



A Connectivity Aware Path Planning for a fleet of UAVs in an Urban Environment: Supplementary Material

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A Connectivity Aware Path Planning for a fleet of UAVs in an Urban Environment

Supplementary Material

Nouman Bashir, Saadi Boudjit, and Gabriel Dauphin

I. INTRODUCTION

In this 7-pages document, we give some supplementary material for the preprint of a paper, titled, A Connectivity Aware Path Planning for a fleet of UAVs in an Urban Environment. Section II shows two illustrations. Section III shows a few complimentary figures. And section IV gives an extensive demonstrations of different theorems formulated and used in the paper.

II. ILLUSTRATION

Figure 1 are two subfigures illustrating the two application scenarios described in section III-A (p. 4).

III. SIMULATIONS

We present here three figures accounting for simulations whose implementation is described in the beginning of section V (p. 11). These simulations use parameters listed in table II (p. 12).

Figure 2 shows reaction time between two neighboring UAVs, when there is a total of 4 UAVs.

Figure 3 shows UAVs' minimum distance variation from any other UAV for a fixed fleet speed of 2, 5, and 10 m/s. Similarly for the same experiments.

Figure 4 represents, among all UAVs and obstacle, the minimum UAV-obstacle distance during entire simulation time with fixed fleet speed of 2, 5, and 10 m/s.

Figure 5 demonstrates the delay faced by data packets (i.e., T_{BS}) for UAV fleet speed of 2, 5, and 10 m/s. This figure demonstrates that the average delay remains the same for all these speed variations.

Figure 6 plots data packet delays (i.e., T_{BS}) for the ground users to reach BS and for different

leaders' locations within the fleet. It shows that leaders' position plays only a slight role in determining the delay incurred by the data packets.

Fig. 7 plots the delay incurred (i.e., T_{BS}) by the data packets for ground users to reach BS under 2, 5, and 10 m/s fleet speed. It can be observed from the figure that the delay faced by the data packets under these conditions is not affected by the variation in the fleet speed.

IV. SET OF PROOFS

A. Proof of theorem 1 (p. 8)

The proof is organized in three parts. First we use analytical formulas of $v(t)$, $S(t)$ and $\Phi[s]$, the latter uses a given value of $s \geq 0$. Then we derive the desired properties. And thirdly we prove that $v(\tau)$, $S(\tau)$ and $\Phi[s'']$ are known to all UAVs at time t if $\tau \leq t$ and $\Phi[s''] \leq t$.

1) We first compute analytic formulas.

- Based on assumption 2, we denote $v(t)$ as a finite linear combination of indicator functions also denoted with 1.

$$v(t) = \sum_{l=0}^{L-1} v_l \mathbf{1}_{[t_l, t_{l+1})}(t) \quad (1)$$

where $[t_l, t_{l+1})$ are the left-closed and right-open intervals of the different indicator functions. Because a speed is non-negative and based on assumption 2, we also have

$$\forall l, v_l \in [0, v_{\max}] \quad (2)$$

$$t_L = +\infty \text{ and } v_{L-1} > 0$$

- Using equation (26) defining the ramp function and integrating equation (1), we get

$$S(t) = \sum_{l=0}^{L-1} v_l \mathcal{R}_{t_l}^{t_{l+1}}(t) \quad (3)$$

- Because $S(t)$ can be constant on some intervals, there are different ways of defining its inverse. We have chosen to define $\Phi[s]$ as

$$\Phi[s] = \max\{t | S(t) \leq s\} \quad (4)$$

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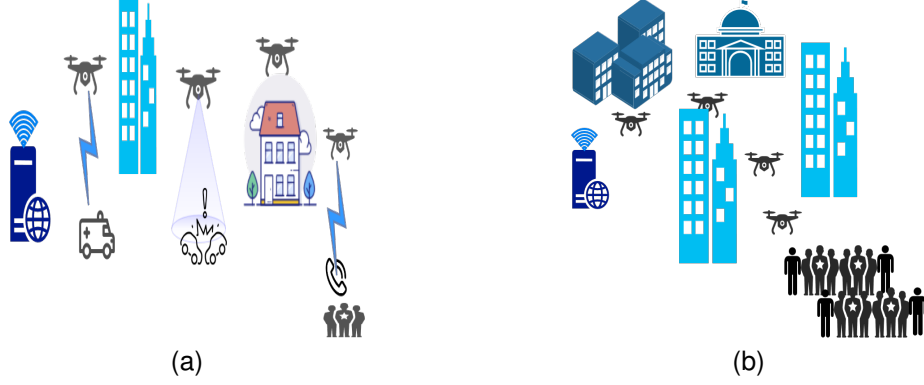


Fig. 1. Application scenarios described in section III-A (p. 4). (a) Natural disaster scenario. (b) Continuous tracking and monitoring of a rally.

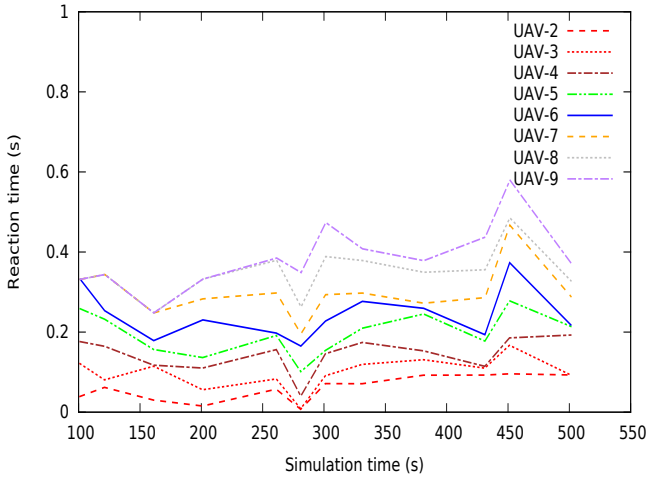


Fig. 2. UAV's reaction time to speed change requests (4 transmitting UAVs)

An equivalent definition is

$$\Phi[s] = t_{l_s} + \frac{s - \mathcal{S}(t_{l_s})}{v_{l_s}} \quad (5)$$

where

$$l_s = \max \{0 \leq l < L \mid \mathcal{S}(t_{l_s}) \leq s\} \quad (6)$$

This definition of l_s ensures that $v_{l_s} \neq 0$, note that it says nothing as to whether $v_{l_{s-1}} = 0$.

We also have

$$\Phi[s] \in [t_{l_s}, t_{l_{s+1}}) \quad (7)$$

2) We now derive the expected properties.

- Equation (5) and the linearity of $\mathcal{S}(t)$ within each intervals $[t_l, t_{l+1})$ show that

$$\mathcal{S}(\Phi[s]) = \mathcal{S}(t_{l_s}) + \mathcal{R}_{\mathcal{S}(t_{l_s})}^{\mathcal{S}(t_{l_{s+1}})}(s) = s \quad (8)$$

- We first assume that we do not have both, $s = \mathcal{S}(t_{l_s})$ and $v_{l_{s-1}} = 0$.

Equation (5) and (7) prove that

$$\frac{d}{ds} \Phi(s) = \frac{1}{v_{l_s}} = \frac{1}{v(\Phi(s))}$$

- We then assume that we do have both $s = \mathcal{S}(t_{l_s})$ and $v_{l_{s-1}} = 0$. Because $s = \mathcal{S}(t_{l_{s-1}}) > \mathcal{S}(t_{l_{s-2}})$, we consider

$$s' \in [\mathcal{S}(t_{l_{s-2}}), \mathcal{S}(t_{l_{s-1}}))$$

and see that

$$\Phi(s') = t_{l_{s-2}} + \frac{s' - \mathcal{S}(t_{l_{s-2}})}{v_{l_s}} \rightarrow t_{l_{s-1}} < t_{l_s}$$

when $s' \rightarrow s^-$.

3) Finally we study the availability of these quantities at time t . Let us assume that $\tau \leq t$ and s'' such that $\Phi(s'') \leq t$.

- By defining $l_\tau = \max\{l \mid t_l \leq \tau\}$, we get $v(\tau) = v_{l_\tau}$ which is known to all UAVs at time t .
- Equation (3) shows that

$$\mathcal{S}(\tau) = \sum_{l=0}^{l_\tau-1} v_l(t_{l+1} - t_l) + v_{l_\tau}(\tau - t_{l_\tau})$$

which is also known to all UAVs at time t .

- Applying equation (5) to s'' , we have

$$\Phi[s''] = t_{l_{s''}} + \frac{s'' - \mathcal{S}(t_{l_{s''}})}{v_{l_{s''}}}$$

$t_{l_{s''}}$, $v_{l_{s''}}$ and $\mathcal{S}(t_{l_{s''}})$ are known to all UAVs at time t since:

- $t_{l_{s''}} \leq \Phi[s''] \leq t$
- $v_{l_{s''}} = v(t_{l_{s''}})$
- $\mathcal{S}(t_{l_{s''}}) = \int_0^{t_{l_{s''}}} v(\tau) d\tau$

Therefore $\Phi[s'']$ is also known to all UAVs at time t .

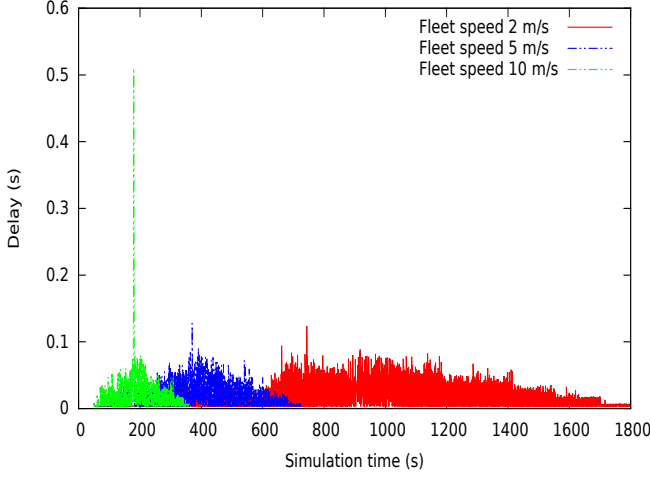


Fig. 7. Data packets delay to reach BS from ground users

Equations (29) and (30) are then easily derived.

C. Proof of theorem 3 (p. 9)

The proposed proof of theorem 3 uses the following lemma.

Lemma 1. Let P_k and P_{k+1} be two points defining a line segment $\overline{P_k P_{k+1}}$ sufficiently far from any obstacle in that

$$\forall i, \mathcal{D}(\overline{P_k P_{k+1}}, \overline{C_i D_i}) \geq \rho_c \quad (10)$$

Let M be a point of this line segment.

$$M \in \overline{P_k P_{k+1}}$$

Then for any point M' , we have

$$MM' < \rho_c \Rightarrow M' \notin \overline{C_i D_i}$$

Proof. We are actually proving here the contrapositive of this lemma.

Let M be a point in $\overline{P_k P_{k+1}}$ and M' another point in $\overline{C_i D_i}$, for some value of i and k .

Equation (10) implies that $MM' \geq \rho_c$. \square

This lemma is used in the following proof of theorem 3.

Proof. Let $t \in [0, T]$. Let $\mathbf{P} = [P_1 \dots P_K]$ be a complete path with Δ_A and Δ_B its departure and arrival areas all of them being sufficiently far from any obstacles (i.e. left of equation (31) holds). Equation (27) proves that there exists $k \in \{0 \dots K\}$ such that

$$V_n(t) \in \overline{P_{k,n} P_{k+1,n}} \quad (11)$$

Equations (13), (16), and (17) prove that

$$\overline{P_{0,n} P_1} \subset \Delta_A \text{ and } \overline{P_K P_{K+1,n}} \subset \Delta_B$$

Therefore $\mathcal{D}(\overline{P_{k,n} P_{k+1,n}}, \overline{C_i D_i}) \geq \rho_c$ holds also for $k \in \{0, K\}$.

Left of equation (5) in assumption 1 tells us that $R_n V_n(t) < \rho_c$. Lemma 1 ends the proof. \square

D. Proof of theorem 4 (p. 9)

Based on equations (9) and (23), we have

$$\Delta \mathcal{S}_r \geq \Delta \mathcal{S}_c \quad (12)$$

Lemma 2. Let α and β be two non-negative real numbers.

$$\sqrt{\alpha^2 + \beta^2} \geq \frac{\alpha + \beta}{\sqrt{2}} \quad (13)$$

Proof.

$$\begin{aligned} \sqrt{\alpha^2 + \beta^2} \geq \frac{\alpha + \beta}{\sqrt{2}} &\Leftrightarrow \alpha^2 + \beta^2 \geq \frac{(\alpha + \beta)^2}{2} \\ &\Leftrightarrow \frac{\alpha^2}{2} + \frac{\beta^2}{2} \geq \alpha\beta \Leftrightarrow \frac{1}{2}(\alpha - \beta)^2 \geq 0 \end{aligned}$$

\square

Lemma 3. Let E, F, G be three points such that

$$|\angle(\overrightarrow{EF}, \overrightarrow{FG})| \leq \frac{\pi}{2}$$

Let $M_e \in \overline{EF}$ and $M_g \in \overline{FG}$

$$M_e M_g \geq \frac{FM_e + FM_g}{\sqrt{2}} \quad (14)$$

This proof uses the scalar product denoted \bullet and its corresponding norm $\| \cdot \|$ derived from the Euclidean distance.

Proof. We first prove that $\overrightarrow{FM_e} \bullet \overrightarrow{FM_g} \leq 0$ then use this result and that of lemma 2 to end the proof.

1) Because $M_e \in \overline{EF}$ and $M_g \in \overline{FG}$, there exist $\lambda, \mu \in [0, 1]$ such that

$$\begin{aligned} \overrightarrow{FM_e} &= \lambda \overrightarrow{FE} \\ \overrightarrow{FM_g} &= \mu \overrightarrow{FG} \end{aligned}$$

We can now conclude on the sign of $\overrightarrow{FM_e} \bullet \overrightarrow{FM_g}$

$$\overrightarrow{FM_e} \bullet \overrightarrow{FM_g} = \lambda \mu \overrightarrow{FE} \bullet \overrightarrow{FG} \cos(\angle(\overrightarrow{FE}, \overrightarrow{FG})) \leq 0$$

2) Thanks to some scalar product operations and to lemma 2, we get

$$\begin{aligned} M_e M_g &= \left\| \overrightarrow{FM_g} - \overrightarrow{FM_e} \right\| = \\ &= \sqrt{FM_g^2 - 2\overrightarrow{FM_e} \bullet \overrightarrow{FM_g} + FM_e^2} \geq \\ &= \sqrt{FM_g^2 + FM_e^2} \geq \frac{FM_e + FM_g}{\sqrt{2}} \end{aligned}$$

□

Before handling the proof itself, we recast equation (9) describing the position of each virtual UAV into two equations, first a description of where each virtual UAV is inside a global path and then a description of this global path.

$$S_n(s) = \mathcal{R}_{L_{0,n}}^{L_{K+1,n}}(s) - (n-1)\Delta\mathcal{S}_c \quad (15)$$

$$\begin{aligned} V(s) = & \mathbf{A} - (N-1)\Delta\mathcal{S}_c \overrightarrow{e_0} + \mathcal{R}_{-(N-1)\Delta\mathcal{S}_c}^0(s) \\ & + \sum_{k=0}^K \mathcal{R}_{L_{k,1}}^{L_{k+1,1}}(s) \overrightarrow{e_k} \end{aligned} \quad (16)$$

Simple computations show that

$$V_n(t) = V(S_n(S(t))) \quad (17)$$

Lemma 4. Let n_1 and n_2 denote two different UAV and let $S_{n_1}(s)$ and $S_{n_2}(s)$ be their relative positions on the global path. We have

$$|S_{n_1}(s) - S_{n_2}(s)| \geq \Delta\mathcal{S}_c \quad (18)$$

Proof. There is no loss of generality in assuming that $n_2 > n_1$. The proof depends on s and we consider three possibilities.

1) When $s \leq L_{0,n_2}$, we have

$$S_{n_2}(s) = -(n_2-1)\Delta\mathcal{S}_c \text{ and } S_{n_1}(s) \geq -(n_1-1)\Delta\mathcal{S}_c$$

By subtracting the two equations, we prove equation (18).

$$S_{n_2}(s) - S_{n_1}(s) \leq (n_1 - n_2)\Delta\mathcal{S}_c \leq -\mathcal{S}_c$$

2) When $s \geq L_{K+1,n_1}$, we have

$$\begin{aligned} S_{n_2}(s) & \leq \mathcal{L}(\mathbf{P}) - (N - n_2)\Delta\mathcal{S}_c \\ S_{n_1}(s) & = \mathcal{L}(\mathbf{P}) - (N - n_1)\Delta\mathcal{S}_c \end{aligned}$$

By subtracting the two equations, we prove again equation (18).

$$S_{n_2}(s) - S_{n_1}(s) \leq (n_1 - n_2)\Delta\mathcal{S}_c \leq -\mathcal{S}_c$$

3) When $s \in [L_{0,n_2}, L_{K+1,n_1}]$, we note that this implies $s \in [L_{0,n_1}, L_{K+1,n_2}]$ and therefore that

$$S_{n_2}(s) - S_{n_1}(s) = (n_1 - n_2)\Delta\mathcal{S}_c \leq -\Delta\mathcal{S}_c \leq -\Delta\mathcal{S}_c$$

□

Below is the proof of theorem 4.

Proof. Let $t \in [0, T]$ and n_1, n_2 denote two different UAVs. We again assume that $n_2 > n_1$, meaning that the n_2^{th} -UAV is following the n_1^{th} -UAV. Let $\mathbf{P} = [P_1 \dots P_K]$ be a complete path with Δ_A and Δ_B its departure and arrival areas, all of which fulfill left of equation (32). First we note that we only need to prove that $V_{n_1}(t)V_{n_2}(t) \geq 2\rho_c$, as left of equation (5) in assumption 1 transforms this statement into $R_{n_1}(t) \neq R_{n_2}(t)$.

Lemma 4 and equation (23) show that

$$|S_n(S(t)) - S_{n_1}(S(t))| \geq 2\rho_c\sqrt{2} \quad (19)$$

And to prove that $V_{n_1}(t)V_{n_2}(t) \geq 2\rho_c$ we consider three possibilities, namely both UAVs are at time t : in the same line segment, in two consecutive line segments or in two distant line segments.

1) When both UAVs are in the same line segment, equations (17) and (16) show that

$$\begin{aligned} V_{n_1}(t)V_{n_2}(t) & = S_{n_2}(S(t)) - S_{n_1}(S(t)) \\ & \geq 2\rho_c\sqrt{2} \geq 2\rho_c \end{aligned}$$

2) When both UAVs are in consecutive line segments, let J denote their intersecting point. Equations (17) and (16) prove that

$$\begin{aligned} V_{n_1}(t)J + JV_{n_2}(t) & = S_{n_2}(S(t)) - S_{n_1}(S(t)) \\ & \geq 2\rho_c\sqrt{2} \end{aligned}$$

Thanks to left of equation (32), consecutive line segments have acute bending

$$\left| \angle \left(\overrightarrow{V_{n_1}(t)J}, \overrightarrow{JV_{n_2}(t)} \right) \right| \leq \frac{\pi}{2}$$

Lemma 3 shows that $V_{n_1}(t)V_{n_2}(t) \geq 2\rho_c$

3) When both UAVs are in non-consecutive line segments, then their mutual distance is greater than the distance between these line segments. This distance is lower bounded by $2\rho_c$, thanks to left of equation (32).

□

E. Proof of theorem 5 (p. 9)

The definition below gives a more formal definitions of \bowtie and \boxtimes .

Definition 1. Let D_1 and D_2 be two points with wireless communication capabilities. The symbol \bowtie denotes here the ability to communicate directly.

$$D_1 \bowtie D_2 \Leftrightarrow D_1 D_2 \leq \rho_R \quad (20)$$

Let D'_j a set of points. The symbol \boxtimes denotes here the ability to communicate using a routing communication path.

$$\begin{aligned} D_1 \boxtimes D_2, \Leftrightarrow \exists D'_{j_1} \dots D'_{j_J}, \text{ such that} \\ D_1 \bowtie D'_{j_1} \bowtie D'_{j_1} \bowtie D'_{j_2}, \dots, D'_{j_{J-1}} \bowtie D'_{j_J}, \text{ and } D'_{j_J} \bowtie D_2 \end{aligned} \quad (21)$$

\bowtie is regarded as a reflexive and symmetric relation. \boxtimes is transitive and also reflexive and symmetric, it is therefore regarded as an *equivalence relation*.

Similarly to lemma 4, we prove that consecutive UAVs are not too far apart.

Lemma 5. Let n and $n+1$ denote two consecutive drones.

$$0 \leq S_n(s) - S_{n+1}(s) \leq \Delta S_r \quad (22)$$

Proof. Left of equation (22) is only claiming that the $(n+1)^{\text{th}}$ is following the n^{th} , which is true. Right of equation (22) depends on the value of s , and three possibilities are considered.

- 1) When $s \leq n(\Delta S_r - \Delta S_c)$ that is smaller than $L_{0,n+1}$, then either both UAVs are on their parking slots or the first one is leaving the parking slot.

In the first outcome, $S_n(s) - S_{n+1}(s)$ is smaller than ΔS_c which is smaller than ΔS_r .

In the second outcome,

$$\begin{aligned} S_n(s) - S_{n+1}(s) &= \{-(n-1)\Delta S_c\} - \{s - n(\Delta S_r - \Delta S_c)\} \\ &\leq \Delta S_r \end{aligned}$$

- 2) When $s \geq \mathcal{L}(\mathbf{P}) + (n-1)\Delta S_r + (N-n)\Delta S_c$ that is greater than $L_{K+1,n+1}$ then either both UAVs are on their parking slots or the last one is reaching the parking slot.

In the first outcome, $S_n(s) - S_{n+1}(s)$ is smaller than ΔS_c which is smaller than ΔS_r .

In the second outcome,

$$\begin{aligned} S_n(s) - S_{n+1}(s) &= \{\mathcal{L}(\mathbf{P}) + (N-n)\Delta S_c\} - \{s - n\Delta S_r\} \\ &\leq \Delta S_r \end{aligned}$$

- 3) When $s \in [L_{0,n+1}, L_{K+1,n}]$, then both UAVs are moving and

$$S_n(s) - S_{n+1}(s) = \Delta S_r$$

□

Lemma 6. Let a_i be an increasing sequence of I numbers and \vec{e}_i be unitary vectors. Let $\vec{f}(x)$ be a function defined as

$$\vec{f}(x) = \sum_{i=1}^{I-1} \mathcal{R}_{a_i}^{a_{i+1}}(x) \vec{e}_i$$

Then $\vec{f}(x)$ is a contraction mapping.

$$\left\| \vec{f}(x_1) - \vec{f}(x_2) \right\| \leq |x_1 - x_2| \quad (23)$$

where $\| \cdot \|$ is the norm.

Proof. With no loss of generality, we assume that $x_1 < x_2$.

- First we consider that both x_1 and x_2 are in $[a_1, a_I]$. Let i, j such that $x_1 \in [a_i, a_{i+1})$ and $x_2 \in [a_j, a_{j+1})$, we get

$$\begin{aligned} \vec{f}(x_2) - \vec{f}(x_1) &= (x_2 - a_j) \vec{e}_j \\ &+ \sum_{k=i+1}^{j-1} (a_{k+1} - a_k) \vec{e}_k + (a_i - x_1) \vec{e}_i \end{aligned}$$

Because the vectors \vec{e}_k are unitary and because of the triangle inequality, we have

$$\begin{aligned} \left\| \vec{f}(x_2) - \vec{f}(x_1) \right\| &\leq (x_2 - a_j) + \sum_{k=i+1}^{j-1} (a_{k+1} - a_k) + (a_i - x_1) \\ &= x_2 - x_1 \end{aligned}$$

- If $x_1 \leq a_1$, $f(x_1) = f(a_1)$ and by applying equation (23) to (a_1, x_2) , we have

$$\left\| \vec{f}(x_2) - \vec{f}(x_1) \right\| \leq x_2 - a_1 \leq x_2 - x_1$$

- If $x_2 \geq a_I$, $f(x_2) = f(a_I)$ and by applying equation (23) to (x_1, a_I) , we have

$$\left\| \vec{f}(x_2) - \vec{f}(x_1) \right\| \leq a_I - x_1 \leq x_2 - x_1$$

□

Below is the proof of theorem 5.

Proof. Let \mathbf{P} be a complete path and $N > \frac{\mathcal{L}(\mathbf{P})}{\Delta S_r} - 1$. The proof is in four parts, dealing first with connectivity among UAVs, then with connectivity first to A, then to B, and finally with A or B.

- 1) To prove that all UAVs are connected to each other, it is sufficient to prove that each UAV is connected to its follower when it has one, that is

$$\forall n \in \{1 \dots N-1\}, R_n(t) \bowtie R_{n+1}(t)$$

Let $n \in \{1 \dots N-1\}$ refer to a UAV. Equation (17), lemmas 5 and 6 yield an upper bound on $V_n(t)V_{n+1}(t)$.

$$\begin{aligned} V_n(t)V_{n+1}(t) &= \left\| \overrightarrow{\text{AV}}(S_{n+1}(S(t))) - \overrightarrow{\text{AV}}(S_n(S(t))) \right\| \\ &\leq |S_{n+1}(S(t)) - S_n(S(t))| \\ &\leq \Delta S_r \end{aligned}$$

The triangle inequality, equation (23) and assumptions 1 and 3 prove that this UAV is directly connected with its follower.

$$\begin{aligned} R_n(t)R_{n+1}(t) &\leq R_n(t)V_n(t) + V_n(t)V_{n+1}(t) + R_{n+1}(t)V_{n+1}(t) \\ &\leq 2\rho_c + \Delta S_r = \rho_r \end{aligned}$$

- 2) We prove here that connectivity to A is ensured when

$$S(t) \in [0, N\Delta S_r] \quad (24)$$

This interval comes from the following calculation.

$$[0, N\Delta\mathcal{S}_r] \subset \bigcup_{n=1}^N [L_{1,n} - \Delta\mathcal{S}_r, L_{1,n} + \Delta\mathcal{S}_r]$$

Hence to prove the statement, it suffices to show for all n that

$$S(t) \in [L_{1,n} - \Delta\mathcal{S}_r, L_{1,n} + \Delta\mathcal{S}_r] \Rightarrow R_n(t) \bowtie A$$

Let $n \in \{1 \dots N\}$ and t such that $S(t)$ belongs to $[L_{1,n} - \Delta\mathcal{S}_r, L_{1,n} + \Delta\mathcal{S}_r]$.

Because $A = V(0) = V(S_n(L_{1,n}))$, equation (17), lemmas 5 and 6 show that

$$\begin{aligned} V_n(t)A &= \|V(S_n(S(t))) - V(S_n(L_{1,n}))\| \\ &\leq |S_n(S(t)) - S_n(L_{1,n})| \\ &\leq |S(t) - L_{1,n}| \leq \Delta\mathcal{S}_r \end{aligned}$$

The triangle inequality, equation (23) and assumptions 1 and 3 prove that $R_n(t) \bowtie A$.

- 3) We prove here that connectivity to B is ensured when

$$S(t) \in [\mathcal{L}(\mathbf{P}) - \Delta\mathcal{S}_r, \mathcal{L}(\mathbf{P}) + N\Delta\mathcal{S}_r] \quad (25)$$

This interval comes from the following calculation.

$$[\mathcal{L}(\mathbf{P}) - \Delta\mathcal{S}_r, \mathcal{L}(\mathbf{P}) + N\Delta\mathcal{S}_r] \subset \bigcup_{n=1}^N [L_{K,n} - \Delta\mathcal{S}_r, L_{K,n} + \Delta\mathcal{S}_r]$$

Hence to prove the statement, it suffices to show for all n that

$$S(t) \in [L_{K,n} - \Delta\mathcal{S}_r, L_{K,n} + \Delta\mathcal{S}_r] \Rightarrow R_n(t) \bowtie B$$

Let $n \in \{1 \dots N\}$ and t such that $S(t)$ belongs to $[\mathcal{L}(\mathbf{P}) - \Delta\mathcal{S}_r, \mathcal{L}(\mathbf{P}) + N\Delta\mathcal{S}_r]$.

Because $B = V(\mathcal{L}(\mathbf{P})) = V(S_n(L_{K,n}))$, equation (17), lemmas 5 and 6 again show:

$$V_n(t)B = \|V(S_n(S(t))) - V(S_n(L_{K,n}))\| \leq \Delta\mathcal{S}_r$$

The triangle inequality, equation (23) and assumptions 1 and 3 prove that $R_n(t) \bowtie B$.

- 4) All-time connectivity (i.e. $\forall n, t, R_n(t) \bowtie A$ or $R_n(t) \bowtie B$) is directly derived from equations (24) and (25) when

$$\begin{aligned} &[0, N\Delta\mathcal{S}_r] \cup [\mathcal{L}(\mathbf{P}) - \Delta\mathcal{S}_r, \mathcal{L}(\mathbf{P}) + N\Delta\mathcal{S}_r] \\ &\supset [0, \mathcal{L}(\mathbf{P}) + (N-1)\Delta\mathcal{S}_r] \end{aligned}$$

that is when

$$N\Delta\mathcal{S}_r \geq \mathcal{L}(\mathbf{P}) - \Delta\mathcal{S}_r$$

which is equivalent to $N \geq \frac{\mathcal{L}(\mathbf{P})}{\Delta\mathcal{S}_r} - 1$.

□