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A particle method for non-local advection-selection-mutation equations

Frank Ernesto Alvarez ^{*} [†] and Jules Guilberteau [‡]

Abstract

The well-posedness of a non-local advection-selection-mutation problem deriving from adaptive dynamics models is shown for a wide family of initial data. A particle method is then developed, in order to approximate the solution of such problem by a regularised sum of weighted Dirac masses whose characteristics solve a suitably defined ODE system. The convergence of the particle method over any finite interval is shown and an explicit rate of convergence is given. Furthermore, we investigate the asymptotic-preserving properties of the method in large times, providing sufficient conditions for it to hold true as well as examples and counter-examples. Finally, we illustrate the method in two cases taken from the literature.

Keywords : Adaptive dynamics, non-local advection, Particle methods, Asymptotic preservation

1 Introduction

Presentation of the model

The goal of this paper is to develop a numerical method allowing to approximate the solutions of equations of the form

$$\begin{cases} \partial_t v(t, x) + \nabla_x \cdot (a(t, x, I_a v(t, x))v(t, x)) = R(t, x, I_g v(t, x))v(t, x) + \int_{\mathbb{R}^d} m(t, x, y, I_d v(t, x))v(t, y)dy, \\ v \in \mathcal{C}([0, T], L^1(\mathbb{R}^d)), \\ v(0, \cdot) = v^0(\cdot) \in W^{1,1}(\mathbb{R}^d), \end{cases} \quad (1)$$

where

$$(I_l u)(t, x) = \int_{\mathbb{R}^d} \psi_l(t, x, y)u(t, y)dy, \quad l = a, g, d$$

are non-local terms and a , R , m and ψ_l are smooth functions.

This general formulation aims to bring together a wide family of PDE models typically used in the field of *adaptive dynamics*. In this context, x represents a phenotypic trait (usually simply called ‘phenotype’ or ‘trait’) which is a characteristic inherent to individuals, and $x \mapsto v(t, x)$ represents the density of the studied population at time $t \geq 0$. One purpose of adaptive dynamics is to understand the combined effect of selection and mutations (which are usually assumed to be rare and small [19, 23, 37]) on living populations [5]. The literature concerning phenotype-structured equations is abundant [3, 15, 38, 20, 34, 39, 30]. The model proposed in this paper (which includes, among others, the equations studied in [33, 6, 4, 24, 22, 18]) takes into account

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- Selection and growth, via the term ‘ $R(t, x, I_g v(t, x))v(t, x)$ ’, where R can be interpreted as the instantaneous growth rate, which depends on the trait x and the whole population.
- Mutation, via the term ‘ $\int_{\mathbb{R}^d} m(t, x, y, I_d v(t, x))v(t, y)dy$ ’, where the function m can be seen as the probability density for a cell of trait y to mutate into a cell of trait x .
- Advection, via the term ‘ $\nabla_x \cdot (a(t, x, I_a v(t, x))v(t, x))$ ’. This term models how the environment drives the individuals towards specific regions, as opposed to random mutations. Among others, this term can be used in order to model a cell differentiation phenomenon.

Mutations can also be modelled through a second order differential operator such as in [9] and [1]. Laplacian-like terms can be approximated by integral operators, as shown in [16], which means that, after choosing an appropriate integral approximation, our analysis could be extended to deal with second order equations. The other two non local-terms (I_a and I_g) allow to take into account the influence of the environment, created by the whole population, over the behaviour of the individuals [42, 22], or competition between individuals [38].

The long time behaviour of models considering only one phenomenon among selection, advection and mutation is well-known: broadly speaking, it has been shown that considering selection alone, or advection alone, leads to concentration phenomena (towards a finite number of traits) [33, 38, 21], meaning that the density converges to a sum of Dirac masses, while mutations by themselves have a smoothing effect [25]. Nevertheless, the combined effects of these terms remains unclear, and may lead to different and non-intuitive behaviours. As an example, considering both selection and advection can lead to convergence either to a Dirac mass or to a continuous function [24, 12], and considering mutation and selection leads either to convergence to a non-smooth measure or to a continuous function [4]. Note that this model also includes the equation studied in [2], which was also approximated with a particle method.

Upon establishing the well-posedness of (1), this paper is concerned with the derivation of a particle method inspired by [16], the analysis of its convergence and asymptotic-preserving properties. However, we must emphasise the two main novelties with respect to that work: First, the use of non local terms, which as we will show, poses technical difficulties and affects the existence of smooth solutions in certain cases. Secondly, the study of the asymptotic preserving property, which guarantees that, under certain hypotheses, the long time behaviour of the solution is conserved. As will be seen, these equations are naturally posed in the space of Radon measures, making particle methods a natural tool to approximate them. Compared to finite volume or finite element methods, they are more easily implemented. Furthermore, a change in model leads to very few changes in the corresponding code, a clear advantage over other methods.

Particle method

Particle methods use ODE resolution in order to approximate the solution of PDEs. This makes them particularly easy to implement, as they only require a classical ODE solver. The main idea is to seek a sum of weighted Dirac masses, called particle solution, which is denoted

$$v^N(t) = \sum_{i=1}^N \alpha_i(t) \delta_{x_i(t)}, \quad (2)$$

where the weights α_i and the points x_i are solutions of a suitable ODE system.

In order to recover a smooth function close to the solution of the studied PDE, the particle solution needs to be regularised: this is usually done by means of a convolution with a so-called ‘cut-off function’ φ_ε which must satisfy some specific properties. We denote this regular solution

$$v_\varepsilon^N(t, x) = \sum_{i=1}^N \alpha_i(t) \varphi_\varepsilon(x - x_i(t)), \quad (3)$$

where the scaling parameter ε is a function of N .

This method is especially adapted for the linear advection equation ‘ $\partial_t v(t, x) + \nabla \cdot (a(x)v(t, x)) = 0$ ’, but has been generalised to many other kinds of equations which mostly come from physics [28], such as diffusion

equations [17, 41, 7, 32, 31], advection-diffusion equations [35, 27], convection-diffusion equations [16], the Navier-Stokes equation [14, 10] or the Vlasov-Poisson equation [40, 13].

We apply the particle method by following its main three steps, as described in [8]:

- (i) **Particle approximation of the initial data.** This first step consists of approaching the initial condition of v^0 with a sum of weighted Dirac masses, *i.e.* choosing $N \in \mathbb{N}$, $x_1^0, \dots, x_N^0 \in \mathbb{R}^d$, $\alpha_1^0, \dots, \alpha_N^0 \in \mathbb{R}$ such that

$$v_0^N := \sum_{i=1}^N \alpha_i^0 \delta_{x_i^0} \sim v^0,$$

in the sense of Radon measures, which means that, for any $\phi \in \mathcal{C}_c^0(\mathbb{R}^d)$,

$$\sum_{i=1}^N \alpha_i^0 \phi(x_i^0) \xrightarrow{N \rightarrow +\infty} \int_{\mathbb{R}^d} \phi(x) v^0(x) dx.$$

Assuming that v^0 has a compact support, a canonical way of choosing these values is to choose a finite collection of subsets $\Omega_i^0 \subset \text{supp}(v^0)$ satisfying

$$\Omega_i^0 \cap \Omega_j^0 = \emptyset, \text{ if } i \neq j, \text{ and } \bigcup_{i \in \{1, \dots, N\}} \Omega_i^0 = \text{supp}(v^0),$$

and to take, for any $i \in \{1, \dots, N\}$

$$x_i^0 \in \Omega_i^0, \quad w_i^0 = |\Omega_i^0|, \quad \nu_i^0 = v^0(x_i^0) \quad \alpha_i = \nu_i w_i.$$

- (ii) **Time-evolution of the particles.** By using a weak formulation of the PDE, we determine the ODE satisfied by the positions (denoted x_i), the volumes (denoted w_i) and the weights (denoted ν_i) associated to each of the N particles, with initial conditions (x_i^0, w_i^0, ν_i^0) given at the previous step. The exact ODE, and the way the particles are correlated with each other depends on the complexity of the PDE. In the case where the advection term is local ($a(t, x, I) = a(t, x)$), the positions and the volumes satisfy

$$\dot{x}_i(t) = a(t, x_i(t)), \quad \dot{w}_i(t) = \nabla_x \cdot (a(t, x_i(t))) w_i(t) \quad (4)$$

and the formula for the ν_i depends on the selection and the mutation terms. The method described in the core of this article also allows to use this method in the case of a non-local advection, which modifies the ODE satisfied by the positions of the particles. Full formulas are given in Section 3.

Rewriting $\alpha_i = \nu_i w_i$, with the volumes w_i which satisfy (4) is required for the approximation of the different integral terms. Indeed, by Liouville's formula [26], for any f smooth enough,

$$\int_{\mathbb{R}^d} f(x) dx \sim \sum_{i=1}^N f(x_i(t)) w_i(t).$$

Formally, v and v^N (as defined by (2)) are both solution of (1) in the weak sense, with $v^N(0) \sim v^0$, which implies that, assuming that the parameters of the PDE are smooth enough, for any given time $T > 0$, $v^N(T) \sim v(T, \cdot)$.

- (iii) **Regularisation.** In order to transform the discrete measure v^N into a smooth function, we use a regularisation process based on convolution, which writes as a sum, shown in (3), since the convoluted measure is a sum of Dirac masses. The function

$$\varphi_\varepsilon := \frac{1}{\varepsilon} \varphi \left(\frac{\cdot}{\varepsilon} \right),$$

depending on a parameter $\varepsilon > 0$, is a scaling of the so-called *cut-off function* φ , which must satisfy some regularity and symmetry properties (which we specify in Section 4.1). The choice of ε , which intimately

depends on the choice of N , is intricate: if ε is too large, then the solution is ‘over-regularised’, and the scheme loses its accuracy. Conversely, if ε is too small, then some of the particles will be neglected, and the scheme does not converge towards the solution. Choosing the optimal ε as a function of N is thus not a trivial question, and it is possible in some cases to optimise the convergence rate by improving this regularisation step [11, 29].

Main results

Well-posedness. We first prove that problem (1) is well-posed, *i.e.* that for any family of parameters satisfying some regularity properties, defined in $\mathcal{C}([0, +\infty), L^1(\mathbb{R}^d))$. The proof heavily relies on the use of the characteristic curves $X_u(t, y)$, solution to the equation

$$\begin{cases} \dot{X}_u(t, y) = a(t, X_u(t, y), (I_a u)(t, X_u(t, y))) =: \mathcal{A}_u(t, X_u(t, y)), & t \in [0, T], \\ X_u(0, y) = y, \end{cases}$$

for $y \in \mathbb{R}^d$ and $u \in \mathcal{C}([0, T], L^1(\mathbb{R}^d))$. The main difference with respect to the approach taken for similar problems, as in [16], is the need for continuity results for $X_u(t, y)$, not only with respect to the trait variable y , but with respect to u as well. The required results are stated in Section 2.1 and proved in Appendix A.

The well posedness of the problem for smooth initial data is proved in Section 2.2 using a standard fixed point argument. Moreover, in Section 2.3, we consider a more general family of initial conditions and we prove that the regularity of $v(t, \cdot)$ is linked to that of v^0 : more precisely, if $v^0 \in W^{k, \infty}(\mathbb{R}^d)$ with compact support, then $v \in \mathcal{C}([0, +\infty), W^{k-1, 1}(\mathbb{R}^d))$. This result can be improved if the advection is local, *i.e.* $a(t, x, I) = a(t, x)$: In this case, for any initial condition in $W^{k, 1}(\mathbb{R}^d)$, the solution v is in $\mathcal{C}([0, +\infty), W^{k, 1}(\mathbb{R}^d))$.

Particle method: definition and well-posedness. In Section 3, we define the particle method corresponding to this PDE, by deriving the ODE system satisfied by the particles $(x_i, w_i$ and $\nu_i)$. For the non-local case, the equation we obtain is a coupled system with infinitely many equations and unknowns. We generalise some classical results from the Cauchy-Lipschitz theory in order to deal with this problem, and prove that the ODE system is well posed.

Convergence of the particle method. Section 4 is structured as follows: in Section 4.1 we prove the following estimate, detailed in Theorem 6:

$$\|v - v_\varepsilon^h\|_{L^1(\mathbb{R}^d)} \leq C(\varepsilon^r + (\frac{h}{\varepsilon})^\kappa + h^\kappa) \|v^0\|_{W^{\mu, 1}(\mathbb{R}^d)}, \quad \text{for all } 0 \leq t \leq T,$$

with $h = \frac{1}{N}$, $C > 0$, $r \in \mathbb{N}^*$ (which depends on the regularity of φ), $\kappa \in \mathbb{N}^*$ (which depends on k and the local or non-local nature of a) and $\mu \in \mathbb{N}^*$ (which depends on κ and r). This result ensures the convergence of the particle method in finite time, upon choosing ε as an suitable function of h which allows to obtain a convergence in $(\varepsilon(h)^r + (\frac{h}{\varepsilon(h)})^\kappa + h^\kappa)$ which yields a convergence rate of $h^{\frac{\kappa r}{\kappa+r}}$ after choosing the optima value of $\varepsilon(h) \sim h^{\frac{\kappa}{\kappa+r}}$. Nevertheless, in general this scheme is not asymptotic preserving. Therefore, in subsection 4.2, we show examples for which the scheme is asymptotic preserving, and others for which it is not. In general, the asymptotic behaviour of a solution is preserved when it converges to a sum of Dirac masses, and is not when it converges to a smooth solution.

Perspectives and open problems

Although a loss of regularity appears to be taking place when advection is non local, we do not know whether such a loss of regularity does happen in certain cases. The construction of such an example or, on the contrary, the improvement of our results in order to prove that, in fact, no regularity is lost could be a

first way to extend our work.

Another open problem is the optimisation of the order of convergence for the numerical solution, improving upon the order $\frac{\kappa T}{\kappa+T}$ obtained in the present work. The approach from [11], where the local averages are viewed as point values of an approximation of the solution, and the regularisation of the solution at time $t > 0$ is performed by interpolation rather than convolution, could be a suitable choice.

Lastly, as mentioned before, another direction could be the extension of our results in order to deal with second order equations, as done in [16], where the Laplacian operator is approximated by an appropriate sequence of mutation kernels.

2 The problem

For $T > 0$ and $k \in \mathbb{N}$ we consider the functions

$$(t, x, I) \mapsto a(t, x, I) \in W^{1,\infty}([0, T], (W^{k+1,\infty}(\mathbb{R}^{d+1}))^d), \quad (5)$$

$$(t, x, I) \mapsto R(t, x, I) \in \mathcal{C}([0, T] \times \mathbb{R}_x^d, W_{loc}^{k+1,\infty}(\mathbb{R}_I)) \cap \mathcal{C}([0, T] \times \mathbb{R}_I, W^{k+1,\infty}(\mathbb{R}_x^d)). \quad (6)$$

We consider as well $(t, x, y, I) \mapsto m(t, x, y, I)$ such that

$$0 \leq m \in \mathcal{C}([0, T] \times \mathbb{R}_x^d \times \mathbb{R}_y^d, W^{k+1,\infty}(\mathbb{R}_I)) \cap \mathcal{C}([0, T] \times \mathbb{R}_x^d \times \mathbb{R}_I, L^\infty(\mathbb{R}_y^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y^d \times \mathbb{R}_I, \mathcal{C}_c^k(\mathbb{R}_x^d)),$$

which is globally Lipschitz with respect to the non local variables and uniformly compactly supported with respect to the x variable. That is, we suppose m to satisfy the following hypotheses:

- There exists $\mu > 0$ such that

$$\sup_{t,x,y} \sum_{i=1}^k \sum_{|\alpha|=i} \sum_{j=1}^k |\partial_x^\alpha \partial_I^j m(t, x, y, I) - \partial_x^\alpha \partial_I^j m(t, x, y, J)| \leq \mu |I - J|. \quad (7)$$

- There exists a compact set \mathcal{X} such that the function

$$M(x) := \sup_{t,y,I} \sum_{i=1}^k \sum_{|\alpha|=i} \sum_{j=1}^k |\partial_x^\alpha \partial_I^j m(t, x, y, I)|, \quad (8)$$

satisfies

$$\text{supp } M(x) \subset \mathcal{X}. \quad (9)$$

Furthermore, we assume that

$$\|M\|_{L^\infty(\mathbb{R}^d)} \leq \bar{M} < \infty. \quad (10)$$

From a modelling point of view, assuming m to be uniformly compactly supported reflects the fact that some traits are realistically out of reach for a given population, and that, in general, mutations are rare and small.

We remark that hypotheses (9) and (10) imply that

$$\|M\|_{L^1(\mathbb{R}^d)} \leq |\mathcal{X}| \bar{M} =: K < \infty. \quad (11)$$

For all functions $u \in \mathcal{C}([0, T], L^1(\mathbb{R}^d))$ we consider the linear mappings I_a , I_g and I_d which satisfy, for all $t \in [0, T]$, $x \in \mathbb{R}^d$,

$$(I_a u)(t, x) := \int_{\mathbb{R}^d} \psi_a(t, x, y) u(t, y) dy, \quad (12)$$

$$(I_g u)(t, x) := \int_{\mathbb{R}^d} \psi_g(t, x, y) u(t, y) dy, \quad (13)$$

$$(I_d u)(t, x) := \int_{\mathbb{R}^d} \psi_d(t, x, y) u(t, y) dy, \quad (14)$$

where

$$\psi_a \in W^{1, \infty}([0, T] \times \mathbb{R}_x^d, L^\infty(\mathbb{R}_y^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y^d, W^{k+1, \infty}(\mathbb{R}_x^d)), \quad (15)$$

$$0 < \underline{\psi}_g \leq \psi_g \in \mathcal{C}([0, T] \times \mathbb{R}_x^d, L^\infty(\mathbb{R}_y^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y^d, W^{k+1, \infty}(\mathbb{R}_x^d)), \quad (16)$$

$$\psi_d \in \mathcal{C}([0, T] \times \mathbb{R}_x^d, L^\infty(\mathbb{R}_y^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y^d, W^{k+1, \infty}(\mathbb{R}_x^d)), \quad (17)$$

for a certain $\underline{\psi}_g > 0$. We remark that $I_a u, I_g u, I_d u \in \mathcal{C}([0, T], L^\infty(\mathbb{R}^d))$. The functions ψ_a and ψ_d do not need to be positive, reflecting this way how different traits have different impacts (which are not always beneficial) on the environment, and ultimately on the population itself. On the other hand, ψ_g has to be bounded away from zero. This hypothesis reflects the fact that, at least for the growth term, all interactions between individuals are of the same type. These interactions may be interpreted as either strictly competitive or strictly cooperative. In particular, this means that only “very” non-local dependence with respect to u are allowed. This excludes partial densities, for instance on part of the traits.

Lastly, we assume that there exist non-negative constants I^* and r^* such that, for all $t \in [0, T]$, $x \in \mathbb{R}^d$, and $I \geq I^*$,

$$R(t, x, I) + K < -r^*, \quad (18)$$

uniformly on t and x . It is somewhat natural to assume that R is negative for a large population: for example, if a carrying capacity is assumed to exist, and the population size is approaching such value, then the growth rate will inevitably drop to levels where no amount of mutations will be able to compensate for it.

For a given function $v^0 \in L^1(\mathbb{R}^d)$, we will study the existence and uniqueness of solution for the problem

$$\begin{cases} \partial_t v(t, x) + \nabla_x \cdot (a(t, x, I_a v(t, x)) v(t, x)) = R(t, x, I_g v(t, x)) v(t, x) + \int_{\mathbb{R}^d} m(t, x, y, I_d v(t, x)) v(t, y) dy, \\ v \in \mathcal{C}([0, T], L^1(\mathbb{R}^d)), \\ v(0, \cdot) = v^0(\cdot). \end{cases} \quad (19)$$

We will show that, under additional hypotheses, either over a or v^0 , we can guarantee the well-posedness of this problem. In particular, we provide the results regarding the cases of local advection ($\partial_I a = 0$) and non-local advection ($\partial_I a \neq 0$). We will see that this distinction directly affects the set of initial data v^0 for which the existence of solutions is guaranteed.

2.1 Some bounds over the characteristics

Consider a satisfying (5) and ψ_a satisfying (15) for some $k \geq 1$. For all $y \in \mathbb{R}^d$ and $u \in \mathcal{C}([0, T], L^1(\mathbb{R}^d))$ we define the characteristic curve $t \mapsto X_u(t, y)$ as the unique solution the following ODE

$$\begin{cases} \dot{X}_u(t, y) = a(t, X_u(t, y), (I_a u)(t, X_u(t, y))) =: \mathcal{A}_u(t, X_u(t, y)), & t \in [0, T], \\ X_u(0, y) = y, \end{cases} \quad (20)$$

where $(I_a u)(t, x)$ is defined in (12). Since the function ψ_a belongs to $W^{1,\infty}([0, T] \times \mathbb{R}_x^d, L^\infty(\mathbb{R}_y^d))$, then $(I_a u)(t, x)$ belongs to $W^{1,\infty}([0, T] \times \mathbb{R}_x^d) \subset \mathcal{C}^{0,1}([0, T] \times \mathbb{R}_x^d)$. The regularity of a then implies that $\mathcal{A}_u(t, x)$ is a Lipschitz function with respect to the x variable, uniformly with respect to t , guaranteeing this way the global existence of solution for (20).

For all $u \in \mathcal{C}([0, T], L^1(\mathbb{R}^d))$, we define the norms

$$\|u\|_1 := \|u\|_{L^1([0, T] \times \mathbb{R}^d)} = \int_0^T \int_{\mathbb{R}^d} |u(t, x)| dx dt \text{ and } \|u\| := \sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^1(\mathbb{R}^d)}.$$

We present some results involving the characteristics. The proofs for such results are given in Appendix A.

The first property we describe is the continuity of the family of characteristics with respect to the spatial variables and the function u .

Lemma 1. *Let ψ_a satisfy (15) and a satisfy (5) for $k = 0$. Consider $y_1, y_2 \in \mathbb{R}^d$ and $u_1, u_2 \in \mathcal{C}([0, T], L^1(\mathbb{R}^d))$. Then there exists a positive constant $C(T, \|u_1\|, \|u_2\|)$, such that the solutions X_{u_1} and X_{u_2} of (20) satisfy for any $t \in [0, T]$,*

$$\sum_{j=1}^d |X_{u_1}^j(t, y_1) - X_{u_2}^j(t, y_2)| \leq C(T, \|u_1\|, \|u_2\|) (|y_1 - y_2| + \|u_1 - u_2\|_1).$$

Secondly, we claim that the spatial derivatives of the characteristics remain bounded by a constant only depending on T and $\|u\|$. We also claim that the spatial derivatives are continuous with respect to the spatial variables and the function u .

Lemma 2. *Let a satisfy (5) and ψ_a satisfy (15) for some $k \geq 1$. Consider $u \in \mathcal{C}([0, T], L^1(\mathbb{R}^d))$. Then there exists a positive constant $C(T, \|u\|)$, such that the solution $X_u(t, y)$ of (20) satisfies, for all $t \in [0, T]$ and $y \in \mathbb{R}^d$,*

$$\sum_{i=1}^k \sum_{|\alpha| \leq i} \sum_{j=1}^d |\partial_y^\alpha X_u^j(t, y)| \leq C(T, \|u\|). \quad (21)$$

Furthermore, for any two points $y_1, y_2 \in \mathbb{R}^d$, and any two functions $u_1, u_2 \in \mathcal{C}([0, T], L^1(\mathbb{R}^d))$, there exists a positive constant $C_2(T, \|u_1\|, \|u_2\|)$, such that the solutions X_{u_1} and X_{u_2} of (20) satisfy for any $t \in [0, T]$

$$\sum_{i=0}^k \sum_{|\alpha| \leq i} \sum_{j=1}^d |\partial_y^\alpha X_{u_1}^j(t, y_1) - \partial_y^\alpha X_{u_2}^j(t, y_2)| \leq C(T, \|u_1\|, \|u_2\|) (|y_1 - y_2| + \|u_1 - u_2\|_1). \quad (22)$$

Since for all $t \in [0, T]$, $y \mapsto X_u(t, y)$ is a \mathcal{C}^1 -diffeomorphism from \mathbb{R}^d onto itself, we may define its inverse as the function satisfying $X_u(t, X_u^{-1}(t, x)) = x$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$. We have the following results for X_u^{-1} .

Lemma 3. *Let a satisfy (5) and ψ_a satisfy (15) for some $k \geq 1$. Consider $u \in \mathcal{C}([0, T], L^1(\mathbb{R}^d))$. Then there exists a positive constant $\tilde{C}(T, \|u\|)$, such that the inverse of the solution $X_u(t, y)$ of (20) satisfies, for all $t \in [0, T]$, $x \in \mathbb{R}^d$*

$$\sum_{i=1}^k \sum_{|\alpha| \leq i} \sum_{j=1}^d |\partial_x^\alpha (X_u^{-1})^j(t, x)| \leq \tilde{C}(T, \|u\|). \quad (23)$$

Lemma 4. *Let a satisfy (5) and ψ_a satisfy (15) for some $k \geq 1$. Consider any two functions $u_1, u_2 \in \mathcal{C}([0, T], L^1(\mathbb{R}^d))$. Then there exists a positive constant $\tilde{C}(T, \|u_1\|, \|u_2\|)$, which satisfies $\lim_{T \rightarrow 0} \tilde{C}(T, \|u_1\|, \|u_2\|) = 0$ and such that the inverses of the solutions X_{u_1} and X_{u_2} of (20) satisfy, for all $t \in [0, T]$ and $x \in \mathbb{R}^d$*

$$\sum_{i=0}^{k-1} \sum_{|\alpha| \leq i} \sum_{j=1}^d |(\partial_x^\alpha X_{u_1}^{-1})^j(t, x) - \partial_x^\alpha (X_{u_2}^{-1})^j(t, x)| \leq \tilde{C}(T, \|u_1\|, \|u_2\|) \|u_1 - u_2\|_1. \quad (24)$$

We remark that thanks to the relation $\|u\|_1 \leq T\|u\|$, the relations (22) and (24) also hold true when replacing $\|u_1 - u_2\|_1$ by $\|u_1 - u_2\|$. Lastly, we give a result regarding the regularity of $X_u(t, x)$ with respect to t .

Lemma 5. *Let a satisfy (5) and ψ_a satisfy (15) for some $k \geq 1$. Consider $u \in \mathcal{C}^1([0, T_1] \times \mathbb{R}^d)$ such that $\sup_{t \in [0, T_1]} (\|u(t, \cdot)\|_{L^1(\mathbb{R}^d)} + \|\partial_t u(t, \cdot)\|_{L^1(\mathbb{R}^d)}) < +\infty$. Then $X_u(t, y) \in \mathcal{C}^1([0, T_1], \mathcal{C}^k(\mathbb{R}^d))$. As a consequence, $X_u^{-1}(t, x) \in \mathcal{C}^1([0, T_1], \mathcal{C}^k(\mathbb{R}^d))$.*

Proof. Thanks to Lemma 2, we know that under these hypotheses, for all $T < T_1$, $X_u(t, y)$ exists and belongs to $\mathcal{C}^1([0, T], \mathcal{C}^k(\mathbb{R}^d))$. Consider $0 < t_1, t_2 < T_1$, then

$$\begin{aligned} \sum_{i=0}^k \sum_{|\alpha| \leq i} \sum_{j=1}^d |\partial_y^\alpha X_u^j(t_1, y) - \partial_y^\alpha X_u^j(t_2, y)| &= \sum_{i=0}^k \sum_{|\alpha| \leq i} \sum_{j=1}^d \left| \int_{t_1}^{t_2} \partial_y^\alpha \dot{X}_u^j(s, y) ds \right| \\ &= \sum_{i=0}^k \sum_{|\alpha| \leq i} \sum_{j=1}^d \left| \int_{t_1}^{t_2} \partial_y^\alpha a_j(s, X_u(s, y), (I_a u)(s, X_u(s, y))) ds \right|. \end{aligned}$$

Thanks to the regularity of a and ψ_a , the bounds given in Lemma 2 for the derivatives of $X_u(t, y)$ and the uniform bound for $\|u\|_{L^1(\mathbb{R}^d)}$ we conclude that there exists a positive constant such that

$$\sum_{i=0}^k \sum_{|\alpha| \leq i} \sum_{j=1}^d |\partial_y^\alpha X_u^j(t_1, y) - \partial_y^\alpha X_u^j(t_2, y)| \leq C|t_1 - t_2|.$$

Similarly, we have

$$\sum_{i=0}^k \sum_{|\alpha| \leq i} \sum_{j=1}^d |\partial_y^\alpha \dot{X}_u^j(t_1, y) - \partial_y^\alpha \dot{X}_u^j(t_2, y)| = \sum_{i=0}^k \sum_{|\alpha| \leq i} \sum_{j=1}^d |\partial_y^\alpha (\mathcal{A}_u)_j(t_1, y) - \partial_y^\alpha (\mathcal{A}_u)_j(t_2, y)|,$$

where

$$(\mathcal{A}_u)_j(t, y) := a_j(t, X_u(t, y), (I_a u)(t, X_u(t, y))).$$

Again, the regularity up to order $k + 1$ of the involved coefficients allow us to conclude that

$$\sum_{i=0}^k \sum_{|\alpha| \leq i} \sum_{j=1}^d |\partial_y^\alpha \dot{X}_u^j(t_1, y) - \dot{X}_u^j(t_1, y)| \leq C|t_1 - t_2|.$$

We have shown that $X_u(t, \cdot)$ is a Cauchy sequence in $\mathcal{C}^k(\mathbb{R}^d)$ when t goes to T_M , therefore, there exists $X^*(x) \in \mathcal{C}^k(\mathbb{R}^d)$ and $Y^*(x) \in \mathcal{C}^k(\mathbb{R}^d)$ such that

$$\lim_{t \rightarrow T_M} \|X_u(t, \cdot) - X^*\|_{\mathcal{C}^k(\mathbb{R}^d)} + \|\dot{X}_u(t, \cdot) - Y^*\|_{\mathcal{C}^k(\mathbb{R}^d)},$$

which is the desired result. □

2.2 Existence of solution for smooth initial data

We first provide the proof of existence and uniqueness of solution for problem (19) when the initial condition v^0 is a smooth enough function. We still assume hypotheses (5) through (18) hold. For a smooth initial condition v^0 we denote by *classical solution* any function $v \in \mathcal{C}^1([0, T] \times \mathbb{R}^d)$ which satisfies problem (19). The following *a priori* estimate will allow us to guarantee the global existence of a classical solution given that such solution exists over a certain interval $[0, T_1]$.

Lemma 6. Let $v^0 \in \mathcal{C}_c^1(\mathbb{R}^d)$ and $T_1 > 0$ be such that a classical solution $v \in \mathcal{C}^1([0, T] \times \mathbb{R}^d)$ exists for problem (19) for all $T < T_1$, which is positive and has compact support with respect to the x variable. Then, such solution satisfies the estimate

$$\sup_{t \in [0, T_1]} \|v(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq \max\{\|v^0\|_{L^1(\mathbb{R}^d)}, \frac{I^*}{\underline{\psi}_g}\}. \quad (25)$$

Proof. Let v be the aforementioned positive solution. Denoting $\rho(t) := \|v(t, \cdot)\|_{L^1(\mathbb{R}^d)}$, we see directly from equation (19) that

$$\dot{\rho}(t) = \int_{\mathbb{R}^d} \left(R(t, y, (I_g v)(t, y)) + \int_{\mathbb{R}^d} m(t, x, y, (I_d v)(t, x)) dx \right) v(t, y) dy.$$

If there exists t such that $\rho(t) > \frac{I^*}{\underline{\psi}_g}$, then $(I_g v)(t, y) \geq \underline{\psi}_g \rho(t) > I^*$, which allows us to use hypothesis (18) in order to conclude

$$\dot{\rho}(t) < -r^* \rho(t) < 0.$$

This way, we see that either $\rho(t)$ is smaller than $\frac{I^*}{\underline{\psi}_g}$ or $\rho(t)$ is decreasing, which in turn implies the bound (25). \square

A fixed point argument together with estimate (25) will allow us to conclude the existence of solution for problem (19) for smooth initial conditions.

Theorem 1. Consider $k \geq 1$ and $T > 0$. Under hypotheses (15) through (18), for all non-negative functions $v^0 \in \mathcal{C}_c^k(\mathbb{R}^d)$, there exists a unique non-negative classical solution $v \in \mathcal{C}^1([0, T], \mathcal{C}_c^k(\mathbb{R}^d))$ to problem (19). Furthermore, such solution satisfies

$$\sup_{t \in [0, T]} \|v(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq \max\{\|v^0\|_{L^1(\mathbb{R}^d)}, \frac{I^*}{\underline{\psi}_g}\}, \quad (26)$$

$$\sup_{t \in [0, T]} \|v(t, \cdot)\|_{W^{k,1}(\mathbb{R}^d)} \leq C_T \|v^0\|_{W^{k,1}(\mathbb{R}^d)}. \quad (27)$$

Proof. Consider $v^0 \in \mathcal{C}_c^k(\mathbb{R}^d)$. For $t \geq 0$ and the function M introduced in (8) we define $r_t := 2\|a\|_{L^\infty} t$, B_{r_t} the open ball centred at 0 and of radius r_t ,

$$O_t := \text{supp}(v^0) \cup \text{supp}(M) + B_{r_t},$$

and for $\alpha > 1$ we define $\bar{\rho}_\alpha := \max\{\alpha \|v^0\|_{L^1(\mathbb{R}^d)}, \frac{I^*}{\underline{\psi}_g}\}$. We consider

$$u \in D_\alpha^T := \left\{ u \in \mathcal{C}([0, T] \times \mathbb{R}^d) : u \geq 0, \text{supp}(u(t, \cdot)) \subset O_t, \int_{\mathbb{R}^d} u(t, x) dx \leq \bar{\rho}_\alpha, \forall t \in [0, T] \right\}.$$

We denote as Φu the mapping defined by $v = \Phi u$, where v is the solution of

$$\begin{cases} \partial_t v(t, x) + \nabla_x \cdot (a(t, x, (I_a u)(t, x)) v(t, x)) - R(t, x, (I_g u)(t, x)) v(t, x) = \int_{\mathbb{R}^d} m(t, x, y, (I_d u)(t, x)) u(t, y) dy \\ v(0, \cdot) = v^0(\cdot). \end{cases}$$

Let us denote, for any $y \in \mathbb{R}^d$, as $X_u(\cdot, y)$ the unique solution of

$$\begin{cases} \dot{X}_u(t, y) = a(t, X_u(t, y), (I_a u)(t, X_u(t, y))) =: \mathcal{A}_u(t, X_u(t, y)) & t \geq 0, \\ X(0, y) = y. \end{cases}$$

As stated before, for any $t \in [0, T]$, $x \mapsto X_u(t, x)$ is a \mathcal{C}^1 -diffeomorphism, therefore, for all $t \geq 0$ and all $x \in \mathbb{R}^d$ there exists a unique $y \in \mathbb{R}^d$ such that $x = X_u(t, y)$, which we denote $y = X_u^{-1}(t, x)$.

We see that,

$$\begin{aligned} \frac{d}{dt}v(t, X_u(t, y)) &= [R(t, X_u(t, y), (I_g u)(t, X_u(t, y))) - \operatorname{div} \mathcal{A}_u(t, X_u(t, y))]v(t, X_u(t, y)) \\ &\quad + \int_{\mathbb{R}^d} m(t, X_u(t, y), z, (I_d u)(t, X_u(t, y)))u(t, z)dz, \end{aligned}$$

and thus, denoting

$$\mathcal{G}_u(t, y) := R(t, X_u(t, y), (I_g u)(t, X_u(t, y))) - \operatorname{div} \mathcal{A}_u(t, X_u(t, y)),$$

we get

$$\begin{aligned} \Phi u(t, X_u(t, y)) &= v(t, X_u(t, y)) \\ &= v^0(y) \exp\left(\int_0^t \mathcal{G}_u(s, y)ds\right) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} m(s, X_u(s, y), z, (I_d u)(s, X_u(s, y)))u(s, z)dz \exp\left(\int_s^t \mathcal{G}_u(\tau, y)d\tau\right) ds \end{aligned}$$

or, equivalently

$$\begin{aligned} \Phi u(t, x) &= v(t, x) \\ &= v^0(X_u^{-1}(t, x)) \exp\left(\int_0^t \mathcal{G}_u(s, X_u^{-1}(t, x))ds\right) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} m(s, X_u(s, X_u^{-1}(t, x)), z, (I_d u)(s, X_u(s, X_u^{-1}(t, x))))u(s, z)dz \exp\left(\int_s^t \mathcal{G}_u(\tau, X_u^{-1}(t, x))d\tau\right) ds. \end{aligned}$$

The solution v is thus non-negative, according to the non-negativity of v^0 , u and m . Thanks to Lemma 3, v belongs to $\mathcal{C}^1([0, T], \mathcal{C}^\kappa(\mathbb{R}^d)) \subset \mathcal{C}([0, T] \times \mathbb{R}^d)$. Furthermore, the fact that for all $t \in [0, T]$, $|X(t, y) - y| \leq \|a\|_{L^\infty t}$ implies that $\operatorname{supp}(v(t, \cdot)) \subset O_t$.

Additionally, directly from its definition, we see that

$$\mathcal{G}_u(s, X_u^{-1}(t, x)) \leq \gamma := \|R\|_{L_{t,x}^\infty} + \|a\|_{W_x^{1,\infty} L_{t,I}^\infty} + \|a\|_{W_I^{1,\infty} L_{t,x}^\infty} \|\psi_a\|_{W_x^{1,\infty} L_y^\infty} \bar{\rho}_\alpha,$$

and consequently, for all $u \in D_\alpha^T$, we have

$$\begin{aligned} \|\Phi u(t, \cdot)\|_{L^1(\mathbb{R}^d)} &\leq e^{\gamma T} \left(\int_{\mathbb{R}^d} v^0(X_u^{-1}(t, x))dx \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m(s, X_u(s, X_u^{-1}(t, x)), z, (I_d u)(s, X_u(s, X_u^{-1}(t, x))))u(s, z)dz dx ds \right) \\ &\leq e^{\gamma T} \left(\int_{\mathbb{R}^d} v^0(X_u^{-1}(t, x))dx + \int_0^t \int_{\mathbb{R}^d} M(s, X_u(s, X_u^{-1}(t, x)))dx \int_{\mathbb{R}^d} u(s, z)dz ds \right). \end{aligned}$$

Making the changes of variables $y = X_u^{-1}(t, x)$ and $y = X_u(s, X_u^{-1}(t, x))$ respectively on each of the integrals on the last expression, recalling that, according to Liouville's formula

$$\begin{aligned} |J_{X_u(t,y)}| &= e^{\int_0^t \operatorname{div} \mathcal{A}_u(s, X_u(s, y))ds}, \\ |J_{X_u(t, X_u^{-1}(s, y))}| &= e^{\int_0^t \operatorname{div} \mathcal{A}_u(\tau, X_u(\tau, X_u^{-1}(s, y)))d\tau - \int_0^s \operatorname{div} \mathcal{A}_u(\tau, y)d\tau}, \end{aligned}$$

and using the hypotheses over a and m we obtain that for all $t \in [0, T]$,

$$\|\Phi u(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq e^{(\gamma+2\tilde{a})T} (\|v^0\|_{L^1(\mathbb{R}^d)} + KT\|u\|_{L^1(\mathbb{R}^d)}),$$

where $\tilde{a} := \|a\|_{W_x^{1,\infty} L_{t,I}^\infty} + \|a\|_{W_I^{1,\infty} L_{t,x}^\infty} \|\psi_a\|_{W_x^{1,\infty} L_y^\infty} \bar{\rho}_\alpha$. Finally, using the hypothesis over v^0 and u we see that

$$\|\Phi u(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq e^{(\gamma+2\tilde{a})T} \left(\frac{1}{\alpha} + KT \right) \bar{\rho}_\alpha.$$

Thanks to the condition $\alpha > 1$ there exists T_α (only depending on α and on the coefficients of the problem) such that $\|\Phi u\|_{L^1(\mathbb{R}^d)} \leq \bar{\rho}_\alpha$ for all $t \in [0, T_\alpha]$. In other words, $\Phi : D_\alpha^{T_\alpha} \rightarrow D_\alpha^{T_\alpha}$.

We now claim that the mapping Φu is a contraction on some $D_\alpha^{T_1}$, $0 < T_1 \leq T_\alpha$, with respect to the usual norm in $\mathcal{C}([0, T] \times \mathbb{R}^d)$. For two functions $u_1, u_2 \in D_\alpha^{T_\alpha}$ and any $t \in [0, T_\alpha]$, thanks to Lemma 1 and Lemma 4, we have

$$\begin{aligned} |\Phi u_1 - \Phi u_2| &\leq C e^{\gamma t} |X_{u_1}^{-1}(t, x) - X_{u_2}^{-1}(t, x)| \\ &\leq C e^{\gamma t} \|u_1 - u_2\|_1 \\ &\leq C e^{\gamma t} O_t |t| \|u_1 - u_2\|_{\mathcal{C}([0, t] \times \mathbb{R}^d)}. \end{aligned}$$

Clearly, for $t = T_1$ small enough, Φu is a contraction, and therefore, thanks to the Banach fixed point theorem, there exists a unique $v \in D_\alpha^{T_1}$ such that $\Phi v = v$. Such v is a solution of (19) over $[0, T_1]$. Furthermore, directly from the relation $v = \Phi v$ we see that $v \in \mathcal{C}^1([0, T], \mathcal{C}^\kappa(\mathbb{R}^d))$.

Let us now assume that there exists T_M , a finite maximal time such that a solution exists in B_α^T for all $T < T_M$, and let v be such solution. Directly from Lemma 6, the solution v satisfies estimate (25) over $[0, T_M)$. Furthermore, thanks to the relation $v = \Phi v$, we are able to show that

$$\sup_{[0, T_M)} (\|v(t, \cdot)\|_{L^1(\mathbb{R}^d)} + \|\dot{v}(t, \cdot)\|_{L^1(\mathbb{R}^d)}) < +\infty.$$

From Lemma 5 we get then that $X_v^{-1}(t, x) \in \mathcal{C}^1([0, T_M], \mathcal{C}^k(\mathbb{R}^d))$, and by composition of functions, so is $v = \Phi v$. We can then iterate the previous ideas using $v(T_M) \in \mathcal{C}^k(\mathbb{R}^d)$ as a starting point in order to obtain the existence of solution over a certain interval $[T_M, T_M + \delta)$, contradicting this way the maximal character of T_M . Hence, there exists a classical solution of (19) for all $t > 0$.

The uniqueness on $\mathcal{C}([0, T], \mathcal{C}_c^\kappa(\mathbb{R}^d))$ comes from the fact that every other solution on D_α^∞ will coincide with v at least over a small interval $(0, t_0)$ and then, by continuity, the same would hold for all t .

To obtain the $W^{k,1}(\mathbb{R}^d)$ estimates, we differentiate the relation $v = \Phi v$ and notice that for all multi-index β such that $|\beta| \leq k$

$$\begin{aligned} \partial_x^\beta v &= \exp\left(\int_0^t \mathcal{G}_v(s, X_v^{-1}(t, x)) ds\right) \sum_{|\gamma| \leq |\beta|} \partial_x^\gamma v^0(X_v^{-1}(t, x)) F_1^\gamma(t, x) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} F_2(s, x, y) v(s, y) dy \exp\left(\int_s^t \mathcal{G}_v(\tau, x) d\tau\right) ds \end{aligned}$$

where the functions F_1^γ and F_2 are combinations of sums and multiplications of the derivatives up to order $|\beta|$ of X_v^{-1} , R , m , I_g and I_d . Taking absolute values, integrating over \mathbb{R}^d and using the boundedness of all the involved coefficients we arrive at

$$\|v(t, \cdot)\|_{W^{k,1}(\mathbb{R}^d)} \leq C_T^1 \|v^0\|_{W^{k,1}(\mathbb{R}^d)} + C_T^2 \int_0^t \|v(s, \cdot)\|_{W^{k,1}(\mathbb{R}^d)} ds$$

and thanks to Grönwall's lemma we get (27). □

2.3 Existence of solution for more general initial data

Depending on whether $\partial_I a = 0$ or $\partial_I a \neq 0$, we will have a different class of initial data for which we are able to guarantee existence of solution for problem (19). Furthermore, the regularity of such solution might also be affected.

We first prove that, when $\partial_I a \neq 0$, a solution exists (in a sense that will be defined below) for any initial condition $v^0 \in W^{k,\infty}(\mathbb{R}^d)$, with compact support. However, we do not prove that the regularity of the solution is preserved over time, even if we did not manage to highlight the existence of cases where a loss of regularity is observed. Secondly, we will show that, when $\partial_I a = 0$, not only the set of initial conditions for which we can claim existence of solution is more general ($v^0 \in W^{k,1}(\mathbb{R}^d)$), but the regularity of such solution is preserved for all $t > 0$.

We introduce the definition of *weak solution* for problem (19). We say that v is a *weak solution* of problem (19) associated to $v^0 \in L^p(\mathbb{R}^d)$ if

$$v \in L^\infty([0, T], L^p(\mathbb{R}^d)),$$

and it satisfies the equation in the following weak sense

$$\int_0^T \int_{\mathbb{R}^d} v L_v^* \varphi dx dt = \int_{\mathbb{R}^d} v^0 \varphi dx,$$

for any $\varphi \in \mathcal{C}_c^1([0, T] \times \mathbb{R}^d)$, where we define the operator L_v^* by

$$L_v^* \varphi(t, x) = -\partial_t \varphi(t, x) - a(t, x, (I_a v)(t, x)) \cdot \nabla \varphi(t, x) - R(t, x, (I_g v)(t, x)) \varphi(t, x) - \int_{\mathbb{R}^d} m(t, y, x, (I_d v)(t, y)) \varphi(t, y) dy,$$

for all $t \in [0, T]$, $y \in \mathbb{R}^d$. We remark that a *classical solution* is always a *weak solution*.

Theorem 2. *Under hypotheses (5) through (18), for all $k \geq 1$ and any non-negative functions $v^0 \in W^{k,\infty}(\mathbb{R}^d)$ with compact support, there exists a unique non-negative weak solution $v \in \mathcal{C}([0, T], \mathcal{C}_c^{k-1}(\mathbb{R}^d))$ to problem (19). Furthermore, such solution satisfies*

$$\sup_{t \in [0, T]} \|v(t, \cdot)\|_{L^1(\mathbb{R}^d)} \leq \max\{\|v^0\|_{L^1(\mathbb{R}^d)}, \frac{I^*}{\psi_g}\}, \quad (28)$$

$$\sup_{t \in [0, T]} \|v(t, \cdot)\|_{W^{k-1,1}(\mathbb{R}^d)} \leq C_T \|v^0\|_{W^{k-1,1}(\mathbb{R}^d)}, \quad (29)$$

and, for $k \geq 2$, $v \in \mathcal{C}^1([0, T], \mathcal{C}_c^{k-1}(\mathbb{R}^d))$.

Proof. Directly from Morrey's inequality, we get the relation $v^0 \in W^{k,\infty}(\mathbb{R}^d) \subset \mathcal{C}^{k-1,1}(\mathbb{R}^d)$. If $k \geq 2$, we are able to apply Theorem 1 in order to get the desired result.

Consider now $k = 1$. The compact support of v^0 and the $W^{1,\infty}(\mathbb{R}^d)$ regularity imply that $v^0 \in W^{k,1}(\mathbb{R}^d)$. This means that, there exists a sequence of compactly supported functions $v_\varepsilon^0 \in \mathcal{C}_c^1(\mathbb{R}^d)$ such that

$$\begin{aligned} \text{supp}(v_\varepsilon^0) &\subset \text{supp}(v^0), \\ \|v_\varepsilon^0\|_{W^{1,\infty}(\mathbb{R}^d)} &\leq \|v^0\|_{W^{1,\infty}(\mathbb{R}^d)}, \\ \lim_{\varepsilon \rightarrow 0} \|v^0 - v_\varepsilon^0\|_{W^{1,1}(\mathbb{R}^d)} &= 0. \end{aligned}$$

We denote as v_ε the solution of problem (19) associated to v_ε^0 , and we claim that v_ε is a Cauchy sequence in $\mathcal{C}([0, T], L^1(\mathbb{R}^d))$.

We recall that for all ε ,

$$\sup_{t \in [0, T]} \|v_\varepsilon\|_{L^1(\mathbb{R}^d)} \leq \bar{\rho}_\varepsilon := \max\{\|v_\varepsilon^0\|_{L^1(\mathbb{R}^d)}, \frac{I^*}{\psi_g}\}.$$

Furthermore, the equality $v_\varepsilon = \Phi v_\varepsilon$ holds true, where Φ was defined on the proof of Theorem 1. Consequently, for $\varepsilon_1, \varepsilon_2 > 0$ we have the relation

$$\begin{aligned}
\Delta V_{\varepsilon_1 \varepsilon_2} &:= v_{\varepsilon_1} - v_{\varepsilon_2} \\
&= \Phi v_{\varepsilon_1} - \Phi v_{\varepsilon_2} \\
&= \left(v_{\varepsilon_1}^0(X_{v_{\varepsilon_1}}^{-1}(t, x)) - v_{\varepsilon_2}^0(X_{v_{\varepsilon_2}}^{-1}(t, x)) \right) \exp\left(\int_0^t \mathcal{G}_1(\tau, t, x) d\tau \right) - v_{\varepsilon_2}^0(X_{v_{\varepsilon_2}}^{-1}(t, x)) \left(\Delta E(0, t, x) \right) \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \left(\mathcal{M}_1(s, t, x, z) - \mathcal{M}_2(s, t, x, z) \right) v_{\varepsilon_1}(s, z) dz \exp\left(\int_s^t \mathcal{G}_1(\tau, t, x) d\tau \right) ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \mathcal{M}_2(s, t, x, z) \left(\Delta V_{\varepsilon_1 \varepsilon_2}(s, z) \right) dz \exp\left(\int_s^t \mathcal{G}_1(\tau, t, x) d\tau \right) ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \mathcal{M}_2(s, t, x, z) v_{\varepsilon_j}(s, z) dz \left(\Delta E(s, t, x) \right) ds,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}_i(\tau, t, x) &= \mathcal{G}_{v_{\varepsilon_i}}(\tau, X_{v_{\varepsilon_i}}^{-1}(t, x)), \\
\Delta E(s, t, x) &:= \exp\left(\int_s^t \mathcal{G}_1(\tau, t, x) d\tau \right) - \exp\left(\int_s^t \mathcal{G}_2(\tau, t, x) d\tau \right), \\
\mathcal{M}_i(s, t, x, z) &:= m(s, X_{v_{\varepsilon_i}}(s, X_{v_{\varepsilon_i}}^{-1}(t, x)), z, (I_d v_{\varepsilon_i})(s, X_{v_{\varepsilon_i}}(s, X_{v_{\varepsilon_i}}^{-1}(t, x)))).
\end{aligned}$$

We write

$$\begin{aligned}
v_{\varepsilon_1}^0(X_{v_{\varepsilon_1}}^{-1}(t, x)) - v_{\varepsilon_2}^0(X_{v_{\varepsilon_2}}^{-1}(t, x)) &= v_{\varepsilon_1}^0(X_{v_{\varepsilon_1}}^{-1}(t, x)) - v_{\varepsilon_2}^0(X_{v_{\varepsilon_1}}^{-1}(t, x)) \\
&\quad + v_{\varepsilon_2}^0(X_{v_{\varepsilon_1}}^{-1}(t, x)) - v_{\varepsilon_2}^0(X_{v_{\varepsilon_2}}^{-1}(t, x)).
\end{aligned} \tag{30}$$

Thanks to the change of variables $y = X_{v_{\varepsilon_1}}^{-1}(t, x)$ and the relations

$$\begin{aligned}
|J_{X_{v_{\varepsilon_1}}^{-1}}(t, y)| &= e^{\int_0^t \operatorname{div} \mathcal{A}_{v_{\varepsilon_1}}(s, X_{v_{\varepsilon_1}}(s, y)) ds}, \\
\mathcal{G}(s, X_{v_{\varepsilon_1}}^{-1}(t, x)) &\leq \gamma := \|R\|_{L_{t,x,I}^\infty} + \|a\|_{W_x^{1,\infty} L_{t,I}^\infty} + \|a\|_{W_I^{1,\infty} L_{t,x}^\infty} \|\psi_a\|_{W_x^{1,\infty} L_y^\infty} \bar{\rho}, \\
\operatorname{div} \mathcal{A}_{v_{\varepsilon_1}}(s, X_{v_{\varepsilon_1}}^{-1}(t, y)) &\leq \tilde{a} := \|a\|_{W_x^{1,\infty} L_{t,I}^\infty} + \|a\|_{W_I^{1,\infty} L_{t,x}^\infty} \|\psi_a\|_{W_x^{1,\infty} L_y^\infty} \bar{\rho},
\end{aligned}$$

we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(v_{\varepsilon_1}^0(X_{v_{\varepsilon_1}}^{-1}(t, x)) - v_{\varepsilon_2}^0(X_{v_{\varepsilon_1}}^{-1}(t, x)) \right) \exp\left(\int_0^t \mathcal{G}_1(\tau, t, x) d\tau \right) dx \\
&= \int_{\mathbb{R}^d} \left(v_{\varepsilon_1}^0(y) - v_{\varepsilon_2}^0(y) \right) \exp\left(\int_0^t \mathcal{G}_{v_{\varepsilon_1}}(\tau, y) d\tau \right) |J_{X_{v_{\varepsilon_1}}^{-1}}^{-1}(t, y)| dy \\
&\leq e^{(\gamma + \tilde{a})T} \|v_{\varepsilon_1}^0 - v_{\varepsilon_2}^0\|_{L^1(\mathbb{R})}.
\end{aligned} \tag{31}$$

On the other hand, the compactness of the support of $v_{\varepsilon_2}^0$, together with the relation

$$|X_{v_\varepsilon}(t, y) - y| \leq \|a\|_{L^\infty} t,$$

implies that

$$v_{\varepsilon_2}^0(X_{v_{\varepsilon_1}}^{-1}(t, x)) - v_{\varepsilon_2}^0(X_{v_{\varepsilon_2}}^{-1}(t, x)) = 0, \text{ for all } x \notin O_t := \operatorname{supp} v^0 + B_{r_t},$$

where B_{r_t} is the ball of radius $\|a\|_{L^\infty} t$. Hence, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^d} \left(v_{\varepsilon_2}^0(X_{v_{\varepsilon_1}}^{-1}(t, x)) - v_{\varepsilon_2}^0(X_{v_{\varepsilon_2}}^{-1}(t, x)) \right) \exp\left(\int_0^t \mathcal{G}_1(\tau, t, x) d\tau \right) dx \\ & \leq |O_t| \|v_{\varepsilon_2}^0\|_{W^{1,\infty}(\mathbb{R}^d)} e^{\gamma T} |X_{v_{\varepsilon_1}}^{-1}(t, x) - X_{v_{\varepsilon_2}}^{-1}(t, x)| \\ & \leq \tilde{C}_T \|v_{\varepsilon_1} - v_{\varepsilon_2}\|_1, \end{aligned} \quad (32)$$

where we have used (24) on the second line¹.

From the definition of $\mathcal{G}_u(t, x)$, we observe that

$$\begin{aligned} |\Delta E(s, t, x)| & \leq e^{\gamma T} \int_s^t |\mathcal{G}_1(\tau, t, x) - \mathcal{G}_2(\tau, t, x)| d\tau \\ & \leq \tilde{C}_T \|v_{\varepsilon_1} - v_{\varepsilon_2}\|_1, \end{aligned} \quad (33)$$

where \tilde{C}_T depends on $\bar{\rho}$, T , the derivatives of R , a and ψ_a , and on the constant appearing in (24). Using the change of variables $y = X_{v_{\varepsilon_2}}(s, X_{v_{\varepsilon_2}}^{-1}(t, x))$ and recalling that

$$|J_{X_{v_{\varepsilon_2}}(t, X_{v_{\varepsilon_2}}^{-1}(s, x))}| = e^{\int_0^t \operatorname{div} \mathcal{A}_u(\tau, X_{v_{\varepsilon_2}}(t, X_{v_{\varepsilon_2}}^{-1}(s, x))) d\tau - \int_0^s \operatorname{div} \mathcal{A}_{v_{\varepsilon_2}}(\tau, x) d\tau},$$

we see that

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{M}_2(s, t, x, z) dx & = \int_{\mathbb{R}^d} m(s, y, z, (I_d v_{\varepsilon_2})(s, y)) |J_{X_{v_{\varepsilon_2}}(t, X_{v_{\varepsilon_2}}^{-1}(s, y))}|^{-1} dy \\ & \leq e^{2\bar{a}T} \int_{\mathbb{R}^d} \sup_{s, z, I} m_{\varepsilon_i}(s, y, z, I) dy \leq e^{2\bar{a}T} |\mathcal{K}| \bar{M}. \end{aligned}$$

Therefore, we have the bounds

$$\int_{\mathbb{R}^d} v^0(X_{v_{\varepsilon_2}}^{-1}(t, x)) |\Delta E(0, t, x)| dx \leq e^{\bar{a}T} \|v^0\|_{L^1(\mathbb{R}^d)} \tilde{C}_T \|v_{\varepsilon_1} - v_{\varepsilon_2}\|_1, \quad (34)$$

$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{M}_2(s, t, x, z) \left(\Delta V_{\varepsilon_1 \varepsilon_2}(s, z) \right) dz \exp\left(\int_s^t \mathcal{G}_1(\tau, t, x) d\tau \right) dx ds \leq e^{(\gamma+2\bar{a})T} |\mathcal{K}| \bar{M} \|v_{\varepsilon_1} - v_{\varepsilon_2}\|_1, \quad (35)$$

$$\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathcal{M}_2(t, x, z) v_{\varepsilon_j}(s, z) dz \left(|\Delta E(s, t, x)| \right) dx ds \leq e^{2\bar{a}T} |\mathcal{K}| \bar{M} \bar{\rho} T \tilde{C}_T \|v_{\varepsilon_1} - v_{\varepsilon_2}\|_1. \quad (36)$$

The function $m(t, x, z, I)$ having a compact support on the x variable, leads to

$$\mathcal{M}_1(s, t, x, z) - \mathcal{M}_2(s, t, x, z) = 0, \text{ for all } x \notin \mathcal{K} + B_{2r_t}.$$

On the other hand, the function m being differentiable and Lipschitz, implies that, for all $x \in \mathcal{K} + B_{2r_t}$

$$\begin{aligned} |\mathcal{M}_1(s, t, x, z) - \mathcal{M}_2(s, t, x, z)| & \leq \|m\|_{W_x^{1,\infty}} |X_{v_{\varepsilon_1}}(s, X_{v_{\varepsilon_1}}^{-1}(t, x)) - X_{v_{\varepsilon_2}}(s, X_{v_{\varepsilon_2}}^{-1}(t, x))| \\ & \quad + \mu |(I_d v_{\varepsilon_1})(s, X_{v_{\varepsilon_1}}(s, X_{v_{\varepsilon_1}}^{-1}(t, x))) - (I_d v_{\varepsilon_2})(s, X_{v_{\varepsilon_2}}(s, X_{v_{\varepsilon_2}}^{-1}(t, x)))| \\ & \leq (\|m\|_{W_x^{1,\infty}} + \|\psi_d\|_{W_x^{1,\infty}} \bar{\rho}) |X_{v_{\varepsilon_1}}(s, X_{v_{\varepsilon_1}}^{-1}(t, x)) - X_{v_{\varepsilon_2}}(s, X_{v_{\varepsilon_2}}^{-1}(t, x))| \\ & \quad + \|\psi_d\|_{L^\infty} \|v_{\varepsilon_1} - v_{\varepsilon_2}\|_{L^1(\mathbb{R}^d)}. \end{aligned}$$

¹This term is responsible for the possible loss of regularity for $t > 0$: In order to prove that v_ε is a Cauchy sequence in $\mathcal{E}([0, T], W^{1,1}(\mathbb{R}^d))$, we would need a $W^{2,\infty}(\mathbb{R}^d)$ estimate over v_ε^0 , which we do not have.

Using first Lemma 1 and then Lemma 4, we conclude that there exists a constant \tilde{C}_T such that

$$\begin{aligned} |\mathcal{M}_1(s, t, x, z) - \mathcal{M}_2(s, t, x, z)| &\leq \tilde{C}_T \left(|X_{v_{\varepsilon_1}}^{-1}(t, x) - X_{v_{\varepsilon_2}}^{-1}(t, x)| + \|v_{\varepsilon_1} - v_{\varepsilon_2}\|_1 + \|v_{\varepsilon_1} - v_{\varepsilon_2}\|_{L^1(\mathbb{R}^d)} \right) \\ &\leq \tilde{C}_T \left(\|v_{\varepsilon_1} - v_{\varepsilon_2}\|_1 + \|v_{\varepsilon_1} - v_{\varepsilon_2}\|_{L^1(\mathbb{R}^d)} \right). \end{aligned}$$

Therefore, we have the bound

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\mathcal{M}_1(s, t, x, z) - \mathcal{M}_2(s, t, x, z) \right) v_{\varepsilon_1}(s, z) dz \exp \left(\int_s^t \mathcal{G}_1(\tau, t, x) d\tau \right) dx ds \\ &\leq |\mathcal{K} + B_{2r_T}| \tilde{\rho} \tilde{C}_T e^{\gamma T} \|v_{\varepsilon_1} - v_{\varepsilon_2}\|_1. \end{aligned} \quad (37)$$

Putting together the bounds (31) through (37), we get

$$\|v_{\varepsilon_1} - v_{\varepsilon_2}\|_{L^1(\mathbb{R}^d)} \leq \tilde{C}_T \left(\|v_{\varepsilon_1}^0 - v_{\varepsilon_2}^0\|_{L^1(\mathbb{R})} + \int_0^T \|v_{\varepsilon_1} - v_{\varepsilon_2}\|_{L^1(\mathbb{R}^d)} ds \right).$$

Thanks to Grönwall's lemma, we have then the relation

$$\sup_{t \in [0, T]} \|v_{\varepsilon_1} - v_{\varepsilon_2}\|_{L^1(\mathbb{R}^d)} \leq \tilde{C}_T \|v_{\varepsilon_1}^0 - v_{\varepsilon_2}^0\|_{L^1(\mathbb{R}^d)}$$

for some \tilde{C}_T independent of ε_1 and ε_2 , which proves that, up to the extraction of a sub-sequence, v_ε is a Cauchy sequence in $\mathcal{C}([0, T], L^1(\mathbb{R}^d))$. Therefore, there exists $v \in \mathcal{C}([0, T], L^1(\mathbb{R}^d))$ such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} \|v_\varepsilon - v\|_{L^1(\mathbb{R}^d)} = 0.$$

Furthermore, such function satisfies the bounds (25) and (27).

We claim now that the sequence $L_{v_\varepsilon}^* \varphi$ converges to $L_v^* \varphi$ in $L^\infty([0, T] \times \mathbb{R}^d)$ for all $\varphi \in \mathcal{C}_c^1([0, T] \times \mathbb{R}^d)$. This is a direct consequence of the relation

$$|L^* v_\varepsilon \varphi - L^* v \varphi| \leq (L_r \|\psi_g\|_{L^\infty} + \mu \|\psi_d\|_{L^\infty}) \|\varphi\|_{L^\infty(\mathbb{R}^d)} \|v^\varepsilon - v\|_{L^1(\mathbb{R}^d)}.$$

In order to conclude, we recall that all classical solutions are weak solutions, and therefore, for all $\varepsilon > 0$

$$\int_0^T \int_{\mathbb{R}^d} v_\varepsilon L_{v_\varepsilon}^* \varphi dx dt = \int_{\mathbb{R}^d} v_\varepsilon^0 \varphi dx,$$

and taking the limit when ε goes to 0 we see that v is a weak solution of problem (19). \square

Considering initial data with compact support might be enough in order to model most of the biological scenarios found in nature. However, the hypothesis $v^0 \in W^{k, \infty}(\mathbb{R}^d)$ might be too restrictive for some real life scenarios. Furthermore, the study of the problem when more general initial conditions are present, is of theoretical interest. We show below that, when $\partial_I a = 0$, a solution exists for any initial data $v^0 \in W^{k, 1}(\mathbb{R}^d)$, $k \geq 1$.

Theorem 3. *Under hypothesis (15) through (18), if $\partial_I a = 0$, for all non-negative functions $v^0 \in W^{k, 1}(\mathbb{R}^d)$, there exists a unique non-negative weak solution $v \in \mathcal{C}([0, T], W^{k, 1}(\mathbb{R}^d))$ of problem (19). Furthermore, such a solution satisfies*

$$\sup_{t \in [0, T]} \|v\|_{L^1(\mathbb{R}^d)} \leq \max \left\{ \|v^0\|_{L^1(\mathbb{R}^d)}, \frac{I^*}{\psi_g} \right\}, \quad (38)$$

$$\sup_{t \in [0, T]} \|v\|_{W^{k, \infty}(\mathbb{R}^d)} \leq C_T \|v^0\|_{W^{k, 1}(\mathbb{R}^d)}. \quad (39)$$

Proof. As in the proof for $k = 1$ when $\partial_I a \neq 0$, we can approximate any function $v^0 \in W^{k,1}$ by a smooth, compactly supported sequence of functions v_ε^0 . The same arguments as in the previous proof will show that v_ε , the sequence of solutions associated to v_ε^0 , is a Cauchy sequence in $\mathcal{C}([0, T], L^1(\mathbb{R}^d))$. Furthermore, given that the second term in (30), which is responsible for the possible loss of regularity in the previous case, is equal 0 when $\partial_I a = 0$, we show that v_ε is a Cauchy sequence in $\mathcal{C}([0, T], W^{k,1}(\mathbb{R}^d))$ as well. We prove as in Theorem 2 that the limit of v_ε is the required weak solution. \square

Given that the regularity of the solution varies depending on whether $\partial_I a = 0$ or $\partial_I a \neq 0$, and that such regularity will be of importance in the upcoming sections, we define the parameter

$$\kappa := \begin{cases} k - 1, & \text{if } \partial_I a \neq 0, \\ k, & \text{if } \partial_I a = 0, \end{cases}$$

which encompasses the information over said regularity.

We remark that if we had $\partial_I m = 0$, we might obtain existence for a larger class of mutation functions m . For conciseness, we will however not consider such cases in the present work.

3 Particle Method

The particle method basically consists in searching for an approximate solution of problem (19) which is a sum of weighted Dirac masses.

Throughout the following section we suppose

$$\psi_a \in \mathcal{C}([0, T] \times \mathbb{R}_x^d, W^{1,\infty}(\mathbb{R}^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y, \mathcal{C}^2(\mathbb{R}_x^d) \cap W^{2,\infty}(\mathbb{R}_x^d)), \quad (40)$$

$$0 < \underline{\psi}_g \leq \psi_g \in \mathcal{C}([0, T] \times \mathbb{R}_x^d, W^{1,\infty}(\mathbb{R}_y^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y^d, \mathcal{C}^1(\mathbb{R}_x^d) \cap W^{1,\infty}(\mathbb{R}_x^d)) \quad (41)$$

$$\psi_d \in \mathcal{C}([0, T] \times \mathbb{R}_x^d, W^{1,\infty}(\mathbb{R}_y^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y^d, \mathcal{C}^1(\mathbb{R}_x^d) \cap W^{1,\infty}(\mathbb{R}_x^d)). \quad (42)$$

Notice that, unlike the set of hypotheses (15)-(17), we have imposed $W^{1,\infty}(\mathbb{R}^d)$ regularity for the y variable, which is needed in order to approximate the integral terms by sums over a countable set.

Consider as well

$$0 \leq m \in \mathcal{C}([0, T] \times \mathbb{R}_x^d \times \mathbb{R}_y^d, W^{1,\infty}(\mathbb{R}_I)) \cap \mathcal{C}([0, T] \times \mathbb{R}_x^d \times \mathbb{R}_I, W^{1,\infty}(\mathbb{R}_y^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y^d \times \mathbb{R}_I, \mathcal{C}_c^1(\mathbb{R}_x^d)) \quad (43)$$

satisfying hypotheses (7) through (18).

For $h > 0$, consider a countable set of indices $\mathcal{F}_h \in \mathbb{Z}^d$, points $x_i^0 \in \mathbb{R}^d$ and weights w_i^0 for $i \in \mathcal{F}_h$. The weights w_i^0 can be regarded as the respective masses of a collection of subsets $\Omega_i^0 \subset \mathbb{R}^d$ satisfying

$$\Omega_i^0 \cap \Omega_j^0 = \emptyset, \text{ if } i \neq j, \text{ and } \bigcup_{i \in \mathcal{F}_h} \Omega_i^0 = \mathbb{R}^d. \quad (44)$$

For example, we may choose the Ω_i^0 as the set of all non intersecting cubes of side length equal h having the points hi as centers, $i \in \mathbb{Z}^d$. This way, $w_i^0 = h^d$, with each of the x_i^0 being a point in Ω_j^0 . In general we assume that there exist positive constants c and C such that

$$ch \leq |x_i^0 - x_j^0| \leq Ch, \quad \forall i \neq j, \quad (45)$$

$$ch^d \leq w_i^0 \leq Ch^d, \quad \forall i \in \mathcal{F}_h. \quad (46)$$

Following [16], the particle method then consists in looking for a measure ν_h of the form

$$\nu_h(t) = \sum_{i \in \mathcal{F}_h} \nu_i(t) w_i(t) \delta_{x_i(t)},$$

where $(\nu := \{\nu_i(t)\}_{i \in \mathcal{J}_h}, w := \{w_i(t)\}_{i \in \mathcal{J}_h}, \bar{x} := \{x_i(t)\}_{i \in \mathcal{J}_h})$, is the solution of the following system

$$\begin{cases} \dot{x}_i(t) &= A_{\nu,w}(t, x_i), \\ \dot{w}_i(t) &= \operatorname{div} A_{\nu,w}(t, x_i(t))w_i(t), \\ \dot{\nu}_i(t) &= (-\operatorname{div} A_{\nu,w}(t, x_i(t)) + R(t, x_i(t), I_g(t, x_i(t), \nu, w)))\nu_i(t) \\ &\quad + \sum_{j \in \mathcal{J}_h} w_j(t)\nu_j(t)m(t, x_i(t), x_j(t), I_d(t, x_i(t), \nu, w)), \\ x_i(0) &= x_i^0, w_i(0) = w_i^0, \nu_i(0) = \nu_i^0(x_i^0), \end{cases} \quad (47)$$

where

$$A_{\nu,w}(t, x) = a(t, x, I_a(t, x, \nu, w)),$$

and

$$I_l(t, x, \nu, w) := \sum_{j \in \mathcal{J}_h} \nu_j(t)w_j(t)\psi_l(t, x, x_j(t)),$$

with $l \in \{a, g, d\}$.

In what follows we assume that h and x_k^0 are chosen in such a way that

$$\|v^0\|_{1,h} := \sum_{i \in \mathcal{J}_h} v^0(x_i^0)w_i^0 < \infty. \quad (48)$$

We define the subset of indices

$$\mathcal{J}_h^m := \{i \in \mathcal{J}_h : x_i^0 \in \operatorname{supp} m + B_{\|a\|_{L^\infty T}}\},$$

where $B_{\|a\|_{L^\infty T}}$ is the ball of radius $\|a\|_{L^\infty T}$. The compact support of m implies that $|\mathcal{J}_h^m| < \infty$.

For a positive value of h and a set of indexes \mathcal{J}_h we define the functional spaces

$$\begin{aligned} \ell^1(\mathcal{J}_h) &:= \{u = \{u_i\}_{i \in \mathcal{J}_h} : \sum_{i \in \mathcal{J}_h} |u_i| < +\infty\}, \\ \ell^\infty(\mathcal{J}_h) &:= \{w = \{w_i\}_{i \in \mathcal{J}_h} : \sup_{i \in \mathcal{J}_h} |w_i| < +\infty\}. \end{aligned}$$

We equip these spaces with the norms

$$\|u\|_{\ell^1} := \sum_{i \in \mathcal{J}_h} |u_i| \quad \text{and} \quad \|w\|_{\ell^\infty} := \sup_{i \in \mathcal{J}_h} |w_i|$$

respectively. It is clear that for all $u \in \ell^1(\mathcal{J}_h)$ and $v \in \ell^\infty(\mathcal{J}_h)$, then $uv \in \ell^1(\mathcal{J}_h)$. For $T > 0$ we define as well the spaces

$$X_h^T := \mathcal{C}([0, T], \ell^1(\mathcal{J}_h)) \quad \text{and} \quad Y_h^T := \mathcal{C}([0, T], \ell^\infty(\mathcal{J}_h)),$$

equipped with the norms

$$\|u\|_{1,h} := \sup_{t \in [0, T]} \|u(t)\|_{\ell^1} \quad \text{and} \quad \|w\|_{\infty,h} := \sup_{t \in [0, T]} \|w(t)\|_{\ell^\infty}.$$

Problem (47) is a strongly coupled system of ODEs, with an infinite number of unknowns and equations. In some cases the system becomes uncoupled (for example if $\partial_J a = 0$) or with a finite number of equations and unknowns (for example if v^0 , a and m have compact support), however, for the sake of generality, we present below the proof of existence of solution in the general case, and later discuss briefly these particular scenarios.

We start by giving two results that will be of great use for the proof of existence for problem (47). First, we deal with the existence of solution for a simpler system of infinite equations with infinitely many unknowns:

Lemma 7. Consider $a \in \mathcal{C}([0, T], (W^{1, \infty}(\mathbb{R}^{d+1})))^d$, $u \in X_h^T$, $w \in Y_h^T$ and ψ_a satisfying hypothesis (40). Then there exists a unique family of functions $x := \{x_i\}_{i \in \mathcal{J}_h}$, $x_i \in \mathcal{C}^1([0, T])$ for all $i \in \mathcal{J}_h$ which is solution of the system of equations

$$\dot{x}_i(t) = A_{u,w}(t, x_i), \quad t \in [0, T], \quad x_i(0) = x_i^0. \quad (49)$$

When $\partial_I a = 0$, system (49) becomes uncoupled, each individual equation has a solution, thanks to the classic Cauchy-Lipschitz theory. The proof of the general case is given in Appendix B.

The second auxiliary result comes from approximation theory, and it will also be of great use in Section 4:

Lemma 8.

$$\forall \varphi \in W^{k,1}(\mathbb{R}^d), \quad \left| \int_{\mathbb{R}^d} \varphi(x) dx - \sum_{i \in \mathcal{J}_h} w_i(t) \varphi(x_i(t)) \right| \leq Ch^k \|\varphi\|_{k,1},$$

where C is a constant which depends on a , ψ_a , $\|vw\|_{1,h}$ and T .

This result is a direct corollary of Lemma 8 in [36]. More details regarding its proof are given in Appendix C.

From now on, we suppose h to be small enough so that for any $t \in [0, T]$,

$$\sum_{i \in \mathcal{J}_h} w_i(t) m(t, x_i(t), y, I_d(t, x_i(t), \nu, w)) < K + \frac{r^*}{2}, \quad (50)$$

where the values of K and r^* are given in (11) and (18) respectively. Such a choice is always possible thanks to Lemma 8.

Theorem 4. Under hypothesis (5) through (18) and (40) through (46), for all $T > 0$ and all non-negative initial data $v^0 \in \ell^1(\mathcal{J}_h, \Omega^0)$ there exists a unique solution $x_i \in \mathcal{C}^1([0, T])$, for all $i \in \mathcal{J}_h$, $w := \{w_i(\cdot)\}_{i \in \mathcal{J}_h} \in \mathcal{C}([0, T], \ell^\infty(\mathcal{J}_h))$ and $0 \leq \nu := \{\nu_i(\cdot)\}_{i \in \mathcal{J}_h} \in \mathcal{C}([0, T], \ell^1(\mathcal{J}_h))$ of problem (47). Furthermore, there exist positive constants c_T and C_T such that the solution satisfies, for all $t \in [0, T]$

$$c_T h \leq |x_i(t) - x_j(t)| \leq C_T h, \quad \forall i, j \in \mathcal{J}_h, \quad i \neq j, \quad (51)$$

$$c_T h^d \leq w_i(t) \leq C_T h^d, \quad \forall i \in \mathcal{J}_h, \quad (52)$$

$$\|\nu w\|_{1,h} \leq \max\{\|v^0 h^d\|_{\ell^1}, \frac{I^*}{\underline{\psi}_g}\}. \quad (53)$$

Proof. Consider $v^0 \in \ell^1(\mathcal{J}_h, \Omega^0)$, satisfying $v^0 \geq 0$. Consider as well $\alpha > 1$, and define

$$\bar{\rho}_\alpha := \max\{\alpha h^d \|v^0\|_{\ell^1}, \frac{I^*}{\underline{\psi}_g}\},$$

$$\tilde{a} := \|a\|_{W_x^{1,\infty} L_{I,t}^\infty} + \|a\|_{W_I^{1,\infty} L_{t,x}^\infty} \|\psi_a\|_{W_x^{1,\infty}} C_\alpha.$$

For $T > 0$ we define the set

$$D_\alpha^T := \{(u, w) \in X_h^T \times Y_h^T : \|uw\|_{1,h} \leq \bar{\rho}_\alpha, \forall t \in [0, T], u(t) \geq 0, w(t) \geq 0, h^d e^{-\tilde{a}t} \leq w_k(t) \leq h^d e^{\tilde{a}t}\}.$$

For any $(u, w) \in D_\alpha^T$ we introduce the problem, for $t \in [0, T]$,

$$\begin{cases} \dot{x}_i(t) &= A_{u,w}(t, x_i), \\ \dot{\omega}_i(t) &= \operatorname{div} A_{u,w}(t, x_i(t)) \omega_i(t), \\ \dot{\nu}_i(t) &= (-\operatorname{div} A_{u,w}(t, x_i(t)) + R(t, x_i(t), I_g(t, x_i(t), u, w))) \nu_i(t) \\ &\quad + \sum_{j \in \mathcal{J}_h} \omega_j(t) u_j(t) m(t, x_i(t), x_j(t), I_d(t, x_i(t), u, w)), \\ x_i(0) &= x_i^0, \quad \omega_i(0) = w_i^0, \quad \nu_i(0) = v^0(x_i^0). \end{cases} \quad (54)$$

We denote $(\nu, \omega) = \Phi(u, w)$.

For each pair (u, w) , the existence and uniqueness of x_i is immediate from Lemma 7. Furthermore, for all values of i , we have the following explicit expression for ω_i

$$\omega_i(t) = w_i^0 e^{\int_0^t \operatorname{div} A_{u,w}(s, x_i(s)) ds},$$

which satisfies, for any $t \in [0, T]$

$$h^d e^{-\tilde{\alpha}t} \leq \omega_i(t) \leq h^d e^{\tilde{\alpha}t}.$$

On the other hand, for all $(u, w) \in D_\alpha^T$ and all values of i , the right-hand side of the differential equation in (54) is well defined, as we have for all $t \in [0, T]$

$$\sum_{j \in \mathcal{F}_h} w_j(t) u_j(t) m(t, x_i(t), x_j(t), I_d(t, x_i(t), u, w)) \leq \bar{M} \sum_{j \in \mathcal{F}_h} w_j(t) u_j(t) \leq \bar{M} \bar{\rho}_\alpha.$$

Therefore, the expression for ν_i is given by

$$\nu_i(t) = v^0(x_i^0) e^{\int_0^t \mathcal{G}_i(s) ds} + \int_0^t \sum_{j \in \mathcal{F}_h} w_j(s) u_j(s) m(s, x_i(s), x_j(s), I_d(s, x_i(s), u, w)) e^{\int_s^t \mathcal{G}_i(\tau) d\tau} ds, \quad (55)$$

where

$$\mathcal{G}_i(t) := -\operatorname{div} A_{u,w}(s, x_i(s)) + R(s, x_i(s), I_g(s, x_i(s), u(s), w(s))),$$

satisfies

$$\sup_{t \in [0, T]} |\mathcal{G}_i(t)| \leq \gamma := \|R\|_{L_{t,x,I}^\infty} + \|a\|_{W_x^{1,\infty} L_{t,I}^\infty} + \|a\|_{W_I^{1,\infty} L_{t,x}^\infty} \|\psi_\alpha\|_{W_x^{1,\infty} L_y^\infty} \bar{\rho}_\alpha.$$

The positiveness of ν is immediate from the positiveness of v^0 and m .

Furthermore, given that $|x_i(t) - x_i^0| \leq \|a\|_{L^\infty} T$, for all $k \in \mathcal{F}_h$ and $t \in [0, T]$, we have

$$m(t, x_i(s), y, I) = 0,$$

for all $i \notin \mathcal{F}_h^m$, $t \in [0, T]$, $y \in \mathbb{R}^d$ and $I \in \mathbb{R}$. As a result, multiplying (55) by $\omega_i(t)$ for each i and adding for all values of i , we obtain

$$\begin{aligned} \sum_{i \in \mathcal{F}_h} \nu_i(t) \omega_i(t) &\leq e^{(\gamma + \tilde{\alpha})T} \left(\sum_{i \in \mathcal{F}_h} v^0(x_i^0) h^d + h^d \int_0^t \sum_{i \in \mathcal{F}_h} \sum_{j \in \mathcal{F}_h} w_j(s) u_j(s) m(s, x_i(s), x_j(s), I_d(s, x_i(s), u, w)) ds \right) \\ &= e^{(\gamma + \tilde{\alpha})T} \left(\sum_{i \in \mathcal{F}_h} v^0(x_i^0) h^d + h^d \int_0^t \sum_{i \in \mathcal{F}_h^m} \sum_{j \in \mathcal{F}_h} w_j(s) u_j(s) m(s, x_i(s), x_j(s), I_d(s, x_i(s), u, w)) ds \right) \\ &\leq e^{(\gamma + \tilde{\alpha})T} \left(\sum_{i \in \mathcal{F}_h} v^0(x_i^0) h^d + h^d |\mathcal{F}_h^m| \bar{M} \int_0^t \sum_{j \in \mathcal{F}_h} w_j(s) u_j(s) ds \right) \\ &\leq e^{(\gamma + \tilde{\alpha})T} \left(\frac{1}{\alpha} + TK_h \right) \bar{\rho}_\alpha, \end{aligned}$$

where $K_h := h^d |\mathcal{F}_h^m| \bar{M}^2$. Thanks to the condition $\alpha > 1$ there exists T_α (only depending on α and on the coefficients of the problem) such that $\Phi : D_\alpha^T \rightarrow D_\alpha^T$, for all $T \leq T_\alpha$.

We now prove that there exists $T \in (0, T_\alpha)$ such that Φ is a contraction over D_α^T .

Step 1: Bounds over $x = \{x_i\}_{i \in \mathcal{F}_h}$

Let (u^1, w^1) and (u^2, w^2) be two pairs in D_α^T , and let x^1, x^2 be the respective solutions of

$$\dot{x}_i^j = A_{u^j, w^j}(t, x_i^j).$$

²Notice that $h^d |\mathcal{F}_h^m| \approx |\operatorname{supp} m + B_{\|a\|_{L^\infty} T}|$

By following the same ideas as in the proof of Lemma 1 (see Appendix A), we obtain that for all $t \in [0, T]$,

$$\|x^1(t) - x^2(t)\|_{\infty, h} \leq C(T, h) (\|w^1 - w^2\|_{\infty, h} + \|u^1 - u^2\|_{1, h}),$$

where the constant $C(T, h)$ satisfies $\lim_{T \rightarrow 0} C(T, h) = 0$.

Step 2: Bounds over $\omega = \{\omega_i\}_{i \in \mathcal{I}_h}$

From the expression for ω , we get, for all $t \in [0, T]$

$$\begin{aligned} |\omega_i^1(t) - \omega_i^2(t)| &\leq h^d T e^{\bar{a}T} \left(\|a\|_{W_{x,T}^{2,\infty}} (1 + |\partial_x I_a(t, x_i^2, u^2, w^2)|) \|x^1 - x^2\|_{\infty, h} \right. \\ &\quad + \|a\|_{W_{x,T}^{2,\infty}} (1 + |\partial_x I_a(t, x_i^2, u^2, w^2)|) |I_a(t, x_i^1, u^1, w^1) - I_a(t, x_i^2, u^2, w^2)| \\ &\quad \left. + \|a\|_{W_T^{1,\infty}} |\partial_x I_a(t, x_i^1, u^1, w^1) - \partial_x I_a(t, x_i^2, u^2, w^2)| \right). \end{aligned}$$

On the other hand we have

$$\begin{aligned} |\partial_x I_a(t, x_i^2, u^2, w^2)| &\leq \|\psi_a\|_{W_x^{1,\infty}} \bar{\rho}_\alpha, \\ |I_a(t, x_i^1, u^1, w^1) - I_a(t, x_i^2, u^2, w^2)| &\leq \bar{\rho}_\alpha \|\psi_a\|_{W_{x,y}^{1,\infty}} \|x^1 - x^2\|_{\infty, h}, \\ &\quad + \|\psi_a\|_{L_{x,y}^\infty} e^{\bar{a}T} (h^d \|u^1 - u^2\|_{1,\infty} + \frac{\bar{\rho}_\alpha}{h^d} \|w^1 - w^2\|_{\infty, h}), \end{aligned}$$

and

$$\begin{aligned} |\partial_x I_a(t, x_i^1, u^1, w^1) - \partial_x I_a(t, x_i^2, u^2, w^2)| &\leq \bar{\rho}_\alpha \|\psi_a\|_{W_{x,y}^{2,\infty}} \|x^1 - x^2\|_{\infty, h} \\ &\quad + \|\psi_a\|_{W_x^{1,\infty}} e^{\bar{a}T} (h^d \|u^1 - u^2\|_{1,\infty} + \frac{\bar{\rho}_\alpha}{h^d} \|w^1 - w^2\|_{\infty, h}). \end{aligned}$$

In conclusion, there exists a constant $C(T, h)$, satisfying $\lim_{T \rightarrow 0} C(T, h) = 0$ such that

$$\|\omega^1 - \omega^2\|_{\infty, h} \leq C(T, h) (\|w^1 - w^2\|_{\infty, h} + \|u^1 - u^2\|_{1, h}).$$

Step 3: Bounds over $\nu = \{\nu_i\}_{i \in \mathcal{I}_h}$

Using the expression for ν , the regularity of m and bounds similar to those used for ω , we see that there exists a constant $C(T, h)$ satisfying $\lim_{T \rightarrow 0} C(T, h) = 0$ such that

$$\|\nu^1 - \nu^2\|_{1, h} \leq C(T, h) (\|w^1 - w^2\|_{\infty, h} + \|u^1 - u^2\|_{1, h}).$$

Consequently, there exists a constant $C(T, h)$ satisfying $\lim_{T \rightarrow 0} C(T, h) = 0$, such that

$$\|\omega^1 - \omega^2\|_{\infty, h} + \|\nu^1 - \nu^2\|_{1, h} \leq C(T, h) (\|w^1 - w^2\|_{\infty, h} + \|u^1 - u^2\|_{1, h}),$$

which implies that, for $0 < T_1 \leq T_\alpha$ small enough, Φ is a contraction over $D_\alpha^{T_1}$, and therefore it has a unique fixed point. Such fixed point is a solution of problem (47) over $[0, T_1]$.

We now claim that the solution exists for T arbitrary, and furthermore, it satisfies the relation (53). Let T_f be the maximal time of existence of solution. Suppose that there exists $t_0 \in (0, T_f]$ such that $\|\nu w\|_{1, h} > \bar{\rho}_\alpha$. This implies that there exist $\delta \geq 0$ and $t^* > 0$ such that for a certain finite subset of \mathcal{I}_h , that we denote as \mathcal{K}_h , the following statements are true:

$$\begin{aligned} \sum_{i \in \mathcal{K}_h} \nu_i(t) w_i(t) &\leq \bar{\rho}_\alpha, \quad \forall t \in [t^* - \delta, t^*], \\ \sum_{i \in \mathcal{K}_h} \nu_i(t) w_i(t) &> \bar{\rho}_\alpha, \quad \forall t \in (t^*, t^* + \delta]. \end{aligned}$$

This implies the existence of $t_1 \in [t^*, t^* + \delta]$ such that the following properties are satisfied simultaneously

$$\sum_{i \in \mathcal{X}_h} \nu_i(t_1) w_i(t_1) \geq \bar{\rho}_\alpha \text{ and } \left(\sum_{i \in \mathcal{X}_h} \nu_i w_i \right)' (t_1) \geq 0. \quad (56)$$

Multiplying the equation satisfied by $\nu_i(t)$ by $w_i(t)$ we get the relation

$$\begin{aligned} \dot{\nu}_i(t) w_i(t) = & (-\operatorname{div} A_{\nu, w}(t, x_i(t)) + R(t, x_i(t), I_g(t, x_i(t), \nu, w))) \nu_i(t) w_i(t) \\ & + w_i(t) \sum_{j \in \mathcal{F}_h} w_j(t) \nu_j(t) m(t, x_i(t), x_j(t), I_g(t, x_i(t), \nu, w)), \end{aligned}$$

while, directly from the equation for $w_i(t)$ we deduce

$$\nu_i(t) \dot{w}_i(t) = \operatorname{div} A_{\nu, w}(t, x_i(t)) \nu_i(t) w_i(t).$$

Therefore, adding both relations for $i \in \mathcal{X}_h$ and using (50), we get

$$\begin{aligned} \left(\sum_{i \in \mathcal{X}_h} \nu_i w_i \right)' (t) &= \left(\sum_{i \in \mathcal{X}_h} R(t, x_i(t), I_g(t, x_i(t), \nu, w)) \nu_i(t) w_i(t) \right) \\ &\quad + \sum_{j \in \mathcal{F}} w_j(t) \nu_j(t) \sum_{i \in \mathcal{X}_h \cap \mathcal{F}_h^m} m(t, x_i(t), x_j(t), I_d(t, x_i(t), \nu, w)) w_i(t) \\ &\leq \left(\sum_{i \in \mathcal{X}_h} R(t, x_i(t), I_g(t, x_i(t))) \nu_i(t) w_i(t) \right) + \left(K + \frac{r^*}{2} \right) \sum_{j \in \mathcal{X}_h} w_j(t) \nu_j(t). \end{aligned}$$

Given that $\|\nu(t_1) w(t_1)\|_{\ell^1} \geq \bar{\rho}_\alpha \geq \frac{I^*}{\underline{\psi}_g}$, then $I_g(t_1, x_i(t_1), \nu(t_1), w(t_1)) \geq I^*$, and consequently

$$R(t_1, x_i(t_1), I_g(t_1, x_i(t_1), \nu(t_1), w(t_1))) < -r^* - K,$$

for all values of i , which in turn implies that

$$\left(\sum_{i \in \mathcal{X}_h} \nu_i w_i \right)' (t_1) \leq -\frac{r^2}{2} \sum_{i \in \mathcal{X}_h} \nu_i(t_1) w_i(t_1) < 0,$$

which contradicts (56). Therefore, $\|\nu w\|_{1,h} \leq \bar{\rho}_\alpha$ for all values of $\alpha > 1$ and for all $t \in (0, T_f)$. We can then iterate the arguments used to prove existence of a solution, and conclude that the solution can be extended to any interval $[0, T]$. As $\|\nu w\|_{1,h} \leq \bar{\rho}_\alpha$ for all t , independently of α , taking the limit when α goes to 1, we obtain (53). \square

4 Convergence of the numerical solution towards a weak solution

We study now the conditions under which a solution of problem (54) converges towards a solution of problem (19), in a certain sense that will be defined later. We split our analysis in two cases: first, the study of convergence on a finite interval of time $[0, T]$. We will see that for any $T > 0$, the solution obtained through the particle method converges towards the solution of the PDE (19). However, the speed of convergence might be affected by the value of T . The second case we study is the asymptotic proximity of both solutions when t goes to ∞ . This is a far more complex and interesting issue, and we show different examples exposing some of the behaviours that can be observed.

Directly from the study of existence of solutions for each problem, we notice that the sets of hypotheses we have used, do not coincide. We give a set of hypotheses which simultaneously guarantees the existence of

solution for both problems, while taking into account the distinction of cases involved in the definition of κ . For a certain $T > 0$ and $k > 0$, we consider the functions

$$\psi_a \in \mathcal{C}([0, T] \times \mathbb{R}_x^d, W^{1, \infty}(\mathbb{R}_y^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y^d, \mathcal{C}^{k+1}(\mathbb{R}_x^d) \cap W^{k+1, \infty}(\mathbb{R}_x^d)), \quad (57)$$

$$0 < \underline{\psi}_g \leq \psi_g \in \mathcal{C}([0, T] \times \mathbb{R}_x, W^{1, \infty}(\mathbb{R}_y^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y, \mathcal{C}^\kappa(\mathbb{R}_x^d) \cap W^{\kappa, \infty}(\mathbb{R}_x^d)), \quad (58)$$

$$\psi_d \in \mathcal{C}([0, T] \times \mathbb{R}_x^d, W^{1, \infty}(\mathbb{R}_y^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y^d, \mathcal{C}^\kappa(\mathbb{R}_x^d) \cap W^{\kappa, \infty}(\mathbb{R}_x^d)). \quad (59)$$

As in Section 2 we introduce

$$a \in \mathcal{C}([0, T], W^{k+1, \infty}(\mathbb{R}^{d+1})), \quad (60)$$

$$R \in \mathcal{C}([0, T] \times \mathbb{R}_x, W_{loc}^{\kappa, \infty}(\mathbb{R}_y^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y, \mathcal{C}^\kappa(\mathbb{R}_x^d) \cap W^{\kappa, \infty}(\mathbb{R}_x^d)) \quad (61)$$

We consider as well

$$0 \leq m \in \mathcal{C}([0, T] \times \mathbb{R}_x^d \times \mathbb{R}_y^d, W^{\kappa, \infty}(\mathbb{R}_I)) \cap \mathcal{C}([0, T] \times \mathbb{R}_x^d \times \mathbb{R}_I^d, W^{\kappa, \infty}(\mathbb{R}_y^d)) \cap \mathcal{C}([0, T] \times \mathbb{R}_y^d \times \mathbb{R}_I^d, \mathcal{C}^\kappa(\mathbb{R}_x^d)). \quad (62)$$

satisfying hypothesis (7) through (18).

Finally, we consider $v^0 \in W^{k,1}(\mathbb{R}^d)$ if $\partial_I a = 0$ and $v^0 \in W^{k,1}(\mathbb{R}^d) \cap W^{k, \infty}(\mathbb{R}^d)$ with compact support otherwise.

4.1 Convergence on a finite time interval

We recall that the function v represents the solution of problem (19) while x_i , w_i and ν_i , $i \in \mathcal{J}_h$ represents that of problem (54). We recall as well that

$$\max\{\|v\|, \|\nu w\|_{1,h}\} \leq \max\{\|v^0\|_{L^1(\mathbb{R}^d)}, \|v^0 h^d\|_{\ell^1}, \frac{I^*}{\underline{\psi}_g}\} =: \bar{\rho}.$$

Let $\varepsilon > 0$, $r \in \mathbb{R}$ and $\varphi \in \mathcal{C}_c(\mathbb{R}^d)$ satisfy the following conditions

$$\int_{\mathbb{R}^d} \varphi(x) dx = 1, \quad (63)$$

$$\int_{\mathbb{R}^d} x^\alpha \varphi(x) dx = 0, \quad \forall \alpha \in \mathbb{N}^n, |\alpha| \leq r - 1. \quad (64)$$

. We define, for all $t \in [0, T]$, $x \in \mathbb{R}^d$

i)

$$v^h(t, x) = \sum_{i \in \mathcal{J}_h} \nu_i(t) w_i(t) \delta(x - x_i(t)), \quad (65)$$

a time dependent measure obtained as a sum of weighted Dirac deltas at $x_i(t)$,

ii)

$$v_\varepsilon^h(t, x) = (v^h(t) * \varphi_\varepsilon)(x) = \sum_{i \in \mathcal{J}_h} \nu_i(t) w_i(t) \varphi_\varepsilon(x - x_i(t)), \quad (66)$$

a regular function obtained as the space convolution of $v^h(t, x)$ and $\varphi_\varepsilon(x)$, where

$$\varphi_\varepsilon := \frac{1}{\varepsilon^d} \varphi\left(\frac{\cdot}{\varepsilon}\right).$$

We also introduce the following operator, for any function $v \in L^\infty(0, T; L^1(\mathbb{R}^d))$:

$$\left(\Pi_\varepsilon^h(t)v\right)(x) = \sum_{i \in \mathcal{J}_h} w_i(t) v(t, x_i(t)) \varphi_\varepsilon(x - x_i(t)).$$

We recall a direct corollary of the **Theorem 3** in [36]:

Proposition 5. *Let k, r be two integers, with $k > d$, and let us assume that $a \in L^\infty(0, T; W^{k+1, \infty}(\mathbb{R}^d)^d)$, and that $\varphi \in \mathcal{C}_c^1(\mathbb{R}^d) \cap W^{k+1, 1}(\mathbb{R}^d)$ satisfies conditions (63) and (64). Then, for any $p \in [1, +\infty]$, there exists $C = C(T) > 0$ such that, for any $u \in W^{\mu, 1}(\mathbb{R}^d)$ ($\mu = \max(r, k)$),*

$$\|u - \Pi_\varepsilon^h(t)u\|_{L^p(\mathbb{R}^d)} \leq C(\varepsilon^r \|u\|_{W^{r, p}(\mathbb{R}^d)} + \left(\frac{\varepsilon}{h}\right)^k \|u\|_{W^{k, p}(\mathbb{R}^d)}).$$

We seek to prove the following approximation result between v_ε^h and v , the solution of problem (19).

Theorem 6. *Assume that hypotheses (57) through (62) are satisfied, and that $\varphi \in \mathcal{C}_c^1(\mathbb{R}^d) \cap W^{k+1, 1}(\mathbb{R}^d)$ satisfies (63) and (64). Then, there exists $C = C(T, a, R, m, \bar{\rho}) > 0$, a positive constant which depends on T, a, R, m and $\bar{\rho}$ such that*

$$\|v - v_\varepsilon^h\|_{L^1(\mathbb{R}^d)} \leq C(\varepsilon^r + \left(\frac{h}{\varepsilon}\right)^\kappa + h^\kappa) \|v^0\|_{W^{\mu, 1}(\mathbb{R}^d)}, \quad \forall 0 \leq t \leq T,$$

where $\mu = \max(r, \kappa)$.

The proof of Theorem 6 strongly relies on Proposition 5 and the following result:

Proposition 7. *Under hypotheses (57) through (62), there exists a constant $C_T > 0$, depending only on T and on the parameters of problems (19) and (54), such that their respective solutions satisfy, for all $t \in [0, T]$,*

$$\sum_{i \in \mathcal{I}_h} |v(t, x_i(t)) - \nu_i(t)| w_i(t) \leq C_T h^{k-1} \|v^0\|_{W^{\mu, 1}(\mathbb{R}^d)}. \quad (67)$$

Proof. Consider β_ε as in (88). We define $e = \{e_i(\cdot)\}_{i \in \mathcal{I}}$ where for all $i \in \mathcal{I}$ and $t \in [0, T]$,

$$\begin{aligned} e_i(t) &:= v(t, x_i(t)) - \nu_i(t), \\ e_{\varepsilon, i}(t) &:= \beta_\varepsilon(e_i(t)) w_i(t), \end{aligned}$$

and compute

$$\dot{e}_{\varepsilon, i}(t) = \beta'_\varepsilon(e_i(t)) \dot{e}_i(t) w_i(t) + \beta_\varepsilon(e_i(t)) \dot{w}_i(t). \quad (68)$$

We recall that

$$\begin{aligned} \dot{e}_i(t) &= \left(G_{\nu, w}(t, x_i(t)) - \mathcal{G}_v(t, x_i(t)) \right) \nu_i(t) + \left(A_{\nu, w}(t, x_i(t)) - \mathcal{A}_v(t, x_i(t)) \right) \nabla v(t, x_i(t)) \\ &\quad - \mathcal{G}_v(t, x_i(t)) e_i(t) + \Delta \mathcal{M}(t, x_i(t)), \end{aligned}$$

where

$$\begin{aligned} G_{\nu, w}(t, x_i(t)) &:= \operatorname{div} A_{\nu, w}(t, x_i(t)) - R(t, x_i(t), I_g(t, x_i(t), \nu, w)), \\ \mathcal{G}_v(t, x_i(t)) &:= \operatorname{div} \mathcal{A}_v(t, x_i(t)) + R(t, x_i(t), (I_g v)(t, x_i(t))), \\ \Delta \mathcal{M}(t, x_i(t)) &:= \int_{\mathbb{R}^d} m(t, x_i(t), y, (I_d v)(t, x_i)) v(t, y) dy - \sum_{j \in \mathcal{I}_h} w_j(t) \nu_j(t) m(t, x_i(t), x_j(t), I_d(t, x_i(t), \nu, w)). \end{aligned}$$

The functions a and R being Lipschitz, there exists a constant C depending on the parameters of the problem and the value $\bar{\rho}$, such that

$$\begin{aligned} \left| G_{\nu, w}(t, x_i(t)) - \mathcal{G}_v(t, x_i(t)) \right| &\leq C \left(|(I_a v)(t, x_i(t)) - I_a(t, x_i(t), \nu, w)| \right. \\ &\quad \left. + |\partial_x (I_a v)(t, x_i(t)) - \partial_x I_a(t, x_i(t), \nu, w)| \right. \\ &\quad \left. + |(I_g v)(t, x_i(t)) - I_g(t, x_i(t), \nu, w)| \right). \end{aligned}$$

Notice that

$$\begin{aligned}
|(I_a v)(t, x_i(t)) - I_a(t, x_i(t), \nu, w)| &= \left| \int_{\mathbb{R}^d} \psi_a(t, x_i(t), y) v(y) dy - \sum_{j \in \mathcal{F}_h} \psi_a(t, x_i(t), x_j(t)) \nu_j(t) w_j(t) \right| \\
&\leq \left| \int_{\mathbb{R}^d} \psi_a(t, x_i(t), y) v(y) dy - \sum_{j \in \mathcal{F}_h} \psi_a(t, x_i(t), x_j(t)) v(t, x_j(t)) w_j(t) \right| \\
&\quad + \left| \sum_{j \in \mathcal{F}_h} \psi_a(t, x_i(t), x_j(t)) \left(v(t, x_j(t)) - \nu_j(t) \right) w_j(t) \right| \\
&\leq C \left(h^\kappa \|v\|_{W^{\kappa,1}(\mathbb{R}^d)} + \sum_{j \in \mathcal{F}_h} |e_j(t)| w_j(t) \right),
\end{aligned}$$

where in the last line we have used Lemma 8 and the $W^{\kappa,1}(\mathbb{R}^d)$ regularity of v . Similar results are true for $|\partial_x(I_a v)(t, x_i(t)) - \partial_x I_a(t, x_i(t), \nu, w)|$ and $|(I_g v)(t, x_i(t)) - I_g(t, x_i(t), \nu, w)|$. In conclusion, thanks to (27), there exists a constant C_T , only depending on T , the parameters of the problem and the value $\bar{\rho}$, such that

$$\left| G_{\nu, w}(t, x_i(t)) - \mathcal{G}_v(t, x_i(t)) \right| \leq C_T \left(h^\kappa \|v^0\|_{W^{\kappa,1}(\mathbb{R}^d)} + \sum_{j \in \mathcal{F}_h} |e_j(t)| w_j(t) \right). \quad (69)$$

Again, using the Lipschitz regularity of a , we see that

$$\begin{aligned}
\left| A_{\nu, w}(t, x_i(t)) - \mathcal{A}_v(t, x_i(t)) \right| &\leq C |(I_a v)(t, x_i(t)) - I_a(t, x_i(t), \nu, w)| \\
&\leq C_T \left(h^\kappa \|v^0\|_{W^{\kappa,1}(\mathbb{R}^d)} + \sum_{j \in \mathcal{F}_h} |e_j(t)| w_j(t) \right). \quad (70)
\end{aligned}$$

The boundedness of a and R implies the existence of a constant \bar{G} such that for all $i \in \mathcal{I}$ and $t \in [0, T]$,

$$|\mathcal{G}_v(t, x_i(t))| \leq \bar{G}. \quad (71)$$

We recall from the previous section that

$$\Delta \mathcal{M}(t, x_i(t)) = 0, \quad \forall i \notin \mathcal{F}_h^m,$$

where the set of indexes \mathcal{F}_h^m has a finite number of elements, which depends on T . On the other hand, for those $i \in \mathcal{F}_h^m$, we have

$$\begin{aligned}
&|\Delta \mathcal{M}(t, x_i(t))| \\
&\leq \left| \int_{\mathbb{R}^d} m(t, x_i(t), y, (I_d v)(t, x_i(t))) v(t, y) dy - \sum_{j \in \mathcal{F}_h} w_j(t) v(t, x_j(t)) m(t, x_i(t), x_j(t), (I_d v)(t, x_i(t))) \right| \\
&\quad + \sum_{j \in \mathcal{F}_h} w_j(t) v(t, x_j(t)) \left| m(t, x_i(t), x_j(t), (I_d v)(t, x_i(t))) - m(t, x_i(t), x_j(t), I_d(t, x_i(t), \nu, w)) \right| \\
&\quad + \sum_{j \in \mathcal{F}_h} w_j(t) |e_j(t)| m(t, x_i(t), x_j(t), (I_d v)(t, x_i(t))) \\
&\leq C \left(h^\kappa \|v\|_{W^{\kappa,1}(\mathbb{R}^d)} + \mu |(I_d v)(t, x_i(t)) - I_d(t, x_i(t), \nu, w)| \sum_{j \in \mathcal{F}_h} w_j(t) v(t, x_j(t)) + \bar{M} \sum_{j \in \mathcal{F}_h} |e_j(t)| w_j(t) \right),
\end{aligned}$$

where we have used again Lemma 8, the $W^{\kappa,1}(\mathbb{R}^d)$ regularity of $m(t, x, y, I)v(t, y)$ with respect to the y variable, and the Lipschitz regularity of m . Furthermore, Lemma 8 gives us the bound

$$\sum_{j \in \mathcal{F}_h} w_j(t) v(t, x_j(t)) \leq \|v\|_{L^1(\mathbb{R}^d)} + Ch^\kappa \|v\|_{W^{\kappa,1}(\mathbb{R}^d)},$$

which together with manipulations similar to those made for $|(I_a v)(t, x_i(t)) - I_a(t, x_i(t), \nu, w)|$, and the bound (27), gives

$$|\Delta \mathcal{M}(t, x_i(t))| \leq C_T \left(h^\kappa \|v^0\|_{W^{\kappa,1}(\mathbb{R}^d)} + \sum_{j \in \mathcal{F}_h} |e_j(t)| w_j(t) \right). \quad (72)$$

We denote as \mathcal{K} an arbitrary finite subset of \mathcal{F}_h . If we add (68) for all values of $i \in \mathcal{K}$, and use bounds (69) through (72), together with the equation for w_i , we get

$$\left(\sum_{i \in \mathcal{K}} e_{\varepsilon,i}(t) \right)' \leq C_T B(t) \left(h^\kappa \|v^0\|_{W^{\kappa,1}(\mathbb{R}^d)} + \sum_{j \in \mathcal{F}_h} |e_j(t)| w_j(t) \right),$$

where

$$B(t) := \sum_{i \in \mathcal{K}} (\nu_i(t) + |\nabla v(t, x_i(t))|) w_i(t) + \bar{G} + |\mathcal{K} \cap \mathcal{F}_h^m| h^d e^{\bar{a}T} + \tilde{a}.$$

Given that

$$\sum_{i \in \mathcal{K}} \nu_i(t) w_i(t) \leq \bar{\rho},$$

and

$$\sum_{i \in \mathcal{K}} w_j(t) |\nabla v(t, x_j(t))| \leq \|\nabla v\|_{L^1(\mathbb{R}^d)} + C h^{\kappa-1} \|\nabla v\|_{W^{\kappa-1,1}(\mathbb{R}^d)},$$

thanks to (27), we conclude that there exists a constant B_T , independent of the choice of \mathcal{K} and h , such that $B(t) \leq B_T$. Consequently, for all values of $t \in [0, T]$, h small enough and any finite subset of \mathcal{F}_h , we have the relation

$$\left(\sum_{i \in \mathcal{K}} e_{\varepsilon,i}(t) \right)' \leq C_T \left(h^\kappa + \sum_{j \in \mathcal{F}_h} |e_j(t)| w_j(t) \right).$$

Integrating between 0 and t , taking the limit when ε goes to zero, and using Grönwall's lemma, we obtain that there exists a constant C_T , independent of \mathcal{K} such that

$$\sum_{i \in \mathcal{K}} |e_i(t)| w_i(t) \leq C_T h^\kappa \|v^0\|_{W^{\kappa,1}(\mathbb{R}^d)}.$$

Being C_T independent of \mathcal{K} and h , (67) is immediate. \square

In other words, we proved in Proposition 7 that the piece-wise constant functions that take values $v(t, x_k(t))$ and $v_k(t)$ respectively over the intervals $\Omega_k(t)$ are close in $L^1(\mathbb{R}^d)$.

Proof of Theorem 6. According to the triangle inequality,

$$\|v - v_\varepsilon^h\|_{L^1(\mathbb{R}^d)} \leq \|v - \Pi_\varepsilon^h(t)v\|_{L^1(\mathbb{R}^d)} + \|\Pi_\varepsilon^h(t)v - v_\varepsilon^h\|_{L^1(\mathbb{R}^d)}, \quad (73)$$

it only remains to bound both terms on the right hand side.

i) According to Proposition 5 with $p = 1$, and bound (27)

$$\|v - \Pi_\varepsilon^h(t)v\|_{L^1(\mathbb{R}^d)} \leq C(\varepsilon^r + \left(\frac{\varepsilon}{h}\right)^\kappa) \|v\|_{\mu,p} \leq C_T(\varepsilon^r + \left(\frac{\varepsilon}{h}\right)^\kappa) \|v^0\|_{\mu,p}.$$

ii) On the other hand, one computes

$$\begin{aligned} \|\Pi_\varepsilon^h(t)v - v_\varepsilon^h\|_{L^1(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \left| \sum_{i \in \mathcal{F}_h} w_i(t) \varphi_\varepsilon(x - x_i(t)) (v(t, x_i(t)) - \nu_i(t)) \right| dx \\ &\leq \sum_{i \in \mathcal{F}_h} \left(w_i(t) |v(t, x_i(t)) - \nu_i(t)| \int_{\mathbb{R}^d} |\varphi_\varepsilon(x - x_i(t))| dx \right). \end{aligned}$$

According to the definition of φ_ε , with the change of variable $x' = \frac{x-x_k(t)}{\varepsilon}$, we note that

$$\int_{\mathbb{R}^d} |\varphi_\varepsilon(x - x_k(t))| dx = \int_{\mathbb{R}^d} |\varphi(x)| dx < +\infty,$$

by hypothesis on φ . We have then, according to Proposition 7, that

$$\|\Pi_\varepsilon^h(t)v - v_\varepsilon^h\|_{L^1(\mathbb{R}^d)} \leq C_T h^\kappa \|v^0\|_{\kappa,1} \leq C h^\kappa \|v^0\|_{\mu,1},$$

which concludes the proof of Theorem 6. □

4.2 Asymptotic preserving properties

The study of the asymptotic behaviour of the solution for adaptive dynamics models, such as (19), is one of the main interests often treated in the literature (see [3, 4, 6, 12, 15, 18, 22, 24, 30, 33]). For this reason, the design of numerical methods which preserve the asymptotic behaviour, or at least, the identification of the problems for which the asymptotics are preserved under a certain numerical scheme, is a priority. In other words, given that $v(t, \cdot)$ converges to a measure μ when t goes to infinity, we expect to identify the conditions under which $\lim_{h \rightarrow 0} \lim_{t \rightarrow +\infty} v_{\varepsilon(h)}^h(t, \cdot) = \mu$, that is, ensuring the commutativity of diagram (74):

$$\begin{array}{ccc} v(t, \cdot) & \xrightarrow[t \rightarrow \infty]{} & \mu \\ \uparrow h \rightarrow 0 & & \uparrow h \rightarrow 0 \\ v_{\varepsilon(h)}^h(t) & \xrightarrow[t \rightarrow \infty]{} & \mu_h \end{array} \quad (74)$$

In what follows, we formally define the concept of an asymptotic preserving approximation, and give examples and counter-examples of this concept.

We recall that, according to the Riesz representation theorem, the space of finite Radon measures can be identified with the topological dual space of $\mathcal{C}_c(\mathbb{R}^d)$. Hence, we say that a sequence of finite Radon measures $\{\mu_n\}_{n \in \mathbb{N}}$ converges weakly to a finite Radon measure μ (denoted $\mu_n \xrightarrow[n \rightarrow +\infty]{} \mu$) if for all $\phi \in \mathcal{C}_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \phi(x) d\mu_n(x) \xrightarrow[n \rightarrow +\infty]{} \int_{\mathbb{R}^d} \phi(x) d\mu(x).$$

This leads us to introduce the following definition:

Definition 1. We say that the particle solution v_ε^h defined in (66) is an *asymptotic preserving approximation* of v , the solution to (19), if for all $\varepsilon : (0, 1] \rightarrow \mathbb{R}_+^*$ which converges to 0 when h goes to 0, and all $\phi \in \mathcal{C}_c(\mathbb{R}^d)$,

$$\limsup_{t \rightarrow +\infty} \left| \int_{\mathbb{R}^d} \phi(x) v_{\varepsilon(h)}^h(t, x) dx - \int_{\mathbb{R}^d} \phi(x) v(t, x) dx \right| \xrightarrow[h \rightarrow 0]{} 0.$$

The following lemma ensures that, in the previous definition, $v_{\varepsilon(h)}^h$, introduced in (66), can be replaced by v^h , introduced in (65).

Lemma 9. *The function v_ε^h is an asymptotic preserving approximation of v if and only if*

$$\limsup_{t \rightarrow +\infty} \left| \int_{\mathbb{R}^d} \phi(x) dv^h(t, x) - \int_{\mathbb{R}^d} \phi(x) v(t, x) dx \right| \xrightarrow[h \rightarrow 0]{} 0.$$

Proof. Let us prove that for all $\varepsilon : (0, 1] \rightarrow \mathbb{R}_+^*$ which converges to 0 as h goes to 0, and all $\phi \in \mathcal{C}_c(\mathbb{R}^d)$,

$$\limsup_{t \rightarrow +\infty} \left| \int_{\mathbb{R}^d} \phi(x) v_{\varepsilon(h)}^h(t, x) dx - \int_{\mathbb{R}^d} \phi(x) dv^h(t, x) \right| \xrightarrow{h \rightarrow 0} 0.$$

Let $h > 0$. According to the definitions of v^h and v_{ε}^h , and since $\int_{\mathbb{R}^d} \varphi_{\varepsilon(h)}(x - x_i(t)) dx = 1$ for all $i \in \mathcal{J}_h$ we get, for all $t \geq 0$,

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \phi(x) v_{\varepsilon(h)}^h(t, x) dx - \int_{\mathbb{R}^d} \phi(x) dv^h(t, x) \right| &= \left| \int_{\mathbb{R}^d} \sum_{i \in \mathcal{J}_h} \nu_i(t) w_i(t) (\phi(x) - \phi(x_i(t))) \varphi_{\varepsilon(h)}(x - x_i(t)) dx \right| \\ &\leq \sum_{i \in \mathcal{J}_h} |\nu_i(t) w_i(t)| \int_{\mathbb{R}^d} |(\phi(x) - \phi(x_i(t))) \varphi_{\varepsilon(h)}(x - x_i(t))| dx. \end{aligned}$$

With the change of variable ' $y = \frac{x - x_i(t)}{\varepsilon(h)}$ ', we get, for all $i \in \mathcal{J}_h$,

$$\int_{\mathbb{R}^d} |(\phi(x) - \phi(x_i(t))) \varphi_{\varepsilon(h)}(x - x_i(t))| dx = \int_K |(\phi(\varepsilon(h)y + x_i(t)) - \phi(x_i(t))) \varphi(y)| dy,$$

where K is the support of φ . Let $\eta > 0$. Since ϕ is continuous with a compact support, and thus uniformly continuous, then $|\phi(\varepsilon(h)x + x_i(t)) - \phi(x_i(t))| \leq \eta$ for all $i \in \mathcal{J}_h$, $x \in K$, $t \geq 0$ and any h small enough. Therefore, for any h small enough,

$$\left| \int_{\mathbb{R}^d} \phi(x) v_{\varepsilon(h)}^h(t, x) dx - \int_{\mathbb{R}^d} \phi(x) dv^h(t, x) \right| \leq \eta \sum_{i \in \mathcal{J}_h} \nu_i(t) w_i(t),$$

which concludes the proof, since there exists $\bar{\rho} > 0$ such that $0 \leq \sum_{i \in \mathcal{J}_h} \nu_i(t) w_i(t) \leq \bar{\rho}$ for all $h > 0$, $t \geq 0$, as proved in Theorem 4. \square

The problem of determining if v_{ε}^h is an asymptotic preserving approximation of v is generally a difficult question. In what follows, we deal with cases where we are able to determine the asymptotic behaviour of both v and v^h , and we check if the necessary and sufficient condition from Lemma 9 holds. From now on, we assume that a is local and not time dependent, *i.e.* $a(t, x, I) = a(x)$ and that the functions m , R , ψ_g and ψ_d are not time-dependent. We assume as well that the function $(x, y, I) \mapsto m(x, y, I)$ is not only uniformly compactly supported as a function of the x variable, but relative to the y variable as well. That is, there exist two compact sets K_x and K_y such that $\sup_{y, I} m(x, y, I) = 0$ for all y outside of K_x and $\sup_{x, I} m(x, y, I) = 0$ for all x outside of K_y . We denote $K_{xy} := K_x \cup K_y$. Finally, we assume ψ_g and ψ_d to be compactly supported as functions of the y variable.

4.2.1 Necessary conditions of convergence towards a Radon measure

With the help of necessary conditions, we would be able to rule out those cases where v does not converge towards certain types of Radon measures, which are the object of our interest. We start by giving a general result, involving the necessary conditions of convergence towards any Radon measure.

Lemma 10. *Let us assume that $v(t, \cdot) \xrightarrow{t \rightarrow +\infty} \mu$ in the weak sense in the space of finite Radon measures.*

Then, for $\gamma \in \{g, d\}$,

$$I_{\alpha}(t, x) \xrightarrow{t \rightarrow +\infty} \int_{\mathbb{R}^d} \psi_{\alpha}(x, y) d\mu(y) =: \bar{I}_{\alpha}(x) \quad \forall x \in \mathbb{R}^d,$$

and for all $\phi \in \mathcal{C}_0^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \left(a(x) \cdot \nabla \phi(x) + R(x, \bar{I}_g(x)) \phi(x) \right) d\mu(x) + \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} m(x, y, \bar{I}_d(x)) \phi(x) dx \right) d\mu(y) = 0.$$

Proof. Let us assume that $v(t, \cdot) \xrightarrow{t \rightarrow +\infty} \mu$. By definition of the weak convergence in the space of finite Radon measure, for any $\phi \in \mathcal{C}_c^1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \phi(x)v(t, x)dx \xrightarrow{t \rightarrow +\infty} \int_{\mathbb{R}^d} \phi(x)d\mu(x).$$

The first identity is thus a direct consequence of the definition of the weak convergence, applied with $\phi = \psi_\gamma(x, \cdot)$, $\gamma \in \{g, d\}$.

Let $\phi \in \mathcal{C}_c^1(\mathbb{R}^d)$. One computes

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \phi(x)v(t, x)dx &= - \int_{\mathbb{R}^d} \phi(x) \nabla \cdot (a(x)v(t, x)) + \int_{\mathbb{R}^d} R(x, I_g(t, x))\phi(x)v(t, x)dx \\ &\quad + \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} m(x, y, I_d(t, x))v(t, y) \right) \phi(x)dx \\ &= \int_{\mathbb{R}^d} \left(a(x) \cdot \nabla \phi(x) + R(x, I_g(t, x))\phi(x) \right) v(t, x)dx \\ &\quad + \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} m(x, y, I_d(t, x))\phi(x)dx \right) v(t, y)dy \\ &\xrightarrow{t \rightarrow +\infty} \int_{\mathbb{R}^d} \left(a(x) \cdot \nabla \phi(x) + R(x, \bar{I}_g(x))\phi(x) \right) d\mu(x) \\ &\quad + \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} m(x, y, \bar{I}_d(x))\phi(x)dx \right) d\mu(y). \end{aligned}$$

Thus, $t \mapsto \int_{\mathbb{R}^d} \phi(x)v(t, x)dx$ is a convergent function with a convergent derivative, which ensures that the limit of its derivative is zero, which concludes the proof. \square

The following proposition provides a necessary condition for the convergence to a sum of Dirac masses.

Proposition 8. *Let us assume that $v(t, \cdot) \xrightarrow{t \rightarrow +\infty} \sum_{i=1}^N C_i \delta_{x_i}$, with $x_1, \dots, x_N \in \mathbb{R}^d$, $C_1, \dots, C_N > 0$. Then,*

for $\alpha = g, d$, $I_\alpha(t, x) \xrightarrow{t \rightarrow +\infty} \sum_{i=1}^N C_i \psi_\alpha(x, x_i) =: \bar{I}_\alpha(x)$. Moreover, for all $i \in \{1, \dots, N\}$, $a(x_i) = 0$, $R(x_i, \bar{I}_g(x_i)) = 0$, and for all $x \in \mathbb{R}^d$, $m(x, x_i, \bar{I}_d(x)) = 0$.

Proof. According to the previous lemma, for all $\phi \in \mathcal{C}_c^1(\mathbb{R}^d)$,

$$\sum_{i=1}^N C_i a(x_i) \cdot \nabla \phi(x_i) + \sum_{i=1}^N C_i R(x_i, \bar{I}_g(x_i)) \phi(x_i) + \sum_{i=1}^N C_i \int_{\mathbb{R}^d} m(x, x_i, \bar{I}_d(x)) \phi(x)dx = 0.$$

Let $\varepsilon > 0$. For any non-negative function $\phi_\varepsilon \in \mathcal{C}_c^1(\mathbb{R}^d)$ with a support on $\mathbb{R}^d \setminus \bigcup_{i=1}^N B(x_i, \varepsilon)$, we have that

$$\int_{\mathbb{R}^d} \sum_{i=1}^N C_i m(x, x_i, \bar{I}_d(x)) \phi_\varepsilon(x)dx = 0,$$

which proves that $x \mapsto \sum_{i=1}^N C_i m(x, x_i, \bar{I}_d(x))$ is 0 on $\mathbb{R}^d \setminus \bigcup_{i=1}^N B(x_i, \varepsilon)$. Since m is a non-negative function, for all $i \in \{1, \dots, N\}$, $x \mapsto m(x, x_i, \bar{I}_d(x))$ is 0 as well on $\mathbb{R}^d \setminus \bigcup_{i=1}^N B(x_i, \varepsilon)$, and therefore on \mathbb{R}^d , since the result holds for any $\varepsilon > 0$. Hence,

$$\sum_{i=1}^N C_i a(x_i) \cdot \nabla \phi(x_i) + \sum_{i=1}^N C_i R(x_i, \bar{I}_g(x_i)) \phi(x_i) = 0,$$

for any $\phi \in \mathcal{C}_c^1(\mathbb{R}^d)$. By choosing ϕ_j^1 such that

$$\phi_j^1(x_i) = \delta_{ij} \text{ and } \nabla \phi_j^1(x_i) = 0, \quad i, j \in \{1, \dots, N\},$$

where δ_{ij} represents the Kronecker delta, one proves that $R(x_i, \overline{I_g}(x_i)) = 0$ for all $i \in \{1, \dots, N\}$. Finally, by choosing ϕ_{ij}^2 such that

$$\nabla \phi_{jl}^2(x_i) = \delta_{ij} e_l, \quad i, j \in \{1, \dots, N\}, \quad l \in \{1, \dots, d\},$$

where $\{e_l\}_{l=1}^d$ represents the euclidean basis in \mathbb{R}^d , one proves that $a(x_i) = 0$, for any $i \in \{1, \dots, N\}$. \square

4.2.2 Limit identification and asymptotic preserving approximations

In some cases, it is possible to guarantee the existence of a limit for v^h and identify it. Assume that there exists $\hat{x} \in \mathbb{R}^d$ an asymptotically stable equilibrium for the ODE ' $\dot{x} = a(x)$ ' and that there exists $C, \delta > 0$ such that

$$\forall y \in \text{supp}(v^0), t \geq 0, \quad \|X(t, y) - \hat{x}\| \leq C e^{-\delta t}. \quad (75)$$

Moreover, let us assume that there exist positive values D, I^m and I^M such that

$$R(x, I^m) \geq 0, \quad R(x, I^M) \leq 0 \text{ and } \partial_I R(x, I) \leq -D, \quad \forall x \in \text{supp}(v^0). \quad (76)$$

Then, we can compute the limit of v^h when t goes to $+\infty$, whatever the value of m , as stated in the following proposition.

Proposition 9. *Let us assume that $\text{supp}(v^0)$ is a compact set such that hypotheses (75) and (76) hold. Then, v^h converges to $\hat{\rho}_h \delta_{\hat{x}}$ in the weak sense in the space of Radon measures, where $\hat{\rho}_h$ is the unique solution of*

$$R(\hat{x}, \psi_g(\hat{x}, \hat{x}) \hat{\rho}_h) = 0.$$

The following lemma, proved in Appendix D, is required in the proof of this result.

Lemma 11. *Let $u \in \mathcal{C}^2(\mathbb{R}_+, \mathbb{R})$ be a bounded function, and let us assume that there exist $p_0 > 0, p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a function which satisfies $p \geq p_0$ and $B \in L^1(\mathbb{R}_+)$ an integrable function such that*

$$\ddot{u}(t) \geq -p(t)\dot{u}(t) + B(t).$$

Then, there exists $u_\infty \in \mathbb{R}$ such that $\lim_{t \rightarrow +\infty} u(t) = u_\infty$.

Proof of Proposition 9. Given that $\text{supp}(v^0) \cup K_{xy}$ is a compact set which is strictly contained in the basin of attraction of \hat{x} we will have the existence of $\mathcal{J}_h^0 \subset \mathcal{J}_h$, with $|\mathcal{J}_h^0| < +\infty$ such that $\nu^h(t) \neq 0$ only for $i \in \mathcal{J}_h^0$.

Let us denote, for all $i \in \mathcal{J}_h^0$, $\alpha_i(t) := \nu_i(t) w_i(t)$, and

$$\rho_h(t) := \sum_{i \in \mathcal{J}_h} \alpha_i(t) = \sum_{i \in \mathcal{J}_h^0} \alpha_i(t).$$

Let us note that, according to the hypotheses on a , for all $i \in \mathcal{J}_h^0$, $x_i(t)$ converges to \hat{x} . Thus,

$$v^h(t) - \rho_h(t) \delta_{\hat{x}} \xrightarrow[t \rightarrow +\infty]{} 0.$$

Hence, it only remains to prove that ρ_h converges to the expected limit. According to the definition of ρ_h ,

$$\dot{\rho}_h(t) = \sum_{i \in \mathcal{J}_h^0} R(x_i(t), I_g(t, x_i(t))) \alpha_i(t) + \underbrace{\sum_{i, j \in \mathcal{J}_h^0} w_i(t) m(x_i(t), x_j(t), I_d(t, x_i(t))) \alpha_j(t)}_{:= \varepsilon(t)}. \quad (77)$$

According to hypothesis 75, there exist $C, \delta > 0$ such that

$$\|x_i(t) - \hat{x}\| \leq C e^{-\delta t}, \quad \forall i \in \mathcal{F}_h^0, \quad \forall t \geq 0.$$

Thus, for all $i \in \mathcal{F}_h^0$,

$$w_i(t) = e^{\int_0^t \operatorname{div} a(x_i(s)) ds} w_i(0) \leq \underbrace{e^{\int_0^t \operatorname{div} a(x_i(s)) - \operatorname{div} a(\hat{x}) ds}}_{\leq \tilde{C}} e^{\operatorname{div} a(\hat{x}) t} w_i(0),$$

which proves, since $\operatorname{div} a(\hat{x}) < 0$, that there exist C', δ' such that for all $t \geq 0$,

$$0 \leq \max_{i \in \mathcal{F}_h^0} w_i(t) \leq C' e^{-\delta' t}. \quad (78)$$

In particular, it proves that $t \mapsto \varepsilon(t)$ defined in (77), converges to zero with an exponential speed, since

$$|\varepsilon(t)| \leq \max_{i \in \mathcal{F}_h} w_i(t) |\mathcal{F}_h^0| \|m\|_{L^\infty} \rho_h(t),$$

and ρ_h is bounded, according to Theorem 4.

Moreover, according to the hypothesis on ψ_g , $\underline{\psi}_g \rho(t) \leq I_g(t, x_i(t)) \leq \|\psi_g\|_\infty \rho(t)$. Thus, according to the hypotheses (76) on R , the relation

$$\dot{\rho}_h(t) \geq \left(\min_{i \in \mathcal{F}_h^0} (R(x_i(t), I_g(t, x_i(t)))) \right) \rho_h(t)$$

implies that, as soon as ρ_h becomes small, and so does I_g , ρ_h becomes increasing, which proves that ρ_h is lower bounded by a positive constant. Moreover, since ρ_h and ε are bounded, and

$$\begin{aligned} \|\dot{\alpha}\|_{\ell_1} &= \sum_{i \in \mathcal{F}_h^0} |\dot{\alpha}(t)| \leq \sum_{i \in \mathcal{F}_h^0} (2|\operatorname{div} a(x_i(t))| + |R(x_i(t), I_g(t, x_i(t)))|) \alpha_i(t) + \varepsilon(t) \\ &\leq (2\|a\|_{W^{1,\infty}(\mathbb{R}^d)} + \bar{R}) \rho_h(t) + \varepsilon(t), \end{aligned}$$

where

$$\bar{R} := \max_{t \in \mathbb{R}, i \in \mathcal{F}_h^0} |R(x_i(t), I_g(t, x_i(t)))|,$$

then $\|\dot{\alpha}\|_{\ell_1}$ is also bounded.

Now, let us prove that ρ_h satisfies the equality of Lemma 11. First, for $\gamma \in \{g, d\}$, we compute

$$\begin{aligned} \left| \frac{d}{dt} I_\gamma(t, x_i(t)) - \psi_\gamma(\hat{x}, \hat{x}) \dot{\rho}_h(t) \right| &\leq \left| \sum_{j \in \mathcal{F}_h^0} \left(a(x_i(t)) \partial_x \psi_\gamma(x_i(t), x_j(t)) + a(x_j(t)) \partial_y \psi_\gamma(x_i(t), x_j(t)) \right) \alpha_j(t) \right| \\ &\quad + \left| \sum_{j \in \mathcal{F}_h^0} (\psi_\gamma(x_i(t), x_j(t)) - \psi_\gamma(\hat{x}, \hat{x})) \dot{\alpha}_j(t) \right| \\ &\leq 2 \max_{i \in \mathcal{F}_h^0} |a(x_i(t))| \|\psi_\gamma\|_{W^{1,\infty}(\mathbb{R}^d)} \rho_h(t) + \max_{i,j \in \mathcal{F}_h^0} |\psi_\gamma(x_i(t), x_j(t)) - \psi_\gamma(\hat{x}, \hat{x})| \|\dot{\alpha}\|_{\ell_1}. \end{aligned}$$

Since hypothesis (75), is satisfied, the functions $t \mapsto \max_{i \in \mathcal{F}_h^0} |a(x_i(t))|$ and $t \mapsto \max_{i,j \in \mathcal{F}_h^0} |\psi_\gamma(x_i(t), x_j(t)) - \psi_\gamma(\hat{x}, \hat{x})|$ converge to zero with an exponential speed. Since $|\mathcal{F}_h^0| < +\infty$, this proves that, for $\gamma \in \{g, d\}$,

$$\sum_{i \in \mathcal{F}_h^0} \left| \frac{d}{dt} I_\gamma(t, x_i(t)) - \psi_\gamma(\hat{x}, \hat{x}) \dot{\rho}_h(t) \right| = O(e^{-\delta t}), \quad (79)$$

for a certain $\delta > 0$.

Thus, by differentiating (77), we get

$$\begin{aligned}\ddot{\rho}_h(t) &= \sum_{i \in \mathcal{F}_h^0} a(x_i(t)) \partial_x R(x_i(t), I_g(t, x_i(t))) \alpha_i(t) + \sum_{i \in \mathcal{F}_h^0} \left(\frac{d}{dt} I_g(t, x_i(t)) \right) \partial_I R(x_i(t), I_g(t, x_i(t))) \alpha_i(t) \\ &+ \sum_{i \in \mathcal{F}_h^0} \dot{\alpha}_i(t) R(x_i(t), I_g(t, x_i(t))) + \sum_{i, j \in \mathcal{F}_h^0} w_i(t) m(x_i(t), x_j(t), I_d(t, x_i(t))) \dot{\alpha}_j(t) \\ &+ \sum_{i, j \in \mathcal{F}_h^0} \dot{w}_i(t) m(x_i(t), x_j(t), I_d(t, x_i(t))) \alpha_j(t) + \sum_{i, j \in \mathcal{F}_h^0} w_i(t) \frac{d}{dt} m(x_i(t), x_j(t), I_d(t, x_i(t))) \alpha_j(t),\end{aligned}$$

where

i) $\sum_{i \in \mathcal{F}_h^0} a(x_i(t)) \partial_x R(x_i(t), I_g(t, x_i(t))) \alpha_i(t) = O(e^{-\delta t})$, since $\max_{i \in \mathcal{F}_h} |a(x_i(t))|$ converges to zero with exponential speed, and $\partial_x R$ and ρ_h are bounded,

ii) according to (79), $\sum_{i \in \mathcal{F}_h^0} \left(\frac{d}{dt} I_g(t, x_i(t)) \right) \partial_I R(x_i(t), I_g(t, x_i(t))) \alpha_i(t) = -p(t) \dot{\rho}_h(t) + O(e^{-\delta t})$, where

$$p(t) = -\psi_g(\hat{x}, \hat{x}) \sum_{i \in \mathcal{F}_h^0} \partial_I R(x_i(t), I_g(t, x_i(t))) \alpha_i(t) \geq D \underline{\psi}_g \min_{t \geq 0} \rho_h(t) > 0,$$

iii)

$$\begin{aligned}\sum_{i \in \mathcal{F}_h^0} \dot{\alpha}_i(t) R(x_i(t), I_g(t, x_i(t))) &= \underbrace{\sum_{i \in \mathcal{F}_h^0} R(x_i(t), I_g(t, x_i(t)))^2 \alpha_i(t)}_{:= P(t) \geq 0} \\ &+ \underbrace{\sum_{i, j \in \mathcal{F}_h^0} w_i(t) R(x_i(t), I_g(t, x_i(t))) \alpha_j(t) m(t, x_i(t), x_j(t), I_d(t, x_i(t)))}_{= O(e^{-\delta t})},\end{aligned}$$

where the relation for the second term was proved thanks to the bound

$$\left| \sum_{i, j \in \mathcal{F}_h^0} w_i(t) R(x_i(t), I_g(t, x_i(t))) \alpha_j(t) m(t, x_i(t), x_j(t), I_d(t, x_i(t))) \right| \leq \overline{M} \overline{R} \rho_h(t) |\mathcal{F}_h^0| \max_{i \in \mathcal{F}_h} w_i(t),$$

the boundedness of ρ_h and inequality (78),

iv)

$$\begin{aligned}\sum_{i, j \in \mathcal{F}_h^0} w_i(t) m(x_i(t), x_j(t), I_d(t, x_i(t))) \dot{\alpha}_j(t) &+ \sum_{i, j \in \mathcal{F}_h^0} \dot{w}_i(t) m(x_i(t), x_j(t), I_d(t, x_i(t))) \alpha_j(t) \\ &+ \sum_{i, j \in \mathcal{F}_h^0} w_i(t) \frac{d}{dt} m(x_i(t), x_j(t), I_d(t, x_i(t))) \alpha_j(t) = O(e^{-\delta t}),\end{aligned}$$

since for all $t \geq 0$,

$$\left| \sum_{i, j \in \mathcal{F}_h^0} w_i(t) m(x_i(t), x_j(t), I_d(t, x_i(t))) \dot{\alpha}_j(t) \right| \leq \overline{M} |\mathcal{F}_h^0| \|\dot{\alpha}\|_{\ell^1} \max_{i \in \mathcal{F}_h^0} w_i(t),$$

$$\left| \sum_{i,j \in \mathcal{F}_h} \dot{w}_i(t) m(x_i(t), x_j(t), I_d(t, x_i(t))) \alpha_j(t) \right| \leq \overline{M} \rho_h(t) |\mathcal{F}_h^0| \|a\|_{W^{1,\infty}(\mathbb{R}^d)} \max_{i \in \mathcal{F}_h^0} w_i(t),$$

and

$$\left| \sum_{i,j \in \mathcal{F}_h} w_i(t) \frac{d}{dt} m(x_i(t), x_j(t), I_d(t, x_i(t))) \alpha_j(t) \right| \leq \overline{M} \left(\|a\|_{L^\infty(\mathbb{R})} + \left| \frac{d}{dt} I_g(t, x_i(t)) \right| \right) \rho_h(t) |\mathcal{F}_h^0| \max_{i \in \mathcal{F}_h^0} w_i(t),$$

where $\frac{d}{dt} I_d(t, x_i(t))$ is bounded thanks to (79). For these three inequalities, we conclude with (78).

Hence, $\ddot{\rho}_h(t) \geq -p(t)\dot{\rho}_h(t) + O(e^{-\delta t})$. According to Lemma 11, ρ_h has a limit when t goes to ∞ , which we denote $\hat{\rho}_h$. Since $\dot{\rho}_h(t) = \sum_{i \in \mathcal{F}_h^0} R(x_i(t), I_g(t, x_i(t))) \alpha_i(t) + \varepsilon(t)$, which converges to $R(\hat{x}, \psi_g(\hat{x}, \hat{x}) \hat{\rho}_h) \hat{\rho}_h$, we deduce that $R(\hat{x}, \psi_g(\hat{x}, \hat{x}) \hat{\rho}_h) = 0$, since $\hat{\rho}_h > 0$. \square

When $m \equiv 0$ and under the same assumptions for the remaining coefficients of the problem as in Proposition 9, we are able to identify the limit of v and prove that it coincides with the limit of v^h . According to Lemma 9, this ensures that v_ε^h is an asymptotic preserving approximation of v .

Theorem 10. *Let us assume that there exists $\hat{x} \in \mathbb{R}^d$ which is an asymptotically stable equilibrium for the ODE $\dot{x} = a(x)$ such that hypothesis (75) holds. We assume as well that $m \equiv 0$ and that hypotheses (76) hold. Then, v converges to $\hat{\rho} \delta_{\hat{x}}$ in the weak sense in the space of Radon measures, where $\hat{\rho}$ is the unique solution of*

$$R(\hat{x}, \psi_g(\hat{x}, \hat{x}) \hat{\rho}) = 0.$$

Consequently, v_ε^h is an asymptotic preserving approximation of v .

Proof. Let us recall that

$$\rho(t) = \int_{\mathbb{R}^d} v(t, x) dx.$$

We recall as well that, the function a being only dependent of x , the characteristic lines $X_t(x) := X(t, x)$ satisfy $X_t^{-1}(x) := X^{-1}(t, x) = X_{-t}(x)$ and $X_t(X_s(x)) = X_{t+s}(x)$. By using the fact that, for any $x \in \mathbb{R}^d$,

$$v(t, x) = v^0(X_{-t}(x)) e^{\mathcal{G}(0,t,x)},$$

where

$$\mathcal{G}(s, t, x) := \int_s^t R(X_{\tau-t}(x), (I_g v)(\tau, X_{\tau-t}(x))) - \operatorname{div} a(X_{\tau-t}(x)) d\tau,$$

and that, by hypothesis, $K := \operatorname{supp}(v^0)$ is a compact set included in the basin of attraction of \hat{x} , one proves that $\operatorname{supp}(v(t, \cdot))$ is the image of $\operatorname{supp}(v^0)$ by $X_t(\cdot)$. Since $X_t(y)$ converges to \hat{x} for all $y \in \operatorname{supp}(v^0)$, we prove that $K_t := \operatorname{supp}(v(t, \cdot)) = X_t(\operatorname{supp}(v^0)) = X_t(K)$ is a compact set included in the basin of attraction of \hat{x} , for all $t \geq 0$. By (75), there exist $C > 0$ and $\delta > 0$ such that

$$\|X_t(y) - \hat{x}\| \leq C e^{-\delta t}, \quad \forall y \in K, \quad \forall t \geq 0.$$

Let $\phi \in \mathcal{C}_c(\mathbb{R}^d)$. By definition of ρ and by using the change of variable ' $x = X_t(y)$ ' we get

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \phi(x) v(t, x) - \rho(t) \phi(\hat{x}) \right| &\leq \int_{K_t} |\phi(x) - \phi(\hat{x})| v(t, x) dx \\ &= \int_K |\phi(X_t(y)) - \phi(\hat{x})| v(t, X_t(y)) e^{\int_0^t \operatorname{div} a(X_t(y)) dy} dy \\ &\leq \max_{y \in K} |\phi(X_t(y)) - \phi(\hat{x})| \int_K v(t, X_t(y)) e^{\int_0^t \operatorname{div} a(X_t(y)) dy} dy \\ &= \max_{y \in K} |\phi(X_t(y)) - \phi(\hat{x})| \rho(t). \end{aligned}$$

Since ρ is bounded, this proves that

$$v(t, \cdot) - \rho(t)\delta_{\hat{x}} \xrightarrow{t \rightarrow +\infty} 0. \quad (80)$$

Hence, it only remains to prove that ρ converges to the expected limit. We see that

$$\dot{\rho}(t) = \int_{\mathbb{R}^d} R(x, (I_g v)(t, x)) v(t, x) dx. \quad (81)$$

First, let us note that ρ has a positive lower bound. Indeed, according to the hypothesis on ψ_g , for all $x \in \mathbb{R}^d$, $\underline{\psi}_g \rho(t) \leq (I_g v)(t, x) \leq \|\psi_g\|_{L^\infty} \rho(t)$. Thus,

$$\dot{\rho}(t) \geq \left(\min_{x \in K_t} (R(x, (I_g v)(t, x))) \right) \rho(t).$$

Hence, as soon as $\rho(t)$ becomes small, and so does $I_g(t, x)$, ρ becomes increasing, which proves that ρ is lower bounded by a positive constant. We get an upper bound for $|\dot{\rho}(t)|$ from the relation

$$|\dot{\rho}(t)| \leq \left(\max_{x \in K_t} (R(x, (I_g v)(t, x))) \right) \rho(t),$$

and the boundedness of $\rho(t)$.

We introduce now the function

$$\tilde{v}(t, y) = v(t, X_t(y)) e^{\int_0^t \operatorname{div} a(X_s(y)) ds},$$

which satisfies

$$\int_K \tilde{v}(t, y) dy = \int_{K_t} v(t, x) dx = \rho(t).$$

Moreover,

$$\partial_t \tilde{v}(t, y) = R(X_t(y), (I_g v)(t, X_t(y))) \tilde{v}(t, y). \quad (82)$$

Before proving that ρ satisfies the equality of Lemma 11, we observe that the following relation holds for all $y \in K$:

$$\begin{aligned} \left| \frac{d}{dt} (I_g v)(t, X_t(y)) - \psi_g(\hat{x}, \hat{x}) \dot{\rho}(t) \right| &\leq \left| \int_{K_t} a(X_t(y)) \cdot \nabla_x \psi_g(X_t(y), z) v(t, z) dz \right| \\ &\quad + \left| \int_{K_t} (\psi_g(X_t(y), z) - \psi_g(\hat{x}, \hat{x})) \partial_t v(t, z) dz \right|. \end{aligned} \quad (83)$$

From the equation satisfied by v , we see that

$$\begin{aligned} &\int_{K_t} (\psi_g(X_t(y), z) - \psi_g(\hat{x}, \hat{x})) \partial_t v(t, z) dz \\ &= \int_{K_t} \nabla_z \psi_g(X_t(y), z) a(z) v(t, z) dz \\ &\quad + \int_{K_t} (\psi_g(X_t(y), z) - \psi_g(\hat{x}, \hat{x})) R(z, (I_g v)(t, z)) v(t, z) dz \\ &= \int_K \nabla_z \psi_g(X_t(y), X_t(\bar{z})) a(X_t(\bar{z})) \tilde{v}(t, \bar{z}) d\bar{z} \\ &\quad + \int_K (\psi_g(X_t(y), X_t(\bar{z})) - \psi_g(\hat{x}, \hat{x})) R(X_t(\bar{z}), (I_g v)(t, X_t(\bar{z}))) \tilde{v}(t, \bar{z}) d\bar{z} \end{aligned}$$

which allows us to conclude that

$$\left| \int_{K_t} (\psi_g(X_t(y), z) - \psi_g(\hat{x}, \hat{x})) \partial_t v(t, z) dz \right| \leq \max_{\bar{z} \in K} \|a(X_t(\bar{z}))\| \|\psi_g\|_{W^{1,\infty}(\mathbb{R}^d)} \rho(t) \\ + \max_{y, \bar{z} \in K} |\psi_g(X_t(y), X_t(\bar{z})) - \psi_g(\hat{x}, \hat{x})| \bar{R} \rho(t).$$

Using this relation in (83) gives

$$\left| \frac{d}{dt} (I_\gamma v)(t, X_t(y)) - \psi_\gamma(\hat{x}, \hat{x}) \dot{\rho}(t) \right| \leq 2 \max_{\bar{z} \in K} \|a(X_t(\bar{z}))\| \|\psi_\gamma\|_{W^{1,\infty}(\mathbb{R}^d)} \rho(t) \\ + \max_{y, \bar{z} \in K} |\psi_\gamma(X_t(y), X_t(\bar{z})) - \psi_\gamma(\hat{x}, \hat{x})| \bar{R} \rho(t),$$

which proves, according to hypothesis (75), that for all $y \in K$

$$\frac{d}{dt} I_g(t, X(t, y)) = \psi_g(\hat{x}, \hat{x}) \dot{\rho}(t) + O(e^{-\delta t}), \quad (84)$$

for a certain $\delta > 0$.

Differentiating (81) we get

$$\begin{aligned} \ddot{\rho}(t) &= \frac{d}{dt} \left(\int_{K_t} R(x, (I_g v)(t, x)) v(t, x) dx \right) \\ &= \frac{d}{dt} \left(\int_K R(X_t(y), (I_g v)(t, X_t(y))) \tilde{v}(t, y) dy \right) \\ &= \int_K a(X_t(y)) \cdot \nabla_x R(X_t(y), (I_g v)(t, X_t(y))) \tilde{v}(t, y) dy \\ &\quad + \int_K \frac{d}{dt} (I_g v)(t, X_t(y)) \partial_I R(X_t(y), (I_g v)(t, X_t(y))) \tilde{v}(t, y) dy \\ &\quad + \int_K R(X_t(y), (I_g v)(t, X_t(y))) \partial_t \tilde{v}(t, y) dy. \end{aligned}$$

Let us note that

i) Since $a(X_t(y))$ converges uniformly to zero with an exponential speed, then

$$\int_{\mathbb{R}^d} a(X_t(y)) \cdot \nabla_x R(X_t(y), (I_g v)(t, X_t(y))) \tilde{v}(t, y) dy = O(e^{-\delta t}),$$

thanks to the boundedness of $\nabla_x R$ and $\rho(t)$.

ii) According to (84),

$$\int_K \frac{d}{dt} (I_g v)(t, X_t(y)) \partial_I R(X_t(y), (I_g v)(t, X_t(y))) \tilde{v}(t, y) dy = -p(t) \dot{\rho}(t) + O(e^{-\delta t}),$$

with

$$p(t) := -\psi_g(\hat{x}, \hat{x}) \int_K \partial_I R(X_t(y), (I_g v)(t, X_t(y))) \tilde{v}(t, y) dy \geq \psi_g(\hat{x}, \hat{x}) D \min_{t \geq 0} \rho(t) > 0.$$

iii) Directly from (82),

$$\begin{aligned} \int_K R(X_t(y), (I_g v)(t, X_t(y))) \partial_t \tilde{v}(t, y) dy &= \int_K \left(R(X_t(y), (I_g v)(t, X_t(y))) \right)^2 \tilde{v}(t, y) dy \\ &=: P(t) \geq 0. \end{aligned}$$

Thus, $\dot{\rho}(t) \geq -p(t)\rho(t) + O(e^{-\delta t})$, hence, ρ converges, thanks to Lemma 11.

Recalling (80), $v(t, \cdot)$ thus converges to $\hat{\rho}\delta_{\hat{x}}$, where $\hat{\rho}$ is the limit of ρ . We conclude, according to Proposition 8, that $\hat{\rho}$ satisfies the expected equality.

Having proved that v and v^h share the same limit is enough then to conclude, thanks to Lemma 9, that v_ε^h is an asymptotic preserving approximation. \square

If m is not 0, under very specific hypotheses over its support, we can extend the result of Theorem 10. The explanation behind this is simple: as long as the population is composed of traits that are not prone to mutations, it will evolve as in the case where mutations are not possible at all.

Theorem 11. *Let us assume that there exists $\hat{x} \in \mathbb{R}^d$ which is an asymptotically stable equilibrium for the ODE $\dot{x} = a(x)$ such that (75) holds. Moreover, let us assume that $\text{supp}(v^0) \cup K_x$ is a compact set such that there exist $C', \delta' > 0$ such that*

$$\forall y \in \text{supp}(v^0) \cup K_x, \quad t \geq 0, \quad \|X(t, y) - \hat{x}\| \leq C' e^{-\delta' t},$$

that $\bigcup_{s \geq 0} (X_s(\text{supp}(v^0) \cup K_x)) \cap K_y = \emptyset$ and that hypothesis (76) holds. Then, v converges to $\hat{\rho}\delta_{\hat{x}}$ in the weak sense in the space of Radon measures, where $\hat{\rho}$ is the unique solution of

$$R(\hat{x}, \psi_g(\hat{x}, \hat{x})\hat{\rho}) = 0.$$

Consequently, v_ε^h is an asymptotic preserving approximation of v .

Proof. By using the fact that, for any $x \in \mathbb{R}^d$,

$$v(t, x) = v^0(X_{-t}(x))e^{\mathcal{G}(0,t,x)} + \int_0^t \int_{\mathbb{R}^d} m(X_{s-t}(x), z, (I_d v)(s, X_{s-t}(x)))v(s, z)dz e^{\mathcal{G}(s,t,x)} ds,$$

where

$$\mathcal{G}(s, t, x) := \int_s^t R(X_{\tau-t}(x), (I_g v)(\tau, X_{\tau-t}(x))) - \text{div } a(X_{\tau-t}(x)) d\tau,$$

we observe that $\text{supp}(v(t, x)) \subset X_t(\text{supp}(v^0)) \cup \bigcup_{0 \leq s \leq t} X_s(K_x) \subset \bigcup_{s \geq 0} (X_s(\text{supp}(v^0) \cup K_x))$, therefore

$$\begin{aligned} v(t, x) &= v^0(X_{-t}(x))e^{\mathcal{G}(0,t,x)} + \int_0^t \int_{\mathbb{R}^d} m(X_{s-t}(x), z, (I_d v)(s, X_{s-t}(x)))v(s, z)dz e^{\mathcal{G}(s,t,x)} ds \\ &= v^0(X_{-t}(x))e^{\mathcal{G}(0,t,x)} + \int_0^t \int_{\text{supp}(v(t,x)) \cap K_y} m(X_{s-t}(x), z, (I_d v)(s, X_{s-t}(x)))v(s, z)dz e^{\mathcal{G}(s,t,x)} ds \\ &= v^0(X_{-t}(x))e^{\mathcal{G}(0,t,x)}. \end{aligned}$$

Therefore, we can replicate the proof for the case $m \equiv 0$. \square

The result of Theorem 10 does not generalize when $\text{supp}(v^0)$ is not strictly contained in the basin of attraction of x_s , as shown in the following result:

Proposition 12. *Let us consider the one-dimensional PDE*

$$\begin{cases} \partial_t v(t, x) + \nabla_x \cdot (a(x)v(t, x)) = (r(x) - \rho(t))v(t, x), \\ \rho(t) = \int_{\mathbb{R}} v(t, x) dx, \\ n(0, \cdot) = n^0(\cdot), \end{cases} \quad (85)$$

which is a particular case of (19), and let us assume that there exist $x_u < x_s$ such that $a(x_u) = a(x_s) = 0$, $a'(x_u) > 0$, $a'(x_s) < 0$, $\text{supp}(n^0) \subset [x_u, x_s]$, $n^0(x_u) = 0$, and that there exists $\alpha > 0$ such that $n^{0'}(x) = O((x - x_u)^\alpha)$ and that $r(x_u) - (1 + \alpha)f'(x_u) > r(x_s)$. Then, v_ε^h is not an asymptotic preserving approximation of v .

Proof. The long-time behaviour of the solution of (85) has been studied in detail in [24], and it has been proved, under the hypotheses of Proposition 12, that v converges to a function in L^1 . Let us now compute the limit of v^h . Since $n^0(x_u) = 0$, we can assume, without loss of generality, that for all $i \in \mathcal{J}_h$, $x_i^0 \in (0, 1]$. Thus, since $a > 0$ on (x_u, x_s) , for all $t \geq 0$, $x_i(t)$ converges to x_s . As seen in the proof of Proposition 9, v^h therefore converges to $r(x_s)\delta_{x_s}$, and v_ε^h is therefore not an asymptotic preserving approximation of v . \square

5 Simulations

In this section, we present some simulations obtained with the particle method developed throughout the article. In Figure 1, we deal with the non-local advection equation presented in [22], which writes

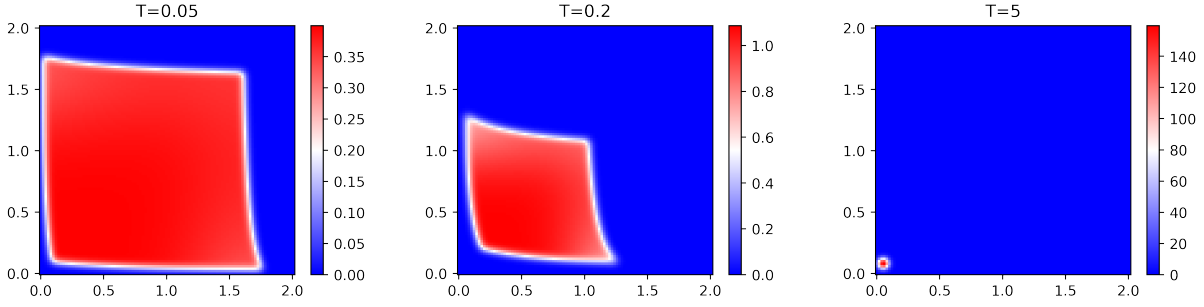
$$\partial_t v(t, x) + \nabla_x (a(t, I_1 v(t, x), I_2 v(t, x))) = 0, \quad x \in \mathbb{R}^d, t \geq 0, \quad (86)$$

with $I_j v(t, x) = \int_{\mathbb{R}^2} x_j v(t, x) dx$, for $j \in \{1, 2\}$. Note that this equation is not exactly a particular case of (1), since there are two non-local terms involved for advection, but the particle method can straightforwardly be adapted to this case. As in this paper, we show that, depending on the parameters, the solution of this PDE can converge to a single Dirac mass, to a sum of two Dirac masses, or to the sum of four Dirac masses. The parameters used for the simulations are the same as the one detailed in Figures 9, 10 and 11 of [22].

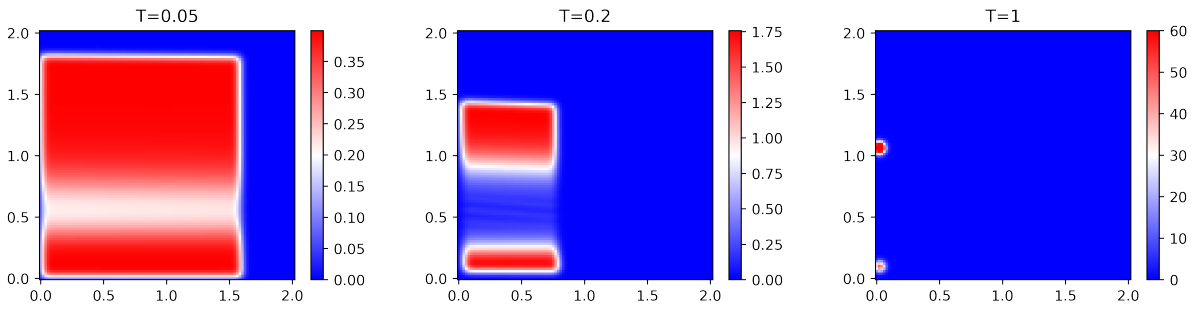
Figure 2 illustrates different scenarios for the equation

$$\begin{cases} \partial_t v(t, x) + \nabla_x (a(x)v(t, x)) = (r(x) - \rho(t)) n(t, x), \\ \rho(t) = \int_{\mathbb{R}} v(t, x) dx, \\ n(0, x) = n^0(x), \end{cases} \quad (87)$$

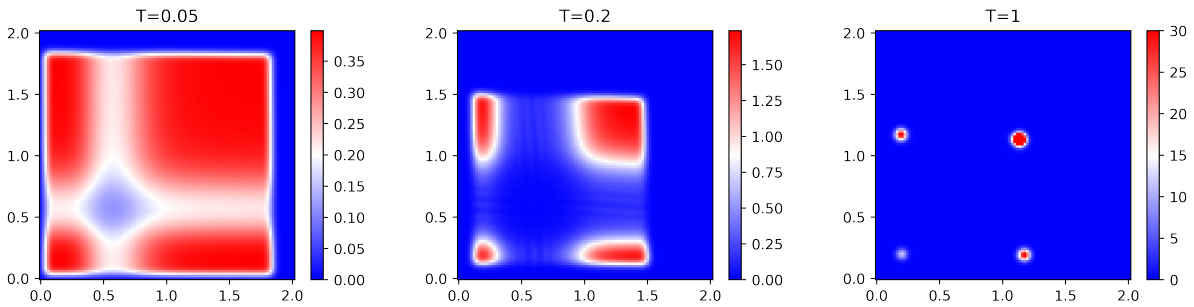
with $a(x) = x(1-x)$, and an initial solution supported in $[0, 1]$. This equation has been studied in [24] where it has been proved that its solution can either converge to a function in L^1 , (which depends on the initial condition), or to a Dirac mass on 1, depending on the functions r and n^0 .



(a) Monostability

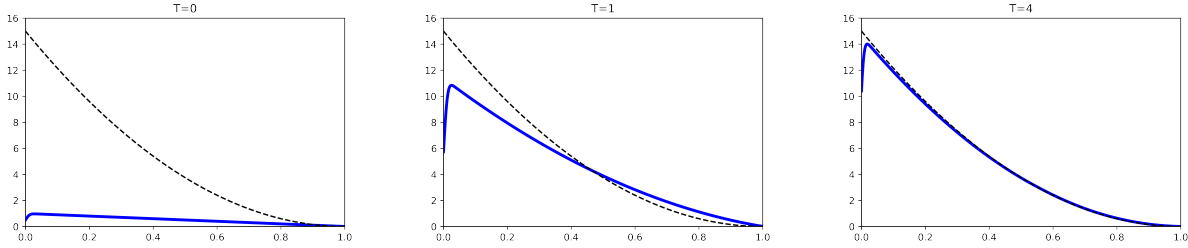


(b) Bistability

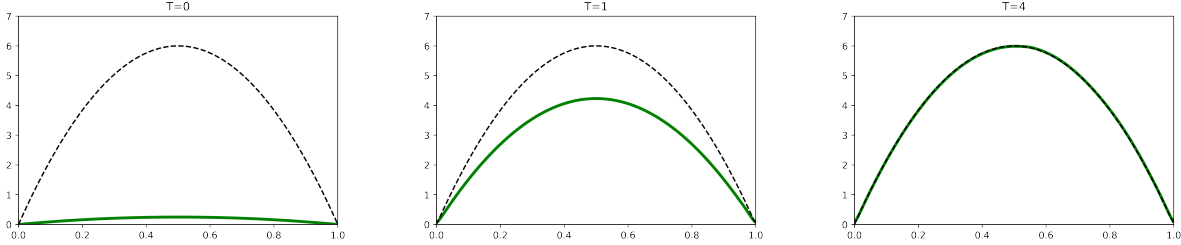


(c) Quadstability

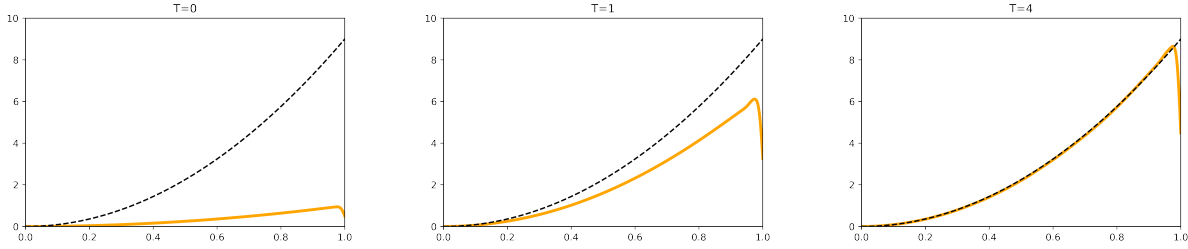
Figure 1: The three possible regimes of convergence for equation (86), obtained with the particle method. The lines (a), (b) and (c) respectively show the convergence to a single Dirac mass, two Dirac masses and four Dirac masses, and have been obtained by choosing the parameters of Figures 9, 10 and 11 of [22]. In the three cases, we have chosen $N = 100$, $h = 2./100$, $\varepsilon = h^{0.8}$, and the cut off function φ is a Gaussian.



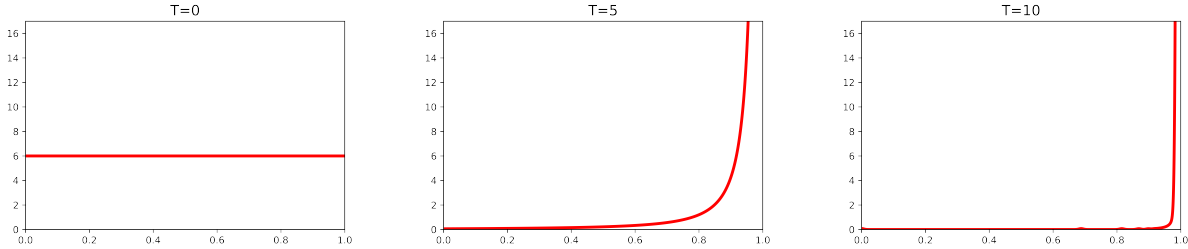
(a) Solution of (87) at different time steps, with $n^0(x) = 1 - x, r(x) = 6 - 4x$.



(b) Solution of (87) at different time steps, with $n^0(x) = x(1 - x), r(x) = 6 - 4x$.



(c) Solution of (87) at different time steps, with $n^0(x) = x^2, r(x) = 6 - 4x$.



(d) Solution of (87) at different time steps, with $n^0(x) = 6, r(x) = 6 - 0.5x$.

Figure 2: Different possible regimes of convergence for the solution of (87). The first three lines (green, blue and orange curves), show the convergence to a function in L^1 , which can be explicitly computed (see [24]), and is represented by a black dashed line. Note that the limit function is different when the initial condition changes. The last line (red curves) shows the convergence to a Dirac mass in 1. In the four cases, we have chosen $a(x) = x(1 - x), N = 5000, h = \frac{1}{N}, \varepsilon = \sqrt{h}$ and the cut-off function φ is a Gaussian.

A Proof of the results over the characteristics

In order to prove results which involve the use of absolute values, we introduce a smooth re-normalizing sequence of functions. Consider a sequence of smooth positive functions β_ε satisfying $\beta_\varepsilon(0) = 0, \beta_\varepsilon(s) > 0$

for all $s \neq 0$, $\beta_\varepsilon(s) \leq |s|$, $\beta_\varepsilon(s) \rightarrow |s|$ almost everywhere, $|\dot{\beta}_\varepsilon(s)| \leq 1$ and $s\dot{\beta}_\varepsilon(s) \rightarrow |s|$ almost everywhere. For example we may choose

$$\beta_\varepsilon(s) = \begin{cases} -s - \varepsilon(1 - \frac{2}{\pi}) & \text{if } s \leq -\varepsilon, \\ \frac{2\varepsilon}{\pi} (1 - \cos(\frac{\pi}{2\varepsilon}s)) & \text{if } -\varepsilon < s < \varepsilon, \\ s - \varepsilon(1 - \frac{2}{\pi}) & \text{if } s \geq \varepsilon. \end{cases} \quad (88)$$

Proof of Lemma 1. We introduce the notation

$$\Delta X_j(t) := X_{u_1}^j(t, y_1) - X_{u_2}^j(t, y_2).$$

For all $t \in [0, T]$, the function

$$U_\varepsilon(t) := \sum_{j=1}^d \beta_\varepsilon(\Delta X_j(t))$$

satisfies then the relation

$$\begin{aligned} \dot{U}_\varepsilon(t) &= \sum_{j=1}^d \dot{\beta}_\varepsilon(\Delta X_j(t)) (a_j(t, X_{u_1}(t, y_1), (I_a u_1)(t, X_{u_1}(t, y_1))) - a_j(t, X_{u_2}(t, y_2), (I_a u_2)(t, X_{u_2}(t, y_2)))) \\ &\leq \sum_{j=1}^d \left(\|a_j\|_{W_x^{1,\infty}} \sum_{i=1}^d |\Delta X_i(t)| + \|a_j\|_{W_I^{1,\infty}} |(I_a u_1)(t, X_{u_1}(t, y_1)) - (I_a u_2)(t, X_{u_2}(t, y_2))| \right) \\ &\leq d \|a\|_{W_{x,I}^{1,\infty}} \left(\sum_{i=1}^d |\Delta X_i(t)| + |(I_a u_1)(t, X_{u_1}(t, y_1)) - (I_a u_2)(t, X_{u_2}(t, y_2))| \right) \\ &\leq d \|a\|_{W_{x,I}^{1,\infty}} \left((1 + \|\psi_a\|_{W_x^{1,\infty} L_y^\infty} \|u_1\|) \sum_{i=1}^d |\Delta X_i(t)| + \|\psi_a\|_{L^\infty} \|u_1 - u_2\|_{L^1(\mathbb{R}^d)} \right). \end{aligned}$$

Integrating between 0 and t we get

$$\begin{aligned} U_\varepsilon(t, y) - U_\varepsilon(0, y) &\leq d \|a\|_{L_t^\infty W_{x,I}^{1,\infty}} \left((1 + \|\psi_a\|_{L_{t,y}^\infty W_x^{1,\infty}} \|u_1\|) \int_0^t \sum_{i=1}^d |\Delta X_i(s)| ds + \|\psi_a\|_{L^\infty} \int_0^t \|u_1 - u_2\|_{L^1(\mathbb{R}^d)} ds \right). \end{aligned}$$

Taking the limit when ε goes to 0 and applying Grönwall's lemma we get the desired result. \square

Proof of Lemma 2. We explicitly give the proof for $k = 1$. The proof for higher values of k follows the same ideas.

Thanks to the hypothesis over a , the function X_u is one time differentiable with respect to y , and directly from (20) we get the system of equations

$$\begin{cases} \partial_{y_i} \dot{X}_u(t, y) = J_a(t, X_u(t, y)) \partial_{y_i} X_u(t, y), & t \in [0, T], \\ \partial_{y_i} X_u(0, y) = e_i, \end{cases} \quad (89)$$

for all values of $i \in \{1, \dots, d\}$, where

$$[J_a(t, x)]_{ij} := \partial_{x_i} a_j(t, x, (I_a u)(t, x)) + \partial_I a_j(t, x, (I_a u)(t, x)) \int_{\mathbb{R}^d} \partial_{x_i} \psi_a(t, x, y) u(t, y) dy, \quad (90)$$

is the Jacobian matrix of the function $a(t, x, (I_a u)(t, x))$ and the e_i represent the canonical basis of \mathbb{R}^d .

The function

$$V_\varepsilon(t, y) := \sum_{i,j=1}^n \beta_\varepsilon(\partial_{y_i} X_u^j(t, y))$$

satisfies then

$$\dot{V}_\varepsilon(t, y) = \sum_{i,j=1}^n \dot{\beta}_\varepsilon(\partial_{y_i} X_u^j(t, y)) \sum_{k=1}^d [J_a(t, X_u(t, y))]_{kj} \partial_{y_i} X_u^k(t, y)$$

and consequently

$$\dot{V}_\varepsilon(t, y) \leq d \|a\|_{W_{x,I}^{1,\infty}} (1 + \|\psi_a\|_{W_x^{1,\infty} L_y^\infty} \|u\|) \sum_{i,j=1}^n |\partial_{y_i} X_u^j(t, y)|.$$

Integrating between 0 and t , we obtain the relation

$$V_\varepsilon(t, y) - V_\varepsilon(0, y) \leq d \tilde{\alpha}_1 \int_0^t \sum_{i,j=1}^n |\partial_{y_i} X_u^j(s, y)| ds$$

with $\tilde{\alpha}_1 := \sup_{t \in [0, T]} \|a\|_{W_{x,I}^{1,\infty}} (1 + \|\psi_a\|_{W_x^{1,\infty} L_y^\infty} \|u\|)$, which after taking the limit when ε goes to 0 leads to

$$\sum_{i,j=1}^n |\partial_{y_i} X_u^j(t, y)| \leq d + d \tilde{\alpha}_1 \int_0^t \sum_{i,j=1}^n |\partial_{y_i} X_u^j(s, y)| ds.$$

We obtain (21) thanks to Grönwall's lemma.

In order to prove (22) we adopt the notation

$$\Delta \partial_{y_k} X^j(t) := \partial_{y_k} X_{u_1}^j(t, y_1) - \partial_{y_k} X_{u_2}^j(t, y_2)$$

and define

$$[J_{X_u}(t, y)]_{jk} := \partial_{y_k} X_u^j(t, y),$$

which satisfies the relation

$$\dot{J}_{X_u}(t, y) = J_a(t, X_u(t, y)) J_{X_u}(t, y), \quad J_{X_u}(0, y) = I_d,$$

with $J_a(t, x)$ as defined on (90). Consequently, we have that

$$D_\varepsilon(t) := \sum_{j=1}^d \sum_{k=1}^d \beta_\varepsilon(\Delta \partial_{y_k} X^j(t))$$

satisfies for all $t \in [0, T]$

$$\begin{aligned} \dot{D}_\varepsilon(t) &= \sum_{j=1}^d \sum_{k=1}^d \dot{\beta}_\varepsilon(\Delta \partial_{y_k} X^j(t)) \sum_{i=1}^d ([J_a(t, X_{u_1}(t, y_1))]_{ij} \partial_{y_k} X_{u_1}^i(t, y_1) - [J_a(t, X_{u_2}(t, y_2))]_{ij} \partial_{y_k} X_{u_2}^i(t, y_2)) \\ &\leq \sum_{j=1}^d \sum_{k=1}^d \sum_{i=1}^d |[J_a(t, X_{u_1}(t, y_1))]_{ij} \partial_{y_k} X_{u_1}^i(t, y_1) - [J_a(t, X_{u_2}(t, y_2))]_{ij} \partial_{y_k} X_{u_2}^i(t, y_2)| \\ &\leq \sum_{j=1}^d \sum_{k=1}^d \sum_{i=1}^d |[J_a(t, X_{u_1}(t, y_1))]_{ij}| |\Delta \partial_{y_k} X^i(t)| \\ &\quad + \sum_{j=1}^d \sum_{k=1}^d \sum_{i=1}^d |[J_a(t, X_{u_1}(t, y_1))]_{ij} - [J_a(t, X_{u_2}(t, y_2))]_{ij}| |\partial_{y_k} X_{u_2}^i(t, y_2)|. \end{aligned}$$

From the hypothesis over a and ψ_a we see that

$$|[J_a(t, X_{u_1}(t, y_1))]_{ij}| \leq \|a\|_{W_{x,I}^{1,\infty}} (1 + \|\psi_a\|_{W_x^{1,\infty} L_y^\infty} \|u_1\|).$$

Furthermore, from the definition of $J_a(t, x)$ we conclude that there exists a constant C , depending only on $\|a\|_{W_{x,T}^{2,\infty}}$, $\|\psi_a\|_{W_x^{2,\infty}L_y^\infty}$ and $\|u_i\|$ such that

$$\begin{aligned} |[J_a(t, X_{u_1}(t, y_1))]_{ij} - [J_a(t, X_{u_2}(t, y_2))]_{ij}| &\leq C \left(\sum_{j=1}^d |X_{u_1}^j(t, y_1) - X_{u_2}^j(t, y_2)| + \|u_1 - u_2\|_{L^1(\mathbb{R}^d)} \right) \\ &\leq C(|y_1 - y_2| + \|u_1 - u_2\|_1 + \|u_1 - u_2\|_{L^1(\mathbb{R}^d)}), \end{aligned}$$

where we have used the results from Lemma 1 on the second line.

Putting all estimates together, we conclude that there exist constants C_1 and C_2 only depending on $\|a\|_{W_{x,T}^{2,\infty}}$, $\|\psi_a\|_{W_x^{2,\infty}L_y^\infty}$ and $\|u_i\|$, such that

$$\dot{D}_\varepsilon(t) \leq C_1 \sum_{j=1}^d \sum_{k=1}^d |\partial_{y_k} X_{u_1}^j(t, y_1) - \partial_{y_k} X_{u_2}^j(t, y_2)| + C_2(|y_1 - y_2| + \|u_1 - u_2\|_1 + \|u_1 - u_2\|_{L^1(\mathbb{R}^d)}).$$

Integrating in time, using Grönwall's lemma and taking the limit when ε goes to zero, we obtain (22). \square

Proof of Lemma 3. We explicitly give the proof for $k = 1$. The proof for higher values of k follows the same ideas.

Differentiating once each component of the equality $X_u(t, X_u^{-1}(t, x)) = x$ with respect to each of the variables x_k , we obtain the family of relations

$$\sum_{i=1}^d \partial_{y_i} X_u^j(t, X_u^{-1}(t, x)) \partial_{x_k} (X_u^{-1})^i(t, x) = \delta_{jk}, \quad j, k = 1, \dots, d,$$

where δ_{jk} represents the Kronecker's delta. Written in matrix form, this equality reads

$$J_{X_u}(t, X_u^{-1}(t, x)) J_{X_u^{-1}}(t, x) = I_d.$$

It is known that the matrix $J_{X_u}(t, y)$ is invertible for all values of x , furthermore, its determinant is given by the expression

$$\det(J_{X_u}(t, y)) = e^{\int_0^t \nabla_x \cdot a(s, y, (I_a u)(s, y)) + \partial_I a(s, y, (I_a u)(s, y)) \cdot \int_{\mathbb{R}^d} \nabla_x \psi_a(s, y, z) u(s, z) dz ds} \geq c_T > 0$$

for all values of $t \in [0, T]$ and $y \in \mathbb{R}^d$.

We conclude by writing

$$J_{X_u^{-1}}(t, x) = J_{X_u}^{-1}(t, X_u^{-1}(t, x)),$$

and noticing that all of the components of $J_{X_u}^{-1}(t, X_u^{-1}(t, x))$ are a combination of sums and multiplications of the components of $J_{X_u}(t, X_u^{-1}(t, x))$, divided by $\det(J_{X_u}(t, y))$. The bound (21) from Lemma 2, together with the lower bound for the determinant of $J_{X_u}(t, y)$ gives the bound (23) over the components of $J_{X_u^{-1}}(t, x)$. \square

Proof of Lemma 4. We explicitly give the proof for $k = 1$. The proof for higher values of k follows the same ideas.

Differentiating with respect to t the relation $X_{u_i}(t, X_{u_i}^{-1}(t, x)) = x$, for $i = 1, 2$, we see that

$$a(t, x, (I_a u_i)(t, x)) + J_{X_{u_i}}(t, X_{u_i}^{-1}(t, x)) \dot{X}_{u_i}^{-1}(t, x) = 0,$$

which gives

$$\dot{X}_{u_i}^{-1}(t, x) = -J_{X_{u_i}}^{-1}(t, X_{u_i}^{-1}(t, x)) a(t, x, (I_a u_i)(t, x)). \quad (91)$$

From now on we adopt the notations

$$\begin{aligned} A^i(t, x) &:= a(t, x, (I_a u_i)(t, x)), \\ K^i(t, x) &:= -J_{X_{u_i}^{-1}}(t, X_{u_i}^{-1}(t, x)), \\ \Delta X_j^{-1}(t, x) &:= (X_{u_1}^{-1})^j(t, x) - (X_{u_2}^{-1})^j(t, x). \end{aligned}$$

The function

$$W_\varepsilon(t, x) := \sum_{j=1}^d \beta_\varepsilon(\Delta X_j^{-1}(t, x))$$

satisfies the relation

$$\begin{aligned} \dot{W}_\varepsilon(t, x) &= \sum_{j=1}^d \sum_{k=1}^d \dot{\beta}_\varepsilon(\Delta X_j^{-1}(t, x)) (K_{jk}^1(t, x) A_k^1(t, x) - K_{jk}^2(t, x) A_k^2(t, x)) \\ &\leq \sum_{j=1}^d \sum_{k=1}^d |K_{jk}^1(t, x) A_k^1(t, x) - K_{jk}^2(t, x) A_k^2(t, x)|. \end{aligned}$$

From Lemma 3 we know that all components of K^i are uniformly bounded by a constant only depending on T and $\|u_i\|$. We deduce from the hypothesis over a that the components of A^i are uniformly bounded by $\tilde{a} := \|a\|_{L^\infty}$. Therefore

$$\dot{W}_\varepsilon(t, x) \leq d\tilde{C}(T, \|u_1\|) \sum_{k=1}^d |A_k^1(t, x) - A_k^2(t, x)| + \tilde{a} \sum_{j=1}^d \sum_{k=1}^d |K_{jk}^1(t, x) - K_{jk}^2(t, x)|.$$

The function a being L -Lipschitz with respect to the I variable, we have that, for all values of k

$$|A_k^1(t, x) - A_k^2(t, x)| \leq L \|\psi_a\|_{L^\infty} \|u_1 - u_2\|_{L^1(\mathbb{R}^d)}.$$

On the other hand, from the definition of K^i and Lemma 2 we conclude that there exists C , depending on T , $\|a\|_{W_{x,I}^{2,\infty}}$, $\|\psi_a\|_{W_x^{2,\infty} L_y^\infty}$ and $\|u_i\|$ such that

$$|K_{jk}^1(t, x) - K_{jk}^2(t, x)| \leq \begin{cases} 0, & \text{if } \partial_I a = 0, \\ C \left(\sum_{j=1}^d |(X_{u_1}^{-1})^j(t, x) - (X_{u_2}^{-1})^j(t, x)| + \|u_1 - u_2\|_1 \right), & \text{if } \partial_I a \neq 0. \end{cases}$$

Putting everything together, integrating between 0 and t , taking the limit when ε goes to 0 and applying Grönwall's lemma we get (24). \square

B Existence of solution for a system of ODEs with infinitely many unknowns and equations

Proof of Lemma 7. For all $u \in X_h^T$, there exists a sequence of elements $u^\delta \in X_h^T$ such that:

- 1) $\mathcal{K}_h^\delta := \{k \in \mathcal{J}_h : u_k^\delta \neq 0\}$ has a finite number of elements.
- 2)

$$\lim_{\delta \rightarrow 0} \|u - u^\delta\|_{1,h} = 0.$$

We denote $K^\delta := |\mathcal{K}_h^\delta|$ and notice that the system

$$\dot{x}_k^\delta(t) = A_{u^\delta, w}(t, x_k^\delta), \quad x_k^\delta(0) = x_k^0, \quad (92)$$

is composed of a coupled system of K^δ equations and unknowns (corresponding to those $k \in \mathcal{K}_h^\delta$), and an uncoupled infinite number of equations, corresponding to those $k \notin \mathcal{K}_h^\delta$. Therefore, thanks to the classic Cauchy-Lipschitz theory, the system (92) has a unique solution $x_k^\delta \in \mathcal{C}^1([0, T])$, $k \in \mathcal{F}_h$.

We claim that for all values of k , the sequence $x_k^{\delta_1} - x_k^{\delta_2}$ is a Cauchy sequence in $\mathcal{C}^1([0, T])$, therefore it has a limit that we will call $x_k(t)$, which is solution to (49).

We first remark that $x^{\delta_1} - x^{\delta_2} \in Y_h^T$ due to the fact that $|x_k^\delta(t) - x_k^0| \leq \|a\|_{L^\infty} T$ for all values of k and δ . Consider now β_ε as defined in (88), then

$$\begin{aligned} \dot{\beta}_\varepsilon(x_k^{\delta_1} - x_k^{\delta_2}) &\leq |A_{u^{\delta_1}, w}(t, x_k^{\delta_1}) - A_{u^{\delta_2}, w}(t, x_k^{\delta_2})| \\ &\quad \|a\|_{W_x^{1, \infty}} |x_k^{\delta_1} - x_k^{\delta_2}| + \|a\|_{W_t^{1, \infty}} |I_a(t, x_k^{\delta_1}, u^{\delta_1}, w) - I_a(t, x_k^{\delta_2}, u^{\delta_2}, w)|. \end{aligned}$$

Noticing that

$$\begin{aligned} |I_a(t, x_k^{\delta_1}, u^{\delta_1}, w) - I_a(t, x_k^{\delta_2}, u^{\delta_2}, w)| &= \left| \sum_{j \in \mathcal{F}_h} \left(u_j^{\delta_1}(t) \psi_a(t, x_k^{\delta_1}, x_j^{\delta_1}(t)) - u_j^{\delta_2}(t) \psi_a(t, x_k^{\delta_2}, x_j^{\delta_2}(t)) \right) w_j(t) \right| \\ &\leq (\|\psi_a\|_{L^\infty} \|u^{\delta_1} - u^{\delta_2}\|_{1, h} + \|u^{\delta_2}\|_{1, h} \|\psi_a\|_{W_{x, y}^{1, \infty}} \|x^{\delta_1} - x^{\delta_2}\|_{\infty, h}) \|w\|_{\infty, h}, \end{aligned}$$

we deduce the existence of two constants³, C_1 and C_2 , only depending on a , ψ_a , u and w , such that

$$\dot{\beta}_\varepsilon(x_k^{\delta_1} - x_k^{\delta_2}) \leq C_1 \|x^{\delta_1} - x^{\delta_2}\|_{\infty, h} + C_2 \|u^{\delta_1} - u^{\delta_2}\|_{1, h}.$$

Integrating between 0 and t , taking the maximum over k and t and using Grönwall's lemma, we conclude that there exists a constant C_T , only depending on T and the coefficients of the problem, such that

$$\|x^{\delta_1} - x^{\delta_2}\|_{\infty, h} \leq C_T \|u^{\delta_1} - u^{\delta_2}\|_{1, h}.$$

Proceeding in a similar way with the absolute value of $\dot{x}^{\delta_1} - \dot{x}^{\delta_2}$ we obtain that

$$\|\dot{x}^{\delta_1} - \dot{x}^{\delta_2}\|_{\infty, h} \leq C_T \|u^{\delta_1} - u^{\delta_2}\|_{1, h}.$$

Recalling that u^δ is a Cauchy sequence on X_h^T , then so it is x_k^δ on $\mathcal{C}^1([0, T])$, for each k .

Let $x := \{x_k\}_{k \in \mathcal{F}_h}$ be the limit of x^δ when δ goes to 0. With a simple continuity argument we conclude that x is a solution of (49) over $[0, T]$. The uniqueness can be obtained by assuming the existence of two solutions, deriving the equation satisfied by the difference and using Grönwall's lemma to conclude that they have to be equal. \square

C A result from approximation theory

As mentioned before, Lemma 8 is a direct corollary of Lemma 8 in [36], that we recall here

Lemma 12. *Let $k > d$ an integer. Assume that*

$$a \in (L^\infty(0, T; W^{k+1, \infty}(\mathbb{R}^d)))^d.$$

Then, there exists a constant $C > 0$ such that for all functions $\varphi \in W^{k, p}(\mathbb{R}^d)$, $1 \leq p \leq +\infty$, and $t \in [0, T]$,

$$\left\| \varphi - \sum_{i \in \mathcal{F}_h} w_i(t) \varphi(x_i(t)) \delta(\cdot - x_i(t)) \right\|_{W^{-k, p}(\mathbb{R}^d)} \leq C h^k \|\varphi\|_{W^{k, p}(\mathbb{R}^d)}.$$

³Notice that in order to obtain the estimate over I_a , we used the hypothesis of differentiability over both variables on ψ_a

Given that for fixed functions $\nu \in X_h^T$ and $w \in Y_h^T$ we have the inclusion

$$A_{\nu,w} : (t, x) \mapsto a(t, x, I_a(t, x, \nu, w)) \in (L^\infty(0, T; W^{k+1, \infty}(\mathbb{R}^d)))^d,$$

then Lemma 12 holds true as well for the values of x_i obtained in Section 3.

Proof of Lemma 8. We recall that $W^{-k,1}(\mathbb{R}^d)$ is the dual space of $W^{k,\infty}(\mathbb{R}^d)$. Thus, for any $\psi \in W^{-k,1}(\mathbb{R}^d)$ we have

$$\|\psi\|_{-k,1} = \sup_{f \in W^{k,\infty}(\mathbb{R}^d)} \frac{|\langle \psi, f \rangle|}{\|f\|_{k,\infty}}.$$

Since the function $f \equiv 1$ belongs to $W^{k,\infty}(\mathbb{R}^d)$, and has norm equal to 1 in this space, we get for all $\varphi \in W^{k,p}(\mathbb{R}^d)$

$$\begin{aligned} \left| \int_{\mathbb{R}^d} \varphi(x) dx - \sum_{i \in \mathcal{J}_h} w_i(t) \varphi(x_i(t)) \right| &= \left| \langle \varphi - \sum_{i \in \mathcal{J}_h} w_i(t) \varphi(x_i(t)) \delta(\cdot - x_i(t)), 1 \rangle \right| \\ &\leq \left\| \varphi - \sum_{i \in \mathcal{J}_h} w_i(t) \varphi(x_i(t)) \delta(\cdot - x_i(t)) \right\|_{-k,1}. \end{aligned}$$

We conclude by applying Lemma 12 with $p = 1$. □

D Proofs of convergence results from ODE theory

This appendix is dedicated to the proofs of lemma 11, used in subsection 4.2. In order to prove this lemma, we use the following result:

Lemma 13. *Let $\alpha > 0$ and $B \in L^1(\mathbb{R}_+)$. Then, all the solutions of the ODE*

$$\dot{u}(t) = -\alpha u(t) + B(t)$$

are in $L^1(\mathbb{R}^+)$.

Proof of lemma 13. The solution of this ODE is explicitly given by

$$u(t) = u(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} B(s) ds.$$

Hence,

$$\int_0^{+\infty} |u(t)| dt \leq |u(0)| \int_0^{+\infty} e^{-\alpha t} dt + \iint_{\mathbb{R}_+^2} e^{-\alpha(t-s)} |B(s)| \mathbb{1}_{\{s \leq t\}} ds dt.$$

With the change of variables $y = s, z = t - s$, we get

$$\iint_{\mathbb{R}_+^2} e^{-\alpha(t-s)} |B(s)| \mathbb{1}_{\{s \leq t\}} ds dt \leq \int_0^{+\infty} |B(y)| dy \int_0^{+\infty} e^{-\alpha z} dz,$$

which concludes the proof. □

Proof of Lemma 11. First, let us note that if \dot{u} is a BV function, *i.e.* if $\int_0^{+\infty} |\dot{u}(t)| dt < +\infty$, then u is a Cauchy function, and thus converges. Let us denote $v := \dot{u}$. Since u is assumed to be bounded, and $|v| = v + 2v^-$, where v^- denotes the negative part of v , it is enough to prove that $v^- \in L^1(\mathbb{R}^d)$. By hypothesis,

$$\dot{v}(t) = -p(t)v(t) + P(t) + B(t),$$

which implies that

$$\dot{v}^-(t) \leq -p_0 v^-(t) + B(t).$$

We conclude, according to lemma 13, that $v^- \in L^1(\mathbb{R}^d)$, which implies that u converges. □

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References

- [1] FE Alvarez, JA Carrillo, and J Clairambault. “Evolution of a structured cell population endowed with plasticity of traits under constraints on and between the traits”. In: *J. Math. Biol.* 85.64 (2022).
- [2] Loic Barbarroux et al. “Multi-scale modeling of the CD8 immune response”. In: *AIP Conference Proceedings*. Vol. 1738. 1. AIP Publishing LLC. 2016, p. 320002.
- [3] Guy Barles, Sepideh Mirrahimi, and Benoît Perthame. “Concentration in Lotka-Volterra parabolic or integral equations: a general convergence result”. In: *Methods and Applications of Analysis* 16.3 (2009), pp. 321–340.
- [4] Olivier Bonnefon, Jérôme Coville, and Guillaume Legendre. “Concentration phenomenon in some non-local equation”. In: *arXiv preprint arXiv:1510.01971* (2015).
- [5] Reinhard Bürger. *The mathematical theory of selection, recombination, and mutation*. John Wiley & Sons, 2000.
- [6] Àngel Calsina et al. “Asymptotics of steady states of a selection–mutation equation for small mutation rate”. In: *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* 143.6 (2013), pp. 1123–1146.
- [7] José Antonio Carrillo, Katy Craig, and Francesco S Patacchini. “A blob method for diffusion”. In: *Calculus of Variations and Partial Differential Equations* 58 (2019), pp. 1–53.
- [8] Alina Chertock. “A practical guide to deterministic particle methods”. In: *Handbook of numerical analysis*. Vol. 18. Elsevier, 2017, pp. 177–202.
- [9] Rebecca H. Chisholm et al. “Emergence of Drug Tolerance in Cancer Cell Populations: An Evolutionary Outcome of Selection, Nongenetic Instability, and Stress-Induced Adaptation”. In: *Cancer Research* 75.6 (2015), pp. 930–939. ISSN: 0008-5472. DOI: 10.1158/0008-5472.CAN-14-2103. eprint: <https://cancerres.aacrjournals.org/content/75/6/930.full.pdf>. URL: <https://cancerres.aacrjournals.org/content/75/6/930>.
- [10] JP Choquin and S Huberson. “Particles simulation of viscous flow”. In: *Computers & fluids* 17.2 (1989), pp. 397–410.
- [11] Albert Cohen and Benoit Perthame. “Optimal approximations of transport equations by particle and pseudoparticle methods”. In: *SIAM Journal on Mathematical Analysis* 32.3 (2000), pp. 616–636.
- [12] Daniel B Cooney and Yoichiro Mori. “Long-time behavior of a PDE replicator equation for multilevel selection in group-structured populations”. In: *Journal of Mathematical Biology* 85.2 (2022), pp. 1–67.
- [13] Georges-Henri Cottet and Pierre-Arnaud Raviart. “On particle-in-cell methods for the Vlasov-Poisson equations”. In: *Transport Theory and Statistical Physics* 15.1-2 (1986), pp. 1–31.
- [14] GH Cottet and S Mas-Gallic. “A particle method to solve the Navier-Stokes system”. In: *Numerische Mathematik* 57.1 (1990), pp. 805–827.
- [15] Jérôme Coville. “Convergence to equilibrium for positive solutions of some mutation-selection model”. In: *arXiv preprint arXiv:1308.6471* (2013).
- [16] Pierre Degond and S Mas-Gallic. “The weighted particle method for convection-diffusion equations. I. The case of an isotropic viscosity”. In: *Mathematics of computation* 53.188 (1989), pp. 485–507.
- [17] Pierre Degond and Francisco-José Mustieles. “A deterministic approximation of diffusion equations using particles”. In: *SIAM Journal on Scientific and Statistical Computing* 11.2 (1990), pp. 293–310.

- [18] Laurent Desvillettes et al. “On selection dynamics for continuous structured populations”. In: *Communications in Mathematical Sciences* 6.3 (2008), pp. 729–747.
- [19] Ulf Dieckmann and Richard Law. “The dynamical theory of coevolution: a derivation from stochastic ecological processes”. In: *Journal of mathematical biology* 34.5 (1996), pp. 579–612.
- [20] Odo Diekmann et al. “The dynamics of adaptation: an illuminating example and a Hamilton–Jacobi approach”. In: *Theoretical population biology* 67.4 (2005), pp. 257–271.
- [21] Ronald J DiPerna and Pierre-Louis Lions. “Ordinary differential equations, transport theory and Sobolev spaces”. In: *Inventiones mathematicae* 98.3 (1989), pp. 511–547.
- [22] Avner Friedman, Chiu-Yen Kao, and Chih-Wen Shih. “Asymptotic phases in a cell differentiation model”. In: *Journal of Differential Equations* 247.3 (2009), pp. 736–769.
- [23] Stefan AH Geritz, Eva Kisdi, Johan AJ Metz, et al. “Evolutionarily singular strategies and the adaptive growth and branching of the evolutionary tree”. In: *Evolutionary ecology* 12.1 (1998), pp. 35–57.
- [24] Jules Guilberteau, Camille Pouchol, and Nastassia Pouradier Duteil. “Long-time behaviour of an advection-selection equation”. In: *arXiv preprint arXiv:2301.02470* (2023).
- [25] Mats Gyllenberg and Géza Meszéna. “On the impossibility of coexistence of infinitely many strategies”. In: *Journal of mathematical biology* 50.2 (2005), pp. 133–160.
- [26] Philip Hartman. *Ordinary differential equations*. SIAM, 2002.
- [27] F Hermeline. “A deterministic particle method for transport diffusion equations: Application to the Fokker-Planck equation”. In: *Journal of Computational Physics* 82.1 (1989), pp. 122–146.
- [28] Roger W Hockney and James W Eastwood. *Computer simulation using particles*. crc Press, 2021.
- [29] Thomas Y Hou. “Convergence of a variable blob vortex method for the Euler and Navier–Stokes equations”. In: *SIAM Journal on Numerical Analysis* 27.6 (1990), pp. 1387–1404.
- [30] Pierre-Emmanuel Jabin and Gaël Raoul. “On selection dynamics for competitive interactions”. In: *Journal of mathematical biology* 63.3 (2011), pp. 493–517.
- [31] Anthony Leonard. “Computing three-dimensional incompressible flows with vortex elements”. In: *Annual Review of Fluid Mechanics* 17.1 (1985), pp. 523–559.
- [32] Anthony Leonard. “Vortex methods for flow simulation”. In: *Journal of Computational Physics* 37.3 (1980), pp. 289–335.
- [33] Tommaso Lorenzi and Camille Pouchol. “Asymptotic analysis of selection-mutation models in the presence of multiple fitness peaks”. In: *Nonlinearity* 33.11 (2020), p. 5791.
- [34] Alexander Lorz et al. “Populational adaptive evolution, chemotherapeutic resistance and multiple anti-cancer therapies”. In: *ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique* 47.2 (2013), pp. 377–399.
- [35] B Lucquin-Desreux and S Mas-Gallic. “A conservative particle approximation for a boundary advection-diffusion problem”. In: *ESAIM: Mathematical Modelling and Numerical Analysis* 26.6 (1992), pp. 757–791.
- [36] S Mas-Gallic and PA Raviart. “A particle method for first-order symmetric systems”. In: *Numerische Mathematik* 51.3 (1987), pp. 323–352.
- [37] Johan AJ Metz et al. “Adaptive dynamics: a geometrical study of the consequences of nearly faithful reproduction”. In: (1995).
- [38] Benoît Perthame. *Transport equations in biology*. Springer Science & Business Media, 2006.
- [39] Benoît Perthame and Guy Barles. “Dirac concentrations in lotka-volterra parabolic pdes”. In: *Indiana University Mathematics Journal* (2008), pp. 3275–3301.
- [40] Pierre-Arnaud Raviart. “An analysis of particle methods”. In: *Numerical methods in fluid dynamics*. Springer, 1985, pp. 243–324.
- [41] Giovanni Russo. “Deterministic diffusion of particles”. In: *Communications on Pure and Applied Mathematics* 43.6 (1990), pp. 697–733.

- [42] Andrew Yates, Robin Callard, and Jaroslav Stark. “Combining cytokine signalling with T-bet and GATA-3 regulation in Th1 and Th2 differentiation: a model for cellular decision-making”. In: *Journal of theoretical biology* 231.2 (2004), pp. 181–196.