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1 On Distances between Words with Parameters

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8 — Abstract —

9 The edit distance between parameterized words is a generalization of the classical edit distance where
10 it is allowed to map particular letters of the first word, called parameters, to parameters of the second
11 word before computing the distance. This problem has been introduced in particular for detection
12 of code duplication, and the notion of words with parameters has also been used with different
13 semantics in other fields. The complexity of several variants of edit distances between parameterized
14 words has been studied, however, the complexity of the most natural one, the Levenshtein distance,
15 remained open.

16 In this paper, we solve this open question and close the exhaustive analysis of all cases of
17 parameterized word matching and function matching, showing that these problems are NP-complete.
18 To this aim, we also provide a comparison of the different problems, exhibiting several equivalences
19 between them. We also provide and implement a MaxSAT encoding of the problem, as well as a
20 simple FPT algorithm in the alphabet size, and study their efficiency on real data in the context of
21 theater play structure comparison.

22 **2012 ACM Subject Classification** Theory of computation → Pattern matching

23 **Keywords and phrases** String matching, edit distance, Levenshtein, parameterized matching,
24 parameterized words, parameter words, instantiable words, NP-completeness, MAX-SAT.

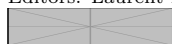
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26 **1** Introduction

27 Measuring the similarity between text strings is a fundamental problem in computer
28 science, and has applications in bioinformatics [23], databases [1, 16] and natural language
29 processing [27]. Among the measures of similarities between strings, the Levenshtein
30 distance [28] is the most commonly used, both for its practicality and its ease of computation.
31 This distance quantifies the minimum number of operations of insertion, deletion, and
32 substitution needed to transform a string into another one. It has a wide range of applications,
33 ranging from biological sequence alignment [33] to dialect pronunciation differences [25] or
34 signature authentication [34]. Computing the edit distance between two strings of length
35 n and m can be achieved in time $O(nm)$, by computing the distance between all their
36 prefixes, and storing the results in a dynamic programming fashion [37]. The success of the
37 Levenshtein distance generated many extensions and generalization on more complex models,
38 such as trees [38] or automata [32].

39 However, a limitation of the Levenshtein distance is that it only captures proximity
40 between strings (or objects) written on the same alphabet. Evaluating the proximity of
41 strings written on different alphabets is a problem that arises in various applications, such as
42 bioinformatics [35], image processing [17] and code duplication [6, 7]. In all those contexts,
43 the technique used is the one of parameterized matching [6, 7]. Instead of using classical
44 strings, parameterized matching uses “parameterized words” written using both constant
45 parts, which are expensive to rename, and parameters, which are meant to be renamed freely.
46 Formally, two equal-length strings u and v over an alphabet Π are said to be parameterized
47 matching if there exists a 1-to-1 function $f : \Pi \rightarrow \Pi$ such that $f(u) = v$, where $f(u)$ is
48 defined as $f(u_1) \dots f(u_{|u|})$.

49 Words with parameters also occur in other frameworks, and are often used in slightly
50 different ways. The first of those frameworks was initially introduced in the context of
51 Ramsey theory in the 80s [36], and is called “parameter words”. In this context, parameters
52 are labelled according to their order of first occurrence. Parameter words are also equipped
53 with a composition operation, where parameters of the first word can be instantiated by
54 characters or parameters of the second word. Parameter words can also be seen as equivalence
55 classes of parameterized words, which are the main focus of this article.

56 A second framework using parameters is the one of parameterized regular expressions
57 introduced in [10], where parameters can only be instantiated by constants, and not by other
58 parameters. The focus in this context is therefore made on the set of all possible valuations
59 of the parameters. Then, when defining algorithmic problems on such objects, two distinct
60 semantics can be studied: either the “certainty semantics”, where all valuations need to
61 have some property, or the “possibility semantics”, where at least one valuation needs to
62 have this property. To make a difference with the parameterized word framework mentioned
63 below, we choose to call these words “instantiable words”. Finally, this notion of words with
64 parameters can also be seen as a refined version of partial words (words containing a wildcard
65 character) [15]. The notion of partial words is also important in the context of databases
66 where paths of incomplete graphs can be interpreted as instantiable words [9].

67 This article aims at studying similarity by using edit distances in the framework of words
68 with parameters. In this framework, the pattern matching problem, which consists in looking
69 for the first string as a subword of the second string, has been extensively studied, either
70 looking for exact occurrences, with efficient algorithms [4, 19, 30] or approximate ones, which
71 is often NP-hard [21, 22]. In the case where we compare the two input strings in their
72 entirety, various exact parameterized matching problems have been studied for parameterized

d	\emptyset	D	I	DI
\emptyset	P [8]	NP-complete (Th. 12)	NP-complete (Cor. 14)	NP-complete [26]
S	P [24]	NP-complete (Cor. 14)	NP-complete (Cor. 14)	NP-complete (Th. 13)

■ **Figure 1** Complexity of the variants of parameterized matching PM^d , depending on the kind of operations (D: deletion, I: insertion, S: substitution) allowed in the edit distance d .

73 pattern matching, namely string parameterized matching [7], single pattern parameterized
74 matching [7, 3], multiple pattern parameterized matching, or 2-dimensional parameterized
75 matching, many of those works being compiled in [29] and [31]. Different approximate variants
76 of parameterized matching using edit distance have already been studied, but the problem
77 has not been completely solved: the first work on the topic is [8], in which Baker introduces
78 a form of approximate parameterized pattern matching in which the replacement of any
79 substring by another one that is in parameterized matching with it is considered as a base
80 edit operation. Parameterized matching under the Hamming distance, i.e., with a distance
81 allowing only substitutions, has been covered in [24], where the authors prove that the string
82 matching problem with at most k mismatches can be solved in time $O(m + k^{1.5})$. The LCPS
83 (Longest Common Parameterized Subsequence) problem, equivalent to the parameterized
84 pattern matching problem with insertions and deletions, is shown to be NP-hard in [26],
85 which also provides an approximation algorithm. Those two different complexity classes for
86 these problems raise the question of the complexity of the problem under the Levenshtein
87 distance. This problem was left as an open question in the conclusion of [24].

88 Our paper establishes that this problem is NP-complete. Moreover, the result also extends
89 to any possible edit distances obtained from deletion, insertion, and substitution as soon
90 as substitution is not the only operation allowed, as summarized in Figure 1. Our main
91 proof also implies the main theorem of [26] with a different NP-completeness reduction.
92 This contrasts with the problems of exact parameterized pattern matching which are all
93 polynomial-time solvable, as well as all variants of the string matching problem with deletions,
94 insertions or substitutions.

95 We also extend these results to function matching, which is the problem obtained by
96 relaxing the 1-to-1 restriction in parameterized matching, as defined in [2]. This generalization,
97 by breaking the symmetry of parameterized matching, actually gives rise to two close but
98 different problems, depending of the order of operations that are considered. We study the
99 links between all these problems and their computational complexity, and study two practical
100 ways to solve them, parameterized complexity and the use of maxSAT solvers.

101 We also make a direct connection with the framework of instantiable words, more precisely
102 with a natural problem of distance between languages. We show how instantiable word
103 problems can be reduced to parameterized matching ones, under the right assumptions. This
104 allows us to open new perspectives on the complexity of several language repair problems.

105 In Section 2, we give basic definitions and notations, and recall the existing formalism
106 of parameterized matching and instantiable words. In Section 3 we discuss approximate
107 parameterized matching and its various generalizations. We also link it to instantiable
108 words. In Section 4, we first prove a collection of technical results that build up to the
109 NP-completeness proofs for parameterized matching and function matching problems defined
110 above. In Section 5, we study two approaches to solve one of the variants of parameterized
111 matching in practice, a simple FPT algorithm parameterized by the alphabet size and a
112 MaxSAT encoding. We show in Section 6 that these implementations can solve real instances
113 of the problem, motivated by structure comparison of theater plays.

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114 Finally, in Section 7, we conclude the paper and give a few perspectives on the notion of
115 distance between parameterized languages.

116 **2** Notations and Definitions

117 **2.1** Basic Notations on Words and Editions

118 **Words**

119 An **alphabet** is a set of letters. A word on an alphabet A is a finite sequence of letters from
120 A , indexed starting from 1. Let u be a word on A . Unless defined differently, we note u_i the
121 i -th letter of u , and $|u|$ is the length of u . If $i \notin [1, |u|]$, u_i is defined as the empty word ε . If x
122 is a letter from A , $|u|_x$ is the number of times x appears in u . Similarly, if X is a set of letters,
123 $|u|_X = \sum_{x \in X} |u|_x$ is the number of occurrences of letters of X in u . If f is a function defined
124 on an alphabet A , we extend it to A^* in the usual way, so that $f(u) = f(u_1) \dots f(u_{|u|})$.
125 If f is a function, we denote by $\mathcal{D}(f)$ the domain of f . Two functions f and g are said
126 to be **compatible** if $f|_{\mathcal{D}(g) \cap \mathcal{D}(f)} = g|_{\mathcal{D}(g) \cap \mathcal{D}(f)}$. The identity function on D is defined as
127 $Id_D(x) = x$ for all x in D .

128 **Edit Operations**

129 In this paper, we consider the three classical **edit operations** which are *deletion*, *substitution*
130 and *insertion*. Let $u = u_1 \dots u_n$ be a word of size n . Let i be an integer between 0 and n and
131 x be a letter of the alphabet, the **insertion** at position i is the operation that transforms u
132 to $u_1 \dots u_i x u_{i+1} \dots u_n$. Let j be an integer between 1 and n , the **deletion** at position j is
133 the operation that transforms u into $u_1 \dots u_{j-1} u_{j+1} \dots u_n$. Let y be a letter of the alphabet
134 and $y \neq u_j$, the **substitution** to y at position j is the operation that transforms u into
135 $u_1 \dots u_{j-1} y u_{j+1} \dots u_n$. A **sequence of operations** or **rewriting sequence** ρ is a sequence
136 of edit operations. We denote by $\rho(u)$ the word obtained by applying the edit operations of
137 ρ one after another, in the order defined by ρ , to u .

138 **Distances**

139 Given a set of edit operations E and two words u and v , the **edit distance** between u and
140 v is defined as the length of a shortest sequence of operations of E changing u into v . We
141 denote by D the distance obtained on words by allowing only deletion operations: that is to
142 say $D(u, v) = k$ iff v can be obtained by deleting k letters from u . Similarly, we note I and S
143 the distances obtained by allowing only insertions and substitutions respectively (note that
144 S is the Hamming distance). We also combine these notations to define DI as the distance
145 with insertions and deletions, and so on. We also denote the Levenshtein distance DIS by L .
146 Note that some of these edit distances are not metrics, because they are not symmetrical.
147 We emphasize this by calling **symmetric edit distances** the distances DI, S , and L .

148 **2.2** Comparing Words with Parameters

149 Conceptually, a word with parameters is a word in which some letters are not yet determined.
150 In order to distinguish the parameters from the constants, we split the alphabet into Σ ,
151 the alphabet of the constants and Π , the alphabet of the parameters. By definition, these
152 alphabets are finite. A word with parameters can either be seen as representing a “word
153 template” (i.e., a word with variable parts), or a set of words (determined by all possible

154 affectations of its parameters). Depending on the definition chosen, comparing two words
 155 w_1 and w_2 is done in two different ways. In the first setting [6, 7, 8, 31, 2, 5, 24, 29, 26, 17],
 156 parameters of w_1 are renamed through a function f that maps the set of parameters to itself,
 157 and acts as identity on the set of constants. It is then possible to compare $f(w_1)$ and w_2 ,
 158 which are written on the same alphabet. In the second setting, constants are seen as the
 159 concrete values parameters can take [11]. Parameters are instantiated through two functions
 160 f_1 and f_2 that map constants to themselves, but also map parameters to constants. The
 161 words $f_1(w_1)$ and $f_2(w_2)$ are then made only of constants, and can be compared. Formally,
 162 this gives rise to the two following different definitions:

163 On the one hand, a **parameterized word** is a word on an alphabet $\Sigma \cup \Pi$. In all that
 164 follows, Σ and Π are two disjoint alphabets, respectively called the **alphabet of constants**
 165 and the **alphabet of parameters**. Alphabets Σ and Π are considered to be finite, unless
 166 specified otherwise.

167 Two parameterized words u and v are said to be in **function matching** if there exist
 168 $f_\Pi : \Pi \rightarrow \Pi$ and $f : \Pi \cup \Sigma \rightarrow \Pi \cup \Sigma$ such that $f|_\Pi = f_\Pi$, $f|_\Sigma = Id_\Sigma$, and $f(u) = v$. In
 169 the classical setting [6], f is also **constrained to be 1-to-1**, and this relationship is called
 170 **parameterized matching**. Note that parameterized matching is an equivalence relation on
 171 parameterized words. Testing if two words are parameterized matching can be achieved in
 172 linear time [7].

173 On the other hand, an **instantiable word** is a word on the alphabet $\Sigma \cup \Pi$. Given
 174 $f : \Pi \rightarrow \Sigma$, we extend it to constants by setting $f(x) = x$ for all $x \in \Sigma$, and we then define
 175 the **language of an instantiable word** u to be $L(u) = \{w \in \Sigma^* \mid \exists f : \Pi \rightarrow \Sigma, f(u) = w\}$.
 176 This definition is akin to the L_\diamond semantic of a parameterized regular expression defined
 177 in [11], but restricted here to a single instantiable word. Two instantiable words w_1 and w_2
 178 describe the same elements if their languages are equal, i.e. $L(w_1) = L(w_2)$.

179 3 Different Definitions for Different Semantics and Problems

180 In this section, we introduce various new approximate variants of parameterized matching,
 181 and compare them, highlighting their differences on examples.

182 3.1 Variants of Parameterized Matching

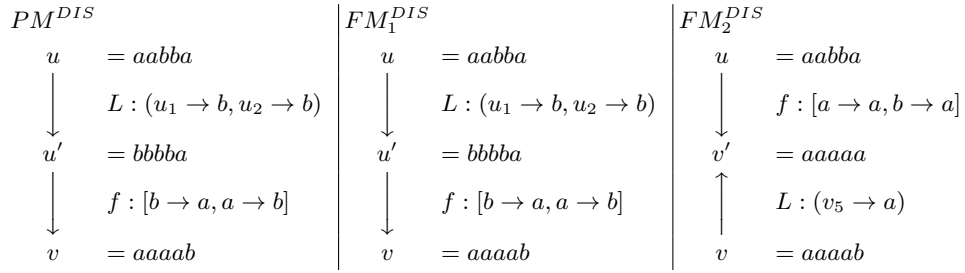
183 In parameterized matching, the function f renaming parameters is generally considered to
 184 be 1-to-1. In this paper, we also consider the **function matching problem**, which is the
 185 case where f is not constrained to be injective anymore, as defined in [2]. We also introduce
 186 multiple approximate variants of the parameterized matching problems, depending on several
 187 edit distances obtained by combining insertion, deletion and substitution operations.

188 3.1.1 Edit distances for parameterized matching between two strings:

189 PM^d

190 ► **Definition 1.** *If d is an edit distance, we denote by PM^d the parameterized matching problem*
 191 *under d , which is the following:*

- 192 ■ **Input:** *two parameterized words u, v , a parameter alphabet Π , an alphabet Σ of constants,*
 193 *and a natural number k .*
- 194 ■ **Problem:** *Does there exist u' such that $d(u, u') \leq k$ and u' and v are parameterized*
 195 *matching, i.e. there exists a 1-to-1 function $f : \Pi \cup \Sigma \rightarrow \Pi \cup \Sigma$ such that $f|_\Sigma = Id_\Sigma$,*
 196 *$f(\Pi) = \Pi$, and $f(u') = v$?*



■ **Figure 2** Side-by-side comparison of PM^{DIS} , FM_1^{DIS} and FM_2^{DIS}

197 In that case, we say that u' and f **realize** the matching between u and v . We sometimes
 198 write that only f or u' realize the matching if the other one is not relevant to a proof.

199 In cases where Σ and Π are already defined, we omit them and simply call $PM^d(u, v, k)$
 200 the result of the decision problem. Furthermore, $PM^d(u, v)$ denotes the minimum integer k
 201 (potentially infinite) such that $PM^d(u, v, k)$ is true.

202 We can note that this problem can be solved in polynomial time adapting the classical
 203 dynamic programming algorithm [33, 37] when the alphabet sizes are fixed.

204 3.1.2 Edit distances for function matching between 2 strings: FM_i^d

205 To denote **function matching problems**, we use FM instead of PM : FM^D denotes the
 206 function matching problems with deletions.

207 Furthermore, if \mathcal{P} is one of the problems defined above, we note \mathcal{P}_1 the problem where
 208 edit operations are only applied to the first argument, and \mathcal{P}_2 the one where they are only
 209 applied to the second argument.

210 ► **Definition 2.** The FM_1^d and FM_2^d problems are defined as follows. For both problems,
 211 the input is the following:

212 ■ **Input:** two parameterized words u, v , a parameter alphabet Π , a constant alphabet Σ ,
 213 and a natural number k .

214 The problems are then:

215 ■ **Problem FM_1^d :** $\exists u'$ such that $d(u, u') \leq k$ and u' and v are function matching?

216 ■ **Problem FM_2^d :** $\exists v'$ such that $d(v, v') \leq k$ and u and v' are in function matching?

217 Note that the renaming function f is always applied to one input only. These definitions are
 218 illustrated on an example in Figure 2.

219 3.2 Comparing Variants of PM

220 In this subsection, we compare the different variants of our problem.

221 Regarding the one-to-one parameterized matching PM , note that the definition we
 222 give above is designed to be easily extended to the different variants when we drop the
 223 one-to-one restriction. In [24], the authors consider that the “correct way for defining the
 224 edit distance problem” is “to allow the operations and then apply the edit distance”. By
 225 extending the definition of FM_1^d and FM_2^d to define PM_1^d and PM_2^d in the case of one-to-one
 226 matching, we see that it is actually possible to switch the order of operations, and to reverse
 227 them (deletions then become insertions and vice versa, and the renaming function f^{-1} is
 228 well-defined), in this case. This makes our definition consistent with the quote from [24]

above. Formally, this gives the following equalities, for all parameterized words u and v :
 $PM_1^I(u, v) = PM_1^D(v, u) = PM_2^D(u, v) = PM_2^I(v, u)$.

More generally, it holds that for every edit distance d , $PM_1^d(u, v) = PM_1^{d^{-1}}(v, u) = PM_2^{d^{-1}}(u, v) = PM_2^d(v, u)$, where d^{-1} denotes the converse distance of d , i.e. d^{-1} contains deletions if d contains insertions, insertions if d contains deletions, and substitutions if d contains substitutions.

However, for function matching, we only have the following equalities: $FM_1^I(u, v) = FM_2^D(u, v)$ and $FM_1^D(u, v) = FM_2^I(u, v)$.

By taking $u = ab$ and $v = cc$, we can notice that $FM_1^I(u, v) = 0$ and $FM_1^D(v, u) = \infty$, so the equality $FM_1^I(u, v) = FM_1^D(v, u)$ does not hold.

Finally, note the following inequalities:

► **Proposition 3.** *Let u and v be parameterized words over $\Sigma \cup \Pi$. Then:*

1. $FM_1^d(u, v) \leq PM^d(u, v)$;
2. If d is a symmetric edit distance, $FM_2^d(u, v) \leq FM_1^d(u, v)$.

Proof. The first point comes from the fact that any solution to PM^d is also a solution to FM_1^d . For the second point, let $k = FM_1^d(u, v)$, and let u' and f realize $FM_1^d(u, v)$. We construct a word v' , obtained by applying to v the same operations applied to u to obtain u' , but "mirrored". That is to say, for every operation used in u , we apply an operation in v , in the following way:

- If a letter a is inserted in u , there exists a position i in u' such that $u'_i = a$, and $f(u'_i) = v_i$. Hence, we delete v_i in v .
- Similarly, if a letter is substituted for another letter a' in u , there exists i such that $u'_i = a$, and we substitute v_i to $f(a)$.
- If a letter a is deleted in u at position i , we insert $f(a)$ in v at position i instead.

It then holds that $f(u) = v'$, and hence $PM_2^d(u, v) \leq k$. ◀

Note that the above proof does not work to prove the converse inequality between FM_1^d and FM_2^d , as it would require to consider an element of $f^{-1}(a)$, which might be empty. This is illustrated in the following example, on the alphabet $\Pi = \{a, b\}$:

► **Example 4.** Let $N \in \mathbb{N}$ and consider $u = a^N b^N b$ and $v = a^N a^N b$. u and v are not in parameterized matching, hence $FM_1^{DIS}(u, v) > 0$ and $FM_2^{DIS}(u, v) > 0$. By substituting the last b in v for a a , and picking a function f such that $f(a) = f(b) = a$, we get $FM_2^{DIS}(u, v) = 1$ (see Figure 2 for an example with $N = 2$). For FM_1^{DIS} , since b appears in v , it holds that for any function f realizing FM_1^{DIS} , $f(a) = b$ or $f(b) = b$. Hence, at least N occurrences of b appear in $f(u)$. Since there is only one occurrence of b in v , it is clear that $FM_1^{DIS}(u, v) \geq N - 1$.

The difference between FM_1^d and FM_2^d comes from the fact that Π is fixed in the input. In the case where Π could be extended, both problems can be shown equivalent (for example if we allow a new letter c in the example of Figure 2, we also get $FM_1^{DIS}(u, v) = 1$ by setting $u_5 \rightarrow c$ and $f : [a \rightarrow a, b \rightarrow a, c \rightarrow b]$), by using the same proof as Proposition 3.

3.3 Instantiable Words versus Parameterized Words

The parameterized word formalism and the instantiable word formalism give rise to two different definitions of distances between words. Given an edit distance d on words, there are two ways to extend it to words with parameters. Let w_1 and w_2 be two words over $\Sigma \cup \Pi$. The two possible extensions are the following:

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273 ■ The distance between w_1 and w_2 is defined as $d(w_1, w_2) = PM^d(w_1, w_2)$. Alternatively,
274 the function distance between w_1 and w_2 is defined as $FM_1^d(w_1, w_2)$.

275 ■ The distance between w_1 and w_2 is the distance between their respective languages
276 seen as sets, that is to say $d(w_1, w_2) = d(L(w_1), L(w_2)) = \sup_{u \in L(w_1)} \inf_{v \in L(w_2)} d(u, v)$.
277 Equivalently, $d(w_1, w_2) \leq k$ if and only if for all $f_1 : \Pi \rightarrow \Sigma$, there exists $f_2 : \Pi \rightarrow \Sigma$ such
278 that $d(f_1(w_1), f_2(w_2)) \leq k$.

279 This second definition stems from the definition of distance between languages, as defined
280 and studied in [12, 13, 14].

281 ► **Example 5.** Consider the words $u = axyb$ and $v = xbbv$, on the alphabets $\Sigma = \{a, b\}$
282 and $\Pi = \{x, y\}$, and consider the distance S . On the one hand, $FM_1^S(u, v) = 4$, because
283 regardless of the matching chosen, every letter of $f(u)$ has to be substituted. On the other
284 hand, for any function $f_1 : \Pi \rightarrow \Sigma$, choosing f_2 such that $f_2(x) = a$ and $f_2(y) = b$ yields a
285 distance $d(f_1(u), f_2(v))$ of at most 2, by substituting the 2 middle letters.

286 Given a big enough alphabet, those two definitions can in fact be shown equivalent:

287 ► **Proposition 6.** Let w_1 and w_2 be words over $\Sigma \cup \Pi$, and let d be a symmetric edit distance
288 on $\Sigma \cup \Pi$. Suppose $|\Sigma| \geq |w_1| + |w_2|$, and let k be an integer. Then, the following are
289 equivalent:

- 290 1. $FM_1^d(w_2, w_1) \leq k$
- 291 2. $d(L(w_1), L(w_2)) \leq k$

292 Notice how w_1 and w_2 change position between the two distances. This is not benign, as
293 FM_1^d is not symmetric.

294 **Proof.** Suppose $FM_1^d(w_2, w_1) \leq k$. There exists $f : \Pi \rightarrow \Pi$ such that $d(f(w_2), w_1) \leq k$. For
295 this proof, we will use the characterization of the distance between languages with f_1 and f_2 .
296 Let $f_1 : \Pi \rightarrow \Sigma$. Define $f_2 = f_1 \circ f$. Since $d(w_1, f(w_2)) \leq k$, we have $d(f_1(w_1), f_1 \circ f(w_2)) \leq k$,
297 by following the same edit operations. Hence $d(f_1(w_1), f_2(w_2)) \leq k$.

298 Suppose now $d(L(w_1), L(w_2)) \leq k$. Let $f_1 : \Pi \rightarrow \Sigma$ be a 1-to-1 function such that for
299 all parameters x in w_1 , $f(x)$ does not appear in w_1 or w_2 . This is possible since Σ is large
300 enough. There exists $f_2 : \Pi \rightarrow \Sigma$ such that $d(f_1(w_1), f_2(w_2)) \leq k$. Let $h : \Sigma \rightarrow \Pi \cup \Sigma$ be such
301 that if $x \in \Pi$, $h(f_1(x)) = x$, and if $x \notin f_1(\Pi)$, $h(x) = x$. We then have $h \circ f_1 = Id$. What
302 is more, since h is injective, $d(f_1(w_1), f_2(w_2)) = d(h(f_1(w_1)), h(f_2(w_2))) = d(h(f_2(w_2)), w_1)$.
303 Hence, $FM_1^d(w_2, w_1) \leq k$. ◀

304 4 Hardness Results for Approximate Parameterized Matching

305 In this section, we study the complexity of the various parameterized matching problems.
306 We show the NP-completeness of the simplest problems using only deletions, which will be
307 sufficient to show the NP-completeness of all the other problems. We start by proving some
308 practical lemmas, and then proceed to the reductions.

309 4.1 “Block by block” Lemmas

310 In this section, we regroup a few useful technical lemmas. We start of by stating two simple
311 results on distance and words, for which the proofs can be found in Appendix A. We then
312 turn to block lemmas, which will later be useful in the proofs of Theorems 12,17 and 15, to
313 combine the various gadgets defined during the reduction.

314 This lemma simply states a commutativity result between the application of a matching
315 f and the rewriting steps.

316 ▶ **Lemma 7.** *Let d be a distance, k an integer and u, v two parameterized words such that*
 317 *$PM^d(u, v) \leq k$, and let f realize this parameterized match. Then: $d(f(u), v) \leq k$. The same*
 318 *result holds for $FM_1^d(u, v)$.*

319 **Proof idea:** The proof is done by induction on k . We discuss whether the $(k + 1)$ -th
 320 operation is an insertion, a deletion, or a substitution, and show that a corresponding
 321 operation can be used in $f(u)$.

322 ▶ **Lemma 8.** *Let z, u and v be (parameterized) words, and let d be a distance. Then*
 323 *$d(zu, zv) = d(u, v)$.*

324 **Proof Idea:** We show that we can consider every rewriting operation to be applied in u only:
 325 if z is modified during an optimal rewriting sequence, the words have some redundancy, and
 326 the same operations could have been carried in u instead. We proceed again by induction,
 327 and focus on the base case by studying the 3 possible cases, one for each type of operation.

328 Next, we turn to prove “block by block” matching lemmas. Those results state that it is
 329 possible to encode multiple parameterized matching instances into a single one. They hold
 330 for every type of problems considered here, but their proofs vary slightly; we present them
 331 in order of increasing complexity. Note that all the constructions given can be achieved in
 332 polynomial time.

333 ▶ **Lemma 9.** *Let u_1, \dots, u_n and v_1, \dots, v_n be parameterized words over $\Sigma \cup \Pi$ such that for*
 334 *$1 \leq i \leq n$, $k_i = |u_i| - |v_i| \geq 0$, and $k = \sum_{i=1}^n k_i$. There exist u and v two parameterized words*
 335 *over $\{\#\} \cup \Sigma \cup \Pi$, where $\#$ is a fresh variable, such that the following are equivalent:*

- 336 1. $PM^D(u, v) = k$
- 337 2. For all $1 \leq i \leq n$, $PM^D(u_i, v_i) = k_i$ and the applications f_i realizing those matchings are
 338 all compatible.

339 **Proof.** The idea behind this proof and all the following ones is that we can introduce a
 340 separator $\#$ to concatenate all the words, and that this separator will never be touched by
 341 any deletions or applications of f .

342 Let $\#$ be a fresh constant. We define $u = u_1\#u_2\#\dots\#u_n$, and $v = v_1\#v_2\#\dots\#v_n$.

343 2. \implies 1.: For every $1 \leq i \leq n$, take u'_i and f_i to realize the matchings. We can obtain
 344 $u' = u'_1\#u'_2\#\dots\#u'_n$ from u by applying the same deletions. Taking f to be the smallest
 345 function extending all the f_i , we get $PM^D(u, v) \leq k$.

346 1. \implies 2.: Assume $PM^D(u, v) \leq k$. Let u' and f realize this parameterized match.
 347 Since the $\#$ symbols are constants, we have $f(\#) = \#$. Since u' is obtained from u by
 348 deletions, we have $|u'|_{\#} \leq |u|_{\#}$. Since f is injective and $f(\#) = \#$, $|f(u')|_{\#} \leq |f(u)|_{\#}$.
 349 Hence, it holds that $|v|_{\#} = |f(u')|_{\#} \leq |f(u)|_{\#} = |u|_{\#}$. Since $|u|_{\#} = |v|_{\#}$, this is an equality,
 350 and $|f(u')|_{\#} = |f(u)|_{\#}$. Hence $|u'|_{\#} = |u|_{\#}$, and no $\#$ character is deleted. The word u'
 351 is then of the form $u'_1\#u'_2\#\dots\#u'_n$, where $|u'_i|_{\#} = 0$ and $D(u_i, u'_i) = k_i$ for all i . Thus,
 352 $f(u') = f(u'_1)\#f(u'_2)\#\dots\#f(u'_n) = v_1\#v_2\#\dots\#v_n$. Since no other $\#$ letter appear in any
 353 $f(u'_i)$ and v_i , we can deduce that $f(u'_i) = v_i$ for all i . Finally, this yields $PM^D(u_i, v_i) = k_i$,
 354 and taking all the $f_i = f$ gives all compatible functions, which concludes the proof. ◀

355 In this proof, we used a constant $\#$. However, it can also be conducted without using
 356 a constant alphabet; indeed, constants can be encoded with parameters, as shown in
 357 Appendix B.

358 Lemma 9 is still valid if PM^D is replaced by FM_2^D . This time, we conduct this proof
 359 without resorting to the use of constants. This result will be used twice : once for the proof

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of theorem 17, and again to show that we can once more encode constants into Π using Lemma 25 in Appendix B.

► **Lemma 10.** *Let u_1, \dots, u_n and v_1, \dots, v_n be parameterized words over Π such that $k_i = |v_i| - |u_i| \geq 0$, and $k = \sum_{i=1}^n k_i$. Then there exist u and v , two parameterized words over $\Pi \cup \{\#\}$, where $\#$ is a fresh variable, such that the following are equivalent:*

1. $FM_2^D(u, v) \leq k$
2. For all $1 \leq i \leq n$, $FM_2^D(u_i, v_i) \leq k_i$, and the applications f_i realizing those matchings are all compatible.

Proof idea: The same technique as Lemma 9 is used but u and v are defined as $u = \#^{k+1}u_1\#u_2\#\dots\#u_n$ and $v = \#^{k+1}v_1\#v_2\#\dots\#v_n$ where $\#^{k+1}$ denotes $k+1$ repetitions of the character $\#$. The full proof can be found in Appendix A.

Finally, the same block result holds for FM_1^D , and will be used in the proof of theorem 15

► **Lemma 11.** *Let u_1, \dots, u_n and v_1, \dots, v_n be parameterized words over Π such that for every $1 \leq i \leq n$, $k_i = |u_i| - |v_i| \geq 0$, and $k = \sum_{i=1}^n k_i$. Then there exist u and v two parameterized words over $\Pi \cup \{\#\}$, where $\#$ is a fresh variable, such that the following are equivalent:*

1. $FM_1^D(u, v) \leq k$
2. For all $1 \leq i \leq n$, $FM_1^D(u_i, v_i) \leq k_i$, and the applications f_i realizing those matchings are all compatible.

Proof idea: The difference with Lemma 10 is that some $\#$ symbols might be deleted, while some base letters could be mapped to $\#$. To ensure this does not happen, we define $u = \#^N u_1 \#^N u_2 \dots \#^N u_n \#^N$ and $v = \#^N v_1 \#^N v_2 \dots \#^N v_n \#^N$. The full proof can be found in Appendix A.

The technique of block-by-block matching will be used in all the reductions, to encode multiple constraints in a single PM or FM instance.

4.2 1-to-1 Parameterized Matching PM

We now focus on the complexity of the PM^d problems. These problems, as well as function matching problems, are all clearly in NP: given the list of deletion, insertion or substitution operations to do and the matching to apply, it is easy to check that the solution is correct.

For the reductions in this paper, we always assume that words are written without constants, that is to say $\Sigma = \emptyset$, since this is sufficient for NP-completeness results. This choice is also motivated by the results of Appendix B, which show that Σ can in most cases be coded into Π .

► **Theorem 12.** *The 1-to-1 Parameterized Matching with deletions PM^D is NP-complete.*

The proof is a reduction from the NP-complete problem **3-coloring**[20]. Given an instance G of **3-coloring**, we will construct two words u and v . The word v will represent the list of vertices and edges of G , while the word u will list the color of each vertex, and the possible coloring of each pair of vertices joined by an edge. By deleting characters from u , we make a choice for the coloring of each vertex, and thus each edge. The function f answering the parameterized matching problem will assign a choice of color to each vertex. The instance that we define should be positive iff G is 3-colorable. More formally:

Proof. The 3-Coloring problem is defined as follows :

- 401 ■ **Input:** $G = (V, E)$ a graph with vertices V and edges E
 402 ■ **Output:** A coloring $c : V \rightarrow \{c_1, c_2, c_3\}$ such that for all $\{u, v\} \in E$, $c(u) \neq c(v)$
 403 Let $G = (V, E)$ be an instance of **3-Coloring**, and let $V = \{x_1, \dots, x_n\}$ be the set of its
 404 n vertices, and $E = \{e_1, \dots, e_m\}$ be the set of its edges. The parameter alphabet Π , of
 405 polynomial size in $O(|G|)$ will contain:
 406 ■ x_1, \dots, x_n , corresponding to the vertices of G ;
 407 ■ n copies of the parameters corresponding to the colors c_1, c_2 and c_3 : c_1^i, c_2^i, c_3^i for $1 \leq i \leq n$;
 408 ■ for every edge e , the delimiters Y^e and $\square_1^e, \dots, \square_6^e$;
 409 ■ $2n$ bottom symbols, \perp_1^i, \perp_2^i for $1 \leq i \leq n$, which will be used to fix the image of some
 410 parameters.

411 First, we define words that will encode the constraint that each vertex is colored, and
 412 we make sure that the unused color variables cannot be assigned elsewhere. For $1 \leq i \leq n$,
 413 $u_1^i = u_{\perp}^i = c_1^i c_2^i c_3^i$, $v_1^i = x_i$ and $v_{\perp}^i = \perp_1^i \perp_2^i$. We then define words that include all
 414 possible colorings of each edge, and we make sure to use enough brackets. For every edge
 415 $e = \{x_i, x_j\}$, we define $u_2^e = \square_1^e c_1^i c_2^j \square_1^e \square_2^e c_1^i c_3^j \square_2^e \square_3^e c_2^i c_1^j \square_3^e \square_4^e c_2^i c_3^j \square_4^e \square_5^e c_3^i c_1^j \square_5^e \square_6^e c_3^i c_2^j \square_6^e$
 416 and $v_2^e = Y^e x_i x_j Y^e$.

417 Applying Lemma 9 to $u_1^1, \dots, u_1^n, u_{\perp}^1, \dots, u_{\perp}^n, u_2^{e_1}, \dots, u_2^{e_m}$ and $v_1^1 \dots v_1^n, v_{\perp}^1, \dots, v_{\perp}^n, v_2^{e_1}, \dots, v_2^{e_m}$,
 418 we obtain u and v . Let $k = |u| - |v| = 3n + 20m$. We now show that G is 3-colorable \Leftrightarrow
 419 $PM^D(u, v) \leq k$.

420 \Rightarrow : Suppose G is 3-colorable. Let $c : V \rightarrow \{c_1, c_2, c_3\}$ be a 3-coloring of G . We define f
 421 in the following way, for $1 \leq y \leq 3$:

$$422 \quad f(c_y^i) = \begin{cases} x_i & \text{if } c(x_i) = c_y, \\ \perp_1^i & \text{if } y \text{ is the smallest integer in } \{1, 2, 3\} \text{ such that } c(x_i) \neq c_y, \\ \perp_2^i & \text{otherwise.} \end{cases}$$

423 For every edge $e = \{x_i, x_j\} \in E$, since c is a valid coloring, and since every allowed
 424 arrangements of the colors is in u_2^e , there exists a unique factor of the form $\square_y^e f^{-1}(x_i) f^{-1}(x_j) \square_y^e$
 425 in u_2^e , for some $1 \leq y \leq 3$. Hence, we define $f(\square_y^e) = Y^e$. The function f can then be
 426 extended in any way to be 1-to-1 (the remaining characters whose image under f are not yet
 427 defined will all be deleted in what follows, so their image doesn't matter).

428 By using f defined in this way:

- 429 ■ For $1 \leq i \leq n$, $PM^D(u_1^i, v_1^i) \leq 2$, by deleting the 2 colors not matching the color of x_i ;
 430 ■ For $1 \leq i \leq n$, $PM^D(u_{\perp}^i, v_{\perp}^i) \leq 1$;
 431 ■ For every edge $e \in E$, $PM^D(u_2^e, v_2^e) \leq 20$, by keeping only the factor delimited by the
 432 \square_y^e symbols defined above.

433 Thus Lemma 9 yields $PM^D(u, v) \leq k$.

434 \Leftarrow : We now suppose u and v are a parameterized match with k deletions. The following
 435 can then be derived about f :

- 436 1. Since the u_1^i and v_1^i are matching for $1 \leq i \leq n$, there exists an element $c \in \{c_1^i, c_2^i, c_3^i\}$
 437 such that $f(c) = x_i$. Each of these matchings is done with exactly 2 deletions, for a total
 438 of $2n$.
 439 2. Since the u_{\perp}^i and v_{\perp}^i are in matching, the two other colors that are not sent to x_i are sent
 440 to \perp_1^i and \perp_2^i . Each of these matchings is done with exactly one deletion, for a total of n .
 441 3. For every edge $e \in E$, u_2^e and v_2^e are in matching. Let $u_2^{e'}$ realize this matching. For every
 442 $1 \leq i \leq n$ and $1 \leq i' \leq 3$ the colors $c_{i'}^e$ have images that are different from Y^e , so there
 443 necessarily exists $1 \leq y \leq 6$ such that $f(\square_y^e) = Y^e$. Furthermore, since f is injective,
 444 $|v_2^e|_{Y^e} = |u_2^{e'}|_{\square_y^e}$. Since $|v_2^e|_{Y^e} = |u_2^e|_{\square_y^e} = 2$, no \square_y^e is deleted from u . Since there are two

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445 characters between the Y^e in v_2^e and none outside, $u_2^{e'}$ has the same structure, and all
 446 other \square_y^e for $y' \neq y$ and all other colors are deleted from u_2^e .

447 Finally, $u_2^{e'}$ is of the form $\square_y^e c_t c_{t'} \square_y^e$, where $t \neq t'$ are elements of $\{1, 2, 3\}$. Each of these
 448 matchings is done with exactly 20 deletions, for a total of $20m$.

449 The function f then implies a coloring of G . Formally, we define $col(c_y^i) = c_y$ for $1 \leq i \leq n$
 450 and $1 \leq y \leq 3$. We can then define $c : V \rightarrow \{c_1, c_2, c_3\}$ such that $c(x_i) = col(f^{-1}(x_i))$.
 451 Point 1 above ensures that this function definition is correct. Furthermore, for every edge
 452 $e = \{x_i, x_j\}$, point 3 ensures that $c(x_i) \neq c(x_j)$, and thus c is a valid coloring of G . ◀

453 This first NP-completeness results yields a few immediate corollary results, and in
 454 particular, the NP-completeness of the problem under the Levenshtein distance:

455 ▶ **Theorem 13.** *The problem PM^{DIS} of parameterized matching under the Levenshtein*
 456 *distance is NP-complete.*

457 **Proof.** We do a simple reduction from PM^D . Let u, v, k be an instance of PM^D . If the
 458 instance is trivially false (that is to say, $k \neq |u| - |v|$), answer negatively. Else, consider
 459 u, v, k as an instance of PM^{DIS} . If this is a negative instance for PM^{DIS} , it is also negative
 460 for PM^D . Furthermore, if it is a positive instance for PM^{DIS} , exactly k deletions should be
 461 applied, and so no substitution or insertion are used in that solution. Hence, that solution is
 462 also a solution to PM^D , and the reduction holds. ◀

463 The same result in fact holds for all other distances, and in particular the longest common
 464 sub-word distance ID . This proves once again the result shown in [26]:

465 ▶ **Corollary 14.** *The problems $PM^I, PM^{DI}, PM^{IS}, PM^{DS}$ are all NP-complete.*

466 **Proof.** From Section 3.2, PM^I and PM^D are equivalent in the 1-to-1 case. For the other
 467 problem, we do an immediate reduction from PM^I or PM^D analog to the proof of Theorem 13.
 468 ◀

469 We now turn to proofs of NP-completeness without the restriction asking f to be injective.

470 4.3 Function Matching FM_1^d

471 The problem considered in this section is the one where both deletions and f are applied to
 472 the first word. A reduction very similar to the one given for PM^D is used.

473 ▶ **Theorem 15.** *FM_1^D is NP-complete.*

474 **Proof idea:** The reduction follows the same idea as in Theorem 12. Since the function f
 475 realizing the matchings is not injective in this version, it will be used to send every vertex to
 476 its color. Moreover, we add more “sink” \perp letters to force the image of every unused letter.
 477 The full proof can be found in Appendix A.

478 This again ensures the NP-completeness of the problem for all edit distances, using the
 479 same proof as for Theorem 13.

480 ▶ **Corollary 16.** *The problem FM_1^{DIS} of function matching under the Levenshtein distance*
 481 *is NP-complete. The problems $FM_1^I, FM_1^{ID}, FM_1^{IS}, FM_1^{DS}$ are all NP-complete too.*

482 We can notice that the problem FM_1^S , where substitution is the only operation allowed,
 483 is polynomial-time solvable. Intuitively, for each parameter, consider the possible parameters
 484 that it could be mapped to, and their respective number of occurrences. Then, choose the
 485 letter with the highest number of occurrences for the mapping. The remaining letters are
 486 then substituted.

487 **4.4 Function Matching** FM_2^d

488 The problem considered in this section is the one where deletions are applied to the second
 489 word, while f is applied to the first word.

490 ► **Theorem 17.** FM_2^D is NP-complete.

491 **Proof Idea:** The proof is very similar to the previous case, but the bracketing has to be
 492 adapted. Separators Y^e are duplicated enough times to ensure that no vertex variable is
 493 mapped to them. The full proof can be found in Appendix A.

494 ► **Corollary 18.** FM_1^I , FM_2^{DI} , and FM_2^L are all NP-complete.

495 **Proof.** FM_1^I is equivalent to FM_2^D . For the two other problems, we use a reduction from
 496 FM_2^D exactly like in Corollary 14. ◀

497 This last result completes the picture of NP-completeness proofs, and indicates that
 498 computing the distances between parameterized words defined in Section 3.3 is in general an
 499 NP-complete problem.

500 Similarly to FM_1^S , FM_2^S is also polynomial-time solvable.

501 **5 Approaches to Solve Parameterized Matching**

502 In this section, we discuss two ways to get around the difficulty of the parameterized matching
 503 problems. The first one is to design an FPT algorithm in the alphabet size, and the second
 504 one is to translate the problem into a SAT formalism, with the intent of using a SAT-solver.

505 **5.1 An FPT Algorithm in the Alphabet Size**

506 The fact that Σ and Π are part of the input is what makes the various parameterized matching
 507 problems NP-hard. When the alphabet size is considered fixed, a simple polynomial algorithm
 508 can be used, which generalizes the “naïve” brute force algorithm of Theorem 1 of [26]:

■ **Algorithm 1** Simple FPT algorithm for FM^d

```

m ← 0
for all functions  $f : \Pi \rightarrow \Pi$  do
  dist ←  $d(f(u), v)$ 
  if  $dist \leq m$  then
    m ← dist
  end if
end for

```

509 ► **Theorem 19.** Let d be a distance. Algorithm 1 computes $FM^d(u, v)$ in time $O(|\Pi|^{|\Pi|}|u||v|)$

510 **Proof.** Algorithm 1 uses an exhaustive search and finds $\min_{f: \Pi \rightarrow \Pi} d(f(u), v)$, which is the
 511 definition of $FM^d(u, v)$. Furthermore, there are $|\Pi|^{|\Pi|}$ functions from Π to Π , and computing
 512 $d(f(u), v)$ is done in time $O(|f(u)||v|) = O(|u||v|)$, hence a total running time in $O(|\Pi|^{|\Pi|}|u||v|)$.
 513 ◀

514 Note that this also leads to a similar algorithm for PM^d by iterating over injective
 515 functions rather than all functions from Π to Π .

516 **5.2 A MaxSat Formulation of Parameterized Matching**

517 In this section, we encode PM^d problems into SAT problems, with the intent of solving them
 518 with a SAT solver. More precisely, we will use the weighted max-SAT variant of SAT, which
 519 is defined in the following way :

- 520 ■ **Input:** a set V of literals, a formula $\varphi = \bigwedge_{i=1}^n \varphi_i$ on V in conjunctive normal form (CNF),
 521 a weight function $w : \llbracket 1, n \rrbracket \rightarrow \mathbb{N}$.
 522 ■ **Output :** a valuation $\nu : V \rightarrow \{0, 1\}$ such that $\sum_{\nu \models \varphi_i} w(i)$ is maximal.

523 Moreover, we will sometimes use a partially weighted variant of Max-SAT, which is
 524 defined in the following way :

- 525 ■ **Input:** a set V of literals, a satisfiable formula φ_c on V in CNF, a formula $\varphi_w = \bigwedge_{i=1}^n \varphi_i$
 526 on V in CNF and a weight function $w : \llbracket 1, n \rrbracket \rightarrow \mathbb{N}$.
 527 ■ **Output :** a valuation $\nu : V \rightarrow \{0, 1\}$ such that $\nu \models \varphi_c$ and $\sum_{\nu \models \varphi_i} w(i)$ is maximal.

528 In that case, clauses of φ_c are called “hard” clauses while clauses of φ_w are called “soft
 529 clauses”. We give a proof of the equivalence in Proposition 26 of Appendix C.

530 We will define an encoding of an instance (u, v) of PM^d such that an assignment of the
 531 variables of V will define an alignment between u and v . First, we make a link between the
 532 ID edit distance and the length of the optimal alignment between two strings.

- 533 ► **Definition 20.** *Let u and v be two words on Π , such that $p = |u|$ and $p' = |v|$. A set*
 534 *$A \subset \llbracket 1, |u| \rrbracket \times \llbracket 1, |v| \rrbracket$ is an alignment between u and v iff the following are true:*
 535 **1.** *Each position of u appears at most once : For all $1 \leq i \leq p$ and $1 \leq j, j' \leq p'$, if $(i, j) \in A$*
 536 *and $(i, j') \in A$, then $j = j'$.*
 537 **2.** *Each position of v appears at most once : For all $1 \leq j \leq p'$ and $1 \leq i, i' \leq p$, if $(i, j) \in A$*
 538 *and $(i', j) \in A$, then $i = i'$.*
 539 **3.** *There are no crossings : if $(i, j) \in A$, $(i', j') \in A$, and $i' > i$, then $j' > j$.*
 540 **4.** *Aligned positions match in u and v : if $(i, j) \in A$, then $u_i = v_j$*

541 An alignment relates to the insertion/deletion distance ID in the following way :

- 542 ► **Theorem 21.** *Let u, v be words on Π and $k \leq |u| + |v|$ be an integer. The following are*
 543 *equivalent :*
 544 **1.** *There exists an alignment A such that $2|A| = |u| + |v| - k$*
 545 **2.** *$ID(u, v) \leq k$.*

546 **Proof.** The proof, which works by induction, can be found in Appendix C. ◀

547 We now turn to the max-SAT encoding of our problem.

548 ► **Theorem 22.** *Let u and v be two words over Π . There exists a formula $\varphi_{u,v} = \varphi_c \wedge \varphi_w$*
 549 *and a weight function w , instance of the partially weighted Max-SAT problem such that the*
 550 *following are equivalent:*

- 551 ■ *ν is a solution to this partially weighted Max-SAT instance and satisfies k clauses of φ_w*
 552 ■ *There exists a function $f : \Pi \rightarrow \Pi$ and an alignment between $f(u)$ and v of size k .*

553 *The formula φ uses $|m||p| + |\Pi|^2$ variables and is of size $O(m^2p^2)$, where $m = |u|$ and*
 554 *$p = |v|$. Moreover, there exists φ^{inj} of size $O(|\Pi|^3)$ such that the above result is true for f*
 555 *injective by replacing φ_c with $\varphi'_c = \varphi_c \wedge \varphi^{inj}$.*

556 In particular, finding the valuation maximizing k gives a maximal alignment between u
557 and v , and with Theorem 21, the distance $ID(u, v)$.

558 **Proof.** For this proof, we fix an ordering on the alphabet $\Pi = \{a_1, \dots, a_n\}$.
559 We define the set of literals V as $V = \{x_{i,j} \mid 1 \leq i \leq |u|, 1 \leq j \leq |v|\} \cup \{y_{a,b} \mid a \in \Pi, b \in \Pi\}$.
560 Intuitively, $x_{i,j}$ represents a match between position i and j in the alignment, and $y_{a,b}$ will
561 represent the fact that $f(a) = b$. We define the following sets of formulas, where all indices i
562 are taken between 1 and m and all j between 1 and p , and a and b are taken in Π :

$$\begin{array}{lll}
563 & \forall i \forall j' \neq j, & \varphi_{i,j,j'}^{A_1} \equiv x_{i,j} \implies \neg x_{i,j'} & \text{(NoDouble i)} \\
564 & \forall j \forall i' \neq i, & \varphi_{i',j}^{A_2} \equiv x_{i',j} \implies \neg x_{i',j} & \text{(NoDouble j)} \\
565 & \forall i' > i \forall j' < j, & \varphi_{i',i,j,j'}^C \equiv x_{i,j} \implies \neg x_{i',j'} & \text{(NoCrossing)} \\
566 & \forall a \forall b \neq b', & \varphi_{a,b,b'}^f \equiv y_{a,b} \implies \neg y_{a,b'} & \text{(Function)} \\
567 & \forall a \neq a' \forall \neq b, & \varphi_{a,a',b}^{inj} \equiv y_{a,b} \implies \neg y_{a',b} & \text{(Injectivity)} \\
568 & \forall i \forall j, & \varphi_{i,j}^M \equiv x_{i,j} \implies y_{u_i,v_j} & \text{(Match)} \\
569 & \forall i, & \varphi_i^\exists \equiv \bigvee_{1 \leq j \leq p} x_{i,j} & \text{(ExistsMatch)} \\
570 & & &
\end{array}$$

571 We then define φ_c as the conjunction of all the formulas (NoDouble i), (NoDouble j),
572 (NoCrossing), (Function), and (Match). Furthermore, we define φ^{inj} as the conjunction of
573 all the (Injectivity) formulas. Lastly, we define $\varphi_w = \bigwedge_{1 \leq i \leq m} \varphi_i^\exists$, and set $w(C) = 1$ for every
574 clause C of φ_w .

575 There are $m \binom{p}{2}$ (NoDouble i) formulas, $p \binom{m}{2}$ (NoDouble j), $\binom{m}{2} \binom{p}{2}$ (NoCrossing), $n \binom{n}{2}$
576 (Function) and (Injectivity) formulas, pm (Match) formulas and n (ExistsMatch) formulas.

577 We now prove both implications of the theorem. Suppose ν is a valuation satisfying φ_c
578 and k clauses of φ_w . We define, for all $a, b \in \Pi$, $f(a) = b$ if and only if $\nu(y_{a,b}) = \top$. Since ν
579 satisfies all the (Function) formulas, this is a correct definition of a (partial) function. We
580 define $A = \{(i, j) \mid \nu(x_{i,j}) = \top\}$. A is an alignment between $f(u)$ and v . Indeed : (NoDouble
581 i) and (NoDouble j) ensures point 1. and 2. of Definition 20, (NoCrossing) ensures point 3.,
582 and Match ensures point 4. The size of A is the number of $x_{i,j}$ instantiated to \top , which is
583 exactly the number of clauses of φ_c satisfied, i.e., k .

584 Suppose now that there exists a function $\Pi \rightarrow \Pi$ and an alignment A between $f(u)$ and
585 v . Similarly, we define $\nu(y_{a,b}) = \top$ if and only if $f(a) = b$, and $\nu(x_{i,j}) = \top$ if and only if
586 $(i, j) \in A$. Since A is an alignment, ν satisfies (NoDouble i), (NoDouble j), and (NoCrossing).
587 Since f is a function, (Function) is satisfied. Finally, if $\nu(x_{i,j}) = \top$, then $(i, j) \in A$, and
588 since A is a matching, $f(u)_i = f(u_i) = v_j$ and $\nu(y_{u_i,v_j}) = \top$.

589 The proof for φ^b is the same, and (Injectivity) ensures the injectivity of f . \blacktriangleleft

590 What is more, this proof can be adapted to change the ID distance to the Levenshtein
591 distance, simply by choosing to consider all the (Match) formulas as soft clauses.

592 **6 Experiments**

593 The two approaches presented in Section 5 were implemented in Python to solve PM^{ID} .
594 They are available under the GPL license at <https://github.com/AaronFive/paramatch>.
595 The FPT algorithm of Section 5.1 is implemented in the function `parameterizedAlignment`

596 of file `fpt_alphabet_size.py`. The MaxSAT-reduction of Section 5.2 is implemented in the
 597 function `make_sat_instance` of file `sat_instance.py`. The MaxHS solver [18] available at
 598 <http://www.maxhs.org> is used by our script to solve the MaxSAT instances derived from
 599 the PM^{ID} instances.

600 Our initial motivation to introduce parameterized matching under various distances
 601 is theater play comparison. To represent the structure of a theater play, we represent
 602 each character by a letter of the alphabet, and create the parameterized word obtained by
 603 considering the succession of all consecutive speakers. To check their adequacy with real data,
 604 we use a corpus of theater plays in which each character is represented by one letter of the
 605 alphabet, and each act of the play is represented by a string corresponding to the sequence of
 606 speaking characters. A letter may be duplicated in this string if the corresponding characters
 607 has lines in the end of a scene and in the beginning of the next one. Therefore, the edit
 608 distance between two parameter words representing acts will be small if both acts have a
 609 similar structure in terms of succession of speaking characters. We selected a corpus of 10
 610 pairs of plays where one inspired the other, and performed 47 comparisons between pairs
 611 of acts. Among those comparisons, 26 were solved by the maxSAT algorithm and all by
 612 the FPT algorithm (detailed results are presented in the supplementary material available
 613 at <https://github.com/AaronFive/paramatch/tree/main/corpus10pairs>), with a 800
 614 second timeout. The computation times are obtained on a XMG laptop running on Windows,
 615 with a 2.60 Ghz processor and 16 Gb RAM. Only the running time of MaxHS is provided,
 616 the encoding into a MaxSAT formula usually runs in approximately 1 second. Note that
 617 all instances are solved faster by the FPT algorithm than by the MaxSAT approach. The
 618 analysis of running times depending on the product of the lengths of the input strings (see
 619 supplementary material) shows that the MaxSAT approach may be relevant for strings with
 620 more than 10 distinct characters, but where the product of the length of input strings may
 621 not exceed 2000.

622 **7 Conclusion**

623 In this paper, we studied the complexity of several variants of the edit distance problem
 624 between parameterized words. We proved the NP-completeness of all previously unsolved cases,
 625 including the Levenshtein distance left open in [24], and provided practical approaches to
 626 solve real instances of those problems. We also studied similar problems for various definitions
 627 of words with parameters, namely parameter words and parameterized expressions, proving
 628 some relationships with parameterized word problems.

629 As future work, we will study the restrictions introduced in [21, 22] for a pattern matching
 630 problem with patterns in the parameter, in order to obtain polynomial time algorithms for
 631 the edit distance between parameterized words. Moreover, we will explore the question of
 632 distance between sets of words, in particular when they are defined through generalizations of
 633 automata. These problems are variants of the notion of distance between regular languages
 634 as defined in [12]. In this context, we can notice that different notions of automata can be
 635 considered: either automata generating parameterized words, or automata using parameters
 636 to define languages over classical words, with two different semantics as defined in [11].

637 **References**

- 638 1 Rakesh Agrawal, Christos Faloutsos, and Arun Swami. Efficient similarity search in sequence
 639 databases. In *International conference on foundations of data organization and algorithms*,
 640 pages 69–84. Springer, 1993.

- 641 2 Amihood Amir, Yonatan Aumann, Richard Cole, Moshe Lewenstein, and Ely Porat. Function
642 matching: Algorithms, applications, and a lower bound. In *Proceedings of the 30th International*
643 *Conference on Automata, Languages and Programming*, pages 929–942, 2003. doi:10.1007/
644 3-540-45061-0_72.
- 645 3 Amihood Amir, Martin Farach, and S. Muthukrishnan. Alphabet dependence in parameterized
646 matching. *Information Processing Letters*, 49(3):111–115, 1994. doi:10.1016/0020-0190(94)
647 90086-8.
- 648 4 Dana Angluin. Finding patterns common to a set of strings. *J. Comput. Syst. Sci.*, 21(1):46–62,
649 1980. doi:10.1016/0022-0000(80)90041-0.
- 650 5 Alberto Apostolico, Péter L. Erdős, and Moshe Lewenstein. Parameterized matching with
651 mismatches. *Journal of Discrete Algorithms*, 5(1):135–140, 2007. URL: <https://www.sciencedirect.com/science/article/pii/S1570866706000256>, doi:<https://doi.org/10.1016/j.jda.2006.03.014>.
- 652 6 Brenda S. Baker. A theory of parameterized pattern matching: Algorithms and applications.
653 In *Proceedings of the Twenty-Fifth Annual ACM Symposium on Theory of Computing*, STOC
654 '93, page 71–80, New York, NY, USA, 1993. Association for Computing Machinery. doi:
655 10.1145/167088.167115.
- 656 7 Brenda S. Baker. Parameterized duplication in strings: Algorithms and an application to
657 software maintenance. *SIAM Journal on Computing*, 26:1343–1362, 1997.
- 658 8 Brenda S. Baker. Parameterized diff. In *Proceedings of the Tenth Annual ACM-SIAM*
659 *Symposium on Discrete Algorithms*, SODA '99, page 854–855, USA, 1999. Society for Industrial
660 and Applied Mathematics.
- 661 9 Pablo Barceló, Leonid Libkin, and Juan L. Reutter. Querying graph patterns. In Maurizio
662 Lenzerini and Thomas Schwentick, editors, *Proceedings of the 30th ACM SIGMOD-SIGACT-*
663 *SIGART Symposium on Principles of Database Systems (PODS 2011)*, pages 199–210. ACM,
664 2011. doi:10.1145/1989284.1989307.
- 665 10 Pablo Barceló, Juan Reutter, and Leonid Libkin. Parameterized regular expressions and their
666 languages. *Theoretical Computer Science*, 474:21–45, 2013. doi:10.4230/LIPIcs.FSTTCS.
667 2011.351.
- 668 11 Pablo Barceló, Leonid Libkin, and Juan Reutter. Parameterized regular expressions and their
669 languages. *Theoretical Computer Science*, 474:21–45, 2011. doi:10.1016/j.tcs.2012.12.036.
- 670 12 Michael Benedikt, Gabriele Puppis, and Cristian Riveros. The cost of traveling between
671 languages. In Luca Aceto, Monika Henzinger, and Jiří Sgall, editors, *Automata, Languages*
672 *and Programming*, pages 234–245, Berlin, Heidelberg, 2011. Springer Berlin Heidelberg.
- 673 13 Michael Benedikt, Gabriele Puppis, and Cristian Riveros. Regular repair of specifications.
674 In *2011 IEEE 26th Annual Symposium on Logic in Computer Science*, pages 335–344, 2011.
675 doi:10.1109/LICS.2011.43.
- 676 14 Michael Benedikt, Gabriele Puppis, and Cristian Riveros. Bounded repairability of word
677 languages. *Journal of Computer and System Sciences*, 79(8):1302–1321, 2013. doi:10.1016/j.
678 jcss.2013.06.001.
- 679 15 Francine Blanchet-Sadri. *Algorithmic Combinatorics on Partial Words (Discrete Mathematics*
680 *and Its Applications)*. Chapman, Hall/CRC, 2007.
- 681 16 William W. Cohen. Integration of heterogeneous databases without common domains using
682 queries based on textual similarity. *SIGMOD Rec.*, 27(2):201–212, 1998. doi:10.1145/276305.
683 276323.
- 684 17 Richard Cole, Carmit Hazay, Moshe Lewenstein, and Dekel Tsur. Two-dimensional
685 parameterized matching. *ACM Trans. Algorithms*, 11(2), oct 2014. doi:10.1145/2650220.
- 686 18 Jessica Davies. *Solving MAXSAT by Decoupling Optimization and Satisfaction*. PhD thesis,
687 University of Toronto, Canada, 2014. URL: <http://hdl.handle.net/1807/43539>.
- 688 19 Arnab Ganguly, Rahul Shah, and Sharma V. Thankachan. PbwT: Achieving
689 succinct data structures for parameterized pattern matching and related problems. In
690 *Proceedings of the 2017 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*,
691 692

- 693 pages 397–407, 2017. URL: [https://epubs.siam.org/doi/abs/10.1137/1.9781611974782.](https://epubs.siam.org/doi/abs/10.1137/1.9781611974782.25)
 694 25, arXiv:<https://epubs.siam.org/doi/pdf/10.1137/1.9781611974782.25>, doi:10.1137/
 695 1.9781611974782.25.
- 696 20 M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of*
 697 *NP-Completeness (Series of Books in the Mathematical Sciences)*. W. H. Freeman, first edition
 698 edition, 1979.
- 699 21 Pawel Gawrychowski, Florin Manea, and Stefan Siemer. Matching patterns with variables
 700 under hamming distance. In Filippo Bonchi and Simon J. Puglisi, editors, *46th International*
 701 *Symposium on Mathematical Foundations of Computer Science, MFCS 2021, August 23-27,*
 702 *2021, Tallinn, Estonia*, volume 202 of *LIPICs*, pages 48:1–48:24. Schloss Dagstuhl - Leibniz-
 703 Zentrum für Informatik, 2021. doi:10.4230/LIPICs.MFCS.2021.48.
- 704 22 Pawel Gawrychowski, Florin Manea, and Stefan Siemer. Matching patterns with variables
 705 under edit distance. In Diego Arroyuelo and Barbara Pobleto, editors, *String Processing*
 706 *and Information Retrieval - 29th International Symposium, SPIRE 2022, Concepción, Chile,*
 707 *November 8-10, 2022, Proceedings*, volume 13617 of *Lecture Notes in Computer Science*, pages
 708 275–289. Springer, 2022. doi:10.1007/978-3-031-20643-6_20.
- 709 23 Dan Gusfield. *Algorithms on Strings, Trees, and Sequences - Computer Science and*
 710 *Computational Biology*. Cambridge University Press, 1997.
- 711 24 Carmit Hazay, Moshe Lewenstein, and Dina Sokol. Approximate parameterized matching.
 712 *ACM Trans. Algorithms*, 3(3):29–es, 2007. doi:10.1145/1273340.1273345.
- 713 25 Wilbert Jan Heeringa. *Measuring dialect pronunciation differences using Levenshtein distance*.
 714 PhD thesis, University of Groningen, 2004.
- 715 26 Orgad Keller, Tsvi Kopelowitz, and Moshe Lewenstein. On the longest common parameterized
 716 subsequence. *Theoretical Computer Science*, 410(51):5347–5353, 2009. doi:10.1016/j.tcs.
 717 2009.09.011.
- 718 27 Lillian Jane Lee. *Similarity-based approaches to natural language processing*. PhD thesis,
 719 Harvard University, 1997.
- 720 28 Vladimir I Levenshtein et al. Binary codes capable of correcting deletions, insertions, and
 721 reversals. In *Soviet physics doklady*, volume 10(8), pages 707–710. Soviet Union, 1966.
- 722 29 Moshe Lewenstein. *Parameterized Matching*, pages 635–638. Springer US, Boston, MA, 2008.
 723 doi:10.1007/978-0-387-30162-4_282.
- 724 30 Florin Manea and Markus L. Schmid. Matching patterns with variables. In Robert Mercas and
 725 Daniel Reidenbach, editors, *Combinatorics on Words - 12th International Conference, WORDS*
 726 *2019, Loughborough, UK, September 9-13, 2019, Proceedings*, volume 11682 of *Lecture Notes*
 727 *in Computer Science*, pages 1–27. Springer, 2019. doi:10.1007/978-3-030-28796-2_1.
- 728 31 Juan Mendivelso, Sharma V. Thankachan, and Yoan Pinzón. A brief history of parameterized
 729 matching problems. *Discrete Applied Mathematics*, 274:103–115, 2020. Stringology Algorithms.
 730 doi:10.1016/j.dam.2018.07.017.
- 731 32 Mehryar Mohri. Edit-distance of weighted automata: General definitions and algorithms.
 732 *International Journal of Foundations of Computer Science*, 14(06):957–982, 2003.
- 733 33 Saul B Needleman and Christian D Wunsch. A general method applicable to the search
 734 for similarities in the amino acid sequence of two proteins. *Journal of Molecular Biology*,
 735 48(3):443–453, 1970.
- 736 34 Sascha Schimke, Claus Vielhauer, and Jana Dittmann. Using adapted levenshtein distance
 737 for on-line signature authentication. In *Proceedings of the 17th International Conference on*
 738 *Pattern Recognition (ICPR 2004)*, volume 2, pages 931–934. IEEE, 2004.
- 739 35 T. Shibuya. Generalization of a suffix tree for RNA structural pattern matching. *Algorithmica*
 740 *(New York)*, 39(1):1–19, 2004. doi:10.1007/s00453-003-1067-9.
- 741 36 Bernd Voigt. The partition problem for finite abelian groups. *Journal of Combinatorial Theory,*
 742 *Series A*, 28(3):257–271, 1980.
- 743 37 Robert A. Wagner and Michael J. Fischer. The string-to-string correction problem. *J. ACM*,
 744 21(1):168–173, 1974. doi:10.1145/321796.321811.

- 745 38 Kaizhong Zhang and Dennis Shasha. Simple fast algorithms for the editing distance between
746 trees and related problems. *SIAM Journal on Computing*, 18(6):1245–1262, 1989.

747 **A** Details of the proofs

748 **Proof of Lemma 7.** We proceed by induction on k . If $k = 0$, then u and v are parameterized
 749 matching, and $f(u) = v$, thus $d(f(u), v) = 0$. Suppose the result holds until a fixed k . Suppose
 750 $PM^d(u, v) = k + 1$. There exist f , u'' and u' such that $d(u, u'') = 1$, $d(u'', u') = k$, and
 751 $f(u') = v$. Hence $PM^d(u'', v) \leq k$, and by induction hypothesis $d(f(u''), v) \leq k$. Moreover,
 752 since $d(u, u'') = 1$, we get u'' from u by applying only one operation. We prove that regardless
 753 of this operation, $d(f(u), f(u'')) = 1$, and thus $d(f(u), v) \leq d(f(u), f(u'')) + d(f(u''), v) \leq k + 1$
 754 which will conclude the proof. There are 3 cases to consider:

- 755 ■ If the operation is a deletion, $u = v_1 x v_2$ and $u'' = v_1 v_2$ for some words v_1 and v_2 and
 756 some letter x . Then $f(u) = f(v_1) f(x) f(v_2)$ and we can obtain $f(v_1) f(v_2) = f(u'')$ by
 757 deleting $f(x)$.
- 758 ■ If it is an insertion, $u = v_1 v_2$ and $u'' = v_1 x v_2$, and we can similarly go from $f(u)$ to $f(u'')$
 759 by inserting $f(x)$.
- 760 ■ If it is a substitution, $u = v_1 x v_2$ and $u'' = v_1 y v_2$, and we can go from $f(u)$ to $f(u'')$ by
 761 replacing $f(x)$ with $f(y)$.

762 Hence $d(f(u), f(u'')) = 1$, which concludes the proof for PM^d .

763 Since this proof does not use the fact that f is 1-to-1, it also stands for FM_1^d . ◀

764 **Proof of Lemma 8.** It is obvious that $d(zu, zv) \leq d(u, v)$, so we only prove $d(u, v) \leq$
 765 $d(zu, zv)$. We prove that any rewriting sequence from zu to zv can be modified such that no
 766 edit operation is applied in z . This will be enough to prove the result, as the edit sequence
 767 obtained can be seen as an edit sequence between u and v . We proceed by induction on
 768 the size of z . Suppose $|z| = 1$. Then $z = a \in \Sigma \cup \Pi$. We can consider that no character is
 769 modified twice in an edit sequence (i.e. no character is inserted and then deleted, or inserted
 770 and then substituted etc.), as that is always sub-optimal. Suppose z is modified. There are 3
 771 possible cases:

- 772 1. There is an insertion in z , hence a word w ends up being inserted before a . Since $zv = av$
 773 starts with a , w must start with an a , hence $w = aw'$. We insert $w'a$ to the right of z
 774 instead with the same operations. If z should be deleted or substituted, we apply the
 775 same operation to the new a instead. These operations yield the same result, and do not
 776 modify z .
- 777 2. There is a deletion in z , and hence a is deleted. Since this an optimal rewriting sequence,
 778 no a is created at that position through insertion or substitution afterwards. Since av
 779 starts with an a , u must be of the form $u = sau'$, where all the characters in s are deleted,
 780 and a isn't. Deleting sa instead of as yields the same result, and doesn't modify z .
- 781 3. There is a substitution in z , hence a is modified into a character $b \neq a$, that will not be
 782 further modified. Since av starts with a , an a has to be inserted in z , which is handled in
 783 case 1.

784 Hence, we can consider that every edit operations is done in u , and $d(au, av) = d(u, v)$.
 785 Suppose now that the result is proven for $|z| = k$, and let $z = az'$, with $|z'| = k$. Using the
 786 base case and the case for $|z| = k$, we have $d(zu, zv) = d(az'u, az'v) = d(zu, zv) = d(u, v)$,
 787 which concludes the proof. ◀

788 **Proof of Lemma 10 .** Let $\#$ be a fresh parameterized letter. Let then $u = \#^{k+1} u_1 \# u_2 \# \dots$
 789 $\# u_n$ and $v = \#^{k+1} v_1 \# v_2 \# \dots \# v_n$, where $\#^{k+1}$ denotes $k + 1$ repetitions of the character $\#$.
 790 The proof of the reverse direction is the same as in Lemma 9, so we only prove the other one.

791 Assume $FM_2^D(u, v) \leq k$. Let v' and f realize this parameterized match.

792 We prove that $f(\#) = \#$, and that no other character is sent to $\#$ by f . Indeed,
 793 v starts with $k + 1$ symbols $\#$, which ensure that v' starts with the letter $\#$. Since u
 794 starts with $\#$ and $f(u) = v'$, $f(\#) = \#$. Furthermore, this implies that since $|u|_{\#} = k + n$,
 795 $|f(u)|_{\#} = |v'|_{\#} \geq k + n$. Since v' is obtained from v by deletions, we have $|v'|_{\#} \leq |v|_{\#} = k + n$.
 796 Hence $|v'|_{\#} = k + n$ and all those inequalities are equalities, which is only the case when no
 797 $\#$ symbols is deleted from v , and that for all $x \neq \#$, $f(x) \neq \#$.

798 Since all the $\#$ symbols are left untouched, the rest of the proof is the same as in Lemma 9,
 799 and all of the factors u_i and v_i are parameterized matching. ◀

800 **Proof of Lemma 11.** Let $\#$ be a fresh parameterized letter, and $N = k + 2$.

801 Let then $u = \#^N u_1 \#^N u_2 \dots \#^N u_n \#^N$ and $v = \#^N v_1 \#^N v_2 \dots \#^N v_n \#^N$. Once again,
 802 we only prove the non-trivial implication.

803 Suppose $FM_1^D(u, v) \leq k$, and let f and u' realize this matching. Since u starts with $k + 1$
 804 copies of $\#$, u' starts with $\#$. Since v starts with $\#$ too, $f(\#) = \#$.

805 We now prove that we can consider that for all $x \neq \#$, $f(x) \neq \#$. This will also imply that
 806 no $\#$ symbol is deleted from u . Let $S = \{a \in \Pi \mid f(a) = \#\}$ be the set of symbols (different
 807 from $\#$) sent to $\#$. Since $|u|_{\#} = |v|_{\#}$, the number of deleted $\#$ symbols from u is exactly
 808 $|u|_S$, hence $|u|_S \leq k$. Let us now consider the leftmost occurrence of an element of S in u' ,
 809 that we denote by a . The letter a appears in u in a factor of the form $\#^N w_1 a w_2 \#^N$. Since
 810 all $\#$ in v appear in blocks of size N , a must contribute to such a block, after deletions and
 811 application of f . We distinguish two cases:

- 812 1. The entirety of the word w_1 is deleted. In this case, at least one symbol $\#$ from the
 813 left $\#^N$ block is deleted; otherwise $f(\#^N)f(a) = \#^{N+1}$ would be a factor of v , which is
 814 impossible. Thus, choosing not to delete $\#$ and to delete a instead yields the same result.
- 815 2. w_1 is not deleted. Since no character from S appears to the left of a , $f(a)$ is the start of
 816 a $\#^N$ block. Furthermore, since $|u|_S \leq k$, it is not possible to form $\#^N$ with only a and
 817 w_2 , and characters from the right $\#^N$ contribute to it. Hence, at least one $\#$ symbol
 818 from this right block is deleted. Like before, the same result can be obtained by not
 819 deleting it, and deleting a instead.

820 Either way, we can repeat this process to eliminate all occurrences of characters of S and of
 821 deletions of $\#$, which proves that we can consider that for all $x \neq \#$, $f(x) \neq \#$. Once again,
 822 we are taken back to the conditions of Lemma 9, and the rest of the proof follows. ◀

Proof of Theorem 15. We define Π like in Theorem 12, and we add the letters $\perp_1, \perp_2, \perp_3, \perp_4$
 and \perp_5 . Similarly, we define $u_1^i, v_1^i, u_{\perp}^i, v_{\perp}^i, u_2^e$, and v_2^e just like in Theorem 12. Additionally,
 we define for every edge e ,

$$u_{\perp}^e = \square_1^e \square_2^e \square_3^e \square_4^e \square_5^e \square_6^e \text{ and } v_{\perp}^e = \perp_1 \perp_2 \perp_3 \perp_4 \perp_5.$$

We then apply Lemma 11 with

$$u_1^1, \dots, u_1^n, u_{\perp}^1, \dots, u_{\perp}^n, u_2^{e_1}, \dots, u_2^{e_m}, u_{\perp}^{e_1}, \dots, u_{\perp}^{e_m}$$

and

$$v_1^1, \dots, v_1^n, v_{\perp}^1, \dots, v_{\perp}^n, v_2^{e_1}, \dots, v_2^{e_m}, v_{\perp}^{e_1}, \dots, v_{\perp}^{e_m}$$

823 to obtain u , v , and k . We show that G is 3-colorable $\Leftrightarrow FM_1^D(u, v) \leq k$.

824 \Rightarrow Suppose G is 3 colorable. Define f like in Theorem 12 on the c_y^i and \square_y^e . Let e be an
 825 edge and $k_e \in [1, 6]$ be the integer such that $f(\square_{k_e}^e)$ is defined. We map every remaining \square_y^e

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826 in the following way:

$$827 \quad f(\square_i^e) = \begin{cases} \perp_i & \text{if } i < k_e, \\ Y^e & \text{if } i = k_e, \\ \perp_{i-1} & \text{if } i > k_e. \end{cases} \quad (1)$$

828 It is then easy to check that $d(f(u), v) = k$, and thus $FM_1^D(u, v) \leq k$.

829 \Leftarrow Suppose $FM_1^D(u, v) \leq k$, and let f and u' realize it. We define a coloring of G based
830 on f . We note, for $1 \leq i \leq n$ and $1 \leq t \leq 3$, $col(c_t^i) = c_t$. If x_i is a vertex of G , define $c(x_i)$
831 to be $col(c_k^i)$, where c_k^i is the only element such that $f(c_k^i) = x_i$. We show in what follows
832 that (1) this function definition is correct and (2) it is a valid coloring, i.e. if $e = \{x_i, x_j\}$ is
833 an edge, $c(x_i) \neq c(x_j)$.

834 (1): The same points 1. and 2. from the proof of Theorem 12 apply, hence for every
835 $1 \leq i \leq n$, exactly one element from $\{c_1^i, c_2^i, c_3^i\}$ is sent to x_i , while the two others are sent to
836 \perp_1^i and \perp_2^i , hence the result.

837 (2): Let e be an edge. The words u_\perp^e and v_\perp^e are in matching, which is done with exactly
838 one deletion. Hence, there exists k_e such that

$$839 \quad f(\square_i^e) = \begin{cases} \perp_i & \text{if } i < k_e, \\ \perp_{i-1} & \text{if } i > k_e. \end{cases} \quad (2)$$

840 Moreover, u_2^e and v_2^e are in matching. Since Y^e appears in v_2^e and all the characters in u_2^e
841 apart from $\square_{k_e}^e$ have an image different from Y^e , $f(\square_{k_e}^e) = Y^e$. Hence, the only characters
842 that are not suppressed from u_2^e are the two characters between the $\square_{k_e}^e$. Denoting them by
843 c and c' , the construction of the word ensures that $col(c) \neq col(c')$. Hence, if $e = \{x_i, x_j\}$,
844 we have proven $c(x_i) \neq c(x_j)$, which is (2).

845 The coloring c is therefore valid, which concludes the proof. \blacktriangleleft

846 **Proof of Theorem 17.** Let $G = (V, E)$, with $V = \{x_1, \dots, x_n\}$ and $\{e_1, \dots, e_m\}$. Like in
847 the 1-to-1 case, we construct factors u_i and v_i to encode vertex coloring. The parameter
848 alphabet contains:

- 849 ■ x_1, \dots, x_n , corresponding to V ,
- 850 ■ the colors c_1, c_2, c_3 ,
- 851 ■ for every $e \in E$, the delimiters Y^e ,
- 852 ■ for every $e \in E$ and every $1 \leq i, j \leq 3$, $i \neq j$, the delimiters $Y_{i,j}^e$.

853 We define for $1 \leq i \leq n$, $u_1^i = x_i$ and $v_1^i = c_1 c_2 c_3$. If e is an edge and c_i and c_j are two colors,
854 we denote $w^e(c_i, c_j) = Y_{i,j}^e Y_{i,j}^e Y_{i,j}^e c_i c_j Y_{i,j}^e Y_{i,j}^e Y_{i,j}^e$. For every edge $e = \{x_i, x_j\}$, we now define
855 $u_2^e = Y^e Y^e Y^e x_i x_j Y^e Y^e Y^e$ and $v_2^e = w^e(c_1, c_2) w^e(c_1, c_3) w^e(c_2, c_1) w^e(c_2, c_3) w^e(c_3, c_1) w^e(c_3,$
856 $c_2)$.

857 We now apply Lemma 10 with $u_1^1, \dots, u_1^n, u_2^{e_1}, \dots, u_2^{e_m}, v_1^1, \dots, v_1^n, v_2^{e_1}, \dots, v_2^{e_m}$, to obtain u and v .

858 \Rightarrow Suppose G is 3-colorable, and let $c : V \rightarrow \{c_1, c_2, c_3\}$ be a valid coloring. Define
859 $f|_V = c$. For every edge $e = \{x_i, x_j\}$, let s and t be such that $c(x_i) = c_s$ and $c(x_j) = c_t$.
860 We then define $f(Y^e) = Y_{s,t}^e$. It is easy to check now that $d(f(u), v) = k$, and hence
861 $FM_2^D(u, v) \leq k$.

862 \Leftarrow Suppose now that $FM_2^D(u, v) \leq k$. We will show that $f|_V$ defines a 3-coloring of G ,
863 by showing that (1) for all $x \in V$, $f(x) \in \{c_1, c_2, c_3\}$ and (2) If $\{x, y\} \in E$, then $f(x) \neq f(y)$.

- 864 ■ Lemma 10 ensures that the words u_i and v_i are in matching, which proves (1).
- 865 ■ Lemma 10 also ensures that the words u^e and v^e are in matching. Let $e \in E$, with
866 $e = x_s, x_t$. We have $|u_2^e|_{Y^e} = 6$, hence $|f(u_2^e)|_{f(Y^e)} \geq 6$. Since c_1, c_2 and c_3 each occur

867 exactly 4 times in v_2^e , they cannot occur 6 times after deletions, and $f(Y_e) \notin \{c_1, c_2, c_3\}$.
 868 Hence, there exist $i \neq j$ with $1 \leq i, j \leq 3$ such that $f(Y^e) = Y_{i,j}^e$. This implies that all
 869 but one of the w^e factors from v_2^e are suppressed, and that the remaining one is $w^e(c_i, c_j)$.
 870 Hence $f(x_s) = c_i$ and $f(x_t) = c_j$, which proves (2).
 871 ◀

872 **B** Encoding Constant Alphabet Σ in Π

873 We show why it is always possible to consider that $\Sigma = \emptyset$ for certain problems. These results
 874 use the lemmas proved in Section 4.1.

875 **► Lemma 23.** *Let d be a distance, k an integer and u and v be two parameterized words*
 876 *over the alphabet of constants Σ and the alphabet of parameters Π . There exist words \tilde{u} and*
 877 *\tilde{v} over the alphabet of constants \emptyset and the alphabet of parameters $\Pi' = \Pi \uplus \Sigma$ such that the*
 878 *following are equivalent:*

- 879 \blacksquare $PM^d(u, v, k)$ is realized by f ;
- 880 \blacksquare $PM^d(\tilde{u}, \tilde{v}, k)$ is realized by f .

881 *In particular, this implies that if $PM^d(\tilde{u}, \tilde{v}) \leq k$, all functions f realizing this matching*
 882 *verify that for all $x \in \Sigma$, $f(x) = x$, and for all $x \in \Pi$, $f(x) \in \Pi$.*

883 **Proof.** Let $N = k + 1$. If $\Sigma = \{a_1, \dots, a_n\}$, we define z to be $a_1^N a_2^N \dots a_n^N u$ and $\tilde{u} = zu$,
 884 $\tilde{v} = zv$. It is clear that if $PM^d(u, v) \leq k$ then $PM^d(\tilde{u}, \tilde{v}) \leq k$, by following the same
 885 operations, and applying the same renaming function.

886 Suppose now that $PM^d(\tilde{u}, \tilde{v}) \leq k$, and let f and u' realize it. Let $i \in [1, n]$. All the letters
 887 of u between position Ni and $N(i + 1)$ are a_i . At most k of these positions can be modified
 888 with an edit operation. Since $N > k$, at least one of these positions is not modified, and thus
 889 there exists $j \in [Ni, N(i + 1)]$ such that $u'_j = a_i$. Since all letters in v between position Ni
 890 and $N(i + 1)$ are a_i , in particular $v_j = a_i$, and hence $f(a_i) = a_i$. This proves that for all
 891 $x \in \Sigma$, $f(x) = x$, and thus $f(z) = z$. Since f is 1-to-1, this entails $f(\Pi) \subseteq \Pi$. By Lemma 7,
 892 $d(f(\tilde{u}), \tilde{v}) \leq k$. Hence $d(f(zu), zv) = d(zf(u), zv) \leq k$ and by Lemma 8, $d(f(u), v) \leq k$.
 893 Hence $PM^d(u, v) \leq k$. ◀

894 **► Remark 24.** Note that the words \tilde{u} and \tilde{v} have a size increased by $N\Sigma$. If less operations
 895 are considered, it is possible to reduce this overhead. For example, in the case of PM^D , we
 896 can take z to be of the form $a_1 \dots a_n z^N$, to reduce the overhead to $N + \Sigma$.

897 Similarly, constants can be encoded in Π in some FM problems. We prove this result for
 898 FM_2^D , with the help of the block decomposition allowed by Lemma 10.

899 **► Lemma 25.** *Let u and v be two parameterized words over the alphabet of constants Σ and*
 900 *the alphabet of parameters Π . There exist words \tilde{u} and \tilde{v} over the alphabet of constants \emptyset*
 901 *and the alphabet of parameters $\Pi' = \Pi \uplus \Sigma$ such that the following are equivalent:*

- 902 \blacksquare $FM_2^D(u, v, |v| - |u|)$ is realized by f ;
- 903 \blacksquare $FM_2^D(\tilde{u}, \tilde{v}, |\tilde{v}| - |\tilde{u}|)$ is realized by f .

904 **Proof.** We write $\Sigma = \{a_1, \dots, a_n\}$ and $\Pi = \{b_1, \dots, b_m\}$. We define $z_\Sigma = a_1 \dots a_n$, and $z_\Pi =$
 905 $b_1 \dots b_m$. Let \tilde{u} and \tilde{v} be the words obtained by applying Lemma 10 to $z_\Sigma, b_1, b_2, \dots, b_m, u$ and
 906 $z_\Sigma, z_\Pi, z_\Pi, \dots, z_\Pi, v$. If $FM_2^D(u, v, k)$ is realized by a function f , it realizes $FM_2^D(\tilde{u}, \tilde{v}, |\tilde{v}| - |\tilde{u}|)$
 907 too. Indeed, it is enough to apply the same operations in v , and to delete all the characters
 908 but $f(b_i)$ in the i -th copy of z_Π .

909 Suppose now that $FM_2^D(\tilde{u}, \tilde{v}) \leq k$, and let f realize it. Then, by Lemma 10, we have:

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- 910 ■ $D(z, f(z)) = 0$, and hence $f(z) = z$, which implies that for all $x \in \Sigma$, $f(x) = x$.
 911 ■ For every $1 \leq i \leq m$, $D(z_\Pi, f(b_i)) = |\Pi| - 1$. Hence $f(b_i)$ is a character of z_Π , which is
 912 some character $b_j \in \Pi$.
 913 ■ $D(v, f(u)) \leq k$.
 914 Hence f verifies $D(f(v), u) \leq k$ and respects the conditions on Π and Σ , which implies that
 915 it also realizes $FM_2^D(u, v, k)$. ◀

916 The overhead to pay for this transformation is $O(|\Sigma| + |\Pi|^2 + k)$, where the term in k
 917 comes from the proof of Lemma 10.

918 Transposing the technique used for Lemma 25 is not sufficient to get a similar result for
 919 FM_1^D . The question thus remains open in this context.

920 C Proofs Regarding the Max-SAT Encoding

921 **Proof of theorem 21.** We proceed by induction on $|u| + |v|$. If $|u| + |v| = 0$, both u and v
 922 are the empty string, and the equivalence is trivial. Fix $n \in \mathcal{N}$ and suppose now that the
 923 result holds up for all words u, v such that $|u| + |v| \leq n - 1$. Let u and v be two words such
 924 that $|u| + |v| \leq n$. Without loss of generality, consider $|u| \geq |v|$.

925 Suppose $ID(u, v) \leq k$. Let ρ be a rewriting sequence between u and v of length k . If there
 926 is no deletion in u in ρ , there are only insertions in v , and v is a sub-word of u , and there
 927 exists another rewriting sequence ρ' only deleting letters from u . Hence, we can consider
 928 that there is at least a deletion in u in ρ . Let p be a position at which such a deletion occur,
 929 and let $a = u_p$. The word u can be written as $u = u'au''$ for some words u' and u'' . Define
 930 $w = u'u''$. It holds that $d(w, v) \leq k - 1$ and $|w| = |u| - 1$. By induction, there exists an
 931 alignment A between w and v such that $2|A| = |w| + |v| - (k - 1) = |u| + |v| - k$. We define

932 $r(i) = \begin{cases} i & \text{if } i < p \\ i - 1 & \text{if } i > p \end{cases}$, and $B = \{(r(i), j) \mid (i, j) \in A\}$. Since A is an alignment, so is B : it

933 satisfies conditions 1 to 3 of Definition 20, and since $w_r(i) = u_i$, it also satisfies condition 4.
 934 Finally, $|B| = |A|$, hence $2|B| = |u| + |v| - k$, hence the result.

935 Suppose now that there exists an alignment A such that $2|A| = |u| + |v| - k$. Similarly,
 936 consider p , a position in u such that there does not exist a j with $(p, j) \in A$. If no such position
 937 exist, since $|u| \geq |v|$, $u = v$ and the result is proven. Consider w the word obtained by deleting
 938 u_p from u . It then holds that $|w| = |u| - 1$ and that $2|A| = |u| + |v| - k = |w| + |v| - (k - 1)$.
 939 Defining B in the same way as above yields an alignment between w and v of the same size,
 940 and thus by induction, $d(w, v) \leq k - 1$, and since $d(u, w) = 1$, $d(u, v) \leq k$. ◀

941 ► **Proposition 26.** *Weighted Max-SAT and partial weighted Max-SAT are equivalent.*

942 **Proof.** Encoding a weighted Max-SAT instance as a partially weighted Max-SAT instance is
 943 straightforward, as we just have to choose φ_c to be empty.

944 Conversely, given a satisfiable CNF formula φ_c , a CNF formula φ_w , and a weight function w
 945 on the clauses of φ_w , we can define a weighted Max-Sat instance in the following way :

- 946 ■ We define $\varphi = \varphi_c \wedge \varphi_w$
 947 ■ We set $W = 1 + \sum_{C_i \text{ clause of } \varphi_c} w(C_i)$, and extend w to clauses of φ_c such that $w(C_j) = W$
 948 for all clauses C_j of φ_c

949 If ν is a valuation, we denote by $w(\nu)$ the sum of the weights of all clauses it satisfies $\sum_{\nu \models C_i} w(C_i)$.

950 Since φ_c is satisfiable, there exists a valuation ν_c such that $\nu_c \models \varphi_c$, and $w(\nu_c) \geq |\varphi_c|W$. Let
 951 now ν be a valuation not satisfying a clause of φ_c . Then $w(\nu) \leq (|\varphi_c| - 1)W + (W - 1) < w(\nu_c)$,
 952 hence ν is not maximal and cannot be a solution to the weighted Max-SAT instance. ◀