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A reconstruction of quantum theory for spinning particles

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Abstract

As part of a probabilistic reconstruction of quantum theory (QT), we show that spin is not a purely quantum mechanical phenomenon, as has long been assumed. Rather, this phenomenon occurs before the transition to QT takes place, namely in the area of the quasi-classical (here better quasi-quantum) theory. This borderland between classical physics and QT can be reached within the framework of our reconstruction by the replacement $\mathbf{p} \rightarrow \mathbf{M}(\mathbf{q}, \mathbf{t})$, where \mathbf{p} is the momentum variable of the particle and $\mathbf{M}(\mathbf{q}, \mathbf{t})$ is the momentum field in configuration space. The occurrence of spin, and its special value $\mathbf{1}/\mathbf{2}$, is a consequence of the fact that $\mathbf{M}(\mathbf{q}, \mathbf{t})$ must have exactly three independent components $M_k(\mathbf{q}, \mathbf{t})$ for a single particle because of the three-dimensionality of space. In the Schrödinger equation for a “particle with spin zero”, the momentum field is usually represented as a gradient of a single function \mathcal{S} . This implies dependencies between the components $M_k(\mathbf{q}, \mathbf{t})$ for which no explanation exists. In reality, $\mathbf{M}(\mathbf{q}, \mathbf{t})$ needs to be represented by three functions, two of which are rotational degrees of freedom. The latter are responsible for the existence of spin. All massive structureless particles in nature must therefore be spin-one-half particles, simply because they have to be described by $\mathbf{4}$ real fields, one of which has the physical meaning of a probability density, while the other three are required to represent the momentum field in three-dimensional space. We derive the Pauli-Schrödinger equation, the correct value $\mathbf{g} = \mathbf{2}$ of the gyromagnetic ratio, the classical limit of the Pauli-Schrödinger equation, and clarify some other open questions in the borderland between classical physics and QT.

MSC Classification: 81P05 , 81S99 , 60A99 , 70H99

1 Introduction

”Niels Bohr brainwashed a whole generation of theorists into thinking that the job (interpreting quantum theory) was done 50 years ago”

Murray Gell-Mann (1929-2019)

Quantum theory (QT) is the subject of controversial discussions since its invention more than ninety years ago. The phenomenon of spin, discovered at about the same time, plays a particularly mysterious role, because of the disturbing fact that we are unable to identify a classical counterpart for this ”classically non-describable two-valuedness” [1]. One suspects that a satisfying explanation of spin will not be possible without a better understanding of QT as a whole. There are two fundamentally different ways to interpret the formalism of QT. The first way, mainly propagated by Bohr, claims that QT is able to describe individual particles. In contrast to this “individuality interpretation”, Einstein’s “ensemble interpretation” asserts that QT is only able to describe statistical ensembles [2]. It may seem strange that a theory whose dynamical predictions are statistical in nature should be able to describe individual particles. As a matter of fact, however, Bohr’s interpretation is predominant at the moment; it is accompanied by ongoing discussions about various internal contradictions.

The reference theory for any interpretation of QT must be part of classical physics. In the context of the individuality interpretation, this reference theory is classical mechanics (CM). Due to the fundamental differences in the mathematical formulation of both theories (ordinary differential equations and observables defined as ordinary functions in CM, as opposed to partial differential equations and observables defined as operators in QT), the similarities between QT and CM are limited to structural (mathematical) similarities between both theories. A satisfactory understanding of QT in this formal sense would be achieved if it were possible to map the structural properties of classical observables in a one-to-one manner to corresponding properties of quantum mechanical operators. However, such a general process called “quantization” does not exist, as Groenewold has shown [3].

In the probabilistic version of CM, which we call probabilistic mechanics (PM), the object to be examined is not one (or several) individual particles, but a statistical ensemble of (one or several) particles. The observable quantities are expectation values, and the fundamental dynamical variable, the probability density ρ , satisfies a partial differential equation (the Liouville equation). Thus, in contrast to CM, the mathematical formulation of PM is very similar to that of QT. This similarity allows for a completely different approach when studying the relation between QT and classical physics, namely a “derivation” or “reconstruction” of one of the two theories from the other. The relevance of the probabilistic component for a proper understanding of

QT was recognized very early by Van Vleck [4], Schiller [5],[6], Bopp [7],[8] and others.

This work is the fourth in a series of papers that attempt to understand QT on the basis of Einstein's ensemble interpretation. These works will be referred to as I [9], II [10], and III [11]. According to the ensemble interpretation, the natural starting point for a reconstruction of QT is PM. In I-III it was shown that it is possible, by choosing this starting point, to derive not only Schrödinger's equation, but also most other fundamental properties of QT, such as the non-commutative structure of operators, Born's rule and others. In the present work it will be shown that even the strange phenomenon of spin may be derived using a refined version (actually a corrected version) of the theory presented in III.

The mathematical similarity between both theories allows a derivation of QT from PM that is very simple, at least in conceptual terms. The main formal difference between PM and QT is the number of independent variables, as can be seen immediately if one descends from Hilbert space to the most concrete formulation of QT in terms of differential equations. Obviously, PM is a theory in phase space with, neglecting time at the moment, $2n$ independent variables, while QT is a theory in configuration space with n independent variables. Thus, a reconstruction of QT must necessarily contain a projection from phase space to configuration space. As a second fundamental step a linearization or randomization must be performed, as explained in detail in III. A theory containing these two steps, regardless of the order, is referred to as "Hamilton-Liouville-Lie-Kolmogorov theory" (HLLK).

In works I and II, the first version of the HLLK was used, in which the linearization is performed first followed by the projection. In III and the present work, the projection is performed first and then the linearization (or randomization) follows as second step. This second version of the HLLK allows for a deeper understanding of the relationship between QT and classical physics. Let us also note, that in II a variety of different observables A was investigated, while in III and the present work we study only the particular sector $A = H$ of HLLK related to spin.

The present work follows the same general scheme as III, but the actual implementation is more involved. In the following section 2 we introduce, as in III, a general momentum field. As was shown in III, it is not possible to use the components M_k of the momentum field themselves as dynamical variables because they are in general not functionally independent from each other. Instead, one must take as dynamical variables certain independent functions, called potentials, which may be used to represent the momenta; the situation is similar as in QT. In III only irrotational momentum fields were dealt with, which means that all components M_k could be derived from a single function S . In this case the transition from the M_k to the new dynamic variable S (which later becomes the quantum mechanical phase) is obvious. In the present work we need a larger number of potentials S, Q_α, P_α in order to represent general momentum fields, that also have vortical components. The

fields Q_α, P_α called Clebsch potentials describe these vortical components and are ultimately responsible for the spin. The equations of motion for the new dynamic variables S, Q_α, P_α were derived by the mathematician H. Rund [12]. In section 3 we give a brief overview of Rund's theory and then reproduce his results within the present formalism.

In the emerging quasi-quantal or quasi-classical theory, which is referred to as QA, particle trajectories do still exist, but are only locally valid. In the following sections we restrict ourselves to the case $N = 1$ of a single particle, where besides S and ρ , only two additional (vortical) variables Q, P occur. The QA (for $N = 1$) is structurally identical to the theory of inviscid barotropic fluids; this correspondence is only of a formal nature and it should be born in mind that the physical meaning of the variables is quite different. We use some results from fluid mechanics concerning vortex lines. In section 4 we implement a topological restriction for the variables Q, P . We assume that the mapping of \mathbb{R}^3 onto the space of the Q, P describes linked vortex lines (the linking number is a topological invariant). The suitable mapping is given by the famous Hopf map, which implies the replacement of the Q, P with new *canonical* Clebsch variables ϑ and φ . This map also provides the definition of a suitable two-component state variable (spinor) ψ which is of central importance for the transition to QT and for the implementation of the minimal-coupling rule. As shown in section 6 the special form of Hopf's map is closely related to the fact that the parameter space of the rotation group is doubly connected.

Using the new state variable ψ , we construct in section 5 a semi-linear form of the evolution equations, which differs from the Pauli-Schrödinger equation (without external electromagnetic field) only by a non-linear term. Only when using this form of the evolution equations is it possible (see section 6) to implement the minimal-coupling-rule in a natural way and to derive the correct g-factor of 2. Section 6 also contains a discussion of the concept of the "magnetic moment of the electron".

The transition to QT is carried out in section 7 by linearization with respect to the variable ψ . This can be done very easily by eliminating the non-linear term in the semi-linear equation. Section 7 also examines which terms have to be added to the original evolution equations for $\rho, S, \vartheta, \varphi$ in order to generate the linearization.

The justification for performing the linearization, discussed in more detail in III, is that a large number of globally valid solutions can only be generated in a linear theory (because of the superposition principle). This point is discussed in more detail in III. While this reasoning is understandable, the process of eliminating a term in a differential equation may seem somewhat formal, if not crude. One wonders what exactly happened during the discontinuous transition from the quasi-classical equations to the quantum equations. In section 8 this discontinuous process is "resolved" and a statistical theory is constructed which is as similar as possible to the quasi-classical theory, but which does not claim to describe individual particles at all. This theory, whose basic assumptions are the same as in III, represents the most detailed version of the HLLK.

In the last section 9 we list the results obtained so far, taking into account both versions of the HLLK, both the first version used in I and II and the second version used in III and here. Then we go into more detail on some important points. In particular, we emphasize that quantum spin cannot be a localizable property of individual particles, but a collective property of the probabilistic single-particle ensemble, which is related to the possibility of rotations in three-dimensional space. The three-dimensionality of space is also the reason for the special value $1/2$ of the spin. Other points that will be discussed are the classical limit of the Pauli equation and the role of potentials in quantum theory. We conclude with a few remarks concerning the interpretation of the quantum theoretical formalism.

2 Basic equations

In this section we briefly recall all those relations from III that we need for our spin theory; detailed explanations may be found in III. The state of a system of N particles is described by $n = 3N$ coordinates $q = q_1, \dots, q_n$ and n conjugate momenta $p = p_1, \dots, p_n$. Particle trajectories $q_k(t), p_k(t)$ are given by the solutions of the canonical equations

$$\dot{q}_k = \frac{\partial H(q, p)}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H(q, p)}{\partial q_k}, \quad (1)$$

with Hamilton's function $H(q, p)$ not depending on time t . A particular trajectory in the $2n$ -dimensional phase space $\Omega = \mathbb{R}_q^n \times \mathbb{R}_p^n$ may be labeled by its state q^0, p^0 at an initial time t_0 . The solutions of (1) are written in the form

$$q_k = Q_k(t, t^0, q^0, p^0), \quad p_k = P_k(t, t^0, q^0, p^0), \quad (2)$$

where the dependence on t_0 will often be suppressed. The variables q, p are "Lagrangian coordinates" representing particle properties. A statistical ensemble is defined as the uncountable set of all solutions (2). Describing ensembles with the help of "Eulerian coordinates", which are denoted by the same symbols q, p but represent points of Ω , is much more convenient. The most important Eulerian dynamic variable is the probability density $\rho(q, p, t)$, ruling the time-dependent distribution of trajectories in phase space, which obeys the Liouville equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial \rho}{\partial p_k} \frac{\partial H}{\partial q_k} = 0. \quad (3)$$

In order to perform the transition from probabilistic mechanics (PM) to QT we need a second Eulerian variable, the action variable $S(q, p, t)$, which obeys

the differential equation

$$\frac{\partial S}{\partial t} + \frac{\partial S}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial S}{\partial p} \frac{\partial H}{\partial q} = \bar{L}, \quad (4)$$

which will be referred to as action equation. The quantity \bar{L} (the Lagrangian) is defined by

$$\bar{L} = \bar{L}(q, p, t) = p \frac{\partial H(q, p)}{\partial p} - H(q, p). \quad (5)$$

As a first step on our way from PM to QT, we have to perform the projection of the basic equations (1), (3), (4) onto configuration space. We replace the particle momentum p at each instant of time t by a n -component momentum field M :

$$p_k \rightarrow M_k(q, t). \quad (6)$$

Thus, $2n$ -dimensional phase space is projected to a n -dimensional subspace $\mathbb{M} = \{(q, p) \in \Omega \mid p = M(q)\}$ which is parametrized by the configuration space coordinates q_k [13].

The projection of the canonical equations (1) leads to the differential equations [12, 14]

$$\dot{q}_k = v_k(q, t), \quad (7)$$

$$\frac{\partial M_i(q, t)}{\partial t} + \left[\frac{\partial M_i(q, t)}{\partial q_l} - \frac{\partial M_l(q, t)}{\partial q_i} \right] v_l(q, t) = - \frac{\partial}{\partial q_i} h(q, t), \quad (8)$$

where the fields $v(q, t)$, $h(q, t)$ are defined in terms of $H(q, p)$, $V(q, p)$ by

$$v_k(q, t) = V_k(q, M(q, t)), \quad (9)$$

$$h(q, t) = H(q, M(q, t)), \quad (10)$$

$$V_k(q, p) = \frac{\partial H(q, p)}{\partial p_k}. \quad (11)$$

Eq. (8) will be referred to as canonical condition. As a result of the projection to configuration space the $2n$ ordinary differential equations (1) are replaced by n ordinary differential equations (7), for the particle positions q , and n partial differential equations (8), for the momentum field M .

A useful quantity, characterizing the purely rotational part of the momentum field, is the vorticity tensor Ω_{ik} , defined by

$$\Omega_{ik}(q, t) := \frac{\partial M_k(q, t)}{\partial q_i} - \frac{\partial M_i(q, t)}{\partial q_k}. \quad (12)$$

Its equation of motion, which may easily be derived from the canonical condition, shows that the time-dependence of Ω_{ik} is completely determined by the

solutions of the particle equations of motion (7). As a consequence, performing a standard calculation from fluid mechanics, one may show that

$$\Omega_{ik}(q, t) = \Omega_{l_j}^0(q) \frac{\partial q_l^0(q, t)}{\partial q_i} \frac{\partial q_j^0(q, t)}{\partial q_k}. \quad (13)$$

Here, the functions $q_l^0(q, t)$ are obtained by inverting the solutions $q_l(t, q^0)$ of (7); the quantities q^0 and $\Omega_{l_j}^0(q)$ are the initial values of $q_l(t)$ and $\Omega_{ik}(q, t)$. This relation shows explicitly that the time dependence of the vorticity tensor is determined by the flow of the particle equations of motion. In particular, $\Omega_{l_j}^0(q) = 0$ implies $\Omega_{ik}(q, t) = 0$ for all future times.

The projection of the Liouville equation to configuration space leads to the continuity relation

$$\frac{\partial \rho(q, t)}{\partial t} + \frac{\partial}{\partial q_k} \rho(q, t) v_k(q, t) = 0 \quad (14)$$

for the probability density in configuration-space $\rho(q, t)$. The latter is defined by the relation $\rho(q, p, t) = \rho(q, t) \delta(p - M(q, t))$, where δ is the n -dimensional delta function. The projection of the action equation to configuration space may be put in the form

$$\frac{\partial s}{\partial t} + v_k \left[\frac{\partial s}{\partial q_k} - M_k(q, t) \right] + h = 0, \quad (15)$$

where the action field on configuration space $s(q, t)$ is defined by

$$s(q, t) = S(q, M(q, t), t). \quad (16)$$

Relation (15) may also be obtained by means of a Lagrangian to Eulerian transition of the action integral in configuration space.

As shown in III, the n components $M_1(q, t), \dots, M_n(q, t)$ of the momentum field are generally not functionally independent from each other. They are therefore unsuitable as dynamic variables and must be replaced by suitable, functionally independent quantities. The problem of finding these new variables for arbitrary vector fields was solved by Pfaff [15]. For the present nonrelativistic problem, we need the part of Pfaff's solution relating to fields with an odd number L of independent functions. In this case there is an integer m given by $L = 2m + 1$ and M may be written in the form

$$M_k(q, t) = \frac{\partial S(q, t)}{\partial q_k} + P_\alpha(q, t) \frac{\partial Q_\alpha(q, t)}{\partial q_k}, \quad (17)$$

where the Clebsch potentials [16] $S(q, t)$, $P_\alpha(q, t)$, $Q_\alpha(q, t)$ are $2m + 1$ independent functions of q_1, \dots, q_n, t (Greek indices α, β, \dots run from 1 to m and double

occurrence of these indices entails a summation from 1 to m). Inserting (17) in the definition (12) we see that the components

$$\Omega_{ij} = \frac{\partial P_\alpha}{\partial q_j} \frac{\partial Q_\alpha}{\partial q_i} - \frac{\partial Q_\alpha}{\partial q_j} \frac{\partial P_\alpha}{\partial q_i}. \quad (18)$$

of the vorticity tensor will generally be different from zero. As a consequence, if we consider a closed loop γ , lying in \mathbb{M} at time t , then the closed path integral

$$\oint_\gamma dq_k M_k(q, t), \quad (19)$$

will be different from zero. Thus, the components $P_\alpha(q, t)$, $Q_\alpha(q, t)$ of a general momentum field ($m > 0$) describe a *vortical* state of motion of the probabilistic ensemble; they will sometimes be referred to as “vortical variables”. In contrast, for the irrotational momentum fields considered in III the integral (19) vanishes for arbitrary t (assuming that $S(q, t)$ is single-valued).

In III the case $m = 0$ of an irrotational momentum field was studied. This case is unphysical but important because it leads to Schrödinger’s equation. The determination of the equation of motion for the single Clebsch potential S in III was simple because both the canonical condition and the action equation reduce in this case to a Hamilton-Jacobi equation. In the following, some basic results for the general case $m > 0$ are reported first; the transition to QT is carried out later for $m = 1$.

3 Evolution equations

Our task is to derive equations of motion for the new variables q_k , $S(q, t)$, $P_\alpha(q, t)$, $Q_\alpha(q, t)$ from the equations of motion (7), (8) for the old variables q_k , $M_k(q, t)$. This is an easy task as regards the particle equation of motion (7); one only has to insert the expansion (17) in Eq. (7),

$$\dot{q}_k = V_k \left(q, \frac{\partial S(q, t)}{\partial q} + P(q, t) \frac{\partial Q(q, t)}{\partial q} \right). \quad (20)$$

The less trivial problem of determining the evolution equations for the Clebsch potentials was solved by the mathematician H. Rund more than four decades ago [12]. Rund’s original work contained an unnecessary restriction which was removed shortly afterwards by Baumeister [17]. This extension will of course be taken into account in the present work. In this section we first give a brief overview and discussion of Rund’s theory and then clarify the relationship between his variables and those used in this work.

3.1 Outline of Rund’s theory

We give a brief outline of Rund’s theory for the convenience of the reader. If the expansion (17) is inserted in the canonical condition (8) the resulting

relation may be put in the form

$$\frac{\partial}{\partial q_i} T(q, t) + \frac{\partial Q_\alpha}{\partial q_i} \frac{DP_\alpha}{Dt} - \frac{\partial P_\alpha}{\partial q_i} \frac{DQ_\alpha}{Dt} = 0, \quad (21)$$

where

$$T(q, t) := H\left(q, \frac{\partial S}{\partial q} + P \frac{\partial Q}{\partial q}, t\right) + \frac{\partial S}{\partial t} + P_\alpha \frac{\partial Q_\alpha}{\partial t}, \quad (22)$$

and the total derivatives of P_α and Q_α with respect to time are defined by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V_i \left(q, \frac{\partial S}{\partial q} + P \frac{\partial Q}{\partial q}, t \right) \frac{\partial}{\partial q_i}. \quad (23)$$

In fluid mechanics these total derivatives are referred to as convective derivatives or material derivatives.

In the next step, one takes advantage of the functional independence of S , P_α , Q_α . As a consequence a one-to-one correspondence between the n variables S , P_α , $Q_\alpha, q_{2m+2}, \dots, q_n$ and the n variables q_1, \dots, q_n must exist at each instant of time. As a further consequence, a function Φ , depending on S , P_α , $Q_\alpha, q_{2m+2}, \dots, q_n$ must exist which fulfills the relation

$$\Phi(t, S, P_\alpha, Q_\alpha, q_{2m+2}, \dots, q_n) = T(q_1, \dots, q_n, t). \quad (24)$$

In two further important steps it is shown that Φ depends neither on S nor on q_{2m+2}, \dots, q_n ; the latter result is required for $2m+1 < n$ [17]. The final result of Rund's theory is given by the dynamic equations

$$\frac{\partial S}{\partial t} + P_\alpha \frac{\partial Q_\alpha}{\partial t} + H\left(q, \frac{\partial S}{\partial q} + P \frac{\partial Q}{\partial q}, t\right) - \Phi(Q, P, t) = 0, \quad (25)$$

$$\frac{DP_\alpha}{Dt} = - \frac{\partial \Phi(Q, P, t)}{\partial Q_\alpha}, \quad (26)$$

$$\frac{DQ_\alpha}{Dt} = \frac{\partial \Phi(Q, P, t)}{\partial P_\alpha}, \quad (27)$$

where the total derivatives are defined by (23). The function $\Phi(Q, P, t)$ is completely arbitrary.

Besides these fundamental equations we quote from Rund's work the following important relation which is obtained by differentiating Eq. (25) with respect to q_k :

$$\left[\frac{DM_k}{Dt} + \frac{\partial H}{\partial q_k} \right] - \frac{\partial Q_\alpha}{\partial q_k} \left[\frac{DP_\alpha}{Dt} + \frac{\partial \Phi}{\partial Q_\alpha} \right] + \frac{\partial P_\alpha}{\partial q_k} \left[\frac{DQ_\alpha}{Dt} - \frac{\partial \Phi}{\partial P_\alpha} \right] = 0. \quad (28)$$

Equation (25) is referred to by Rund as "generalized Hamilton-Jacobi equation". Indeed, if we set $Q_\alpha = P_\alpha = \Phi = 0$, Eq. (25) reduces to the

Hamilton-Jacobi equation. Equations (26), (27) are referred to as "associated canonical equations", as they may be interpreted as ordinary differential equations of canonical form, with a "Hamiltonian" $\Phi(Q, P, t)$, for the "particle variables" $P_\alpha(t) = P_\alpha(q(t), t)$, $Q_\alpha(t) = Q_\alpha(q(t), t)$.

As a first step towards a physical interpretation of these equations we note the following points:

- The "Hamiltonian" Φ is arbitrary; no physical result can depend on its functional form. We are therefore allowed to set $\Phi = 0$. Rund has already shown, using his theory of Clebsch gauge transformations, that a gauge with vanishing Φ may be introduced [12]; see also [18].
- The relations (26), (27) reduce for $\Phi = 0$ to the condition that the potentials Q_α, P_α are constant if transported along the velocity field $V\left(q, \frac{\partial S}{\partial q} + P \frac{\partial Q}{\partial q}, t\right)$.
- Equation (28) shows that the generalized Hamilton-Jacobi equation (25) makes sure that the validity of (26), (27) implies the validity of the second canonical equation [see Eqs. (1)] and vice versa.

In view of the important role of the Hamilton-Jacobi equation for the quantum-classical transition of the Schrödinger equation, the physical meaning of the generalized Hamilton-Jacobi equation, in particular the meaning of the new variables Q_α, P_α , is of great interest. Its possible role with regard to QT has, however, not been clarified, neither by Rund himself nor by authors elaborating later on his theory [17, 19]. Samuel, who constructed an elegant phase-space version of Rund's theory wrote [19]:

It is not clear at the moment whether the generalized theory has any relevance to quantum mechanics. It does seem safe to say however that analogies with quantum mechanics, if they exist, are not straightforward and will require some unearthing.

In section 5 it will be shown that Rund's theory, as completed by the continuity equation, leads for $N = 1$ to the quasi-classical counterpart of the quantum theory of a single particle (ensemble) with spin.

3.2 Implementing the Clebsch potentials in the ensemble equations

We now use Pfaff's expansion (17) to rewrite the ensemble equations in configuration space (14), (15) in terms of the new variables S, Q_α, P_α . We expect to reproduce Rund's results and also to find relations between the quantities $S(q, p, t)$, $S(q, t)$, and $s(q, t)$.

The continuity equation (14) can be simply rewritten by inserting the expansion (17),

$$\frac{\partial \rho(q, t)}{\partial t} + \frac{\partial}{\partial q_k} \rho(q, t) V_k \left(q, \frac{\partial S}{\partial q} + P \frac{\partial Q}{\partial q}, t \right) = 0. \quad (29)$$

The continuity equation (29), the generalized Hamilton Jacobi equation (25), and the associated canonical equations (26), (27) represent a closed system of $2m + 2$ equations for the $2m + 2$ variables ρ , S , Q_α , P_α . This set of extended Rund's equations (extended by the continuity equation) is a precursor of the quasi-quantal approximation (QA) of PM (in the actual QA the variables Q , P will be replaced by other variables due to a topological restriction; see section 4). This is of course still a classical system of equations; this means that the equations for S , Q_α , P_α form a closed system that does not contain the variable ρ . Given a solution of these "deterministic" equations the motion of particles may be determined with the help of (20).

The projected action equation is given by Eq. (15), where $s(q, t)$ is now defined by

$$s(q, t) = S \left(q, \frac{\partial S}{\partial q} + P \frac{\partial Q}{\partial q}, t \right), \quad (30)$$

in terms of the phase space action $S(q, p, t)$. In order to rewrite (15) we use Pfaff's expansion (17) and introduce the symbol $\chi(q, t) = s(q, t) - S(q, t)$ for the difference between s and S . The projected action equation may then be written in the form

$$\frac{D\chi}{Dt} - P_\alpha \frac{DQ_\alpha}{Dt} + \frac{\partial S}{\partial t} + P_\alpha \frac{\partial Q_\alpha}{\partial t} + H \left(q, \frac{\partial S}{\partial q} + P \frac{\partial Q}{\partial q}, t \right) = 0, \quad (31)$$

We calculate the derivative of (15) with respect to q_i and change the order of derivatives to obtain

$$\frac{\partial}{\partial t} \frac{\partial s}{\partial q_i} + \frac{\partial}{\partial q_i} v_k \left[\frac{\partial s}{\partial q_k} - M_k(q, t) \right] + \frac{\partial}{\partial q_i} h = 0. \quad (32)$$

The canonical condition (8) and the action equation in the form (32) describe basically the same physics. However, Eq. (8) is an initial value problem for $M_k(q, t)$ while Eq. (32) describes the relation between $M_k(q, t)$ and $s(q, t)$. Both equations should agree if an appropriate representation of $M_k(q, t)$ is chosen. This requirement leads to the following condition for the Clebsch potentials:

$$\frac{\partial}{\partial q_i} \left[\frac{D\chi}{Dt} - P_\alpha \frac{DQ_\alpha}{Dt} \right] - \frac{\partial Q_\alpha}{\partial q_i} \frac{DP_\alpha}{Dt} + \frac{\partial P_\alpha}{\partial q_i} \frac{DQ_\alpha}{Dt} = 0. \quad (33)$$

This condition is fulfilled if χ , Q_α , P_α are solutions of the associated canonical equations Eqs. (26), (27) and χ fulfills the relation

$$\frac{D\chi}{Dt} - P_\alpha \frac{\partial \Phi}{\partial P_\alpha} + \Phi = 0, \quad (34)$$

where Φ is an arbitrary function of Q , P , t . The generalized Hamilton-Jacobi equation follows from (31) if Eq. (34) is taken into account. The results of

Rund's theory are thus reproduced if the ensemble equations are combined with Pfaff's expansion.

Equation (34) clarifies the relation between the Clebsch potential $S(q, t)$ and the projected action $s(q, t)$. The difference $\chi = s - S$ plays the role of an action defined for a "Hamiltonian" Φ in a $2m$ -dimensional phase space with coordinates Q_α, P_α . We may set $\Phi = 0$, thereby destroying the canonical structure of Eqs. (26), (27), (34). Condition (34) reduces for $\Phi = 0$ formally to the corresponding condition in III for the irrotational case; note however that the velocity field is defined differently. Then, for this most important gauge, χ, Q_α, P_α become quantities "moving with the flow", i.e. $\frac{D\chi}{Dt} = 0, \frac{DQ_\alpha}{Dt} = 0, \frac{DP_\alpha}{Dt} = 0$. The simplest solution of $\frac{D\chi}{Dt} = 0$ is again $\chi = 0$, just as in the irrotational case.

3.3 Interpretation of Clebsch potentials

Let us ask if the new dynamical degrees of freedom $S(q, t), P_\alpha(q, t), Q_\alpha(q, t)$ may be understood in terms of any standard concept of physics. This question is not answered in Rund's theory. With regard to the Clebsch variable S , which is also responsible for the $U(1)$ gauge mechanism, we can answer this question in the affirmative. We derived in section 3.2 the relation between $S(q, t)$ and the projected action variable $s(q, t)$. The variable $S(q, t)$ can therefore be traced back to a standard concept in phase space, namely the action variable $S(q, p, t)$. We mention without proof that this connection can also be established within the framework of the theory of canonical transformations; if we are given a complete solution of the Hamilton-Jacobi equation, we can use it to construct a projection onto the configuration space with prescribed initial values.

The question arises if the vortical variables $P_\alpha(q, t), Q_\alpha(q, t)$, may be understood in an analogous way. Can we find a structure in phase space, using a possibly extended theory of canonical transformations, which allows us to understand these variables in a similar way as $S(q, t)$? A closer look at the theory of canonical transformations shows that such an extension does probably not exist. This implies that the variables Q, P describe probably a structure which has its origin in configuration space. A different concept, valid only in configuration space, must be found if we want to understand the physical meaning of the variables Q, P .

The following two hints, as to the physical origin of Q, P , may be obtained by looking more closely at the basic equations (25)-(28) of Rund's theory:

- There are two different terms (both sums over α) in Eq. (25) containing the Clebsch potentials. These terms appear exactly at the positions where the electrodynamic scalar and vector potentials are located according to the minimal coupling rule.
- Relation (28) shows that the appearance of the new variables does not lead to new forces in the particle equations of motion. Introducing Q, P just means introducing new degrees of freedom, which do not interact with the particle coordinates.

The first of these hints tells us that the vortical terms due to Q , P may possibly be interpreted as electrodynamic potentials, giving rise to nonvanishing electrodynamic fields. The second hint tells us that these fields must be constructed in such a way that they do *not* exert any forces on the particles. In a future publication it will be shown that such “internal potentials” actually exist, and that the vortical Clebsch potentials may be understood with the help of this concept.

4 Implementing a topological constraint

In the remaining part of this paper we will restrict ourselves to the most important case $N = 1$, $n = 3$ of a single particle ensemble. The number m pairs of functions $P_\alpha(q, t)$, $Q_\alpha(q, t)$ is equal to 1 and the particle momentum is specified at each space-time point q, t by three numbers $S(q, t)$, $Q(q, t)$, $P(q, t)$, corresponding to the fact that the momentum must have three independent components at each point of three-dimensional space. Further, the Clebsch gauge will be fixed according to $\Phi = 0$ and the simplest Hamiltonian $H^0(q, p) = p_k p_k / 2m$ without external electromagnetic fields will be used (this important point will be taken into account in section 6). The basic equations for the four fields $S(q, t)$, $Q(q, t)$, $P(q, t)$, $\rho(q, t)$ are then given by

$$\frac{\partial S}{\partial t}(q, t) + P(q, t) \frac{\partial Q}{\partial t}(q, t) + H^0 \left(q, \frac{\partial S}{\partial q} + P \frac{\partial Q}{\partial q} \right) = 0, \quad (35)$$

$$\left[\frac{\partial}{\partial t} + V_k^0 \left(q, \frac{\partial S}{\partial q} + P \frac{\partial Q}{\partial q} \right) \frac{\partial}{\partial q_k} \right] Q(q, t) = 0, \quad (36)$$

$$\left[\frac{\partial}{\partial t} + V_k^0 \left(q, \frac{\partial S}{\partial q} + P \frac{\partial Q}{\partial q} \right) \frac{\partial}{\partial q_k} \right] P(q, t) = 0, \quad (37)$$

$$\frac{\partial \rho}{\partial t}(q, t) + \frac{\partial}{\partial q_k} \rho(q, t) V_k^0 \left(q, \frac{\partial S}{\partial q} + P \frac{\partial Q}{\partial q} \right) = 0. \quad (38)$$

The particle equations of motion $\dot{q}_k = v_k(q, t)$, with the velocity field defined by (20), are still valid in their limited range of validity. In this section we consider a topological property of the mapping from \mathbb{R}^3 into the space of the S, Q, P , which implies a more specific form of the Q, P .

For the simplest case of a single free particle considered now, the relationship between momentum, Clebsch potentials, and velocity is given by

$$M_k(q, t) = \partial_k S(q, t) + P(q, t) \partial_k Q(q, t) = m v_k(q, t). \quad (39)$$

The vorticity tensor Ω_{ik} [see Eq. (18)] may conveniently be replaced by an axial vorticity vector $\Omega_i = \frac{1}{2} \epsilon_{ikl} \Omega_{kl} = \epsilon_{ikl} \partial_k M_l$, which is closely related to the vorticity $\omega_i = \epsilon_{ikl} \partial_k v_l$ defined by the velocity field v . The relationship between these vorticities and the Clebsch potentials is given by

$$\Omega_i(q, t) = \epsilon_{ikl} \partial_k P(q, t) \partial_l Q(q, t) = m \omega_i(q, t), \quad (40)$$

and the equation of motion for the three-vector $\boldsymbol{\Omega}$ may be written in the form

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \nabla \times (\boldsymbol{\Omega} \times \mathbf{v}) = 0, \quad (41)$$

which is familiar from fluid mechanics. As mentioned already, there is a strong formal analogy between this part of the present theory and the theory of inviscid (ideal) barotropic fluids. This allows us to use several important results from fluid mechanics in the present context; bearing always in mind the completely different physical meaning of the variables.

The vorticity field $\boldsymbol{\Omega}$ is by definition solenoidal. Surface effects do not exist in our theory as we are considering an infinitely extended medium with sufficiently rapidly decreasing variables. The field lines of $\boldsymbol{\Omega}$, referred to as vortex lines, are therefore closed curves. The topology of a vector field is basically determined by the mutual position of its field lines, and in particular by the interlinking of its field lines. This structure remains invariant under the time evolution given by Eq. (41). Helmholtz's and Kelvin's theorems, which also apply in the present theory, can be interpreted in terms of the invariance of the topological structure.

The case of linked vortex lines may be explained in terms of simple physical concepts [see [20], [21] and references therein]. For example, in fluid mechanics Moffatt considers two idealized vortex filaments C_1 and C_2 with vanishing vorticity outside the filaments [20]. Stokes's theorem leads then immediately to the conclusion that the closed path integral along filament C_1 differs only from zero if filament C_2 penetrates the surface spanned by C_1 . Generalizing this consideration to more realistic distributions he arrived at the conclusion that the quantity

$$H_\omega = \int d^3q \mathbf{v} \omega, \quad (42)$$

referred to as (total) helicity, is a temporal and topological invariant, characterizing the degree of linkage of vortex lines.

Using the present notation it is more convenient to work with the helicity $H_\Omega = m^2 H_\omega$, which is defined by (42) with v, ω replaced by M, Ω . In order to study the variation of H_Ω with time, we need the evolution equation for the helicity density $h_\Omega = M_k \Omega_k$. It is obtained by multiplying the canonical equation (8) and the vorticity equation (41) by Ω and M respectively, and adding both equations. The result may be put in the form

$$\frac{\partial}{\partial t} h_\Omega + \nabla \cdot (h_\Omega \mathbf{V} - (\mathbf{M}\mathbf{V})\boldsymbol{\Omega}) = 0 \quad (43)$$

which shows explicitly the invariance of H_Ω . The total helicity vanishes if linked vortex lines do nowhere exist. On the other hand, if linked vortex lines exist M cannot be single valued; in this case we expect topological singularities of some kind. In order to examine this point more closely we calculate H_Ω using Pfaff's formula (17), expressing M in terms of the Clebsch potentials

S, P, Q . Neglecting a surface integral at infinity the helicity may be written as

$$H_\Omega = - \int d^3q S \epsilon_{ikl} \left(\frac{\partial^2 P}{\partial q_i \partial q_k} \frac{\partial Q}{\partial q_l} + \frac{\partial^2 Q}{\partial q_i \partial q_l} \frac{\partial P}{\partial q_k} \right). \quad (44)$$

This formula shows that H_Ω vanishes if P, Q are both C^2 . Linked vortex lines are therefore, as expected, related to singular, typically multivalued, behavior of the P, Q . In order to obtain a nonvanishing H_Ω it is not required that the first order derivatives of both Clebsch potentials do not commute; it suffices if only one of the P, Q is singular.

4.1 The Hopf map

The mapping of \mathbb{R}^3 in the space of the Clebsch variables P, Q , which is suitable for describing the quantum mechanical ensembles occurring in nature, belongs to a topologically non-trivial class, with linked vortex lines and non-vanishing helicity. It is given by the so-called Hopf map or Hopf bundle [22]. We use the complex form of the Hopf map (see e.g. [23]), which also provides us with an appropriate definition for the new (spinorial) state variable, which will later be used to perform the transition to QT. Let \mathbf{z} denote the elements of the two-dimensional complex vector space \mathbb{C}^2 ,

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad (45)$$

with inner product (\mathbf{z}, \mathbf{w}) defined according to $(\mathbf{z}, \mathbf{w}) = \mathbf{z}^+ \mathbf{w} = z_1^* w_1 + z_2^* w_2$. The norm $\|\mathbf{z}\|$ of \mathbf{z} is defined by $\|\mathbf{z}\|^2 = (\mathbf{z}, \mathbf{z})$. Writing $z_1 = a + ib, z_2 = c + id$ the points of \mathbb{C}^2 may be used to assign coordinates a, b, c, d to the points of \mathbb{R}^4 . The 3-sphere S_3 is the subset of \mathbb{R}^4 defined by $\|\mathbf{z}\|^2 = a^2 + b^2 + c^2 + d^2 = 1$,

$$S_3 = \{ \mathbf{z} \in \mathbb{C}^2 \mid \|\mathbf{z}\|^2 = 1 \}. \quad (46)$$

In order to characterize the topological (homotopy) class of the mapping from \mathbb{R}^3 to our two-dimensional manifold Q, P , the subset S_3 may be identified with \mathbb{R}^3 (it is in fact a compactified version of \mathbb{R}^3 [24]). The Hopf map is a many to one map from S_3 to the 2-sphere S_2 , defined by

$$h^i(x) = z_\alpha^* \sigma_{\alpha\beta}^i z_\beta, \quad (47)$$

where x stands for a, b, c, d , the indices α and β run from 1 to 2, and σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (48)$$

In order to verify this we use the formula $\sigma_{\alpha\beta}^i \sigma_{\gamma\delta}^i = 2\delta_{\alpha\delta} \delta_{\gamma\beta} - \delta_{\alpha\beta} \delta_{\gamma\delta}$ to obtain $h^i h^i = (\mathbf{z}_-^{-1}{}''^2 + \mathbf{z}_-^{-2}{}''^2)^2$. Thus, the image of the Hopf map is indeed a point of S_2 given that $\mathbf{z}_-^{-1}{}''^2 + \mathbf{z}_-^{-2}{}''^2 = a^2 + b^2 + c^2 + d^2 = 1$. The points of S_2 have coordinates

$$h^1 = 2(ac + bd), \quad (49)$$

$$h^2 = 2(ad - bc), \quad (50)$$

$$h^3 = a^2 + b^2 - c^2 - d^2. \quad (51)$$

Two points \mathbf{z} , \mathbf{w} in S_3 , which differ from each other by a complex number of magnitude 1, are mapped to the same point of S_2 . The set of all points $e^{i\chi} \mathbf{u}$ on S_3 that results from a fixed \mathbf{u} is called a fiber. Each fiber is the preimage of a point of S_2 . Thus S_3 has the structure of a fiber bundle.

The points on S_3 can now be represented in the form

$$\mathbf{z} = e^{i\frac{\chi}{2}} \begin{pmatrix} a + ib \\ c + id \end{pmatrix}, \quad a^2 + b^2 + c^2 + d^2 = 1. \quad (52)$$

It will become clear soon why it is convenient to use the definition $\frac{\chi}{2}$ in the prefactor. It makes sense to introduce new variables that eliminate the surface constraint in (52). We use spherical coordinates, where the angle φ is measured from the y -axis in clockwise direction, to represent the three-vector \mathbf{h} in the form

$$\mathbf{h} = \sin \vartheta \sin \varphi \mathbf{e}_1 + \sin \vartheta \cos \varphi \mathbf{e}_2 + \cos \vartheta \mathbf{e}_3. \quad (53)$$

Using now Eqs. (49)-(51) the dependence of a, b, c, d on ϑ, φ may be determined and we obtain

$$\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = e^{i\frac{\chi}{2}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = e^{i\frac{\chi}{2}} \begin{pmatrix} \cos \frac{\vartheta}{2} e^{i\frac{\varphi}{2}} \\ i \sin \frac{\vartheta}{2} e^{-i\frac{\varphi}{2}} \end{pmatrix}, \quad (54)$$

as a possible representation of \mathbf{z} . In the literature, essentially two different but equivalent representations are in use, which are due to Takabayasi [25] and Bohm [26], respectively. Both were introduced in order to rewrite Pauli's equation in "hydrodynamic form". The state vector derived here is associated with the description of rotations in terms of Euler angles and agrees with Bohm's representation; it was already used in Ref. [27],

In order to better understand the physical meaning of the quantity χ , we introduce a three-vector \mathbf{t} , with components defined by $t^i = \{\mathbf{z}, \sigma^i \mathbf{z}\}$. The curly brackets denote here an antisymmetric product, defined by $\{\mathbf{z}, \mathbf{w}\} = z_1 w_2 - z_2 w_1$. This vector fulfills the relations $\mathbf{t}^2 = 1$ and $\mathbf{t}\mathbf{h} = 0$, i.e. \mathbf{t} , as a unit vector perpendicular to \mathbf{h} , lies in the tangential plane determined by \mathbf{h} .

Using the representation (54) we obtain \mathbf{t} in terms of the angles χ , ϑ , φ :

$$\begin{pmatrix} t^1 \\ t^2 \\ t^3 \end{pmatrix} = \begin{pmatrix} \cos \chi \cos \varphi - \sin \chi \cos \vartheta \sin \varphi \\ -\cos \chi \sin \varphi - \sin \chi \cos \vartheta \cos \varphi \\ \sin \chi \sin \vartheta \end{pmatrix}. \quad (55)$$

This relation shows that χ may be interpreted as an angle of rotation around an axis determined by ϑ and φ . Remarkably, the rotation angle of the vector \mathbf{t} is 2α if the phase of \mathbf{z} changes by α . In particular, \mathbf{z} changes its sign when χ changes by 2π and returns to its original value only when χ changes by 4π . An equivalent form of a ‘‘spinor’’, as a directed quantity that describes a rotation, was derived by Payne using intuitive geometric methods[28].

4.2 Helicity and Hopf invariant

The preimage of each point P of S_2 is a circle. The topological nontriviality of the Hopf map is given by the fact that the linking number for every 2 circles, that are mapped to different points P_1, P_2 , is not 0 but 1 ; an explicit proof may e.g. be found in [29]. This linking number is a topological invariant referred to as Hopf invariant γ . Whitehead [30] found that γ may be written as an integral,

$$\gamma = \frac{1}{16\pi^2} \int d^3q C_i D_i, \quad (56)$$

where C , the canonical connection of the Hopf map [31], is given by $C_i = -2\iota z_a^* \partial_i z_a$, and $D_i = \epsilon_{ijk} \partial_j C_k$. The value of the Hopf invariant γ is a property of the divergence-free field D . The field C plays the role of a vector potential for D and is not gauge-invariant. Conversely, one may choose any suitable C from the class leading to D in order to calculate γ , as will be done here.

The basic fields χ , ϑ , φ that determine (56) are dimensionless quantities while the fields C and D have dimensions cm^{-1} and cm^{-2} . If one wants to use the Whitehead integral in physics, the abstract fields C and D have to be replaced by suitable physical fields (e.g. velocity and vorticity, or vector potential and magnetic field), and appropriate dimensions and pre-factors must be introduced. In our case we have to establish the relation between the field C and the momentum field M ,

$$C_i = \partial_i \chi + \cos \vartheta \partial_i \varphi \quad M_i = \partial_i S + P \partial_i Q, \quad (57)$$

in order to be able to find the relation between S, P, Q and χ, ϑ, φ . If we accept \mathbf{z} , as defined by (54), as our new state variable (apart from an amplitude which we will be introduced later) then it is obvious to associate the two-component quantity in (54), with components u_1, u_2 , with the new rotational degrees of freedom Q, P . As a consequence we identify the prefactor $e^{i\frac{S}{\hbar}}$ with the earlier phase factor $e^{i\frac{S}{\hbar}}$, associated with the irrotational momentum fields

studied in III. This leads to the relations

$$\frac{S}{\hbar} = \frac{\chi}{2}, \quad C_i = \frac{2}{\hbar} M_i, \quad P \partial_i Q = \frac{\hbar}{2} \cos \vartheta \partial_i \varphi, \quad (58)$$

As for the identification of P, Q we could, remembering that the Clebsch potentials are canonical variables [12], interpret P as momentum and Q as position. This could be achieved by replacing $\hbar/2$ by $\hbar/2R$ and ϕ by $R\phi$, where R is a length. The numerical value of R is arbitrary since P, Q always occur in pairs. Thus we may set $R = 1$:

$$P = \frac{\hbar}{2} \cos \vartheta, \quad Q = \varphi. \quad (59)$$

The topological meaning of the ‘‘canonical Clebsch potentials’’ defined by (59), was clarified by Kutnetsov and Mikhailov [32]. These authors studied ideal fluids which are, however, described by essentially the same mathematical structure as the present problem. In their work the constant $\hbar/2$ is replaced by an undetermined constant - let us recall that we were only able to fix the value of this constant by anticipating the quantization process.

If we replace P, Q in (40) by the canonical Clebsch potentials (59) the vorticity Ω_i and the helicity H_Ω become quantized variables,

$$\Omega_i = \frac{\hbar}{2} \epsilon_{ikl} (\partial_k \cos \vartheta) (\partial_l \varphi), \quad H_\Omega = (2\pi\hbar)^2 \gamma. \quad (60)$$

The linking number γ takes integer values in general, and is 1 in the present theory. The Hopf map is sometimes introduced by restricting the vorticity according to the relation $\Omega_i = \frac{\hbar}{2} T_i$, where

$$T_i = \frac{1}{2} \epsilon_{ijk} \epsilon_{lmn} h^l (\partial_j h^m) (\partial_k h^n). \quad (61)$$

The vector \mathbf{T} was first introduced by Takabayasi [33] and later rediscovered in other contexts by Faddeev [34] and Mermin and Ho [35]. Topological methods have been used successfully to classify phases in superfluid He and other many-body systems [36]. If \mathbf{v} in Eq. (42) is replaced by the superfluid velocity [37] one obtains the same \hbar -dependent prefactor as in Eq. (60).

4.3 Invariance of circulation and quantization condition

As shown in III, after projection to the n -dimensional subspace defined by M , the Poincaré integral invariant takes the form

$$I_{\bar{C}_t}(t) = \oint_{\bar{C}_t} M_i(q, t) dq_i, \quad (62)$$

where \bar{C}_t is a closed path in the subset $(q, M(q, t))$ of phase space. This formula expresses the invariance of the circulation $I_{\bar{C}}$; in a fluid-dynamical context it is referred to as Kelvin's theorem. It remains true if M is expressed in terms of S, Q, P according to (17), or in terms of S, ϑ, φ according to (58).

We expect, in analogy to the helicity, that the invariant $I_{\bar{C}}$ becomes "quantized" for topological reasons. This is indeed the case as found already by Takabayasi, who used - contrary to the present work - the basic equations of QT as his starting point[38]. Using (57)-(61) the circulation may be written as

$$\oint_{\bar{C}_t} M_i(q, t) dq_i = h \left(n + \frac{1}{4\pi} \int_{S_t} dS_i T_i \right), \quad (63)$$

where S_t is a surface [a cross section of the tube formed by the solutions of (7)] with boundary \bar{C}_t . The first term on the r.h.s. is due to the fact, that S may change by $2\pi\hbar n$ (where n is an integer) when going around \bar{C}_t . This multi-valuedness, and its associated topological singularity, does not affect the uniqueness of the state function \mathbf{z} . This term corresponds to the usual quantization condition. The second term on the r.h.s. is due to the additional non-singular vorticity of the momentum field.

To interpret this second term, we note that the vortical part of the momentum field may be interpreted, apart from a constant of proportionality, as a kind of internal vector potential, with components $A_k^I(q, t)$. When expressed in terms of the canonical Clebsch potentials, A_k^I is given by

$$A_k^I = -\frac{\hbar c}{2e} \cos \vartheta \partial_k \varphi. \quad (64)$$

The associated inner magnetic field is proportional to the Takabayasi vector or to the vorticity, $B_i^I = \epsilon_{ikl} \partial_k A_l^I = \frac{e}{c} \Omega_i$. It is useful to introduce, for comparison, an external vector potential with associated magnetic field B_i^E (This does not cover the complete influence of the external magnetic field, the full theory will be given in section 6). The circulation extended this way may be written as

$$\oint_{\bar{C}_t} M_i(q, t) dq_i + \frac{e}{c} \int_{S_t} dS_i (B_i^E + B_i^I) = nh. \quad (65)$$

We see that the vortical part of the momentum field may be interpreted as an additional, "internal" contribution to the magnetic field. In superconducting many-particle systems one observes the phenomenon of flux quantization, which is described by Eq. (65) *without* this vortical contribution. Its absence is, however, not really unexpected, since the superconducting state is not generated by single electrons but by spinless paired electrons (Cooper pairs).

5 Semilinear evolution equation

After introducing the canonical Clebsch variables, one obtains the updated set of basic equations for the dynamical variables $\rho, S, \vartheta, \varphi$ by replacing P, Q in Eqs. (35)-(38) by $\cos \vartheta, \varphi$ according to (59):

$$\partial_t S + e\Phi + \frac{\hbar}{2} \cos \vartheta \partial_t \varphi + \frac{1}{2m} \sum_k \left(\partial_k S + \frac{\hbar}{2} \cos \vartheta \partial_k \varphi \right)^2 = 0, \quad (66)$$

$$\left[\partial_t + \left(\partial_k S + \frac{\hbar}{2} \cos \vartheta \partial_k \varphi \right) \partial_k \right] \varphi = 0, \quad (67)$$

$$\left[\partial_t + \left(\partial_k S + \frac{\hbar}{2} \cos \vartheta \partial_k \varphi \right) \partial_k \right] \vartheta = 0, \quad (68)$$

$$\partial_t \rho + \partial_k \rho \left(\partial_k S + \frac{\hbar}{2} \cos \vartheta \partial_k \varphi \right) = 0. \quad (69)$$

Instead of the two dynamical equations for ϑ and φ the relation $D_t \mathbf{h} = 0$ for the unit vector \mathbf{h} , defined by Eq. (53), may equivalently be used. The particle equations of motion $\dot{q}_k = v_k(q, t)$ remain true; they define the ranges of integration in the quantization condition (65).

The above evolution equations, as given by (66)-(69), are neither suitable for the transition to QT by linearization, nor for the introduction of a gauge field. To achieve these goals, these four equations must first be replaced by a single equation for a single variable with two complex components. A suitable quantity is provided by the Hopf map. In order to perform this transformation, it is convenient to rewrite (66)-(69) in the following form:

$$D_t S = \bar{L}, \quad (70)$$

$$D_t \vartheta = 0, \quad (71)$$

$$D_t \varphi = 0, \quad (72)$$

$$D_t \rho^{\frac{1}{2}} = -\frac{1}{2} \rho^{\frac{1}{2}} \partial_k v_k, \quad (73)$$

where $D_t = \partial_t + v_k \partial_k$, $\bar{L} = \frac{1}{2} v_k M_k - V$, and

$$v_k = \frac{1}{m} \left(\partial_k S + \frac{\hbar}{2} \cos \vartheta \partial_k \varphi \right). \quad (74)$$

The basic equations of QA reformulated this way differ from the equations used in III only with regard to the additional equations of motion (71) and (72) and the additional vortical term in the definition (74). The structural differences between these equations and the original phase space equations used in I, II were mentioned in III.

Let us stress once again, that these basic equations do essentially not belong to QT, despite the occurrence of the constant \hbar . The occurrence of \hbar

is explained as follows: Due to topological considerations, the introduction of a quantity with the dimension of an action became necessary. For the sake of simplicity, we have already assigned the special numerical value of Planck's constant to this quantity; without this assignment the transition to QT, to be performed later, cannot be realized. This assignment of a numerical value may therefore be interpreted as a first step in the quantization process. However, the very occurrence of this constant is a consequence of classical (topological) considerations.

We obtain an appropriate two-component variable ψ , suitable for linearization, by multiplying the variable \mathbf{z} defined by the Hopf map [see Eq. (54)] by the factor $\sqrt{\rho}$,

$$\psi_a = \sqrt{\rho} e^{\frac{i}{\hbar} S} u_a, \quad a = 1, 2. \quad (75)$$

Then, as a consequence of the physical meaning of ρ , the integral of $|\psi|^2$ over the entire space must be 1.

Let us write the projected Liouville equation (73) in the form $\mathbb{L}\rho^{\frac{1}{2}} = 0$, with the differential operator $\mathbb{L} = \frac{\hbar}{i} [D_t + \frac{1}{2} (\partial_i v_i)]$. Let us next consider the two-component quantity $T_a = \mathbb{L}\sigma_{ab}^0 \psi_b$. The terms in the differential equation to be constructed are necessarily 2×2 matrices. It turns out that it is sufficient to use the 2×2 identity matrix σ^0 , at least in the present field-free case. If we let the differential operator \mathbb{L} act on ψ , then T_a takes the form

$$T_a = \frac{\hbar}{i} \sigma_{ab}^0 \psi_b \left[\rho^{-\frac{1}{2}} \left(D_t + \frac{1}{2} (\partial_i v_i) \right) \rho^{\frac{1}{2}} + \frac{i}{\hbar} D_t S + \frac{1}{u_{(b)}} D_t u_{(b)} \right], \quad (76)$$

whithout summation over b in the bracket. If we now use the evolution equations (70)-(73), the first and third terms vanish and $D_t S$ is replaced by \bar{L} . If we equate the resulting expression with the original definition of T_a , we obtain the differential equation

$$\left[\frac{\hbar}{i} D_t + \frac{\hbar}{2i} (\partial_i v_i) - \frac{1}{2} v_k M_k + V \right] \psi_a = 0, \quad (77)$$

which is equivalent to Eqs. (70)- (73), but is already more similar to the desired form. In order to proceed with the linearization we use the relation $(\partial_k S) \psi_b = \frac{\hbar}{i} \left[\partial_k - \rho^{-\frac{1}{2}} \left(\partial_k \rho^{\frac{1}{2}} \right) + u_{(b)}^{-1} (\partial_k u_{(b)}) \right] \psi_b$ which follows directly from the definition of ψ [see Eq. (75)]. Using this last relation as well as the definition of M_k one obtains the equation $v_i M_i \psi_b = \frac{\hbar}{i} v_i (\partial_i - f_{i(b)}) \psi_b$. The function

$$f_{kb} = \frac{1}{\rho^{\frac{1}{2}}} \left(\partial_k \rho^{\frac{1}{2}} \right) + \frac{1}{u_{(b)}} (\partial_k u_{(b)}) - \frac{i}{2} \cos \vartheta (\partial_k \varphi), \quad (78)$$

does not depend on the variable S which carries the gauge degree of freedom. Repeated application of this last equation leads, after a number of further

elementary rearrangements, to the following “semilinear” differential equation

$$\left[\frac{\hbar}{i} \partial_t - \frac{\hbar^2}{2m} \partial_k \partial_k + V \right] \psi_a = -\frac{\hbar^2}{2m} [f_{k(a)} f_{k(a)} + (\partial_k f_{k(a)})] \psi_a. \quad (79)$$

This differential equation, together with the particle equation of motion $\dot{q}_k = v_k(q, t)$, represents that form of the QA which is most similar to QT - and which accordingly enables a particularly simple transition to QT. In I a differential equation in phase space was derived (Eq. 19 of I), which has been called the “classical counterpart of Schrödinger’s equation”. If we follow this naming scheme, we can call Eq. (79) the “quasiclassical counterpart of Schrödinger’s equation”. It represents the *completion* of the non-linear Schrödinger equation reported in III (see Eq. 40 of III), where two of the three possible degrees of freedom of the momentum field were unjustifiably neglected.

6 Introducing a gauge field

Let us now “switch on” an electromagnetic field. This is usually done using the principle of minimal coupling

$$\frac{\hbar}{i} \partial_k \Rightarrow \frac{\hbar}{i} \partial_k - \frac{e}{c} A_k, \quad -\frac{\hbar}{i} \partial_t \Rightarrow -\frac{\hbar}{i} \partial_t + e\Phi, \quad (80)$$

where A, Φ is the vector and scalar potential respectively. In the (quasi) classical equation (79) application of this rule to the derivatives on the right-hand side does not lead to meaningful results. We therefore use the more fundamental version of the principle of minimal coupling formulated by Dirac [39] (see also [40], [41]). With this method, the wave function ψ is multiplied by a non-integrable phase factor:

$$\exp \left\{ -\frac{i e}{\hbar c} \int^{x,t} [dq'_k A_k(q', t') - c dt' \Phi(q', t')] \right\}, \quad (81)$$

This phase factor is then shifted to the left of the differential operators, creating the potentials, and can afterwards be eliminated. The final wave function is again single-valued as it should be. This version of the minimal coupling rule can be applied to Eq. (79), since the right-hand side does not contain any derivatives of ψ . We obtain in this way the standard result

$$\left[\frac{\hbar}{i} \partial_t - e\Phi - \frac{\hbar^2}{2m} \sum_{k=1}^3 \left(\partial_k - i \frac{e}{\hbar c} A_k \right)^2 + V \right] \psi_a = -\frac{\hbar^2}{2m} [f_{k(a)} f_{k(a)} + (\partial_k f_{k(a)})] \psi_a. \quad (82)$$

The linear part of Eqs. (79) and (82) apparently plays a decisive role in the transition to QT. This part, in which, based on our assumptions, the kinetic energy is proportional to the identity matrix, does not contain any coupling between the two components of ψ , neither in the field-free case nor in the

presence of an electromagnetic field. Assuming that Eq. (79) is true, this would lead to the strange conclusion that the vortical components of the momentum field have no observable consequences at all after transition to QT. Instead, we suspect that Eq. (79) is incomplete.

6.1 Two representations of Euclidean space

The geometric objects occurring in physics may be characterized by their behavior under certain groups of transformations. The relevant group in the present context is the rotation group $SO(3)$, the group of orthogonal 3×3 matrices with determinant 1. A geometric object is associated with a specific representation of $SO(3)$. For example, in the Schrödinger equation for spinless particles studied in III, a one-component complex quantity occurs as a geometric object. In this case, only the space of independent variables is transformed under $SO(3)$ and the representation associated with this geometric object is the identical representation.

The geometric object that arises in the present problem is the spinor ψ . At every point of space this is an element of a Hilbert space \mathbb{H}^2 , which is defined (as in section 4.1) by \mathbb{C}^2 with the inner product $(\psi, \phi) = \psi_1^* \phi_1 + \psi_2^* \phi_2$. The group of automorphisms of \mathbb{H}^2 is given by the linear transformations that leave the inner product invariant. This is the group $U(2)$, but it can be restricted to $SU(2)$, the group of two-dimensional unitary transformations with determinant 1. A general element $U \in SU(2)$ has the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \tag{83}$$

where α and β are complex numbers that satisfy $|\alpha|^2 + |\beta|^2 = 1$. This means that the group space of $SU(2)$ is given by the 3-sphere S_3 (the spinorial part of ψ is also determined by a point on S_3 , as was shown in section 4.1). Thus, a spinor is the geometric object transforming according to the natural representation of the group $SU(2) \cong S_3$.

Let us recall the relation between the groups $SU(2)$ and $SO(3)$. This point is well-known from the quantum mechanical theory of spin; but of course this relation is primarily a “classical” matter of group theory and associated topological considerations. To find this relation we introduce a matrix representation of Euclidean space following Eberlein [42]. We start from the observation that every Hermitian operator in \mathbb{H}^2 may be represented as a linear combination, with real coefficients, of the three Pauli matrices σ^k and the identity matrix σ^0 . If we associate in our non-relativistic theory the identity matrix σ^0 with a time-like coordinate, then the remaining three Pauli matrices σ^k , with trace 0, may be used to represent an arbitrary three-dimensional Hermitian 2×2 matrix, say Q , in the form

$$Q = q_1 \sigma^1 + q_2 \sigma^2 + q_3 \sigma^3, \tag{84}$$

where the q_i are real numbers. Let us denote the Euclidean space with the usual inner product by E_3 and the real vector space of the Hermitian matrices (84), with basis elements σ^k , by \mathbb{E}_3 . There is a one-to-one correspondence between the vectors $q_i \mathbf{e}_i \in E_3$, where \mathbf{e}_i is an orthonormal basis, and the matrices $q_k \sigma^k \in \mathbb{E}_3$,

$$q_i \mathbf{e}_i \leftrightarrow q_k \sigma^k. \quad (85)$$

This mapping from E_3 to \mathbb{E}_3 also preserves the inner product $(p, q) = p_i q_i$. If P and Q are the elements of \mathbb{E}_3 corresponding to p and q , then their inner product, which is denoted by $(P \cdot Q)$, is defined by $PQ + QP = 2(P \cdot Q)\sigma^0$. The invariance can easily be verified with the help of the relation $\sigma^i \sigma^k + \sigma^k \sigma^i = 2\delta^{ik}\sigma^0$. Thus, the space \mathbb{E}_3 may be called the matrix model, or the spin model, of Euclidean space [24],[42].

Let us consider an element $Q \in \mathbb{E}_3$ and a unitary matrix $U \in SU(2)$. The transformed element

$$R_U(Q) = UQU^{-1} \quad (86)$$

is hermitian and has vanishing trace, so it is again an element of \mathbb{E}_3 which may be written in the form $Q' = q'_k \sigma^k$. If we now insert U as given by (83) and evaluate Eq. (86) we find the relation

$$q'_i = R_{ik}(\alpha, \beta)q_k, \quad (87)$$

where $R_{ik}(\alpha, \beta)$ are the elements of a (real) orthogonal matrix with determinant 1 which is uniquely determined by the matrix U defined by (83). These elements satisfy the relationship $R_{ik}(-\alpha, -\beta) = R_{ik}(\alpha, \beta)$. As a consequence, Eq. (86) defines a mapping $SU(2) \rightarrow SO(3)$ which is not one-to-one, since the two elements $\pm U$ are mapped onto the same element of $SO(3)$. One can also show that $R_{U_1 U_2} = R_{U_1} R_{U_2}$ and that every rotation in \mathbb{E}_3 may be represented in the form (86) [42],[24]. The mapping $SU(2) \rightarrow SO(3)$, which is often referred to as ‘‘spinor mapping’’, is therefore a two-to-one group homomorphism.

The difference between the groups $SU(2)$ and $SO(3)$ is essentially due to the different topological structure of their parameter spaces. The two-to-one group homomorphism from the simply connected $SU(2)$ to the double connected $SO(3)$ means that $SU(2)$ is the covering group (double cover) of $SO(3)$. As already mentioned, the parameter space of $SU(2)$ is the 3-sphere S_3 . The parameter space of $SO(3)$ may be identified with a filled sphere in three-dimensional space with radius π ; this choice of the group manifold allows an easy intuitive understanding of the topological structure [43]. Alternatively, one may also use the surface of the 3-sphere S_3 with antipodal points identified, as group manifold; this latter choice is particularly plausible in view of the two-to-one homomorphism from $SU(2)$ to $SO(3)$.

To choose between the two representations E_3 and \mathbb{E}_3 of the abstract Euclidean space, we note that the transformation behavior of the Hermitian operator $Q \in \mathbb{E}_3$, as given by (86), is consistent with the general condition that the action of an operator before and after a basis transformation U must be

the same. The spin model \mathbb{E}_3 of Euclidean space is therefore the appropriate choice for vectors that act as operators on the Hilbert space \mathbb{H}^2 .

The representation E_3 would be suitable if we had three-vectors as geometric objects, in which the effect of $SO(3)$ is realized by means of the usual rotation matrices. The spin group $SU(2)$, as discussed above, has a richer structure compared to the $SO(3)$. Its group manifold is larger, as is the number of its representations; any representation of $SO(3)$ can be extended to a representation of the $SU(2)$ (by composition with the spinor mapping), but not the other way around. [However, in the physical literature the term “two-valued representation of $SO(3)$ ” is often used for $SU(2)$]. The fact that the simply connected parameter space of $SU(2)$ contains two elements $\pm U$ for every rotation R may be related to the corresponding property of the full group $O(3)$ of the automorphisms of Euclidean space. The statement

The orthogonal transformations are the automorphisms of Euclidean vector space. Only with the spinors do we strike this level in the theory of its representations...

made by Hermann Weyl emphasises the fundamental role of the spin representation [44].

The spinorial geometric object ψ associated with $SU(2)$ is essentially (except for a subsequently added amplitude $\sqrt{\rho}$) given by the Hopf map, defined in section 4.1. The introduction of the Hopf map was based on the assumption that the statistical ensembles occurring in nature belong to a topological class describing linked vortex lines. This choice, which may have seemed a bit arbitrary, is now justified by the fact that the Hopf map describes the topology of rotations in \mathbb{H}^2 . The property of linked vortex lines seems to correspond to a particular topological property of rotations, namely the fact that ψ does not return to the starting point until it has been rotated by 4π .

6.2 Minimal coupling in \mathbb{E}_3

According to the discussion of the last section, the transition from E_3 to \mathbb{E}_3 should be performed simultaneously with the introduction of the spinor ψ . The transition is defined by the simple replacement rule $v_i \mathbf{e}_i \rightarrow v_i \sigma^i$. This rule can only be applied in the quantum-like version (83) of the equation of motion, and here again only in the Laplace operator, which can be written in the form $\partial_i \mathbf{e}_i \partial_k \mathbf{e}_k$. One obtains

$$\partial_k \partial_k = \partial_i \mathbf{e}_i \partial_k \mathbf{e}_k \rightarrow \partial_i \sigma^i \partial_k \sigma^k = \partial_k \partial_k + \imath \epsilon^{lik} \sigma^l \partial_i \partial_k, \quad (88)$$

where the relation $\sigma^i \sigma^k = \delta^{ik} + \imath \epsilon^{lik} \sigma^l$ was used to obtain the last term on the right-hand side. Eq. (88) shows that this replacement rule can only lead to a new contribution, as compared with the original Laplace operator, if the first order derivatives are not interchangeable, that is, if they operate on a singular, typically multi-valued ψ . This is the case if the minimal coupling

rule (81) is applied. The evolution equation then takes the form

$$\left[\frac{\hbar}{i} \partial_t - e\Phi - \frac{\hbar^2}{2m} \sum_{k=1}^3 \left(\partial_k - i \frac{e}{\hbar c} A_k \right)^2 + V \right] \psi_a + \mu_B \sigma_{ab}^k B_k \psi_b = -\frac{\hbar^2}{2m} [f_{k(a)} f_{k(a)} + (\partial_k f_{k(a)} + (\partial_k f_{k(a)})) \quad (89)$$

where $\mu_B = -e\hbar/2mc$ is the Bohr magneton, and B_k are the components of the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. The B -dependent coupling term was introduced by Pauli and is sometimes referred to as ‘‘Zeemann term’’, as it played an important role in the explanation of the anomalous Zeeman effect. In mathematical terms \mathbf{A} is called connection and \mathbf{B} is called curvature of the connection [24].

We brought the quasi-classical evolution Eqs (66)-(69) in the quantum-like form (79) and then introduced an external electromagnetic field with the help of the principle of minimal coupling. If we now go back from the result (99) to the original form, we obtain the equations

$$\partial_t S + e\Phi + \frac{\hbar}{2} \cos \vartheta \partial_t \varphi + \frac{m}{2} \sum_k v_k^2 + \mu_i B_i = 0, \quad (90)$$

$$D_t \varphi = \frac{e}{mc} \frac{1}{\sin \vartheta} (B_3 \sin \vartheta - B_2 \cos \vartheta \cos \varphi - B_1 \cos \vartheta \cos \varphi), \quad (91)$$

$$D_t \vartheta = \frac{e}{mc} (B_1 \cos \varphi - B_2 \sin \varphi), \quad (92)$$

$$\partial_t \rho + \partial_k \rho v_k = 0, \quad (93)$$

where the velocity field v and the total derivative D_t are now defined by

$$v_k = \frac{1}{m} \left(\partial_k S - \frac{e}{c} A_k + \frac{\hbar}{2} \cos \vartheta \partial_k \varphi \right), \quad D_t = \partial_t + v_k \partial_k. \quad (94)$$

The ‘‘magnetic moment of the electron’’ is defined by $\mu_i = -\frac{e}{mc} s_i$, where \mathbf{s} is a three-vector of constant length $\frac{\hbar}{2}$ which is parallel to the unit vector \mathbf{h} defining the Hopf map,

$$\mathbf{s} = \frac{\hbar}{2} (\sin \vartheta \sin \varphi \mathbf{e}_1 + \sin \vartheta \cos \varphi \mathbf{e}_2 + \cos \vartheta \mathbf{e}_3). \quad (95)$$

The electromagnetic field leads to two types of new terms in Eqs. (89)-(93), namely on the one hand to the usual extensions of the energy and momentum fields by potentials and on the other hand to terms depending on the magnetic field \mathbf{B} . The latter terms may be interpreted with the help of the vector \mathbf{s} , using the fact that Eqs. (91) and (92) may be written in the form

$$D_t \mathbf{s} = -\frac{e}{mc} \mathbf{B} \times \mathbf{s}, \quad (96)$$

which is the field-theoretical version of the equation of motion of a classical magnetic dipole in a magnetic field.

6.3 The magnetic moment of the electron

According to classical electrodynamics, a rotating charge distribution with mass m , total charge q , and angular momentum \mathbf{L} leads to a magnetic dipole moment $\boldsymbol{\mu} = \frac{q}{2mc}\mathbf{L}$. The pre-factor $\frac{q}{2mc}$ is called the gyromagnetic ratio. The interaction with an external magnetic field is taken into account by a term $\boldsymbol{\mu}\mathbf{B}$ in the Hamiltonian function. The new B -dependent terms in Eqs. (90) and (96) agree in form with this particle image, differ however with regard to the gyromagnetic ratio between the “intrinsic angular momentum” \mathbf{s} and the associated magnetic moment $\boldsymbol{\mu}_s$. The gyromagnetic factor g defined by $\boldsymbol{\mu}_s = \frac{g}{2}\frac{q}{mc}\mathbf{s}$ takes the value $g = 2$ and is consequently twice as large as would be expected in the particle image.

If we want to interpret the Zeemann term in the quantum-like form (89) of the evolution equations, we must first identify the quantum-like quantity \mathbf{S} corresponding to \mathbf{s} . The formula $\rho\mathbf{s} = \psi_a^* \frac{\hbar}{2} \boldsymbol{\sigma}_{ab} \psi_b$ which results directly from the definitions of \mathbf{h} and ψ , suggests the choice $\mathbf{S}_{ab} = \frac{\hbar}{2} \boldsymbol{\sigma}_{ab}$; the same result for the “spin operator” \mathbf{S} is obtained with the help of the angular momentum commutation relations in QT. The quantum-like formulation (89) of the evolution equations leads of course to the same result for g as (90), (93), (96): The Zeemann term in (89) may be written in the form $\boldsymbol{\mu}_s^{op}\mathbf{B}$, and the relation between \mathbf{S} and the operator of the magnetic moment $\boldsymbol{\mu}_s^{op}$ is given by $\boldsymbol{\mu}_s^{op} = -\frac{g}{2}\frac{e}{mc}\mathbf{S}$ with $g = 2$. This value of the gyromagnetic factor has been experimentally confirmed to a good approximation (there are corrections to this value that are not of interest here).

The correct gyromagnetic factor $g = 2$ was obtained by Dirac in 1928, in the course of the derivation of the relativistic wave equation that bears his name [45]. It has long been assumed that the deviation from $g = 1$ is to be regarded as a relativistic (as well as quantum) effect. About four decades later Levy-Leblond showed, however, that the same correct value of g may also be derived from Schrödinger’s equation if a *linearization* with regard to the differential operator ∂_k is performed [46]. One of the assumptions made by Dirac is that the basic quantum equations for a particle should be linear in all first order derivatives. This assumption may be partly motivated by the relativistic space-time structure considered by Dirac. It’s physical meaning is, however, not invariably linked to this space-time structure. Lévy-Leblond made the same assumption as Dirac for the non-relativistic space-time and obtained a differential equation for a four-component spinor, just as in Dirac’s theory. This four-component spinor is composed of two two-component spinors $\hat{\psi}$ and $\hat{\chi}$ which obey the equations

$$\left[\frac{\hbar}{i} \frac{\partial}{\partial t} + V(q) \right] \hat{\psi} - \sigma_k \frac{\hbar}{i} \frac{\partial}{\partial q_k} \hat{\chi} = 0, \quad (97)$$

$$\sigma_l \frac{\hbar}{i} \frac{\partial}{\partial q_l} \hat{\psi} + 2m\hat{\chi} = 0. \quad (98)$$

The two-spinor $\hat{\chi}$, defined as a linear combination of derivatives of $\hat{\psi}$ with respect to q_l , plays obviously the role of an auxiliary variable. If $\hat{\chi}$ is inserted in (97) one obtains the field free Pauli-Schrödinger equation if the derivatives of $\hat{\psi}$ with respect to q_k commute with each other. If, on the other hand, a gauge field is turned on with the help of the minimal coupling rule, one obtains Pauli's equation with the Zeemann term including the correct factor $g = 2$. Lévy-Leblond's derivation shows that spin is not a relativistic phenomenon. This fact is still not widely known. You can still find statements like: "Quantum spin arises from the combination of special relativity with quantum mechanics" [47].

The value $g = 2$, which is "anomalous" in the sense of a deviation from the result of the particle image, cannot therefore be explained as a relativistic (quantum mechanical) effect. This convenient explanation breaks down. Instead, the question now arises why the linearity in ∂_k should be responsible for this deviation. The answer may be found in an unjustly forgotten work [42] by Eberlein from 1962. In this work, which provides the basis for our above considerations, the correct value $g = 2$ was found for the first time in the framework of non-relativistic QT.

As discussed in more detail in section 6.1, Eberlein introduced a matrix representation of Euclidean space that is suitable for describing the behavior of spinors. The deeper reason for this lies in the topology of the rotation group. The result of these topological considerations agrees with the result of the "linearization" (with regard to the differential operator), which was carried out by Levy-Leblond following Dirac's method. A comparison of these two processes shows that this is no coincidence: The conditions for a meaningful "linearization" lead to the same commutation relations as the conditions for a consistent spinor representation of \mathbb{R}^3 . This means that the unclear requirement of "linearity" may be replaced by the obvious requirement that the topological structure of the rotation group has to be correctly taken into account. But this means that actually no assumption at all is required in order to be able to derive the correct g -factor (it is tempting to associate the factor 2 with the ratio $4\pi/2\pi$).

This insight automatically leads to the next question. If the deviation of the gyromagnetic factor from 1 is a purely topological effect why should we then need QT to derive it. The answer given by the present theory is that we actually do *not* need QT to derive it; according to the present derivation, it may be classified as a semiclassical (or semiquantal) effect. The fact that the Zeeman term is not a quantum effect may also be realized from the very absence of \hbar in the constant of proportionality between μ_s^{op} and \mathbf{S} (the \hbar that appears in the definition of \mathbf{S} is irrelevant, as discussed in section 4.2). It should also be mentioned in advance that the transition to QT carried out below does not lead to any modification of the Zeemann term. The latter is a

consequence of the minimal coupling rule if the momentum field is displayed correctly - that is, taking into account its vortical degrees of freedom.

Let us now ask for possible interpretations of the “magnetic moment of the electron”. In the particle picture it was suggested by Uhlenbeck and Goudsmit [48] that an “intrinsic” angular momentum of the electron, called *spin*, is responsible for the observed effects. The magnitude of the spin vector is assumed to be a constant equal to $\frac{\hbar}{2}$. As already mentioned, the form of the relevant coupling term is compatible with this assumption, but not the magnitude of the prefactor ($g = 2$ instead of $g = 1$). The second, much more serious shortcoming is the universally accepted fact that such a classical intrinsic spin cannot exist because it is in conflict with fundamental physical principles [49]. While the theories of Dirac [45], Eberlein [42], and Levy-Leblond [46] provide an explanation for the value $g = 2$, the fundamental second difficulty remains. In a relativistic world the construction of a classical model for a spinning electron is just as impossible as in the non-relativistic case. Thus, while the existence of quantum spin is experimentally extremely well confirmed, a classical counterpart of this property - if interpreted as a property of individual electrons - does not exist [50]. Nevertheless, the intuitive ideas of Uhlenbeck and Goudsmit dominate our thinking about spin even today. The reason is that there is no better explanation, at least in the framework of the individuality interpretation (particle picture) of QT. This represents a painful gap in our understanding of nature, especially in view of the fundamental importance of spin for the stability of matter.

This gap in our understanding disappears if one renounces the claim to be able to describe the behavior of individual particles with the help of QT. In the ensemble interpretation on which the present work is based, it is possible to explain the origins of quantum spin in a simple way: The momentum field, which describes a collective of particles and assigns a momentum to every point of configuration space, must have three independent components, as space is three dimensional. This simple fact leads to the appearance of two additional fields describing an *internal degree of freedom of the ensemble* (so one can dispense with the strange idea that a point particle has internal degrees of freedom). After linearization one obtains the complete basic equation of non-relativistic QT for a single particle, derived by Eberlein and Levy-Leblond.

It is of course also possible to introduce additional spin degrees of freedom in phase space. The problem here, however, is that you then make assumptions that do not correspond to reality. In contrast, the existence of spin in the ensemble theory follows automatically from the basic assumption that the dynamic variables of quantum theory only depend on the space-time coordinates as independent variables.

The two difficulties of the individuality interpretation mentioned above do not arise in the ensemble interpretation. The difficulty, or better impossibility, to understand spin as an intrinsic angular momentum of a single particle does not exist in the ensemble theory, since the spin in this interpretation is a collective property that results from a vortical component in the momentum

distribution of the probabilistic ensemble of *all* particles. Of course, there is then no difficulty in accepting a deviation of the gyromagnetic ratio from the single particle value.

7 Transition to quantum theory by linearization

Let us briefly recap at this point. Our first step was the projection from phase space to configuration space, transforming Eqs. (1)-(4) to Eqs. (7), (8), (14), (15). With this first, most fundamental step, the globally valid theory PM was converted to the only locally valid theory QA. The next two steps were the introduction of independent variables (potentials) and the transition to canonical Clebsch potentials. Of course, these two steps, that led to Eqs. (66)-(69) do not change anything in the fundamentally unacceptable nature (only local validity) of the above theory. This also applies to the next step that we have carried out, namely the rewriting of the basic equations in the quantum-like form (79).

The form (79) of the QA, in which the quantum variable ψ is already used, is still a theory in which particle trajectories exist - albeit subject to "local validity" as discussed in III. Using this form of the basic equations of QA, the influence of an external electromagnetic field could be taken into account in a simple way, with the help of the minimal coupling rule. This form also allows a particularly simple transition to QT. This transition must be performed in such a way that the global validity, which has been lost during the projection, emerges again from the locally valid QA. This requires either linearizing Eq. (79) or randomizing Eqs. (66)-(69), as discussed in detail in III. It will be shown that both kinds of "quantization" can be carried out quite analogously to the spinless case. This final transition to QT is a process that is completely independent from the presence or absence of an electromagnetic field. We therefore omit the electrodynamic terms in (89) for the sake of clarity.

The transition to QT by linearization takes place in a simple manner by omitting the non-linear term on the right-hand side of Eq. (79). In the resulting quantum theoretical evolution equation

$$\left[\frac{\hbar}{i} \partial_t - \frac{\hbar^2}{2m} \partial_k \partial_k + V \right] \psi_a = 0, \quad (99)$$

the role of S , ϑ , φ as functions defining the momentum field is destroyed and the particle equations of motion $\dot{q}_k = v_k(q, t)$ become meaningless. All theorems (such as those of Helmholtz type) that are based on deterministic laws for particle motion become invalid. This conclusion is of course the same as in the irrotational case studied in III. Probabilistic theories of this kind, in which probabilistic statements about particles can be made, while no statements can be made about the orbits of the particles, were referred to as Type 3 theories in an earlier work of the present author [51]. Thus, the restoration of the global validity of our theory leads to a radical change in its physical meaning. Just as radical is the change in the mathematical description that is now made

possible by linearity. The new probabilistic structure, the role of eigenvalues and non-commuting observables, Born's rule, and other characteristics of QT were derived in I and II.

The equation of motion (99) is a doubling of the Schrödinger equation and at first glance appears to be equivalent to it. But this is not the case; the new discrete degree of freedom has important physical consequences even in the absence of an external electromagnetic field. The latter may be introduced in the same way as in section 6.2 and leads to Eq. (89) with vanishing right hand side.

7.1 Alternative description of the linearization

Let us ask which modifications Eqs. (66)-(69) will undergo as a consequence of the transition to QT. The answer provides us with a reformulation of Schrödinger's equation in terms of the variables $S, \varphi, \vartheta, \rho$ which will be useful later. We assume that the new equations take the form

$$\frac{\partial S}{\partial t}(q, t) + \frac{\hbar}{2} \cos \vartheta(q, t) \frac{\partial \varphi}{\partial t}(q, t) + H^0 \left(q, \frac{\partial S}{\partial q} + \frac{\hbar}{2} \cos \vartheta \frac{\partial \varphi}{\partial q} \right) = L_S, \quad (100)$$

$$\left[\frac{\partial}{\partial t} + V_k^0 \left(q, \frac{\partial S}{\partial q} + \frac{\hbar}{2} \cos \vartheta \frac{\partial \varphi}{\partial q} \right) \frac{\partial}{\partial q_k} \right] \varphi(q, t) = L_\varphi, \quad (101)$$

$$\left[\frac{\partial}{\partial t} + V_k^0 \left(q, \frac{\partial S}{\partial q} + \frac{\hbar}{2} \cos \vartheta \frac{\partial \varphi}{\partial q} \right) \frac{\partial}{\partial q_k} \right] \frac{\hbar}{2} \cos \vartheta(q, t) = L_\vartheta, \quad (102)$$

$$\frac{\partial \rho}{\partial t}(q, t) + \frac{\partial}{\partial q_k} \rho(q, t) V_k^0 \left(q, \frac{\partial S}{\partial q} + \frac{\hbar}{2} \cos \vartheta \frac{\partial \varphi}{\partial q} \right) = L_\rho, \quad (103)$$

with new terms L_A , where $A = S, \varphi, \vartheta, \rho$, replacing the zeros on the right-hand sides of Eqs. (66)-(69).

We will here determine the quantum terms L_A in a purely formal way, as a consequence of the linearization, postponing questions concerning deeper physical meaning. We denote the left-hand sides of (66)-(69) by T_A . The four basic equations $T_A = 0$ of QA are equivalent to the nonlinear equations

$$\mathbb{L}\psi + \mathbb{M} = 0, \quad (104)$$

where \mathbb{L} and \mathbb{M} are given by $\mathbb{L} = \frac{\hbar}{i} \partial_t - \frac{\hbar^2}{2m} \partial_k \partial_k + V$ and $\mathbb{M}_a = \frac{\hbar^2}{2m} [f_{k(a)} f_{k(a)} + (\partial_k f_{k(a)})] \psi_a$. On the other hand, the four basic equations $T_A = L_A$ of QT are equivalent to the linear equations

$$\mathbb{L}\psi = 0. \quad (105)$$

We multiply the spinor (104) from the left by ψ^+ and ϕ^+ , where the spinor ϕ , which is orthogonal to ψ , is defined by $\phi = \sqrt{\rho} e^{\frac{i}{\hbar} S} (u_2^*, -u_1^*)^T$. Taking advantage of the fact that all equations are evolution equations, containing

only a single first-order derivative with respect to time, we can identify the real and imaginary parts of the resulting expressions and obtain the relations

$$\rho T_S + \iota \frac{\hbar}{2} T_\rho = \psi^+ \mathbb{M} + \psi^+ \mathbb{L} \psi \quad (106)$$

$$-\frac{\hbar}{2} \rho T_\vartheta + \iota \frac{\hbar}{2} \rho \sin \vartheta T_\varphi = \phi^+ \mathbb{M} + \phi^+ \mathbb{L} \psi. \quad (107)$$

We proceed in the same way with the spinor (105) and obtain the relations

$$\rho (T_S - L_S) + \iota \frac{\hbar}{2} (T_\rho - L_\rho) = \psi^+ \mathbb{L} \psi \quad (108)$$

$$-\frac{\hbar}{2} \rho (T_\vartheta - L_\vartheta) + \iota \frac{\hbar}{2} \rho \sin \vartheta (T_\varphi - L_\varphi) = \phi^+ \mathbb{L} \psi. \quad (109)$$

Comparison of (106), (107) and (108), (109) shows that the terms L_A obey the equations $\rho L_S + \iota \frac{\hbar}{2} L_\rho = \psi^+ \mathbb{M}$ and $-\frac{\hbar}{2} \rho L_\vartheta + \iota \frac{\hbar}{2} \rho \sin \vartheta L_\varphi = \phi^+ \mathbb{M}$. It is of course reasonable that the additional quantum terms may be determined from the nonlinear term \mathbb{M} which makes the difference between QA and QT. A longer calculation leads to the result

$$L_S = \frac{\hbar^2}{8m} \left\{ 4\rho^{-\frac{1}{2}} \left(\partial_k \partial_k \rho^{\frac{1}{2}} \right) - \sin^2 \vartheta (\partial_k \varphi) (\partial_k \varphi) - (\partial_k \vartheta) (\partial_k \vartheta) \right\}, \quad (110)$$

$$L_\varphi = \frac{\hbar}{2m} \left\{ \cos \vartheta (\partial_k \varphi) (\partial_k \varphi) - \frac{1}{\sin \vartheta} (\partial_k \partial_k \vartheta) - \frac{1}{\rho \sin \vartheta} (\partial_k \rho) (\partial_k \vartheta) \right\}, \quad (111)$$

$$L_\vartheta = \frac{\hbar}{2m} \left\{ 2\rho^{-\frac{1}{2}} \left(\partial_k \rho^{\frac{1}{2}} \right) \sin \vartheta (\partial_k \varphi) + 2 \cos \vartheta (\partial_k \varphi) (\partial_k \vartheta) + \sin \vartheta (\partial_k \partial_k \varphi) \right\}, \quad (112)$$

$$L_\rho = 0. \quad (113)$$

Equations (100)- (103), with the terms L_A given by (110)- (113) are equivalent to the spinorial Schrödinger Eq. (99). Apart from the continuity equation (103), which remains unchanged, the new quantum terms lead to a coupling of the variables S , ρ , φ , ϑ which makes the concept of individual particle trajectories meaningless. The situation is basically the same as in the case of irrotational momentum fields treated in III; in this case only the first term of L_S survives, which is sometimes (misleadingly) referred to as “quantum potential”. Our next task, performed in section 8, is to understand the terms L_A with the help of statistical concepts.

The form (100)- (103) of the Schrödinger equation, with the terms L_A given by (110)- (113), was derived many years ago in pioneering work by Bohm and co-workers [26], [52] and Takabayasi [38]. In earlier work by Takabayasi [33] and also in works by Bialynicki-Birula [53] hydrodynamic variables (velocity field or momentum field) in addition to spin variables were used as dynamic variables instead of the three quantum mechanical potentials used in the present theory. The basic equations of this “hydrodynamic formulation” of QT [53]

may be derived from the above equations by means of a tedious differentiation. The derivation of the above equations presented here can be seen as a continuation and completion of the theories of Bohm and Takabayasi. It has now been possible to derive the same equations starting from the well-understood classical theory PM, and give also plausible reasons for the individual steps leading from PM to QT.

8 Transition to quantum theory by randomization

The zeroing of the nonlinear term in Eq. (79), that creates QT is, in physical terms, a randomization. However, it is an unusual kind of randomization. With the standard concept of randomization, as used in classical probabilistic physics, only the initial conditions are random while the particle movement itself is ruled by deterministic laws. This standard type of probabilistic theory was referred to as type 2 theory in a previous work of the present author [51]. In contrast, the above transition from (79) to (99) makes the equations of motion themselves “random” (which means nonexistent), while statistical predictions about the behavior of the particles are still possible. This (quantum) type of probabilistic theory was referred to as type 3 theory [51]. The question arises as to whether we can better understand this transition from type 2 to type 3. Are there statistical assumptions that are equivalent to the linearization process, explaining, in an alternative way, the transition from (79) [or (66)-(69)] to (99) ? In III it was shown that this question can be answered in the affirmative in the case of irrotational momentum fields. Here we show that the same holds true in the present completed theory. The following construction is an expanded and improved version of an earlier theory of the present author [27].

8.1 Definition of Ehrenfest-like relations

There should be as close a relationship as possible between the statistical equations of type 3 that we want to construct, and the equations of classical mechanics. We assume therefore the validity of Ehrenfest-like relations of the form

$$d_t \bar{q}_k(t) = \frac{1}{m} \bar{p}_k(t), \tag{114}$$

$$d_t \bar{p}_k(t) = \overline{F_k(q, t)}, \tag{115}$$

$$d_t \bar{s}_k(t) = \overline{T_k(q, t)}, \tag{116}$$

where $F_k(q, t)$ is the external force and the “macroscopic variables” \bar{q} , \bar{p} , \bar{s} are defined as average values

$$\bar{q}_k(t) = \int dq \rho(q, t) q_k, \tag{117}$$

$$\bar{p}_k(t) = \int dq \rho(q, t) M_k(q, t), \quad (118)$$

$$\bar{s}_k(t) = \int dq \rho(q, t) s_k(q, t), \quad (119)$$

of the “microscopic” field variables q , M , s . The “statistical conditions” (114)-(116) ensure that the mean values of $M_k(q, t)$ and $s_k(q, t)$ obey particle-like relations. We switch on an external electromagnetic field by taking into account the terms containing Φ , \mathbf{A} and \mathbf{B} [see Eqs. (90), (94) and (96)] which are induced by the process of minimal coupling. We do this because in the course of the following derivation of Schrödinger’s equation, it will be possible to establish a connection between the “potentials” Φ , \mathbf{A} and \mathbf{B} on the one hand and associated external forces \mathbf{F} on the other hand. It is instructive to write the energy and momentum components M_0 and M_k in the form

$$M_0 = \partial_t S' + R_0, \quad M_k = \partial_k S' + R_k, \quad (120)$$

where

$$\partial_t S' = \partial_t S + e\Phi, \quad \partial_k S' = \partial_k S - \frac{e}{c} A_k, \quad (121)$$

$$R_0 = \frac{\hbar}{2} \cos \vartheta \partial_t \varphi, \quad R_k = \frac{\hbar}{2} \cos \vartheta \partial_k \varphi. \quad (122)$$

Both the multivalued phase S' (in contrast to S the first derivatives of S' with respect to q_k and t are not interchangeable) and the terms R_k contain rotational components of M_k . The terms that come from S' describe the external field, and the terms that come from R_k describe the internal part of the rotational component of the momentum field. The fundamental law of conservation of probability

$$\partial_t \rho + \frac{1}{m} \partial_k \rho M_k = 0, \quad (123)$$

with the probability current v_k defined by $mv_k = M_k$, completes the set of our basic equations. Let us note that the forces \mathbf{F} and \mathbf{T} on the right-hand sides of (115) and (116) have a completely different character. The form of \mathbf{T} is given by the minimal coupling mechanism [see Eq. (96)]. In contrast, the form of \mathbf{F} need not be specified, as will be shown in the next section.

8.2 Implementing the statistical conditions

The first statistical condition (114) is automatically satisfied due to the continuity equation and the definition of $\bar{q}_k(t)$ and $\bar{p}_k(t)$. Using the continuity equation again and performing some rearrangements, the second statistical condition (115) takes the form

$$- \int dq (\partial_k \rho) \left[M_0 + \frac{1}{2m} M_i M_i \right]$$

$$+ \int dq\rho \left[\frac{1}{m} M_i \Omega_{ik} + (\partial_t M_k - \partial_k M_0) \right] = \int dq\rho F_k, \quad (124)$$

where Ω_{ik} is the vorticity tensor defined by Eq. (12). We have not specified the external forces that appear in our statistical theory. The reason is that the form of these forces may be *derived* as a consequence of the special structure of our theory. The Ehrenfest-like relations (114)-(116) are integral equations. Our task is to derive one or more differential equations from these integral equations. The forces \mathbf{F} on the right-hand side of (115) must be designed to *allow* this. That means there is a “statistical constraint” on the forces in the present theory.

The concrete form of this restriction can already be seen in Eq. (124): The permitted forces - that are compatible with the minimal coupling rule - must already appear on the left-hand side of (124), namely in the form of statistical mean values. Only such forces can be real, because they may cancel with the same forces on the right-hand side, thus disappearing from the integral equation and making the derivation of one or more differential equations possible.

We now use the decomposition of M_0 and M_k according to Eq. (120). The square bracket in the second term of Eq. (124) takes the form:

$$\begin{aligned} & \left[\frac{1}{m} M_i \Omega_{ik} + (\partial_t M_k - \partial_k M_0) \right] = \\ & \frac{1}{m} (\partial_i S' + R_i) [\partial_i, \partial_k] S' + [\partial_t, \partial_k] S' \\ & + \frac{1}{m} (\partial_i S' + R_i) (\partial_i R_k - \partial_k R_i) + \partial_t R_k - \partial_k R_0, \end{aligned} \quad (125)$$

where $[\partial_i, \partial_k] S' = \frac{e}{c} (\partial_k A_i - \partial_i A_k)$ and $[\partial_t, \partial_k] S' = -\frac{e}{c} (\partial_t A_k + c \partial_k \Phi)$. Because of the well-known relations $B_k = \epsilon_{kij} \partial_i A_j$ and $E_k = -\frac{1}{c} \partial_t A_k - \partial_k \Phi$ the first term on the right hand side of (125) is given by the Lorentz force

$$F_k^{(L)} = e E_k + \frac{e}{c} v_i (\partial_k A_i - \partial_i A_k). \quad (126)$$

So we write $\mathbf{F} = \mathbf{F}^{(L)} + \mathbf{F}^{(2)}$ and skip the Lorentz force from both sides of Eq. (124). The second statistical condition now takes the form

$$\begin{aligned} & - \int dq(\partial_k \rho) \left[M_0 + \frac{1}{2m} M_i M_i \right] \\ & + \int dq\rho \left[+ \frac{1}{m} (\partial_i S' + R_i) (\partial_i R_k - \partial_k R_i) + \partial_t R_k - \partial_k R_0 \right] = \int dq\rho F_k^{(2)}, \end{aligned} \quad (127)$$

We now express the vortical components R_0 , R_k according to (122) in terms of the Clebsch potentials and obtain after a few rearrangements the representation

$$-\int dq(\partial_k \rho) \left[M_0 + \frac{1}{2m} M_i M_i \right] - \frac{\hbar}{2} \int dq \rho \sin \vartheta \left\{ (\partial_k \varphi) D_t \vartheta - (\partial_k \vartheta) D_t \varphi \right\} = \int dq \rho F_k^{(2)}, \quad (128)$$

for the second statistical condition. At this point the third statistical condition must be taken into account.

If we identify the minimal coupling force \mathbf{T} with the right hand side of (96) and use the continuity equation (123), the integral equation (116) takes the form

$$\int dq \rho \left[D_t s_k + \frac{e}{mc} \epsilon_{kij} B_i s_j \right] = 0. \quad (129)$$

The trivial solution of this equation (vanishing of the square bracket) agrees with the quasi-classical field equation (96). The simplest nontrivial differential equation to be associated with the integral equation (129) has the form

$$D_t s_k + \frac{e}{mc} \epsilon_{kij} B_i s_j = \frac{\hbar}{2} G_k, \quad (130)$$

where the G_k are three functions with vanishing average value. We split off the factor $\frac{\hbar}{2}$ so that the functions G_k do not depend on the parameter defining the length of the spin vector; the quantities G_k should only describe the influence of the randomization. If we insert s_k [see Eq. (95)] in (130) we obtain the differential equations for the two independent field variables ϑ and φ ,

$$D_t \vartheta = \frac{e}{mc} (B_1 \cos \varphi - B_2 \sin \varphi) - \frac{G_3}{\sin \vartheta}, \quad (131)$$

$$D_t \varphi = \frac{e}{mc} \frac{1}{\sin \vartheta} (-B_1 \cos \vartheta \sin \varphi - B_2 \cos \vartheta \cos \varphi + B_3 \sin \vartheta) + \frac{1}{\sin \vartheta} (G_1 \cos \varphi - G_2 \sin \varphi). \quad (132)$$

The form of these differential equations is still undetermined since the functions G_k are not known. What we do know is that the G_k have to obey the conditions

$$\int dq \rho G_k = 0, \quad G_k s_k = 0. \quad (133)$$

The first of these says that the average value of G_k has to vanish, the second is a solvability condition that takes into account the fact that \mathbf{s} is a vector of constant length. With the help of (131) and (132) we can now eliminate the total derivatives $D_t \vartheta$ and $D_t \varphi$ from the second statistical condition (128). Performing some rearrangements and a partial integration Eq. (128) takes the form

$$-\int dq(\partial_k \rho) \left[M_0 + \frac{1}{2m} M_i M_i + \mu_i B_i \right] - \int dq \rho \mu_i \partial_k B_i$$

$$+\frac{\hbar}{2} \int dq\rho \left\{ G_1 \cos \varphi \partial_k \vartheta - G_2 \sin \varphi \partial_k \vartheta + G_3 \partial_k \varphi \right\} = \int dq\rho F_k^{(2)}, \quad (134)$$

We see that the vortical part of the momentum field leads to the potential energy term $\mu_i B_i$, the force $\mu_i \partial_k B_i$, and to the integral depending on the G_i . Except for the prefactor, the newly derived force agrees with the electrodynamic force exerted by an inhomogeneous magnetic field on a magnetic dipole. As is well known, it plays a central role in the interpretation of the Stern-Gerlach experiment. We eliminate the new force by setting

$$F_k^{(2)} = -\mu_i \partial_k B_i + F_k^{(3)}. \quad (135)$$

The remaining force $F_k^{(3)}$ can only have the form $-\partial_k V$. This leads to the usual mechanical potential V in the field equation. Then, the second statistical condition takes the form

$$\begin{aligned} & - \int dq(\partial_k \rho) \left[M_0 + \frac{1}{2m} M_i M_i + \mu_i B_i + V \right] \\ & + \frac{\hbar}{2} \int dq\rho \left\{ G_1 \cos \varphi \partial_k \vartheta - G_2 \sin \varphi \partial_k \vartheta + G_3 \partial_k \varphi \right\} = 0, \end{aligned} \quad (136)$$

The second integral in Eq. (136) must be a contribution to the field equations. This implies a relationship of the form

$$\frac{\hbar}{2} \int dq\rho \left\{ G_1 \cos \varphi \partial_k \vartheta - G_2 \sin \varphi \partial_k \vartheta + G_3 \partial_k \varphi \right\} = \int dq(\partial_k \rho) L'_0, \quad (137)$$

whereby L'_0 is an unknown function. Then the second statistical condition takes the form

$$- \int dq(\partial_k \rho) \left[M_0 + \frac{1}{2m} M_i M_i + \mu_i B_i + V - L'_0 \right] = 0. \quad (138)$$

The simplest nontrivial differential equation whose solutions solve this integral equation is given by

$$M_0 + \frac{1}{2m} M_i M_i + \mu_i B_i + V = L_0, \quad (139)$$

where the function L_0 obeys the conditions

$$L_0 = L'_0 + \Delta L_0, \quad \int dq(\partial_k \rho) \Delta L_0 = 0. \quad (140)$$

We have thus transformed the second statistical condition, originally formulated as an integral equation, into a partial differential equation. The latter contains, however, 4 unknown functions G_i , L_0 which are still to be determined.

8.3 Determination of the quantum terms G_i, L_0

The first fundamental assumption that we made when constructing our type 3 theory was the validity of the Ehrenfest-like relations (114)-(116). This assumption led us to the field equations (123), (139), (131), (132) for our dynamic variables $\rho, S, \vartheta, \varphi$. These new equations differ from the field equations of our original type 2 theory by terms G_i, L_0 , which are unknown functions of our dynamic variables and their derivatives. The conditions listed above are not sufficient to determine the G_i, L_0 . We need a second assumption, presumably of a statistical nature. A most fundamental statistical principle says that all states that are unknown must occur with the same probability. In statistical mechanics (a type 2 theory) this principle is implemented through the requirement for maximum entropy. In the problem at hand, we are faced with the task of determining certain terms in a differential equation in accordance with this principle. In this case, the same general principle leads to the requirement for minimal Fisher information [54],[55],[51],[27]. In the following we will derive the terms implied by the requirement of minimal Fisher information with the help of a variational problem. Large parts of the related calculations have already been reported[27]. In this regard, we may be brief here.

We introduce the abbreviation L for the left hand side of Eq. (139) so that this equation takes the form $L - L_0 = 0$. We assume that the function L_0 we are looking for depends only on the variables ρ, ϑ, φ , and their first and second derivatives with respect to q_k [27]. Our fundamental second statistical assumption is that the spatial and temporal average value of $L - L_0$ is extremal with respect to the variation of $S, \rho, \vartheta, \varphi$,

$$\delta \int dt \int dq \rho (L - L_0) = 0. \quad (141)$$

Furthermore, the four field equations (123), (139), (131), (132) must also be fulfilled; we write this condition symbolically for the sake of brevity in the form

$$E_a = 0, \quad a = \rho, S, \vartheta, \varphi. \quad (142)$$

The two conditions (141) and (142) lead to differential equations for G_i, L_0 . As shown in detail in [27], the physically relevant solution of these equations is given by

$$L_0 = \frac{\hbar^2}{2m} \left[\frac{1}{\sqrt{\rho}} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_k} \sqrt{\rho} - \frac{1}{4} \sum_k \left\{ \sin^2 \vartheta \left(\frac{\partial \varphi}{\partial q_k} \right)^2 + \left(\frac{\partial \vartheta}{\partial q_k} \right)^2 \right\} \right], \quad (143)$$

$$\hbar G_1 = \frac{\hbar^2}{2m} \frac{1}{\rho} \frac{\partial}{\partial q_k} \rho \left(\frac{1}{2} \sin 2\vartheta \sin \varphi \frac{\partial \varphi}{\partial q_k} - \cos \varphi \frac{\partial \vartheta}{\partial q_k} \right), \quad (144)$$

$$\hbar G_2 = \frac{\hbar^2}{2m} \frac{1}{\rho} \frac{\partial}{\partial q_k} \rho \left(\frac{1}{2} \sin 2\vartheta \cos \varphi \frac{\partial \varphi}{\partial q_k} + \sin \varphi \frac{\partial \vartheta}{\partial q_k} \right), \quad (145)$$

$$\hbar G_3 = -\frac{\hbar^2}{2m} \frac{1}{\rho} \frac{\partial}{\partial q_k} \left(\rho \sin^2 \vartheta \frac{\partial \varphi}{\partial q_k} \right). \quad (146)$$

A new adjustable parameter appears on the right-hand-sides of these expressions which has been identified with $\hbar^2/2m$. Let us recall here that two different adjustable parameters appeared in the course of our developments both of which were identified with \hbar . The first \hbar was associated with the length of the spin vector \mathbf{s} . The second \hbar is associated with the quantum-mechanical principle of maximal disorder. The physical meaning of these two adjustable parameters is different, but they must both be identified with Planck's constant in order to enable the transition to QT.

As may be easily checked, the conditions (133), (137), (140) for the L_0 , G_i are all fulfilled. In particular, $L_0 = L'_0 + \Delta L_0$, where

$$L'_0 = -\frac{\hbar^2}{8m} \left[\sum_k \left\{ \sin^2 \vartheta \left(\frac{\partial \varphi}{\partial q_k} \right)^2 + \left(\frac{\partial \vartheta}{\partial q_k} \right)^2 \right\} \right], \quad \Delta L_0 = \frac{\hbar^2}{2m} \frac{1}{\sqrt{\rho}} \frac{\partial}{\partial q_k} \frac{\partial}{\partial q_k} \sqrt{\rho}, \quad (147)$$

and L'_0 fulfills (137). One can also show that the mean value of L_0 agrees with the Fisher functional [55], [27]. If the above solution for L_0 is inserted, then the variational principle (141) leads to the correct field equations (142). We did not use the principle of variation here in the usual way, as a mathematical tool to derive field equations from a given Lagrangian function, but we use it instead to construct the Lagrangian itself.

Finally, we should compare the above field theory, which is based on statistical postulates, with the earlier one generated by linearization. The easily verifiable relations

$$L_S = L_0, \quad (148)$$

$$L_\vartheta = -\frac{1}{\sin \vartheta} G_3, \quad (149)$$

$$L_\varphi = \frac{\cos \varphi}{\sin \vartheta} G_1 - \frac{\sin \varphi}{\sin \vartheta} G_2. \quad (150)$$

show that both theories are identical. We have thus shown that one may obtain the Pauli equation

$$\left[\frac{\hbar}{i} \partial_t - e\Phi - \frac{\hbar^2}{2m} \sum_{k=1}^3 \left(\partial_k - i \frac{e}{\hbar c} A_k \right)^2 + \mu_B \sigma^k B_k + V \right] \psi = 0. \quad (151)$$

either through the “formal” process of linearization or through the implementation of some plausible statistical postulates. The first way is a discontinuous process that destroys the possibility of describing particle motion and thus has the physical meaning of a randomization. With the second way, we have replaced this discontinuous process with a continuous one. This enabled us to understand the detailed nature of this randomization. The second way is

longer and “less elegant” than the first, but allows a more precise insight into the transition from QA to QT.

9 Discussion

The second version of the HLLK reported here starts from PM, generates the theory QA by projection onto configuration space, and then realizes the transition to QT by a linearization or randomization. The theory QA, appearing only in this second version of the HLLK, represents the transition area, the “borderland”, between classical physics and QT [56]. This theory is unphysical in that the particle trajectories that occur are only locally valid and are not realized in nature (although they may be a good approximation in certain situations). The importance of this theory is that it represents a clearly defined and justifiable intermediate step in the construction of the QT.

Let us first summarize what has been achieved so far by looking at the list of essential properties of QT already given in II:

1. Schrödinger’s equation as fundamental dynamical law - and eigenvalues as observable numbers.
2. The nonstandard probabilistic structure of QT - in particular non-commuting observables.
3. Born’s rule - the law which tells us how to extract probabilistic predictions from the theory.
4. The minimal-coupling rule - the way interactions are formulated in QT.
5. The existence of spin - a particularly mysterious phenomenon believed to belong to QT exclusively
6. The anomalous value of the magnetic moment of the electron - a spin related phenomenon
7. The spin-statistics connection - a spin related multi-particle phenomenon

Points 1, 2 and 3 were derived in papers I and II. Without going into detail, we mention that the generalization of Born’s rule to degenerate states may be obtained in a straightforward way (the classical counterpart of degenerate states are non-connected level sets). In contrast to I and II, only a single observable, namely the Hamilton function $H(q, p)$, is studied in III and the present work. Our research in the “borderland” created a new derivation of the Schrödinger equation, which allows for a deeper understanding of the relationship between QT and classical physics. We found, as most important result of the present work, that spin is not a purely quantum mechanical phenomenon and that the value $1/2$, which is experimentally observed for the spin of all massive structureless particles, is a consequence of the three-dimensionality of space. An associated result is the value $g = 2$ for the gyromagnetic ratio of the electron. We have thus essentially understood points 5 and 6. In the course of our second derivation of the Schrödinger equation (in which statistical assumptions were used) we were also able to understand why in QT the influence of external force fields must be described with the help of potentials. Of course,

we do not claim to have explained all the details related to points 1 – 6. The last point 7, the spin-statistics connection for massive particles, remains open at the moment, but we have little doubt that this point can also be explained in the framework of the HLLK. In the remainder of this section, we continue to discuss some relevant points in more detail.

9.1 The meaning of spin

“Pls tell me why is spin of an electron $+1/2$ or $-1/2$ and not something like $+1$ or -1 ?”

Tridib Banerjee (age 16)

In this work it was possible to identify the origin of spin. This property is not, as has long been assumed, to be regarded as a purely quantum mechanical phenomenon. Rather, its origin may be found in the quasi-classical (or quasi-quantum) theory QA, which represents the borderland between classical physics and QT. As this theory is not realized in nature it would not be correct to claim that spin has a classical origin. There is no such phenomenon in phase space (although it is formally possible to introduce corresponding degrees of freedom) since no self-rotation of individual particles exists in nature.

The possibility of discovering the origin of spin arises by replacing the particle momenta p by momentum fields $M(q, t)$, and thus halving the number of degrees of freedom. A necessary prerequisite for the occurrence of spin is the simple fact that all degrees of freedom of the momentum fields $M(q, t)$ have to be taken into account, in particular those which result in a non-vanishing rotation. The origin of the quantum spin is thus a collective property of the probabilistic ensemble in configuration space, which is associated with the rotational degrees of freedom. The idea that spin should better be understood as a non-localized phenomenon has already been expressed several times in the literature [50], [57].

A simple and extremely important property of nature is that spin-1/2 particles are fermions, which means they obey Fermi-Dirac statistics. When asked about this fact, Richard Feynman suggested that there should be a simple explanation for such a fundamental fact. And that if we cannot find such a simple explanation, we should admit that we have not understood the phenomenon. We cannot give an answer to Feynman’s question in the present work. But we can answer an even more fundamental question, a question so fundamental that it is hardly ever asked by professionals. Namely the question, why all massive structureless particles in nature have spin $1/2$. Based on the present reconstruction of QT for the case of a single particle, the reason is the three-dimensionality of space. As a consequence of this three-dimensionality we have to introduce *three* functionally independent potentials in order to represent the momentum field in a correct way. Together with the probability density, we have four independent real functions that is a 2-component spinor,

in other words a spin 1/2 particle. This answer is hard to beat in terms of simplicity.

According to the current state of knowledge in the theory of elementary particles, (almost) all structureless massive particles are actually spin 1/2 particles. One would justifiably object that reality has a relativistic space-time structure that is not correctly described by our non-relativistic theory. But it is also a fact that our conclusion is essentially based only on the number of spatial dimensions. It is therefore not unreasonable to assume that our conclusion remains valid even when moving to a relativistic theory.

9.2 The limit $\hbar \rightarrow 0$ of the Pauli-Schrödinger equation

In this section we discuss the question of which theory the Pauli-Schrödinger theory “reduces” to in the limit $\hbar \rightarrow 0$. There are two types of reduction which Rosaler called formal and empirical reduction [58]; a similar distinction was made by Berry [56]. The important difference between these two concepts was discussed in detail in III. Here we are only examining the concept of formal reduction, that is, we are examining the behavior of the basic equations of our theory in the limit of small \hbar .

The above question was decided for a long time on the basis of philosophical postulates. Also, no distinction was made between formal and empirical reduction. The first mathematically precise investigation of the question of formal reduction was carried out in [59] and II and led to the conclusion that QT (without spin) cannot be reduced to either CM or PM. This conclusion meant that - quite contrary to the prevailing opinion - QT cannot (formally) be reduced to classical physics, but, on the contrary, it must be possible to derive it from classical physics. This conclusion thus provided the basis for the reconstruction of QT carried out in I-III and here. The general conclusion that QT reduces for small \hbar neither to CM nor to PM but to a (quasi-classical) probabilistic theory in configuration space, does not depend on whether we take spin into account or not. The presence of spin leads however to some peculiarities, which we want to summarize in the following.

The version of the Pauli-Schrödinger equation that is best suited for examining the limit of small \hbar is given by Eqs. (139), (130), (123) for the potentials S , ϑ , φ and the probability density ρ . To make the dependence on \hbar visible, we write Eqs. (139), (130) in the form

$$\partial_t S + e\Phi + \frac{\hbar}{2} \cos \vartheta \partial_t \varphi + \frac{1}{2m} \sum_k \left(\partial_k S - \frac{e}{c} A_k + \frac{\hbar}{2} \cos \vartheta \partial_k \varphi \right)^2 - \frac{\hbar}{2} \frac{e}{mc} s_i B_i = L_0, \quad (152)$$

$$\left[\partial_t + \frac{1}{m} \left(\partial_i S - \frac{e}{c} A_i + \frac{\hbar}{2} \cos \vartheta (\partial_i \varphi) \right) \partial_i \right] h_k + \frac{e}{mc} \epsilon_{kij} B_i h_j = G_k, \quad (153)$$

where L_0 and G_k are of the order \hbar^2 and \hbar , respectively and the mechanical potential has been omitted. We do not write down the continuity equation,

which behaves in an obvious way. If we set $\hbar = 0$ in these equations, we obtain the equations

$$\partial_t S + e\Phi + \frac{1}{2m} \sum_k \left(\partial_k S - \frac{e}{c} A_k \right)^2 = 0, \quad (154)$$

$$\left[\partial_t + \frac{1}{m} \left(\partial_i S - \frac{e}{c} A_i \right) \partial_i \right] h_k + \frac{e}{mc} \epsilon_{kij} B_i h_j = 0, \quad (155)$$

which, together with the continuity equation, represent the classical limit of the Pauli-Schrödinger equation. This is a classical (deterministic) field theory defined by the Hamilton-Jacobi equation (154) for a spinless charged particle in an electromagnetic field and the equation of motion (155) of the variable h_k which is associated with the vortical component of the momentum field. The change of h_k with time is determined by the solution S of (154), while conversely (154) does not depend on h_k .

The “survival” of spin variables in the case $\hbar = 0$ is no surprise in our theory, since we have identified the vortical components of the momentum field as origin of quantum spin. In all works in which Eqs. (152), (153) were derived so far, the starting point was the quantum mechanical Pauli-Schrödinger equation (151), which was then rewritten, using a representation like (75) (see [38], [26], [60]). The limiting case $\hbar = 0$ was rarely dealt with in these theories, see however [60]. One reason for this might be that this limiting case is not compatible with the prevailing interpretation of spin as a purely quantum mechanical phenomenon. Due to this interpretation, all spin variables (or the corresponding terms in a Lagrangian function) should disappear from the theory in the limit $\hbar = 0$. The fact that this is not the case led Yahalom to the conclusion that the Pauli theory “has no standard classical limit” [60]. In fact, one could have concluded from this fact that spin cannot be a purely quantum mechanical phenomenon.

A second difficulty is as follows. If one wants to eliminate only terms of order \hbar^2 in Eqs. (152), (153) and keep all terms of order \hbar , then one finds that this is not possible, since it destroys the structure of the kinetic energy. A meaningful theory, namely the Eqs. (90)-(94) which were the starting point for our last step to QT, can only be obtained if one eliminates only the terms $L_0 \hbar G_k$ and keeps terms of any order in \hbar that come from the spin amplitude. The present derivation from classical physics provides us automatically with a physically meaningful distinction between these two kinds of \hbar^2 terms.

9.3 The role of potentials

Why does the minimal coupling rule apply in QT? The point where it appears in our formalism is the projection from phase space to configuration space. This projection introduces an energy field M_0 and three components of the momentum field M_k [we may write these in the simplest case as derivatives with respect to t and q_k of an action variable $s(q, t)$, see Eq. (15)]. An external field can then no longer be taken into account by introducing a force field,

as in Newton's equations. Such an external influence can actually only be taken into account by modifying the fields M_0 and M_k . The minimal coupling rule represents the simplest possible modification of these fields, namely the *addition* of externally specified fields (potentials) $e\Phi$ and $-\frac{e}{c}A_k$ to M_0 and M_k . Konopinski has already pointed out that potentials provide field energy and momentum for exchange with charged matter [61]. In the context of the present derivation, there seems to be almost no alternative to this interpretation.

In the present framework of the ensemble interpretation of QT, it is clear that a local argument based on potentials is questionable. Rather, the non-gauge-invariant quantities $e\Phi$ and $-\frac{e}{c}A_k$ should be interpreted as describing the influence of an external electromagnetic field on a statistical ensemble as a whole. This fact requires a fundamental reconsideration of the Aharonov-Bohm effect, which will not be undertaken here.

9.4 Concerning the interpretation

Large parts of the scientific community are still dominated by the idea that the “single-particle Schrödinger equation” describes the behavior of a single particle. This idea is incompatible with the fact that one can only derive *statistical predictions* about the behavior of a single particle from the single-particle Schrödinger equation. All attempts to remove this logical contradiction with the help of complicated constructions have been unsuccessful and always will be. It can only be removed by abandoning certain philosophical principles, such as the belief that any “complete” description of nature must be deterministic. Then one can accept that the one-particle Schrödinger equation describes only a statistical ensemble of individual particles. This statistical interpretation forms the basis for the theory described in this series of works, which allows for an almost complete reconstruction of QT. This reconstruction includes not only the formal aspects but also all essential questions of the interpretation of the formalism. This high degree of agreement is a strong argument in favor of the ensemble interpretation.

References

- [1] Pauli, W.: In: Kronig, R., Weisskopf, V. (eds.) *Collected Scientific Papers by Wolfgang Pauli*, Vols. 1-2 vol. 2, pp. 201–213. Interscience Publishers, New York (1964)
- [2] Ballentine, L.E.: The statistical interpretation of quantum mechanics. *Reviews of Modern Physics* **42**, 358–381 (1970)
- [3] Groenewold, H.J.: On the principles of elementary quantum mechanics. *Physica* **12**, 405–460 (1946)
- [4] Van Vleck, J.H.: The correspondence principle in the statistical interpretation of quantum mechanics. *Proc. Natl. Acad. Sci. U.S.* **14**, 178–188

(1928)

- [5] Schiller, R.: Quasi-classical theory of the nonspinning electron. *Phys. Rev.* **125**(3), 1100–1108 (1962)
- [6] Schiller, R.: Quasi-classical theory of the spinning electron. *Phys. Rev.* **125**(3), 1116–1123 (1962)
- [7] Bopp, R.: La mécanique quantique est-elle une mécanique statistique classique particulière ? *Ann. Inst. H. Poincaré* **15**, 81–112 (1956)
- [8] Bopp, F.: The principles of the statistical equations of motion on quantum theory. In: Körner, S. (ed.) *Observation and Interpretation*, p. 189. Butterworths Scientific Publications, London (1957)
- [9] Klein, U.: From Koopman-von Neumann theory to quantum theory. *Quantum Stud.: Math. Found.* **5**, 219–227 (2018)
- [10] Klein, U.: From probabilistic mechanics to quantum theory. *Quantum Stud.: Math. Found.* **7**, 77–98 (2020)
- [11] Klein, U.: A reconstruction of quantum theory for nonspinning particles. arXiv:2202.13356, submitted to *Foundations of Physics*
- [12] Rund, H.: Clebsch potentials and variational principles in the theory of dynamical systems. *Arch. Ration. Mech. Anal.* **65**, 305 (1977)
- [13] Mukunda, N.: Phase space methods and the Hamilton-Jacobi form of dynamics. *Proc. Indian Acad. Sci.* **87**, 85–105 (1987)
- [14] Kozlov, V.V.: General Theory of Vortices. In: *Dynamical Systems X. Encyclopaedia of Mathematical Sciences*, vol. 67,. Springer, Berlin (2003)
- [15] Caratheodory, C.: *Calculus of Variations and Partial Differential Equations of the First Order, Part I*. Holden-Day, Inc, San Francisco (1965)
- [16] Clebsch, A.: Über eine allgemeine Transformation der hydrodynamischen Gleichungen. *J. reine angew. Math.* **54**, 293–312 (1857)
- [17] Baumeister, R.: Generalized Hamilton-Jacobi theories. *J. Math. Phys.* **19**(11), 2377 (1978)
- [18] Sudarshan, E.C.G., Mukunda, N.: *Classical Dynamics: A Modern Perspective*. Wiley, New York (1974)
- [19] Samuel, J.: A phase space approach to generalized Hamilton-Jacobi theory. *J. Math. Phys.* **21**, 56 (1980)

- [20] Moffatt, H.K.: The degree of knottedness of tangled vortex lines. *J. Fluid Mech.* **35**, 117–129 (1969)
- [21] Ranada, A.F.: On the magnetic helicity. *Eur. J. Phys.* **13**, 70–76 (1992)
- [22] Hopf, H.: Über die Abbildungen der dreidimensionalen Sphäre auf die kugelfläche. *Math. Ann.* **104**, 637–665 (1931)
- [23] Socolovsky, M.: On the geometry of spin 1/2. *Advances in Applied Clifford Algebras* **11**, 109–127 (2001)
- [24] Naber, G.L.: *Topology, Geometry and Gauge Fields*. Springer, New York (2011)
- [25] Takabayasi, T., Vigier, J.-P.: Description of Pauli matter as a continuous assembly of small rotating bodies. *Progress in Theoretical Physics.* **18**(6), 573–590 (1957)
- [26] Bohm, D., Schiller, R., Tiomno, J.: A causal interpretation of the Pauli equation (A). *Supplemento al Nuovo Cimento I Serie X*, 48–66 (1955)
- [27] Klein, U.: A statistical derivation of non-relativistic quantum theory. In: Pahlavani, M.R. (ed.) *Measurements in Quantum Mechanics*, pp. 141–174. ISBN:978-953-51-0058-4, Rijeka, Croatia (2012). see also arXiv:1109.6244 [quant-ph]
- [28] Payne, W.T.: Elementary spinor theory. *Am. J. Phys.* **20**(5), 253–262 (1952)
- [29] Lyons, D.W.: An elementary introduction to the Hopf fibration. *Mathematics Magazine* **76**(2), 87–98 (2003)
- [30] Whitehead, J.H.C.: An expresseion of Hopf’s invariant as an integral. *Proc. Nat. Acad. Sci. USA* **33**, 117–123 (1947)
- [31] Urbantke, H.K.: The Hopf fibration – seven times in physics. *J. Geom. Phys.* **46**, 125–150 (2003)
- [32] Kuznetsov, E.A., Mikhailov, A.V.: On the topological meaning of canonical Clebsch variables. *Physics Letters* **77A**(1), 37–38 (1980)
- [33] Takabayasi, T.: The vector representation of spinning particles in the quantum theory I. *Progress in Theoretical Physics.* **14**, 283–302 (1955)
- [34] Faddeev, L.D.: Some comments on the many-dimensional solitons. *Lett. Math. Phys.* **1**, 289–293 (1976)
- [35] Mermin, N.D., Ho, T.-L.: Circulation and angular momentum in the a

- phase of superfluid he-3. *Phys. Rev. Lett.* **36**, 594–597 (1976)
- [36] Mermin, N.D.: The topological theory of defects in ordered media. *Rev. Mod. Phys.* **51**(3), 591–648 (1979)
- [37] Volovik, G.E., Mineev, V.P.: Particle-like solitons in superfluid ^3He phases. *Sov. Phys. JETP* **46**(2), 401–404 (1977)
- [38] Takabayasi, T.: Vortex, spin and triad for quantum mechanics of spinning particle. i. *Progress of Theoretical Physics.* **70**(1), 1–17 (1983)
- [39] Dirac, P.A.M.: Quantised singularities in the electromagnetic field. *Proc. R. Soc. London, Ser. A* **133**, 60–72 (1931)
- [40] Kaempfer, F.A.: *Concepts in Quantum Mechanics*. Academic Press, New York (1965)
- [41] Klein, U.: Schrödinger’s equation with gauge coupling derived from a continuity equation. *Foundations of Physics* **39**, 964 (2009)
- [42] Eberlein, W.F.: The spin model of Euclidean 3-space. *Am. Math. Monthly* **69**, 587–598 (1962)
- [43] Sattinger, D.H., Weaver, O.L. (eds.): *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics*. Springer, New York Berlin Heidelberg Tokyo (1986)
- [44] Weyl, H.: *The Classical Groups*. Princeton University Press, Princeton (1978). p. 273
- [45] Dirac, P.A.M.: The quantum theory of the electron. *Proc. R. Soc. London, Ser. A* **117**, 610–624 (1928)
- [46] Levy-Leblond, J.M.: Nonrelativistic particles and wave equations. *Comm. Math. Physics* **6**, 286–311 (1967)
- [47] Lambiase, G., Papini, G.: *The Interaction of Spin with Gravity in Particle Physics*. Springer, Heidelberg (2021)
- [48] Uhlenbeck, G.E., Goudsmit, S.: Spinning electrons and the structure of spectra. *Nature* **117**, 264–265 (1926)
- [49] de L. Kronig, R.: Spinning electrons and the structure of spectra. *Nature* **117**, 550 (1926)
- [50] Ohanian, H.C.: What is spin ? *Am. J. Phys.* **54**, 500–505 (1986)
- [51] Klein, U.: The statistical origins of quantum mechanics. *Physics Research*

- International **2010**, 18. <http://downloads.hindawi.com/journals/phys/2010/808424.pdf>
- [52] Bohm, D., Schiller, R.: A causal interpretation of the Pauli equation (B). *Supplemento al Nuovo Cimento I Serie X*, 68–91 (1955)
- [53] Bialynicki-Birula, I.: Hydrodynamic form of the Weyl equation. *Acta Physica Polonica B* **26**(7), 1201–1208 (1995)
- [54] Frieden, B.R.: Fisher information as the basis for the Schrödinger wave equation. *Am. J. Phys.* **57**(11), 1004–1008 (1989)
- [55] Reginatto, M.: Derivation of the Pauli equation using the principle of minimum Fisher information. *Physics Letters A* **249**, 355–357 (1998)
- [56] Berry, M.: Classical limits. In: *The Theory of the Quantum World*, pp. 52–54. World Scientific, Singapore (2013)
- [57] Chuu, C.-P., Chang, M.-C., Niu, Q.: Semiclassical dynamics and transport of the Dirac spin. *Solid State Communications* **150**, 533–537 (2010)
- [58] Rosaler, J.: Formal vs. Empirical Approaches to Quantum-Classical Reduction. *Topoi* **34**, 325–338 (2015)
- [59] Klein, U.: What is the limit $\hbar \rightarrow 0$ of quantum theory? *Am. J. Phys.* **80**, 1009 (2012)
- [60] Yahalom, A.: The fluid dynamics of spin. *Molecular Physics* (2018). <https://doi.org/10.1080/00268976.2018.1457808>
- [61] Konopinski, E.J.: What the electromagnetic potential describes. *Am. J. Phys.* **46**, 499–502 (1978)