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Sensitivity conjecture and signed hypercubes

Sophie Laplante, Reza Naserasr, Anupa Sunny and Zhouningxin Wang*

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Abstract

Using spectral techniques, H. Huang proved that every subgraph of the hypercube of dimension n induced on more than half the vertices has maximum degree at least \sqrt{n} . Combined with earlier work, this completed a proof of the sensitivity conjecture. In this work we show how to derive Huang's result using linear dependency and independence of vectors associated with the vertices of the hypercube. Our approach leads to several improvements of Huang's result. In particular we prove that in any induced subgraph of H_n with more than half the number of vertices, there are two vertices, one of odd parity and the other of even parity, each with at least n vertices at distance at most 2. As an application, we show that for any Boolean function f , the polynomial degree of f is bounded above by $s_0(f)s_1(f)$, a strictly stronger statement which implies the sensitivity conjecture. Using these linear dependencies, we show structural relations about the neighborhoods on the induced subgraph at distance at most three.

A key implement in Huang's proof is to assign signs $(+, -)$ to the edges of H_n such that the product of the signs on each 4-cycle is $-$. With the set of negative edges being called a signature, one may observe that there are a total of $2^{2^n - 1}$ such signatures on H_n satisfying this condition and that the symmetric difference of any two such signatures is an edge cut. A question of high interest then is to find the smallest size among all these signatures. This is known as the frustration index in the study of signed graphs. Here we provide lower and upper bounds for this parameter, observing that the two bounds match when n is a power of 4. We then establish a strong connection with other studies: On the one hand with a question of Erdős on the number of edges of the largest 4-cycle free subgraph of the hypercube. On the other hand with Ambainis functions which are used to show a separation between degree and adversary lower bounds on query complexity.

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1 Introduction

The notion of sensitivity was first introduced by Cook and Dwork and independently by Reischuk [7, 19] to bound below the time taken to compute a function in a CREW (Concurrent Read Exclusive Write) PRAM (Parallel Random Access Machines) model. The time complexity in this model was later shown to be exactly characterized by the logarithm of block sensitivity in a paper by Noam Nisan [16]. It follows from their definitions that block sensitivity is larger than sensitivity, i.e., $\text{bs}(f) \geq s(f)$. The natural question of whether block sensitivity is bounded above by some polynomial in sensitivity was then posed as the sensitivity conjecture as the largest separation found between these two measures was quadratic¹. A major breakthrough in this direction was due to Nisan and Szegedy where they showed an upper bound on block sensitivity in terms of polynomial degree [18]. This famous problem was reduced by Gotsman and Linial in 1992 to the graph theoretic question posed by Chung et al. in 1988. Numerous other formulations of the sensitivity conjecture also appeared over the years and a detailed survey can be found in the work by Hatami et al. [10]. Chung, Füredi, Graham and Seymour [6] posed the following question: how small can the maximum degree of an induced subgraph of a hypercube H_n with strictly more than 2^{n-1} vertices be? They showed an upper bound of \sqrt{n} on this quantity and a lower bound of $\Omega(\log n)$. They also noted that the lower bound (and its proof) is similar to the one on sensitivity of non-degenerate functions on n bits, where a non-degenerate function is one that depends on all n bits.

In 2019, Hao Huang showed a surprisingly simple proof of the sensitivity conjecture using signed hypercubes and Cauchy's Interlace theorem [12].

Theorem 1.1 (Huang). *Any induced subgraph of the n -dimensional hypercube with more than 2^{n-1} vertices has at least one vertex of degree larger than or equal to \sqrt{n} .*

Using a signed hypercube, Huang proved that every subgraph of the hypercube H_n of dimension n induced on more than half the vertices has maximum degree at least \sqrt{n} . Combined with earlier work, this completed a proof of the sensitivity conjecture. Knuth [13] exhibited a collection of eigenvectors of the signed hypercube used by Huang and provided a proof using a basis for the eigenspace corresponding to the larger eigenvalue. The idea of using the linear dependence and a basis of the eigenspace for proving the sensitivity conjecture has been attributed to a comment by Shalev Ben-David [13].

We show an alternative proof of Huang's result using only linear dependency of vectors associated with the vertices of the hypercube. Our approach helps gain deeper insight on structural properties of the induced subgraph in addition to the largest degree. In particular, we prove that in any induced subgraph of H_n with more than half the number of vertices, there are two vertices, one of odd parity and the other of even parity, each with at least n vertices at distance at most 2. As an application, we show that for any Boolean function f , the polynomial degree is bounded above by the product of 0-sensitivity and 1-sensitivity, $s_0(f)s_1(f)$, a strictly stronger statement which implies Huang's theorem. Similarly, we prove that given an induced subgraph F of H_n with $|V(F)| > 2^{n-1}$, there exists a vertex $v \in V(F)$ such that the number of vertices of F at distance 3 from v (in F) is at least $n^{3/2} - \Delta_F(2\Delta_F - 1)$.

¹This separation was achieved by the Rubinstein function which is a function on n bits with sensitivity \sqrt{n} and block sensitivity $n/2$.

A key implement in Huang’s proof was a signed hypercube with the property that every cycle of length 4 is assigned a negative sign. We take a detailed look at this signature and give a nearly optimal signature that uses the minimum number of negative edges while ensuring that every 4-cycle is negative. This problem of optimizing the number of negative edges turns out to be related to one of Erdős’ problems on the largest 4-cycle-free subgraph of the hypercube. We prove that the construction of a large 4-cycle-free subgraph by Brass, Harborth and Nienborg [4] leads to a signed hypercube with only negative 4-cycles. Surprisingly, we see that the Ambainis function naturally comes up in the analysis of this construction.

2 Preliminaries

Graphs. A graph G is a pair (V, E) where V is the vertex set and E , the edge set, is an unordered subset of $\binom{V}{2}$. The *degree* of a vertex x in a graph G , denoted $d_G(x)$, is the number of edges incident to x . The *maximum degree* of G is the maximum degree over all vertices in G and is denoted by Δ_G . Given a subset V' of V , the *subgraph induced* by V' , denoted $G[V']$, is the graph (V', E') where E' consists of all the pairs in E whose both elements are in V' . In particular, a cycle C of G is an *induced cycle* if its vertices induce the cycle C . The distance between two vertices u and v of G is the length of a shortest path connecting u to v .

Boolean functions. A *Boolean function* f takes as inputs n -bit strings and produces an output of either 0 or 1, i.e., $f : \{0, 1\}^n \rightarrow \{0, 1\}$. We denote by x^i the input obtained from x with the i th bit flipped for an $i \in [n]$, i.e., $x^i = x + e_i$.

Hamming weight and distance. We denote the Hamming weight of a string x , which is the number of 1s in x , by $|x|$. The *support* of x is the set of indices at which x is 1. The Hamming distance between two strings $x, y \in \{0, 1\}^n$ is the number of positions where the two strings differ, that is the Hamming weight of $x - y$, i.e., $|x - y|$.

Hypercubes. The hypercube of dimension n , denoted H_n , is a graph whose vertices are the binary vectors (or strings) of length n , in other words elements of $\{0, 1\}^n$. Two vertices are adjacent if their Hamming distance is 1, equivalently, if their binary sum is one of the elements of the standard basis, i.e., $u \sim v$ if $u \oplus v = e_i$ for some $i \in \{1, 2, \dots, n\}$. More generally, the distance of two vertices x and y of H_n is their Hamming distance as binary strings. The natural bipartition of H_n is by considering the parity of the Hamming weight of vertices: The vertices with an odd number of 1’s, called *odd vertices*, form one part denoted U_n^{odd} , and the vertices with an even number of 1’s, called *even vertices*, form the other part, denoted U_n^{even} .

Any Boolean function can be viewed as a 2-coloring of the vertices of H_n (not necessarily a proper coloring).

Signed graphs. A signed graph (G, σ) is a graph together with an assignment σ of signs to the edges of G . A key operation in signed graphs is *switching*. A switching at a vertex set X of $V(G)$ is to multiply the signs of all the edges with exactly one end in X by a $-$. Two signed graphs (G, σ_1) and (G, σ_2) are said to be *switching equivalent* if one can be obtained from the other by switching at a subset X of vertices.

The *sign* of a cycle, a closed walk, or in general a structure W in a signed graph (G, σ) is the

product of the signs of its edges considering the multiplicity. A basic theorem in the study of signed graphs is the following theorem of Zaslavsky.

Theorem 2.1 ([21, Theorem 3.2]). *Given two signatures σ_1 and σ_2 of a graph G , (G, σ_1) is switching equivalent to (G, σ_2) if and only if the sets of positive (or equivalently negative) cycles of (G, σ_1) and (G, σ_2) are the same.*

In practice, when using this theorem, one does not need to consider all the cycles of G but rather just a set of cycles that generates all other cycles (by taking symmetric differences).

Signed hypercubes. In this work we will consider a special class of equivalent signatures on H_n . Those are the signatures where all 4-cycles of H_n are negative. Let us first see why such a signature exists noting that this was implicit in [1, 12].

Lemma 2.2 ([1, 12]). *For each $n \geq 2$, there exists a signature σ_n on H_n such that in (H_n, σ_n) all 4-cycles are negative.*

Proof. For $n = 2$, as H_2 is just a 4-cycle, it would be enough to choose a signature which assigns $-$ to only one edge. Assuming σ_n exists, to find σ_{n+1} we use the inductive construction of H_{n+1} as the Cartesian product of H_n and H_2 . Thus H_{n+1} is built from two disjoint copies of H_n with a perfect matching connecting the corresponding vertices in the two copies. We assign σ_n to the edges of one copy, $-\sigma_n$ to the edges of the other copy and $+$ to the edges of the matching. It is straightforward to check that all 4-cycles in H_{n+1} have the required property now. \square

The signature on H_n , obtained from this proof, will be denoted by σ_n^* . Since the 4-cycles of H_n generate all the cycles, following the remark on Theorem 2.1, any other signature on H_n where all C_4 's are negative is switching equivalent to σ_n^* . To denote a typical member of this switching equivalent class, when the choice of signature is of no importance, we use \hat{H}_n . We point out that there are exactly $2^{2^n - 1}$ signatures with the property that all 4-cycles on H_n are negative, each obtained from σ_n^* by switching on a subset of vertices which does not contain the vertex $\mathbf{0}$.

Recall that the adjacency matrix of a signed graph (G, σ) , denoted $A(G, \sigma)$, is a $|V(G)| \times |V(G)|$ matrix whose rows and columns are labeled by the vertices of G where the entry xy is 0 if x and y are nonadjacent, it is $+1$ if $\sigma(xy) = +$, and it is -1 if $\sigma(xy) = -$.

Observe that to switch at a vertex v of (G, σ) is the same as multiplying both the column and the row of $A(G, \sigma)$ associated to v by a $-$. Furthermore, we shall note that in the power k of the adjacency matrix A , $A^k(G, \sigma)$, the entry uv is the algebraic sum of the number of u - v walks of length k , that is the number of positive u - v walks of length k minus the number of negative u - v walks of length k .

Observing that each pair of vertices at distance 2 of H_n belongs to a unique 4-cycle, we have the following key lemma.

Lemma 2.3. *For any $n \geq 2$ we have $A^2(\hat{H}_n) = nI$.*

Proof. Noting that each edge incident to a vertex u provides a u - u walk of length 2, as the sign of each such walk is $+$ and as there are exactly n of them, we have n for the entry uu . For an adjacent pair u and v , since H_n is bipartite, there is no u - v walk of length 2. If $|u - v| = 2$, then, as mentioned, they are in a unique C_4 . As all C_4 's are negative in \hat{H}_n , one of the u - v walks of length 2 is positive and the other is negative, thus the corresponding entry is 0. For all other pairs there is no walk of length 2 connecting them, thus the corresponding entry is 0. \square

Sensitivity of a Boolean function. Given a Boolean function f , a vector x is said to be *sensitive* at coordinate i if $f(x^i) \neq f(x)$. The *sensitivity* of f at x , denoted $s(f, x)$, is the number of coordinates at which it is sensitive. With f viewed as a 2-coloring of H_n , $s(f, x)$ is the number of vertices adjacent to x which are not of the same color as x . The *sensitivity of the function* f , denoted $s(f)$, is then defined to be the maximum sensitivity over all vectors in $\{0, 1\}^n$ with respect to f , i.e., $s(f) = \max_x s(f, x)$. Given a subset B of $[n]$, a vector x is said to be *B-sensitive* with respect to f , if simultaneously flipping all the bits in B changes the value of the function to $1 - f(x)$. The *block sensitivity* of f at x is the maximum number of disjoint subsets B_i of $[n]$ such that x is B_i -sensitive for each i . The *block sensitivity of f* is the maximum block sensitivity of f over all vectors in $\{0, 1\}^n$.

The *0-sensitivity* of Boolean function f , denoted $s_0(f)$, is the maximum sensitivity of f over binary vectors which evaluate to 0 on f . The *1-sensitivity* of Boolean function f , denoted $s_1(f)$, is defined similarly.

Definition 2.4. A Boolean function is said to be *parity-balanced* if the number of even vectors that evaluate to 1 is the same as the number of odd vectors that evaluate to 1 (see Figure 1).

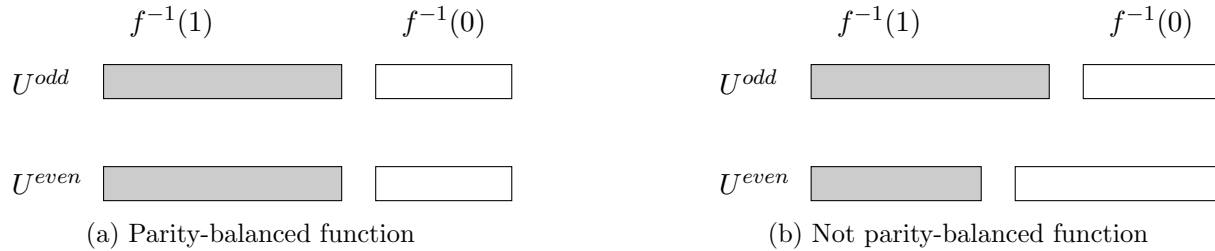


Figure 1: Parity-balance in a Boolean function can be thought of as the equality of the shaded regions in the figure above. A function has full degree if and only if it is not parity-balanced.

Polynomials and degree. For a vector $v \in \{0, 1\}^n$ whose support is B , we define the multilinear polynomial $P_v : \mathbb{R}^n \rightarrow \mathbb{R}$ as follows:

$$P_v = \prod_{i \in B} x_i \prod_{j \in \bar{B}} (1 - x_j).$$

The polynomial has the property that when restricted to the elements of $\{0, 1\}^n$, it takes the value 1 on v and 0 anywhere else. This polynomial has degree n where the coefficient of the term $x_1 x_2 \dots x_n$ is either $+1$ or -1 depending on the parity of v .

A polynomial $p : \mathbb{R}^n \mapsto \mathbb{R}$ *represents* a Boolean function f if for all $x \in \{0, 1\}^n$, $p(x) = f(x)$. Every Boolean function can be represented by a multilinear polynomial P_f which is the sum of polynomials P_v such that $f(v) = 1$, i.e.,

$$P_f = \sum_{v: f(v)=1} P_v.$$

We may now define the *degree* of a Boolean function f , denoted $\deg(f)$, as the degree of the multilinear polynomial P_f . Observe that a Boolean function f has full degree (i.e., degree n) if

and only if it is not parity-balanced. This can also be found in [5] and was attributed to Yao and Shi.

Composition of multivariate Functions. For any two Boolean functions $f : \{0, 1\}^n \rightarrow \{0, 1\}$ and $g : \{0, 1\}^m \rightarrow \{0, 1\}$, we define the composed function $f \circ g$ on an input $x \in \{0, 1\}^{nm}$ as follows:

$$f \circ g(x) = f(g(x_1, \dots, x_m), \dots, g(x_{(n-1)m+1}, \dots, x_{nm})).$$

Partial composition of multivariate functions. Let I be an m -subset of the coordinates of $\{0, 1\}^{n+m-1}$ and let ϑ be a function from the coordinates of $\{0, 1\}^{n+m-1}$ to the coordinates of $\{0, 1\}^n$ such that the coordinates in I are all mapped to the same coordinate, say i , and all other coordinates are mapped to distinct coordinates. Then, given two Boolean functions $f : \{0, 1\}^n \rightarrow \{0, 1\}$ and $g : \{0, 1\}^m \rightarrow \{0, 1\}$, we define the partial composed function $f \circ_{I, \vartheta} g$ on an input $x \in \{0, 1\}^{n+m-1}$ as follows. First we build the string $x' \in \{0, 1\}^n$ whose i^{th} -coordinate is $g(X|I)$ and whose coordinate $\vartheta^-(j)$, $j \neq i$, is x_j . Then we set $f \circ_{I, \vartheta} g(x) = f(x')$.

The Ambainis function. Ambainis constructed a function to show that the positive adversary bound can give better lower bounds on quantum query complexity than the polynomial method.

Definition 2.5 (f_A [2]). *The function f_A is defined on inputs with 4-bits. The function evaluates the following inputs 0000, 0001, 0011, 0111, 1111, 1110, 1100, 1000 to 1 and the rest to 0. In other words, $f_A(x) = 1$ if $x_1 \leq x_2 \leq x_3 \leq x_4$ or $x_1 \geq x_2 \geq x_3 \geq x_4$. The function is composed with itself for larger inputs, i.e., $f_A^d : \{0, 1\}^{4^d} \rightarrow \{0, 1\}$*

$$f_A^d = f_A \circ f_A^{d-1},$$

where $f_A^1 = f_A$.

Some of the important properties of the Ambainis function are: $s(f_A^d) = 2^d$, approximate polynomial degree $\widetilde{\deg}(f_A^d) = 2^d$ while the positive adversary bound $\text{MM}(f_A^d) = 2.5^d$ and quantum query complexity $\text{Q}(f_A^d) = 2.5135^d$ [2, 14, 11].

3 The sensitivity conjecture

The sensitivity conjecture as posed by Nisan and Szegedy [17] was the following problem,

Problem 3.1. *For every Boolean function f , $bs(f) \leq (s(f))^{O(1)}$.*

The last piece of the proof of this conjecture was proved in 2019 by Huang [12]. Below, we state and review some of the works on the sensitivity conjecture leading up to this paper.

Nisan and Szegedy [17, 20] established a strong connection between degree and block sensitivity of a Boolean function:

Theorem 3.2. (Nisan-Szegedy, Tal) *For any Boolean function f , $bs(f) \leq \text{deg}(f)^2$.*

This was proved using symmetrization and lower bounds on the degree of single-variable polynomials. An excellent exposition on the proof of this theorem and those that relate other measures of complexity of Boolean functions such as certificate complexity, decision tree complexity to block sensitivity can be found in [5]. The theorem in the presented form is due to Tal [20].

In 1992, Gotsman and Linial [9] showed the equivalence of the problem of showing that the polynomial degree of a function is at most some polynomial in the sensitivity of the function and the graph theoretic problem stated in Theorem 1.1.

Theorem 3.3. (Gotsman-Linial) *For any monotone function $h : \mathbb{N} \mapsto \mathbb{R}$, the following statements are equivalent :*

1. *For any induced subgraph G of H_n such that $|V(G)| \neq 2^{n-1}$, there exists a vertex of degree at least $h(n)$ in either G or $H_n \setminus G$.*
2. *For any Boolean function f , $s(f) \geq h(\deg(f))$.*

A rough sketch of this proof is as follows: consider a monomial of the largest degree in the multilinear polynomial representing the Boolean function and discard all variables that do not appear in this monomial (i.e., set them to zero). The sensitivity of the modified function f' might be smaller than that of the original function, but the polynomial degree of the function is preserved and f' now has full polynomial degree. The range of f' is taken to be $\{+1, -1\}$ instead of $\{0, 1\}$. This function is further modified as $g(x) = f'(x)p(x)$ where $p(x) = (-1)^{\sum x_i}$ is the parity function.

Notice that if we consider the graph induced by the vertices in $g^{-1}(0)$ (similarly for $g^{-1}(1)$), the graph degree of a vertex x in this subgraph is the sensitivity of x with respect to f . Equivalently, we can bypass g and consider the bipartite graph induced by the vertices in $(f^{-1}(0) \cap U^{\text{even}}) \cup (f^{-1}(1) \cap U^{\text{odd}})$, (similarly for $(f^{-1}(1) \cap U^{\text{even}}) \cup (f^{-1}(0) \cap U^{\text{odd}})$).

Since the function f' has full degree, f' is not parity-balanced and without loss of generality assume $|f^{-1}(1) \cap U^{\text{odd}}| > |f^{-1}(1) \cap U^{\text{even}}|$. This implies that there are more than 2^{n-1} vertices in $(f^{-1}(1) \cap U^{\text{odd}}) \cup (f^{-1}(0) \cap U^{\text{even}})$. By applying Theorem 1.1, we are sure that the subgraph induced on this set of vertices has a vertex of degree (sensitivity) at least $\sqrt{n} = \sqrt{\deg(f)}$. A clear presentation of this proof and a survey on the results leading up to the recent proof of the sensitivity conjecture are provided in [10]. Putting together Theorem 1.1, Theorem 3.2 and Theorem 3.3, we now have that $\text{bs}(f) \leq s(f)^4$.

4 Structural properties in linear dependency graphs

We have observed that $A^2(\hat{H}_n) = nI$ in Lemma 2.3, applying conjugate identity we have

$$(A(\hat{H}_n) + \sqrt{n}I)(A(\hat{H}_n) - \sqrt{n}I) = \mathbf{0} \tag{1}$$

Following the labeling of columns and rows of $A^2(\hat{H}_n)$, for each vertex v of H_n we define v^+ , respectively v^- to be the column of $(A(\hat{H}_n) + \sqrt{n}I)$, respectively $(A(\hat{H}_n) - \sqrt{n}I)$, corresponding to v . Moreover let V^+ , respectively V^- , be the subspace (of $\mathbb{R}^{|V(H_n)|}$) generated by v^+ vectors, respectively v^- vectors.

A subset U of the vertices of H_n is said to be *linearly dependent* if for every $u \in U$ there exists $c_u \neq 0$ such that

$$\sum_{u \in U} c_u u^+ = 0.$$

Observe that changing c_u to $-c_u$ for only odd vertices we will have a linear dependency formula based on v^- vectors. Recall that $H_n[U]$ is the subgraph of H_n induced by a set of vertices U .

Huang's result is a consequence the following.

Lemma 4.1. *Given a linearly dependent set U of vertices in H_n , the subgraph $H_n[U]$ has a vertex of degree at least $\lceil\sqrt{n}\rceil$.*

Proof. Let $\sum_{u \in U} c_u u^+ = 0$ be the linear dependency formula on U . Let w be a vertex in U such that $|c_w| = \max\{|c_u| : u \in U\}$. Consider the row corresponding to w in the linear dependency formula. On the coordinate w of this row we have $\sqrt{n}c_w$. For this value to vanish, there must be at least $\lceil\sqrt{n}\rceil$ nonzero element in the coordinates w . But each such entry corresponds to a neighbor of w . \square

Observe that if the equality is to happen, i.e., U induces a subgraph of maximum degree exactly \sqrt{n} , then $c_w = \sigma(wu)c_u$ for each neighbor u of w . Thus when a linearly dependent set U of vertices induced a subgraph of maximum degree \sqrt{n} , then the induced (signed) subgraph is \sqrt{n} -regular and can be switched so that all edges are positive.

Corollary 4.2 (Huang's degree theorem). *The subgraph induced by any $2^{n-1} + 1$ vertices of H_n has maximum degree at least \sqrt{n} .*

Proof. Since V^+ and V^- are orthogonal subspaces of R^{2^n} , at least one has dimension at most 2^{n-1} . Thus given a set of $2^{n-1} + 1$ vertices a linearly dependency is guaranteed on some subset of vertices. \square

It is however of interest to determine the exact dimension of V^+ and V^- .

Proposition 4.3. *We have $\dim(V^+) = \dim(V^-) = 2^{n-1}$.*

Proof. The set $\{u^+ \mid u \in U^{\text{odd}}\}$ is a set of size 2^{n-1} which is linearly independent. That is because for each element u^+ in this set, in the coordinate corresponding to u , u^+ is the only vector with a nonzero entry, all others, being neighbors of u , are in U^{even} . Similarly V^- is of dimension at least 2^{n-1} . Since they are orthogonal, the sum of their dimensions is at most 2^n which implies that 2^{n-1} is the exact dimension of both. \square

The most basic example of a linearly dependent set is $N[w]$ which consists of a vertex w and all its neighbors. For a linear dependency formula take $c_w = \sqrt{n}$ and $c_x = \sigma(wx)$ for every neighbor x of w . One can build larger linear dependencies starting from this basic one: in the basic linear dependency formula for w for some element x , $x \neq w$, replace x^+ with the formula of x^+ obtained from the basic linear dependency at x .

Observe that in a basic linearly dependent set, the maximum degree of the induced subgraph is n . To deviate from this value one must create a fair number of vertices at distance 2. To put a limit on this set of vertices at distance two, one has create even more vertices of distance 3 and so on. In the next subsection we explore this idea to get almost best possible upper bound for the number of distance 2 and distance 3 neighbors in a given linearly dependent set.

4.1 Structural Relations at distance 2

Extending Lemma 4.1, here we show that in the subgraph induced by a linearly dependent set U of vertices in H_n , there are vertices with a large number of neighbors at distance 1 or 2 from it.

Theorem 4.4. *Given a linearly dependent set U of vertices in H_n , there exist vertices $u \in U^{\text{odd}}$ and $v \in U^{\text{even}}$ in $H_n[U]$ such that $|N^U(u)| + |N_2^U(u)| \geq n$ and $|N^U(v)| + |N_2^U(v)| \geq n$.*

Proof. Let \mathbf{c} be the column vector in \mathbb{R}^{2^n} , presenting linear dependency of U based on u^- vectors. Here coordinates are labeled by the vertices of H_n and nonzero coordinates of \mathbf{c} are among coordinates corresponding to the elements of U , noting that there must be at least one nonzero coordinate. The linearly dependency formula can be written as: $(\hat{A}_n - \sqrt{n}I)\mathbf{c} = 0$.

Let A_U be the adjacency matrix of $\hat{H}_n[U]$. Let us analyze A_U^2 . As mentioned before, since A_U is the adjacency matrix of a signed graph, the entry xy of this matrix is the algebraic sum of the number of x - y walks of length 2. Thus the diagonal entry xx is $d_U(x)$ as each edge incident to x contributes a value of 1 to this sum regardless of its sign. Noting that H_n and thus $\hat{H}_n[U]$ is bipartite, the entry xy is 0 if they are in different parts. The only other potentially nonzero entries of A_U^2 then are entries xy where $|x - y| = 2$. This entry must be either +1 or -1 because if there are more than one x - y walk of length 2 in $\hat{H}_n[U]$, then there are exactly two and they form a negative 4-cycle, thus the corresponding entry is 0.

Let \mathbf{c}' be the vector obtained from \mathbf{c} by keeping only the coordinates corresponding to the vertices in U . The relation $(\hat{A}_n - \sqrt{n}I)\mathbf{c} = 0$ implies that $(A_U - \sqrt{n}I)\mathbf{c}' = 0$, or equivalently $A_U\mathbf{c}' = \sqrt{n}I\mathbf{c}'$. Multiplying by A_U and using the same identity we have

$$A_U^2\mathbf{c}' = nI\mathbf{c}'. \quad (2)$$

Let u be a vertex in U^{odd} for which $|\mathbf{c}'_u|$ is maximum among the entries corresponding to vertices in U^{odd} , noting that this is a nonzero value. Consider the row corresponding to u in the identity (2): on the right we have $n\mathbf{c}'_u$; on the left we have $d_U(u) + \sum \mathbf{c}'_x A_U^2(u, x)$. In this sum, the contribution is from vertices x for which the xy -entry of A_U^2 is nonzero. Those are the vertices that are at distance 2 from u (thus in U^{odd}), with the extra property that each has a unique path to u in $H_n(U)$ (but we do not use this extra property in this proof). Since $|\mathbf{c}'_u|$ is chosen to be the maximum, for this identity to hold, there must be at least $n - d_U(u)$ nonzero values among $A_U^2(u, x)$. This completes the proof of the claim for U^{odd} . The proof works the same for U^{even} . \square

By taking a vertex u given by the theorem and considering one of its neighbors which has the largest degree in $H_n[U]$ among all neighbors of u we conclude the following.

Corollary 4.5. *If U is a linearly dependent set of vertices of H_n , then there exists an edge uv in $H_n[U]$ such that $d_U(u) \times d_U(v) \geq n$.*

Observe that this corollary is already stronger than Huang's degree theorem as $d_U(u) \times d_U(v) \geq n$ implies that at least one of $d_U(u)$ or $d_U(v)$ is greater or equal to \sqrt{n} .

4.1.1 Application to odd and even sensitivity

We have the following corollary relating the degree of a function to its 0-sensitivity and 1-sensitivity which gives a stronger upper bound on the polynomial degree of a Boolean function.

Corollary 4.6. *For any Boolean function f , we have $\deg(f) \leq s_0(f)s_1(f)$.*

Proof. Assume $\deg(f)$ is d . We can concentrate on a function f' of degree d on d variables by setting the variables outside of the largest monomial to be 0. Let x' be a binary string in the domain of f' and x be the binary vector in the domain of f whose coordinates outside those in x' are set to 0. Then sensitivity of f' at x' is at most the sensitivity of f at x i.e., $s(f', x') \leq s(f, x)$. In particular, this implies that $s_0(f') \leq s_0(f)$ and $s_1(f') \leq s_1(f)$.

We partition the domain of f' (which are binary strings of length d) into two sets $U = (f'^{-1}(0) \cap U^{\text{even}}) \cup (f'^{-1}(1) \cap U^{\text{odd}})$ and $U' = (f'^{-1}(1) \cap U^{\text{even}}) \cup (f'^{-1}(0) \cap U^{\text{odd}})$. As f' is not parity-balanced, these sets are not of the same size (see Figure 1) and hence one of them, say U , has strictly more than 2^{d-1} . This implies that U is a linearly dependent set of vertices.

Since adjacent pairs x, y in $H_d[U]$ are assigned distinct values by f' , one of $s(f', x)$ and $s(f', y)$ is bounded above by $s_0(f)$ and the other by $s_1(f)$. By choosing xy according to Corollary 4.5, we have $s_0(f')s_1(f') \geq d$. The claim follows by noting that $s_0(f') \leq s_0(f)$, $s_1(f') \leq s_1(f)$ and $\deg(f) = d$. \square

Notice that we have proven something stronger, which is that the lower bound on 0-sensitivity and 1-sensitivity of a function is achieved on inputs at Hamming distance 1.

Since by Theorem 3.2 $bs(f) \leq \deg(f)^2$, we get the following polynomial relation between sensitivity and block sensitivity.

Corollary 4.7. *For any Boolean function f , $bs(f) \leq s_0(f)^2 s_1(f)^2$.*

4.2 Structural Relations at distance 3

We have the following theorem describing structural relations at distance 3 in the subgraph $H_n[U]$.

Theorem 4.8. *Given a linearly dependent set of vertices U in H_n , there exists a vertex $x \in U$ such that,*

$$|N_3^U(x)| \geq n^{\frac{3}{2}} - \sum_{y:|x-y|=1} c(x, y)$$

where $c(x, y)$ denotes the number of coordinates $j \in [n]$ such that either x^j or y^j belongs to $H_n[U]$ i.e.,

$$c(x, y) := |\{j : x^j \in U \vee y^j \in U\}|.$$

and $N_3^U(x)$ is the set of vertices $y \in U$ at exactly distance 3 from x .

We now analyze the structure of an induced subgraph corresponding to a linear dependency at distance at most 3. In other words, for studying the structure of $H_n[U]$ at distance 3, we will analyze A_U^3 as it contains information about walks of length 3 in the subgraph $H_n[U]$. In particular for any two vertices x and y in $H_n[U]$, $A_U^3(x, y)$ is the sum of the signs of all the walks of length 3 from x to y .

The following lemmas will be useful in proving Theorem 4.8.

Lemma 4.9. *Every induced 6-cycle in \hat{H}_n is negative.*

Proof. There are two types of 6-cycles in \hat{H}_n : Ones that are the symmetric difference of two 4-cycles (see Figure 2b) and ones that are the symmetric difference of three 4-cycles (see Figure 2a). As all the 4-cycles are negative in \hat{H}_n , the parity of the number of 4-cycles that are used to define a 6-cycle (as the symmetric difference of 4-cycles) determines its sign. What remains to note is that 6-cycles that are the symmetric difference of two 4-cycles are not induced. \square



Figure 2: 6-cycles in \hat{H}_n

Lemma 4.10. *Given a linearly dependent set of vertices U in H_n , for vertices $x, y \in U$,*

$$A_U^3(x, y) = \begin{cases} 0 & \text{if } |x - y| > 3 \text{ or } |x - y| \text{ is even,} \\ 0/ -1/ +1 & \text{if } |x - y| = 3, \\ \sigma(x, y) c(x, y) & \text{if } |x - y| = 1, \end{cases}$$

where $c(x, y)$ denotes the number of coordinates $j \in [n]$ such that either x^j or y^j belongs to $H_n[U]$.

Proof. We have the following cases for $A_U^3(x, y)$ based on the Hamming distance between x and y where $x, y \in U$.

Case 1: $|x - y| > 3$ or $|x - y|$ is even

In this case there exists no walk of length 3 that connects x and y , and we have $A_U^3(x, y) = 0$.

Case 2: $|x - y| = 3$

If $y = x^{\{i,j,k\}}$ where $i, j, k \in [n]$, we can write $A_U^3(x, y)$ as the following

$$A_U^3(x, y) = (A_U \times A_U^2)(x, y) = \sum_{\ell \in \{i,j,k\}} A_U(x, x^\ell) A_U^2(x^\ell, y).$$

In the proof of Theorem 4.4 for structural relations at distance 2, we had observed that for any two vertices u and v at distance 2 from each other, $A_U^2(u, v) \in \{0, -1, +1\}$. If there were two paths of length 2 between u and v , the signs of the two paths would cancel each other, due to the property of the signature we have chosen. Thus there can be at most three walks of length 3 from x to y with a non-zero contribution to the term $A_U^3(x, y)$. Moreover, if there are more than one paths of length 3 between x and y , they will form an induced 6-cycle. By Lemma 4.9, every induced 6-cycle is negative. This gives us that $|A_U^3(x, y)| \leq 1$ when $|x - y| = 3$.

Case 3: $|x - y| = 1$

Let us suppose that $y = x^i$ for an $i \in [n]$. There are four types of possible walks of length 3 from x to y which are shown in Figure 3. The walks of length 3 from x to y are marked in red in the figure.

To count all of these possible walks from x to y in $H_n[U]$, we look at all the walks between them using edges corresponding to an index $j \in [n]$. For each contributing index j , we have the following sub-cases: when both x^j and y^j are both vertices in the subgraph $H_n[U]$, and when only one of x^j and y^j belong to $H_n[U]$.

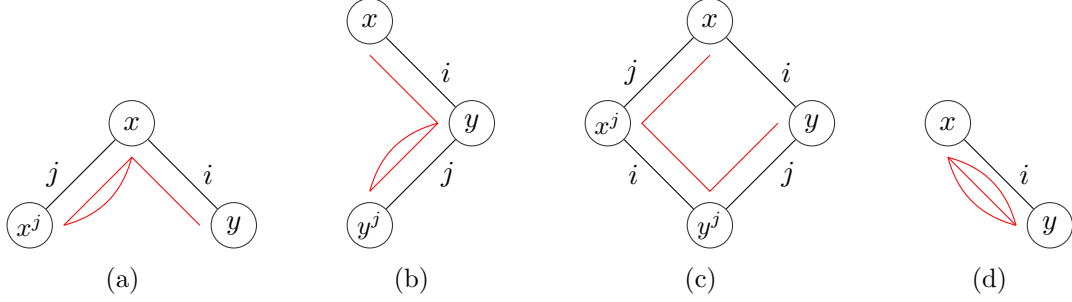


Figure 3: All walks of length 3 from x to y when $|x, y| = 1$. The edges are labelled by the coordinate on which the endpoints differ.

Case 3a: Both x^j and y^j belong to U

Assume $j \neq i$. In this case, there are three $x - y$ walks using coordinates i and j (see Figure 4 (a), (b), (c)). Of the three, two are of the same sign as $\sigma(x, y)$ and the third is of the opposite

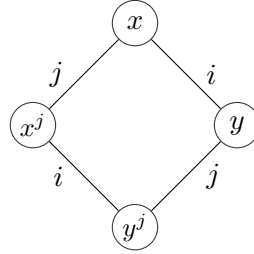


Figure 4: The subgraph when x, y, x^j and y^j belong to U . Edge labels are the indices at which the endpoints differ.

sign. Thus the total contribution is $\sigma(x, y)$.

When the index $j = i$, we have a single $x - y$ walk using only coordinate i which is of the sign $\sigma(x, y)$ (see Figure 4 (d)). This implies that we have a contribution of $\sigma(x, y)$ from each index j such that both x^j and y^j belong to U .

Case 3b: Only one of x^j and y^j belong to U

In this case, we have one of the following walks of length 3 from x to y using edges corresponding to the index j : either $x - x^j - x - y$ or $x - y - y^j - y$ (see Figure 5). Since each of these walks has a signature $\sigma(x, y)$, an index j such that only one of x^j and y^j belong to U contributes $\sigma(x, y)$ to $A_U^3(x, y)$.

In conclusion every index in the set $\{j : x^j \in U \vee y^j \in U\}$ contributes $\sigma(x, y)$ to $A_U^3(x, y)$ when $|x - y| = 1$ which completes the proof of the lemma. \square

We can now prove Theorem 4.8 which shows the structural relations between vertices in $H_n[U]$ that are at distance 3.

Proof of Theorem 4.8. As observed in the proof of Theorem 4.4, a linear dependency among a set of vertices U in H_n implies that $(A_U - \sqrt{n}I)\mathbf{c} = 0$ for a non-zero column vector \mathbf{c} , in other words,

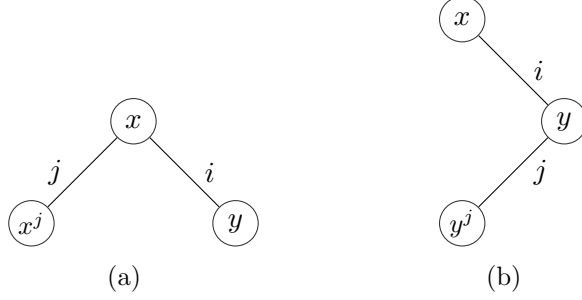


Figure 5: The subgraph when only one of x^j and y^j belong to U . Edge labels are the indices on which the endpoints differ.

$A_U \mathbf{c} = \sqrt{n} \mathbf{c}$, thus

$$A_U^3 \mathbf{c} = n^{\frac{3}{2}} \mathbf{c}.$$

Let \mathbf{c}_x be the largest coefficient in the linear dependency U , i.e., $\mathbf{c}_x = \max_i \{|\mathbf{c}_i|\}$. At the row labelled by x we have,

$$\begin{aligned} n^{\frac{3}{2}} \mathbf{c}_x &= (A_U^3 \times \mathbf{c})(x) = \sum_{y \in U} A_U^3(x, y) \mathbf{c}_y \\ n^{\frac{3}{2}} |\mathbf{c}_x| &\leq \sum_{y \in U} |A_U^3(x, y) \mathbf{c}_y| \leq \sum_{y \in U} |A_U^3(x, y)| |\mathbf{c}_x|. \end{aligned}$$

Based on Lemma 4.10 we get

$$n^{\frac{3}{2}} \leq \sum_{y \in U} |A_U^3(x, y)| \leq \sum_{y: |x-y|=3} 1 + \sum_{y: |x-y|=1} c(x, y).$$

This shows that the size of the neighborhood of x at distance 3, $N_3^U(x) \geq n^{\frac{3}{2}} - \sum_{y: |x-y|=1} c(x, y)$

which completes the proof of the theorem. \square

We have the following corollary from Theorem 4.8 that gives a lower bound on the number of neighbors of a vertex at distance 3 in $H_n[U]$ in terms of Δ_U , where Δ_U denote the maximum degree in $H_n[U]$.

Corollary 4.11. *Given a linear dependency U , there exists a vertex in $H_n[U]$ that has at least $n^{\frac{3}{2}} - \Delta_U(2\Delta_U - 1)$ neighbors at distance 3 from it.*

Proof. From Theorem 4.8, we get a vertex x that has at least $n^{\frac{3}{2}} - \sum_{y: |x-y|=1} c(x, y)$ neighbors at distance 3 from it. The number of coordinates i such that x^i belongs to $H_n[U]$ is at most Δ_U for any vertex $x \in U$. For any two vertices x and y in $H_n[U]$ such that $x \sim_i y$ for some $i \in [n]$, the size of the set of coordinates j such that either x^j or y^j belongs to $H_n[U]$ is at most $2\Delta_U - 1$. This holds as there is a vertex i such that both x^i and y^i belong to $H_n[U]$. This implies that $c(x, y) \leq 2\Delta_U - 1$ for any x and y such that $|x - y| = 1$. Since the number of y that are at distance 1 from x is bounded above by Δ_U , we have the above corollary. \square

5 Signatures of Hypercubes with negative 4-cycles

The proof of the sensitivity conjecture relied crucially on signed hypercubes with the property that each 4-cycle is negative, i.e., the product of the signs on the edges of the 4-cycle is negative. A signature with this property was used to show the existence of a signed hypercube with exactly two eigenvalues for the corresponding weighted adjacency matrix by Ahmadi et al. [1, Theorem 6.7 and Corollary 6.9]. This was the same signature σ_n^* given by Huang inductively. As it was pointed out in Section 2, there are 2^{2^n-1} signatures each equivalent to σ_n^* and each of them can be employed for these arguments.

While σ_n^* is the easiest signature to present, it is not necessarily the best one. In this section we will see that to find one with the minimum (or equivalently, the maximum) number of negative edges is a nontrivial task of high interest, relating to both some extremal graph theory questions and the existence of some binary function of importance in computational complexity theory. To this end we first give formal definition in the language of signed graphs.

Definition 5.1 (Frustration Index). *The frustration index of a signed graph (G, σ) , denoted $F(G, \sigma)$, is the smallest number of negative edges of (G, σ') over all signatures σ' that are a switching of σ .*

Recall that (H_n, σ^*) is defined by viewing H_n as two copies of H_{n-1} with a perfect matching between them. Each edge of H_{n-1} then is negative exactly in one copy and edges of the matching are positive. Then there are $(n-1)2^{n-2}$ negative edges in (H_n, σ^*) . This gives us a trivial upper bound on the frustration index $F(H_n, \sigma^*) \leq (n-1) \cdot 2^{n-2}$. In what follows, we will first give a lower bound on the frustration index $F(H_n, \sigma^*)$ and then construct a signature that tries to match the lower bound.

5.1 A lower bound on the frustration index

Although the problem of finding the frustration index of (H_n, σ^*) has not been looked at in the literature to the best of our knowledge, a related problem of finding the minimum number of edges that need to be removed from an H_n to make it C_4 -free has been studied extensively. A lower bound on this problem would easily translate to a lower bound on $F(H_n, \sigma^*)$. First such lower bound was implied by a result of Bialostocki [3]. We present one proof of this lower bound, different from the original proof.

Theorem 5.2 (implied by [3]). *For a signed hypercube (H_n, σ^*) which satisfies the property that every C_4 is negative, the frustration index $F(H_n, \sigma^*) \geq (n - \sqrt{n}) \cdot 2^{n-2}$.*

We use the following definition of negative and positive degrees for vertices in our proof.

Definition 5.3 (Positive and negative degree). *Given a signed graph (G, σ) and a vertex $v \in V(G)$, we define the positive degree of v , denoted by $d^+(v)$, as the number of positive edges incident on v , and negative degree of v , denoted by $d^-(v)$, as the number of negative edges incident on v . The average negative degree of (G, σ) , denoted by \bar{d}^- , is the average of the negative degree over all vertices in G , i.e., $\bar{d}^- = \sum_{v \in V(G)} \frac{d^-(v)}{|V(G)|}$.*

Proof of Theorem 5.2. Let us consider the following types of vertices in a 4-cycle in (H_n, σ^*) :

- *type 00* point: if it has two positive edges in a 4-cycle.
- *type 11* point: if it has two negative edges in a 4-cycle.
- *type 10* point: if it has one negative edge and one positive edge in a 4-cycle.

A vertex could be counted repeatedly as different types of points with respect to different 4-cycles. Every 4-cycle in (H_n, σ^*) , regardless of it having one or three negative edges, has two points of type 10. Thus the number of points of type 10 is equal to twice the number of 4-cycles in H_n , i.e., $\sum_v (d^+(v) \cdot d^-(v)) = 2 \binom{n}{2} \cdot 2^{n-2}$. Hence, we have

$$\sum_v d^-(v)(n - d^-(v)) = n \sum_v d^-(v) - \sum_v d^-(v)^2 = n(n-1)2^{n-2}.$$

On applying the Cauchy-Schwarz inequality,

$$\sum_v d^-(v)^2 \geq \frac{(\sum_v d^-(v))^2}{2^n},$$

we get

$$n \cdot \left(\frac{\sum_v d^-(v)}{2^n} \right) - \left(\frac{\sum_v d^-(v)}{2^n} \right)^2 \geq \frac{n(n-1)}{4}$$

which simplifies as, $4\bar{d}^{-2} - 4n\bar{d}^{-} + (n^2 - n) \leq 0$. This gives $\bar{d}^{-} \in [\frac{n-\sqrt{n}}{2}, \frac{n+\sqrt{n}}{2}]$, and hence $F(H_n, \sigma^*) \geq \bar{d}^{-} \cdot 2^{n-1} = (n - \sqrt{n}) \cdot 2^{n-2}$. \square

Interestingly, both the lower bound and the upper bound on the frustration index of (H_n, σ^*) are closely related to the problem of largest C_4 -free subgraph of H_n . This was posed by Erdős as one of the interesting open problems in combinatorics [8]. In fact, he conjectured that the number of edges in the largest C_4 -free subgraph of H_n is at most $(n + c) \cdot 2^{n-2}$ for some constant c but this was later disproved in a work by Brass, Harborth and Nienborg [4]. Brass et al. showed that it is at most $(n + 0.9\sqrt{n}) \cdot 2^{n-2}$ for $n \geq 9$ and showed a construction that achieves $(n + \sqrt{n}) \cdot 2^{n-2}$ for $n = 4^k$ where $k \geq 1$. We will now present the construction used in a slightly different language and obtain a signature on the hypercube that satisfies every 4-cycle being negative. In particular, we will prove the following theorem.

Theorem 5.4. *For a signed hypercube (H_n, σ^*) , $F(H_n, \sigma^*) \leq 2^{n-2} \left(n - \frac{(n+2 \cdot 4^k - \frac{m}{2})}{3 \cdot 2^k} \right)$ where $4^k \leq n < 4^{k+1}$ and $n + m = 1 \pmod{3}$ with $m \in \{0, 1, 2\}$.*

We make note that the construction is the same as that of Brass et al., but we present a proof of the fact that it leads to a signature with only negative 4-cycles as that does not follow from the original work. In addition, the new outlook on the construction naturally leads to the introduction of a sequence of highly interesting binary functions, which on dimension $n = 4^k$ are known as the Ambainis functions (see Section 5.3).

5.2 Construction of a signed hypercube nearly achieving optimal frustration index

The edges of H_n are 3-colored, say red, blue, and green, such that every 4-cycle has an odd number of red edges, an odd number of blue edges, and hence an even number of green edges. Thus the set of red (or blue) edges would form a signature of nearly minimum size, whereas, the set of red and green edges together would form a signature of nearly maximum size. For the inductive construction to work we must also color the vertices with two colors, say black and white (this is implicit in [4]). The details are as follows:

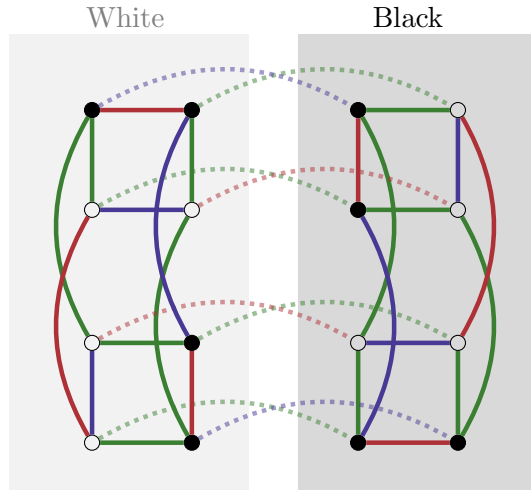


Figure 6: White and Black halves of a fully colored \tilde{H}_4

A fully colored hypercube of dimension k is a triplet (H_k, f, g) where the function f is a 2-coloring of the vertices of the hypercube H_k . This vertex-coloring induces two types of edges: monochromatic edges and 2-colored edges. The function g is a 3-coloring of the edges using (green), (red), (blue). They satisfy the following constraints:

1. Red and blue edges are monochromatic under f .
2. Each 4-cycle of H_k has an odd number of red edges, an odd number of blue edges, and hence an even number of green edges.

Let S_1 and S_2 be sets of indices such that S_1 is of order k , S_2 is of order m and they have a single element i in common, i.e., $S_1 \cap S_2 = \{i\}$. Let (H_k, f_1, g_1) be a fully colored hypercube of dimension k whose coordinates are labeled by S_1 , and let (H_m, f_2, g_2) be a fully colored hypercube of dimension m whose coordinates are labeled by S_2 . We define a fully colored $(H_{m+k-1}, f, g) := (H_m, f_2, g_2) \square (H_k, f_1, g_1)$ as follows:

- H_{m+k-1} is a hypercube of dimension $m+k-1$ and the coordinates of its vertices are labeled by elements in $S_1 \cup S_2$.
- To define the vertex-coloring f on the new hypercube for a vertex $x \in V(H_{m+k-1})$, we take x_2 to be the restriction of x to the coordinates labelled by S_2 . Next define x_1^* to be the vector

obtained by restricting x to the coordinates labelled by S_1 and setting the coordinate labelled by i to be $f_2(x_2)$. The vertex-coloring f on x is defined as $f(x) := f_1(x_1^*)$.

We can think of this as replacing each vertex of H_m by a half of the hypercube H_k that is split into two halves across the index i . For simplicity, we can think of H_m as the outer hypercube and H_k as the inner hypercube. The color of the vertex in the outer hypercube decides which half of the inner hypercube replaces it, and the new vertex in the larger hypercube inherits the color from the vertex corresponding to it in this half of the inner hypercube.

- The edge-coloring g is defined for strings $x, y \in V(H_{m+k-1})$ such that x and y differ only at an index j i.e., $x \sim_j y$ where $j \in S_1 \cup S_2$ as follows:
 - **If $j \in S_1 \setminus \{i\}$:** we define the edge-coloring as $g(x, y) = g_1(x_1^*, y_1^*)$, i.e., if two vertices differ at an index corresponding to the inner hypercube (other than the index i which is common to both), they lie in the same half of the inner hypercube and the color of the edges is inherited from that of the inner hypercube.
 - **If $j \in S_2$:** we decide on the coloring based on $g_2(x_2, y_2)$ which is the edge-coloring of the projection of x and y on the outer hypercube H_m . If $g_2(x_2, y_2) = \text{green}$, and since green edges are bichromatic (i.e., one of its endpoints is white and the other black), they are replaced by the two halves of the inner hypercube. In this case, the edge xy inherits the color from the edge class that split the inner hypercube into two halves, i.e., $g(x, y) = g_1(x_1^*, y_1^*)$. Otherwise, if $g_2(x_2, y_2) = \text{red}$ (and resp. blue) and if the restriction of x to the coordinates labelled by $S_1 - \{i\}$, denoted x_1 , is of an even weight we set $g(x, y) = \text{blue}$ (and resp. red). Note that the restriction of x or that of y onto $S_1 - \{i\}$ are the same since x and y do not differ on any index in $S_1 - \{i\}$. Otherwise, we set $g(x, y) = \text{red}$ (and resp. blue).

We will now see that this vertex-coloring and edge-coloring satisfy the conditions that blue and red colored vertices are monochromatic and green edges are bichromatic. We will also prove that every 4-cycle of H_{m+k-1} has an odd number of red and an odd number of blue edges.

Proposition 5.5. *The construction of the hypercube (H_{m+k-1}, f, g) from fully colored hypercubes (H_m, f_2, g_2) and (H_k, f_1, g_1) is well defined, and the result is a fully colored hypercube that satisfies properties (1) and (2).*

Proof. Let us first check if the edges between two vertices of the same color are colored either red or blue and that the edges between two vertices of different colors are colored green, i.e., they satisfy (1). We consider $x, y \in V(H_{m+k-1})$ such that x and y differ only at an index j , i.e., $x \sim_j y$ and look at the following cases:

- **$j \in S_1 \setminus \{i\}$:** In this case, both x and y correspond to the same vertex in the outer hypercube which has been replaced by a half of the inner hypercube (H_k, f_1, g_1) . Since both the vertex and edge colors are inherited from the inner hypercube and since it was originally properly colored, the coloring constraints (1) are satisfied in this case.
- **$j \in S_2$:** In this case, if the vertices belong to different halves of the inner hypercube, i.e., $f_2(x_2) \neq f_2(y_2)$, the coloring constraints are satisfied since the outer edge is colored green, and both the edge-coloring and the vertex-coloring are inherited from the inner hypercube. If the two vertices belong to the same half of the hypercube, they are colored the same and the edges between them are colored either red or blue in our construction.

The only thing that is left to show is that each 4-cycle of (H_{m+k-1}, f, g) has an odd number of red and an odd number of blue edges. Let $C = xywz$ be a 4-cycle in this fully colored hypercube. Let p and q be the indices corresponding to the edges of C . We consider three possibilities:

- **p and q are both in $S_1 \setminus \{i\}$:** In this case, since the vertices and edges in C inherit their colors from the inner hypercube (H_k, f_1, g_1) and C is a 4-cycle in it, property (2) holds.
- **p is in $S_1 \setminus \{i\}$, q is in S_2 :** Without loss of generality, let us assume that $y = x^q$. Here we consider two cases: when the edge $g_2(x_2, y_2)$ corresponding to the outer hypercube is colored green and when it is not colored green. If $g_2(x_2, y_2) = \text{Green}$, C is a 4-cycle in an isomorphic copy of (H_k, f_1, g_1) in this construction in which the number of red and blue edges remain odd, and the green edges are even in number. If $g_2(x_2, y_2) \neq \text{Green}$, there is one red edge and one blue edge in C corresponding to the index q . Since the edge corresponding to the outer hypercube $g_2(x_2, y_2)$ is either red or blue, the end points x_2 and y_2 must be of the same color, i.e., $f_2(x_2) = f_2(y_2)$. The edges corresponding to index p have the same color since x_2 and y_2 have been replaced by the same copy of H_{k-1} . Property (2) is satisfied in this case as there is an odd number of green and blue edges and the number of green edges is either 0 or 2.
- **p and q are both in S_2 :** The 4-cycle C corresponds to a 4-cycle C' with vertices x_2, y_2, w_2 and z_2 in (H_m, f_2, g_2) . Since all the vertices in C are the same on all the coordinates labelled by $S_1 \setminus \{i\}$, the parity of their restriction onto $S_1 \setminus \{i\}$ is the same. The red and blue edges in the 4-cycle C' remain the same as in C if the parity of their restriction onto $S_1 \setminus \{i\}$ is odd and are swapped otherwise. In any case, the number of red and the number of blue edges both remain odd. Note that for vertices x, y, w and z , only the coordinate at i changes and remaining coordinates are the same. The green edges in the cycle C' contribute edges of the same color in C since the colors of the edges are inherited from the same edge in (H_k, f_1, g_1) . Since the number of green edges in C' are even, the number of edges it contributes (of any color) remains even.

This completes the proof. □

We will now show an explicit construction of a fully colored hypercube of dimension k , denoted \tilde{H}_k , from a fully colored hypercube in 4 dimensions. Let us denote the fully colored hypercube given in Figure 7 as \tilde{H}_4 . For each of the indices $i \in [4]$ representing coordinates of vertices in \tilde{H}_4 , there are 4 green edges, 2 blue and 2 red edges.

From two copies of \tilde{H}_4 , using the construction described above, we get \tilde{H}_7 in which the number of green edges is 2^4 for four of the indices and 2^5 for the three other indices.

In general, we use \tilde{H}_4 to build a fully colored hypercube \tilde{H}_{3l+1} using \tilde{H}_{3l-2} and \tilde{H}_4 iteratively as follows:

$$\tilde{H}_{3l+1} = \tilde{H}_4 \square \tilde{H}_{3l-2}$$

for $l \geq 2$. In other words, each vertex of \tilde{H}_4 is replaced by a half of \tilde{H}_{3l-2} depending on the color of the vertex in \tilde{H}_4 .

For a fully colored hypercube \tilde{H}_k constructed as above, let $r_k[j]$, $b_k[j]$ and $g_k[j]$ be the number of edges colored red, blue and green, respectively for an index j . Let the total number of red, total number of blue and total number of green edges in \tilde{H}_k be denoted r_k, b_k and g_k respectively. It follows from the construction of $\tilde{H}_{3l+1} = \tilde{H}_4 \square \tilde{H}_{3l-2}$ that if the number of the red edges equals

the number of the blue edges in every index of the constituent \tilde{H}_4 and \tilde{H}_{3l-2} , the same would hold for the resulting \tilde{H}_{3l+1} . Similarly, one can also see that if the constituent hypercubes \tilde{H}_4 and \tilde{H}_{3l-2} have equal number of black and white vertices, then \tilde{H}_{3l+1} also has an equal number of black and white vertices.

Observation 5.6. *For the fully colored hypercube \tilde{H}_{3l+1} built using \tilde{H}_{3l-2} and \tilde{H}_4 :*

1. *The number of white vertices equals the number of black vertices.*
2. *Given any index j , the number of blue edges equals the number of red edges corresponding to that index, i.e., $b_{3l+1}[j] = r_{3l+1}[j]$.*

This follows since fully colored hypercubes are constructed iteratively from \tilde{H}_4 which has an equal number of white and black vertices, and has an equal number of red and blue edges along every coordinate.

Recall that all the edges of \tilde{H}_k are colored either red, blue or green, and that there are 2^{k-1} edges corresponding to any given index. Thus, calculating $g_k[j]$ would also determine $r_k[j]$ and $b_k[j]$ when $k = 3l + 1$ for some integer $l \geq 1$. The number of green edges in a fully colored hypercube using the construction above is given by the following theorem.

Theorem 5.7. *Given a fully colored hypercube \tilde{H}_{3l+1} there are two types of indices:*

- *indices in \mathcal{L}_n each with $2^{n-\lceil \log_4 n \rceil - 1}$ green edges and there are $\frac{4}{3}(n - 4^{\lceil \log_4 n \rceil - 1})$ many of them.*
- *indices in \mathcal{M}_n each with $2^{n-\lceil \log_4 n \rceil}$ green edges and there are $\frac{1}{3}(4^{\lceil \log_4 n \rceil} - n)$ many of them.*

We prove this theorem by induction. The importance of the choice of the index i along which the inner hypercube is split into two halves is illustrated by the following lemma.

Lemma 5.8. *In the construction of a fully colored hypercube $(H_{m+k-1}, f, g) = (H_m, f_2, g_2) \square (H_k, f_1, g_1)$ from two fully colored hypercubes \tilde{H}_m and \tilde{H}_k , if we choose to split the inner hypercube H_k along an index i , the number of green edges in the new hypercube along an index $j \in S_1 \cup S_2$ is given as follows.*

$$g_{m+k-1}[j] = \begin{cases} 2^{m-1} \times g_k[j], & \text{if } j \in S_1 \setminus \{i\} \\ g_k[i] \times g_m[j] & \text{otherwise,} \end{cases}$$

where $g_l[j]$ is the number of green edges in a hypercube \tilde{H}_l along an index j .

Proof. We consider two cases:

- For a $j \in S_1 \setminus \{i\}$, all the edges along this coordinate inherit their colors from the corresponding half of the fully colored hypercube \tilde{H}_k by construction. Since there are an equal number of white and black vertices in \tilde{H}_m (by Observation 5.6), there are 2^{m-1} copies of \tilde{H}_k split into two which replace every vertex of \tilde{H}_m . Thus the number of green edges corresponding to this index is $g_{m+k-1}[j] = 2^{m-1} \times g_k[j]$.
- For a $j \in S_2$, notice that a fully colored hypercube \tilde{H}_{m+k-1} can have a green edge along an index j only if the corresponding edge in the outer hypercube (when restricted to indices in S_2) is colored green. If the outer edge is colored red or blue, the resulting edge could only be of one of those colors. In the construction, the edges corresponding to a green outer edge

inherit their colors from the edges along the index i of \tilde{H}_k . Combined with the fact that every green edge of the outer hypercube contributes to $g_k[i]$ green edges in \tilde{H}_{m+k-1} , this gives that the number of green edges along the index j is $g_{m+k-1}[j] = g_k[i] \times g_m[j]$.

□

With the above lemma, we may now prove Theorem 5.7.

Proof of Theorem 5.7. Let us choose the outer hypercube to be \tilde{H}_4 with 4 green edges along each of its indices. The proof of the theorem is by induction on l where $n = 3l + 1$.

Base Case: When $l = 1$ (i.e., $n = 4$), we have shown an example of a fully colored hypercube with 4 green edges along each index which trivially satisfies the theorem.

Induction Case: By induction hypothesis, let us assume that there exists a fully colored hypercube \tilde{H}_n such that there is a set of indices \mathcal{L}_n of size $\frac{4}{3}(n - 4^{\lceil \log_4 n \rceil - 1})$ with $2^{n - \lceil \log_4 n \rceil - 1}$ green edges, and a set of indices \mathcal{M}_n of size $\frac{1}{3}(4^{\lceil \log_4 n \rceil} - n)$ with $2^{n - \lceil \log_4 n \rceil}$ green edges. We will now show that we can construct an $\tilde{H}_{n'}$ where $n' = n + 3$ from \tilde{H}_n and \tilde{H}_4 . We pick \tilde{H}_4 as the outer hypercube and pick the index i from the inner hypercube \tilde{H}_n such that it has a large number of green edges. The set S_2 is chosen to work for our choice of i , i.e., $S_1 = \mathcal{L}_n \cup \mathcal{M}_n$ and $S_1 \cap S_2 = \{i\}$. By Lemma 5.8, the number of green edges along an index $j \in S_1 \setminus \{i\}$ equals $2^3 \times g_n[j]$ for our choice of outer and inner hypercubes and the index i . We also have $4 \times g_n[i]$ green edges along an index $j \in S_2$. We have the following cases:

Case 1: n is a power of 4.

In this case, \tilde{H}_n has $\mathcal{M}_n = \emptyset$ and $|\mathcal{L}_n| = n$ with each index in \mathcal{L}_n having $2^{n - \lceil \log_4 n \rceil - 1}$ green edges by induction hypothesis. We pick any one of the indices in \mathcal{L}_n as the index i to split \tilde{H}_n into two halves. By Lemma 5.8, the number of green edges along an index $j \in S_1 \setminus \{i\}$ (or equivalently $j \in \mathcal{L}_n \setminus \{i\}$) is $2^{n - \lceil \log_4 n \rceil - 1 + 3} = 2^{n + 3 - \lceil \log_4(n + 3) \rceil}$ since $\lceil \log_4 n \rceil + 1 = \lceil \log_4(n + 3) \rceil$ when n is a power of 4. This gives $n - 1$ coordinates with $2^{n' - \lceil \log_4 n' \rceil}$ green edges. Since $\frac{1}{3}(4^{\lceil \log_4 n' \rceil} - n') = \frac{1}{3}(4n - n - 3) = n - 1$, we have a set of indices $\mathcal{M}_{n'} = S_1 \setminus \{i\}$ with $2^{n' - \lceil \log_4 n' \rceil}$ green edges. Note that in this case every index in \mathcal{L}_n except i is in $\mathcal{M}_{n'}$.

By Lemma 5.8, the number of green edges along each index $j \in S_2$ is $4 \times 2^{n - \lceil \log_4 n \rceil - 1} = 2^{n' - \lceil \log_4 n' \rceil - 1}$ and there are 4 indices in S_2 . Since $\frac{4}{3}(n' - 4^{\lceil \log_4 n' \rceil - 1}) = \frac{4}{3}(n + 3 - 4^{\lceil \log_4 n \rceil}) = 4$, we have a set of indices $\mathcal{L}_{n'} = S_2$ with $2^{n' - \lceil \log_4 n' \rceil - 1}$ green edges. This proves that the induction case holds when $n' = n + 3$ is a power of 4. We note here that the chosen index i now belongs to $\mathcal{L}_{n'}$.

Case 2: When $n = 3l + 1$ for some integer $l > 1$ but is not a power of 4.

In this case, we assume the induction hypothesis for n and show that the induction case holds for $n' = n + 3$. Since n is not a power of 4 and since powers of 4 are 1 modulo 3, we have $\lceil \log_4 n \rceil = \lceil \log_4 n' \rceil$. We choose to split \tilde{H}_n into two halves across an index $i \in \mathcal{M}_n$ that has the largest number of green edges (which is $2^{n - \lceil \log_4 n \rceil}$). There are $\frac{1}{3}(4^{\lceil \log_4 n \rceil} - n) - 1 = \frac{1}{3}(4^{\lceil \log_4 n' \rceil} - n')$ indices in $\mathcal{M}_n \setminus \{i\}$ with $2^{n - \lceil \log_4 n \rceil}$ green edges in \tilde{H}_n . By Lemma 5.8, these indices have $2^{n - \lceil \log_4 n \rceil + 3} = 2^{n' - \lceil \log_4 n' \rceil}$ green edges and they form $\mathcal{M}_{n'}$.

Since \mathcal{L}_n has $\frac{4}{3}(n - 4^{\lceil \log_4 n \rceil - 1})$ indices with $2^{n - \lceil \log_4 n \rceil - 1}$ green edges and the index i

chosen does not belong to \mathcal{L}_n , these indices have $2^{n-\lceil \log_4 n \rceil - 1 + 3} = 2^{n' - \lceil \log_4 n' \rceil - 1}$ green edges in $\tilde{H}_{n'}$ by Lemma 5.8. These indices now belong to $\mathcal{L}_{n'}$. We also have $4g_n[i] = 2^{n-\lceil \log_4 n \rceil + 2} = 2^{n' - \lceil \log_4 n' \rceil - 1}$ green edges along an index $j \in S_2$ which are also in $\mathcal{L}_{n'}$. In total, we have a set of indices $\mathcal{L}_{n'}$ of size $\frac{4}{3}(n - 4^{\lceil \log_4 n \rceil - 1}) + 4 = \frac{4}{3}(n' - 4^{\lceil \log_4 n' \rceil - 1})$ with $2^{n' - \lceil \log_4 n' \rceil - 1}$ green edges and a set of indices $\mathcal{M}_{n'}$ of size $\frac{1}{3}(4^{\lceil \log_4 n' \rceil} - n')$ with $2^{n' - \lceil \log_4 n' \rceil}$ green edges which proves that the theorem holds for this case. We note that the indices in \mathcal{M}_n except the chosen index i are now in $\mathcal{M}_{n'}$ and that the indices in \mathcal{L}_n and S_2 form the set $\mathcal{L}_{n'}$. In particular, the chosen index i belongs to $\mathcal{L}_{n'}$.

This proves the theorem by induction for all $n = 3l + 1$ for an integer $l \geq 1$.

□

We compile the following observations about the sets $\mathcal{L}_{n'}$ and $\mathcal{M}_{n'}$ made in the above proof that will prove to be useful later to characterize the vertex-coloring of the fully colored hypercubes.

Observation 5.9. *In a fully colored hypercube $\tilde{H}_{n'} = \tilde{H}_n \square \tilde{H}_4$ constructed as in the proof of Theorem 5.7, we have the following:*

- *The index i , chosen to split \tilde{H}_n into two halves, belongs to $\mathcal{L}_{n'}$.*
- *Every index $j \in \mathcal{M}_n$ except i belongs to $\mathcal{M}_{n'}$.*
- *Every index $j \in S_2$ belongs to $\mathcal{L}_{n'}$.*
- *When n is a power of 4, the set $S_1 \setminus \{i\}$ belongs to $\mathcal{M}_{n'}$.*

We use Theorem 5.7 to obtain an upper bound on the frustration index of a signature that assigns a negative sign to every 4-cycle in a hypercube.

Proof of Theorem 5.4. Since every 4-cycle in the fully colored hypercube constructed in Theorem 5.7 has an odd number of red and blue edges, setting the signs of all the edges of a color, say red, to negative and the rest to positive ensures that every 4-cycle will be negative under this signature. Since the number of red edges equals the number of blue edges along each index as seen in Observation 5.6, we will count the number of green edges to find the number of red edges. We analysis this in various cases as below:

Case 1: n is a power of 4.

Let $n = 4^k$ for an integer $k \geq 1$. From Theorem 5.7, we have n indices with 2^{n-k-1} green edges each. Since $\sqrt{n} = 2^k$, the total number of green edges is $\sqrt{n} (2^{n-1})$. Each of the edges in a fully colored hypercube is colored green, red or blue and there are an equal number of red and blue edges. Hence the total number of edges colored red is $\frac{n2^{n-1} - \sqrt{n}2^{n-1}}{2}$. The frustration index when n is a power of 4 is,

$$F(H_n, \sigma^*) \leq (n - \sqrt{n}) \cdot 2^{n-2}.$$

By Theorem 5.2, in this case, $F(H_n, \sigma^*) = (n - \sqrt{n}) \cdot 2^{n-2}$.

Case 2: $n = 3l + 1$ for an integer $l \geq 1$.

Let $4^k < n \leq 4^{k+1}$. From Theorem 5.7, the number of green edges in \tilde{H}_n is

$$g_n = \frac{4}{3} (n - 4^k) 2^{n-k-2} + \frac{1}{3} (4^{k+1} - n) 2^{n-k-1} = \frac{2^{n-1}}{3 \cdot 2^k} (n + 2 \cdot 4^k).$$

The number of red edges is

$$b_n = \frac{n \cdot 2^{n-1} - \frac{2^{n-1}}{3 \cdot 2^k} (n + 2 \cdot 4^k)}{2} = 2^{n-2} \left(n - \frac{(n + 2 \cdot 4^k)}{3 \cdot 2^k} \right).$$

Case 3: $n \not\equiv 1 \pmod{3}$.

In this case, we take the fully colored hypercube \tilde{H}_{n+m} constructed as above such that $n + m = 3l + 1$ for an integer $l \geq 1$ and $m \in \{1, 2\}$. The main idea is to remove all the edges corresponding to m coordinates in \tilde{H}_{n+m} , which results in 2^m components and we choose the component with the least number of blue edges. We will now formalize this intuition and count the number of red edges.

Let $n + m \equiv 1 \pmod{3}$ be such that $4^k < n + m \leq 4^{k+1}$ for an $m \in \{1, 2\}$. From Theorem 5.7, \tilde{H}_{n+m} has $\frac{4}{3}(n + m - 4^k)$ indices with $2^{n+m-k-2}$ green edges and $\frac{1}{3}(4^{k+1} - n - m)$ indices with $2^{n+m-k-1}$ green edges. We delete all the edges from m indices that have $2^{n+m-k-2}$ green edges. This is possible since $\frac{4}{3}(n + m - 4^k) \geq 4 > m$. The total number of green edges along n indices, after m coordinates have been deleted is

$$\frac{1}{3} (4n + m - 4^{k+1}) 2^{n+m-k-2} + \frac{1}{3} (4^{k+1} - n - m) 2^{n+m-k-1} = \frac{2^{n+m-k-2}}{3} (2n - m + 4^{k+1}).$$

This subgraph of \tilde{H}_{n+m} , formed after deleting all the edges in m indices, consists of 2^m components each of which is a fully colored hypercube of dimension n . We pick a component of this subgraph with the largest number of green edges which is at least

$$g_n \geq \frac{1}{2^m} \left(\frac{2^{n+m-k-2}}{3} (2n - m + 4^{k+1}) \right) = \frac{2^{n-k-2}}{3} (2n - m + 4^{k+1}).$$

The number of red edges is bounded as follows:

$$b_n \leq \frac{n \cdot 2^{n-1} - \frac{2^{n-k-2}}{3} (2n - m + 4^{k+1})}{2} = 2^{n-2} \left(n - \frac{(n + 2 \cdot 4^k - \frac{m}{2})}{3 \cdot 2^k} \right).$$

□

We now turn our attention to the function that is used for vertex-coloring in the construction of \tilde{H}_n . We see that this function corresponds to the well-studied Ambainis function [2, 14, 11] when $n = 4^k$.

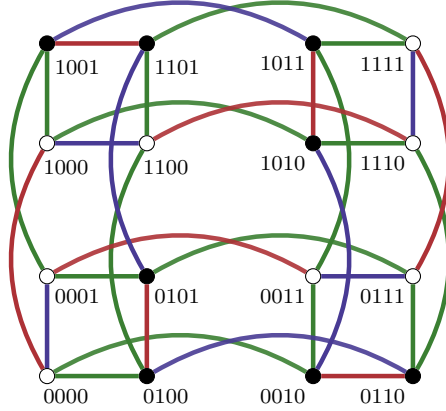


Figure 7: A fully colored \tilde{H}_4 with vertices being colored according to Ambainis function

5.3 Ambainis function

We observe that the recursive function we used to determine the vertex-coloring in \tilde{H}_{3l+1} for any integer l is an extension of the Ambainis function (which were only defined for powers of 4). This is easy to see in \tilde{H}_4 (Figure 7) where a vertex x is colored “white” if the corresponding string is evaluated to 1 by the Ambainis function (Definition 2.5), i.e., $f_A(x) = 1$.

We now show that the Ambainis function is composed with itself to get the vertex-coloring for fully colored hypercubes with larger dimensions. Let the vertex-coloring of \tilde{H}_k be denoted by f_k . Recall that a fully colored hypercube \tilde{H}_{3l+4} is constructed from \tilde{H}_{3l+1} and \tilde{H}_4 for any integer $l \geq 1$ by splitting \tilde{H}_{3l+1} across an index i and replacing each vertex of \tilde{H}_4 by a half of \tilde{H}_n which depends on the vertex-coloring f_4 . We get f_{3l+4} from f_{3l+1} after replacing the i -th input by an f_4 . This is because f_4 determines the half of the hypercube from which the vertex inherit colors and the two halves differ only at the i th bit. We show that such a construction yields a vertex-coloring function that can be represented by a 4-ary tree with each internal node replaced by the Ambainis function f_A which acts on its 4 input bits.

Theorem 5.10. *For a fully colored hypercube \tilde{H}_k of dimension k where $k = 3l + 1$ for an integer $l \geq 1$, the vertex-coloring function f_k can be represented by a 4-ary tree whose leaves correspond to input bits of the function. Every internal node in the tree corresponds to the Ambainis function f_A and has exactly 4 children. The leaves lie at either the last or the penultimate level and the height of this tree is $\lceil \log_4 k \rceil + 1$.*

This tree corresponds to the Ambainis function f_A^d when $k = 4^d$ for any integer $d \geq 1$. We denote such a tree with k leaves by T_k . The tree T_{73} is given in Figure 8.

Proof of Theorem 5.10. Our proof is based on induction on $l \geq 1$. The induction hypothesis is as follows.

Induction Hypothesis: For a $k = 3l + 1$, the vertex-coloring of \tilde{H}_k is represented by the tree T_k . If the leaf corresponding to an index j lies in the last level of the tree T_k , $j \in \mathcal{L}_k$ and if it lies in the penultimate level, $j \in \mathcal{M}_k$.

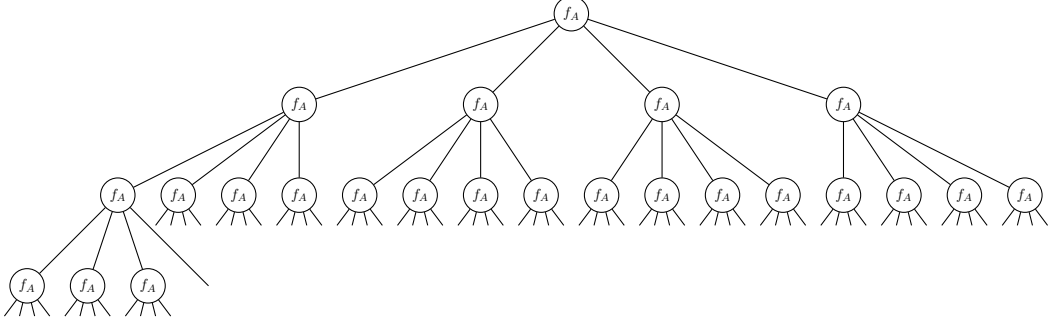


Figure 8: A vertex-coloring function for \tilde{H}_{73}

Base Case: For the base case, the hypercube of dimension $k = 4$ when $l = 1$ is given in Figure 7 whose vertex-coloring is according to the Ambainis function f_A on 4 bits. This is represented by a T_4 that takes as input 4 bits and outputs according to f_A . In this case, all the leaves lie in the same level and we also know that all the indices lie in \mathcal{L}_4 since k is a power of 4.

Induction Case: For the induction case, let us assume that the theorem holds for $k = 3l + 1$. The hypercube corresponding to $3l + 4$ is constructed from \tilde{H}_{3l+1} and \tilde{H}_4 where \tilde{H}_{3l+1} is split into halves across an index i . The vertex-coloring f_{3l+4} is given by f_{3l+1} with the i -th input bit replaced by an f_4 by construction. By induction hypothesis, f_{3l+1} is given by a 4-ary tree T_{3l+1} with the internal nodes being f_A . By the construction of \tilde{H}_{3l+4} , the tree corresponding to f_{3l+4} is T_{3l+1} with the leaf corresponding to the i -th index replaced by an f_A node with 4 children. We will make use of Observation 5.9 to show that the tree corresponding to f_{3l+4} increases in height from that of T_{3l+1} only if all the leaves of T_{3l+1} are at the same level, i.e., the last level. We consider the following two cases:

- $3l + 1$ is a power of 4: all the indices in S_1 lie in \mathcal{L}_{3l+1} and are in the last level of T_{3l+1} by induction hypothesis and Observation 5.9. In the construction, one of these indices is chosen as i and if T_{3l+1} is modified by changing the i -th leaf to f_A , this tree matches the description of T_{3l+4} . Note that the new leaves of the f_A node that replaced a leaf of T_{3l+1} are now indices in S_2 . From Observation 5.9, we have that every index in $S_1 \setminus \{i\}$ lies in the penultimate level and are in \mathcal{M}_{3l+4} . We also have that every index in S_2 that lies in the last level belongs to \mathcal{L}_{3l+4} , thus proving the induction case.
- $3l + 1$ is not a power of 4: the set \mathcal{M}_{3l+1} is non-empty and by the induction hypothesis, there are leaves in the penultimate level of T_{3l+1} . In the construction of \tilde{H}_{3l+4} , the tree representing the vertex-coloring f_{3l+4} can be obtained by modifying T_{3l+1} by taking the leaf corresponding to an index $i \in \mathcal{M}_{3l+1}$ and replacing it by a node f_A with its leaves now being labelled by S_2 . Such a tree is consistent with the description of T_{3l+4} as the leaf being replaced lies in the penultimate level. Since all the other leaves in the penultimate level remain at the same level and since the leaves corresponding to S_2 now lie in the last level, these are consistent with the new sets of indices \mathcal{M}_{3l+4} and \mathcal{L}_{3l+4} by Observation 5.9. This proves the induction case.

This shows that the tree T_k represents the vertex-coloring for \tilde{H}_k and that the vertex-coloring is given by the Ambainis function. \square

6 Further discussion

In this section we look at some further implications of techniques developed in the previous sections. We look at how they can be used to prove analogous results in weighted hypercubes and what they imply for cases where the maximum degree of the subgraph is close to \sqrt{n} .

6.1 Weighted version

Let $\{a_i\}$ for $i = 1, 2, \dots, n$ be a sequence of nonzero real values. Consider the following weight assignment to the edges of H_n : if the edge (x, y) corresponds to the coordinate i (i.e., x and y differ only at the coordinate i), then the edge (x, y) is assigned a weight a_i . Furthermore, we multiply the weight of the edge (x, y) to the signature of the edge $\sigma(xy)$ which was defined in the previous sections. The adjacency matrix of the corresponding signed weighted graphs has exactly two eigenvalues: $\pm\sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$. This would follow from Theorem 6.7 of [1] if one follows the proof steps to complete the proof of Corollary 6.9 starting with K_2 . However, without the use of this fact and with a modification of our proof of Theorem 1.1, we can give a proof of the following weighted version of Huang's theorem, first proved by Mathews[15] using Clifford algebras.

Theorem 6.1. *Given a weighted hypercube H_n where the edges corresponding to coordinate i are given a weight a_i with $a_i \neq 0$, in the induced subgraph corresponding to any set of $2^{n-1} + 1$ vertices, there exists a vertex x whose sum of weights of incident edges is at least $\sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$.*

Proof. Once the definitions of the vectors x^+ and x^- are modified, the rest of the proof is exactly the same. The x^{th} coordinate of x^+ is set as $\sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$ and that of x^- is set as $-\sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$. If y is adjacent to x and $y = x + e_i$, then the y^{th} coordinate of both x^+ and x^- is set as $\sigma(xy)a_i$. All other coordinates are set to 0. \square

6.2 Tightness

To summarize our result, what we have proved here is that in a subgraph induced by a set of vertices with a linear dependency, the maximum degree lies between n and \sqrt{n} , with both extremities being tight. Furthermore, we proved that the closer we are to the lower bound the more vertices we must have from the second and the third neighborhood of a vertex with maximum degree. However, this is not the limit of our approach, it can imply more vertices from the fourth neighborhood and so on. While we do not yet have the strongest claim to present, we have the following observation in the case where the maximum degree is exactly \sqrt{n} (assuming that n is a perfect square).

Theorem 6.2. *If U is a nontrivial linear dependency relation and $H_n[U]$ has maximum degree exactly \sqrt{n} , then the vertices in $H_n[U]$ of degree \sqrt{n} induce a \sqrt{n} -regular subgraph.*

This theorem is corollary of our proof of Lemma 4.1 and its details is left to the reader.

6.3 Linear dependency

We remark that the vectors x^+ and x^- were built by further investigating Huang's proof using eigenvalues. The set V^+ generated by $\{x^+ \mid x \in V(H_n)\}$ is the eigenspace corresponding to the eigenvalues \sqrt{n} of the incidence matrix of the signed graph (H_n, σ) and the set V^- generated by $\{x^- \mid x \in V(H_n)\}$ is eigenspace corresponding to the eigenvalue $-\sqrt{n}$. This provides an alternate proof for the fact that V^+ is orthogonal to V^- and that each is of dimension (at most) 2^{n-1} .

Our approach to this problem suggests a strong connection between linear dependency of the vectors x_i^+ in the study of the sensitivity of a function. It is, therefore, intriguing to ask:

Problem 6.3. *What are the (minimal) subsets of V^+ that are linearly dependent?*

The smallest linear dependency is among a vertex and all its neighbors:

$$x^+ = \frac{1}{\sqrt{n}} \sum_{y \sim x} \sigma(xy) y^+.$$

On the other hand for linearly independent sets, the easiest examples are sets I of vectors x^+ 's where for every vector $x^+ \in I$ there exists a coordinate $u \in V(H_n)$ such that x^+ is the only vector in I that is nonzero at u . We call such a linearly independent set a *basic linearly independent set*. The main example of a basic linearly independent set is the set $\{u^+ \mid u \in U^{\text{odd}}\}$ or $\{v^+ \mid v \in U^{\text{even}}\}$. Each of these sets provides an orthogonal basis for the V^+ .

Another example of basic linearly independent set is the set of all u^+ where, for a fixed i , the i^{th} coordinate of u is 1. Then for each u^+ of this set, the vector u^+ is the only vector of the set that is not 0 at the coordinate $u + e_i$. Thus taking all such vectors provides another basis for V^+ , but this basis is no longer an orthogonal one. The proof of Huang's result given by Knuth in [13] uses one such basis with $i = n$.

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