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Deciding under ignorance: in search of meaningful extensions of the Hurwicz criterion to decision trees

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Abstract. The major paradigm for sequential decision under uncertainty is expected utility. This approach has many good features that qualify it for posing and solving decision problems, especially dynamic consistency and computational efficiency via dynamic programming. However, when uncertainty is due to sheer lack of information, and expected utility is no longer a realistic criterion, the approach collapses because dynamic consistency becomes counterintuitive and the global non-expected utility criteria are no longer amenable to dynamic programming. In this paper we argue against Resolute Choice strategies, following the path opened by Jaffray, and suggest that the dynamic programming methodology may lead to more intuitive solutions respecting the Consequentialism axiom, while a global evaluation of strategies relying on lottery reduction is questionable.

1 Introduction

The traditional approach to multiple stage decision processes under the probabilistic approach [9] is based on decision trees. A decision tree is a graphical structure containing chance nodes and decision nodes. A strategy is the assignment of a decision (i.e. a chance node) to each decision node and each strategy turns the decision tree into a probability tree. This probability tree characterizes a unique probability distribution on the space of final states, and the (global) expected utility of the strategy over the state space can be computed. The optimal strategy is then the one with maximal expected utility.

This model has several features that make computations tractable [9]: Any substrategy of an optimal strategy is optimal with respect to the corresponding decision subtree, the optimal strategy can be computed by means of dynamic programming from the leaves to the root of the decision tree.

The appeal of this approach is also due to three properties that it verifies:

- *Dynamic Consistency*: When reaching a decision node by following an optimal strategy, the best decision at this node is the one that had been considered so when computing this strategy, i.e. prior to applying it.
- *Consequentialism*: the best decision at each step of the decision tree only depends on potential consequences at this point.

- *Tree Reduction*: The result of the dynamic programming procedure on the decision tree comes down to optimizing the criteria defined on the state space via the probability distribution obtained from each strategy via the reduction of lotteries.

More recently, with the emergence of non-additive uncertainty theories, the decision tree approach has been adapted to new decision criteria that differ from expected utility [10], but generalize known but less reputed criteria such as Wald maximin criterion or Hurwicz criterion: for instance lower expected utility with respect to a set of priors [2] or Jaffray’s belief function extension of Hurwicz criterion [6]. These criteria turn out to be incompatible with the three above assumptions in the sequential decision setting [4]. In particular, they violate Dynamic Consistency, and optimizing the non-expected utility criterion cannot be carried out using dynamic programming [3]. Some authors tend to privilege Dynamic Consistency and Tree Reduction and are ready to give up Consequentialism (e.g., the Resolute Choice approach [1]). Another approach called Veto-process has been proposed by Jaffray [7]. It insists on the fact that Resolute Choice is not acceptable since a normally behaved decision-maker is consequentialist.

The aim of this paper is to provide more arguments in favor of Consequentialism as a natural property to be preserved when uncertainty accounts for incomplete information rather than frequentist probability, while questioning Resolute Choice. We follow the line initiated by Jaffray [7] who introduced the so-called Veto-process in the frame of decision under total uncertainty. First, we present the background on decision trees under pure uncertainty and the Hurwicz criterion. Then, we illustrate Resolute Choice, showing its paradoxical behavior on an example. Then we present and discuss two alternatives to Resolute Choice, inspired by the Veto-process philosophy.

2 Background

In this section, we first recall the definition of the Hurwicz criterion and decision trees under uncertainty.

Consider first simple, non sequential decision problems under complete uncertainty: each decision δ is characterized by the multi set of consequences E_δ it can lead to - or equivalently a simple, non probabilistic lottery. Given a utility function ($u(s)$) capturing the attractiveness of each of these consequences, a usual way to taking into account the optimism of the decision-maker under total uncertainty is to use the Hurwicz criterion [5]. The worth of a simple lottery δ is then:

$$H(\delta) = \alpha \times \min_{s \in E_\delta} u(s) + (1 - \alpha) \times \max_{s \in E_\delta} u(s).$$

where $\alpha \in [0, 1]$ is the degree of optimism.

When the decision problem is sequential and fully observable, we shall use decision trees [9] a graphical representations of the problem. This framework proposes an explicit modeling, representing each possible scenario by a path

from the root to the leaves of the tree. Formally, the graphical component of a decision tree \mathcal{T} is composed of a set of nodes \mathcal{N} and a set of edges \mathcal{E} such that the set \mathcal{N} contains three kinds of nodes:

- $\mathcal{D} = \{d_0, \dots, d_m\}$ is the set of decision nodes (represented by rectangles).
- $\mathcal{LN} = \{ln_1, \dots, ln_k\}$ is the set of leaves, that represent final states in $\mathcal{S} = \{s_1, \dots, s_k\}$; such states can be evaluated thanks to a utility function: $\forall s_i \in \mathcal{S}$, $u(s_i)$ is the utility of being eventually in state s_i (in node ln_i). For the sake of simplicity we assume that only leave nodes lead to utilities.
- $\mathcal{X} = \{x_1, \dots, x_n\}$ is the set of chance nodes represented by circles.
For any node $n_i \in \mathcal{N}$, $Succ(n_i) \subseteq \mathcal{N}$ denotes the set of its children. Moreover, for any $d_i \in \mathcal{D}$, $Succ(d_i) \subseteq \mathcal{X}$: $Succ(d_i)$ corresponds to the set of actions that can be decided when d_i is observed. For any $x_i \in \mathcal{X}$, $Succ(x_i) \subseteq \mathcal{LN} \cup \mathcal{D}$: $Succ(x_i)$ is indeed the set of outcomes of the action x_i - either a leaf node is observed, or a decision node is reached (and then a new action should be executed).

In the present paper, we are interested in the simple case of total ignorance, where information at chance nodes is just a list of potential outcomes without probability distribution.

Solving a decision tree amounts to building a *strategy* δ that selects an action (i.e. a chance node) $\delta(d_i) \in Succ(d_i)$ for each reachable decision node $d_i \in \mathcal{D}$ that associates a chance node $\delta(d_i) \in Succ(d_i)$ to each decision node d_i : $\delta(d_i)$ is the action to be executed when a decision node d_i is reached.

Let Δ be the set of strategies that can be built from the decision tree. Any strategy in Δ can be viewed as a connected subtree of the decision tree where there is exactly one decision arc left at each decision node - skipping the decision nodes, we get a chance tree or, using von Neumann and Morgenstern's terminology, a compound lottery.

3 Resolute Choice

To evaluate/compare strategies with the Hurwicz criterion, we shall first follow a resolute choice approach. The idea is that any compound lottery is equivalent to a simple one, using a principle of lottery reduction. In our context of decision under total ignorance, no probability distribution is available and reducing a lottery \mathcal{T}_δ of a strategy δ comes down to computing the multiset of possibly reached states E_δ . We shall now simply compare the strategies by computing $H(\delta) = \alpha \times \min_{s \in E_\delta} u(s) + (1 - \alpha) \times \max_{s \in E_\delta} u(s)$ for each of them. This decision-maker behavior is called Resolute Choice and consists in making a strategic decision now and keeping the same strategy over time.

In the literature, the Hurwicz criterion has been generalized to decision trees pervaded with imprecise probabilities by applying it to the reduced lottery [8]. It follows that this criterion will violate Consequentialism in the probabilistic case, and this is also the case in our simple non probabilistic framework. For instance, consider the following decision tree under incomplete information :

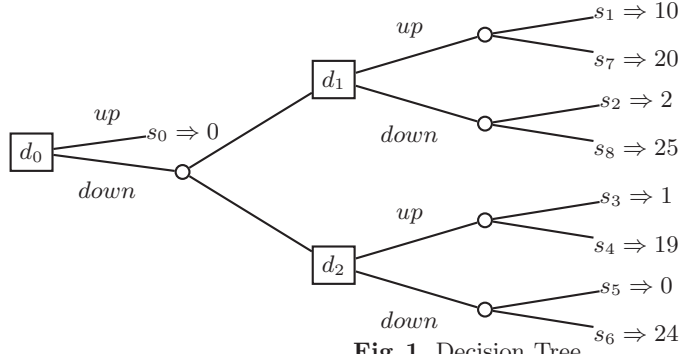


Fig. 1. Decision Tree.

There are 5 strategies : $(d_0 = up)$, $(d_0 = down, d_1 = up, d_2 = down)$, $(d_0 = down, d_1 = down, d_2 = up)$, $(d_0 = down, d_1 = up, d_2 = up)$, $(d_0 = down, d_1 = down, d_2 = down)$

The problem is to find the best strategy for a Hurwicz decision-maker with degree of optimism α , if there is no other information at x_1, x_2, x_3, x_4, x_5 .

The Resolute Choice approach consists in noticing that each strategy yields a different set of possible rewards and computing the best strategy using the prescribed decision criterion.

- Reachable states : $E_{(d_0=down, d_1=up, d_2=up)} = \{s_1, s_7, s_3, s_4\} : H(d_0 = down, d_1 = up, d_2 = up) = \alpha + 20(1 - \alpha)$
- Reachable states : $E_{(d_0=down, d_1=up, d_2=down)} = \{s_1, s_7, s_5, s_6\} : H(d_0 = down, d_1 = up, d_2 = down) = 24(1 - \alpha)$
- Reachable states : $E_{(d_0=down, d_1=down, d_2=up)} = \{s_2, s_8, s_3, s_4\} : H(d_0 = down, d_1 = down, d_2 = up) = \alpha + 25(1 - \alpha)$
- Reachable states : $E_{(d_0=down, d_1=down, d_2=down)} = \{s_2, s_8, s_5, s_6\} : H(d_0 = down, d_1 = down, down) = 25(1 - \alpha)$
- Reachable states : $E_{d_0=up} = \{s_0\} : H(d_0 = up) = 0$

So the optimal strategy ex ante consists in deciding for "down" at node d_0 , for "down" at node d_1 and "up" at node d_2 what ever $\alpha \in [0, 1]$.

However assume Consequentialism. Suppose the decision-maker reaches decision node d_1 , because she/he choses "down" at d_0 : Now the sets of remaining possible rewards and the corresponding evaluations are for each remaining strategy (the boldface decision is a past one):

- $E_{(\mathbf{d_0=down}, d_1=up)} = \{s_1, s_7\} : H(\mathbf{d_0 = down}, up) = 10\alpha + 20(1 - \alpha)$
- $E_{(\mathbf{d_0=down}, d_1=down)} = \{s_2, s_8\} : H(\mathbf{d_0 = down}, down) = 2\alpha + 25(1 - \alpha)$

If the decision-maker is pessimistic enough ($\alpha > \frac{5}{13}$) the best decision d_1 is "up" (and not "down" as found using Resolute Choice). Suppose now that the decision-maker reaches decision node d_2 due to the outcome of chance node x_1 :

- $E_{(\mathbf{d_0=up}, d_2=up)} = \{s_3, s_4\} : H((\mathbf{d_0 = up}, d_2 = up)) = \alpha + 19(1 - \alpha)$

$$- E_{(d_0=up, d_2=down)} = \{s_5, s_6\}: H(d_0 = up, d_2 = down) = 24(1 - \alpha)$$

If the decision-maker is enough optimistic $\alpha < \frac{5}{6}$ the best decision d_2 is "down" and not "up" as found using Resolute Choice. In this example is easy to see that for a decision-maker which is not strictly optimistic nor pessimistic (more precisely if $\alpha \in]\frac{5}{13}, \frac{5}{6}[$), both decisions proposed by the Resolute Choice approach are in conflict with the decision-maker's preference at the moments he has to decide.

4 An alternative to Resolute Choice under pure uncertainty

Jaffray [7] starts from the psychological implausibility of the Resolute Choice: an optimal strategy chosen at time t can become unacceptable in the future. This is because in some sense, the decision-maker now is not the same as the decision-maker in the future. Jaffray speaks of different *egos*. He assigns a different ego to each decision node and tries to build a strategy now that is not dominated for the future egos. The question of Jaffray is: how can egos collaborate? Contrary to Resolute Choice where the present ego enforces his preferences to the future ones, we present two alternatives to the Resolute Choice. The first is the direct application of the Veto-process proposed by Jaffray [7]. The second approach is based on the idea that the satisfaction of the present depends on the one of his future egos.

4.1 A Veto-process under pure uncertainty

Let $\overline{\Delta_N}$ be a set of possible strategies from node N . The application of the algorithm proposed by Jaffray to our case of pure uncertainty comes down to letting each ego N select those of its possible strategies $\delta \in$ that are optimal according to the Hurwicz criterion applied on the reduction of δ , i.e. on the the leaves of $\delta\overline{\Delta_N}$. So the algorithm consists in selecting the best substrategy (strict Veto-process), from the last decision nodes to the root decision node by applying lottery reduction and Hurwicz criterion.

Back to our example: suppose $\alpha \in]\frac{5}{13}, \frac{5}{6}[$; the ego for decision d_1 will choose "up" and the ego of decision d_2 will choose "down". So the ego of decision d_0 will have to decide between strategies:

$$\begin{aligned} - \delta_1 &= (d_0 = down, d_1 = up, d_2 = down) : E_{\delta_1} = \{s_1, s_7, s_5, s_6\} \text{ and } H(\delta_1) = 24(1 - \alpha) \\ - \delta_2 &= (d_0 = up) : E_{d_0=up} = \{s_0\} \text{ and } H(d_0 = up) = 0 \end{aligned}$$

Then the best strategy is δ_1

Algorithm 1: Veto-process under pure uncertainty

Input: decision tree \mathcal{T} of depth $p > 1$, optimism coefficient α

Output: A strategy δ

foreach node N from the depth $p - 1$ to 0 in \mathcal{T} **do**

if $N \in \mathcal{X}$ **then**

$\overline{\Delta}_N \leftarrow \cup_{N' \in Succ(N)} \{(N, N')\} \cup \delta : \delta \in \overline{\Delta}_{N'}$

if $N \in \mathcal{D}$ **then**

$\overline{\Delta}_N \leftarrow \cup_{N' \in Succ(N)} \{\overline{\Delta}_{N'}\}$

foreach $\delta \in \overline{\Delta}_N$ **do**

$H_\delta \leftarrow \alpha \times \min_{l_{n_i} \in \delta \cap \mathcal{LN}} u(s_i) + (1 - \alpha) \times \max_{l_{n_i} \in \delta \cap \mathcal{LN}} u(s_i)$

$V_{\max} \leftarrow \max_{\delta \in \overline{\Delta}_N} H_\delta$

$\overline{\Delta}_N \leftarrow \{argmax_{\delta \in \overline{\Delta}_N} H_\delta\};$

In this approach, the Veto of future egos enforces Consequentialism. Since each ego is responsible from the choice of its strategy, the algorithm ensures that a future ego will not deviate from the chosen strategy.

In the Veto-process, the egos are like independent players, so that each player tries to optimize its own criterion ; one can even imagine that each ego N works with its own degree of optimism α_N over the rewards it finally gets, letting $H_\delta \leftarrow \alpha_N \times \min_{l_{n_i} \in \delta \cap \mathcal{LN}} u(s_i) + (1 - \alpha_N) \times \max_{l_{n_i} \in \delta \cap \mathcal{LN}} u(s_i)$ in the algorithm.

But in sequential decision under uncertainty, all players are dependent since they participate of the same decision-maker. Moreover, only one player gets the final reward. So, what is the relevance of assuming independent egos? In the next section we propose a alternative process where the egos are dependent.

4.2 Ego-dependent process under pure uncertainty

Another possibility could be to consider that the preference degrees of one ego is a function the satisfaction degree of its future egos - and eventually of the egos that receive the final rewards, considering that like decision nodes, leaves are more or less satisfied egos. Moreover the current decision

must put the future egos in the best position to be satisfied, until the last ego (i.e., the ego of a leaf) obtains the final reward.

So in this section, we propose a new criterion that combines Hurwicz's idea and the philosophy of dependent satisfaction of the egos. For the ego of the decision node d_i , the worth of decision N (a given posterior chance node) at this node is recursively obtained as:

$$H(N) = \alpha \times \min_{u \in \mathcal{H}(N)} u + (1 - \alpha) \times \max_{u \in \mathcal{H}(N)} u. \quad (1)$$

where $\mathcal{H}(N) = \{H(N') : \forall d' \in Succ(N)\}$.

In other words, this approach comes down to recursively replacing any simple lottery by a certainty equivalent $H(L)$ that would provide utility $H(L)$ for sure. In a sense, we can say that we use a kind of lottery reduction that depends on

the parameters of the criterion used (Hurwicz), and more particularly on α . To reduce a compound lottery, we replace all final simple lotteries by their Hurwicz values, and carry on recursively.

Algorithm 2: Ego-dependent process under pure uncertainty

Input: decision tree \mathcal{T} of depth $p > 1$, optimism coefficient α

Output: A strategy δ^*

foreach node N from the depth p to 0 in \mathcal{T} **do**

if $N \in \mathcal{LN}$ **then**

$V_N \leftarrow \{u(N)\}$

if $N \in \mathcal{X}$ **then**

$V_N \leftarrow \cup_{N' \in Succ(N)} V_{\Delta_{N'}}$

if $N \in \mathcal{D}$ **then**

foreach $N' \in Succ(N)$ **do**

$H(N') \leftarrow \alpha \times \min_{v \in V_{N'}} v + (1 - \alpha) \times \max_{v \in V_{N'}} v$

$V_N \leftarrow \{\max_{N' \in Succ(N)} H(V_{N'})\}$

$\delta^* \leftarrow \delta^* \cup (N, \text{argmax}_{N' \in Succ(N)} H(V_{N'}))$

Under this approach, it follows that the optimal strategy δ^* under this criterion is the strategy which maximizes $H_N^*, \forall N \in Succ(d_0)$. It yields an optimal worth H^* , which is the maximal value of H_N over the substrategies starting by chance nodes in $Succ(d_0)$. For any chance node N in $Succ(d_0)$, H_N is the utility value such that all the future decisions are optimal for all the future egos; so this model has several nice properties :

- Any substrategy of an optimal strategy is optimal with respect to the corresponding decision subtree.
- The optimal strategy can be computed by means of dynamic programming from the leaves to the root of the decision tree.

This approach is appealing because it verifies two properties (*Dynamic Consistency* and *Consequentialism*) and fails the other one (*Tree Reduction*). Indeed, we do not consider that a decision tree is necessarily equivalent to a set of reduced lotteries associated to strategies, in the face of total uncertainty: the structure of the decision tree must influence the choice of strategies. Our Algorithm 2 is thus based on dynamic programming, where V_N , and $H(V_N)$ are respectively sets of evaluations and the worth of node N .

Back to our example with $\alpha = \frac{6}{13}$. The egos of leaves get the utility value associated to their states, so no choice is needed. The ego of decision node d_1 will choose "up" ($H_{d_1=up} = \frac{6}{13} \times 10 + (1 - \frac{6}{13}) \times 20 \sim 15.38$ and $H_{d_1=down} \sim 14.38$), the ego of decision node d_2 will choose "down" ($H_{d_2=up} \sim 10.69$ and $H_{d_2=down} \sim 12.92$). The ego of decision node d_0 only has to compare the decision "up" with the decision "down" when $d_1 = up$, and $d_2 = down$ using Hurwicz criterion on aggregated values $\{15.38, 12.92\}$; so the ego of d_0 choice "down". In the example, the two methods give the same optimal strategy, but they would not if $H_{d_0=up} = 14$ since then the would be optimal using the Veto-Process (it actually considers only the ego at d_2).

5 Conclusion

In this paper we investigate the problem of sequential decision under pure uncertainty. We argue that the resolute Choice is not always psychologically acceptable. Then we study Jaffray's Veto-process in the context of pure uncertainty. We point out a weakness since it considers that preferences of each ego are independent while there is only one decision-maker over the whole sequential decision process. Hence a new rational approach is proposed, under the assumption that egos try to help the next egos in the sequence. An approach which satisfies this rationality requirement is presented along with an algorithm. Our finding suggest that defining the optimal criterion on the basis of a single reduced lottery induced by the whole decision tree is not satisfactory and looks like a debatable fiction that neglects the structure of decision trees. This is in line with the causal approach to probability revived by Shafer [11] who points out that probability trees were considered more expressive than probability distributions in early times of probability theory. In fact, the thesis of this paper is that computing the decision criterion on the reduced lotteries induced by a decision tree is generally not faithful to what is expected from a best strategy in the face of total uncertainty. Indeed, the optimal worth computed by non-expected utility criteria corresponds to no actual reward, it just reflects the decision maker attitude in front of uncertainty. It contrasts with the classical case, where the optimal expected utility of strategies accounts for the actual satisfaction of the decision-maker after playing the optimal strategy a sufficient number of times, if nature acts according to the prescribed probabilities.

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