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# Stochastic Covariant Derivatives in a (Curved) Space-Time: a Glimpse into the Fractoid Spaces

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**STOCHASTIC COVARIANT DERIVATIVES  
IN A (CURVED) SPACE-TIME:  
A GLIMPSE INTO THE FRACTOID SPACES**

EDOARDO NICCOLAI

ABSTRACT. A study on the notion of covariant derivatives in flat and curved space-time via Itô–Wiener processes, when subjected to stochastic processes, is presented. Going into details, there is an analysis of the following topics: (i) Besov space, (ii) Schrödinger operators, (iii) Klein–Gordon and Dirac equations, (iv) Dirac operator via Clifford connection, (v) semi-martingale and Stratonovich integral, (vi) stochastic geodesics, (vii) white noise on a (4+)D space-time  $\mathfrak{S}$ -geometry (with the Paley–Wiener integral), and (viii) torsion of the covariant derivative. In the background stands the scale relativity theory, together with a sketch of the concept of fractoid spaces.

KEYWORDS:  $\sigma$ -(sub)algebra(s), Besov space,  $\mathcal{C}^2$  smooth vector fields, Clifford connection, covariant derivative of stochastic type in  $\mathbb{R}_{1,3}^4$  (Minkowski space-time) and in curved space-time (GR), Dirac equation(s), Dirac operator, forward and backward mean derivatives, fractal space-time, fractoid spaces, Itô–Wiener processes, Klein–Gordon equation(s), Littlewood–Paley operator, martingale, Newton–Nelson equation(s), Paley–Wiener integral, quantum-like fluctuations, random fluctuations, relativistic generalization of Nelson’s stochastic mechanics, scale relativity, Schrödinger operators, Schwartz space, torsion.

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## Thierry's Question

Que devient la dérivée covariante, que je note ici  $D^*$ , de Laurent Nottale, obtenue en supposant un espace temps fractal, qui revient à un opérateur de type Schrödinger en  $i\frac{d}{dt} + \text{Laplacien}$  dans un espace temps plat dans le cas non relativiste, dans le cas de la relativité général avec une métrique  $g_{ij}$  qui fluctuerait de manière stochastique?

— T. LEHNER, via email to me

Patently, I do not offer a *réponse univoque* to my sodalis Thierry, great and συμπαθητικός friend, also because our starting perspectives are different (me with a mathematical priority, he with a physical priority), but a series of *pièces de puzzle*, nay, a 七巧板 (qīqiǎobǎn), or *tangram*, with a set of pieces, which, assembled together, can help and contribute to offering a *réponse* to his question.

### OUVERTURE

## 1. Scale Relativity and Non-differentiability of a Continuous Fractal Space-Time

### 1.1. A Quick Summary

To begin with, a few words about the scale relativity by L. Nottale & collaborators [29] [30] [33] [31] [32] [34]. This theory is made up of the following postulates.

(1) The concept of space-time is *continuous* but *non-differentiable*, that is to say, *fractal*. The space contemplated in the theory of scale relativity is thereby originated in the Mandelbrot set,

$$M_{\mathbb{C}} = \{c \in \mathbb{C} \mid \varphi_c^n(0) \not\rightarrow \infty \text{ as } n \rightarrow \infty\}. \quad (1)$$

This set is related to a complex quadratic polynomial  $\varphi_c$ ,

$$\mathbb{C} \xrightarrow{\varphi_c} \mathbb{C}, \quad (2a)$$

$$\varphi_c(z) = z^2 + c, \quad (2b)$$

under the map (2a) plus the equation (2b), hence a quadratic family  $\varphi_c: z \mapsto z^2 + c$ .

Consequently,

- (i) any covariant derivative is a math-construct of the non-differentiable and fractal geometry;
  - (ii) the baggage of coordinate transformations are continuous but can be non-differentiable.
- (2) Let us try to find out what that means.

(i) A continuous curve line  $\gamma_c$ , or rather, a geodesic, is non-differentiable when its length  $\ell(\gamma_c)$  is dependent on a scale variable  $\varepsilon_s$ , and it diverges,  $\ell(\varepsilon_s(\gamma_c)) \rightarrow \infty$ , if  $\varepsilon_s \rightarrow 0$ , which constitutes its fractality; the scale divergence of continuous and almost nowhere-differentiable curves turns up as a counterpart to the extension of the fundamental theorem of calculus, of Lebesgueian memory, on the basis of which a curve, of finite length, is almost everywhere differentiable. For a continuous fractal  $\alpha$ -function, namely a scale-dependent function,  $\varphi^\alpha(x)$  viz.  $x(t, \delta t)$ , all this is exemplified by two equality procedures—via nonstandard analysis, in which  $\delta t$  is replaced with  $dt$ —enunciating the variation of the position vector  $x$  of a particle, along a geodesic, between  $t - dt$  and  $t$ , and between  $t$  and  $t + dt$ ,

$$x(t + dt)dt - x(t, dt) = v_+(x, t)dt + \sigma_s^+(t, dt) \left( \frac{dt}{\tau_{\text{EB}}} \right)^{\frac{1}{D_f}}, \quad (3)$$

$$x(t, dt) - x(t - dt)dt = v_-(x, t)dt + \sigma_s^-(t, dt) \left( \frac{dt}{\tau_{\text{EB}}} \right)^{\frac{1}{D_f}}, \quad (4)$$

where  $\sigma_s^+$  and  $\sigma_s^-$  are *stochastic variables* exhibiting finite *velocity fluctuations*,  $\tau_{\text{EB}} = \frac{\hbar}{mc^2}$  is the Einstein–de Broglie time, and  $D_f$  is the fractal dimension (or Hausdorff or Hausdorff–Besicovitch dimension) of the path.

If one wants, Eqq. (3) (4) stand in a statistical explication, under diffusion coefficients.

(ii) The same rule of the previous point (i) applies to the notion of manifold, thus to a topological space.

(iii) Regarding the fractal space-time, it entails the scale dependence of the reference frames, with internal scale variables: obviously, it will consist of non-differentiable but also differentiable parts (in any case, we are dealing with a *continuum* which, historiquement et par définition, is differentiable); the non-differentiable part, the fractal one, will be characterized by *stochastic*

*fluctuations* (the two parts are later combined together). It is conjectured that fluctuations in the fractal space-time lead to  $\delta D_m(x, t)$ -fluctuations of the *coefficient of diffusion*

$$D_m = \left(\frac{1}{2t}\right) \int_{-\infty}^{+\infty} \Delta^2 \mathcal{P}(\Delta, t) d\Delta = \left(\frac{1}{t}\right) \int_{-\infty}^{+\infty} \frac{\Delta^2}{2} \mathcal{P}(\Delta, t) d\Delta, \quad (5)$$

where  $\mathcal{P}(\Delta, \tau)$  is the probability density function, and  $d\Delta$  is the displacement, or the length of path, from  $\Delta$  to  $\Delta + d\Delta$ , in an interval of time  $t$ . Cf. Eq. (14.20) in [27]; from such a context to the theory of Brownian motion, and interconnected Chapman–Kolmogorov equation, Fokker–Planck equation, etc., is a flash.

NB. There is also another fact that brings stochasticity in evidence: since the geodesics of fractal space-time are infinite, its interpretation must be stochastic.

(3) The scale relativity is conceived in such a way as to reject a *discrete* space-time, as it is asserted in the *poussière de Cantor* [21, pp. 55-62],<sup>a</sup> or in the fractal *caillé*.

## 1.2. The Head-scratcher of the Covariant Derivative

The quandary of the covariant derivative in scale relativity is due to two conditions.

### 1.2.1. Lack of Differentiability

The first is that, where there is a lack of differentiability, owing to fractality, of space-time, one is witnessing a divergence that cannot be managed, except with stochastic tools—this is a clue of the *mathematical skeleton* of the scale-dependent properties, which have a random bosom.<sup>b</sup>

### 1.2.2. The Obstacle of the Two Derivatives, and the Arrangement with Sobolev and Besov Spaces

The second condition is that *two derivatives* arise, the classical one and the scale covariant derivative, the latter inserted in the fractal structure, once one accepts the hypothesis that the space-time is continuous, as the theory of scale relativity requires.

That explains why one of the modes of enunciating *fractional derivatives*, in favor of a good mathematical rigour, is to utilize the standard Sobolev space [40] [41]

$$W^{k,2}(Q) = \mathfrak{S}^k(Q), L^2(Q) = \mathfrak{S}^0(Q), \quad k \in \mathbb{Z}_*, \text{ or } k \in \mathbb{N}, Q \subset \mathbb{R}^n,$$

easily tractable; for  $p = 2$ , the Sobolev space  $W^{k,p}(Q)$  is a Hilbert space.

An analogous goal, toward the *fractal Laplacian*, is achieved with the Besov space [1] [2]<sup>c</sup>

$$B_{p,\infty}^s(\mathbb{R}^2), \quad s \in \mathbb{R}^n, 0 < p \leq \infty,$$

which generalizes the Sobolev space, when the  $B$ -space is a collection of every regular tempered (or Schwartz) distribution  $v_t \in \mathcal{D}^*(\mathbb{R}^2)$ ,  $\mathcal{D}^* \stackrel{\text{eqv}}{=} \mathcal{S}_t^*$  being the Schwartz space [12] [7] [39] of distributions, such that the norms

$$\|\varrho | L^p(\mathbb{R}^2)\| + \sup_{0 < |v_t| \leq 1} |v_t|^{-s} \left\| \Delta_{v_t}^k \varrho | L^p(\mathbb{R}^2)\right\|, \quad \frac{2(1-p)}{p} < s < k \in \mathbb{N}, \quad (6a)$$

$$\|\varrho | L^p(\mathbb{R}^2)\| + \left( \int_0^1 |v_t|^{-sq} \left\| \Delta_{v_t}^k \varrho | L^p(\mathbb{R}^2)\right\|^q \frac{dv_t}{|v_t|^2} \right)^{\frac{1}{q}}, \quad 0 < q < \infty, \quad (6b)$$

are finite, id est Eqq. (6a) (6b)  $< \infty$ , where  $\varrho$  is a continuous monotonically increasing function, and  $L^p$  the Lebesgue space. The operator  $\Delta$  can be recognized as the Littlewood–Paley operator [18] [19] [20].

<sup>a</sup> See, in this respect, the interesting notion of *fractal lacunarity* [22].

<sup>b</sup> It is not a coincidence that the adjective  $\sigma\tau\omicron\chi\alpha\sigma\tau\iota\kappa\acute{o}\varsigma$  means “able to hit”, “skilful in aiming”, “guessing”.

<sup>c</sup> For a detailed bibliography on Besov spaces, see [38, pp. 898-901].

### 1.2.3. Do Not Forget the Discretum

I disagree with maintaining the continuity of space as the womb of space-time. Actually, the continuum seems to be an *approximation of the discretum*, cf. [27, Margo 9.2.1]. This is the *original sin* inherent in the theory of scale relativity, as far as I am able to judge.

## PARTS OF THE ANSWER: A FAB-TANGRAM

## 2. Some Tips. Let us Do a Refresh: Schrödinger Operators

Let us first give some definitions, just to cross some T's and dot some I's.

### 2.1. Linear Schrödinger Operator for a 1-parameter Unitary Group

The simple equality

$$\ddot{S}_t \stackrel{\text{viz}}{=} U_t = e^{it\Delta} \quad (7)$$

establishes the Schrödinger operator, whose notation here is a letter  $S$  with diaeresis, for a 1-parameter unitary group. The symbol  $\Delta$  designates the Laplacian.

For an equality

$$\ddot{S}_t\varphi = \psi(\hat{x}, t),$$

the linear Schrödinger operator (7) is determined by

$$\ddot{S}_t\varphi(\hat{x}) = \left(\frac{1}{4\pi it}\right)^{\frac{n}{2}} \int e^{i\frac{|\hat{x}-\dot{\hat{x}}|^2}{4t}} \varphi(\dot{\hat{x}}) d\dot{\hat{x}}, \quad \hat{x} \in \mathbb{R}^n. \quad (8)$$

### 2.2. Schrödinger Operator on Continuous $L^2(\mathbb{R}^n)$ -Space Structures

The Schrödinger operator

$$\ddot{S} = -\frac{\hbar^2}{2} \Delta + a[\mathbb{R}]. \quad (9)$$

is a linear partial differential operator in the Hilbert space—and in fact  $\mathfrak{S} \stackrel{\text{viz}}{=} L^2(\mathbb{R}^n)$ ; in Eq. (9) the Laplacian is  $n$ -dimensional, and  $a$  is a smooth  $\mathbb{R}$ -potential.

### 2.3. Covariant Schrödinger Operator on Riemannian $n$ -Manifolds (Groupal Algebra): $\ddot{S}_\mathfrak{S}^\nabla$ in $\Gamma_\mathfrak{S}^{(L^2)}$

We define a covariant Schrödinger operator  $\ddot{S}_\mathfrak{S}^\nabla$ , where  $\nabla$  is a (metric) covariant derivative on a (metric) vector bundle  $\mathring{\mathcal{E}}$  over  $\mathcal{M}$ , i.e.  $\mathring{\mathcal{E}} \rightarrow \mathcal{M}$ , and  ${}_{\text{s-adj}}(\mathfrak{a}) \stackrel{\text{viz}}{=} \mathfrak{a}$  is a self-adjoint endomorphism, on a smooth Riemannian pair  $(\mathcal{M}, g)$  in the Hilbert space  $L^2$  of square-integrable sections, denoted by

$$\Gamma_\mathfrak{S}^{(L^2)}(\mathcal{M}, \mathring{\mathcal{E}}),$$

so to get

$$\left(e^{-t\ddot{S}_\mathfrak{S}^\nabla}(x)\right)_{t \geq 0} \subset \Gamma_\mathfrak{S}^{(L^2)}(\mathcal{M}, \mathring{\mathcal{E}}), \quad t > 0, x \in \mathcal{M}. \quad (10)$$

Let  $d\mu$  be the Riemannian volume element ( $\mu$  is for a Borel measure); for a function

$$\varphi_\mu \in \Gamma_\mathfrak{S}^{(L^2)}(\mathcal{M}, \mathring{\mathcal{E}}),$$

one has

$$e^{-t\ddot{S}_\mathfrak{S}^\nabla}\varphi(x) = \int_{\mathcal{M}} e^{-t\ddot{S}_\mathfrak{S}^\nabla}(x, y) \varphi_\mu(y) d\mu(y). \quad (11)$$

Here the Schrödinger operator  $\ddot{S}_\mathfrak{S}^\nabla$  is but a *covariant Schrödinger bundle*, that is,

$$\left(\mathring{\mathcal{E}}, \nabla, \mathfrak{a}\right) \xrightarrow{(\cdot)} \mathcal{M},$$

bearing in mind that the map

$$\mathfrak{a}: \mathcal{M} \rightarrow \text{end}\left(\mathring{\mathcal{E}}\right) \quad (12)$$

is Borel  $\mathbb{P}$ -measurable, having a linear self-adjoint map  $\vartheta(x): \mathring{\mathcal{E}}_x \rightarrow \mathring{\mathcal{E}}_x$ , for each  $x \in \mathcal{M}$ . The self-adjoint endomorphism  ${}_{s\text{-adj}}(\vartheta)$  in (12) counts as a potential on  $\mathring{\mathcal{E}} \rightarrow \mathcal{M}$ .

#### 2.4. Random Schrödinger Operator on Discrete $L^2(\mathbb{Z}^n)$ -Space Structures

If one needs to use a random Schrödinger operator can quietly go from a  $L^2(\mathbb{R}^n)$ -space to  $L^2(\mathbb{Z}^n)$ -space, in such a way that the  $\Delta$ -operator from continuous on the  $\mathbb{R}$ -field becomes discrete on the  $\mathbb{Z}$ -field, so as to have this distinctness:

$$-\Delta_{\mathbb{Z}} = \sum_{\lambda=1}^n \left( {}^2\varphi(x)_{\mathbb{Z}} - \varphi(x - e_{\lambda})_{\mathbb{Z}} - \varphi(x + e_{\lambda})_{\mathbb{Z}} \right). \quad (13)$$

### 3. Some Answers—Let Us Just Cut to the Chase: Covariant Derivatives

From this Section, until the end of the articles, I will sketch some solutions to the opening question.

#### 3.1. Covariant Derivative of Stochastic Type in $\mathbb{R}_{1,3}^4$ (c.-à-d. on a Flat Lorentz–Minkowski Space-Time) via Itô–Wiener Processes

Here we look for the covariant derivative, within the stochastic realm, in Minkowski/Lorentz–Minkowski space-time  $\mathbb{M}^4 = \mathbb{R}_{1,3}^4$ .

The first thing to do is recover Itô's formula, for  $\mathbb{M}^4 \stackrel{\text{viz}}{=} \mathcal{M}^4$ , together with the Wiener processes, for  $\mathbb{R}_{1,3}^4$ .

We choose

$$\phi_t = \phi_0 + \int_0^t \beta_s ds + \epsilon_{\mathbb{R}} W_t \quad (14)$$

as an *Itô diffusion-type process*, where

$\phi_t$  is a stochastic process,

$\beta_s$  is a process almost surely with bounded variation of some path,

$\epsilon_{\mathbb{R}} > 0$  is a real constant,

$W_t$  is the Wiener stochastic process, which is almost surely continuous in  $t$ , and square-integrable martingale regarding a non-decreasing family  $\mathcal{U}_t$ ,  $t \in [0, \infty)$  of  $\sigma$ -subalgebras of the  $\sigma$ -algebra  $\mathcal{B}$ .

*Marginalia 3.1.* A martingale is a stochastic process governed by a sequence of *random fluctuations*.

Consider that in my formalism—cf. [27, Sec. 12.4.3.2, and Definition 16.1.9]—the triple  $(\hat{\Omega}, \mathcal{B}_{\sigma}, \mathbb{P})$  denotes a *probability space* with a Borel  $\sigma$ -algebra on  $\hat{\Omega}$ .

Let

$\tau$  be a proper time (an invariant parameter),

$\mathcal{O}$  viz.  $\mathcal{O}(\mathcal{U}_{\tau}^{\phi})$  be the conditional expectation on  $(\hat{\Omega}, \mathcal{B}_{\sigma}, \mathbb{P})$  of  $\phi_{(t)}$  concerning the  $\sigma$ -algebra generated by some Borel sets in a  $n$ -dimensional  $\mathbb{R}$ -field, with the map  $v: \hat{\Omega} \rightarrow \mathbb{R}^n$  (that is why the expectation covers the  $\mathcal{U}_t$ ),

$N$  be the present state,<sup>a</sup> viz. the now, of  $\phi_{\tau}$ , scilicet the present  $\sigma$ -algebra for  $\phi_{\tau}$ .

Suppose that  $\phi_{\tau}$  (the stochastic process) has values in a specific Riemannian manifold, in compliance with the map  $\phi_{\tau}: \hat{\Omega} \rightarrow \mathcal{M}$ .

Then we can write, à la Dohrn–Guerra–Ruggiero [10] [14],<sup>b</sup> the *relativistic forward and backward mean derivative of stochastic type*, indicated with  $D^+_{\phi_{\tau}}$  and  $D^-_{\phi_{\tau}}$ , respectively, for a flat pseudo-Euclidean (Minkowskian-like) Lorentzian space-time:

$$\begin{aligned} D^+_{\phi_{\tau}} &= \lim_{\Delta\tau \downarrow 0} \mathcal{O} \left\{ \frac{\phi(\tau + \Delta\tau) - \phi_{\tau}}{\Delta\tau} \left| N_t^{\phi} \left( \phi(\tau + \Delta\tau) - \phi_{\tau} \right)^2 \leq 0 \right. \right\} \\ &\quad + \lim_{\Delta\tau \downarrow 0} \mathcal{O} \left\{ \frac{\phi_{\tau} - \phi(\tau - \Delta\tau)}{\Delta\tau} \left| N_t^{\phi} \left( \phi_{\tau} - \phi(\tau - \Delta\tau) \right)^2 \geq 0 \right. \right\} \end{aligned} \quad (15)$$

<sup>a</sup> It is not a La. letter. It is a capital Gr. letter, from the adverb  $\nu\upsilon\nu$ , “now”.

<sup>b</sup> See also A.B. Cruzeiro and J.-C. Zambrini [5, sec. 5].

and

$$D^- \phi_\tau = \lim_{\Delta\tau \downarrow 0} \mathfrak{D} \left\{ \frac{\phi_\tau - \phi(\tau - \Delta\tau)}{\Delta\tau} \left| N_t^\phi \left( \phi_\tau - \phi(\tau - \Delta\tau) \right)^2 \leq 0 \right. \right\} \\ + \lim_{\Delta\tau \downarrow 0} \mathfrak{D} \left\{ \frac{\phi(\tau + \Delta\tau) - \phi_\tau}{\Delta\tau} \left| N_t^\phi \left( \phi(\tau + \Delta\tau) - \phi_\tau \right)^2 \geq 0 \right. \right\}, \quad (16)$$

marking with  $\Delta\tau$  the relativistic displacements a/o increments of  $\phi_\tau$ .

The relativistic forward mean derivative  $D^+ \phi_\tau$  (15) and the relativistic backward mean derivative  $D^- \phi_\tau$  (16) are *covariant* under the Lorentz transformations of the reference systems given by the tetrads.

It will of course be useful to note that one has  $D^+ \phi_\tau = \bar{\mathfrak{H}}^+(\tau, \phi_\tau)$ , for a  $\mathcal{C}^2$  smooth vector field

$$\bar{\mathfrak{H}}^+(\tau, x) = \lim_{\Delta\tau \downarrow 0} \mathfrak{D} \left\{ \frac{\phi(\tau + \Delta\tau) - \phi_\tau}{\Delta\tau} \left| \phi_\tau = x \left( \phi(\tau + \Delta\tau) - \phi_\tau \right)^2 \leq 0 \right. \right\} \\ + \lim_{\Delta\tau \downarrow 0} \mathfrak{D} \left\{ \frac{\phi_\tau - \phi(\tau - \Delta\tau)}{\Delta\tau} \left| \phi_\tau = x \left( \phi_\tau - \phi(\tau - \Delta\tau) \right)^2 \geq 0 \right. \right\}, \quad (17)$$

and  $D^- \phi_\tau = \bar{\mathfrak{H}}^-(\tau, \phi_\tau)$ , for a  $\mathcal{C}^2$  smooth vector field

$$\bar{\mathfrak{H}}^-(\tau, x) = \lim_{\Delta\tau \downarrow 0} \mathfrak{D} \left\{ \frac{\phi_\tau - \phi(\tau - \Delta\tau)}{\Delta\tau} \left| \phi_\tau = x \left( \phi_\tau - \phi(\tau - \Delta\tau) \right)^2 \leq 0 \right. \right\} \\ + \lim_{\Delta\tau \downarrow 0} \mathfrak{D} \left\{ \frac{\phi(\tau + \Delta\tau) - \phi_\tau}{\Delta\tau} \left| \phi_\tau = x \left( \phi(\tau + \Delta\tau) - \phi_\tau \right)^2 \geq 0 \right. \right\}. \quad (18)$$

### 3.2. Covariant Derivative of Stochastic Type in Curved Space-Time (c.-à-d. on a Lorentz Hyperbolic Manifold) via Itô–Wiener Processes

We analyze the context of general relativity (GR). Let  $\bar{\mathfrak{H}}(\tau, m)$  be a vector field on a Lorentz 4-manifold, symbolized by  $\mathbb{L}^4$ , with a metric signatures  $(1,3)^-$  viz.  $(-, +, +, +)$ , or, which is the same, to use the usual Riemannian notation, on a 4-manifold, symbolized by  $\mathcal{M}^4$ , of type  $\mathcal{C}^2$  smooth, considering within this scheme a Lorentzian orthonormal frame in the tangent space  $\mathcal{T}_x \mathcal{M}^4$ ,  $x \in \mathcal{M}^4$ .

Let us say that  $\Gamma_{\tau,s}$  is an operator of parallel translation—derived from the Levi-Civita-like connection—on the Lorentz bundle  $\mathcal{H}(\mathcal{M}^4)$ , along a stochastic Itô-process [15] [16],<sup>a</sup> from a *random point*  $\phi_s$  to another *random point*  $\phi_\tau$ . Therefore the *displacements a/o deviations of the geodesic* are taken into account.

For the general relativity, we define the *covariant relativistic mean derivatives of stochastic type*,

$$D^+ \bar{\mathfrak{H}}(\tau, \phi_\tau) \text{ and } D^- \bar{\mathfrak{H}}(\tau, \phi_\tau),$$

over a  $\mathbb{L}^4$ -manifold, on the guideline of these equalities:

$$D \bar{\mathfrak{H}}(\tau, \phi_\tau) = \lim_{\Delta\tau \rightarrow +0} \mathfrak{D}_\tau^\phi \left\{ \frac{\Gamma_{\tau, \tau + \Delta\tau} \bar{\mathfrak{H}}(\tau + \Delta\tau, \phi(\tau + \Delta\tau)) - \bar{\mathfrak{H}}(\tau, \phi_\tau)}{\Delta\tau} \right\}, \quad (19)$$

and

$$D^* \bar{\mathfrak{H}}(\tau, \phi_\tau) = \lim_{\Delta\tau \rightarrow +0} \mathfrak{D}_\tau^\phi \left\{ \frac{\bar{\mathfrak{H}}(\tau, \phi_\tau) - \Gamma_{\tau, \tau - \Delta\tau} \bar{\mathfrak{H}}(\tau - \Delta\tau, \phi(\tau - \Delta\tau))}{\Delta\tau} \right\}, \quad (20)$$

after specifying the expressions

$$D \bar{\mathfrak{H}} = \frac{\partial \bar{\mathfrak{H}}}{\partial \tau} + \nabla_\beta \bar{\mathfrak{H}} + \left( \frac{1}{2} \right) \nabla^2 \bar{\mathfrak{H}} = \frac{\partial \bar{\mathfrak{H}}}{\partial \tau} + \nabla_\beta \bar{\mathfrak{H}} + \left( \frac{\sigma_{\mathbb{R}}^2}{2} \right) \nabla^2 \bar{\mathfrak{H}}, \quad (21)$$

$$D^* \bar{\mathfrak{H}} = \frac{\partial \bar{\mathfrak{H}}}{\partial \tau} + \nabla_{\beta^*} \bar{\mathfrak{H}} - \left( \frac{1}{2} \right) \nabla^2 \bar{\mathfrak{H}} = \frac{\partial \bar{\mathfrak{H}}}{\partial \tau} + \nabla_{\beta^*} \bar{\mathfrak{H}} - \left( \frac{\sigma_{\mathbb{R}}^2}{2} \right) \nabla^2 \bar{\mathfrak{H}}, \quad (22)$$

<sup>a</sup> See the finishing touches in [9].



where  $\nabla$  is the covariant derivative of the Levi-Civita connection,  $\nabla^2$  is the Laplace–Beltrami operator, and

$$\frac{\epsilon_{\mathbb{R}}^2}{2} \stackrel{\text{eqv}}{=} \frac{\hbar}{2m}. \quad (23)$$

The reduced Planck constant,  $\hbar = \frac{h}{2\pi}$ , pops up on the assignment of the value

$$\epsilon_{\mathbb{R}} = \sqrt{\left(\frac{\hbar}{m}\right) \stackrel{\text{eqv}}{=} 1} \quad (24)$$

to the number  $\epsilon_{\mathbb{R}}$ , embracing the Itô diffusion-type process, subordinated to the statement

$$\phi_{[\mathcal{M}^4]} = \int_0^\tau \beta_s ds + \epsilon_{\mathbb{R}} W_\tau, \quad (25)$$

to be compared with Eq. (14).

The aforementioned decomposition is *covariant* referring to Lorentz transformations in  $\mathcal{T}_x \mathcal{M}^4$  for (15) and (16) but not for  $D_\tau^\phi$  and  $D^* \phi_\tau$ . There is however another covariance. Let  $\mathcal{X}$  be a frame under which the time-like component and the space-like 3D components hold, so

$$D^+ \phi_\tau = \mathcal{X}_+^0(\tau, \phi_\tau), \quad (26)$$

$$D^- \phi_\tau = \mathcal{X}_-^0(\tau, \phi_\tau). \quad (27)$$

Let  $w^{\phi_\tau}$  be a vector field. Putting

$$\tilde{w}_1^\phi(\tau, x) = \frac{1}{2} \left( \mathcal{X}_+^0(\tau, x) + \mathcal{X}_-^0(\tau, x) \right), \quad (28)$$

$$\tilde{w}_2^\phi(\tau, x) = \frac{1}{2} \left( \mathcal{X}_+^0(\tau, x) - \mathcal{X}_-^0(\tau, x) \right). \quad (29)$$

We set

$$\tilde{w}_1^\phi(\tau, \phi_\tau) = \tilde{D}_s \phi_\tau \quad (30)$$

as the GR-relativistic current 4-velocity of  $\phi_\tau$ ,<sup>a</sup> where

$$\tilde{D}_s = \frac{1}{2}(D^+ + D^-), \quad (31)$$

is the GR-relativistic symmetric (s) mean derivative, and

$$\tilde{w}_2^\phi(\tau, \phi_\tau) = \tilde{D}_A \phi_\tau \quad (32)$$

as the GR-relativistic velocity (cf. footnote a on p. 8) of osmotic determination of  $\phi_\tau$ , where

$$\tilde{D}_A = \frac{1}{2}(D^+ - D^-) \quad (33)$$

is the GR-relativistic antisymmetric (A) mean derivative.

The current vector  $w_1^{\phi_\tau}$  is clearly covariant, since

$$w_1^{\phi_\tau} = \tilde{w}_1^{\phi_\tau} \quad (34)$$

(they have the the same decomposition-coordinate); whilst about the osmotic vector, it is decreed that

$$w_2^{\phi_\tau} = (D_A^\phi \phi_\tau^0, D_A^\phi \tilde{\phi}_\tau), \quad (35)$$

$$\tilde{w}_2^{\phi_\tau} = (D_A^\phi \phi_\tau^0 - D_A^\phi \tilde{\phi}_\tau). \quad (36)$$

The vector  $\frac{1}{2}(D^+ D^- + D^- D^+) \phi_\tau$  of (the stochastic process)  $\phi_\tau$ , is called the *4-acceleration*. Under the Itô formula in  $\mathcal{M}^4$  emerges that

$$\frac{1}{2}(DD^* + D^*D) \phi_\tau = \frac{1}{2}(D^+ D^- + D^- D^+) \phi_\tau, \quad (37)$$

which is proven by  $D^+ D^- \phi_\tau = (D_\phi^* D_\phi \phi_\tau^0, D_\phi D_\phi^* \tilde{\phi}_\tau)$  and  $D^- D^+ \phi_\tau = (D_\phi D_\phi^* \phi_\tau^0, D_\phi^* D_\phi \tilde{\phi}_\tau)$ .

<sup>a</sup> When we talk about velocity, a *stochastic fluctuation* takes place here.

### 3.3. A Step Back

Let us proceed with an additional conceptualization. The 4-vector acceleration  $\frac{1}{2}(DD^* + D^*D)\phi_\tau$  has a precise stochastic origin:

$$\frac{1}{2}(DD^* + D^*D)\phi_t = (D_s D_s - D_\Lambda D_\Lambda)\phi_t = D_s w^{\phi_t} - D_\Lambda w^{\phi_t}, \quad (38)$$

constituting a Borel vector on a  $\mathbb{R}$ -field. From here, it is agile to have classical formulæ,

$$D_s w_1^{\phi_t} = \frac{\partial}{\partial t} w_1^{\phi_t} + \nabla_{w_1^{\phi_t}} w_1^{\phi_t}, \quad (39)$$

$$D_\Lambda w_2^{\phi_t} = \nabla_{w_2^{\phi_t}} w_2^{\phi_t} + \frac{1}{2} \epsilon_{\mathbb{R}}^2 \nabla^2 w_2^{\phi_t}. \quad (40)$$

NB. Before arriving at an explicitly relativistic picture, the vector  $w_1$  and  $w_1$  can be written as

$$w_1^{\phi_t} = w_1^{\phi_t}(m, t) = \frac{1}{2} [\mathcal{A}^0(m, t) + \mathcal{A}_*^0(m, t)], \quad (41)$$

$$w_2^{\phi_t} = w_2^{\phi_t}(m, t) = \frac{1}{2} [\mathcal{A}^0(m, t) - \mathcal{A}_*^0(m, t)], \quad (42)$$

respectively.

We can thereupon formalize the stochastic acceleration as follows:

$$\frac{1}{2}(DD^* + D^*D)\phi_t = \left( \frac{\partial}{\partial t} w_1^{\phi_t} + \nabla_{w_1^{\phi_t}} w_1^{\phi_t} \right) - \left( \nabla_{w_2^{\phi_t}} w_2^{\phi_t} + \frac{1}{2} \epsilon_{\mathbb{R}}^2 \nabla^2 w_2^{\phi_t} \right). \quad (43)$$

With the support of the latter equation, we are close to describing the accelerated motion  $\ddot{x}(t)$  of a particle of mass  $m$  and velocity  $\dot{x}(t)$  along a curve  $x(t)$ , according to Newtonian mechanics:

$$\ddot{x} = 1/mF(x(t), t, \dot{x}(t)), \quad (44)$$

for a vector force field  $F$ . If the system is conservative, Newton's second law,  $F = m\vec{a} = \frac{d\vec{v}}{dt} \stackrel{\text{eqv}}{=} F(m(t), t, \dot{m}(t)) = \frac{D^{\hat{T}}}{dt} \dot{m}(t)$ , becomes

$$\frac{D^{\hat{T}}}{dt} \dot{m}(t) = -\text{grad } E_u, \quad (45)$$

where  $D^{\hat{T}}/dt$  is the (covariant derivative of the) Levi-Civita connection, and  $E_u$  is the potential energy.

From Eqq. (38) (43) we can extract this new equalization of the stochastic acceleration:<sup>a</sup>

$$\frac{1}{2}(DD^* + D^*D)\phi_t = \left( \frac{\partial}{\partial t} w_1^{\phi_t} + \nabla_{w_1^{\phi_t}} w_1^{\phi_t} \right) - \frac{1}{2} \left( 2\nabla_{w_2^{\phi_t}} w_2^{\phi_t} - \Delta_{\text{RKH}} w_2 \right), \quad (46)$$

with the presence of the Laplace–de Rham operator, aka the Kodaira–Hodge Laplacian [6, p. 196],  $\Delta_{\text{RKH}} = (d + \delta^2) = dd + \delta d$ , including the Cartanian differential (namely, the exterior derivative)  $d$ , and the codifferential  $\delta$ .

## 4. Relativistic Newton–Nelson Equations

What we have seen in the previous Section leads to a double formula à la Nelson [23] [24] [25, chap. I, sec. 10. *Stochastic Parallel Translation*] [26, chap. 12. *Dynamics of stochastic motion*], or à la Newton–Nelson,

$$\frac{1}{2} (D^+ D^- + D^- D^+) \phi_\tau = F(\phi_\tau, \tilde{w}_1^\phi(\tau, \phi_\tau)), \quad (47)$$

$$D^2 \phi_\tau = \frac{\hbar}{m} \mathbb{I}_n, \quad (48)$$

where

<sup>a</sup> Compare with Eqq. (50).

$F$  is a linear operator  $F(x): \mathcal{T}_x \mathcal{M}^4 \rightarrow \mathcal{T}_x \mathcal{M}^4$ , physically interpretable as a 4-force (it is, again, a vector force field), imagining a stochastic mechanics in space-time of general relativity, with a relativistic particle endowed with rest mass  $m$ , under an Itô-like process  $\phi_\tau$  in a 4-manifold,

$D^2 \phi_\tau$  can be re-equalized (and clarified) like this,

$$D^2 \phi_\tau = \left\{ \left[ \lim_{\Delta\tau \downarrow 0} \mathfrak{D}_\tau^\phi \left( \frac{\Delta^+ \phi_\tau \otimes \Delta^+ \phi_\tau}{\Delta\tau} \right) = \lim_{\Delta\tau \downarrow 0} \mathfrak{D}_\tau^\phi \left( \frac{\Delta^- \phi_\tau \otimes \Delta^- \phi_\tau}{\Delta\tau} \right) \right] \middle| \mathcal{U}_\tau^\phi \right\}, \quad (49)$$

$\mathbb{I}_n$  is the identity matrix.

For the appearance of  $\hbar$ , see Eqq. (23) (24). Consult T. Zastawniak [44].

Two remarks.

(1) In (47) (48) the *Nelson's mean acceleration* (NMA), which is a stochastic acceleration in the Nelsonian view, is shown; it can be established in various mathematical guises:

$$F_{\phi_\tau}^a = \begin{cases} \frac{1}{2}(DD^* + D^*D)\phi_{\tau[\mathcal{M}^4]} = \frac{1}{2}(D^+D^- + D^-D^+)\phi_{\tau[\mathcal{M}^4]},^a & (50a) \\ \frac{1}{2}[DD^*x(\tau) + D^*Dx(\tau)]\phi_\tau, & (50b) \\ -\frac{1}{2}[\mathfrak{B}_\tau \mathfrak{B}_\tau(x) + \mathfrak{B}_t \mathfrak{B}_t(x)], & (50c) \\ -\text{grad } \varrho_\alpha/m, & (50d) \end{cases}$$

where

$\mathfrak{B}(t)$  and  $\mathfrak{B}(t)$  are differential operators inherent in Brownian motion, or Wiener process,

$\varrho_\alpha$  is the potential under the Ornstein–Uhlenbeck process [42] addressed to the theory of Brownian motion.

It is very captivating to emphasize that Nelson [23] produces the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = i \frac{\epsilon_{\mathbb{R}}^2}{2} \nabla^2 \psi - i \frac{1}{\hbar} E_u \psi, \quad (51a)$$

$$= -\frac{\hbar^2}{2} \nabla^2 \psi + \underbrace{\text{grad } E_u \psi}_{\text{grad } \varrho_\alpha},^b \nabla^2 = \Delta \text{ (Laplace–Beltrami operator)}, \quad (51b)$$

from diffusion theory, but he ends up developing a stochastic Newtonian equation. In other words, Eq. (51) is deductible from the stochastic Newton's equation. (And with that we reconnect to Section 2). This is verifiable because from a small modification of Eq. (43) and from

$$\frac{1}{2}(DD^* + D^*D)\phi_t = 1/m \cdot F(\phi_\tau, w_1^\phi(t, \phi_t)), \quad (52)$$

$$D^2 \phi_t = \frac{\hbar}{m} \mathbb{I}_n,^c \quad (53)$$

revealing the trajectory of a particle under stochastic laws of motion, one sees that

$$\frac{\partial w_1^\phi}{\partial t} = -\text{grad } \varrho_\alpha - (w_1^\phi \nabla) w_1^\phi + (w_2^\phi \nabla) w_2^\phi + \left( \frac{\epsilon_{\mathbb{R}}^2}{2} \nabla^2 \right) w_2^\phi, \quad (54)$$

for  $F = -\text{grad } \varrho_\alpha$ .

(2) In (47) (48) we can also express the Ricci curvature, designated with  $\text{Ric}$ , in the form of  $\mathcal{C}^\infty$ -smooth  $\binom{1}{1}$ -tensor,

$$\frac{1}{2}(DD^* + D^*D)\phi_\tau = 1/m \cdot F(\phi_\tau, w_1^{\phi_\tau}, \tau) + \frac{\hbar}{2m} \text{Ric}(\phi_\tau) \circ w_2^{\phi_\tau}, \quad (55)$$

$$D^2 \phi_\tau = \frac{\hbar}{m} g_{\{2,0\}}, \quad (56)$$

where  $g_{\{2,0\}}$  is an algebraic symmetric object in the form of  $\binom{2}{0}$ -tensor. Pay attention to the fact that, in the latter double formula, the mean derivatives and the Ricci curvature tensor have a

<sup>a</sup> Cf. [8, p. 123].

<sup>b</sup> To be meticulous,  $i\hbar \frac{\partial \psi}{\partial t}(x, t) = \text{et cetera}$ .

<sup>c</sup> Compare with Eqq. (47) (48).

definition according to the metric connection, viz. the Levi-Civita (Riemannian) connection

$$\omega_{\mathfrak{h}} \stackrel{\text{eqv}}{=} \nabla \quad (57)$$

of a  $\binom{0}{2}$ -tensor field, taking for granted that  $\omega_{\mathfrak{h}} \in \Omega^1(\mathring{\mathcal{P}}, \mathfrak{h})$  is a  $\mathfrak{h}$ -valued 1-form on  $\mathring{\mathcal{P}}$ , or rather, a principal connection equivalent to the Levi-Civita connection (it is sometimes referred to as *Ehresmann connection*).

#### 4.1. Parallel Translation of Random Vectors from One Fiber Bundle to Another

Let  $(\mathring{\mathcal{E}}, \pi, \mathcal{M}, \mathring{\mathcal{F}})$  be a fiber bundle, where  $\mathring{\mathcal{E}}$ ,  $\mathcal{M}$  and  $\mathring{\mathcal{F}}$  are the total space, the base space, and the fiber of the bundle, respectively. The vector bundle can be represented by the map  $\pi: \mathring{\mathcal{E}} \rightarrow \mathcal{M}$ .

Presupposing a connection  $\Gamma_{\tau,s}^{\mathring{\mathcal{E}}}$ , the quartet of Eqq. (47) (48) (55) (56) can nimbly become a set of formulæ on fiber bundles.

*Example 4.1.* Let us confine ourselves to a paradigmatic equalities, with a

$$\left( \Gamma_{\tau,s}^{\mathring{\mathcal{E}}} \right)\text{-translation from } \mathring{\mathcal{E}}_{\phi_s} \text{ to } \mathring{\mathcal{E}}_{\phi_\tau},$$

where the terms  $\mathring{\mathcal{E}}_{\phi_s}$  and  $\mathring{\mathcal{E}}_{\phi_\tau}$  are two fiber bundles related to stochastic processes with the occurrence of random variables. If we impose that

$$\phi_\tau \stackrel{\text{eqv}}{=} \phi_\tau$$

is our stochastic process in the bundle  $\mathring{\mathcal{E}}$ , then Eqq. (15) and (16), articulating the covariant mean derivatives, take this look:

$$\begin{aligned} D^+ \phi_\tau^{\mathring{\mathcal{E}}} &= \lim_{\Delta\tau \downarrow 0} \mathfrak{O} \left\{ \frac{\Gamma_{\tau,\tau+\Delta\tau}^{\mathring{\mathcal{E}}} \phi^{\mathring{\mathcal{E}}}(\tau + \Delta\tau) - \phi_\tau^{\mathring{\mathcal{E}}}}{\Delta\tau} \left| N_t^\phi \left( \phi(\tau + \Delta\tau) - \phi_\tau \right)^2 \leq 0 \right. \right\} \\ &+ \lim_{\Delta\tau \downarrow 0} \mathfrak{O} \left\{ \frac{\phi_\tau^{\mathring{\mathcal{E}}} - \Gamma_{\tau,\tau-\Delta\tau}^{\mathring{\mathcal{E}}} \phi^{\mathring{\mathcal{E}}}(\tau - \Delta\tau)}{\Delta\tau} \left| N_t^\phi \left( \phi_\tau - \phi(\tau - \Delta\tau) \right)^2 \geq 0 \right. \right\} \end{aligned} \quad (58)$$

and

$$\begin{aligned} D^- \phi_\tau^{\mathring{\mathcal{E}}} &= \lim_{\Delta\tau \downarrow 0} \mathfrak{O} \left\{ \frac{\phi_\tau^{\mathring{\mathcal{E}}} - \Gamma_{\tau,\tau-\Delta\tau}^{\mathring{\mathcal{E}}} \phi^{\mathring{\mathcal{E}}}(\tau - \Delta\tau)}{\Delta\tau} \left| N_t^\phi \left( \phi_\tau - \phi(\tau - \Delta\tau) \right)^2 \leq 0 \right. \right\} \\ &+ \lim_{\Delta\tau \downarrow 0} \mathfrak{O} \left\{ \frac{\Gamma_{\tau,\tau+\Delta\tau}^{\mathring{\mathcal{E}}} \phi^{\mathring{\mathcal{E}}}(\tau + \Delta\tau) - \phi_\tau^{\mathring{\mathcal{E}}}}{\Delta\tau} \left| N_t^\phi \left( \phi(\tau + \Delta\tau) - \phi_\tau \right)^2 \geq 0 \right. \right\}. \end{aligned} \quad (59)$$

#### 4.2. Stochastic Quantization of a Particle in a Non-Abelian Gauge Field

In light of what has just been said in Section 4.1, with an addition of the vector bundle  $\mathring{\mathcal{E}}$ , Eqq. (47) (48) are nothing more than a version of the *quantum formulæ of the motion* of a certain particle *in a gauge field*, under a *stochastic process*. Here is how:

$$\frac{1}{2} \left( D^{\mathring{\mathcal{E}^+} D^- + D^{\mathring{\mathcal{E}^-} D^+} \right) \phi_t^{\mathring{\mathcal{E}}} = F_{t,\phi_t^{\mathring{\mathcal{E}}}} \left( w_{\phi_t^{\mathring{\mathcal{E}}}}^{\Gamma(\vartheta,\Omega)} \right), \quad (60)$$

$$D^2 \phi_t = \frac{\hbar}{m} \mathbb{I}_n, \quad (61)$$

postulating a connection  $\Gamma(\vartheta, \Omega)$  having

- a (sub)connection  $\vartheta$ -form, or a vector-valued 1-form,
- a curvature  $\Omega$ -form, or a 2-form of the Cartan connection,
- by virtue of which

$$\Omega = D\vartheta, \quad \Omega \stackrel{\text{eqv}}{=} \Omega^{\mathring{\mathcal{E}}}, \quad \vartheta \stackrel{\text{eqv}}{=} \vartheta^{\mathring{\mathcal{E}}} \quad (62)$$

holds on  $\mathring{\mathcal{E}}$ .

NB. A kindred road-solution to this one is taken by Y.E. Gliklikh and N.V. Vinokurova [11, p. 77].

## 5. Klein–Gordon & Dirac Equations

The quartet of Eqq. (47) (48) (55) (56) is plainly related to Klein–Gordon equation, which I will divide into three versions,

$$\left. \begin{aligned} & \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right) \psi \\ & \left[ \left( \omega_{\mathfrak{h}} - \frac{i}{\hbar} \tilde{w}, \omega_{\mathfrak{h}} - \frac{i}{\hbar} \tilde{w} \right) - \frac{1}{\hbar^2} \right] \psi \\ & \square + m^2 \psi \end{aligned} \right\} = 0, \quad (63)$$

where  $\psi$  is a scalar/wave function, and  $\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$  is the d'Alembertian, constituting the substratum for the Dirac equation, also divided into three versions,

$$\left. \begin{aligned} & (i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - mc) \psi \\ & (i\gamma^\mu \partial_\mu - m) \psi \\ & (i\rlap{/}\partial - m) \psi \end{aligned} \right\} = 0,^a \quad (64)$$

where the Dirac 4-spinor, id est a 4-component wave function,  $\psi = \begin{pmatrix} \zeta^\alpha \\ \tilde{\chi}_{\dot{\alpha}} \end{pmatrix}$ , appears, with a left-handed spinor  $\zeta^\alpha$ , and a right-handed spinor  $\tilde{\chi}_{\dot{\alpha}}$ .

## 6. Dirac Operator via Clifford Connection

In these circumstances, we are able to find a link between the covariant derivatives of Sections 3.1 plus 3.2 and the Dirac operator via Clifford algebra. Let

$\pi: \mathring{\mathcal{E}} \rightarrow \mathcal{M}$  be a vector bundle over a 4-manifold  $\mathcal{M}$ ,

$\Gamma_{\mathcal{S}}(\mathcal{M}, \mathring{\mathcal{E}})$  be the space of smooth sections of  $\mathring{\mathcal{E}}$ ,

$\Gamma_{\mathcal{S}}(\mathcal{M}, \bigwedge^k \mathring{\mathcal{T}}^* \mathcal{M} \otimes \mathring{\mathcal{E}}) \stackrel{\text{viz}}{=} \mathbb{Z}_{\mathcal{S}}(\mathcal{M}, \mathring{\mathcal{E}})$  be the space of sections of the bundle  $(\mathcal{M}, \bigwedge^k \mathring{\mathcal{T}}^* \mathcal{M} \otimes \mathring{\mathcal{E}})$ , or the space of differential  $k$ -forms in  $\mathring{\mathcal{E}}$ , stated that  $\mathring{\mathcal{T}}^* \mathcal{M}$  is a cotangent bundle (more particularly, a disjoint union of the cotangent spaces  $\mathcal{T}_x^* \mathcal{M}$ ).

So, assume that  $\mathring{\mathcal{E}}$  is a complex  $\mathbb{Z}_2$ -bundle

$$\left\{ \mathring{\mathcal{E}} \stackrel{\text{viz}}{=} (\mathring{\mathcal{E}})_{\mathbb{Z}_2} \right\} = \mathring{\mathcal{E}}^+ \oplus \mathring{\mathcal{E}}^-. \quad (67)$$

Which allows us to imagine that  $\mathring{\mathcal{E}}$  is a bundle of Clifford modules, with a bundle map  $\varphi_{C\ell}: \mathring{\mathcal{T}}^* \mathcal{M} \rightarrow \text{end}(\mathring{\mathcal{E}})$  such that

$$\varphi_{C\ell}(\omega_1)\varphi_{C\ell}(\omega_2) + \varphi_{C\ell}(\omega_2)\varphi_{C\ell}(\omega_1) = -2(\omega_1, \omega_2), \quad (68)$$

letting  $\omega \in \mathcal{T}_x^* \mathcal{M}$  be a 1-form (acting as a cotangent vector) on  $\mathcal{M}$ .

Given a Cliffordian bundle of a Riemannian manifold  $C\ell(\mathcal{M})$ , and identified with  $\text{end}_{C\ell(\mathcal{M})}$  an endomorphism of Clifford  $\mathring{\mathcal{E}}$ -valued module over  $C\ell(\mathcal{M})$ , the space of sections

$$\Gamma_{\mathcal{S}}\{\mathcal{M}, \text{end}(\mathring{\mathcal{E}})\} \cong \Gamma_{\mathcal{S}}\{\mathcal{M}, C\ell(\mathcal{M}) \otimes \text{end}_{C\ell(\mathcal{M})}(\mathring{\mathcal{E}})\} \quad (69)$$

is isomorphic to the space of  $\mathring{\mathcal{E}}$ -valued differential  $k$ -forms,

$$\mathbb{Z}_{\mathcal{S}}\{\mathcal{M}, \text{end}_{C\ell(\mathcal{M})}(\mathring{\mathcal{E}})\} \cong \Gamma_{\mathcal{S}}\left(\mathcal{M}, \bigwedge^k \mathring{\mathcal{T}}^* \mathcal{M} \otimes \text{end}_{C\ell(\mathcal{M})}(\mathring{\mathcal{E}})\right). \quad (70)$$

<sup>a</sup> In Eq. (64)  $\gamma^\mu$  viz.  $\gamma_b^\mu = \{\gamma^0, \gamma^1, \gamma^2, \gamma^3\}$  are the  $[M]^{4 \times 4}$  Dirac gamma matrices,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \quad (65)$$

plus  $\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ;  $\gamma^0$  is the time-like matrix,  $1 = \mathbb{I}_2$ , and  $\sigma_{1,2,3}$  the Pauli matrices; its extended writing is:

$$\gamma^0 = \begin{Bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Bmatrix}, \gamma^1 = \begin{Bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{Bmatrix}, \gamma^2 = \begin{Bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{Bmatrix}, \gamma^3 = \begin{Bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Bmatrix}, \quad (66)$$

plus  $\gamma^5 = i\gamma^0, \gamma^1, \gamma^2, \gamma^3$ , in the anti-commutation relation  $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbb{I}_{4 \times 4}$ ,

$\partial_\mu = \frac{\partial}{\partial x^\mu}$ , and  $\rlap{/}\partial = \gamma^\mu \partial_\mu$  is the partial derivative under the Feynman slash notation, with which Feynman's Dirac operator,  $D = \rlap{/}\partial \stackrel{\text{eqv}}{=} \gamma^\mu \partial_\mu$ , is accessed.

The endomorphic decomposition is

$$\text{end}(\mathring{\mathcal{E}}) \cong \text{Cl}(\mathcal{M}) \otimes \text{end}_{\text{Cl}(\mathcal{M})}(\mathring{\mathcal{E}}).$$

A connection on the vector bundle via Clifford  $\mathring{\mathcal{E}}$ -module, simply called *Clifford connection*, is  $\nabla^{\mathring{\mathcal{E}}}$ , according to which

$$\left\{ \nabla_{\vec{X}}^{\mathring{\mathcal{E}}}, \varphi_{\text{Cl}}(\omega) \right\} = \varphi_{\text{Cl}}(\nabla_{\vec{X}}\omega), \quad (71)$$

where  $\vec{X}$  is a vector field, and  $\nabla_{\vec{X}}\omega$  is the Levi-Civita derivative of  $\omega$ . Which means that the Levi-Civita connection  $\nabla^{\mathring{\mathcal{T}}}$  (to wit, the linear connection on the tangent bundle) on  $\bigwedge^k \mathring{\mathcal{T}}^*\mathcal{M}$  is but a Clifford connection,

$$\nabla^{\mathring{\mathcal{T}}} \stackrel{\text{equiv}}{=} \nabla^{\mathring{\mathcal{E}}}. \quad (72)$$

The *Dirac operator* affiliated with the Clifford connection  $\nabla^{\mathring{\mathcal{E}}}$  can take these two forms:

(1) for a local frame field, namely an orthonormal basis,  $e_j = \{e_1, \dots, e_n\}$ ,

$$D = \sum_{j=1}^n \varphi_{\text{Cl}}^j \nabla_{e_j}^{\mathring{\mathcal{E}}}, \quad (73)$$

or, alternatively,

$$D = \sum_{j=1}^n \varphi_{\text{Cl}}(dx^j) \nabla_{\partial_j}^{\mathring{\mathcal{E}}}, \quad (74)$$

(2) for a sequential configuration,

$$\Gamma_{\zeta}(\mathcal{M}, \mathring{\mathcal{E}}) \xrightarrow{\nabla^{\mathring{\mathcal{E}}}} \Gamma_{\zeta}(\mathcal{M}, \mathring{\mathcal{T}}^*\mathcal{M} \otimes \mathring{\mathcal{E}}) \xrightarrow{\varphi_{\text{Cl}}} \Gamma_{\zeta}(\mathcal{M}, \mathring{\mathcal{E}}).$$

*Marginalia 6.1 (Smoothness, differentiability, and non-differentiability).* We were talking earlier on about *space of sections*. We start from the surmise that our sections are *smooth*. It should be remembered that by *sections*—of vector bundles—we mean differentiable relations designed to colleague a certain vector in the corresponding vector space to each point of the manifold under consideration. This is in accordance with the scale relativity, which is a physico-geometric theory mixing differentiable and non-differentiable parts. The non-differentiable parts are certainly the *fractal* ones.

*Marginalia 6.2 (Spinor connection).* About Eq. (72), the same goes for a Levi-Civita connection  $\nabla^{\mathring{\mathcal{P}}_{\beta}}$  on a spinor bundle  $\mathring{\mathcal{P}}_{\beta}$  of a spin manifold  $\mathcal{M}$ . Here too we have a coincidence with the Clifford connection,

$$\nabla^{\mathring{\mathcal{P}}_{\beta}} \stackrel{\text{equiv}}{=} \nabla^{\mathring{\mathcal{E}}}. \quad (75)$$

Recall that a spinor bundle  $\mathring{\mathcal{P}}_{\beta}$  is

a component of the principal  $SL_2(\mathbb{C})$ -bundle over  $\mathbb{R}_{1,3}^4$ , or

a complex vector bundle  $\zeta: \mathring{\mathcal{P}}_{\beta} \rightarrow \mathcal{M}$  (see Atiyah–Singer index theorem).

## 7. A Dive into the Fractoid Spaces

In this Section, we will evaluate the notion of stochastic geodesic together with that of *energy functional*, and of *stochastic diffusions*, which is *neither smooth nor time-differentiable*. (Since the geodesic paths, or the spatial trajectories of the manifold, are not differentiable anent the time, it is indispensable to substitute the *derivative in time* with a *mean derivative*). There will also be a nexus to Brownian motion. I call *fractoid spaces* all non-time-differentiable fractal spaces that, wherever it happens, can also be devoid of smoothness.

### 7.1. Derivatives Coupled with a I-Type Semi-martingale and the Stratonovich Integral

Let  $\mathcal{M}_t$  be

- a semi-martingale in  $\mathcal{T}_{\mathcal{M}_0}\mathcal{M}$ , in accordance with a filtration  $\Phi_t$ ,  $t \in [0, T]$ , and
- a stochastic process with

$$\mathcal{M}_t = \mathcal{M}_t^{\text{loc}} + \chi_t, \quad (76)$$

where  $\mathcal{M}_t^{\text{loc}}$  is a local martingale, and  $\chi_t$  is a stochastic, or random, process having paths almost surely bounded with variation in  $t$ .

Let  $\phi_t$  be a stochastic process outlined with a stochastic differential equation in Stratonovich form, and  $\ell_s$  the arc length (the distance between two points along a section of a curve),

$$\phi_t = \int_0^t \left( \tau_{0 \leftarrow \ell_s}^{\mathcal{M}} \circ d\mathcal{M}_{\ell_s} \right), \quad (77)$$

estimating that

$$\tau_{t \leftarrow \ell_s}^{\mathcal{M}} : \mathcal{T}_{\mathcal{M}_{\ell_s}}\mathcal{M} \rightarrow \mathcal{T}_{\mathcal{M}_t}\mathcal{M}$$

is the stochastic parallel transport along  $\phi_t$  paired with the connection  $\nabla^{\vec{T}}$ . The *time derivative* on the right-hand side of (77) can be demarcated by *conditional expectations under a change of probability measure*  $\mu$  (on every Borel set, cf. Section 3.1),

$$D_{\mu}\phi_t \stackrel{\text{eqv}}{=} D\phi_t = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} E_k^{\phi} \left( \phi(t + \varepsilon) - \phi_t \mid \Phi_t \right), \quad \phi_t = \phi(t + \varepsilon), \quad (78)$$

where  $D_{\mu}$  is the derivative with reference to probability  $\mu$ -measures, and  $E_k^{\phi}$  is the stochastic kinetic energy of  $\mathcal{M}$ .

The derivative, in its *forward version*, is, instead,

$$D_{\mu}^{\nabla}\mathcal{M}_t \stackrel{\text{eqv}}{=} D^{\nabla^{\vec{T}}}\mathcal{M}_t = \tau_{t \leftarrow 0}^{\mathcal{M}} D\phi_t. \quad (79)$$

To *generalize* the derivative, merely subjoin a vector field, say  $\vec{X}$ ,

$$D_{\mu}^{\nabla}\vec{X}_t \stackrel{\text{eqv}}{=} D^{\nabla^{\vec{T}}}\vec{X}_t = \lim_{\varepsilon \rightarrow 0} E_k^{\phi} \Phi_t \left\{ \frac{\tau_{t \leftarrow t + \varepsilon}^{\mathcal{M}} \vec{X}(t + \varepsilon, \mathcal{M}(t + \varepsilon)) - \vec{X}(t, \mathcal{M}_t)}{\varepsilon} \right\}. \quad (80)$$

NB. Compare the above results with those in X. Chen and A.B. Cruzeiro [3, sec. 2].

### 7.2. The Concept of $\mathcal{E}$ -Stochastic Geodesics and of $v^{\phi}$ -Diffusion Process

The concept of geodesic can be characterized by the notion of set of critical points. The most direct manner is to use the *random Brownian motion* (Fig. 1), on the flat space, along certain directions, say  $\vartheta_s$  ( $s$  is for **sign**). Let  $x$  reveal a path of a  $\mathbb{R}$ -valued Brownian motion,  $x_t, t \in [0, T], x_0 = 0$ . We can *generalize* the derivative  $D_{\mu} \stackrel{\text{eqv}}{=} D$  on a flat space, which is that of random motion, to hold onto the stochastic schema.

**Proposition 7.1** (*Geodesic as a set of critical points*). *Let*

$$\mathcal{E}_k^{\phi}(\mathcal{M}) = E_k^{\phi} \int_0^T \left\| D_{\mu}^{\nabla}\mathcal{M}_t \stackrel{\text{eqv}}{=} D^{\nabla^{\vec{T}}}\mathcal{M}_t \right\|^2 dt \quad (81)$$

*be the energy functional,  $w$  a vector field, and  $v^{\phi}$  the diffusion process (see Section 7.3.1), produced by the operator  $L_w$ ,  $w \in \mathcal{C}^2$ , see Eq. (97). Iff*

$$\left. \begin{aligned} \nabla^{\vec{T}} w + \partial_t w + \frac{\Delta_w + \text{Ric}_w}{2} \\ D^{\nabla^{\vec{T}}} w(t, v_t^{\phi}) \end{aligned} \right\} = 0 \quad (82)$$

*almost everywhere, said that  $\Delta_w$  and  $\text{Ric}_w^a$  are the Laplacian operator, see Eq. (95), and the Ricci curvature w.r.t.  $w$ , then  $v^{\phi}$  is a critical path for  $\mathcal{E}_k^{\phi}$ , that is, a geodesic is nothing more than a set*

<sup>a</sup> A definition of the Ricci curvature tensor for the stochastic geodesic curves inside the variational principle and the Euler–Lagrange dynamics, is in A.B. Cruzeiro [4, p. 90].

of critical points of

$$\mathcal{E}_k^\phi(\gamma_{c(t)}) = \int_0^T \left( g_{\mu\nu}(\dot{\gamma}_{c(t)}) \dot{\gamma}_{c(t)}^\mu \dot{\gamma}_{c(t)}^\nu \right) dt = \int_0^T \|\dot{\gamma}_{c(t)}\|^2 dt, \quad (83)$$

$g_{\mu\nu}$  being the Riemannian metric tensor (field).

Eqq. (78) (79) (80) (81) (82) (83) can be placed into the *fractoid spaces*.

*Proof.* Firstly, please note that the critical path satisfies the Euler–Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \right) \quad \text{or} \quad - \frac{\partial E_u}{\partial x^\mu} = \frac{d}{dt} \left( \frac{\partial E_k}{\partial \dot{x}^\mu} \right), \quad 1 \leq \mu \leq n, \quad (84a)$$

$$\frac{\partial \mathcal{L}}{\partial x^\mu}(\gamma_{c(t)}, \dot{\gamma}_{c(t)}, t) = \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial v^\mu} \right) (\gamma_{c(t)}, \dot{\gamma}_{c(t)}, t), \quad \gamma_{c(t)} = \{x_t^1, \dots, x_t^n\}, \dot{\gamma}_{c(t)} = \{\dot{x}_t^1, \dots, \dot{x}_t^n\}.^a \quad (84b)$$

Let  $\pi: \mathring{O}(\mathcal{M}) \rightarrow \mathcal{M}$ ,  $\pi(x, p) = x$ , be a fiber map advantageous for a Euclidean isometry, and  $\mathring{O}(\mathcal{M}) = \{(x \in \mathcal{M}, p)\}$  the orthonormal frame bundle on  $\mathcal{M}$  (see Section 7.3.2). If one handles the action functional of a Lagrangian system [27, Eqq. (1.76) (1.77)], and sets out that

$$\mathcal{E}_k^\phi = E_k^\phi \int_0^T \left\| D\pi(p_{x(t)}) \right\|^2 dt, \quad (85)$$

one can obtain

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} E_k^\phi \int_0^T \left\| D\pi(p_{x+\varepsilon\vartheta(t)}) \right\|^2 dt &= 2E_k^\phi \int_0^T \left\{ D\pi(p_{x(t)}), D\dot{\pi} \left( \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} (p_{x+\varepsilon\vartheta(t)}) \right) \right\} dt \\ &= 2E_k^\phi \int_0^T \left\{ D\pi(p_{x(t)}), \dot{\vartheta} - \frac{\text{Ric}(\vartheta)}{2} - \mathcal{L}(\vartheta_t) \right\} dt \\ &= \llbracket : 2E_k^\phi \int_0^T \left\{ D\pi(p_{x(t)}), D(w_t) \right\} : \rrbracket dt, \end{aligned} \quad (86)$$

where

$\vartheta$  nominates processes of bounded variation, for which

$$\vartheta_0 = \vartheta_T = 0, \quad (87)$$

$\mathcal{L}$  is the Lie derivative, cf. Eq. (95),

the notations  $\llbracket : \text{plus} : \rrbracket$  signify that the combination of symbols within them must be repeated, stimulated by the beginning and ending repeat signs in music, and

$w$  is a semi-martingale, with initial conditions  $w_0 = 0$ , see Eq. (94).

By virtue of the Eq. (87), it follows that the second expression of (86)  $\llbracket : \dots : \rrbracket$  can be put in equivalence to  $-2E_k^\phi \int_0^T \left\{ (D) D\pi(p_x), \vartheta_t \right\}$ , à savoir

$$-2E_k^\phi \int_0^T \left\{ (D) D\pi(p_x), \vartheta_t \right\} \stackrel{\text{eqv}}{=} 2E_k^\phi \int_0^T \left\{ D\pi(p_{x(t)}), D(w_t) \right\}, \quad (88)$$

and we are done. In Eqq. (85) (86) (87) the equivalence  $D_{\mathbb{p}} \stackrel{\text{eqv}}{=} D$  holds.  $\square$

### 7.3. Marginalia of Clarification

#### 7.3.1. The $v^\phi$ -Diffusion Process and $\mathbb{R}$ -valued Random Brownian Motion

The diffusion process  $v^\phi$  by  $\mathcal{L}_{w \in \mathcal{C}^2}$  is built on

$$dv_t^{\phi(\nu)} = y_\xi^\nu dx_t^\xi + dt \left\{ w^\nu - \frac{g^{\varrho, \varsigma} \Gamma_{\varrho, \varsigma}^\nu}{2} \right\}; \quad (89)$$

$y$  and  $x_t$  are path  $\mathbb{R}$ -elements belonging to Brownian motion, accompanied by the Christoffel  $\Gamma$ -symbols.

<sup>a</sup> Consider a particle of mass  $m$  moving along  $\gamma_{c(t)}$  with a velocity  $v$ . The function  $\mathcal{L}(x, \dot{x}, t)$  viz.  $(x, v, t)$  is the Lagrangian on  $\mathring{T}\mathcal{M} \times [0, 1] \rightarrow \mathbb{R}$ , and it matches with the difference between the kinetic energy ( $E_k$ ) and the potential energy ( $E_u$ ).



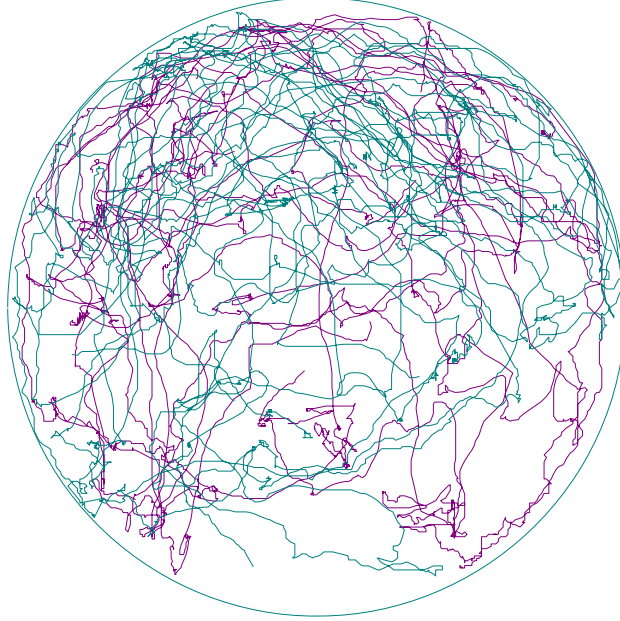


FIGURE 1. To give an idea with a simulation: Brownian-like  $y$ -#800080 paths & Brownian-like  $x_t$ -#008080 paths in a circle, or a subset of the real 2-space,  $\mathbb{S}^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$

Thinking back to the lesson of A.N. Kolmogorov [17, p. 281], a  $\mathbb{R}$ -valued random variable  $y = \varphi(x)$  on a probability space  $\mathcal{P}_\mu = (\hat{\Omega}, \mathcal{B}_\sigma, \mu)$  has a standard *Gaussian distribution* with zero mean, where it takes the form

$$\mu(x \in \mathcal{P}_\mu \mid \varphi(x) < u) = \frac{1}{\sqrt{2\pi \left( \int_{\mathcal{P}_\mu} \varphi^2 d\mu \right)}} \int_{-\infty}^u \exp \left\{ -\frac{u^2}{2 \left( \int_{\mathcal{P}_\mu} \varphi^2 d\mu \right)} \right\} du, \quad -\infty < u < +\infty, \quad (90)$$

under a density function  $\varphi(x)$ .

The same formula applies to a space of (pseudo-)Riemannian type, replacing in (90)  $\mathcal{P}_\mu$  with the conventional mathematical letter  $\mathcal{M}$ , ou seja  $x \in \mathcal{M}$ , and  $\int_{\mathcal{M}}$ . This gives birth to what can be called  *$\mathbb{R}$ -valued Brownian random measure*.

NB. From here it is possible to provide a subsequent definition of space decomposition of Hilbert space

$$\mathfrak{H} \stackrel{\text{eqv}}{=} L^2 \left( \mathcal{M} \stackrel{\text{eqv}}{=} \mathbb{R}^n, \mu \right) \quad (91)$$

into Hermite-Itô polynomial subspaces.

### 7.3.2. Natural Isomorphism Allied with the Orthonormal Frame Bundle

Apropos of  $\hat{\mathcal{O}}(\mathcal{M})$ , the orthonormal basis is  $e(x)$  in  $\mathcal{T}_x \mathcal{M}$ , so that there is a natural isomorphism, in the Euclidean  $n$ -dimensional space,

$$\mathcal{T}_x \hat{\mathcal{O}}(\mathcal{M}) \cong \mathbb{R}^n \oplus \mathfrak{o}_n(\mathbb{R}), \quad (92)$$

thru the medium of the Levi-Civita connection, and the orthogonal Lie algebra

$$\mathfrak{o}_n(\mathbb{R}) = \{X \in \mathfrak{gl}_n(\mathbb{R}) \mid X = -X^T\},^a \quad (93)$$

for a skew symmetric matrix  $X = -X^T$  in the Lie algebra  $\mathfrak{gl}_n(\mathbb{R})$  of the general linear group  $GL_n(\mathbb{R})$ , set of all real  $(n \times n)$ -matrices.

<sup>a</sup>  $X^T$  is the transpose of  $X$ .

### 7.3.3. The II-Type Semi-martingale, and the Laplacian Operator

The semi-martingale  $w$  is calculated by the method of Itô–Stratonovich, and it is definable as

$$w = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} (p_{x+\varepsilon\vartheta}), \vartheta_1^1, \quad (94)$$

once it is reported that  $\vartheta_1^1$  ( $\vartheta^1$  or  $\vartheta_1$ ) is the habitual  $\binom{1}{0}$ -tensor valued 1-form (1 is for **letter**). Ergo (Section 7.3.2) the Laplacian is so explicable, in relation to the Lie  $\mathcal{L}$ -derivative,

$$(\Delta_w)_{\dot{\mathcal{O}}(\mathcal{M})} = \sum_{\lambda=1}^n (\mathcal{L}_{w_\vartheta})^2. \quad (95)$$

Implementing a  $\mathcal{C}^\infty$  smooth function  $z$ , the Laplace–Beltrami operator on  $\mathcal{M}$  is

$$\nabla^2 = (\Delta_w)_{\mathcal{M}}(z)_{\mathcal{C}^\infty} = g^{\mu\nu} \left( \frac{\partial^2 z}{\partial x^\mu \partial x^\nu} - \Gamma_{\mu,\nu}^\xi \frac{\partial z}{\partial x^\xi} \right), \quad x \in \mathcal{M}. \quad (96)$$

For a 4-dimensionality, in order to reconnect with Sections 3.2 and 4, we can straightforwardly write  $\nabla^2 = (\Delta_w)_{\mathcal{M}^4}$ .

And to complete the picture, the above-mentioned  $L_w$  operator is so representable,

$$L_w z = \frac{1}{2} (\Delta_w)_{\mathcal{M}} z + \partial_w z, \quad w \in \mathcal{C}^2. \quad (97)$$

## 8. White Noise on a (4+)D Space-Time $\mathfrak{S}$ -Geometry: the Paley–Wiener Integral

Section 7.3.1 sets the tone for bringing up Paley–Wiener’s paraphernalia (stochastic calculus for Brownian motion) [37] [36, chap. IX] [43]. We may surmise that the Hilbert  $\mathfrak{S}$ -space has the following equivalence:

$$\mathfrak{S} \stackrel{\text{eqv}}{=} L^2(\mathbb{R}_{1,3}^{4+}).^a \quad (98)$$

So let

$$\mathbb{R}_{1,3}^{4+} = [0, \infty) \times \mathbb{R}^{4+} \quad (99)$$

be a space-time, such as that in [28], but still lacking the torsion, with a Minkowski–Lorentzian inner product  $g(v, w)$  viz.  $g\langle v, w \rangle$ ,<sup>b</sup>

$$g(v, w)_{\langle v, w \rangle \in \mathfrak{S}} = \int_0^{+\infty} \int_{\mathbb{R}_{1,3}^{4+}} dt dx \{v_{t,x} w_{t,x}\}, \quad t \geq 0, x \in \mathbb{R}_{1,3}^{4+}. \quad (100)$$

Fix

(1) the process  $W_t = \{W_w\}_{w \in \mathfrak{S}}$ <sup>c</sup> as a (continuous time) mean-zero gaussian distribution of the type of Eq. (90), having a *covariance function*

$$\varphi_{\text{cov}}(W_w, W_v) \stackrel{\text{eqv}}{=} g(w, v)_{\langle w, v \rangle \in \mathfrak{S}}, \quad (101)$$

(2) and the white noise on  $\mathbb{R}_{1,3}^{4+}$ , as

$$\Theta_{t,x}^w = \frac{\partial^{4+} W_{t,x}}{\partial_t \partial x_1 \cdots \partial x_4}. \quad (102)$$

For a map  $w \mapsto W_w$ , the *stochastic Paley–Wiener integration against*  $\Theta^w$  is

$$W_w = \int_0^{+\infty} \int_{\mathbb{R}_{1,3}^{4+}} dt dx \{w_{t,x} \Theta_{t,x}^w\}, \quad w \in \mathfrak{S}. \quad (103)$$

<sup>a</sup> Pay attention to the comma after the number 3.

<sup>b</sup> Isto é  $g(v, w) = v^0 w^0 + v^1 w^1 + v^2 w^2 + \cdots + v^{n-z_*} w^{n-z_*} - v^{n-z_*+1} w^{n-z_*+1} - \cdots - v^n w^n$ , where  $z_*$  is a non-negative integer.

<sup>c</sup> Be careful not to confuse this letter  $W$  (via `\$W\$` command) with the letter  $W$  (via `\Sobolev` command), for the Sobolev vector space, in Sec. 1.2: there is no correlation.

## 9. Torsion of the Covariant Derivative

To insert the spin-torsion tensor in GR, à savoir the Einstein–Cartan theory of gravity, see [28]. In detail, the Einstein–Cartan equations are those marked with the numbers (60) (61a) (61b) in [28], whilst the Einstein–Cartan Lagrangian, for a 4D space-time with spin-torsion (described by a Clifford  $k$ -form field) via Clifford bundles, is the one with the number (82).

Pertaining to the topic examined so far, we can write the *torsion of the covariant derivative* in a variety of ways. I will highlight some of these.

### 9.1. Torsion of $\nabla$ -Connections thru Holonomy

We move on to the study of the holonomy of connections with skew symmetric torsion, as torsion is the skew symmetric contribution to the affine connection. Interestingly, the Einstein–Cartan theory can be deciphered as a theory of defects in a space-time with curvature and torsion, or in a 4-dimensional continuum containing defects. In fact, how much distance we keep, conceptually, between *torsion* and *deformation*? That is the question M.L. Ruggiero and A. Tartaglia [35] are asking.

Let  $\hat{\tau}$  be a torsion  $\binom{1}{2}$ -tensor of the  $\nabla$ -connection on a (pseudo-)Riemannian manifold, so we will take two  $\mathcal{C}^\infty$  vector fields  $\vec{X}$  and  $\vec{Y}$ , for which it holds that

$$\hat{\tau}(\vec{X}, \vec{Y}) = \nabla_{\vec{X}}\vec{Y} - \nabla_{\vec{Y}}\vec{X} - [\vec{X}, \vec{Y}], \quad (104)$$

where

$\nabla_{\vec{X}}\vec{Y}$  is the covariant derivative of  $\vec{Y}$  along  $\vec{X}$ ,

$\nabla_{\vec{Y}}\vec{X}$  is the covariant derivative of  $\vec{X}$  along  $\vec{Y}$ ,

$[\vec{X}, \vec{Y}]$  is the commutator of  $\vec{X}$  and  $\vec{Y}$ .

Going to apply  $\hat{\tau}$  to  $\vec{X}$ , one acquires a  $\binom{1}{1}$ -tensor  $\hat{\tau}(\vec{X})$ ; if  $\hat{\tau}(\vec{X})$  is applied to  $\vec{Y}$ , one acquires a  $\binom{1}{0}$ -tensor  $\hat{\tau}(\vec{X}, \vec{Y})$ .

In the presence of a scalar field a/o scalar function  $f^{\hat{\tau}}$ , which is distinctly differentiable, the equivalence is

$$\nabla_{\vec{X}}[f^{\hat{\tau}}\vec{Y}] = \vec{X}(f^{\hat{\tau}})\vec{Y} + f^{\hat{\tau}}\nabla_{\vec{X}}\vec{Y}. \quad (105)$$

### 9.2. $D$ -differentiation in Gravity Spin-Torsion Interaction thru Cartan–Einstein Model (in Riemann–Cartan Geometry)

Let

$\mathfrak{u}^4$  be a 4-form (which is the exterior derivative of a projective 3-form),

$\Theta_{\hat{\tau}}$  be the torsion form, or the vector-valued 2-form,

$\hat{\tau}$  be the torsion tensor (in this paper I use the notation  $\backslash\mathfrak{tau}$  plus an inverted breve, to avoid confusion with the notation  $\backslash\mathfrak{tau}$  connoting the proper time, see above).

Since

$$D\mathfrak{u}_{\mu\nu\xi}^4 = \mathfrak{u}_{\mu\nu\xi\varrho}^4 \Theta_{\hat{\tau}}^{\varrho}, \quad (106)$$

adopting the algebraic Bianchi identities, one has initially these *covariant exterior derivatives*

$$8\pi D_{\mathfrak{T}_\mu} = \frac{1}{2} \mathfrak{u}_{\mu\nu\xi\varrho}^4 \Theta_{\hat{\tau}}^{\nu} \wedge \Omega^{\xi\varrho}, \quad (107)$$

$$8\pi D_{\hat{S}_{\mu\nu}} = \mathfrak{u}_{\nu\varrho}^4 \wedge \Omega^{\varrho\mu} - \mathfrak{u}_{\mu\varrho}^4 \wedge \Omega^{\varrho\nu}, \quad (108)$$

where

$\mathfrak{T}^\mu = \mathfrak{T}^{\mu\nu} \mathfrak{u}_\nu^4$  is the tensor-valued 3-form, implying the energy-momentum tensor,

$\Omega^{11}$ ,  $\Omega^1_1$  is the curvature 2-form on space-time (1 is for **letter**, as already seen above),

$\hat{S}_{\mu\nu}$  is the spin (density) tensor.

By selecting some suitable field equations, such as

$$\frac{1}{2} g^{\xi\varrho} \mathfrak{u}_{\mu\nu\xi}^4 \wedge \Omega^{\nu\varrho} = -8\pi \mathfrak{T}_\mu, \quad (109)$$

$$\mathfrak{u}_{\mu\nu\xi}^4 \wedge \Omega^{\xi} = 8\pi \hat{S}_{\mu\nu}, \quad (110)$$

it is possible to draw the following rewriting of the previous Eqq. (107) (108),

$$D_{T_\mu} = \hat{r}^\xi_{\mu\nu} \theta^\nu \wedge T_\xi - \frac{1}{2} R^\xi_{\rho\mu\nu} \vartheta^\nu \wedge \hat{S}^e_\xi, \quad (111)$$

$$D_{\hat{S}^e_{\mu\nu}} = \vartheta_\nu \wedge T_\mu - \vartheta_\mu \wedge T_\nu, \quad (112)$$

where  $\theta_1^1$  is a 1-form, or a vector-valued 1-form.

### 9.3. Quantum-like Fluctuations: a Stochastically Gravitational Fabric of Space-Time

In this background it is subsequently permissible to include in our discussion

(1) quantum-like fluctuations of the (pseudo-Euclidean) metric tensor  $\eta_{\mu\nu}^{(1,3)+}$ ,  $\eta_{\mu\nu}^{(1,3)-}$  of Minkowski space-time, and of the metric tensor  $g_{\mu\nu}$  in Einstein's general relativity theory,

(2) quantum-like fluctuations of the energy-momentum tensor  $T^{\mu\nu}$ ,  $T_{\mu\nu}$ , aka stress-energy tensor, or stress-energy-momentum tensor,<sup>a</sup> depicting the fluctuations of quantum matter fields in curved space-times,

(3) quantum-like fluctuations of the whole gravitational field, taken in small (but how small?) pieces.

#### CODA: INSPIRING SNIPPET

This paper has an essential bibliography as it is (was) a private communication, an *échange de vues*, in order to be able to work with a certain lightness. Which results in a prehensibility of the *cerveau rêveur*. Rigor betwixt fantasy & imagination. Is there a mathematical stream of consciousness? If so, this writing is a tiny epiphany of it.

There's a passage from A. Grothendieck [13, 6.2. (6). *Le Rêveur*, p. 12 otm] that reads:

Si nous pouvons communiquer avec nous-mêmes par le truchement du rêve, nous révélant à nous-mêmes, sûrement il doit être possible de façon toute aussi simple de communiquer à autrui le message nullement intime du rêve mathématique [...]. Et à vrai dire, qu'ai-je fait d'autre dans mon passé de mathématicien, si ce n'est suivre, "rêver" jusqu'au bout, jusqu'à leur manifestation la plus manifeste, la plus solide: irrécusable, des lambeaux de rêve se détachant un à un d'un lourd et dense tissu de brumes? Et combien de fois ai-je trépillé d'impatience devant ma propre obstination à polir jalousement jusqu'à sa dernière facette chaque pierre précieuse ou précieuse à demi en quoi se condensaient mes rêves — plutôt que de suivre une impulsion plus profonde: celle de suivre les arcanes multiformes du tissu-mère — aux confins indécis du rêve et de son incarnation patente, "publiable" en somme, suivant les canons en vigueur! J'étais d'ailleurs sur le point de suivre cette impulsion-là, de me lancer dans un travail de "science-fiction mathématique", "une sorte de rêve éveillé".

<sup>a</sup>  $T^{\mu\nu}$ , or  $T_{\mu\nu}$ , composes the kinetic energy of matter, since it acts as a matter-energy flow.



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