after only $T' = 10^4$ iterations the median estimation error is only approximately 0.13 for the two estimators, and that increasing the number of iterations to $T = 50\,000$ approximately divides this value by 2.75 for $\tilde{\theta}_t^N$ and by 2.1 for $\tilde{\theta}_t^N$. In this figure we also remark that for 15 runs of G-PFSO the estimation error of $\tilde{\theta}_T^N$ is between 5.10 and 5.51. A close look at the values obtained for $\tilde{\theta}_T^N$ reveals that these large estimation errors arise because G-PFSO occasionally converges to an

element of the set

$$\Theta_{\text{loc},\star} = \{\theta^{(v_1, v_2)}, (v_1, v_2) \in \{5.5, -5.5, 0\}^2 \setminus \{(0, 0)\}\},$$

where for $(v_1, v_2) \in \mathbb{R}^2$ the notation $\theta^{(v_1, v_2)}$ is used for a $\theta = (a_1, a_2, b_1, b_2, g_1, g_2, k_1, k_2, \rho) \in \Theta$ such that $(g_1, g_2) = (v_1, v_2)$. By contrast, under $\hat{\theta}_{\text{mle},n}$ both g_1 and g_2 are close to zero. As illustrated in the right panel of Figure 4a, when $v_j \approx \pm 5.5$ the probability density function $f_{\theta^{(v_1, v_2)}}$ tries to capture the large spike at zero that is present in the empirical distribution of the observations $\{\tilde{y}_{i,j}\}_{i=1}^n$, and thus $\Theta_{\text{loc},\star}$ is a natural set of local maxima of the log-likelihood function. Due to the large distance $d(\hat{\theta}_{\text{mle},n}, \Theta_{\text{loc},\star}) = \min_{\theta \in \Theta_{\text{loc},\star}} \|\theta - \hat{\theta}_{\text{mle},n}\|_\infty \approx 5.5$ between the maximum likelihood estimator and this set, with only $N = 500$ particles and a fast learning rate $h_t = t^{-0.8}$ it is difficult for G-PFSO to escape from one element of $\Theta_{\text{loc},\star}$ and to reach a small neighbourhood of $\hat{\theta}_{\text{mle},n}$. This explains why, in 15 of the 100 runs of Algorithm 1, the two estimators are still stuck around an element of $\Theta_{\text{loc},\star}$ after 50 000 iterations. Unreported results obtained by maximising $l_n(\theta)$ with a quasi-Newton algorithm suggest that the log-likelihood function has stationary points that are not in $\Theta_{\text{loc},\star}$, and thus that $l_n(\theta)$ may have local maxima other than those belonging to this set.

We now repeat the above experiment with $\alpha = 0.5$. The 100 resulting values of $\|\bar{\theta}_T^N - \hat{\theta}_{\text{mle},n}\|_\infty$, summarized in Figure 4b, fifth boxplot, are all smaller than 0.17, showing that reducing α improves the ability of G-PFSO to quickly find the highest mode of the objective function, as discussed in Section 2.4. On the other hand, when $\alpha = 0.8$ the estimate $\tilde{\pi}_T^N$ of $\tilde{\pi}_T$ computed by Algorithm 1 can be more concentrated around $\hat{\theta}_{\text{mle},n}$ than when $\alpha = 0.5$ and, for this reason, we see in Figure 4b that $\|\bar{\theta}_T^N - \hat{\theta}_{\text{mle},n}\|_\infty$ can be much smaller for the former value of α than for the latter. In Section 4.3 we saw that for a given choice of α reducing c_Σ can reduce the estimation error of G-PFSO. This point is illustrated further in the sixth boxplot in Figure 4b, which summarizes the values of $\|\bar{\theta}_T^N - \hat{\theta}_{\text{mle},n}\|_\infty$ obtained in 100 runs of Algorithm 1 with $(\alpha, c_\Sigma) = (0.5, 1)$. We however observe that, for reasons explained in Section 4.3, reducing c_Σ from 10 to 1 increases the probability of having a ‘large’ estimation error, since for $c_\Sigma = 1$ we have $\|\bar{\theta}_T^N - \hat{\theta}_{\text{mle},n}\|_\infty > 0.24$ in 4 runs of the algorithm while, as mentioned above, for the same value of α this event never happens in Figure 4b when $c_\Sigma = 10$.

Finally, in Figure 4c we report the evolution of $\|\bar{\theta}_t^N - \hat{\theta}_{\text{mle},n}\|$ as the number of iterations t increases, averaged over 5 runs of G-PFSO with $(\alpha, c_\Sigma) = (0.5, 1)$. As in the previous three examples, we observe that for $h_t = t^{-1/2}$ the estimator $\bar{\theta}_t^N$ converges to the target parameter value at the optimal $t^{-1/2}$ rate.

5. Future work and practical recommendations

The full theoretical justification for G-PFSO is still in progress but the already obtained result provides an important preliminary step towards a precise analysis of this algorithm. Notably, future work should aim at validating, or not, the $t^{-1/2}$ convergence rate observed for the estimator $\bar{\theta}_t^N$, with $N \in \mathbb{N}$ fixed. In particular, we conjecture that for $h_t = t^{-1/2}$ and N large enough, $\bar{\theta}_t^N$ has this convergence behaviour with probability at least $p_N > 0$,

where $p_N \rightarrow 1$ as $N \rightarrow \infty$. In addition, following the discussion in Section 2.5 we conjecture that under the additional constraint that $\nu \leq 2$ we have $\|\bar{\theta}_t^N - \theta_\star\| = \mathcal{O}_{\mathbb{P}}(t^{-1/2})$, if N is sufficiently large.

Iterative optimization methods are usually run until a given stopping criterion, guaranteeing some control on the estimation error, is fulfilled. Ideally, a similar approach should be followed when G-PFSO is applied to observations $(Y_t)_{t \geq 1}$ that are sampled by the user, as in Section 4.5. Unfortunately, defining a good stopping criterion for global stochastic optimization is known to be a hard task and “most stochastic global optimization users just let their algorithm run until some time limit is exhausted” (Schoen, 2009). Future research should aim at designing a better strategy for stopping G-PFSO.

G-PFSO requires the user to specify a few ingredients, and we end this paper by proposing some default choices for them. For reasons given in Section 2.4 our recommended default choice for the learning rate is $h_t = t^{-1/2}$. Depending on the estimation problem at hand, when the number of degrees of freedom ν of the Student’s t -distributions appearing in (1) is small, and notably when $\nu \leq 2$, that is when the Student’s t -distributions have an infinite variance, G-PFSO may, as in Section 4.3, or may not, as in Section 4.4, work better than when ν is large. Since in all the challenging problems of Section 4 we observe that G-PFSO performs well when using Student’s t -kernels with thin tails, we recommend by default to choose a value for ν which is not too small, e.g. to let $\nu \geq 5$. Finally, the covariance matrix Σ of the Markov kernels used to generate the new particles is another important ingredient of G-PFSO. Unfortunately, the ‘optimal’ choice for this parameter is problem-dependent, as it should depend on the landscape of the function to be optimized. Focussing on the case where $\Sigma = c_\Sigma I_d$, to escape more easily from a local mode c_Σ should be large if the objective function has a lot of local optima, or if its modes are far apart, and small otherwise to improve the concentration of the particle system around θ_\star , and thus to reduce the estimation error. Since G-PFSO is designed to address difficult optimization tasks, where finding the global optimum is the main challenge, it is sensible to choose by default a value for c_Σ which is not too small, e.g. to let $c_\Sigma \geq 1$.

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A. Appendix: Proofs

A.1. Roadmap

In Section A.2 we introduce a more general definition of the sequence $(\tilde{\pi}_t)_{t \geq 1}$, which notably does not assume that $h_t > 0$ is positive for all t . Then, Theorem 1 follows

from Theorem 2 and from Lemma 1, which are derived for this more general definition of $(\tilde{\pi}_t)_{t \geq 1}$ and proven in Section A.5 and in Section A.6, respectively. Proposition 2 is a direct consequence of the slightly more general Proposition 3, proven in Section A.7, while Proposition 1 is proven in Section A.4. A complement to Section 2.5 of the paper is given in Section B and additional information for the numerical experiments is provided in Section C.

A.2. A more general definition of the sequence $(\tilde{\pi}_t)_{t \geq 1}$

The definition of the sequence $(\tilde{\pi}_t)_{t \geq 1}$ requires to specify $\nu \in (0, \infty)$ and three sequences, namely a sequence $(h_t)_{t \geq 0}$ in $[0, \infty)$, a strictly increasing sequence $(t_p)_{p \geq 0}$ in $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ and a sequence $(\Sigma_t)_{t \geq 0}$ of $d \times d$ covariance matrices verifying

$$\sup_{t \geq 0} (\|\Sigma_t\| \vee \|\Sigma_t^{-1}\|) < \infty. \quad (9)$$

Next, given a distribution $\tilde{\pi}_0 \in \mathcal{P}_L(\Theta)$, and defining

$$\mu_t(d\theta) = \begin{cases} \delta_{\{0\}}, & h_t = 0 \\ \mathcal{N}_d(0, h_t^2 \Sigma_t), & h_t > 0 \text{ and } t \notin (t_p)_{p \geq 0}, \\ t_{d,\nu}(0, h_t^2 \Sigma_t), & h_t > 0 \text{ and } t \in (t_p)_{p \geq 0} \end{cases}, \quad \forall t \geq 0, \quad (10)$$

we let

$$\tilde{\pi}_t(d\theta) = \frac{f_\theta(Y_t)(\mu_{t-1} * \tilde{\pi}_{t-1})|_\Theta(d\theta)}{\int_{\mathbb{R}^d} f_\theta(Y_t)(\mu_{t-1} * \tilde{\pi}_{t-1})|_\Theta(d\theta)} \in \mathcal{P}_L(\Theta), \quad t \geq 1. \quad (11)$$

A.3. Theoretical results for $(\tilde{\pi}_t)_{t \geq 1}$ as defined in Section A.2

Theorem 2. *Let $(h_t)_{t \geq 0}$ and $(t_p)_{p \geq 0}$ be such that*

1. $h_{t_p} > 0$ for all $p \geq 0$,
2. $(t_{p+1} - t_p) \rightarrow \infty$ and $(t_p - t_{p-1}) \sum_{s=t_{p-1}+1}^{t_p-1} h_s^2 \rightarrow 0$,
3. $\limsup_{p \rightarrow \infty} \frac{t_{p+1} - t_p}{t_p - t_{p-1}} < \infty$ and $\log(h_{t_{p-1}})/(t_p - t_{p-1}) \rightarrow 0$.

Moreover, assume that

$$\exists(\beta, c) \in (1, \infty)^2 \quad \text{such that} \quad \mathbb{P}(\tilde{\pi}_{t_p}(V_{c/h_{t_p}^\beta}) \geq \beta^{-1}) \rightarrow 0. \quad (12)$$

Then, under Assumptions A1-A5, $\tilde{\pi}_t \Rightarrow \delta_{\{\theta_\star\}}$ in \mathbb{P} -probability.

Condition (12) of Theorem 2 holds when the parameter space Θ is bounded, since in this case we have $\tilde{\pi}_t(\Theta^c) = 0$ for all $t \geq 0$, \mathbb{P} -a.s. Lemma 1 below provides sufficient conditions for (12) to hold when Θ is unbounded. We recall that for two strictly positive sequences $(a_t)_{t \geq 1}$ and $(b_t)_{t \geq 1}$ the notation $a_t = \Theta(b_t)$ means that $\liminf_{t \rightarrow \infty} (a_t/b_t + b_t/a_t) > 0$.

Lemma 1. Assume Assumptions A1, A2, A5 and A6. For every $C \in (0, \infty)$ let

$$\zeta(C) = \begin{cases} \mathbb{E}[\sup_{\theta \in V_C} \log(\tilde{f}_\theta/f_{\theta_*})], & \text{if Assumption A6.1 holds} \\ \log(\sup_{\theta \in V_C} \mathbb{E}[(\tilde{f}_\theta/f_{\theta_*})]), & \text{if Assumption A6.2 holds} \\ \log(\sup_{\theta \in V_C} \mathbb{E}[\tilde{f}_\theta]), & \text{if Assumption A6.3 holds} \end{cases}$$

and let $k_* \in \{1/2\} \cup \mathbb{N}$ be as in Assumption A6. Let $(h_t)_{t \geq 0}$ and $(t_p)_{p \geq 0}$ be such that Conditions 1-2 of Theorem 2 hold and such that

1. $h_{t_p} < h_{t_{p-1}}$ for all $p \geq 0$,
2. $h_{t_p} = \Theta(t_p^{-\alpha})$ for some $\alpha > 0$,
3. There exists a constant $\beta_* \in (0, \infty)$ such that

$$|\zeta(h_{t_p}^{-\beta_*})|^{-2k_*} \sum_{i=1}^p (t_i - t_{i-1})^{-k_* \mathbf{1}_{\mathbb{N}(k_*)}} \rightarrow 0. \quad (13)$$

Then, Condition (12) of Theorem 2 holds.

Remark 7. We note that the conclusions of Theorem 2 and of Lemma 1 remain valid when $(\Sigma_t)_{t \geq 0}$ is a random sequence of covariance matrices, provided that for all $t > t_2$, Σ_t is independent of $(Y_s)_{s > t_{p_t-2}}$, with $p_t = \sup\{p \in \mathbb{N}_0 : t_p < t\}$.

The following proposition provides sufficient conditions on $(h_t)_{t \geq 0}$ and on $(t_p)_{p \geq 0}$ to ensure that Conditions 1-3 of Theorem 2 and Conditions 1-3 of Lemma 1 hold.

Proposition 3. Let $C \in [1, \infty)$, $\alpha \in (0, \infty)$, $\varrho \in (0, \alpha \wedge 1)$ and $c \in (0, 1)$ be some constants. Let $(t_p)_{p \geq 0}$ be defined by

$$t_0 \in \mathbb{N}_0 \quad C_{p-1} \in [ct_{p-1}^\varrho, t_{p-1}^\varrho/c], \quad t_p = t_{p-1} + \lceil C_{p-1} \log(t_{p-1}) \vee C \rceil, \quad p \geq 1 \quad (14)$$

and let $(h_t)_{t \geq 0}$ be such that $h_{t_p} = \Theta(t_p^{-\alpha})$, such that $h_{t_p} < h_{t_{p-1}}$ for all $p \geq 1$ and such that, for all $t \geq 1$, either $h_{t-1} > h_t$ or $h_{t-1} = 0$. Then, the sequences $(h_t)_{t \geq 0}$ and $(t_p)_{p \geq 0}$ verify Conditions 1-3 of Theorem 2 and Conditions 1-2 of Lemma 1. In addition, the sequences $(h_t)_{t \geq 0}$ and $(t_p)_{p \geq 0}$ also verify Condition 3 of Lemma 1 if Assumption A6 holds for a $k_* > (1 + \varrho)/\varrho$.

A.4. Proof of Proposition 1

Proof. The proof of the second part of the proposition is similar to that of the first part and, to save place, only this latter is given below.

For every $C \in \mathbb{R}_{>0}$ let $A_C = \{(\mu, \sigma^2) \in \mathbb{R}^2 : |\sigma^2| < C, |\mu| < C\}$. We first show that if C is large enough then Assumption A4 holds for $A_* = A_{C_*}$.

Let $B_{1,C} = \{(\mu, \sigma^2) \in \Theta : \sigma^2 \geq C\}$ and $B_{2,C} = \{(\mu, \sigma^2) \in \Theta : |\mu| \geq C\}$ so that $A_C^c \cap \Theta = B_{1,C} \cup B_{2,C}$. Then,

$$\mathbb{E}[\sup_{\theta \notin A_C} \log(\tilde{f}_\theta)] \leq \mathbb{E}[\sup_{\theta \in B_{1,C}} \log(f_\theta)] + \mathbb{E}[\sup_{\theta \in B_{2,C}} \log(f_\theta)] \quad (15)$$

where

$$\mathbb{E}[\sup_{\theta \in B_{1,C}} \log(\tilde{f}_\theta)] \leq -\frac{1}{2} \log(2\pi C). \quad (16)$$

To proceed further let $\theta \in B_{2,C}$, $y \in \mathbb{R}$ and note that

$$\begin{aligned} f_\theta(y) &= f_\theta(y)\mathbf{1}(|y| \geq C/2) + f_\theta(y)\mathbf{1}(|y| < C/2) \\ &\leq \frac{1}{\sqrt{2\pi\sigma^2}}\mathbf{1}(|y| \geq C/2) + \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(y-C)^2}{2\sigma^2}}\mathbf{1}(|y| < C/2) \\ &\leq \frac{1}{\sqrt{2\pi\sigma^2}}\mathbf{1}(|y| \geq C/2) + \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{C^2}{8\sigma^2}}\mathbf{1}(|y| < C/2) \\ &\leq \frac{1}{\sqrt{2\pi\sigma^2}}\mathbf{1}(|y| \geq C/2) + \frac{1}{\sqrt{\pi C^2/2}}e^{-\frac{1}{2}}\mathbf{1}(|y| < C/2). \end{aligned}$$

Therefore,

$$\mathbb{E}[\sup_{\theta \in B_{2,C}} \log(f_\theta)] \leq -\frac{1}{2} \log(2\pi\sigma^2)\mathbb{P}(|Y_1| \geq C/2) - \frac{1}{2} \log(\pi C^2/2)\mathbb{P}(|Y_1| < C/2)$$

which, together with (15) and (16), shows that

$$\mathbb{E}[\sup_{\theta \notin A_C} \log(\tilde{f}_\theta)] \leq -\frac{1}{2} \log(2\pi C) - \frac{1}{2} \log(2\pi\sigma^2)\mathbb{P}(|Y_1| \geq C/2) - \frac{1}{2} \log(\pi C^2/2)\mathbb{P}(|Y_1| < C/2).$$

Since the r.h.s. converges to $-\infty$ as $C \rightarrow \infty$ it follows that Assumption A4.a)) holds for $A_\star = A_{C_\star}$, for a sufficiently large constant $C_\star \in \mathbb{R}_{>0}$.

To show the second part of Assumption A4 let \tilde{A}_\star be an arbitrary compact set that contains a neighbourhood of A_{C_\star} . Note that $\tilde{A}_\star \cap \Theta$ is compact and that the mapping $\theta \mapsto f_\theta(y)$ is continuous on $\tilde{A}_\star \cap \Theta$ for all $y \in \mathbb{R}$. Then, since for all $(\tilde{\theta}, \theta) \in \Theta^2$ we have

$$\log((f_{\tilde{\theta}}/f_\theta)(y)) = \frac{1}{2} \log(\sigma^2/\tilde{\sigma}^2) - \frac{1}{2} \left(\frac{\sigma^2 - \tilde{\sigma}^2}{\sigma^2\tilde{\sigma}^2} y^2 + \frac{\tilde{\mu}^2}{\tilde{\sigma}^2} - \frac{\mu^2}{\sigma^2} - 2y \frac{\tilde{\mu}\sigma^2 - \mu\tilde{\sigma}^2}{\sigma^2\tilde{\sigma}^2} \right)$$

it follows that the second part of Assumption A4 holds since $\mathbb{E}[Y_1^4] < \infty$ by assumption. This concludes to show that Assumption A4 holds

To show that Assumption A5.3 holds it suffices to remark that $f_\theta(y) \leq (2\pi\sigma^2)^{-1/2}$ for all $\theta \in \Theta$ and $y \in \mathbb{Y}$.

We now show that Assumption A6 holds. To this aim let $\theta_\star = (\mu_\star, \sigma_\star^2) \in \Theta$ and note that, for all $\sigma^2 \in \mathbb{R}_{>0}$ and $\delta \neq 0$, we have

$$\begin{aligned} \mathbb{E} \left[\exp \left(- \frac{(Y_1 - (\mu_\star + \delta))^2}{2\sigma^2} \right) \right] &\leq e^{-\delta^2/(8\sigma^2)} + \mathbb{P}(|Y_1 - \mu_\star| \geq |\delta|/2) \\ &\leq e^{-\delta^2/(8\sigma^2)} + \frac{2\mathbb{E}[|Y_1 - \mu_\star|]}{|\delta|}. \end{aligned} \quad (17)$$

where the last inequality uses Markov's inequality.

Let $\theta_C \in V_C$,

$$\epsilon_1 = \frac{|\sigma_C^2 - \sigma_\star^2|}{C}, \quad \epsilon_2 = \frac{|\mu_C - \mu_\star|}{C}$$

and remark that $\epsilon_1 \vee \epsilon_2 \geq 1/\sqrt{2}$. Assume first that $\epsilon_2 \geq 1/\sqrt{2}$. Then, using (17) with $\delta = \mu_C - \mu_\star$, we have

$$\begin{aligned} \mathbb{E}[f_{\theta_C}(Y_1)] &\leq \frac{1}{\sqrt{2\pi\sigma^2}} \left(\exp\left(-\frac{\epsilon_2^2 C^2}{8\sigma_C^2}\right) + \frac{2\mathbb{E}[|Y_1 - \mu_\star|]}{\epsilon_2 C} \right) \\ &\leq \frac{1}{\sqrt{2\pi\sigma^2}} \left(\exp\left(-\frac{C^2}{16\sigma^2}\right) + \frac{2^{3/2}\mathbb{E}[|Y_1 - \mu_\star|]}{C} \right). \end{aligned} \quad (18)$$

Assume now that $\epsilon_2 < 1/\sqrt{2}$, so that $\epsilon_1 \geq 1/\sqrt{2}$. Note that, for $C > 2^{3/2}\sigma_\star^2$ we have

$$\sigma_C^2 = |\sigma_\star^2 - \sigma_C^2 - \sigma_\star^2| \geq \epsilon_1 C - \sigma_\star^2 \geq C/\sqrt{2} - \sigma_\star^2 \geq \sigma_\star^2 > 0.$$

Therefore, for $C > 2^{3/2}\sigma_\star^2$ we have

$$\mathbb{E}[f_{\theta_C}(Y_1)] \leq \left(2\pi(\sigma_\star^2 - \sigma_C^2 - \sigma_\star^2)\right)^{-1/2} \leq \left(2\pi(C/\sqrt{2} - \sigma_\star^2)\right)^{-1/2}. \quad (19)$$

Then, using (18)-(19), it follows that there exists a constant $C' \in \mathbb{R}_{>0}$ such that, for $C > 0$ large enough we have

$$\begin{aligned} \sup_{\theta \in V_C} \mathbb{E}[f_\theta(Y_1)] &\leq \frac{1}{\sqrt{2\pi\sigma^2}} \left(\exp\left(-\frac{C^2}{16\sigma^2}\right) + \frac{2^{3/2}\mathbb{E}[|Y_1 - \mu_\star|]}{C} \right) + \left(2\pi(C/\sqrt{2} - \sigma_\star^2)\right)^{-1/2} \\ &\leq \frac{C'}{C^{1/2}} \end{aligned}$$

showing that

$$\limsup_{C \rightarrow \infty} \frac{\log\left(\sup_{\theta \in V_C} \mathbb{E}[f_\theta(Y_1)]\right)}{\log(C)} \leq \limsup_{C \rightarrow \infty} \frac{\log(C') - \frac{1}{2}\log(C)}{\log(C)} = -\frac{1}{2} < 0.$$

Hence the second part of Assumption A6.3 holds. In addition, since

$$|\log(f_{\theta_\star}(y))|^p \leq 2^{p-1} \left| \frac{1}{2} \log(2\pi\sigma_\star^2) \right|^p + \frac{|y - \mu_\star|^{2p}}{2(\sigma_\star^2)^p}, \quad \forall p \in \mathbb{N},$$

it follows that the first part of Assumption A6.3 holds for every $k_\star \in \mathbb{N}$ since, by assumption, $\mathbb{E}[e^{c|Y_1|}] < \infty$ for some constant $c > 0$. The proof is complete. \square

A.5. Proofs of Theorem 2

A.5.1. Additional notation and conventions

Below we use the convention that empty sums equal zero and empty products equal one, and let $(\tilde{U}_t)_{t \geq 0}$ be a sequence of independent random variables such that

$$\tilde{U}_t \sim \begin{cases} \mathcal{N}_d(0, \Sigma_t), & t \notin (t_p)_{p \geq 0}, \\ t_{d,\nu}(0, \Sigma_t), & t \in (t_p)_{p \geq 0} \end{cases}, \quad \forall t \geq 0.$$

For all $t \geq 0$ we let $U_t = h_t \tilde{U}_t$; notice that $U_t \sim \mu_t$.

Next, for every integers $0 \leq k < t$ we define

$$u_{k:t} = (u_k, \dots, u_t), \quad [u_{k:t}] = \max_{k \leq s < t} \left\| \sum_{i=s}^{t-1} u_i \right\|, \quad (u_k, \dots, u_t) \in \mathbb{R}^{d(t-k+1)}$$

and let

$$\Theta_{\epsilon, k:t} = \{u_{k:t} \in \mathbb{R}^{d(t-k+1)} : [u_{(k+1):t}] < \epsilon\}, \quad \epsilon > 0.$$

Lastly, for every $t \geq 0$ we let \mathcal{F}_t be the σ -algebra generated by (Y_1, \dots, Y_t) (with the convention $\mathcal{F}_0 = \emptyset$) and, for every $0 \leq k < t$ and $A \in \mathcal{B}(\mathbb{R}^d)$, we let

$$\pi'_{k,t}(A) = \frac{\int_A (\mu_k * \tilde{\pi}_k)(\theta - \sum_{s=k+1}^{t-1} U_s) \prod_{s=k+1}^t \tilde{f}_{\theta - \sum_{i=s}^{t-1} U_i}(Y_s) d\theta}{\int_{\Theta} \mathbb{E}[(\mu_k * \tilde{\pi}_k)(\theta - \sum_{s=k+1}^{t-1} U_s) \prod_{s=k+1}^t \tilde{f}_{\theta - \sum_{i=s}^{t-1} U_i}(Y_s) | \mathcal{F}_t] d\theta}.$$

A.5.2. Preliminary results

The following result (see Ghosal and Van der Vaart, 2017, Proposition 6.2 and above comment, page 124) provides the following necessary and sufficient condition for checking that a sequence $(\nu_t)_{t \geq 1}$ of probability measures on Θ (implicitly indexed by random variables) is such that $\nu_t \Rightarrow \delta_{\{\theta_\star\}}$ in \mathbb{P} -probability.

Proposition 4. $\nu_t \Rightarrow \delta_{\{\theta_\star\}}$ in \mathbb{P} -probability if and only if $\mathbb{E}[\nu_t(V_\delta)] \rightarrow 0$ for all $\delta > 0$.

Lemma 2. Assume Assumption A4 and let $A_\star \in \mathcal{B}(\mathbb{R}^d)$ and $\tilde{A}_\star \in \mathcal{B}(\mathbb{R}^d)$ be as in Assumption A4. Then, there exists a set $A'_\star \in \mathcal{B}(\mathbb{R}^d)$, with $A'_\star \subsetneq \tilde{A}_\star$, that contains a neighbourhood of A_\star and such that, for every sequence $(\gamma'_t)_{t \geq 1}$ in $\mathbb{R}_{\geq 0}$ such that $\gamma'_t \rightarrow 0$ and every sequence $(s_t)_{t \geq 1}$ in \mathbb{N}_0 such that $\inf_{t \geq 1} (t - s_t) \geq 1$ and such that $(t - s_t) \rightarrow \infty$, there exists a sequence $(\delta_t)_{t \geq 1}$ in $\mathbb{R}_{> 0}$ such that $\delta_t \rightarrow 0$ and such that

$$\mathbb{P} \left(\sup_{(u_{s_t:t}, \theta) \in \Theta_{\gamma'_t, s_t:t} \times (A'_\star \cap \Theta)} \prod_{s=s_t+1}^t (\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i} / f_\theta)(Y_s) < e^{(t-s_t)\delta_t} \right) \rightarrow 1.$$

The proof of this result is given in Section A.8.1.

The following results rewrite the probability measure $\tilde{\pi}_t$ in a more convenient way.

Lemma 3. With \mathbb{P} -probability one we have, for all $t \geq 0$ and all $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\tilde{\pi}_t(A) = \frac{\int_A \mathbb{E}[(\mu_0 * \tilde{\pi}_0)(\theta - \sum_{s=1}^{t-1} U_s) \prod_{s=1}^t \tilde{f}_{\theta - \sum_{i=s}^{t-1} U_i}(Y_s) | \mathcal{F}_t] d\theta}{\int_{\Theta} \mathbb{E}[(\mu_0 * \tilde{\pi}_0)(\theta - \sum_{s=1}^{t-1} U_s) \prod_{s=1}^t \tilde{f}_{\theta - \sum_{i=s}^{t-1} U_i}(Y_s) | \mathcal{F}_t] d\theta}.$$

The proof of this result is given in Section A.8.2.

The next result builds on Ghosal et al. (2000, Lemma 8.1) and will be used to control the denominator of $\tilde{\pi}_t(\theta)$.

Lemma 4. *Assume Assumptions A1-A2 and let $\delta_\star > 0$ be as in these two assumptions. Then, there exists a constant $\tilde{C}_\star \in (0, \infty)$ such that, for every $\epsilon \geq 0$, sequence $(s_t)_{t \geq 1}$ in \mathbb{N}_0 with $\inf_{t \geq 1} (t - s_t) \geq 1$, every constants $\delta \geq \tilde{\delta} > 0$ such that $\delta + \tilde{\delta} < \delta_\star$ and every probability measure $\eta \in \mathcal{P}(\mathbb{R}^d)$ we have, for all $t \geq 1$ and with $C_{\delta, \tilde{\delta}}^\eta = \inf_{v \in B_{\tilde{\delta}}(0)} \eta(B_\delta(\theta_\star - v))$,*

$$\begin{aligned} \mathbb{P} \left(\int_{\Theta} \mathbb{E} \left[\eta \left(\theta - \sum_{s=s_t+1}^{t-1} U_s \right) \prod_{s=s_t+1}^t (f_{\theta - \sum_{i=s}^{t-1} U_i} / f_{\theta_\star})(Y_s) \middle| \mathcal{F}_t \right] d\theta \leq \frac{\mathbb{P}(U_{s_t:t} \in \Theta_{\tilde{\delta}, s_t:t})}{e^{(t-s_t)(2(\tilde{C}_\star \delta)^2 + \epsilon)}} C_{\delta, \tilde{\delta}}^\eta \right) \\ \leq ((t - s_t)((\tilde{C}_\star \delta) + (\tilde{C}_\star \delta)^{-1} \epsilon)^2)^{-1}. \end{aligned}$$

The proof of this result is given in Section A.8.3.

The next result will be used to control the numerator of $\tilde{\pi}_t(\theta)$.

Lemma 5. *Assume Assumptions A3-A4 and let $(\gamma_t)_{t \geq 1}$ be a sequence in $[0, \infty)$ such that $\gamma_t \rightarrow 0$ and $(s_t)_{t \geq 1}$ be a sequence in \mathbb{N}_0 such that $\inf_{t \geq 1} (t - s_t) \geq 1$ and $(t - s_t) \rightarrow \infty$. Then, for every $\epsilon > 0$ there exist a constant $\tilde{D}_\star \in \mathbb{R}_{>0}$ and a sequence of measurable functions $(\phi_t)_{t \geq 1}$, $\phi_t : \mathbf{Y}^t \rightarrow \{0, 1\}$, such that $\mathbb{E}[\phi_t(Y_{1:t})] \rightarrow 0$ and, for t large enough,*

$$\sup_{(\theta, u_{s_t:t}) \in V_\epsilon \times \Theta_{\gamma_t, s_t:t}} \mathbb{E} \left[(1 - \phi_t(Y_{1:t})) \prod_{s=s_t+1}^t (f_{\theta - \sum_{i=s}^{t-1} u_i} / f_{\theta_\star})(Y_s) \middle| \mathcal{F}_{s_t} \right] \leq e^{-(t-s_t)\tilde{D}_\star}.$$

The proof of this result is given in Section A.8.4.

The next lemma builds on Kleijn and van der Vaart (2012, Theorem 3.1)

Lemma 6. *Assume Assumptions A1-A4. Let $\epsilon > 0$, $(\gamma_t)_{t \geq 1}$ be a sequence in $[0, \infty)$ such that $\gamma_t \rightarrow 0$ and $(s_t)_{t \geq 1}$ be a sequence in \mathbb{N}_0 such that $\inf_{t \geq 1} (t - s_t) \geq 1$ and $(t - s_t) \rightarrow \infty$. Then, there exist constants $(C_1, C_2) \in \mathbb{R}_{>0}^2$ such that, for every constants $\delta \geq \tilde{\delta} > 0$ such that $\delta + \tilde{\delta} < \delta_\star$ (with $\delta_\star > 0$ as in Lemma 4), there exists a sequence of measurable functions $(\phi'_t)_{t \geq 1}$, $\phi'_t : \mathbf{Y}^t \rightarrow \{0, 1\}$, such that $\mathbb{E}[\phi'_t(Y_{1:t})] \rightarrow 0$ and such that, for t large enough,*

$$\begin{aligned} \mathbb{E}[(1 - \phi'_t(Y_{1:t})) \mathbb{1}_{\Theta_{\gamma_t, s_t:t}}(U_{s_t:t}) \pi'_{s_t, t}(V_\epsilon) \middle| \mathcal{F}_{s_t}] \\ \leq \frac{e^{-(t-s_t)(C_1^{-1} - C_2 \delta^2)}}{\mathbb{P}(U_{s_t:t} \in \Theta_{\tilde{\delta}, s_t:t}) \inf_{v \in B_{\tilde{\delta}}(0)} (\mu_{s_t} * \tilde{\pi}_{s_t})(B_\delta(\theta_\star - v))}, \quad \mathbb{P} - a.s. \end{aligned}$$

The proof of this result is given in Section A.8.5.

A.5.3. Proof of the theorem

Theorem 2 is a direct consequence of Proposition 4 and of the following three lemmas.

Lemma 7. *Consider the set-up of Theorem 2. Then, $\mathbb{E}[\tilde{\pi}_{t_p}(V_\epsilon)] \rightarrow 0$ for all $\epsilon > 0$.*

See Section A.5.4 for the proof.

Lemma 8. Consider the set-up of Theorem 2 and assume that the conclusion of Lemma 7 holds. Let $(v_p)_{p \geq 1}$ be a sequence in \mathbb{N} such that $t_{p-1} \leq v_p < t_p$ for all $p \geq 1$ and such that $(v_p - t_{p-1}) \rightarrow \infty$, and let $(\tau_k)_{k \geq 1}$ be a strictly increasing sequence in \mathbb{N} such that $(\tau_k)_{k \geq 1} = \{t \in \mathbb{N} : \exists p \geq 1, v_p \leq t < t_p\}$. Then, $\mathbb{E}[\tilde{\pi}_{\tau_k}(V_\epsilon)] \rightarrow 0$ for all $\epsilon > 0$.

See Section A.5.5 for the proof.

Lemma 9. Consider the set-up of Theorem 2 and assume that the conclusion of Lemma 7 holds. Then, there exists a sequence $(v_p)_{p \geq 1}$ verifying the conditions of Lemma 8 such that, with $(\tau'_q)_{q \geq 1}$ the strictly increasing sequence in \mathbb{N} verifying

$$(\tau'_q)_{q \geq 1} = \{t \in \mathbb{N} : \exists p \geq 1, t_{p-1} < t \leq v_p\},$$

we have $\mathbb{E}[\tilde{\pi}_{\tau'_q}(V_\epsilon)] \rightarrow 0$ for all $\epsilon > 0$.

See Section A.5.6 for the proof.

A.5.4. Proof of Lemma 7

Proof. Below $C \in (0, \infty)$ is a constant whose value can change from one expression to another.

We first remark that, by Doob's martingale inequality and under the assumptions on $(\mu_t)_{t \geq 0}$,

$$\limsup_{p \rightarrow \infty} \mathbb{P}([U_{(t_{p-1}+1):t_p}] \geq \gamma) \leq C\gamma^{-2} \limsup_{p \rightarrow \infty} \sum_{s=t_{p-1}+1}^{t_p-1} h_s^2 = 0, \quad \forall \gamma > 0$$

showing that there exists a sequence $(\gamma_t)_{t \geq 1}$ in $\mathbb{R}_{>0}$ such that

$$\gamma_t \rightarrow 0, \quad \mathbb{P}([U_{(t_{p-1}+1):t_p}] \geq \gamma_{t_p}) \rightarrow 0. \quad (20)$$

Let $(\gamma_t)_{t \geq 1}$ be as in (20) and $(s_t)_{t \geq 1}$ be a sequence in \mathbb{N}_0 such that $\inf_{t \geq 1} (t - s_t) \geq 1$, $(t - s_t) \rightarrow \infty$ and $s_{t_p} = t_{p-1}$ for every $p \geq 1$. Remark that such a sequence $(s_t)_{t \geq 1}$ exists under the assumptions of the lemma.

To proceed further let $(C_1, C_2) \in \mathbb{R}_{>0}^2$ be as in Lemma 6. Without loss of generality we assume below that $2\sqrt{1/(2C_1C_2)} < \delta_\star$, with $\delta_\star > 0$ as in Lemma 4. Let $\tilde{\delta} = \delta = \sqrt{1/(2C_1C_2)}$, $(\phi'_t)_{t \geq 1}$ be as in Lemma 6 and, for every $t \geq 1$, let $\tilde{\phi}_t(Y_{1:t})$ be such that $\phi_t(Y_{1:t}) = 1$ whenever $\tilde{\pi}_t(V_{c/h_t^\beta}) \geq \beta^{-1}$ and such that $\tilde{\phi}_t(Y_{1:t}) = 0$ otherwise, with $(c, \beta) \in (1, \infty)$ as in (12). Notice that $\mathbb{E}[\tilde{\phi}_{t_p}(Y_{1:t_p})] \rightarrow 0$ by (12) while $\mathbb{E}[\phi'_{t_p}(Y_{1:t_p})] \rightarrow 0$ by Lemma 6.

Therefore, using Lemma 3, Tonelli's theorem and the shorthand $\Theta_{t_p} = \Theta_{\gamma_{t_p}, t_{p-1}:t_p}$,

$$\begin{aligned} & \limsup_{p \rightarrow \infty} \mathbb{E}[\tilde{\pi}_{t_p}(V_\epsilon)] \\ & \leq \limsup_{p \rightarrow \infty} \mathbb{E}[(1 - \phi'_{t_p}(Y_{1:t_p}))(1 - \tilde{\phi}_{t_{p-1}}(Y_{1:t_{p-1}}))\tilde{\pi}_{t_p}(V_\epsilon)] \\ & \leq \limsup_{p \rightarrow \infty} \mathbb{E}[(1 - \phi'_{t_p}(Y_{1:t_p}))(1 - \tilde{\phi}_{t_{p-1}}(Y_{1:t_{p-1}}))\mathbf{1}_{\Theta_{t_p}}(U_{t_{p-1}:t_p})\pi'_{t_{p-1}, t_p}(V_\epsilon)] \\ & + \limsup_{p \rightarrow \infty} \mathbb{E}[(1 - \phi'_{t_p}(Y_{1:t_p}))(1 - \tilde{\phi}_{t_{p-1}}(Y_{1:t_{p-1}}))\mathbf{1}_{\Theta_{t_p}^c}(U_{t_{p-1}:t_p})\pi'_{t_{p-1}, t_p}(V_{\epsilon_{t_p}})] \end{aligned} \quad (21)$$

where, by Lemma 6 and for p large enough we have, \mathbb{P} -a.s.,

$$\begin{aligned} & \mathbb{E}[(1 - \phi'_{t_p}(Y_{1:t_p})) \mathbf{1}_{\Theta_{t_p}}(U_{t_{p-1}:t_p}) \pi'_{t_{p-1},t_p}(V_{\epsilon_{t_p}}) | \mathcal{F}_{t_{p-1}}] \\ & \leq \frac{e^{-(t_p-t_{p-1})C^{-1}}}{\mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\delta,t_{p-1}:t_p}) \inf_{v \in B_\delta(0)} (\mu_{t_{p-1}} * \tilde{\pi}_{t_{p-1}})(B_\delta(\theta_\star - v))}. \end{aligned} \quad (22)$$

To proceed further let $v \in B_\delta(0)$, $f_{t_{p-1}}$ be the density of $\mu_{t_{p-1}}$ and remark that for all $p \geq 1$ we have (using Tonelli's theorem for the second equality)

$$\begin{aligned} & (\mu_{t_{p-1}} * \tilde{\pi}_{t_{p-1}})(B_\delta(\theta_\star - v)) \\ & = \int_{B_\delta(\theta_\star - v)} \int_{\Theta} f_{t_{p-1}}(\theta - u) \tilde{\pi}_{t_{p-1}}(du) d\theta \\ & = \int_{\Theta} \int_{B_\delta(\theta_\star - v)} f_{t_{p-1}}(\theta - u) d\theta \tilde{\pi}_{t_{p-1}}(du) \\ & \geq \tilde{\pi}_{t_{p-1}}(B_{ch_{t_{p-1}}^{-\beta}}(\theta_\star)) \inf_{u \in B_{ch_{t_{p-1}}^{-\beta}}(\theta_\star)} \mu_{t_{p-1}}(B_\delta(\theta_\star - v + u)). \end{aligned} \quad (23)$$

Recall that $\mu_{t_{p-1}}(du)$ is the $t_{d,\nu}(0, h_{t_{p-1}}^2 \Sigma_{t_{p-1}})$. Then, under the assumptions on $(\Sigma_t)_{t \geq 0}$, and using (23), it is easily checked that, with the shorthand $\nu_1 = \beta(\nu + d) + \nu$,

$$\mathbb{P}\left(\inf_{v \in B_\delta(0)} (\mu_{t_{p-1}} * \tilde{\pi}_{t_{p-1}})(B_\delta(\theta_\star - v)) \geq C^{-1} h_{t_{p-1}}^{\nu_1} | \tilde{\phi}_{t_{p-1}}(Y_{1:t_{p-1}}) = 0\right) = 1, \quad \forall p \geq 1. \quad (24)$$

Consequently, using (22) and for p large enough, we have

$$\begin{aligned} & \mathbb{E}[(1 - \phi'_{t_p}(Y_{1:t_p})) (1 - \tilde{\phi}_{t_{p-1}}(Y_{1:t_{p-1}})) \mathbf{1}_{\Theta_{t_p}}(U_{t_{p-1}:t_p}) \pi'_{t_{p-1},t_p}(V_\epsilon)] \\ & \leq C h_{t_{p-1}}^{-\nu_1} \frac{e^{-(t_p-t_{p-1})C^{-1}}}{\mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\delta,t_{p-1}:t_p})}. \end{aligned} \quad (25)$$

To proceed further remark that, by Lemma 4 and using (24), we can without loss of generality assume that $(\phi'_t)_{t \geq 1}$ is such that, for all $p \geq 1$,

$$\begin{aligned} & \mathbb{P}\left(\int_{\Theta} \mathbb{E}\left[(\mu_{t_{p-1}} * \tilde{\pi}_{t_{p-1}})\left(\theta - \sum_{s=t_{p-1}+1}^{t_p-1} U_s\right) \prod_{s=t_{p-1}+1}^{t_p} (f_{\theta - \sum_{i=s}^{t_p-1} U_i} / f_{\theta_\star})(Y_s) | \mathcal{F}_{t_p}\right] d\theta\right. \\ & \left. > C^{-1} h_{t_{p-1}}^{\nu_1} \mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\delta,t_{p-1}:t_p}) e^{-(t_p-t_{p-1})C\delta^2} | \phi'_{t_p}(Y_{1:t_p}) \vee \tilde{\phi}_{t_{p-1}}(Y_{1:t_{p-1}}) = 0\right) \\ & = 1 \end{aligned} \quad (26)$$

while, by the law of large numbers, we can also without loss of generality assume that $(\phi'_t)_{t \geq 1}$ is such that

$$\mathbb{P}\left(-\frac{1}{t-s_t} \sum_{s=s_t+1}^t \log(f_{\theta_\star}(Y_s)) \leq 1 - \mathbb{E}[\log(f_{\theta_\star})] | \phi'_t(Y_{1:t}) = 0\right) = 1, \quad \forall t \geq 1. \quad (27)$$

We now show that, under Assumption A5 we have, for p large enough,

$$\begin{aligned} & \mathbb{E}[(1 - \phi'_{t_p}(Y_{1:t_p}))(1 - \tilde{\phi}_{t_{p-1}}(Y_{1:t_{p-1}}))\mathbb{1}_{\Theta_{t_p}^c}(U_{t_{p-1}:t_p})\pi'_{t_{p-1},t_p}(V_\epsilon)] \\ & \leq \frac{Ce^{(t_p-t_{p-1})C}}{\mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\delta,t_{p-1}:t_p})} h_{t_{p-1}}^{-\nu_1} \mathbb{P}(U_{t_{p-1}:t_p} \notin \Theta_{t_p}). \end{aligned} \quad (28)$$

Assume first that Assumption A5.1 holds. In this case, by the law of large numbers, we can without loss of generality assume that $(\phi'_t)_{t \geq 1}$ is such that

$$\mathbb{P}\left(\frac{1}{t-s_t} \sum_{s=s_t+1}^t \sup_{\theta \in \Theta} \log(f_\theta(Y_s)) \leq 1 + \mathbb{E}[\sup_{\theta \in \Theta} \log(f_\theta)] \mid \phi'_t(Y_{1:t}) = 0\right) = 1, \quad \forall t \geq 1 \quad (29)$$

in which case (28) directly follows from (26), (27) and (29).

Assume now that Assumption A5.2 holds. Then, (28) holds since, for all $p \geq 1$,

$$\begin{aligned} & \mathbb{E}[(1 - \phi'_{t_p}(Y_{1:t_p}))(1 - \tilde{\phi}_{t_{p-1}}(Y_{1:t_{p-1}}))\mathbb{1}_{\Theta_{t_p}^c}(U_{t_{p-1}:t_p})\pi'_{t_{p-1},t_p}(V_\epsilon)] \\ & \leq \frac{Ce^{(t_p-t_{p-1})C}}{\mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\delta,t_{p-1}:t_p})} h_{t_{p-1}}^{-\nu_1} \mathbb{E}\left[\mathbb{1}_{\Theta_{t_p}^c}(U_{t_{p-1}:t_p})\right. \\ & \quad \times \left. \int_{V_\epsilon} (\mu_{t_{p-1}} * \tilde{\pi}_{t_{p-1}})\left(\theta - \sum_{s=t_{p-1}+1}^{t_p-1} U_s\right) \prod_{s=t_{p-1}+1}^{t_p} \mathbb{E}[\tilde{f}_{\theta - \sum_{i=s}^{t_p-1} U_i} / f_{\theta_*}](Y_s) \mid U_{t_{p-1}:t_p}\right] d\theta \\ & \leq \frac{Ce^{(t_p-t_{p-1})C}}{\mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\delta,t_{p-1}:t_p})} h_{t_{p-1}}^{-\nu_1} \left(\sup_{\theta \in \Theta} \mathbb{E}[f_\theta / f_{\theta_*}]\right)^{t_p-t_{p-1}} \mathbb{P}(U_{t_{p-1}:t_p} \notin \Theta_{t_p}) \\ & \leq \frac{Ce^{(t_p-t_{p-1})C}}{\mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\delta,t_{p-1}:t_p})} h_{t_{p-1}}^{-\nu_1} \mathbb{P}(U_{t_{p-1}:t_p} \notin \Theta_{t_p}) \end{aligned}$$

where the last inequality holds since $\sup_{\theta \in \Theta} \mathbb{E}[f_\theta / f_{\theta_*}] < \infty$ under Assumption A5.2.

Lastly, assume that Assumption A5.3 holds. Then, (28) holds since, for all $p \geq 1$,

$$\begin{aligned} & \mathbb{E}[(1 - \phi'_{t_p}(Y_{1:t_p}))(1 - \tilde{\phi}_{t_{p-1}}(Y_{1:t_{p-1}}))\mathbb{1}_{\Theta_{t_p}^c}(U_{t_{p-1}:t_p})\pi'_{t_{p-1},t_p}(V_\epsilon)] \\ & \leq \frac{Ce^{(t_p-t_{p-1})C}}{\mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\delta,t_{p-1}:t_p})} h_{t_{p-1}}^{-\nu_1} \mathbb{E}\left[\mathbb{1}_{\Theta_{t_p}^c}(U_{t_{p-1}:t_p})\right. \\ & \quad \times \left. \int_{V_\epsilon} (\mu_{t_{p-1}} * \tilde{\pi}_{t_{p-1}})\left(\theta - \sum_{s=t_{p-1}+1}^{t_p-1} U_s\right) \prod_{s=t_{p-1}+1}^{t_p} \mathbb{E}[\tilde{f}_{\theta - \sum_{i=s}^{t_p-1} U_i}(Y_s) \mid U_{t_{p-1}:t_p}\right] d\theta \\ & \leq \frac{Ce^{(t_p-t_{p-1})C}}{\mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\delta,t_{p-1}:t_p})} h_{t_{p-1}}^{-\nu_1} \left(\sup_{\theta \in \Theta} \mathbb{E}[f_\theta]\right)^{t_p-t_{p-1}} \mathbb{P}(U_{t_{p-1}:t_p} \notin \Theta_{t_p}) \\ & \leq \frac{Ce^{(t_p-t_{p-1})C}}{\mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\delta,t_{p-1}:t_p})} h_{t_{p-1}}^{-\nu_1} \mathbb{P}(U_{t_{p-1}:t_p} \notin \Theta_{t_p}) \end{aligned}$$

where the last inequality holds since $\sup_{\theta \in \Theta} \mathbb{E}[f_\theta] < \infty$ under Assumption A5.3. This concludes to show that (28) holds under Assumption A5.

Using (21), (25) and (28), to conclude the proof it is enough to show that, as $p \rightarrow \infty$,

$$h_{t_{p-1}}^{-\nu_1} e^{-(t_p-t_{p-1})C^{-1}} \rightarrow 0 \quad (30)$$

$$h_{t_{p-1}}^{-\nu_1} e^{(t_p-t_{p-1})C} \mathbb{P}(U_{t_{p-1}:t_p} \notin \Theta_{\gamma_{t_p}, :t_{p-1}, t_p}) \rightarrow 0 \quad (31)$$

$$\liminf_{p \rightarrow \infty} \mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\delta, t_{p-1}:t_p}) > 0. \quad (32)$$

By assumptions, $\log(1/h_{t_{p-1}})(t_p - t_{p-1})^{-1} \rightarrow 0$ and therefore (30) holds.

To establish (31) note that, using Doob's martingale inequality and the fact that $(\tilde{U}_s)_{s=t_{p-1}+1}^{t_p-1}$ are independent Gaussian random variables, we have

$$\mathbb{P}([U_{(t_{p-1}+1):t_p}] \geq \gamma) \leq C \exp\left(-\frac{\gamma^2}{C \sum_{s=t_{p-1}+1}^{t_p-1} h_s^2}\right), \quad \forall \gamma > 0, \quad \forall p \geq 1 \quad (33)$$

so that

$$\mathbb{P}(U_{t_{p-1}:t_p} \notin \Theta_{\gamma_{t_p}, t_{p-1}:t_p}) = \mathbb{P}([U_{(t_{p-1}+1):t_p}] \geq \gamma_{t_p}) \leq C \exp\left(-\frac{\gamma_{t_p}^2}{C \sum_{s=t_{p-1}+1}^{t_p-1} h_s^2}\right), \quad \forall p > 1.$$

Consequently, for all $p > 1$,

$$\begin{aligned} & \log\left(h_{t_{p-1}}^{-\nu_1} e^{(t_p-t_{p-1})C} \mathbb{P}(U_{t_{p-1}:t_p} \notin \Theta_{\gamma_{t_p}, t_{p-1}:t_p})\right) \\ & \leq -\nu_1 \log(h_{t_{p-1}}) + C(t_p - t_{p-1}) + \log(C) - C^{-1} \gamma_{t_p}^2 \left(\sum_{s=t_{p-1}+1}^{t_p-1} h_s^2\right)^{-1} \\ & = -(t_p - t_{p-1}) \left(-\frac{\log(C)}{t_p - t_{p-1}} - C + \frac{C^{-1} \gamma_{t_p}^2 - \nu_1 \log(1/h_{t_{p-1}}) \sum_{s=t_{p-1}+1}^{t_p-1} h_s^2}{(t_p - t_{p-1}) \sum_{s=t_{p-1}+1}^{t_p-1} h_s^2}\right) \end{aligned}$$

where, under the assumptions of the lemma and by taking $(\gamma_t)_{t \geq 1}$ such that $\gamma_{t_p} \rightarrow 0$ sufficiently slowly

$$\left(-\frac{\log(C)}{t_p - t_{p-1}} - C + \frac{C^{-1} \gamma_{t_p}^2 - \nu_1 \log(1/h_{t_{p-1}}) \sum_{s=t_{p-1}+1}^{t_p-1} h_s^2}{(t_p - t_{p-1}) \sum_{s=t_{p-1}+1}^{t_p-1} h_s^2}\right) \rightarrow \infty$$

so that (31) holds.

Lastly, to show (32) it suffices to remark that, since $\gamma_p \rightarrow 0$,

$$\liminf_{p \rightarrow \infty} \mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\delta, t_{p-1}:t_p}) \geq \liminf_{p \rightarrow \infty} \mathbb{P}(U_{t_{p-1}:t_p} \in \Theta_{\gamma_p, t_{p-1}:t_p}) = 1$$

where the equality holds since (31) holds. The proof is complete. \square

A.5.5. Proof of Lemma 8

Proof. Below $C \in (0, \infty)$ is a constant whose value can change from one expression to another.

We first remark that under the assumptions of the lemma there exists a $p_1 \in \mathbb{N}$ such that we have both $t_p - t_{p-1} > 1$ and $v_p - t_{p-1} > 0$ for all $p \geq p_1$. To simplify the presentation of the proof we assume without loss of generality that $p_1 = 1$ in what follows.

For every $k \geq 1$ let $p_k = \sup\{p \geq 0 : t_p < \tau_k\}$ and let $(s_t)_{t \geq 1}$ be a sequence in \mathbb{N}_0 such that $\inf_{t \geq 1} (t - s_t) \geq 1$, $(t - s_t) \rightarrow \infty$ and $s_{\tau_k} = t_{p_k}$ for every $k \geq 1$. Remark that such a sequence $(s_t)_{t \geq 1}$ exists since $(\tau_k - s_{\tau_k}) \geq 1$ for all $k \geq 1$ by construction while, by assumption,

$$\liminf_{k \rightarrow \infty} (\tau_k - s_{\tau_k}) \geq \liminf_{k \rightarrow \infty} (v_{p_{k+1}} - t_{p_k}) = \infty.$$

Remark now that, by Doob's martingale inequality and under the assumptions on $(\mu_t)_{t \geq 0}$ we have, for every $\gamma > 0$,

$$\limsup_{k \rightarrow \infty} \mathbb{P}([U_{(s_{\tau_k}+1):\tau_k}] \geq \gamma) \leq C \limsup_{k \rightarrow \infty} \gamma^{-2} \sum_{s=t_{p_k}+1}^{\tau_k-1} h_s^2 \leq C \limsup_{k \rightarrow \infty} \sum_{s=t_{p_k}+1}^{t_{p_{k+1}}-1} h_s^2 = 0$$

showing that there exists a sequence $(\gamma_t)_{t \geq 1}$ in $\mathbb{R}_{>0}$ such that

$$\gamma_t \rightarrow 0, \quad \mathbb{P}([U_{(s_{\tau_k}+1):\tau_k}] \geq \gamma_{\tau_k}) \rightarrow 0. \quad (34)$$

Let $(\gamma_t)_{t \geq 1}$ be as in (34) and $(C_1, C_2) \in (0, \infty)^2$ be as in Lemma 6. Without loss of generality we assume below that $2\sqrt{1/(2C_1C_2)} < \delta_*$, with $\delta_* > 0$ as in Lemma 4.

Let $\kappa = \sqrt{1/(12C_1C_2)}$, $\delta = 3\kappa$, $\bar{\delta} = \kappa$ and remark that, for every $t \geq 1$ and $v \in B_{\bar{\delta}}(0)$ we have, with $\theta_{s_t} : \Omega \rightarrow \Theta$ such that $\theta_{s_t} \sim \tilde{\pi}_{s_t}(d\theta)$ under \mathbb{P} ,

$$\begin{aligned} \mathbb{P}(\|\theta_{s_t} + U_{s_t} - \theta_* + v\| \geq \delta \mid \theta_{s_t} \in B_\kappa(\theta_*)) \\ \leq \mathbb{P}(\|\theta_{s_t} - \theta_*\| + \|U_{s_t}\| + \|v\| \geq \delta \mid \theta_{s_t} \in B_\kappa(\theta_*)) \\ \leq \mathbb{P}(U_{s_t} \notin B_{\delta - \bar{\delta} - \kappa}(0)) \\ \leq \mathbb{P}(U_{s_t} \notin B_\kappa(0)). \end{aligned} \quad (35)$$

Then, since $h_{s_{t-1}} \rightarrow 0$, it is easily checked using (35) that, under the assumptions on $(\mu_t)_{t \geq 0}$ and with $C_{\delta, \bar{\delta}}^{\mu_{s_t} * \tilde{\pi}_{s_t}}$ as defined in Lemma 4, for t large enough we have

$$C_{\delta, \bar{\delta}}^{\mu_{s_t} * \tilde{\pi}_{s_t}} \geq \mathbb{P}(U_{s_t} \in B_\kappa(0) \mid \tilde{\pi}_{s_t}(B_\kappa(\theta_*))) \geq C^{-1} \tilde{\pi}_{s_t}(B_\kappa(\theta_*)), \quad \mathbb{P} - a.s. \quad (36)$$

Let $(\phi'_t)_{t \geq 1}$ be as in Lemma 6, $\beta \in (0, 1)$ and, for every $t \geq 1$, $\tilde{\phi}_t(Y_{1:t})$ be such that $\tilde{\phi}_t(Y_{1:t}) = 1$ whenever $\tilde{\pi}_t(V_\kappa) \geq \beta^{-1}$ and such that $\tilde{\phi}_t(Y_{1:t}) = 0$ otherwise. Notice that $\mathbb{E}[\phi'_{\tau_k}(Y_{1:\tau_k})] \rightarrow 0$ by Lemma 6 while $\mathbb{E}[\tilde{\phi}_{s_{\tau_k}}(Y_{1:s_{\tau_k}})] \rightarrow 0$ by Lemma 7.

Therefore, using the shorthand $\Theta_{\tau_k} = \Theta_{\gamma_{\tau_k}, s_{\tau_k} : \tau_k}$,

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \mathbb{E}[\tilde{\pi}_{\tau_k}(V_\epsilon)] \\ & \leq \limsup_{k \rightarrow \infty} \mathbb{E}[(1 - \phi'_{\tau_k}(Y_{1:\tau_k}))(1 - \tilde{\phi}_{s_{\tau_k}}(Y_{1:s_{\tau_k}}))\mathbf{1}_{\Theta_{\tau_k}}(U_{s_{\tau_k}:\tau_k})\pi'_{s_{\tau_k}, \tau_k}(V_\epsilon)] \\ & + \limsup_{k \rightarrow \infty} \mathbb{E}[(1 - \phi'_{\tau_k}(Y_{1:\tau_k}))(1 - \tilde{\phi}_{s_{\tau_k}}(Y_{1:s_{\tau_k}}))\mathbf{1}_{\Theta_{\tau_k}^c}(U_{s_{\tau_k}:\tau_k})\pi'_{s_{\tau_k}, \tau_k}(V_\epsilon)]. \end{aligned} \quad (37)$$

We now remark that, using Lemma 7, (36) and Lemma 4, and because $s_{\tau_k} = t_{p_k}$ for all $k \geq 1$, we can without loss of generality assume that $(\phi'_t)_{t \geq 1}$ is such that, for all $k \geq 1$,

$$\begin{aligned} & \mathbb{P}\left(\int_{\Theta} \mathbb{E}\left[(\mu_{t_{p_k}} * \tilde{\pi}_{t_{p_k}})(\theta - \sum_{s=s_{\tau_k}+1}^{\tau_k-1} U_s) \prod_{s=s_{\tau_k}+1}^{\tau_k} (\tilde{f}_{\theta - \sum_{i=s}^{\tau_k-1} U_i} / f_{\theta_*})(Y_s) | \mathcal{F}_{\tau_k}\right] d\theta\right) \\ & > C^{-1} \mathbb{P}(U_{s_{\tau_k}:\tau_k} \in \Theta_{\tilde{\delta}, s_{\tau_k}:\tau_k}) e^{-C(\tau_k - s_{\tau_k})\delta^2} | \phi'_{\tau_k}(Y_{1:\tau_k}) = 0 = 1. \end{aligned} \quad (38)$$

Then, following the computations in the proof of Lemma 7, with (38) used in place of (26), we obtain for k large enough

$$\begin{aligned} & \mathbb{E}[(1 - \phi'_{\tau_k}(Y_{1:\tau_k}))(1 - \tilde{\phi}_{s_{\tau_k}}(Y_{1:s_{\tau_k}}))\mathbf{1}_{\Theta_{\tau_k}}(U_{s_{\tau_k}:\tau_k})\pi'_{s_{\tau_k}, \tau_k}(V_\epsilon)] \\ & \leq \frac{C e^{-(\tau_k - s_{\tau_k})C^{-1}}}{\mathbb{P}(U_{s_{\tau_k}:\tau_k} \in \Theta_{\tilde{\delta}, s_{\tau_k}:\tau_k})} \end{aligned} \quad (39)$$

and

$$\begin{aligned} & \mathbb{E}[(1 - \phi'_{\tau_k}(Y_{1:\tau_k}))(1 - \tilde{\phi}_{s_{\tau_k}}(Y_{1:s_{\tau_k}}))\mathbf{1}_{\Theta_{\tau_k}^c}(U_{s_{\tau_k}:\tau_k})\pi'_{s_{\tau_k}, \tau_k}(V_\epsilon)] \\ & \leq \frac{C e^{(\tau_k - s_{\tau_k})C}}{\mathbb{P}(U_{s_{\tau_k}:\tau_k} \in \Theta_{\tilde{\delta}, s_{\tau_k}:\tau_k})} \mathbb{P}(U_{s_{\tau_k}:\tau_k} \notin \Theta_{\tau_k}). \end{aligned} \quad (40)$$

Therefore, using (37), (39) and (40), to conclude the proof it is enough to show that

$$e^{-(\tau_k - s_{\tau_k})C^{-1}} \rightarrow 0 \quad (41)$$

$$e^{(\tau_k - s_{\tau_k})C} \mathbb{P}(U_{s_{\tau_k}:\tau_k} \notin \Theta_{\gamma_{\tau_k}, s_{\tau_k}:\tau_k}) \rightarrow 0. \quad (42)$$

Since $\liminf_{k \rightarrow \infty} (\tau_k - s_{\tau_k}) \geq \liminf_{k \rightarrow \infty} (v_{p_k+1} - t_{p_k}) = \infty$ it follows that (41) holds.

To show (42) remark that

$$\begin{aligned} e^{(\tau_k - s_{\tau_k})C} \mathbb{P}(U_{s_{\tau_k}:\tau_k} \notin \Theta_{\gamma_{\tau_k}, s_{\tau_k}:\tau_k}) & = e^{(\tau_k - s_{\tau_k})C} \mathbb{P}([U_{(s_{\tau_k}+1):\tau_k}] \geq \gamma_{\tau_k}) \\ & \leq C e^{(t_{p_k+1} - t_{p_k})C} \exp\left(-\frac{\gamma_{\tau_k}}{C \sum_{s=t_{p_k}+1}^{\tau_k-1} h_s^2}\right) \\ & \leq C e^{(t_{p_k+1} - t_{p_k})C} \exp\left(-\frac{\gamma_{\tau_k}}{C \sum_{s=t_{p_k}+1}^{t_{p_k+1}-1} h_s^2}\right) \end{aligned}$$

where the first inequality uses (33). As shown in the proof of Lemma 7, the term on the r.h.s. of the last inequality sign converges to 0 as $k \rightarrow \infty$ when $\gamma_{\tau_k} \rightarrow 0$ sufficiently slowly, and thus (42) holds. The proof is complete. \square

A.5.6. Proof of Lemma 9

Below $C \in (0, \infty)$ is a constant whose value can change from one expression to another.

For every $p \geq 1$ let

$$\begin{aligned}\xi_p &= 1 \wedge \left(\frac{\log(h_{t_p}^{-\nu})^{1/2} \wedge (t_p - t_{p-1})^{1/2}}{\log(h_{t_p}^{-\nu})} \right) \\ v_p &= t_{p-1} + \lfloor \log(h_{t_{p-1}}^{-\nu \xi_{p-1}}) \rfloor \wedge (t_p - t_{p-1} - 1)\end{aligned}$$

so that

$$\xi_p \rightarrow 0, \quad \log(h_{t_{p-1}}^{-\nu \xi_{p-1}}) \rightarrow \infty, \quad \frac{\log(h_{t_{p-1}}^{-\nu \xi_{p-1}})}{t_p - t_{p-1}} \rightarrow 0 \quad (43)$$

while $(v_p)_{p \geq 1}$ verifies the conditions of Lemma 8.

For every $q \geq 1$ let $p_q = \sup\{p \geq 0 : t_p < \tau'_q\}$ and note that, using (43),

$$\begin{aligned}\liminf_{q \rightarrow \infty} (\tau'_q - v_{p_q}) &\geq \liminf_{q \rightarrow \infty} (t_{p_q} - v_{p_q}) \\ &= \liminf_{p \rightarrow \infty} (t_p - v_p) \\ &\geq \liminf_{p \rightarrow \infty} (t_p - t_{p-1}) \left(1 - \frac{\log(h_{t_{p-1}}^{-\nu \xi_{p-1}})}{t_p - t_{p-1}} \right) \\ &= \infty.\end{aligned} \quad (44)$$

Note also that, under the assumptions of the lemma, there exists a $p_1 \in \mathbb{N}$ such that

$$\log(h_{t_{p-1}}^{-\nu})^{1/2} > 1, \quad \forall p \geq p_1. \quad (45)$$

To simplify the presentation of the proof we assume without loss of generality that $p_1 = 1$ in what follows.

We now let $(s_t)_{t \geq 0}$ be a sequence in \mathbb{N}_0 such that $\inf_{t \geq 1} (t - s_t) \geq 1$, $(t - s_t) \rightarrow \infty$ and

$$s_{\tau'_q} = (\tau'_q - \lfloor \log(h_{t_{p_q}}^{-\nu})^{1/2} \rfloor) \vee v_{p_q}, \quad \forall q \geq 1.$$

Notice that such a sequence $(s_t)_{t \geq 0}$ exists by (44) and because we are assuming that (45) holds with $p_1 = 1$. Note also that $(s_t)_{t \geq 1}$ is such that $v_{p_q} \leq s_{\tau'_q} \leq t_{p_q}$ for all $q \geq 1$.

We now show that

$$\mathbb{P}([U_{(s_{\tau'_q}+1):\tau'_q}] \geq \gamma) \rightarrow 0, \quad \forall \gamma > 0. \quad (46)$$

To this aim let $\gamma > 0$ and remark that

$$\begin{aligned}\mathbb{P}([U_{(s_{\tau'_q}+1):\tau'_q}] \geq \gamma) &\leq \mathbb{P}\left(\max_{s_{\tau'_q} < s < \tau'_q} \left\| \sum_{i=s}^{\tau'_q-1} U_i - U_{t_{p_q}} \mathbf{1}(s \leq t_{p_q}) \right\| \geq \gamma/2 \right) \\ &\quad + \mathbb{P}(\|U_{t_{p_q}}\| \geq \gamma/2).\end{aligned} \quad (47)$$

where, using the fact that

$$\int_a^\infty \left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}} dx \leq \nu^{\frac{\nu+1}{2}} \int_a^\infty x^{-(\nu+1)} dx = \nu^{\frac{\nu-1}{2}} a^{-\nu} \quad \forall a > 0, \quad (48)$$

we have $\mathbb{P}(\|U_{t_{p_q}}\| \geq \gamma/2) \rightarrow 0$ since $h_t \rightarrow 0$. In addition, by Doob's martingale inequality and under the assumptions on $(\mu_t)_{t \geq 0}$,

$$\limsup_{q \rightarrow \infty} \mathbb{P}\left(\max_{s_{\tau'_q} < s < \tau'_q} \left\| \sum_{i=s}^{\tau'_q-1} U_i - U_{t_{p_q}} \mathbf{1}(s \leq t_{p_q}) \right\| \geq \frac{\gamma}{2}\right) \leq C \limsup_{q \rightarrow \infty} \frac{\sum_{s=t_{p_{q-1}+1}}^{t_{p_q+1}-1} h_s^2}{(\gamma/2)^2} = 0.$$

Hence, (46) holds showing that there exists a sequence $(\gamma_t)_{t \geq 1}$ in $\mathbb{R}_{>0}$ such that

$$\gamma_t \rightarrow 0, \quad \mathbb{P}([U_{(s_{\tau'_q}+1):\tau'_q}] \geq \gamma_{\tau'_q}) \rightarrow 0. \quad (49)$$

Let $(\gamma_t)_{t \geq 1}$ be as in (49), $(\tau_k)_{k \geq 1}$ be as defined in Lemma 8, $(\tilde{\tau}_r)_{r \geq 1}$ be a strictly increasing sequence in \mathbb{N}_0 such that $(\tilde{\tau}_r)_{r \geq 1} = (\tau_k)_{k \geq 1} \cup (t_p)_{p \geq 0}$ and note that, by Lemmas 7-8 ,

$$\mathbb{E}[\tilde{\pi}_{\tilde{\tau}_r}(V_\kappa)] \rightarrow 0, \quad \forall \kappa > 0. \quad (50)$$

Remark also that, by construction, $(s_{\tau'_q})_{q \geq 1} \subset (\tilde{\tau}_r)_{r \geq 1}$ so that we can now follow the computations in the proof of Lemma 8. As in this latter let $\tilde{\delta} = \kappa$ for some sufficiently small $\kappa > 0$ (see the proof of Lemma 8 for the expression of κ). Then, as shown in the proof of Lemma 8 we have,

$$\begin{aligned} \limsup_{q \rightarrow \infty} \mathbb{E}[\tilde{\pi}_{\tau'_q}(V_\epsilon)] &\leq \limsup_{q \rightarrow \infty} \frac{C e^{-(\tau'_q - s_{\tau'_q})C^{-1}}}{\mathbb{P}(U_{s_{\tau'_q}:\tau'_q} \in \Theta_{\tilde{\delta}, s_{\tau'_q}:\tau'_q})} \\ &\quad + \limsup_{q \rightarrow \infty} \frac{C e^{(\tau'_q - s_{\tau'_q})C}}{\mathbb{P}(U_{s_{\tau'_q}:\tau'_q} \in \Theta_{\tilde{\delta}, s_{\tau'_q}:\tau'_q})} \mathbb{P}(U_{s_{\tau'_q}:\tau'_q} \notin \Theta_{\gamma_{\tau'_q}, s_{\tau'_q}:\tau'_q}) \end{aligned}$$

so that to conclude the proof it is enough to show that

$$e^{(\tau'_q - s_{\tau'_q})C^{-1}} \rightarrow 0 \quad (51)$$

$$e^{(\tau'_q - s_{\tau'_q})C} \mathbb{P}(U_{(s_{\tau'_q}+1):\tau'_q} \notin \Theta_{\gamma_{\tau'_q}, s_{\tau'_q}:\tau'_q}) \rightarrow 0. \quad (52)$$

Since $(\tau'_q - s_{\tau'_q}) \rightarrow \infty$ it follows that (51) holds.

To show (52) remark that

$$\begin{aligned} &e^{(\tau'_q - s_{\tau'_q})C} \mathbb{P}(U_{s_{\tau'_q}:\tau'_q} \notin \Theta_{\gamma_{\tau'_q}, s_{\tau'_q}:\tau'_q}) \\ &= e^{(\tau'_q - s_{\tau'_q})C} \mathbb{P}([U_{(s_{\tau'_q}+1):\tau'_q}] \geq \gamma_{\tau'_q}) \\ &\leq e^{(\tau'_q - s_{\tau'_q})C} \mathbb{P}\left(\max_{s_{\tau'_q} < s < \tau'_q} \left\| \sum_{i=s}^{\tau'_q-1} U_i - U_{t_{p_q}} \mathbf{1}(s \leq t_{p_q}) \right\| \geq \gamma_{\tau'_q}/2\right) \\ &\quad + e^{(\tau'_q - s_{\tau'_q})C} \mathbb{P}(\|U_{t_{p_q}}\| \geq \gamma_{\tau'_q}/2). \end{aligned} \quad (53)$$

In addition,

$$\tau'_q - s_{\tau'_q} \leq \log(h_{t_{p_q}}^{-\nu \xi_{p_q}}) \leq 2(t_{p_{q+1}} - t_{p_q}), \quad (54)$$

where the second inequality holds by (43) and for q large enough.

Therefore, using (33),

$$\begin{aligned} & \limsup_{q \rightarrow \infty} e^{(\tau'_q - s_{\tau'_q})C} \mathbb{P}\left(\max_{s_{\tau'_q} < s < \tau'_q} \left\| \sum_{i=s}^{\tau'_q-1} U_i - U_{t_{p_q}} \mathbf{1}(s \leq t_{p_q}) \right\| \geq \gamma_{\tau'_q}/2\right) \\ & \leq C \limsup_{q \rightarrow \infty} e^{(t_{p_{q+1}} - t_{p_q})C} \exp\left(-\frac{\gamma_{\tau'_q}}{C \sum_{s=t_{p_q-1}+1}^{t_{p_{q+1}}-1} h_s^2}\right) \\ & = 0 \end{aligned} \quad (55)$$

where the equality holds assuming without loss of generality that $\gamma_{\tau'_q} \rightarrow 0$ sufficiently slowly and uses the fact that, by assumption,

$$\limsup_{p \rightarrow \infty} \frac{t_{p+2} - t_{p+1}}{t_{p+1} - t_p} < \infty.$$

Using (48) we have

$$\mathbb{P}(\|U_{t_p}\| \geq \gamma) \leq C \left(\frac{h_{t_p}}{\gamma}\right)^\nu, \quad \forall p \geq 1, \quad \forall \gamma > 0.$$

Remark also that, using the first inequality in (54) and recalling that $\xi_p \rightarrow 0$,

$$\limsup_{p \rightarrow \infty} h_{t_{p_q}}^\nu e^{(\tau'_p - s_{\tau'_p})C} \leq \limsup_{p \rightarrow \infty} h_{t_{p_q}}^{\nu(1-C\xi_{p_q})} = 0$$

and therefore, assuming without loss of generality that $\gamma_{\tau'_q} \rightarrow 0$ sufficiently slowly,

$$\limsup_{p \rightarrow \infty} e^{(\tau'_q - s_{\tau'_q})C} \mathbb{P}(\|U_{t_{p_q}}\| \geq \gamma_{\tau'_q}/2) = 0.$$

Together with (53) and (55) this last result shows that (52) holds. The proof is complete.

A.6. Proof of Lemma 1

A.6.1. Preliminary results

We first show the following simple result.

Lemma 10. *Let $(t_p)_{p \geq 0}$ be a subsequence of \mathbb{N}_0 and $g : Y \rightarrow \mathbb{R}$ be a measurable function such that $\mathbb{E}[|g(Y_1)|] < \infty$. Then, as $p \rightarrow \infty$,*

$$\mathbb{P}\left(\max_{0 < i \leq p} \frac{1}{t_i - t_{i-1}} \left| \sum_{s=t_{i-1}+1}^{t_i} (g(Y_s) - \mathbb{E}[g(Y_1)]) \right| \geq t_p^\delta\right) \rightarrow 0, \quad \forall \delta > 1.$$

Proof. Let $\delta > 1$ so that, using Markov's inequality for the second inequality,

$$\begin{aligned}
& \limsup_{p \rightarrow \infty} \mathbb{P} \left(\max_{0 < i \leq p} \frac{1}{t_i - t_{i-1}} \left| \sum_{s=t_{i-1}+1}^{t_i} (g(Y_s) - \mathbb{E}[g(Y_1)]) \right| \geq t_p^\delta \right) \\
& \leq \limsup_{p \rightarrow \infty} \sum_{i=1}^p \mathbb{P} \left(\frac{1}{t_i - t_{i-1}} \left| \sum_{s=t_{i-1}+1}^{t_i} (g(Y_s) - \mathbb{E}[g(Y_1)]) \right| \geq t_p^\delta \right) \\
& \leq \limsup_{p \rightarrow \infty} \frac{2\mathbb{E}[|g(Y_1)|]}{t_p^{\delta-1}} \\
& = 0.
\end{aligned}$$

□

We also recall the following result (Ferber, 2014, Theorem 1.2).

Lemma 11. *Let $(X_i)_{i \geq 1}$ be a sequence of i.i.d. random variables such that $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^{2p}] < \infty$ for some $p \in \mathbb{N}$. Then $\mathbb{E}[(\sum_{i=1}^n X_i)^{2p}] = \mathcal{O}(n^p)$.*

A.6.2. Proof of the lemma

Proof. Let $\beta > \max(\beta_\star, 1/\alpha)$ and $D_p = h_p^{-\beta}$ for all $p \geq 1$.

We first establish the result of the theorem under Assumption A6.1, and start by showing that

$$\mathbb{P} \left(\max_{0 < i \leq p} e^{-(t_i - t_{i-1})\zeta(D_p)/2} \prod_{s=t_{i-1}+1}^{t_i} \sup_{\theta_s \in V_{D_p}} (\tilde{f}_{\theta_s}/f_{\theta_\star})(Y_s) < 1 \right) \rightarrow 1. \quad (56)$$

Let $p_1 \in \mathbb{N}$ be such that $\zeta(D_p) < 0$ for all $p \geq p_1$; notice that such a p_1 exists under Assumption A6.1. Then, for every $p \geq p_1$ we have

$$\begin{aligned}
& \mathbb{P} \left(\max_{0 < i \leq p} e^{-(t_i - t_{i-1})\zeta(D_p)/2} \prod_{s=t_{i-1}+1}^{t_i} \sup_{\theta_s \in V_{D_p}} (\tilde{f}_{\theta_s}/f_{\theta_\star})(Y_s) \geq 1 \right) \\
& = \mathbb{P} \left(\max_{0 < i \leq p} \sum_{s=t_{i-1}+1}^{t_i} \left\{ \sup_{\theta_s \in V_{D_p}} \log \left(\frac{\tilde{f}_{\theta_s}}{f_{\theta_\star}}(Y_s) \right) - \zeta(D_p) + \frac{\zeta(D_p)}{2} \right\} \geq 0 \right) \\
& \leq \sum_{i=1}^p \mathbb{P} \left(\sum_{s=t_{i-1}+1}^{t_i} \left(\sup_{\theta_s \in V_{D_p}} \log \left(\frac{\tilde{f}_{\theta_s}}{f_{\theta_\star}}(Y_s) \right) - \zeta(D_p) \right) > (t_i - t_{i-1})|\zeta(D_p)|/2 \right) \\
& = \sum_{i=1}^p \mathbb{P} \left(\frac{1}{t_i - t_{i-1}} \sum_{s=t_{i-1}+1}^{t_i} \left(\sup_{\theta_s \in V_{D_p}} \log \left(\frac{\tilde{f}_{\theta_s}}{f_{\theta_\star}}(Y_s) \right) - \zeta(D_p) \right) > |\zeta(D_p)|/2 \right).
\end{aligned} \quad (57)$$

Using Markov's inequality, Assumption A6.1 and Lemma 11, there exists a constant $C \in (0, \infty)$ such that, for all $i \in 1 : p$,

$$\begin{aligned} & \mathbb{P}\left(\frac{1}{t_i - t_{i-1}} \sum_{s=t_{i-1}+1}^{t_i} \left(\sup_{\theta_s \in V_{D_p}} \log \left(\frac{\tilde{f}_{\theta_s}}{f_{\theta_*}}(Y_s) \right) - \zeta(D_p) \right) > |\zeta(D_p)|/2\right) \\ & \leq 2^{2k_*} \frac{\mathbb{E}\left[\left| \frac{1}{t_i - t_{i-1}} \sum_{s=t_{i-1}+1}^{t_i} \left(\sup_{\theta_s \in V_{D_p}} \log \left(\frac{\tilde{f}_{\theta_s}}{f_{\theta_*}}(Y_s) \right) - \zeta(D_p) \right) \right|^{2k_*}\right]}{\zeta(D_p)^{2k_*}} \\ & \leq \frac{C}{(t_i - t_{i-1})^{k_* \mathbb{1}_{\mathbb{N}}(k_*)} |\zeta(D_p)|^{2k_*}}. \end{aligned} \quad (58)$$

By assumptions, $|\zeta(D_p)|^{-2k_*} \sum_{i=1}^p (t_i - t_{i-1})^{-k_* \mathbb{1}_{\mathbb{N}}(k_*)} \rightarrow 0$ which, together with (57) and (58), implies (56).

To proceed further let $\tilde{C}_* \in (0, \infty)$ be as in Lemma 4, $\delta_p = t_p^{-\gamma}$ for some $\gamma > \max(\alpha, 1/2)$, $\delta > 0$ and, for all $i \geq 1$ and $s \in t_{i-1} : (t_i - 1)$, let $V_{i,s} = \sum_{j=s}^{t_i-1} U_j$.

Under Assumptions A1-A2 and by Lemma 4, for p large enough (i.e. for δ_p small enough) and all $i \in 1 : p$, we have

$$\begin{aligned} g_i(Y_{1:t_i}) & := \int_{\Theta} \mathbb{E}[(\mu_{t_{i-1}} * \tilde{\pi}_{t_{i-1}})(\theta - V_{i,t_{i-1}+1}) \prod_{s=t_{i-1}+1}^{t_i} (\tilde{f}_{\theta - V_{i,s}} / f_{\theta_*})(Y_s) | \mathcal{F}_{t_i}] d\theta \\ & \leq \frac{\mathbb{P}(U_{t_{i-1}:t_i} \in \Theta_{\delta_p, t_{i-1}:t_i})}{e^{(t_i - t_{i-1})(2(\delta_p \tilde{C}_*)^2 + \delta)}} \inf_{v \in B_{\delta_p}(0)} (\mu_{t_{i-1}} * \tilde{\pi}_{t_{i-1}})(B_{\delta_p}(\theta_* - v)) \\ & =: \tilde{g}_{i,p}(Y_{1:t_i}) \end{aligned}$$

with probability at most $p_{i,p} := ((t_i - t_{i-1})\{(\tilde{C}_* \delta_p) + (\tilde{C}_* \delta_p)^{-1} \delta\}^2)^{-1}$.

Then, noting that $\sum_{i=1}^p (t_i - t_{i-1})^{-1} \leq t_p$, it follows that

$$\begin{aligned} \limsup_{p \rightarrow \infty} \mathbb{P}\left(\min_{1 \leq i \leq p} \frac{g_i(Y_{1:t_i})}{\tilde{g}_{i,p}(Y_{1:t_i})} \leq 1\right) & \leq \limsup_{p \rightarrow \infty} \sum_{i=1}^p p_{i,p} \\ & \leq \limsup_{p \rightarrow \infty} \frac{\delta_p^2 \tilde{C}_*^2}{\delta^2} \sum_{i=1}^p (t_i - t_{i-1})^{-1} \\ & \leq \frac{\tilde{C}_*^2}{\delta^2} \limsup_{p \rightarrow \infty} t_p^{-2\gamma} t_p \\ & = 0 \end{aligned} \quad (59)$$

where the last equality holds since $\gamma > 1/2$.

Therefore, by (56) and (59), there exists a sequence $(Y_p)_{p \geq 1}$, with $Y_p \subset \mathcal{Y}^{\otimes t_p}$ for all $p \geq 1$, such that $\mathbb{P}(Y_{1:t_p} \in Y_p) \rightarrow 1$ and such that, for all $p \geq 1$,

$$\max_{0 < i \leq p} e^{-(t_i - t_{i-1})\zeta(D_p)/2} \prod_{s=t_{i-1}+1}^{t_i} \sup_{\theta_s \in V_{D_p}} \frac{\tilde{f}_{\theta_s}}{f_{\theta_*}}(y_s) < 1, \quad \min_{1 \leq i \leq p} \frac{g_i(y_{1:t_i})}{\tilde{g}_{i,p}(y_{1:t_i})} > 1, \quad \forall y_{1:t_p} \in Y_p.$$

To proceed further let $\epsilon \in (0, 1)$ and note that, since $\mathbb{P}(Y_{1:t_p} \in Y_p) \rightarrow 1$,

$$\limsup_{p \rightarrow \infty} \mathbb{P}(\tilde{\pi}_{t_p}(V_{2D_p}) \geq \epsilon) \leq \limsup_{p \rightarrow \infty} \mathbb{P}(\{\tilde{\pi}_{t_p}(V_{2D_p}) \geq \epsilon\} \cap \{Y_{1:t_p} \in Y_p\}) \quad (60)$$

so that to prove the theorem under Assumption A6.1 it remains to show that

$$\limsup_{p \rightarrow \infty} \mathbb{P}(\{\tilde{\pi}_{t_p}(V_{2D_p}) \geq \epsilon\} \cap \{Y_{1:t_p} \in Y_p\}) = 0.$$

To this aim let $p_0 \in \mathbb{N}$ and, for every $p > p_0$, let $(Y_{i,p})_{i=0}^p$ be a sequence in $\mathcal{Y}^{\otimes t_p}$ such that $Y_{0,p} = Y^{t_p}$, $Y_{p,p} = Y_p$ and such that, for every $i \in 1 : p$, $Y_{i,p} \subset \cap_{j=0}^{i-1} Y_{j,p}$ and

$$\max_{0 < j \leq i} e^{-(t_j - t_{j-1})\zeta(D_p)/2} \prod_{s=t_{j-1}+1}^{t_j} \sup_{\theta_s \in V_{D_p}} \frac{\tilde{f}_{\theta_s}}{f_{\theta_*}}(y_s) < 1, \quad \min_{1 \leq j \leq i} \frac{g_j(y_{1:t_j})}{\tilde{g}_{j,p}(y_{1:t_j})} > 1, \quad \forall y_{1:t_p} \in Y_{i,p}.$$

Let $A_{i,p} = \{\tilde{\pi}_{t_i}(V_{2D_p}) \geq \epsilon\} \cap \{Y_{1:t_p} \in Y_{i,p}\}$ and $\tilde{A}_{i,p} = \{\tilde{\pi}_{t_i}(V_{2D_p}) \mathbb{1}_{Y_{i,p}}(Y_{1:t_i}) \geq \epsilon\}$ for all $p > p_0$ and all $i \in 1 : p$.

Then, for every $p > p_0$ we have

$$\begin{aligned} \mathbb{P}(\{\tilde{\pi}_{t_p}(V_{2D_p}) \geq \epsilon\} \cap \{Y_{1:t_p} \in Y_p\}) &= \mathbb{P}(A_{p,p}) \\ &\leq \mathbb{P}(A_{p,p} | A_{p-1,p}^c) + \mathbb{P}(A_{p-1,p}) \\ &\leq \sum_{i=p_0+1}^p \mathbb{P}(A_{i,p} | A_{i-1,p}^c) + \mathbb{P}(A_{p_0,p}) \quad (61) \\ &= \sum_{i=p_0+1}^p \mathbb{P}(\tilde{A}_{i,p} | A_{i-1,p}^c) + \mathbb{P}(A_{p_0,p}) \end{aligned}$$

and we now study $\mathbb{P}(\tilde{A}_{i,p} | A_{i-1,p}^c)$ for all $i \in (p_0 + 1) : p$.

Let $i \in (p_0 + 1) : p$ and

$$\begin{aligned} X_{i,p}^{(1)} &= \int_{V_{2D_p}} \mathbb{E} \left[\mathbb{1}([U_{(t_{i-1}+1):t_i}] < D_p) (\mu_{t_{i-1}} * \tilde{\pi}_{t_{i-1}})(\theta - V_{i,t_{i-1}+1}) \prod_{s=t_{i-1}+1}^{t_i} \frac{\tilde{f}_{\theta - V_{i,s}}(Y_s) | \mathcal{F}_{t_i}}{f_{\theta_*}} \right] d\theta \\ X_{i,p}^{(2)} &= \int_{V_{2D_p}} \mathbb{E} \left[\mathbb{1}([U_{(t_{i-1}+1):t_i}] \geq D_p) (\mu_{t_{i-1}} * \tilde{\pi}_{t_{i-1}})(\theta - V_{i,t_{i-1}+1}) \prod_{s=t_{i-1}+1}^{t_i} \frac{\tilde{f}_{\theta - V_{i,s}}(Y_s) | \mathcal{F}_{t_i}}{f_{\theta_*}} \right] d\theta \end{aligned}$$

so that, by Lemma 3,

$$\tilde{\pi}_{t_i}(V_{2D_p}) = \frac{X_{i,p}^{(1)} + X_{i,p}^{(2)}}{g_i(Y_{1:t_i})}, \quad \mathbb{P} - a.s.$$

Then, by Markov's inequality and using the definition of $Y_{i,p}$, we have

$$\begin{aligned}\mathbb{P}(\tilde{A}_{i,p}|A_{i-1,p}^c) &\leq \epsilon^{-1}\mathbb{E}\left[\frac{(X_{i,p}^{(1)} + X_{i,p}^{(2)})\mathbb{1}_{Y_{i,p}}(Y_{1:t_p})}{g_i(Y_{1:t_i})}\middle|A_{i-1,p}^c\right] \\ &\leq \epsilon^{-1}\mathbb{E}\left[\frac{(X_{i,p}^{(1)} + X_{i,p}^{(2)})\mathbb{1}_{Y_{i,p}}(Y_{1:t_p})}{\tilde{g}_{i,p}(Y_{1:t_i})}\middle|A_{i-1,p}^c\right] \\ &= \epsilon^{-1}\mathbb{E}\left[\frac{(X_{i,p}^{(1)} + X_{i,p}^{(2)})\mathbb{1}_{Y_{i,p}}(Y_{1:t_p})}{\tilde{g}_{i,p}(Y_{1:t_i})}\middle|\tilde{\pi}_{t_{i-1}}(V_{2D_p}) < \epsilon\right]\end{aligned}$$

where the equality uses the definition of $A_{i-1,p}$ and the fact that $Y_{i,p} \subset Y_{i-1,p}$.

Following similar computations as in the proof of Lemma 7 (see (24)), it is readily checked that there exists a constant $C_1 \in (0, \infty)$ such that

$$\mathbb{P}\left(\inf_{v \in B_{\delta_p}(0)} (\mu_{t_{i-1}} * \tilde{\pi}_{t_{i-1}})(B_{\delta_p}(\theta_* - v)) \geq C_1^{-1} \delta_p^d h_{t_{i-1}}^\nu h_{t_{p-1}}^{\beta(\nu+d)} \middle| \tilde{\pi}_{t_{i-1}}(V_{2D_p}) < \epsilon\right) = 1$$

and thus

$$\begin{aligned}\mathbb{P}(\tilde{A}_{i,p}|A_{i-1,p}^c) &\leq \frac{C_1 \delta_p^{-d} h_{t_{i-1}}^{-\nu} h_{t_{p-1}}^{-\beta(\nu+d)} e^{(t_i - t_{i-1})(2(\delta_p \tilde{C}_*)^2 + \delta)}}{\epsilon \mathbb{P}(U_{t_{i-1}:t_i} \in \Theta_{\delta_p, t_{i-1}:t_i})} \\ &\quad \times \mathbb{E}[(X_{i,p}^{(1)} + X_{i,p}^{(2)})\mathbb{1}_{Y_{i,p}}(Y_{1:t_i}) \middle| \tilde{\pi}_{t_{i-1}}(V_{2D_p}) < \epsilon].\end{aligned}\tag{62}$$

Next, noting that $X_{i,p}^{(1)} \leq \prod_{s=t_{i-1}+1}^{t_i} \sup_{\theta_s \in V_{D_p}} (\tilde{f}_{\theta_s}/f_{\theta_*})(Y_s)$, \mathbb{P} -a.s., it follows, by the definition of $Y_{i,p}$, that

$$\mathbb{E}[X_{i,p}^{(1)}\mathbb{1}_{Y_{i,p}}(Y_{1:t_i}) \middle| \tilde{\pi}_{t_{i-1}}(V_{2D_p}) < \epsilon] \leq e^{(t_i - t_{i-1})\zeta(D_p)/2}.\tag{63}$$

We now show that under Assumption A5 there exists a constant $C_2 \in (0, \infty)$ such that

$$\mathbb{E}[X_{i,p}^{(2)}\mathbb{1}_{Y_{i,p}}(Y_{1:t_i}) \middle| \tilde{\pi}_{t_{i-1}}(V_{2D_p}) < \epsilon] \leq C_2 e^{-C_2^{-1}(t_i - t_{i-1})D_p^2}.\tag{64}$$

As preliminary computations to establish (64) remark that, under the assumptions on $(\mu_t)_{t \geq 0}$ and for some constant $C_3 \in (0, \infty)$ we have, by (33),

$$\begin{aligned}\mathbb{P}([U_{(t_{i-1}+1):t_i}] \geq D_p) &\leq C_3 \exp\left(-\frac{D_p^2}{C_3 \sum_{j=t_{i-1}+1}^{t_i-1} h_j^2}\right) \\ &= C_3 \exp\left(-\frac{(t_i - t_{i-1})D_p^2}{C_3(t_i - t_{i-1}) \sum_{j=t_{i-1}+1}^{t_i-1} h_j^2}\right) \\ &\leq C_4 \exp(-C_4^{-1}(t_i - t_{i-1})D_p^2)\end{aligned}\tag{65}$$

where the last inequality holds for some constant $C_4 \in (0, \infty)$ since, by assumption, $(t_p - t_{p-1}) \sum_{i=t_{p-1}+1}^{t_p-1} h_i^2 \rightarrow 0$.

Assume first Assumption A5.1 and recall that β is such that $\beta\alpha > 1$. Then, by Lemma 10, for p_0 large enough, we can without loss of generality assume that, for all $p' > p_0$ and $j \in (p_0 + 1) : p'$, the set $\mathcal{Y}_{j,p'}$ is such that

$$\sum_{s=t_{j-1}+1}^{t_j} \sup_{\theta \in \Theta} \log((\tilde{f}_\theta/f_{\theta_*})(y_s)) \leq (t_j - t_{j-1})D_{p'}, \quad \forall y_{1:t_{j-1}} \in \mathcal{Y}_{j,p'}.$$

Then, using (65),

$$\begin{aligned} \mathbb{E}[X_{i,p}^{(2)} \mathbf{1}_{\mathcal{Y}_{i,p}}(Y_{1:t_i}) | \tilde{\pi}_{t_{i-1}}(V_{2D_p}) < \epsilon] &\leq \mathbb{P}([U_{(t_{i-1}+1):t_i}] \geq D_p) e^{(t_i - t_{i-1})D_p} \\ &\leq C_4 e^{-(t_i - t_{i-1})(C_4^{-1}D_p^2 - D_p)} \end{aligned}$$

which establishes (64).

Assume now Assumption A5.2 and let $c_\star = \log(\sup_{\theta \in \Theta} \mathbb{E}[f_\theta/f_{\theta_*}]) < \infty$. Then, using (65) we have, \mathbb{P} -a.s.

$$\begin{aligned} \mathbb{E}[X_{i,p}^{(2)} | \mathcal{F}_{t_{i-1}}] &\leq \mathbb{P}([U_{(t_{i-1}+1):t_i}] \geq D_p) \left(\sup_{\theta \in \Theta} \mathbb{E}[f_\theta/f_{\theta_*}] \right)^{t_i - t_{i-1}} \\ &= C_4 e^{-(t_i - t_{i-1})(C_4^{-1}D_p^2 - c_\star)} \end{aligned}$$

which shows (64).

Lastly, assume Assumption A5.3. Let $c_\star = \log(\sup_{\theta \in \Theta} \mathbb{E}[f_\theta]) < \infty$ and note that, by Lemma 10, for p_0 large enough we can without loss of generality assume that, for all $p' > p_0$ and $j \in (p_0 + 1) : p'$, the set $\mathcal{Y}_{j,p'}$ is such that

$$\sum_{s=t_{j-1}+1}^{t_j} \log(f_{\theta_*}(y_s)) \geq -(t_j - t_{j-1})D_{p'}, \quad \forall y_{1:t_{j-1}} \in \mathcal{Y}_{j,p'}.$$

Then, using (65) we have, \mathbb{P} -a.s.

$$\begin{aligned} \mathbb{E}[X_{i,p}^{(2)} | \mathcal{F}_{t_{i-1}}] &\leq e^{D_p(t_i - t_{i-1})} \mathbb{P}([U_{(t_{i-1}+1):t_i}] \geq D_p) \left(\sup_{\theta \in \Theta} \mathbb{E}[f_\theta] \right)^{t_i - t_{i-1}} \\ &= C_4 e^{-(t_i - t_{i-1})(C_4^{-1}D_p^2 - D_p - c_\star)} \end{aligned}$$

showing (64). This concludes to show that (64) holds under Assumption A5.

Then, using (62)-(64), we have

$$\begin{aligned} \mathbb{P}(\tilde{A}_{i,p} | A_{i-1,p}^c) &\leq C_1 \delta_p^{-d} h_{t_{i-1}}^{-\nu} h_{t_{p-1}}^{-\beta(\nu+d)} \frac{e^{(t_i - t_{i-1})(2(\delta_p \tilde{C}_\star)^2 + \delta + \zeta(D_p)/2)}}{\epsilon \mathbb{P}(U_{t_{i-1}:t_i} \in \Theta_{\delta_p, t_{i-1}:t_i})} \\ &\quad + C_1 C_2 \delta_p^{-d} h_{t_{i-1}}^{-\nu} h_{t_{p-1}}^{-\beta(\nu+d)} \frac{e^{(t_i - t_{i-1})(2(\delta_p \tilde{C}_\star)^2 + \delta - C_2^{-1}D_p^2)}}{\epsilon \mathbb{P}(U_{t_{i-1}:t_i} \in \Theta_{\delta_p, t_{i-1}:t_i})}. \end{aligned} \tag{66}$$

We now find a lower bound for $\mathbb{P}(U_{t_{i-1}:t_i} \in \Theta_{\delta_p, t_{i-1}:t_i})$. To this aim remark that, using the shorthand $\tilde{\delta}_p := d^{-1/2}\delta_p$,

$$\begin{aligned}
& \mathbb{P}(U_{t_{i-1}:t_i} \in \Theta_{\delta_p, t_{i-1}:t_i}) \\
&= \mathbb{P}\left(\left\|\sum_{j=s}^{t_i-1} U_s\right\| < \delta_p, \forall s \in (t_{i-1} + 1) : (t_i - 1)\right) \\
&\geq \mathbb{P}\left(\left\|\sum_{j=s}^{t_i-1} U_s\right\|_\infty < \tilde{\delta}_p, \forall s \in (t_{i-1} + 1) : (t_i - 2)\right) \\
&= \mathbb{P}\left(\|U_{t_{i-1}}\|_\infty < \tilde{\delta}_p\right) \prod_{s=t_{i-1}+1}^{t_i-2} \mathbb{P}\left(\left\|\sum_{j=s}^{t_i-1} U_s\right\|_\infty < \tilde{\delta}_p \mid \left\|\sum_{j=s+1}^{t_i-1} U_s\right\|_\infty < \tilde{\delta}_p\right).
\end{aligned} \tag{67}$$

Recall now that $\delta_p = t_p^{-\gamma}$ with $\gamma > \alpha$, implying that $\delta_p/h_{t_p} \rightarrow 0$. Hence, under the assumptions on $(\mu_t)_{t \geq 0}$ there exists a constant $c_1 > 0$ such that, for every $s \in 1 : p$ such that $h_s \neq 0$ and every $v \in \mathbb{R}^d$ such that $\|v\|_\infty < \tilde{\delta}_p$, we have

$$\begin{aligned}
\mathbb{P}\left(\left\|\sum_{j=s}^{t_i-1} U_s\right\|_\infty < \tilde{\delta}_p \mid \left\|\sum_{j=s+1}^{t_i-1} U_s\right\|_\infty = v\right) &= \mathbb{P}(\|U_s + v\|_\infty < \tilde{\delta}_p) \\
&= \mathbb{P}(-\tilde{\delta}_p \leq U_s + v < \tilde{\delta}_p) \\
&= \mathbb{P}\left(-\frac{\tilde{\delta}_p + v}{h_s} \leq \tilde{U}_s < \frac{\tilde{\delta}_p - v}{h_s}\right) \\
&\geq c_1 (\tilde{\delta}_p/h_s)^d.
\end{aligned}$$

Consequently, for every $s \in (t_{i-1} + 1) : (t_i - 2)$ such that $h_s \neq 0$ we have

$$\mathbb{P}\left(\left\|\sum_{j=s}^{t_i-1} U_s\right\|_\infty < \tilde{\delta}_p \mid \left\|\sum_{j=s+1}^{t_i-1} U_s\right\|_\infty < \tilde{\delta}_p\right) \geq c_1 (2\tilde{\delta}_p^2/h_s)^d. \tag{68}$$

The above computations also show that if $h_{t_{i-1}} > 0$ then $\mathbb{P}(\|U_{t_{i-1}}\|_\infty < \tilde{\delta}_p) \geq c'_1 (\tilde{\delta}_p/h_{t_{i-1}})^d$ for some constant $c'_1 > 0$ which, together with (67)-(68), shows that for some constant $c_2 > 0$,

$$\mathbb{P}(U_{t_{i-1}:t_i} \in \Theta_{\delta_p, t_{i-1}:t_i}) \geq (c_2 \delta_p^{2d})^{t_i - t_{i-1} - 2} \prod_{s=t_{i-1}+1}^{t_i-1} (h_s^{-d} \wedge 1/c_2) \geq \delta_p^{2d(t_i - t_{i-1})} \tag{69}$$

where the second inequality assumes without loss of generality that p_0 is such that $h_s^d < c_2$ for all $s > t_{p_0}$.

Combining (66) and (69), and recalling that $\delta_p \rightarrow 0$, that $|\zeta(D_p)| \rightarrow \infty$ and that $h_{t_p} < h_{t_{p-1}}$, it follows that for p_0 large enough we have, for all $p > p_0$ and all $i \in (p_0 + 1) : p$,

$$\begin{aligned}
\mathbb{P}(\tilde{A}_{i,p} | A_{i-1,p}^c) &\leq C_1 \epsilon^{-1} h_{t_{i-1}}^{-\nu} h_{t_p}^{-\beta(\nu+d)} e^{-(t_i - t_{i-1})(|\zeta(D_p)|/4 - 3d \log(\delta_p))} \\
&\quad + C_1 C_2 \epsilon^{-1} h_{t_{i-1}}^{-\nu} h_{t_p}^{-\beta(\nu+d)} e^{-(t_i - t_{i-1})(C_2^{-1} D_p^2/2 - 3d \log(\delta_p))}.
\end{aligned} \tag{70}$$

To proceed further we first note that, under Assumption A6 and the assumptions on $(h_t)_{t \geq 0}$, there exist constants $c'_\zeta > c_\zeta > 0$ such that, for p_0 is large enough,

$$|\zeta(D_p)| \geq c'_\zeta \log(D_p) = -c'_\zeta \beta \log(h_{t_p}) \geq c_\zeta \beta \alpha \log(t_p). \quad (71)$$

Therefore, recalling that $\delta_p = t_p^{-\gamma}$ with $\gamma > \max(1/2, \alpha)$, for p_0 large enough we have, for all $p > p_0$ and all $i \in (p_0 + 1) : p$,

$$-(t_i - t_{i-1})(|\zeta(D_p)|/4 - 3d \log(1/\delta_p)) \leq -\log(t_p)(t_i - t_{i-1})(c_\zeta \beta \alpha / 4 - 3d\gamma).$$

Next, let $\tilde{c} > 4\alpha\nu$ be sufficiently large so that

$$\beta' := \frac{4(\tilde{c} + 3d\gamma)}{c_\zeta \alpha} \geq \max(\beta_\star, 1/\alpha).$$

Then, the above computations show that, for every $\beta \geq \beta'$ and p_0 large enough we have, for all $p > p_0$ and all $i \in (p_0 + 1) : p$,

$$-(t_i - t_{i-1})(|\zeta(D_p)|/4 - 3d \log(1/\delta_p)) \leq -\tilde{c} \log(t_p)(t_i - t_{i-1}), \quad \forall i \in (p_0 + 1) : p. \quad (72)$$

We now take p_0 sufficiently large so that

$$t_i - t_{i-1} > \frac{\alpha\beta(\nu + d) + 1}{\tilde{c}} + \frac{1}{2}, \quad \forall i > p_0.$$

Then, using (72) and under the assumptions on $(h_t)_{t \geq 0}$, and for p_0 large enough, there exists a constant $C_3 \in (0, \infty)$ such that, for all $p > p_0$,

$$h_{t_p}^{-\beta(\nu+d)} e^{-(t_i - t_{i-1})(|\zeta(D_p)|/4 - 3d \log(1/\delta_p))} \leq C_3 t_p^{-\tilde{c}/2-1}, \quad \forall i \in (p_0 + 1) : p. \quad (73)$$

On the other hand, if p_0 is large enough then, using (71), $C_2^{-1} D_p^2 > c_\zeta \beta \alpha \log(t_p)/2$ for all $p > p_0$ and therefore the above computations imply that, for p_0 large enough and all $p > p_0$,

$$h_{t_p}^{-\beta(\nu+d)} e^{-(t_i - t_{i-1})(C_2^{-1} D_p^2 / 2 - 3d \log(\delta_p))} \leq C_3 t_p^{-\tilde{c}/2-1}, \quad \forall i \in (p_0 + 1) : p. \quad (74)$$

Then, by combining (70), (73) and (74), and recalling that $\tilde{c} > 4\alpha\nu$, it follows that, for p_0 large enough and some constant $C'_3 \in (0, \infty)$, for all $p > p_0$ we have

$$\begin{aligned} \sum_{i=p_0+1}^p \mathbb{P}(\tilde{A}_{i,p} | A_{i-1,p}^c) &\leq C_3 C_1 (1 + C_2) \epsilon^{-1} t_p^{-\tilde{c}/2-1} \sum_{i=p_0+1}^p h_{t_{i-1}}^{-\nu} \\ &\leq C'_3 \epsilon^{-1} t_p^{-\tilde{c}/2+\nu\alpha} \\ &\leq C'_3 \epsilon^{-1} t_p^{-\nu\alpha} \end{aligned}$$

and therefore

$$\limsup_{p \rightarrow \infty} \sum_{i=p_0+1}^p \mathbb{P}(\tilde{A}_{i,p} | A_{i-1,p}^c) \leq \limsup_{p \rightarrow \infty} C'_3 \epsilon^{-1} t_p^{-\nu\alpha} = 0. \quad (75)$$

We now show that $\mathbb{P}(A_{p_0,p}) \rightarrow 0$. To this aim remark that

$$0 \leq \mathbf{1}_{V_{2D_p}}(\theta) \mathbb{E}[\tilde{\pi}_{t_{p_0}}(\theta)] \leq \mathbb{E}[\tilde{\pi}_{t_{p_0}}(\theta)], \quad \forall \theta \in \mathbb{R}^d, \quad \forall p \geq 1$$

and that, using Tonelli's theorem,

$$\int_{\mathbb{R}^d} \mathbb{E}[\tilde{\pi}_{t_{p_0}}(\theta)] d\theta = \mathbb{E}[\tilde{\pi}_{t_{p_0}}(\mathbb{R}^d)] = 1.$$

Therefore, by the reverse Fatou lemma we have (and using Tonelli's theorem for the first equality)

$$\begin{aligned} \limsup_{p \rightarrow \infty} \mathbb{E}[\tilde{\pi}_{t_{p_0}}(V_{2D_p})] &= \limsup_{p \rightarrow \infty} \int_{\mathbb{R}^d} \mathbf{1}_{V_{2D_p}}(\theta) \mathbb{E}[\tilde{\pi}_{t_{p_0}}(\theta)] d\theta \\ &\leq \int_{\mathbb{R}^d} \limsup_{p \rightarrow \infty} \left(\mathbf{1}_{V_{2D_p}}(\theta) \mathbb{E}[\tilde{\pi}_{t_{p_0}}(\theta)] \right) d\theta \\ &= 0 \end{aligned}$$

and thus, using Markov's inequality,

$$\limsup_{p \rightarrow \infty} \mathbb{P}(A_{p_0,p}) \leq \limsup_{p \rightarrow \infty} \mathbb{P}(\tilde{\pi}_{t_{p_0}}(V_{2D_p}) \geq \epsilon) \leq \limsup_{p \rightarrow \infty} \frac{\mathbb{E}[\tilde{\pi}_{t_{p_0}}(V_{2D_p})]}{\epsilon} = 0. \quad (76)$$

Then, combining (60), (61), (75) and (76) proves the theorem under Assumption A6.1.

We now prove the theorem under Assumption A6.3. Let p be such that $\zeta(D_p) < 0$ and remark first that, for some constant $C \in (0, \infty)$,

$$\begin{aligned} &\mathbb{P}\left(\min_{1 \leq i \leq p} e^{-(t_i - t_{i-1})\zeta(D_p)/2} \prod_{s=t_{i-1}+1}^{t_i} f_{\theta_*}(Y_s) \leq 1\right) \\ &= \mathbb{P}\left(\min_{1 \leq i \leq p} \sum_{s=t_{i-1}+1}^{t_i} (\log(f_{\theta_*}(Y_s)) - \zeta(D_p)/2) \leq 0\right) \\ &= \mathbb{P}\left(-\min_{1 \leq i \leq p} \sum_{s=t_{i-1}+1}^{t_i} (\log(f_{\theta_*}(Y_s)) - \zeta(D_p)/2) \geq 0\right) \\ &= \mathbb{P}\left(\max_{1 \leq i \leq p} \sum_{s=t_{i-1}+1}^{t_i} (\zeta(D_p)/2 - \log(f_{\theta_*}(Y_s))) \geq 0\right) \\ &\leq \sum_{i=1}^p \mathbb{P}\left(\frac{1}{t_i - t_{i-1}} \sum_{s=t_{i-1}+1}^{t_i} (\mathbb{E}[\log(f_{\theta_*})] - \log(f_{\theta_*}(Y_s))) \geq |\zeta(D_p)|/2 + \mathbb{E}[\log(f_{\theta_*})]\right) \\ &\leq \frac{C}{(t_i - t_{i-1})^{k_*} \mathbb{1}_{\mathbb{N}}(k_*) |\zeta(D_p)|^{2k_*}} \end{aligned}$$

where the second inequality uses Assumption A6.3, Lemma 11 and Markov's inequality.

Therefore, under the assumptions of the theorem, there exists a sequence $(Y_p)_{p \geq 1}$, with $Y_p \subset \mathcal{Y}^{\otimes t_p}$ for all $p \geq 1$, such that $\mathbb{P}(Y_{1:t_p} \in Y_p) \rightarrow 1$ and such that

$$\min_{1 \leq i \leq p} e^{-(t_i - t_{i-1})\zeta(D_p)/2} \prod_{s=t_{i-1}+1}^{t_i} f_{\theta_*}(y_s) > 1, \quad \min_{1 \leq i \leq p} \frac{g_i(y_{1:t_i})}{\tilde{g}_{i,p}(y_{1:t_i})} > 1, \quad \forall y_{1:t_p} \in Y_p, \quad \forall p \geq 1.$$

Then, using the computations done to prove the theorem under Assumption A6.1, to prove the theorem under A6.3 we only need to show that, for $p_0 \in \mathbb{N}$ large enough and every $p > p_0$,

$$\mathbb{E}[X_{i,p}^{(1)} \mathbb{1}_{Y_{i,p}}(Y_{1:t_i}) | \tilde{\pi}_{t_{i-1}}(V_{2D_p}) < \epsilon] \leq e^{(t_i - t_{i-1})\zeta(D_p)/2}, \quad \forall i \in (p_0 + 1) : p, \quad (77)$$

where, for every $p > p_0$, $(Y_{i,p})_{i=0}^p$ is a sequence in $\mathcal{Y}^{\otimes t_p}$ such that $Y_{0,p} = Y^{t_p}$, $Y_{p,p} = Y_p$ and such that, for every $i \in 1 : p$, we have $Y_{i,p} \subset \bigcap_{j=0}^{i-1} Y_{j,p}$ and

$$\min_{1 \leq j \leq i} e^{-(t_j - t_{j-1})\zeta(D_p)/2} \prod_{s=t_{j-1}+1}^{t_j} f_{\theta_*}(y_s) \geq 1, \quad \min_{1 \leq j \leq i} \frac{g_j(y_{1:t_j})}{\tilde{g}_{j,p}(y_{1:t_j})} > 1, \quad \forall y_{1:t_p} \in Y_{i,p}.$$

Using the definitions of $\zeta(D_p)$ under Assumption A6.3 and the above definition of $Y_{i,p}$ under Assumption A6.3, we have for p_0 large enough and all $p > p_0$,

$$\begin{aligned} \mathbb{E}[X_{i,p}^{(1)} \mathbb{1}_{Y_{i,p}}(Y_{1:t_i}) | \tilde{\pi}_{t_{i-1}}(V_{2D_p}) < \epsilon] &\leq e^{-\zeta(D_p)/2(t_j - t_{j-1})} \left(\sup_{\theta \in V_{D_p}} \mathbb{E}[\tilde{f}_\theta] \right)^{t_i - t_{i-1}} \\ &= e^{(t_i - t_{i-1})\zeta(D_p)/2}. \end{aligned}$$

This shows (77) and the proof of the theorem under Assumption A6.3 is complete.

Lastly, we prove the theorem under Assumption A6.2, where $\zeta(D_p) = \log(\sup_{\theta \in V_{D_p}} \mathbb{E}[\tilde{f}_\theta / f_{\theta_*}])$. Following the computations of the proof of the theorem under Assumption A6.1, to prove the theorem under Assumption A6.2 we only need to show that, $p_0 \in \mathbb{N}$ large enough we have, for all $p > p_0$,

$$\mathbb{E}[X_{i,p}^{(1)} \mathbb{1}_{Y_{i,p}}(Y_{1:t_i}) | \tilde{\pi}_{t_{i-1}}(V_{2D_p}) < \epsilon] \leq e^{(t_i - t_{i-1})\zeta(D_p)/2}, \quad \forall i \in (p_0 + 1) : p \quad (78)$$

where, for every $p > p_0$, $(Y_{i,p})_{i=0}^p$ is a sequence in $\mathcal{Y}^{\otimes t_p}$ such that $Y_{0,p} = Y^{t_p}$, $Y_{p,p} = Y_p$ and such that, for every $i \in 1 : p$, $Y_{i,p} \subset \bigcap_{j=0}^{i-1} Y_{j,p}$ and

$$\min_{1 \leq j \leq i} \frac{g_j(y_{1:t_j})}{\tilde{g}_{j,p}(y_{1:t_j})} > 1, \quad \forall y_{1:t_p} \in Y_{j,p}.$$

Using the definition of $\zeta(D_p)$ and the above definition of $Y_{i,p}$ under Assumption A6.2 we have

$$\mathbb{E}[X_{i,p}^{(1)} \mathbb{1}_{Y_{i,p}}(Y_{1:t_i}) | A_{i-1,p}^c] \leq \left(\sup_{\theta \in V_{D_p}} \mathbb{E}[\tilde{f}_\theta / f_{\theta_*}] \right)^{t_i - t_{i-1}} = e^{(t_i - t_{i-1})\zeta(D_p)}.$$

This shows (78) and the proof of the theorem under Assumption A6.2 is complete. \square

A.7. Proof of Propositions 3

Proof. Under the assumptions of the proposition there exist constants $C_1, C_2 \in (0, \infty)$ such that, for p large enough,

$$\begin{aligned} (t_p - t_{p-1}) \sum_{s=t_{p-1}+1}^{t_p-1} h_s^2 &\leq (t_p - t_{p-1})^2 h_{t_{p-1}}^2 \\ &\leq C_1 (t_p - t_{p-1})^2 t_{p-1}^{-2\alpha} \\ &\leq C_2 \log(t_{p-1})^2 t_{p-1}^{-2(\alpha-\varrho)}. \end{aligned}$$

Therefore, since $\varrho \in (0, \alpha)$, we have $(t_p - t_{p-1}) \sum_{s=t_{p-1}+1}^{t_p-1} h_s^2 \rightarrow 0$ as required. Together with the fact that $(t_{p+1} - t_p) \rightarrow \infty$, this shows that Condition 2 of Theorem 2 holds.

To show that Condition 3 of Theorem 2 holds as well remark first that $(t_p - t_{p-1})^{-1} \log(1/h_{t_{p-1}}) \rightarrow 0$ as required. In addition, with $c \in (0, 1)$ as in the statement of the proposition, we have for p large enough

$$\frac{t_{p+2} - t_{p+1}}{t_{p+1} - t_p} \leq \frac{C_{p+1} \log(t_{p+1})}{C_p \log(t_p) - 1} \leq \frac{t_{p+1}^\varrho \log(t_{p+1})}{c^2 t_p^\varrho \log(t_p) - c} \leq \frac{2}{c^2} \frac{t_{p+1}^\varrho \log(t_{p+1})}{t_p^\varrho \log(t_p)} \quad (79)$$

where, for p large enough,

$$\frac{\log(t_{p+1})}{\log(t_p)} \leq 1 + \frac{t_{p+1} - t_p}{t_p \log(t_p)} \leq 1 + \frac{t_{p+1}}{t_p}. \quad (80)$$

Next, recalling that $\varrho \in (0, 1)$, for p large enough we have

$$\frac{t_{p+1}}{t_p} \leq 1 + \frac{t_p^\varrho \log(t_p)}{c t_p} \leq 1 + (2/c)$$

with, together with (79) and (80), shows that $\limsup_{p \rightarrow \infty} (t_{p+2} - t_{p+1}) / (t_{p+1} - t_p) < \infty$. This concludes to show that Condition 3 of Theorem 2 holds.

To show the second part of the proposition note that there exists a constant $c' > 0$ such that $(t_p - t_{p-1}) \geq c' t_{p-1}^\varrho$ for all $p \geq 1$. By assumption, $k_\star > 1/\varrho + 1$, and thus

$$\limsup_{p \rightarrow \infty} \sum_{i=1}^p (t_i - t_{i-1})^{-k_\star} \leq (1/c') \limsup_{p \rightarrow \infty} \sum_{i=1}^p t_{i-1}^{-k_\star \varrho} \leq (1/c') \limsup_{p \rightarrow \infty} \sum_{i=1}^\infty t_i^{-(1+\varrho)} < \infty.$$

The result follows. \square

A.8. Proof of the preliminary results given in Section A.5.2

A.8.1. Proof of Lemma 2

We first show the following simple lemma.

Lemma 12. Let $(s_t)_{t \geq 1}$ be a sequence in \mathbb{N}_0 such that $\inf_{t \geq 1} (t - s_t) \geq 1$ and $(t - s_t) \rightarrow \infty$, $(X_t)_{t \geq 1}$ be a sequence of independent real-valued random variables such that $\mathbb{E}[X_n] \rightarrow m$ for some $m \in \mathbb{R}$ and such that $\sup_{s \geq 1} \text{Var}(X_s) < \infty$. Then, we have $(t - s_t)^{-1} \sum_{s=s_t+1}^t X_s \rightarrow m$ in \mathbb{P} -probability.

Proof. For every $t \geq 1$ we have

$$\frac{1}{t - s_t} \sum_{s=s_t+1}^t X_s - m = \frac{1}{t - s_t} \sum_{s=s_t+1}^t (X_s - \mathbb{E}[X_s]) + \left(\frac{1}{t - s_t} \sum_{s=s_t+1}^t \mathbb{E}[X_s] - m \right). \quad (81)$$

Let $\epsilon > 0$ and $s_\epsilon \in \mathbb{N}$ be such that $|\mathbb{E}[X_s] - m| \leq \epsilon$ for all $s \geq s_\epsilon$. Then, for all t such that $s_t \geq s_\epsilon$ we have

$$\left| \frac{1}{t - s_t} \sum_{s=s_t+1}^t \mathbb{E}[X_s] - m \right| \leq \frac{1}{t - s_t} \sum_{s=s_t+1}^t |\mathbb{E}[X_s] - m| \leq \epsilon$$

showing that $(t - s_t)^{-1} \sum_{s=s_t+1}^t \mathbb{E}[X_s] \rightarrow m$. Hence, by (81), to complete the proof it remains to show that $(t - s_t)^{-1} \sum_{s=s_t+1}^t (X_s - \mathbb{E}[X_s]) \rightarrow 0$ in \mathbb{P} -probability. Using Markov's inequality, for every $\epsilon > 0$ we have

$$\begin{aligned} \limsup_{t \rightarrow \infty} \mathbb{P} \left(\left| \frac{1}{t - s_t} \sum_{s=s_t+1}^t (X_s - \mathbb{E}[X_s]) \right| \geq \epsilon \right) &\leq \limsup_{t \rightarrow \infty} \frac{\sum_{s=s_t+1}^t \text{Var}(X_s)}{\epsilon^2 (t - s_t)^2} \\ &\leq \limsup_{t \rightarrow \infty} \frac{\sup_{s \geq 1} \text{Var}(X_s)}{\epsilon^2 (t - s_t)} \\ &= 0 \end{aligned}$$

and the proof is complete. \square

Proof of Lemma 2. Let A_\star and \tilde{A}_\star be as in Assumption A4 and $A'_\star \in \mathcal{B}(\mathbb{R}^d)$ be such that A'_\star contains a neighbourhood of A_\star and such that \tilde{A}_\star contains a neighbourhood of A'_\star . Let $B'_\star = A'_\star \cap \Theta$, $\tilde{B}_\star = \tilde{A}_\star \cap \Theta$, and remark that we can without loss of generality assume that the sequence $(\gamma'_t)_{t \geq 1}$ is non-increasing. Below we denote by \bar{B} the closure of the set $B \subset \mathbb{R}^d$.

Let $(\delta_t)_{t \geq 1}$ be a sequence in $\mathbb{R}_{>0}$ and let $W_s = \overline{B_{\gamma'_s}(0)}$ for all $s \geq 1$. Then, for all $t \geq 1$ we have

$$\begin{aligned} \mathbb{P} \left(\sup_{(u_{s_t:t}, \theta) \in \Theta_{\gamma'_t, s_t:t} \times B'_\star} \prod_{s=s_t+1}^t (\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i} / f_\theta)(Y_s) \geq e^{(t-s_t)\delta_t} \right) \\ \leq \mathbb{P} \left(\sup_{\theta \in B'_\star} \prod_{s=s_t+1}^t \sup_{v_s \in \bar{B}_{\gamma'_t}(0)} (\tilde{f}_{\theta+v_s} / f_\theta)(Y_s) \geq e^{(t-s_t)\delta_t} \right) \\ \leq \mathbb{P} \left(\prod_{s=s_t+1}^t \sup_{(\theta, w_s) \in B'_\star \times W_s} (\tilde{f}_{\theta+w_s} / f_\theta)(Y_s) \geq e^{(t-s_t)\delta_t} \right) \end{aligned}$$

where the last inequality uses the fact that the sequence $(\gamma'_t)_{t \geq 1}$ is non-increasing. Consequently, to prove the lemma it is enough to show that

$$\frac{1}{t - s_t} \sum_{s=s_t+1}^t \sup_{(\theta, w_s) \in B'_* \times W_s} \log((\tilde{f}_{\theta+w_s}/f_\theta)(Y_s)) \rightarrow 0, \quad \text{in } \mathbb{P}\text{-probability.} \quad (82)$$

To establish (82) we define, for every $s \geq 1$,

$$X_s = \sup_{(\theta, w_s) \in B'_* \times W_s} \log((\tilde{f}_{\theta+w_s}/f_\theta)(Y_s)), \quad m_s^{(1)} = \mathbb{E}[X_s], \quad m_s^{(2)} = \mathbb{E}[X_s^2]$$

and show below that there exists a $\underline{t} \in \mathbb{N}$ such that

$$\lim_{t \rightarrow \infty} m_{\underline{t}+t}^{(1)} \rightarrow 0, \quad \sup_{s \geq \underline{t}} m_s^{(2)} \leq m_{\underline{t}}^{(2)} < \infty. \quad (83)$$

Then, (82) will follow by Lemma 12.

To show (83) remark first that for all $s \geq 1$ we have $m_s^{(1)} = \mathbb{E}[\tilde{X}_s]$, where

$$\tilde{X}_s = \sup_{(\theta, w_s) \in B'_* \times W_s} \log((\tilde{f}_{\theta+w_s}/f_\theta)(Y_1)).$$

Next, let $\tilde{\Omega} \in \mathcal{F}$ be such that $\mathbb{P}(\tilde{\Omega}) = 1$ and such that the mapping $\theta \mapsto f_\theta(Y_1(\omega))$ is continuous on the compact set \tilde{B}_* , for all $\omega \in \tilde{\Omega}$, notice that such a set $\tilde{\Omega}$ exists under Assumption A4. Let $\underline{s} \in \mathbb{N}$ be such that $\theta + w \in \tilde{B}_*$ for all $(\theta, w) \in B'_* \times W_{\underline{s}}$ such that $\theta + w \in \Theta$. Then, recalling that $\gamma'_{s+1} \leq \gamma'_s$ for all $s \geq 1$, it follows that

$$\tilde{X}_s \leq \sup_{(\theta, \theta') \in \tilde{B}_*^2: \|\theta - \theta'\| \leq \gamma'_s} \log((f_{\theta'}/f_\theta)(Y_1)), \quad \forall s \geq \underline{s}.$$

Then, by Weierstrass's theorem we have, for all $\omega \in \tilde{\Omega}$ and $s \geq \underline{s}$ we have

$$\tilde{X}_s(\omega) \leq \log\left(\left(f_{h_{\gamma'_s}(\omega)}/f_{g_{\gamma'_s}(\omega)}\right)(Y_1(\omega))\right)$$

for some (measurable) functions $h_{\gamma'_s} : \tilde{\Omega} \rightarrow \tilde{B}_*$ and $g_{\gamma'_s} : \tilde{\Omega} \rightarrow \tilde{B}_*$ such that we have $\|h_{\gamma'_s}(\omega) - h_{\gamma'_s}(\omega)\| \leq \gamma'_s$ for all $\omega \in \tilde{\Omega}$.

By the maximum theorem, we can assume that, for all $s \geq \underline{s}$ and every $\omega \in \tilde{\Omega}$ the mappings $\gamma \mapsto h_\gamma(\omega)$ and $\gamma \mapsto g_\gamma(\omega)$ are continuous on $[0, \gamma'_s]$. Therefore, since $h_0(\omega) = g_0(\omega)$ for all $\omega \in \tilde{\Omega}$, we have

$$\begin{aligned} 0 \leq \limsup_{s \rightarrow \infty} \tilde{X}_{\underline{s}+s}(\omega) &\leq \limsup_{s \rightarrow \infty} \log\left(\left(f_{h_{\gamma'_s}(\omega)}/f_{g_{\gamma'_s}(\omega)}\right)(Y_1(\omega))\right) \\ &= \log\left(\left(f_{h_0(\omega)}/f_{g_0(\omega)}\right)(Y_1(\omega))\right) \\ &= 0, \quad \forall \omega \in \tilde{\Omega}. \end{aligned} \quad (84)$$

To proceed further remark that since $W_s \subseteq W_{\underline{s}}$ for all $s \geq \underline{s}$, it follows that for all $s \geq \underline{s}$ we have $\tilde{X}_s \leq \tilde{X}_{\underline{s}}$, \mathbb{P} -a.s. Let $\tilde{\delta} > 0$ be as in (A4). Then, under this latter assumption,

and taking $\underline{s} \in \mathbb{N}$ sufficiently large so that $\gamma'_{\underline{s}} \leq \tilde{\delta}$, we have $\mathbb{E}[\tilde{X}_{\underline{s}}] < \infty$. Then, by the dominated convergence theorem, $\lim_{s \rightarrow \infty} m_{\underline{s}+s}^{(1)} = 0$, showing the first part of (83). To show the second part of (83) it suffices to remark that $m_s^{(2)} \leq m_{\underline{s}}^{(2)}$ for all $s \geq \underline{s}$ where, under (A4), $m_{\underline{s}}^{(2)} < \infty$. \square

A.8.2. Proof of Lemma 3

Proof. Let $t \geq 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$. Then, if $\mu_t \neq \delta_{\{0\}}$ we have, \mathbb{P} -a.s.,

$$(\mu_t * \tilde{\pi}_t)(A) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_A(\theta) \tilde{\pi}_t(\theta - u_t) \mu_t(du_t) d\theta = \int_A \mathbb{E}[\tilde{\pi}_t(\theta - U_t) | \mathcal{F}_t] d\theta$$

while, if $\mu_t = \delta_{\{0\}}$ we have, \mathbb{P} -a.s.,

$$(\mu_t * \tilde{\pi}_t)(A) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbf{1}_A(\theta + u_t) \mu_t(du_t) \tilde{\pi}_t(\theta) d\theta = \int_A \tilde{\pi}_t(\theta) d\theta.$$

Recall that $\mathbb{P}(\cap_{t \geq 0} \Omega_t) = 1$ if $\mathbb{P}(\Omega_t) = 1$ for all $t \geq 0$ and that two probability measures $\nu_1, \nu_2 \in \mathcal{P}(\mathbb{R}^d)$ are such that $\nu_1 = \nu_2$ if $\nu_1(E_i) = \nu_2(E_i)$ for all $i \geq 1$, with $(E_i)_{i \geq 1}$ a dense subset of \mathbb{R}^d such that $E_i \in \mathcal{B}(\mathbb{R}^d)$ for all $i \geq 1$.

Therefore, the above computations imply that

$$(\mu_t * \tilde{\pi}_t)(A) = \int_A \mathbb{E}[\tilde{\pi}_t(\theta - U_t) | \mathcal{F}_t] d\theta, \quad \forall t \geq 0, \quad \forall A \in \mathcal{B}(\mathbb{R}^d), \quad \mathbb{P} - a.s. \quad (85)$$

We now prove the result of the lemma by induction on $t \geq 1$.

The result trivially holds for $t = 1$ and we now assume that it holds for some $t \geq 1$. Then, \mathbb{P} -a.s.,

$$\begin{aligned} \tilde{\pi}_{t+1}(d\theta) &\propto \tilde{f}_\theta(Y_{t+1})(\mu_t * \tilde{\pi}_t)(d\theta) \\ &= \tilde{f}_\theta(Y_{t+1}) \mathbb{E}[\tilde{\pi}_t(\theta - U_t) | \mathcal{F}_t] d\theta \\ &\propto \tilde{f}_\theta(Y_{t+1}) \mathbb{E}\left[(\mu_0 * \tilde{\pi}_0)\left(\theta - \sum_{s=1}^t U_s\right) \prod_{s=1}^t \tilde{f}_{\theta - \sum_{i=s}^t U_i}(Y_s) | \mathcal{F}_t\right] d\theta \\ &= \mathbb{E}\left[(\mu_0 * \tilde{\pi}_0)\left(\theta - \sum_{s=1}^t U_s\right) \prod_{s=1}^{t+1} \tilde{f}_{\theta - \sum_{i=s}^t U_i}(Y_s) | \mathcal{F}_{t+1}\right] d\theta \end{aligned}$$

where the first equality uses (85) and the second line the inductive hypothesis. The proof is complete. \square

A.8.3. Proof of Lemma 4

Proof. Let $t \geq 1$, $p_t = \mathbb{P}(U_{s:t} \in \Theta_{\tilde{\delta}, s:t})$, $C_* \in \mathbb{R}_{>0}$ be as in Assumption A2, $\tilde{C}_* = 2(\mathbb{E}[m_*^2] + C_*)^{1/2}$ and note that, under Assumptions A1-A2 and for all $\theta \in B_{\tilde{\delta}_*}(\theta_*)$,

$$\max \left(-\mathbb{E}[\log(\tilde{f}_\theta / f_{\theta_*})], \mathbb{E}[(\log(\tilde{f}_\theta / f_{\theta_*}))^2] \right) \leq (\mathbb{E}[m_*^2] + C_*) \|\theta - \theta_*\|^2. \quad (86)$$

Remark now that if $p_t = 0$ then the result of the lemma trivially holds and henceforth we therefore assume that $p_t > 0$. To simplify the notation let $\mathbb{E}_{Y_1}[\cdot]$ denote expectations w.r.t. the distribution of Y_1 , $\mathbb{E}_{t,\mu}^{\tilde{\delta}}[\cdot]$ denote expectations w.r.t. the restriction of $\otimes_{s=s_t}^t \mu_t$ to the set $\Theta_{\tilde{\delta},s_t:t}$, and let $V_t := (U_s)_{s=s_t+1}^t$.

For every $u_{(s_t+1):t} \in \Theta^{t-s_t}$ let $\tilde{\eta}(d\theta, u_{s_t:t})$ be the probability measure on $B_\delta(\theta_\star)$ with density function $\tilde{\eta}(\cdot, u_{(s_t+1):t})$ defined by

$$\tilde{\eta}(\theta, u_{(s_t+1):t}) = \frac{\eta(\theta - \sum_{s=s_t+1}^{t-1} u_s)}{\eta(B_\delta(\theta_\star - \sum_{s=s_t+1}^{t-1} u_s))}, \quad \theta \in B_\delta(\theta_\star).$$

Then, using the shorthand $a_t = (t - s_t)(2(\tilde{C}_\star\delta)^2 + \epsilon)$, we have

$$\begin{aligned} \tilde{Y}_t &:= \left\{ y_{s_t:t} : \int_{\Theta} \mathbb{E} \left[\eta \left(\theta - \sum_{s=s_t+1}^{t-1} U_s \right) \prod_{s=s_t+1}^t (\tilde{f}_{\theta - \sum_{i=s}^{t-1} U_i} / f_{\theta_\star})(y_s) \right] d\theta \leq p_t C_{\delta,\tilde{\delta}}^\eta e^{-a_t} \right\} \\ &= \left\{ y_{s_t:t} : \mathbb{E} \left[\int_{\Theta} \eta \left(\theta - \sum_{s=s_t+1}^{t-1} U_s \right) \prod_{s=s_t+1}^t (\tilde{f}_{\theta - \sum_{i=s}^{t-1} U_i} / f_{\theta_\star})(y_s) d\theta \right] \leq p_t C_{\delta,\tilde{\delta}}^\eta e^{-a_t} \right\} \\ &\subset \left\{ y_{s_t:t} : \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} \eta \left(\theta - \sum_{s=s_t+1}^{t-1} U_s \right) \prod_{s=s_t+1}^t (\tilde{f}_{\theta - \sum_{i=s}^{t-1} U_i} / f_{\theta_\star})(y_s) d\theta \right] \leq C_{\delta,\tilde{\delta}}^\eta e^{-a_t} \right\} \\ &\subset \left\{ y_{s_t:t} : \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} \prod_{s=s_t+1}^t (\tilde{f}_{\theta - \sum_{i=s}^{t-1} U_i} / f_{\theta_\star})(y_s) \tilde{\eta}(d\theta, V_t) \right] \leq e^{-a_t} \right\} \\ &\subset \left\{ y_{s_t:t} : \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\sum_{s=s_t+1}^t \int_{B_\delta(\theta_\star)} \log \left((\tilde{f}_{\theta - \sum_{i=s}^{t-1} U_i} / f_{\theta_\star})(y_s) \right) \tilde{\eta}(d\theta, V_t) \right] \leq -a_t \right\} \end{aligned} \tag{87}$$

where the equality uses Tonelli's theorem, the second inclusion uses the definition of $C_{\delta,\tilde{\delta}}^\eta$ and the last inclusion uses twice Jensen's inequality.

To simplify the notation in what follows we define, for every $t \geq s \geq 0$,

$$g_\theta(u_{s:t}, y) = \log \left((\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i} / f_{\theta_\star})(y) \right), \quad (\theta, u_{s:t}) \in \Theta^{t-s+2}, \quad y \in \mathcal{Y}. \tag{88}$$

Remark now that, by (86) and using the inequality $\|a + b\|^2 \leq 2(\|a\|^2 + \|b\|^2)$ for all $a, b \in \mathbb{R}^d$,

$$\begin{aligned} - \sum_{s=s_t+1}^t \mathbb{E}[g_\theta(u_{s:t}, Y_1)] &\leq (t - s_t)(\tilde{C}_\star\delta)^2, \quad \forall \theta \in B_\delta(\theta_\star), \quad \forall u_{s_t:t} \in \Theta_{\tilde{\delta},s_t:t} \\ \sum_{s=s_t+1}^t \mathbb{E}[g_\theta(u_{s:t}, Y_1)^2] &\leq (t - s_t)(\tilde{C}_\star\delta)^2, \quad \forall \theta \in B_\delta(\theta_\star), \quad \forall u_{s_t:t} \in \Theta_{\tilde{\delta},s_t:t}. \end{aligned} \tag{89}$$

Therefore, using (87) and (89), we have

$$\begin{aligned}
& \mathbb{P}(Y_{s_t:t} \in \tilde{Y}_t) \\
& \leq \mathbb{P}\left(\mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\sum_{s=s_t+1}^t \int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_s) \tilde{\eta}(d\theta, V_t) \right] \leq -a_t\right) \\
& = \mathbb{P}\left(\mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\frac{1}{t-s_t} \sum_{s=s_t+1}^t \left\{ \int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_s) - \int_{B_\delta(\theta_\star)} \mathbb{E}_{Y_1} [g_\theta(U_{s:t}, Y_1)] \right\} \tilde{\eta}(d\theta, V_t) \right. \right. \\
& \quad \left. \left. + \frac{1}{t-s_t} \sum_{s=s_t+1}^t \int_{B_\delta(\theta_\star)} \mathbb{E}_{Y_1} [g_\theta(U_{s:t}, Y_1)] \tilde{\eta}(d\theta, V_t) \right] \leq -2(\tilde{C}_\star \delta)^2 - \epsilon \right) \\
& \leq \mathbb{P}\left(\mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\frac{1}{t-s_t} \sum_{s=s_t+1}^t \left\{ \int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_s) - \int_{B_\delta(\theta_\star)} \mathbb{E}_{Y_1} [g_\theta(U_{s:t}, Y_1)] \right\} \tilde{\eta}(d\theta, V_t) \right] \right. \\
& \quad \left. \leq -(\tilde{C}_\star \delta)^2 - \epsilon \right).
\end{aligned} \tag{90}$$

We now show that

$$\begin{aligned}
& \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} \mathbb{E}_{Y_1} [g_\theta(U_{s:t}, Y_1)] \tilde{\eta}(d\theta, V_t) \right] \\
& = \mathbb{E}_{Y_1} \left[\mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_1) \tilde{\eta}(d\theta, V_t) \right] \right].
\end{aligned} \tag{91}$$

By (89), there exists a constant $C_t \in [0, \infty)$ such, that for every $u_{s_t:t} \in \Theta_{\tilde{\delta}, s_t:t}$,

$$\begin{aligned}
& \int_{B_\delta(\theta_\star)} \mathbb{E}_{Y_1} [|g_\theta(u_{s_t:t}, Y_1)|] \tilde{\eta}(d\theta, u_{s_t:t}) \leq \int_{B_\delta(\theta_\star)} \mathbb{E}_{Y_1} [g_\theta(u_{s_t:t}, Y_1)^2]^{\frac{1}{2}} \tilde{\eta}(d\theta, u_{s_t:t}) \\
& \leq C_t
\end{aligned} \tag{92}$$

and thus by, Fubini-Tonelli's theorem,

$$\begin{aligned}
& \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} \mathbb{E}_{Y_1} [g_\theta(U_{s:t}, Y_1)] \tilde{\eta}(d\theta, V_t) \right] \\
& = \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\mathbb{E}_{Y_1} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_1) \tilde{\eta}(d\theta, V_t) \right] \right].
\end{aligned} \tag{93}$$

Using (92), we also have

$$\begin{aligned}
& \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\mathbb{E}_{Y_1} \left[\left| \int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_1) \tilde{\eta}(d\theta, V_t) \right| \right] \right] \leq \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\mathbb{E}_{Y_1} \left[\int_{B_\delta(\theta_\star)} |g_\theta(U_{s:t}, Y_1)| \tilde{\eta}(d\theta, V_t) \right] \right] \\
& = \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} \mathbb{E}_{Y_1} [|g_\theta(U_{s:t}, Y_1)|] \tilde{\eta}(d\theta, V_t) \right] \\
& \leq C_t,
\end{aligned}$$

where the equality uses Tonelli's theorem. By Fubini-Tonelli's theorem we therefore have

$$\begin{aligned} \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\mathbb{E}_{Y_1} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_1) \tilde{\eta}(d\theta, V_t) \right] \right] \\ = \mathbb{E}_{Y_1} \left[\mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_1) \tilde{\eta}(d\theta, V_t) \right] \right] \end{aligned} \quad (94)$$

and (91) follows from (93) and (94).

Consequently, using (90) and (91), we have

$$\begin{aligned} \mathbb{P}(Y_{s_t:t} \in \tilde{Y}_t) &\leq \mathbb{P} \left(\frac{1}{t-s_t} \sum_{s=s_t+1}^t \left\{ \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_s) \tilde{\eta}(d\theta, V_t) \right] \right. \right. \\ &\quad \left. \left. - \mathbb{E}_{Y_1} \left[\mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_1) \tilde{\eta}(d\theta, V_t) \right] \right] \right\} \leq -(\tilde{C}_\star \delta)^2 - \epsilon \right) \\ &\leq \mathbb{P} \left(\left| \frac{1}{t-s_t} \sum_{s=s_t+1}^t \left\{ \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_s) \tilde{\eta}(d\theta, V_t) \right] \right. \right. \right. \\ &\quad \left. \left. \left. - \mathbb{E}_{Y_1} \left[\mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_1) \tilde{\eta}(d\theta, V_t) \right] \right] \right\} \right| \geq (\tilde{C}_\star \delta)^2 + \epsilon \right) \end{aligned} \quad (95)$$

and we finally upper bound the last term using Markov's inequality.

To this aim remark that

$$\begin{aligned} \sum_{s=s_t+1}^t \mathbb{E}_{Y_1} \left[\mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_1) \tilde{\eta}(d\theta, V_t) \right]^2 \right] \\ \leq \sum_{s=s_t+1}^t \mathbb{E}_{Y_1} \left[\mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\left(\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_1) \tilde{\eta}(d\theta, V_t) \right)^2 \right] \right] \\ \leq \sum_{s=s_t+1}^t \mathbb{E}_{Y_1} \left[\mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_1)^2 \tilde{\eta}(d\theta, V_t) \right] \right] \\ = \sum_{s=s_t+1}^t \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\mathbb{E}_{Y_1} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_1)^2 \tilde{\eta}(d\theta, V_t) \right] \right] \\ = \sum_{s=s_t+1}^t \mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} \mathbb{E}_{Y_1} [g_\theta(U_{s:t}, Y_1)^2] \tilde{\eta}(d\theta, V_t) \right] \\ \leq (t-s_t)(\tilde{C}_\star \delta)^2 \end{aligned} \quad (96)$$

where the last inequality uses (89), the first two inequalities use Jensen's inequality and the two equalities hold by Tonelli's theorem.

Therefore, using (95), (96) and Markov's inequality,

$$\begin{aligned} \mathbb{P}(Y_{s_t:t} \in \tilde{Y}_t) &\leq \frac{\frac{1}{(t-s_t)^2} \sum_{s=s_t+1}^t \text{Var} \left(\mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_s) \tilde{\eta}(d\theta, V_t) \right] \right)}{((\tilde{C}_\star \delta)^2 + \epsilon)^2} \\ &\leq \frac{\frac{1}{(t-s_t)^2} \sum_{s=s_t+1}^t \mathbb{E}_{Y_1} \left[\mathbb{E}_{t,\mu}^{\tilde{\delta}} \left[\int_{B_\delta(\theta_\star)} g_\theta(U_{s:t}, Y_1) \tilde{\eta}(d\theta, V_t) \right]^2 \right]}{((\tilde{C}_\star \delta)^2 + \epsilon)^2} \\ &\leq \frac{1}{(t-s_t)((\tilde{C}_\star \delta) + (\tilde{C}_\star \delta)^{-1} \epsilon)^2}. \end{aligned}$$

The proof is complete. \square

A.8.4. Proof of Lemma 5

We first recall the following result due to Kleijn and van der Vaart (2012, Lemma 3.3).

Lemma 13. *Assume Assumption A3. Then, for every compact set $W \in \mathcal{B}(\Theta)$ such that $\theta_\star \in W$ and every $\epsilon > 0$ there exist a sequence of measurable functions $(\psi_t)_{t \geq 1}$, with $\psi_t : \mathcal{Y}^t \rightarrow \{0, 1\}$, and a constant $D_\star \in (0, \infty)$ such that $\mathbb{E}[\psi_t(Y_{1:t})] \rightarrow 0$ and such that, for t large enough,*

$$\sup_{\theta \in V_\epsilon \cap \Theta} \mathbb{E} \left[(1 - \psi_t(Y_{1:t})) \prod_{s=1}^t (f_\theta / f_{\theta_\star})(Y_s) \right] \leq e^{-tD_\star}.$$

Proof of Lemma 5. Let A_\star and \tilde{A}_\star be as in Assumption A4, $A'_\star \subsetneq \tilde{A}_\star$ be as in Lemma 2 and $W = \tilde{A}_\star \cap \Theta$. Remark that W is a compact set under Assumption A4 and that $A'_\star \cap \Theta \subset W$. Let $(\psi_t)_{t \geq 1}$ be as in Lemma 13 and $(\delta_t)_{t \geq 1}$ be as in Lemma 2 for the sequence $(\gamma'_t)_{t \geq 1}$ defined by $\gamma'_t = 2\gamma_t, \forall t \geq 1$. Without loss of generality we assume below that $\epsilon > 0$ is such that $B_\epsilon(\theta_\star) \subset A'_\star$. Indeed, since $\theta_\star \in A_\star$ and A'_\star contains an open set that contains A_\star , it follows that $B_\delta(\theta_\star) \subset A'_\star$ for $\delta > 0$ small enough. Let $t_1 \in \mathbb{N}$ be such that for all $t \geq t_1$ we have $\theta' - \sum_{i=s}^{t-1} u'_i \notin A_\star$ for all $\theta' \notin A'_\star$ and all $u'_{s_t:t} \in \Theta_{\gamma_t, s_t:t}$. Notice that such a $t_1 \in \mathbb{N}$ exists since A'_\star contains a neighbourhood of A_\star .

We first show the lemma assuming Assumption A4.a). To this aim, for every, $t \geq 1$ we define $\mathcal{Y}_t = \mathcal{Y}_t^{(1)} \cap \mathcal{Y}_t^{(2)}$ where

$$\begin{aligned} \mathcal{Y}_t^{(1)} &= \left\{ y_{1:t} \in \mathcal{Y}^t : \sup_{(u_{s_t:t}, \theta) \in \Theta_{\gamma_t, s_t:t} \times (A'_\star \cap \Theta)} \prod_{s=s_t+1}^t (\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i} / f_\theta)(y_s) < e^{(t-s_t)\delta_t} \right\} \\ \mathcal{Y}_t^{(2)} &= \left\{ y_{1:t} \in \mathcal{Y}^t : \sum_{s=s_t+1}^t \sup_{\theta \notin A_\star} \log(\tilde{f}_\theta(y_s)) < \sum_{s=s_t+1}^t \log(f_{\theta_\star}(y_s)) - c_{A4} \right\} \end{aligned}$$

where $c_{A4} > 0$ is such that $\mathbb{E}[\sup_{\theta \notin A_\star} \log(\tilde{f}_\theta)] < \mathbb{E}[\log(f_{\theta_\star})] - 2c_{A4}$. Notice that such a constant $c_{A4} > 0$ exists under Assumption A4.a).

For every $t \geq 1$, let $\phi_t(Y_{1:t}) = \mathbb{1}_{Y_t^c}(Y_{1:t}) + \psi_{t-s_t}(Y_{(s_t+1):t})\mathbb{1}_{Y_t}(Y_{1:t})$ and remark that under Assumption A4.a), and by Lemmas 2 and 13, $\mathbb{E}[\phi_t(Y_{1:t})] \rightarrow 0$, as required.

To show the second part of the lemma let $\theta \in V_\epsilon$, $u_{s_t:t} \in \Theta_{\gamma_t, s_t:t}$. Remark that $1 - \phi_t(Y_{1:t}) = (1 - \psi_{t-s_t}(Y_{(s_t+1):t}))\mathbb{1}_{Y_t}(Y_{1:t})$ for all $t \geq 1$, and assume first that $\theta \in (A'_\star \cap \Theta)$. Then,

$$\begin{aligned} \mathbb{E}\left[(1 - \phi_t(Y_{1:t})) \prod_{s=s_t+1}^t (\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i} / f_{\theta_\star})(Y_s) \mid \mathcal{F}_{s_t}\right] \\ \leq e^{(t-s_t)\delta_t} \mathbb{E}\left[(1 - \psi_{t-s_t}(Y_{(s_t+1):t})) \prod_{s=s_t+1}^t (f_\theta / f_{\theta_\star})(Y_s)\right] \quad (97) \\ \leq e^{-(t-s_t)(D_\star - \delta_t)} \\ \leq e^{-(t-s_t)\frac{D_\star}{2}} \end{aligned}$$

where the first inequality uses the definition of $Y_t^{(1)}$, the second inequality uses Lemma 13 and holds for t large enough, while the last inequality holds for t sufficiently large since $\delta_t \rightarrow 0$. Notice that if $\theta \notin \Theta$ we have $\mathbb{E}[(1 - \phi_t(Y_{1:t})) \prod_{s=s_t+1}^t (\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i} / f_{\theta_\star})(Y_s) \mid \mathcal{F}_{s_t}] = 0$ and thus (97) also holds if $\theta \in (A'_\star \cap \Theta^c)$.

Assume now that $\theta \notin A'_\star$. Then,

$$\begin{aligned} \mathbb{E}\left[(1 - \phi_t(Y_{1:t})) \prod_{s=s_t+1}^t \frac{\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i}}{f_{\theta_\star}}(Y_s) \mid \mathcal{F}_{s_t}\right] \leq \mathbb{E}\left[\mathbb{1}_{Y_t^{(2)}}(Y_{1:t}) \prod_{s=s_t+1}^t \sup_{\theta \notin A'_\star} \frac{\tilde{f}_\theta}{f_{\theta_\star}}(Y_s)\right] \quad (98) \\ \leq e^{-(t-s_t)c_{A4}} \end{aligned}$$

where the first inequality holds for $t \geq t_1$. Together with (97), (98) shows that the result of the lemma holds under Assumption A4.a) with $\tilde{D}_\star = D_\star/2 \vee c_{A4}$.

We now show the result of the lemma under Assumption A4.b) and under Assumption A4.c). To do so remark that, using the above computations, we only need to find a set $Y_t^{(2)} \in \mathcal{Y}^{\otimes t}$ such that $\mathbb{P}(Y_{1:t} \in Y_t^{(2)}) \rightarrow 1$ and such that there exists a constant $c_{A4} > 0$ for which, for t large enough,

$$\sup_{\theta \notin A'_\star} \mathbb{E}\left[(1 - \phi_t(Y_{1:t})) \prod_{s=s_t+1}^t \frac{\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i}}{f_{\theta_\star}}(Y_s) \mid \mathcal{F}_{s_t}\right] \leq e^{-(t-s_t)c_{A4}}, \quad \forall u_{s_t:t} \in \Theta_{\gamma_t, s_t:t}. \quad (99)$$

Assume Assumption A4.b), let $Y_t^{(2)} = Y^t$ (so that $\mathbb{P}(Y_{1:t} \in Y_t^{(2)}) = 1$ for all t), $\theta \notin A'_\star$ and $u_{s_t:t} \in \Theta_{\gamma_t, s_t:t}$. Then,

$$\begin{aligned} \mathbb{E}\left[(1 - \phi_t(Y_{1:t})) \prod_{s=s_t+1}^t \frac{\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i}}{f_{\theta_\star}}(Y_s) \mid \mathcal{F}_{s_t}\right] \leq \mathbb{E}\left[\prod_{s=s_t+1}^t \frac{\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i}}{f_{\theta_\star}}(Y_s) \mid \mathcal{F}_{s_t}\right] \\ = \prod_{s=s_t+1}^t \mathbb{E}\left[\frac{\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i}}{f_{\theta_\star}}(Y_s) \mid \mathcal{F}_{s_t}\right] \quad (100) \\ \leq \left(\sup_{\theta \notin A'_\star} \mathbb{E}[\tilde{f}_\theta / f_{\theta_\star}]\right)^{t-s_t} \end{aligned}$$

where the last equality holds $t \geq t_1$. Under Assumption A4.b), $\sup_{\theta \notin A_\star} \mathbb{E}[\tilde{f}_\theta / f_{\theta_\star}] < 1$ and therefore (100) shows that (99) holds with $c_{A4} = -\log(\sup_{\theta \notin A_\star} \mathbb{E}[\tilde{f}_\theta / f_{\theta_\star}])$.

Lastly, assume Assumption A4.c) and remark that under this condition there exists a constant $c > 0$ such that $\log(\sup_{\theta \notin A_\star} \mathbb{E}[\tilde{f}_\theta]) < \mathbb{E}[\log f_{\theta_\star}] - c$, and let

$$\mathcal{Y}_t^{(2)} = \left\{ y_{1:t} \in \mathcal{Y}^t : \frac{1}{t-s_t} \sum_{s=s_t+1}^t \log(f_{\theta_\star}(y_s)) > \mathbb{E}[\log(f_{\theta_\star})] - c \right\}.$$

Then, by the law of large numbers, $\mathbb{P}(Y_{1:t} \in \mathcal{Y}_t^{(2)}) \rightarrow 1$ while, for every $\theta \notin A'_\star$ and $u_{s_t:t} \in \Theta_{\gamma_t, s_t:t}$,

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_{\mathcal{Y}_t^{(2)}}(Y_{1:t}) \prod_{s=s_t+1}^t (\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i} / f_{\theta_\star})(Y_s) \mathbb{1}_{\mathcal{Y}_t^{(2)}}(Y_{1:t}) \right] \\ \leq e^{(t-s_t)(c - \mathbb{E}[\log(f_{\theta_\star})])} \prod_{s=s_t+1}^t \mathbb{E}[\tilde{f}_{\theta - \sum_{i=s}^{t-1} u_i}(Y_s) | \mathcal{F}_{s_t}] \\ \leq e^{(t-s_t)(c - \mathbb{E}[\log(f_{\theta_\star})])} \left(\sup_{\theta \notin A_\star} \mathbb{E}[\tilde{f}_\theta] \right)^{t-s_t} \\ = e^{-(t-s_t)(\mathbb{E}[\log f_{\theta_\star}] - c - \log(\sup_{\theta \notin A_\star} \mathbb{E}[\tilde{f}_\theta]))} \end{aligned}$$

where the second equality holds for $t \geq t_1$. This shows that (99) holds with $c_{A4} = \mathbb{E}[\log(f_{\theta_\star})] - c - \log(\sup_{\theta \notin A_\star} \mathbb{E}[\tilde{f}_\theta]) > 0$.

The proof of the lemma is complete. \square

A.8.5. Proof of Lemma 6

Let $(\phi_t)_{t \geq 1}$ and $\tilde{D}_\star \in \mathbb{R}_{>0}$ be as Lemma 5, $\tilde{C}_\star \in \mathbb{R}_{>0}$ be as in Lemma 4. For every $t \geq 1$, let $C_{\delta, \tilde{\delta}}^{(\mu_{s_t} * \tilde{\pi}_{s_t})}$ be as defined in Lemma 4 and let

$$\begin{aligned} A_t = \left\{ y_{1:t} \in \mathcal{Y}^t : \int_{\Theta} \mathbb{E} \left[(\mu_{s_t} * \tilde{\pi}_{s_t})(\theta - \sum_{s=s_t+1}^{t-1} U_s) \prod_{s=s_t+1}^t \frac{\tilde{f}_{\theta - \sum_{i=s}^{t-1} U_i}(Y_s)}{f_{\theta_\star}}(Y_s) | Y_{1:t} = y_{1:t} \right] d\theta \right. \\ \left. \leq \mathbb{P}(U_{s_t:t} \in \Theta_{\tilde{\delta}, s_t:t}) C_{\delta, \tilde{\delta}}^{(\mu_{s_t} * \tilde{\pi}_{s_t})} e^{-2(t-s_t)(\tilde{C}_\star \delta)^2} \right\}. \end{aligned}$$

Let $\phi'_t(Y_{1:t}) = \mathbb{1}_{A_t}(Y_{1:t}) \vee \phi_t(Y_{1:t})$ and note that by, Lemmas 4-5,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \mathbb{E}[\phi'_t(Y_{1:t})] &\leq \limsup_{t \rightarrow \infty} \mathbb{E}[\phi_t(Y_{1:t})] + \limsup_{t \rightarrow \infty} \mathbb{P}(Y_{1:t} \in A_t) \\ &\leq \limsup_{t \rightarrow \infty} \frac{1}{\delta^2(t-s_t)\tilde{C}_\star^2} \\ &= 0 \end{aligned} \tag{101}$$

as required.

On the other hand we have, \mathbb{P} -a.s.

$$\begin{aligned}
& \mathbb{E} \left[\frac{(1 - \phi'_t(Y_{1:t})) \mathbb{1}_{\Theta_{\gamma_t, s_t:t}}(U_{s_t:t}) \pi'_{s_t,t}(V_\epsilon) | \mathcal{F}_{s_t}}{e^{2(t-s_t)} (\tilde{C}_* \delta)^2} \right] \\
& \leq \frac{1}{\mathbb{P}(U_{s_t:t} \in \Theta_{\tilde{\delta}, s_t:t}) C_{\tilde{\delta}, \tilde{\delta}}^{(\mu_{s_t} * \tilde{\pi}_{s_t})}} \\
& \times \mathbb{E} \left[\mathbb{1}_{\Theta_{\gamma_t, s_t:t}}(U_{s_t:t}) \int_{V_\epsilon} (1 - \phi_t(Y_{1:t})) (\mu_{s_t} * \tilde{\pi}_{s_t}) \left(\theta - \sum_{s=s_t+1}^{t-1} U_s \right) \prod_{s=s_t+1}^t \frac{\tilde{f}_{\theta - \sum_{i=s}^{t-1} U_i}(Y_s)}{f_{\theta_*}}(Y_s) d\theta \middle| \mathcal{F}_{s_t} \right] \\
& \leq \frac{e^{-(t-s_t)} (\tilde{D}_* - 2\tilde{C}_*^2 \delta^2)}{\mathbb{P}(U_{s_t:t} \in \Theta_{\tilde{\delta}, s_t:t}) C_{\tilde{\delta}, \tilde{\delta}}^{(\mu_{s_t} * \tilde{\pi}_{s_t})}}
\end{aligned}$$

where the second inequality holds for t large enough and uses Tonelli's theorem and Lemma 5. This completes the proof of the lemma with $C_1 = \tilde{D}_*^{-1}$ and $C_2 = 2\tilde{C}_*^2$.

B. Complement to Section 2.5 of the paper

B.1. G-PFSO based on other Markov kernels $(\tilde{M}_t)_{t \geq 1}$

It should be clear that if in (1) we define $(\tilde{M}_t)_{t \geq 1}$ using Gaussian and Student's t -distributions only our theoretical analysis of $\tilde{\pi}_t$ applies more generally for Markov kernels $(\tilde{M}_t)_{t \geq 1}$ whose tails verify certain conditions.

Notably, from the proof of Theorem 2 and of Lemma 1 it is direct to see that a sufficient condition on $(\tilde{M}_{t_p})_{p \geq 0}$ for the conclusion of these two results (and hence of Theorem 1) to hold is that there exist constants $(C, \bar{\nu}, \underline{\nu}) \in (0, \infty)^3$ such that, for all $p \in \mathbb{N}_0$ and $\gamma > 0$, we have

$$\frac{1}{C} \mathbb{P}(\|\theta'_p\| \geq \gamma) \leq \mathbb{P}(\|\theta_p\| \geq \gamma) \leq C \mathbb{P}(\|\theta''_p\| \geq \gamma)$$

where $\theta_p \sim \tilde{M}_{t_p}(0, d\theta)$, $\theta'_p \sim t_{d, \underline{\nu}}(0, h_{t_p}^2 \Sigma)$ and $\theta''_p \sim t_{d, \bar{\nu}}(0, h_{t_p}^2 \Sigma)$.

In particular, from this observation it follows that Theorem 1 also applies if, for some $\mathfrak{w} \in [0, 1)$, for all $p \in \mathbb{N}_0$ the Markov kernel \tilde{M}_{t_p} is defined by

1. $\tilde{M}_{t_p}(\theta', d\theta) = \mathfrak{w} \mathcal{N}_d(\theta', h_{t_p}^2 \Sigma) + (1 - \mathfrak{w}) t_{d, \nu}(\theta', h_{t_p}^2 \Sigma)$,
2. $\tilde{M}_{t_p}(\theta', d\theta) = \mathfrak{w} \delta_{\{\theta'\}}(d\theta) + (1 - \mathfrak{w}) t_{d, \nu}(\theta', h_{t_p}^2 \Sigma)$,
3. $\tilde{M}_{t_p}(\theta', d\theta) = \mathfrak{w} t_{d, \nu'}(\theta', h_{t_p}^2 \Sigma) + (1 - \mathfrak{w}) t_{d, \nu}(\theta', h_{t_p}^2 \Sigma)$ for some $\nu' \in (0, \nu)$.

Remark that in the above definitions of $(\tilde{M}_{t_p})_{p \geq 0}$ we recover the definition (1) of this sequence when $\mathfrak{w} = 0$. As explained in Section B.3, being able to take $\mathfrak{w} > 0$ in the above definitions 1-3 of $(\tilde{M}_{t_p})_{p \geq 0}$ may be useful in practice when a small value of ν is used.

The constraint that $(\tilde{M}_t)_{t \notin (t_p)_{p \geq 0}}$ is a sequence of Gaussian kernels can be relaxed as well. Informally speaking, for Theorem 1 to hold it is enough that, for every $t \notin (t_p)_{p \geq 0}$,

the tails of $\widetilde{M}_t(\theta', d\theta)$ are of the same size as those of the $\mathcal{N}_d(\theta', h_t^2 I_d)$ distribution. There is however no clear practical interest of such a generalization of the sequence $(\widetilde{M}_t)_{t \notin (t_p)_{p \geq 0}}$ and, for this reason, we do not discuss further this extension of G-PFSO.

B.2. G-PFSO with a provable non-vanishing ability to escape from a mode

Consider Algorithm 1 with $c_{\text{ESS}} = 1$ and, for every $(y, \theta^1, \dots, \theta^N) \in \mathcal{Y} \times \Theta^N$, let $p_N(y, \{\theta^n\}_{n=1}^N) \in [0, 1]$ be the probability to choose θ^1 at least once under the resampling algorithm $\mathcal{R}(\cdot)$ when the observation is y and the particles are $\{\theta^n\}_{n=1}^N$, that is

$$p_N(y, \{\theta^n\}_{n=1}^N) = \mathbb{P}\left(\theta_1 \in \mathcal{R}\left(\left\{\theta^n, \frac{f_{\theta^n}(y)}{\sum_{m=1}^N f_{\theta^m}(y)}\right\}_{n=1}^N\right)\right).$$

The following proposition shows that if $\alpha\nu \leq 1$ then there is a zero probability to see, for t large enough, all the particles of Algorithm 1 stuck in a given mode of the objective function.

Proposition 5. *Consider Algorithm 1 with $c_{\text{ESS}} = 1$ and $N \geq 2$. Let $A, B \in \mathcal{B}(\Theta)$ be such that $A \subset B$ and such that $\inf_{(\theta, \theta') \in A \times B} \|\theta - \theta'\| > 0$. Assume that $f_\theta(y) = 0$ for all $(\theta, y) \in (B \setminus A) \times \mathcal{Y}$ and that there exists a set $C \in \mathcal{B}(\Theta)$ such that $C \cap A = \emptyset$ and such that, for some constant $c > 0$,*

$$\mathbb{P}(\widetilde{\mathcal{Y}}) > 0, \quad \widetilde{\mathcal{Y}} := \left\{ p_N(Y_1, \{\theta^n\}_{n=1}^N) \geq c, \quad \forall \theta^1 \in C \text{ and } \forall (\theta^2, \dots, \theta^N) \in \Theta^{N-1} \right\}.$$

Assume also that there exists a set $S \in \mathcal{B}(\mathbb{R}^d)$ such that $\int_S d\theta > 0$ and such that $\theta + \epsilon \in C$ for all $(\theta, \epsilon) \in A \times S$. Let $h_t = t^{-\alpha}$ for some $\alpha > 0$ and $(\widetilde{M}_t)_{p \geq 0}$ be as defined in Section B.1 with $\nu \leq 1/\alpha$. Then,

$$\mathbb{P}\left(\liminf_{t \rightarrow \infty} \{\hat{\theta}_t^n \in A, \forall n \in \{1, \dots, N\}\}\right) = 0.$$

Remark 8. *In Proposition 5 the conditions on A imply that A contains a local or the global maximizer of the objective function $\theta \mapsto \mathbb{E}[\log f_\theta(Y_1)]$.*

Remark 9. *In practice the assumption $\mathbb{P}(\widetilde{\mathcal{Y}}) > 0$ will typically hold if C is a bounded set and $\sup_{(\theta, y) \in \Theta \times \mathcal{Y}} f_\theta(y) < \infty$, while the existence of a set S as defined in Proposition 5 requires that C is sufficiently large compared to A .*

Proof of Proposition 5. Let $1 : N = \{1, \dots, N\}$, $(U_t)_{t \geq 1}$ be a sequence of i.i.d. $\mathcal{U}((0, 1))$ random variables (independent of $(Y_t)_{t \geq 1}$) and $(\epsilon_t^1)_{t \geq 1}$ be a sequence of independent random variables (independent of $(Y_t)_{t \geq 1}$ and of $(U_t)_{t \geq 1}$) such that $\epsilon_t^1 \sim \widetilde{M}_t(0, d\theta)$ for all $t \geq 1$. Then, for all $t \geq 2$ we have

$$\begin{aligned} \left\{ \exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \in C \right\} &\supset \left\{ \exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \in C, \hat{\theta}_{t-1}^1 \in A \right\} \\ &\supset \left\{ \epsilon_t^1 \in S, Y_t \in \widetilde{\mathcal{Y}}, U_t \leq c, \hat{\theta}_{t-1}^1 \in A \right\} \\ &= \left\{ \epsilon_t^1 \in S, Y_t \in \widetilde{\mathcal{Y}}, U_t \leq c \right\} \cap \left\{ \hat{\theta}_{t-1}^1 \in A \right\}. \end{aligned} \tag{102}$$

We now let

$$C_\nu = \frac{\Gamma((\nu + d)/2)}{\Gamma(\nu/2)(\pi\nu)^{d/2}|\Sigma|^{1/2}}$$

so that, for some non-empty interval $[a, b] \subset S$ and every $p \geq 0$ we have

$$\begin{aligned} \mathbb{P}(\epsilon_{t_p}^1 \in S) &\geq \mathbb{P}(\epsilon_{t_p}^1 \in [a, b]) \\ &= C_\nu h_{t_p}^{-d} \int_{[a, b]} \left(1 + \frac{1}{h_{t_p}^2 \nu} x^\top \Sigma^{-1} x\right)^{-\frac{\nu+d}{2}} \\ &= C_\nu h_{t_p}^\nu \int_{[a, b]} \left(h_{t_p}^2 + \frac{1}{\nu} x^\top \Sigma^{-1} x\right)^{-\frac{\nu+d}{2}} \\ &\geq C_\nu h_{t_p}^\nu \int_{[a, b]} \left(1 + \frac{1}{\nu} x^\top \Sigma^{-1} x\right)^{-\frac{\nu+d}{2}}. \end{aligned}$$

Since by assumption we have $\alpha\nu \leq 1$, we have

$$\begin{aligned} \sum_{t \geq 1} \mathbb{P}(\epsilon_t^1 \in S, Y_t \in \tilde{Y}, U_t \leq c) &= \sum_{t \geq 1} c \mathbb{P}(\epsilon_t^1 \in S) \mathbb{P}(Y_t \in \tilde{Y}) \\ &= c \mathbb{P}(Y_1 \in \tilde{Y}) \sum_{t \geq 1} \mathbb{P}(\epsilon_t^1 \in S) \\ &\geq c \mathbb{P}(Y_1 \in \tilde{Y}) \sum_{p \geq 0} \mathbb{P}(\epsilon_{t_p}^1 \in S) \\ &= \infty \end{aligned}$$

and thus, by the second Borel-Cantelli lemma, there exists a $\Omega' \in \mathcal{F}$ such that $\mathbb{P}(\Omega') = 1$ and such that, for all $\omega \in \Omega'$, the event $\{\epsilon_t^1(\omega) \in S, Y_t(\omega) \in \tilde{Y}, U_t(\omega) \leq c\}$ occurs for infinitely many $t \in \mathbb{N}$.

To complete the proof remark first that if the event $\{\exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \in C\}$ occurs for infinitely many $t \in \mathbb{N}$ then, since $C \cap A = \emptyset$, the event $\{\exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \notin A\}$ also occurs for infinitely many $t \in \mathbb{N}$. Therefore,

$$\begin{aligned} &\mathbb{P}\left(\limsup_{t \rightarrow \infty} \{\exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \notin A\}, \limsup_{t \rightarrow \infty} \{\exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \in C\}\right) \\ &= \mathbb{P}\left(\limsup_{t \rightarrow \infty} \{\exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \in C\}\right). \end{aligned} \tag{103}$$

Next, remark that if $w \in \Omega'$ is such that the event $\{\exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n(w) \in C\}$ occurs only for finitely many $t \in \mathbb{N}$ then, by (102), the event $\{\hat{\theta}_t^1(w) \notin A\}$ occurs for infinitely many t . Hence,

$$\begin{aligned} &\mathbb{P}\left(\limsup_{t \rightarrow \infty} \{\exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \notin A\}, \left(\limsup_{t \rightarrow \infty} \{\exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \in C\}\right)^c\right) \\ &= \mathbb{P}\left(\limsup_{t \rightarrow \infty} \{\exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \notin A\}, \left(\limsup_{t \rightarrow \infty} \{\exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \in C\}\right)^c, \Omega'\right) \\ &= \mathbb{P}\left(\left(\limsup_{t \rightarrow \infty} \{\exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \in C\}\right)^c\right). \end{aligned} \tag{104}$$

Therefore, by (103)-(104),

$$\mathbb{P}\left(\limsup_{t \rightarrow \infty} \{\exists n \in 1 : N \text{ s.t. } \hat{\theta}_t^n \notin A\}\right) = 1 \Leftrightarrow \mathbb{P}\left(\liminf_{t \rightarrow \infty} \{\hat{\theta}_t^n \in A, \forall n \in 1 : N\}\right) = 0.$$

The proof is complete. \square

B.3. Discussion

Fulfilling with $\alpha = 0.5$ the condition $\nu \leq 1/\alpha$ imposed in Proposition 5 requires to let $\nu \leq 2$, that is to use in the definition of $(\widetilde{M}_{t_p})_{p \geq 0}$ Student's t -distributions having an infinite variance. As illustrated in Section 4.4 of the paper, G-PFSO may not work well when these kernels are as defined in (1) for some $\nu \leq 2$, the reason being that when ν is small there may be a large probability that at time $t \in (t_p)_{p \geq 0}$ the estimator $\tilde{\theta}_{t_p}^N$ is pushed away from θ_* . For such values of ν , defining $(\widetilde{M}_{t_p})_{p \geq 0}$ as in Section B.1 for some $\mathfrak{w} > 0$ both leads to kernels that verify the conditions Proposition 5 (assuming $h_t = t^{-1/2}$) and reduces the probability of this undesirable event, since as \mathfrak{w} increases the probability to have $\theta_{t_p}^n$ close to $\hat{\theta}_{t_p-1}^n$ becomes larger, for all $t \in (t_p)_{p \geq 0}$ and $n \in \{1, \dots, N\}$.

We finally recall that the condition $\nu \leq 1/\alpha$ imposed in Proposition 5 can be fulfilled with $\nu > 2$, provided that $\alpha < 0.5$. However, as illustrated in Section 4.2, when $\alpha < 0.5$ the estimator $\tilde{\theta}_t^N$ is expected to converge at a sub-optimal rate.

C. Additional information for the numerical experiments

C.1. Code

The code used to produce all the figures of the paper is available on GitHub at

<https://github.com/mathieugerber/gpfso-paper>

C.2. Censored quantile regression model

Let $(Y_t := (Z_t, X_t))_{t \geq 1}$ be a sequence of i.i.d. random variables taking values in $[0, \infty) \times \mathbb{R}^d$ for some $d \geq 1$. Then, with $\Theta = \mathbb{R}^d$, the censored quantile regression (CQR) model assumes that, for every $\tau \in (0, 1)$ and $x \in \mathbb{R}^d$, the τ -th conditional quantile function of Z_t given $X_t = x$ does not depend on t and belongs to the set $\{\max\{x^\top \theta, 0\}, \theta \in \Theta\}$. For every $\tau \in (0, 1)$ we let $\rho_\tau : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\rho_\tau(u) = (|u| + (2\tau - 1)u)/2$, $u \in \mathbb{R}$, and $\theta_\star^{(\tau)} = \operatorname{argmin}_{\theta \in \Theta} \mathbb{E}[\rho_p(Z_1 - \max\{X_1^\top \theta, 0\})]$ be the parameter value of interest.

In order to cast this estimation problem into the general set-up of this paper, following Yu and Moyeed (2001) we let $f_{\tau, \theta}(\cdot | x)$ be the density of the asymmetric Laplace distribution with location parameter $\max\{x^\top \theta, 0\}$, scale parameter $\sqrt{\tau(1 - \tau)}$ and asymmetry parameter $\sqrt{\tau/(1 - \tau)}$; that is

$$f_{\tau, \theta}(z | x) = \tau(1 - \tau) \exp\left\{-\rho_p(z - \max\{x^\top \theta, 0\})\right\}, \quad \forall z \in \mathbb{R}. \quad (105)$$

Then, $\theta_\star^{(\tau)}$ can be redefined as $\theta_\star^{(\tau)} = \operatorname{argmax}_{\theta \in \Theta} \mathbb{E}[\log f_{\tau, \theta}(Z_1 | X_1)]$.

We let $d = 5$ and simulate $T = 10^7$ independent observations $\{(z_t, x_t)\}_{t=1}^T$ according to

$$Z_t = \max(\tilde{Z}_t, 0), \quad \tilde{Z}_t | X_t \sim \mathcal{N}_1(X_t^\top \theta_\star^{(0.5)}, 4), \quad X_t \sim \delta_{\{1\}} \otimes \mathcal{N}_{d-1}(0, \Sigma_X), \quad t \geq 1$$

where Σ_X^{-1} is a random draw from the Wishart distribution with $d - 1$ degrees of freedom and scale matrix I_{d-1} , while $\theta_\star^{(0.5)} = (3, \theta_{\star,2:d}^{(0.5)})$ with $\theta_{\star,2:d}^{(0.5)}$ a random draw from the $\mathcal{N}_{d-1}(0, I_{d-1})$ distribution. The resulting sample $\{(z_t, x_t)\}_{t=1}^T$ is such that about 13% of the observations are censored.

C.3. Smooth adaptive Gaussian mixture model

For this example we generate $T = 2 \times 10^6$ independent observations $\{(z_t, x_t)\}_{t=1}^T$ using

$$Z_t | X_t \sim \int f_{\theta_\star}(z | X_t) dz, \quad X_t \sim \delta_{\{1\}} \otimes \mathcal{N}_{d_x-1}(0, I_{d_x-1}), \quad (106)$$

where $\theta_\star = (\beta_\star^w, \beta_\star^\mu, \beta_\star^\sigma)$ with $\beta_\star^w = (1, 0.1, 0.1, -0.1)$, $\beta_{\star,(1)}^\mu = (1, 1, 1, -1)$, $\beta_{\star,(2)}^\mu = (-1, 1, 1, 1)$, $\beta_{\star,(1)}^\sigma = (0, 1, 1, 1)$ and $\beta_{\star,(2)}^\sigma = (0.5, -1, -1, 1)$. For this choice of θ_\star , and in average over the x_t 's, the k -th component of $f_{\theta_\star}(\cdot | x_t)$ has a mean and a variance¹ respectively equal to 1 and to 7.28 for $k = 1$, and respectively equal to -1 and to 5.1 for $k = 2$, while the first component of $f_{\theta_\star}(\cdot | x_t)$ has a weight approximatively equal to 0.27.

¹To remove the effect of a few extreme variances these numbers are actually the mean values of the variances which are smaller than 100.