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Ikechukwu Udema. Desirability of Reduced mass Concept, Conflicting CODATA Values of some Constants: Variants of Fine structure constant and Rydberg constant Equations. 2023. ⟨hal-04078431⟩

HAL Id: hal-04078431

<https://hal.science/hal-04078431v1>

Preprint submitted on 23 Apr 2023

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1 **Title: Desirability of Reduced mass Concept, Conflicting CODATA**
2 **Values of some Constants: Variants of Fine structure constant and**
3 **Rydberg constant Equations**

4 **Subtitle: Reduced mass, fine structure constant and Rydberg**
5 **constant.**

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11 **ABSTRACT**

12 **Background:** The desirability of the use of the reduced mass concept as opposed to the rest mass
13 seems to give conflicting results. **Objectives:** To derive variants of the equation of the Rydberg constant,
14 fine structure constant, and wave number that relate with the Rydberg constant; **Methods:** The research
15 is theoretical in nature. **Results:** The Rydberg constant (R_∞) equations derived in this research are
16 variants of each other and of those known in the literature; this is also applicable to the fine structure
17 constant (α). The values of R_∞ range between 10973731.6 and 1097373.89 /m for the rest mass case,
18 while for the reduced mass case, the range is between 10966253.06 and 1096784.63 /m; regarding α , the
19 derived equations gave results that are the same. The wave number was related to R_∞ at the atomic level
20 for the first time. The number showed a trend with the square of the atomic number, similar to the square
21 of the average ionisation energy. **Conclusion:** The use of rest mass and reduced mass gives conflicting
22 values for physical constants. Other variants of the equation for R_∞ may be determined in the future.

23 **Keywords:** Average ionisation energy; fine structure constant; reduced mass; rest mass; Rydberg
24 constant; variants of Rydberg constant and fine constant equations

25 **Significance Statement:** The deployment of rest mass and reduced mass can give conflicting results if
26 applied to the determination of other physical constants. Notable among the physical constants are Bohr's

27 radius, the theoretically calculated average ionisation energies of hydrogenic atoms (H), and multi-
28 electron ions. Deploying the parameters by substituting them into different equations for the Rydberg
29 constant (R_∞) can give different results. The goal of the research is to show evidence of this possibility by
30 embarking on the following objectives: 1) Derive variants of the equation of the Rydberg constant; 2)
31 derive variants of the equation relating wave number with the Rydberg constant at the atomic level; 3)
32 show that there could be multiple variants of the equation of the fine structure constant. With rest mass
33 and reduced mass, the values range between 10973731.6 and 10973733.89/m and between
34 10966253.06 and 1096784.63/m, respectively. The former are more similar to the Committee on Data for
35 Science and Technology (CODATA) value than the latter. This result can motivate CODATA to review
36 current data in light of the application of R_∞ in spectroscopy and other uses and to reexamine the issue of
37 stark broadening of the spectral lines attributed to reduced mass effect.

38 **1.0 Introduction**

39 The desirability of the use of reduced mass concept as opposed to the rest mass seems to give
40 conflicting results. "Reduced mass" is a term for the appropriate mass corresponding to a linear
41 combination of coordinates involving two or more individual particles such as atoms in the context of the
42 investigation carried by the authors [1]; a simpler definition is that in physics the reduced mass is the
43 effective inertial mass appearing in the two-body problem of Newtonian mechanics (wikipedia).

44 In deriving the equation of reduced mass concept, the equations of motion of two mutually
45 interacting bodies is reduced to a single equation describing the motion of one body in a reference frame
46 centred in the other body. (Britannica.com). Barbosa [2] who sees the reduced mass as a quantity studied
47 in a two-particle system problem in mechanics have attempted to explain the origin of the reduced mass
48 concept. According to the author's note [2], the reduced mass of a two-particle system has its origin in the
49 combination of noninertial nature of an observer attached to particle 1 leading to an inertial force (F_{in}); the
50 force ($F_{1(2)}$) which drives the observer is intimately related to the force applied to particle 2 ($F_{2(1)}$) through
51 the Newtonian 3rd law of motion as they arise from the mutual interaction.

52 The reduced mass is known to cause the stark broadening of the spectral lines of the hydrogen
53 atoms; it has also been implicated in the deformation of the central part of the line shape of the spectrum

54 [2]. Meanwhile, the Rydberg constant has a role in spectroscopy and is related to the fine structure
 55 constant. Hence the desirability of the reduced mass concept, the conflicting CODATA [3, 4] values of
 56 some constants, and the variants of the fine structure constant and Rydberg constant equations have to
 57 be examined or investigated. The goal is to reveal a possible conflict in some fundamental physical
 58 constants by embarking on the following objectives: 1) Derive variants of the equation of Rydberg
 59 constant and compare the resulting value with literature; 2) derive variants of the equation relating wave
 60 number with Rydberg constant at the atomic level; 3) show that there could be multiple variants of the
 61 equation of fine structure constant, which gives values that are the mass of an electron invariant.

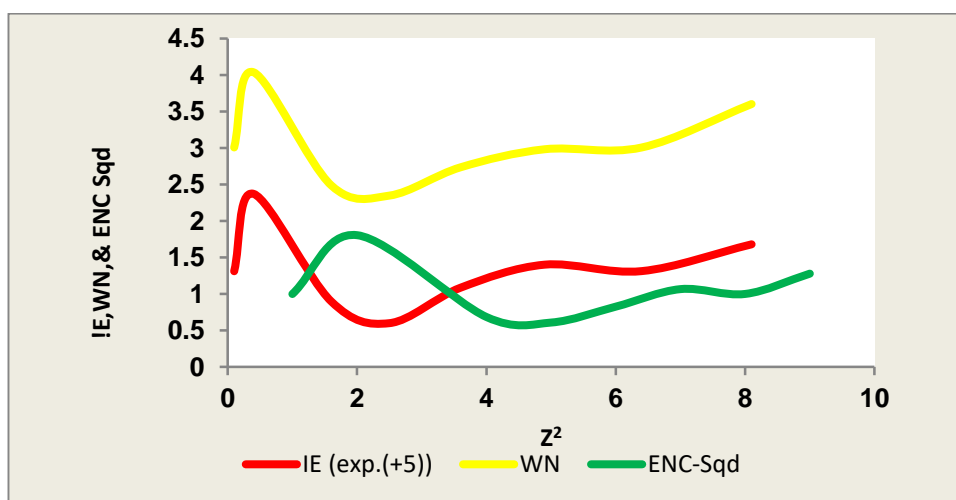
62 2. Materials and Methods

63 The research is theoretical without materials and experiments of any kind.

64 3. Results

65 Worthy of note in this research is the observation that the physical variables, the wave number,
 66 average ionisation energies, and the square of the effective nuclear charge vary with the square of
 67 nuclear charge in the same way (Figure 1). The square of the nuclear charge is chosen because it best
 68 correlates with the physical variable determined based on the average ionisation energy.

69



70
 71 **Figure 1:** The plot illustrating the trend of all related physical variables with the square of atomic
 72 number. The legends, IE, WN, and ENC-Sqd represent the average ionisation energy, wave number, effective
 73 nuclear charge squared. The IE, WN, and ENC-Sqd values were separately plotted versus the square of atomic
 74 number/10. Division by ten was to enhance resolution of the curves despite sum points of contact or intersections.

75 The values of the Rydberg constant calculated in this research ranges between 10973731.6-
76 10973733.89/m for the rest mass case while, for the reduced mass case, the range is between
77 10966253.06-1096784.63/m (Table 1) which compares with the CODATA [3] value of 10973731.568/m.
78 Rest mass values are therefore, more similar to the 2016, CODATA [4] value. This is against the
79 backdrop of higher values of the average ionisation energy of hydrogen and Bohr's radius calculated
80 using rest mass and reduced mass respectively.

81 Variants of the equations for the fine structure constant (α), Eq. (33) and Eq. (34), in particular,
82 are not known in the literature. The values are exactly the same for the reduced mass case (7.294715912
83 exp. (-3)) and 7.297352577 exp. (-3), for the rest mass case. The α achieved with accuracy of 81 PPT is
84 equal to 7.297352563 exp. (-3) [2] and the CODATA [4] value (7.2973525664 exp. (-3)) (Table 1). The
85 rest mass value is higher than CODATA and Morel et al values.

86 **Table 1: Values of the Rydberg constant and the fine structure constant based on derived**
87 **equations**

89	S/N	Equations	Based on rest mass	Based on reduced mass
91	1	$R_{\infty} = 8 \pi \epsilon_0 a_0 E_H^2 / h c e^2$	10973731.6	10966253.06
92				
93	2	$R_{\infty} = E_H / h c$ (Here & in the lit.)	10973731.6	10967784.63
94				
95	3.	$R_{\infty} = \alpha^2 m_e c / 2 h$ (CODATA-I)	10973731.57	10967758.834
96				
97	4.	$R_{\infty} = e^4 m_e / 8 \epsilon_0^2 c h^3$ (CODATA-II)	10973733.89	10967760.69
98				
99	5.	Eq. (33)	7.297352577 exp. (-3)	7.294715912 exp.(-3)
100				
101	6.	Eq. (34)	7.297352577 exp. (-3)	7.294715912 exp.(-3)
102				

103 Serial number (S/N) 1 equation was derived in this research and S/N 2 equation, was a rederivation in
104 this research, which reproduced the literature (lit.) version. The S/N 3 (CODATA-I) and 4 (CODATA-II)
105 equations are CODATA records and derived equation respectively, in which fine structure constant
106 equation was substituted into S/N 3 equation to give S/N 4 equation. The ionisation energy-based radii of
107 atoms used can be found in the literature [5].

108 The values of the wave number calculated based on Eq. (40) are not influenced by the mass of
109 the electron if the ionisation based radii were calculated using experimentally determine average
110

111 ionisation energies. An exception may apply to hydrogen if either the rest mass or reduced mass were
 112 applied for the calculation of Bohr's radius. The calculations based on Eq. (37) are influenced by the use
 113 of either rest mass or reduced mass. The values ranges between 2.735 and 4.036 exp. (+9)/m for Eq.
 114 (37); with Eq. (40) it is between 3.006 and 4.038 exp. (+9)/m (Table 2).

115 **Table 2: Calculated wave numbers based on reduced mass and rest mass of the electron**

117 Elements	H	He	Be	B	C	N	O	F
120 ∇^* (Eq. (37)) 121 exp. (9)/m	3.006	4.038	2.490	2.348	2.736	2.983	3.007	3.403
122 ∇ (Eq. (37)) 123 exp. (9)/m	3.005	4.036	2.495	2.347	2.735	2.982	3.006	3.401
124 ∇^* (Eq. (40)) 125 exp. (9)/m	3.006	4.038	2.490	2.348	2.736	2.983	3.007	3.402
126 ∇ (Eq. (40)) 127 exp. (9)/m	3.008							

128
 129 ∇^* is the wave number affected by the use of rest mass. ∇ is the wave number affected by the use of reduced mass.
 130 Values are approximations to the 3rd decimal places.

131
 132 The values of the wave number calculated based on Eq. (40) are not influenced by the mass of
 133 the electron if the ionisation based radii were calculated using experimentally determine average
 134 ionisation energies. An exception may apply to hydrogen if either the rest mass or reduced mass was
 135 applied for the calculation of Bohr's radius. The calculations based on Eq. (37) are influenced by the use
 136 of either rest mass or reduced mass. The values ranges between 2.735 and 4.036 exp. (+9)/m for Eq.
 137 (37); with Eq. (40) it is between 3.006 and 4.038 exp. (+9)/m (Table 2).

138
 139
 140
 141
 142
 143

144 3.1 Theory

145 3.1.1 The ratio of the product of the ionisation energy dependent atomic radius and the 146 momentum of the orbital electron to the energy level as a function of the composite 147 fundamental constant

148 In this section, the relationship between the ratio of the product of the momentum and the
149 ionisation energy-dependent radius of the atom to the energy level and a composite fundamental physical
150 constant are derived. In line with pedagogical principle, one begins from the known to the unknown. The
151 Rydberg constant (R_∞) is given as:

$$152 R_\infty = \alpha^2 m_e c / 2 h, \quad (1)$$

153 where α , c , and h are the fine structure constant, rest mass of an electron, velocity of light in a vacuum,
154 and Planck constant respectively.

$$155 \alpha = e^2 / 2 h \epsilon_0 c, \quad (2)$$

156 where ϵ_0 is the permittivity in free space; α is the fine structure constant. Substitute Eq. (2) into Eq. (1) to
157 give:

$$158 R_\infty = m_e e^4 / 8 h^3 \epsilon_0^2 c, \quad (3)$$

159 Equation (3) is definitely not a novelty, but with Eqs (1) and (2), it serves an immediate reference.

160 It has been shown that the effective nuclear charge (Z_{eff}) is inversely proportional to α , directly
161 proportional to the principal quantum number (n), and directly proportional to the square root of the
162 average ionisation energy (E_i) [7]; thus,

$$163 Z_{\text{eff}} = n \left(\frac{2E_i}{m_e} \right)^{1/2} / \alpha c, \quad (4)$$

164 Therefore,

$$165 c = n (2E_i / m_e)^{1/2} / Z_{\text{eff}} \alpha, \quad (5)$$

166 Substitute Eq. (5) into Eq. (5) into Eq. (3) to give:

$$167 R_\infty = (Z_{\text{eff}} m_e e^4 \alpha / 8 h^3 \epsilon_0^2 n) \left(\frac{m_e}{2E_i} \right)^{1/2}, \quad (6)$$

168 From Eq. (6) is given:

$$169 E_i = (Z_{\text{eff}} m_e e^4 \alpha / 8 h^3 \epsilon_0^2 n R_\infty)^2 / 2, \quad (7)$$

170 Substitute Eq. (2) into Eq. (7) to get

$$171 E_i = m_e (Z_{\text{eff}} m_e e^6 \alpha / 16 h^4 \epsilon_0^3 n c R_\infty)^2 / 2, \quad (8)$$

172 Solving for Z_{eff} gives:

$$173 \quad Z_{\text{eff}} = \left(\frac{512E_i}{m_e^3} \right)^{1/2} \varepsilon_0^3 h^4 n_i R_{\infty} c / e^6, \quad (9)$$

174 Substitute Eq. (9) into Bohr's equation for ionisation energy to give:

$$175 \quad \left(\frac{512E_i}{m_e^3} \right)^2 (\varepsilon_0^3 h^4 n_i R_{\infty} c)^2 e^4 m_e / 8 \varepsilon_0^2 h^2 n^2 e^{12} = \frac{Z_{\text{eff}}}{8 \pi \varepsilon_0 a_i}, \quad (10a)$$

176 where a_i is the average ionisation energy-dependent radius of any atom.

177 Simplification and solving for Z_{eff} gives:

$$178 \quad Z_{\text{eff}} = \frac{512E_i \varepsilon_0^5 h^6 R_{\infty}^2 c^2 a_i \pi}{m_e^2 e^{10}}, \quad (10b)$$

179 From Eq. (10b) is given $\varepsilon_0^2 h^4 \pi^2 / \pi^2 m_e^2 e^4$ which is $\equiv a_0^2 \pi^2$ where a_0 is the Bohr's radius for hydrogen;

180 Therefore, Eq. (10b) becomes:

$$181 \quad Z_{\text{eff}} = 512E_i a_i \left(\frac{\varepsilon_0 \pi}{e^2} \right)^3 (h R_{\infty} c a_0)^2, \quad (11)$$

182 Since the atomic radius is inversely proportional to the average ionisation energy-a kinetic energy- on the
183 basis of Coulomb's law, one can multiply both sides of Eq. (11) by a_i and convert E_i to its relation to
184 momentum (P) to give:

$$185 \quad Z_{\text{eff}} a_i = 512P^2 a_i^2 \left(\frac{\varepsilon_0 \pi}{e^2} \right)^3 (h R_{\infty} c a_0)^2 / 2 m_e, \quad (12)$$

186 In Eq. (12), $a_i Z_{\text{eff}}$ is equal to $n^2 h^2 \varepsilon_0 / \pi e^2 m_e = a_0 n^2$; thus,

$$187 \quad p^2 a_i^2 = \frac{a_0 n^2}{2^8} \left(\frac{e^2}{\varepsilon_0 \pi} \right) \left(\frac{1}{h R_{\infty} c a_0} \right)^2, \quad (13)$$

$$188 \quad \frac{p a_i}{n} = \frac{1}{16 h c R_{\infty}} \left(\frac{m_e e^6}{a_0 \varepsilon_0^3 \pi^3} \right)^{1/2}, \quad (14)$$

189
190 Given that $(8E_H a_0)^3 = (e^2 / \pi \varepsilon_0)^3$, one can rewrite, Eq. (14) for the purpose of calculational simplicity, as
191 follows:

$$192 \quad p a_i = n a_0 (2E_H^3 m_e)^{1/2} / h R_{\infty} c, \quad (15)$$

193 If $a_i = a_0$ and $n = 1$, then, hydrogen is referred to:

$$194 \quad R_{\infty} = (2E_H^3 m_e)^{1/2} / h c p, \quad (16a)$$

195 Equation (16a) can be simplified to obtain equation similar to a literature version [7]:

$$196 \quad R_{\infty} = E_H / h c, \quad (16b)$$

197 There are two clear corollaries:

198
$$R_{\infty} = n a_0 (2E_H^3 m_e)^{1/2} / h c p a_i \quad , \quad (17a)$$

199 Equation (17a) can be simplified further if one solves mathematically p (i.e. $2m_e E_i)^{1/2}$ and a_i ($=$
 200 $\frac{e^2 n}{8 \pi \epsilon_0 (E_H E_i)^{1/2}}$); after rearrangement the results is:

201
$$R_{\infty} = \frac{8 \pi \epsilon_0 a_0 E_H^2}{h c e^2} \quad (17b)$$

202 Like Eq. (2), one can derive a similar equation for effective nuclear charge as follows: First recall
 203 that [7]:

204
$$\frac{1}{\alpha} = \left(\frac{m_e c^2}{2E_H} \right)^{1/2} \quad (18)$$

205 Also, E_H can be expressed as [7-9]:

206
$$E_H = n^2 E_i / Z_{\text{eff}}^2 \quad (19)$$

207 Substitute Eq. (19) into Eq. (18) to give after rearrangement:

208
$$m_e c^2 = \frac{2 n^2 E_i}{\alpha^2 Z_{\text{eff}}^2} \quad (20)$$

209 Solving for Z_{eff} gives [7]:

210
$$Z_{\text{eff}} = \frac{n}{\alpha c} \left(\frac{2E_i}{m_e} \right)^{1/2} \quad (21)$$

211 Substitute Eq. (2) into Eq. (21) to give:

212
$$Z_{\text{eff}} = \frac{2 n v h \epsilon_0}{e^2} \quad (22)$$

213 where v is the velocity of the electron and n may be ≥ 1 ; this is determined as the result of $(2E_i/m_e)^{1/2}$.

214 Thus, given the reciprocal of Eq. (2), one sees the similarity in the mathematical structure as follows:

215
$$1/\alpha = \frac{2 n_0 c h \epsilon_0}{e^2} \quad (23)$$

216 where n_0 is = 1. The obvious corollary is that:

217
$$\frac{Z_{\text{eff}}}{n v} = \frac{1}{n_0 c \alpha} = \frac{2 h \epsilon_0}{e^2} \quad (24)$$

218
$$Z_{\text{eff}} = \frac{n v}{n_0 c \alpha} \quad (25)$$

219 Equation (14) can be rewritten to give two equations.

220
$$v = \frac{n}{16 a_i h c R_{\infty}} \left(\frac{e^6}{m_e a_0 \epsilon_0^3 \pi^3} \right)^{1/2} \quad (26)$$

221 The values of a_i can range between a_0 and $a_0 n^2 / Z_{\text{eff}}$ where Z_{eff} is between 1 and < 137 and n can range
 222 between 1 and 7.

223 3.1.2 Fine structure constant–based equations for the determination of the radius of 224 any atom

225 As in the literature [9], the radius of any atom is expressed as:

$$226 a_i = \frac{e^2 n}{8 \pi \epsilon_0 (E_H E_i)^{1/2}}, \quad (27)$$

227 Note that if hydrogen is referred to E_H is $= E_i$ and $n = 1$; if it is multi-proton hydrogenic ion, E_H is $\neq E_i$,
 228 though n remains $= 1$. One can then re-express a_i in terms of v as follows: Given that E_i is $= E_H Z_{\text{eff}}^2 / n^2$,
 229 Eq. (27) becomes:

$$231 a_i = \frac{e^2 n^2}{8 \pi \epsilon_0 E_H Z_{\text{eff}}}, \quad (28a)$$

232 Then substituting Eq. (25) into Eq. (28a) gives:

$$233 a_i = \frac{e^2 n n_0 c \alpha}{8 \pi \epsilon_0 E_H v}, \quad (28b)$$

234 Taking v as $(2Z_{\text{eff}} e^2 / 8 \pi \epsilon_0 a_i m_e)^{1/2}$ and substitute into Eq. (28b) to give:

$$235 a_i = \frac{e^2 n n_0 c \alpha (8 \pi \epsilon_0 a_i m_e)^{1/2}}{8 \pi \epsilon_0 E_H (2e^2 Z_{\text{eff}})^{1/2}}$$

236 Simplification gives:

$$237 a_i = \frac{e^2 n_0^2 c^2 \alpha^2 m_e n^2}{16 E_H^2 \pi \epsilon_0 Z_{\text{eff}}}, \quad (29)$$

238 Re-substitution of Eq. (25) into Eq. (29) gives:

$$239 a_i = \frac{e^2 n_0^3 c^3 \alpha^3 m_e n}{16 E_H^2 \pi \epsilon_0 v}, \quad (30)$$

240 One can repeat the operation leading to Eq. (29) to give:

$$241 a_i = \frac{(c \alpha n_0)^6 m_e^3 n^2 e^2}{64 E_H^4 \pi \epsilon_0 Z_{\text{eff}}}, \quad (31)$$

242 Ultimately, one can further re-substitute Eq. (25) into Eq. (31) as in Eq. (29) to give:

$$243 a_i = \frac{(c \alpha n_0)^7 m_e^3 n e^2}{64 E_H^4 \pi \epsilon_0 v}, \quad (32)$$

244 Clearly, it can be seen that none of the equations is final because there is no end to substitutions.
 245 Experimental information is needed in order to proceed with Eqs (28a), (28b), (29), (30), (31), and (32) all
 246 of which give the same result.

247 **3.1.3 Deductions with respect to fine structure constant**

248 Beginning from Eqs (28b), (30), and (32), there are possibilities of reproducing Eq. (18) in terms
 249 of the fine structure constant if all the fundamental constants are experimental variables, the average
 250 ionisation energies, are substituted into the equations. Beginning from Eqs (28b), (30), and (32), there are
 251 possibilities of reproducing Eq. (18) in terms of the fine structure constant if all the fundamental constants
 252 are experimental variables, the average ionisation energies, are substituted into the equations. The
 253 resulting fine structure constant is approximately $7.2935 \exp. (-3)$. In all cases the equation is: $\alpha =$
 254 $\frac{1}{c n_0} \left(\frac{2E_H}{m_e} \right)^{1/2}$ where n_0 is always equal to one. The equation became a reality after substituting Eq. (27) and
 255 $(2E_i/m_e)^{1/2}$ into the respective equations and solving for α . Rederiving the equation based on Eqs (29) and
 256 (31) requires substituting Bohr's equation. This is despite information about ionisation energies, a
 257 deterministic variable first discovered by a Russian scientist, yet most western scientists (or rather not to
 258 be too general, "my west") but not limited to them, have been vehement in criticising Bohr's equation
 259 ($a_i = n^2 h^2 \epsilon_0 / \pi m_e e^2 Z_{\text{eff}}$): It appears no motivation is seen in the Russian scientist, yet there is no better
 260 alternative that could be seen as a motivation for rejecting the possibility that ionisation energies that are
 261 experimentally generated anywhere in the global space can be used to determine the effective nuclear
 262 charge. How effective nuclear charge can be calculated based on ionisation energies and enabling
 263 equations are available in the literature [7]. With Eqs (29) and (31), the new equations after the
 264 substitution and solving for the fine structure constant are respectively:

$$265 \quad \alpha = \frac{4 h \epsilon_0 E_H}{e^2 m_e c}, \quad (33)$$

266 If Bohr's equation for ionisation energy is substituted into Eq. (33), it gives: $\alpha = e^2/2\epsilon_0 hc$.

$$267 \quad \alpha = \frac{2}{c n_0} \left(\frac{E_H^2 h \epsilon_0}{m_e^2 e^2} \right)^{1/3}, \quad (34)$$

268 It must be made known that other variants of the equation can be derived if the series beyond Eq. (32) is
 269 continued. The values the fine structure constant expected from Eqs (33) and (34) is approximately

270 7.2945 exp. (-3). Equation (34) is a major innovation as far as the variants of the equation of fine
 271 structure constant is concerned.

272 3.1.4 Deductions with respect to wave number.

273 The core atomic physicists, Bohr, Rydberg-Ritz *etc* have concern for wave number though such
 274 concern is restricted to hydrogenic atoms and ions only. Moreover it is all about mainly emission
 275 spectrum. It is pointless being in doubt because, a_i and p can be separately calculated as along as the
 276 average ionisation energy and energy level are known; this is usually so for most elements. Here,
 277 deductively or as a corollary, one can relate the wave number of multi-electron atoms to the Rydberg
 278 constant. In this regard, refer to Eq. (26) where the momentum p is equal to $m_e v$ which, in turn, is equal
 279 to h/λ , the well-known de Broglie equation where the reciprocal of the wave length ($1/\lambda$) is equal to the
 280 wave number (∇). Therefore, given Eq. (15), derivation is as follows:

$$281 \quad \frac{h a_i}{\lambda} = n a_0 (2E_H^3 m_e)^{1/2} / h R_\infty c, \quad (35)$$

282 “No beating the gun”, just step-by-step, until one obtains the following:

$$283 \quad \frac{a_i}{n \lambda} = a_0 (2E_H^3 m_e)^{1/2} / h^2 R_\infty c, \quad (36)$$

$$284 \quad \nabla = n a_0 (2E_H^3 m_e)^{1/2} / h^2 R_\infty c a_i, \quad (37)$$

285 Two equations attributed to Bohr are: 1) $E_i = Z_{\text{eff}}^2 e^4 m_e / 8 \epsilon_0^2 n^2 h^2$ and 2) $a_i = n^2 h^2 \epsilon_0 / \pi m_e e^2 Z_{\text{eff}}$.

286 Substituting the latter into the former and solving for E_i gives:

$$287 \quad E_i = n^2 h^2 \epsilon_0 / 8 \pi^2 m_e a_i^2, \quad (38)$$

288 Based on Eq. (38), the momentum of the electron is

$$289 \quad p = nh/2\pi a_i, \quad (39)$$

290 Consequently, the wave number is:

$$291 \quad \nabla = n/2 \pi a_i, \quad (40)$$

292 With Eqs (17a) and (40) the following can be derived. The equations contain n/a_i ; therefore,

$$293 \quad \frac{1}{2 \pi} = \frac{a_0 (2E_H^3 m_e)^{1/2}}{R_\infty h^2 c}, \quad (41a)$$

$$294 \quad R_\infty = \frac{a_0 \pi (8E_H^3 m_e)^{1/2}}{h^2 c}, \quad (41b)$$

295

296 4. Discussion

297 Various equations were derived, notable of which is the equation unknown in the literature for the
298 calculation of the Rydberg constant (R_a) and fine structure constant (α) as well. With the various
299 equations, it has been possible to show that the use of reduced mass as opposed to rest mass leads to
300 lower values of physical constants compared to CODATA values. Higher values of R_a were obtained from
301 all equations in which Bohr's radius and the average ionisation energy of hydrogen were calculated using
302 the rest mass of the electron. These revelations are made possible by the application of the derived
303 equations, which are therefore analysed and discussed as follows: Given Eq. (17a), it could be seen that
304 R_a is inversely proportional to the product of the momentum, p , of the orbital electron and its distance
305 from the nucleus, a_i , for multi-electron atoms, whereas it is inversely proportional to the p (Eq. (16a)) for
306 hydrogen and hydrogenic ions as well. Incidentally, R_a is also directly proportional to the square of the
307 average ionisation energy (E_H) of a hydrogen atom (Eq. 17b) for hydrogen atom only. Also, Eq. (37)
308 shows that R_a is inversely proportional to the product of the wave number and the ionisation energy-
309 dependent atomic radius and directly proportional to the square root of the 3rd power of the E_H for all
310 multi-electron atoms; it is strictly directly proportional to the square root of the 3rd power of the E_H for the
311 hydrogen atom only. With this apparent analysis, one can now x-ray the physical implications.

312 It is not certain whether or not the concept of reduced mass is applied in the spectroscopic
313 determination of ionisation energies. If not, all the equations where R_∞ is E_H -dependent should give the
314 same results. On the contrary, if theoretically calculated hydrogenic average ionisation energy is the
315 case, the application of either the rest mass or the reduced mass in Bohr's equation for the equation of
316 average ionisation energy would obviously give different results. Recall that Bohr's equation for the radius
317 is inversely proportional to the mass of the electron; using rest mass gives a lower value of the radius
318 ($\approx 5.289643 \text{ exp. } (-11) \text{ m}$), while the reduced mass ($9.104424487 \text{ exp. } (-31) \text{ kg}$) calculated by deploying
319 all 2012 CODATA [3] values of relevant physical constants gives $5.292523949 \text{ exp. } (-11) \text{ m}$, which is
320 slightly higher than the 2012 CODATA value of $5.2917721092 \text{ exp. } (-11) \text{ m}$ [3]. If these values are
321 substituted into the Coulomb equation for hydrogen atoms, different values of the kinetic energies (E_H
322 values) are obtainable. In a similar way, if the rest mass and the reduced mass are separately substituted

323 into Bohr's equation ($E_i = Z_{\text{eff}}^2 e^4 m_e / 8\epsilon_0^2 n^2 h^2$) for ionisation energy, different results are expected. One
324 is higher than the other ($2.179872176 \exp. (-18)$ and $2.178690844 \exp. (-18)$ J). If these values are
325 substituted again into the Coulomb equation for the hydrogen atom, the radii above, the first and the
326 second are expected. Yet all calculations of R_∞ involving "physical constants" where rest mass featured
327 gave values that were higher than those where reduced mass was the case. Besides, the results from the
328 former (the rest mass case) are very similar to the CODATA value. Therefore, it seems the rest mass
329 played a role in the CODATA value of R_∞ . Then the question is: what is the desirability of the use of the
330 reduced mass concept as opposed to the rest mass, which seems to give a conflicting result? In any
331 case, if E_H is experimentally generated and gives a consistent result, then R_∞ may be regarded as a
332 fundamental physical constant, considering the fact that the rest mass (an expression of the quantity of
333 matter in any specific object) is also constant.

334 With a clear presence of the ratio of average ionisation energy to any kind of electron mass in all
335 the variants of the equation of fine structure constant (Eqs (33) and (34)), the results of calculation should
336 be electron mass of any kind invariant. A recent experimental study showed the value ($7.297352563 \exp.$
337 (-3)) of the fine structure constant with an accuracy of 81 PPT [5], which is lower than the calculated
338 value in this study and the CODATA value, giving the impression that one or more of the fundamental
339 physical terms in the equation of the fine structure constant may be less accurate. However, one may
340 wish to know if a lower value from any future experiment results in higher accuracy. This has become a
341 pertinent question because the CODATA value or any other unknown in the course of this research
342 seems not to be a standard reference for comparison. Be it as it may, the fine structure constant is
343 directly proportional to E_H (Eq. (33)), and it is directly proportional to the cube root of the square of E_H .

344 Equation (36), once again, implies that the ratio of the ionisation energy-dependent radius of any
345 atom to the product of the wave length and principal quantum number is equal to the ratio of composite
346 fundamental constants. Ultimately, the wave number for any multi-electron atom is now related to the
347 Rydberg. Note that Eq. (37) is generalisable to all atoms, both hydrogenic and nonhydrogenic atoms and
348 ions. The only experimental information needed, for instance, is the average ionisation energy coupled
349 with the enabling equation, Eq. (27) for instance. In order to evaluate Eq. (37), the first members of the

350 groups in the first period of the period table were used as examples whose first average ionisation
351 energies are known. If there is any way by which a_i can be determined either experimentally or
352 theoretically, it is not just the average ionisation energy that can be calculated according to the
353 conventional Bohr's equation but the wave number as well. The wave number is directly proportional to
354 the square root of the cube of E_H and the principal quantum number and inversely proportional to the
355 radius, as shown by equations (37) and (40).

356 There seems to be more interest in fine structure constant than Rydberg constant despite the fact
357 that the latter is related to the former in the equation given as follows [3, 4]: $R_\infty = \alpha^2 m_e c / 2h$. The α has
358 been studied for various reasons including reevaluation of the standard model [10, 11]. Some studies are
359 also devoted to measurements in order to achieve better results in terms of higher precision [12-14]; what
360 looks like theoretical method entailed the application of quartic equations to which is related the much
361 talked about (but yet to be fully elucidated) golden ratio and a-not-too clear classical "harmonic
362 proportions" [15, 16]. In this research, however, variants of the equation of α were derived giving results
363 that are always not more than 1/137. These are in addition to the derived equation relating effective
364 nuclear charge to the α [7] and the equations that confirm the universality of R_∞ ; this is so because such
365 equation can be used to either determine it or be used to determine nuclear property such as mass radius
366 of any nucleon [8]. Recent mention of R_∞ is in connection with Rydberg atom explored in the
367 determination of the angle-of-arrival of an incident radio-frequency (RF) wave or signal [17]; this seems to
368 be an application of Rydberg concept in technology besides spectroscopic facilities. The α has application
369 in the derivation of the equation of the mass radius of the nucleon and larger subatomic particles [18].

370 It is interesting to note that the physical variables exhibit a similar trend with an increasing value
371 of the square of the atomic number. It is akin to the chemical periodicity of elements. Different atoms are
372 characterised by different average ionisation energies, which is another name for the average kinetic
373 energy needed to expel an electron from an uncertain position in its ground state to infinity. It is not
374 unlikely that an electron may, under unnatural conditions (or such natural conditions within the stars, the
375 sun being the nearest of such), move to unspecified lower energy with the release of an amount of energy
376 equal to its ionisation energy at its ground-state energy level. Therefore, wave number may not be

377 restricted to the spectra of different elements; any elementary particle in motion has a de Broglie wave
378 length and, as such, a wave number.

379 **5. Conclusion**

380 The use of rest mass and reduced mass gives conflicting values for physical constants. The
381 question as to the desirability of the reduced mass concept is valid considering the observed value
382 calculated in this research based on rest mass is very similar to the CODATA value. Other variants of the
383 equations for the fine structure constant and, consequently, the Rydberg constant may be determined in
384 the future.

385 **Author contributions:** The sole author designed, analysed, interpreted and prepared the manuscript.

386 **Funding:** "This research received no external funding." However, Science International, under the aegis
387 of LiveDNA, promised a waiver of the article processing charge if the manuscript was accepted for
388 publication.

389 **Acknowledgments:** The management of the Royal Court Yard Hotel in Agbor, Delta State, Nigeria, is
390 immensely appreciated for the supply of electricity during the preparation of the manuscript. The provider
391 of the QuillBot grammar checker is thanked for improving the English language quality of the manuscript.

392 **Conflicts of Interest:** There is no competing interest; specifically, there is no financial interest with any
393 government, corporate body, or individual except the awaited unpaid retirement benefits by the Delta
394 state government of Nigeria for about four years after retirement.

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