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# Ihara's Lemma in higher dimension: the limit case

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## Abstract

Clozel, Harris and Taylor proposed in [CHT08] conjectural generalizations of the classical Ihara's lemma for  $\mathrm{GL}_2$ , to higher dimensional similitude groups. We prove these conjectures in the so called *limit case* under some mild hypothesis coming from a level raising theorem of Gee in [Gee11].

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Ihara's original Lemma: origin and proofs . . . . .	1
1.2	Generalisations of Ihara's Lemma . . . . .	3
1.3	Main result . . . . .	5
<b>2</b>	<b>Preliminaries</b>	<b>7</b>
2.1	Representations of $\mathrm{GL}_d(L)$ . . . . .	7
2.2	Weil–Deligne inertial types . . . . .	9
2.3	Kottwitz–Harris–Taylor Shimura varieties . . . . .	11
2.4	Filtrations of stratification . . . . .	14
<b>3</b>	<b>Genericity for KHT-Shimura varieties</b>	<b>19</b>
3.1	Level raising . . . . .	20
3.2	Local and global monodromy . . . . .	22
3.3	Typicity and monodromy . . . . .	24
	<b>References</b>	<b>27</b>

# 1 Introduction

## 1.1 Ihara's original Lemma: origin and proofs

In the Taylor-Wiles method Ihara's lemma is the key ingredient to extend a  $R = T$  property from the minimal case to a non minimal one. It is usually formulated by the injectivity of some map as follows.

Let  $\Gamma = \Gamma_0(N)$  be the usual congruence subgroup of  $SL_2(\mathbb{Z})$  for some  $N > 1$ , and for a prime  $p$  not dividing  $N$  let  $\Gamma' := \Gamma \cap \Gamma_0(p)$ . We then have two degeneracy maps

$$\pi_1, \pi_2 : X_{\Gamma'} \longrightarrow X_{\Gamma}$$

between the compactified modular curves of levels  $\Gamma'$  and  $\Gamma$  respectively, induced by the inclusion

$$\Gamma' \hookrightarrow \Gamma \text{ and } \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma' \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \hookrightarrow \Gamma.$$

For  $l \neq p$ , we then have a map

$$\pi^* := \pi_1^* + \pi_2^* : H^1(X_{\Gamma}, \mathbb{F}_l)^2 \longrightarrow H^1(X_{\Gamma'}, \mathbb{F}_l).$$

**Theorem 1.1.1.** *Let  $\mathfrak{m}$  be a maximal ideal of the Hecke algebra acting on these cohomology groups which is non Eisenstein, i.e. that corresponds to an irreducible Galois representation. Then after localizing at  $\mathfrak{m}$ , the map  $\pi^*$  is injective.*

Diamond and Taylor in [DT94] proved an analogue of Ihara's lemma for Shimura curves over  $\mathbb{Q}$ . For a general totally real number field  $F$  with ring of integers  $\mathcal{O}_F$ , Manning and Shotton in [MS] succeeded to prove it under some large image hypothesis. Their strategy is entirely different from those of [DT94] but rather consists roughly

- to carry Ihara's lemma for a compact Shimura curve  $Y_{\bar{K}}$  associated to a definite quaternion algebra  $\bar{D}$  ramified at some auxiliary place  $v$  of  $F$ , in level  $\bar{K} = \bar{K}^v \bar{K}_v$  an open compact subgroup of  $D \otimes \mathbb{A}_{F,f}$  unramified at  $v$ ,
- to the indefinite situation  $X_K$  relatively to a quaternion division algebra  $D$  ramified at all but one infinite place of  $F$ , and isomorphic to  $\bar{D}$  at all finite places of  $F$  different to  $v$ , and with level  $K$  agreeing with  $\bar{K}^v$  away from  $v$ .

Indeed in the definite case Ihara's statement is formulated by the injectivity of

$$\pi^* = \pi_1^* + \pi_2^* : H^0(Y_{\bar{K}}, \mathbb{F}_l)_{\mathfrak{m}} \oplus H^0(Y_{\bar{K}}, \mathbb{F}_l)_{\mathfrak{m}} \longrightarrow H^0(Y_{\bar{K}_0(w)}, \mathbb{F}_l)_{\mathfrak{m}}$$

where both  $\bar{D}$  and  $\bar{K}$  are unramified at the place  $w$  and  $\bar{K}_0(w)_w$  is the subgroup of  $GL_2(F_w)$  of elements which are upper triangular modulo  $p$ .

The proof goes like this, cf. [MS] theorem 6.8. Suppose  $(f, g) \in \ker \pi^*$ . Regarding  $f$  and  $g$  as  $K^v$ -invariant function on  $\bar{G}(F) \backslash \bar{G}(\mathbb{A}_{F,f})$ , then  $f(x) =$

$-g(x\omega)$  where  $\omega = \begin{pmatrix} \varpi_w & 0 \\ 0 & 1 \end{pmatrix}$ ,  $\varpi_w$  being an uniformizer for  $F_w$  and  $\overline{G}$  being the algebraic group over  $\mathcal{O}_F$  associated to  $\mathcal{O}_D^\times$  the invertible group of the maximal order  $\mathcal{O}_{\overline{D}}$  of  $\overline{D}$ : note that  $\overline{G}(F_w) \cong \mathrm{GL}_2(F_w)$ . Then  $f$  is invariant under  $K^v$  and  $\omega^{-1}K^v\omega$  so that, using the strong approximation theorem for the subgroup of  $\overline{G}$  of elements of reduced norm 1, then  $f$  factors through the reduced norm map, and so is supported on Eisenstein maximal ideals.

The link between  $X_K$  and  $Y_{K^v}$  is given by the geometry of the integral model of the Shimura curve  $X_{K_0(v)}$  with  $\Gamma_0(v)$ -level structure. The main new ingredient of [MS] to carry this geometric link to Ihara's lemma goes through the patching technology which allows to obtain maximal Cohen-Macaulay modules over deformation rings. Using a flatness property and Nakayama's lemma, there are then able to extend a surjective property, dual to the injectivity in the Ihara's lemma, from the maximal unipotent locus on the deformation space to the whole space, and recover the Ihara's statement reducing by the maximal ideal of the deformation ring.

Recently Caraiani and Tamiozzo following closely [MS] also obtained Ihara's lemma for Hilbert varieties essentially because Galois deformation rings are the same and so regular which is not the case outside the case of  $\mathrm{GL}_2$ .

## 1.2 Generalisations of Ihara's Lemma

To generalize the classical Ihara's lemma in higher dimension, there are essentially two approaches.

- The first natural one developed by Clozel, Harris and Taylor in their first proof of Sato-Tate theorem [CHT08], focuses on the  $H^0$  with coefficients in  $\mathbb{F}_l$  of a zero dimensional Shimura variety associated to higher dimensional definite division algebras. More precisely consider a totally real field  $F^+$  and a imaginary quadratic extension  $E/\mathbb{Q}$  and define  $F = F^+E$ . We then consider  $\overline{G}/\mathbb{Q}$  a unitary group with  $\overline{G}(\mathbb{Q})$  compact so that  $\overline{G}$  becomes an inner form of  $\mathrm{GL}_d$  over  $F$ . This means, cf. §2.3, we have fixed a division algebra  $\overline{B}$  with center  $F$ , of dimension  $d^2$ , provided with an involution of the second kind such that its restriction to  $F$  is the complex conjugation. We moreover suppose that at every place  $w$  of  $F$ , either  $\overline{B}_w$  is split or a local division algebra.

Let  $v$  be a place of  $F$  above a prime number  $p$  split in  $E$  and such that  $\overline{B}_v^\times \cong \mathrm{GL}_d(F_v)$  where  $F_v$  is the associated local field with ring of integers  $\mathcal{O}_v$  and residue field  $\kappa(v)$ .

**Notation 1.2.1.** Let  $q_v$  be the order of the residue field  $\kappa(v)$ .

Consider then an open compact subgroup  $\overline{K}^v$  infinite at  $v$  in the following sense:  $\overline{G}(\mathbb{Q}_p) \cong \mathbb{Q}_p^\times \times \prod_{v_i^+} \overline{B}_{v_i^+}^{op, \times}$  where  $p = \prod_i v_i^+$  in  $F^+$  and we identify places of  $F^+$  over  $p = uu^c \in E$  with places of  $F$  over  $u$ . We then ask  $\overline{K}_p^v = \mathbb{Z}_p^\times \times \prod_{w|u} \overline{K}_w$  to be such that  $\overline{K}_v$  is restricted to the identity element.

The associated Shimura variety in level  $\overline{K} = \overline{K}^v \overline{K}_v$  for some finite level  $\overline{K}_v$  at  $v$ , denoted by  $\overline{\mathrm{Sh}}_{\overline{K}}$ , is then such that its  $\mathbb{C}$ -points are  $\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A}_{\mathbb{Q}}^\infty) / \overline{K}$  and

for  $l$  a prime not divisible by  $v$ , its  $H^0$  with coefficients in  $\overline{\mathbb{F}}_l$  is then identified with the space

$$S_{\overline{G}}(\overline{K}, \overline{\mathbb{F}}_l) = \{f : \overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A}_{\mathbb{Q}}^{\infty}) / \overline{K} \longrightarrow \overline{\mathbb{F}}_l \text{ locally constant}\}.$$

It is naturally an admissible smooth representation of  $\mathrm{GL}_d(F_v)$  and of the Hecke algebra  $\mathbb{T}(\overline{K}^v)$  defined as the image of the abstract unramified Hecke algebra, cf. definition 2.4.1, inside  $\mathrm{End}(S_{\overline{G}}(\overline{K}^v, \overline{\mathbb{F}}_l))$ .

To a maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}(\overline{K}^v)$  is associated a Galois  $\overline{\mathbb{F}}_l$ -representation  $\overline{\rho}_{\mathfrak{m}}$ , cf. §3.1. We consider the case where this representation is irreducible. Note in particular that such an  $\mathfrak{m}$  is then not pseudo-Eisenstein in the usual terminology.

**Conjecture 1.2.2.** (cf. conjecture B in [CHT08])

Any irreducible  $\mathrm{GL}_d(F_v)$ -submodule of  $S_{\overline{G}}(\overline{K}^v, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$  is generic.

- For rank 2 unitary groups, we recover the previous statement as the characters are exactly those representations which do not have a Whittaker model, i.e. are the non generic ones.

- For  $d \geq 2$ , over  $\overline{\mathbb{Q}}_l$ , the generic representations of  $\mathrm{GL}_d(F_v)$  are the irreducible parabolically induced representations  $\mathrm{st}_{t_1}(\pi_{v,1}) \times \cdots \times \mathrm{st}_{t_r}(\pi_{v,r})$  where for  $i = 1, \dots, r$ ,

- $\pi_{v,i}$  is an irreducible cuspidal representation of  $\mathrm{GL}_{g_i}(F_v)$ ,
- $\mathrm{st}_{t_i}(\pi_{v,i})$  is a Steinberg representations, cf. definition 2.1.2,
- $\sum_{i=1}^r t_i g_i = d$  where the Zelevinsky segments  $[\pi_{v,i}\{\frac{1-t_i}{2}\}, \pi_{v,i}\{\frac{t_i-1}{2}\}]$  are not linked in the sense of [Zel80].

- Over  $\overline{\mathbb{F}}_l$  every irreducible generic representation is obtained as the unique generic subquotient of the modulo  $l$  reduction of a generic representation. It can also be characterized intrinsically using representation of the mirabolic subgroup, cf. §2.1.

Here we will be mainly interested in the following weak form of Ihara's lemma.

**Definition 1.2.3.** (cf. definition of [CHT08] 5.1.9)

An admissible smooth  $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$ -module  $M$  is said to have the Ihara property if for every  $m \in M^{\mathrm{GL}_d(\mathcal{O}_v)}$  which is an eigenvector of  $\overline{\mathbb{F}}_l[\mathrm{GL}_d(\mathcal{O}_v) \backslash \mathrm{GL}_d(F_v) / \mathrm{GL}_d(\mathcal{O}_v)]$ , every irreducible submodule of the  $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$ -module generated by  $m$ , is generic.

- The second approach is to find a map playing the same role as  $\pi^* = \pi_1^* + \pi_2^*$ . It is explained in section 5.1 of [CHT08] with the help of the element

$$\theta_v \in \mathbb{Z}_l[K_1(v^n) \backslash \mathrm{GL}_d(F_v) / \mathrm{GL}_d(\mathcal{O}_{F_v})]$$

constructed by Russ Mann, cf. proposition 5.1.7 of [CHT08], where  $F_v$  is here a finite extension of  $\mathbb{Q}_p$  with ring of integers  $\mathcal{O}_v$ .

**Definition 1.2.4.** An admissible smooth  $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$ -module  $M$  is said to have the almost Ihara property if  $\theta_v : M^{\mathrm{GL}_d(\mathcal{O}_v)} \rightarrow M$  is injective.

Recall that  $l$  is called quasi-banal for  $\mathrm{GL}_d(F_v)$  if either  $l \nmid \#\mathrm{GL}_d(\kappa_v)$  (the banal case) or  $l > d$  and  $q_v \equiv 1 \pmod{l}$  (the limit case).

**Proposition 1.2.5.** (cf. [CHT08] lemma 5.1.10)

Suppose that  $l$  is quasi-banal and  $M$  is a  $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$ -module verifying the Ihara property. If  $\ker(\theta_v : M^{\mathrm{GL}_d(\mathcal{O}_v)} \rightarrow M)$  is a  $\overline{\mathbb{F}}_l[\mathrm{GL}_d(\mathcal{O}_{F_v}) \setminus \mathrm{GL}_d(F_v) / \mathrm{GL}_d(\mathcal{O}_{F_v})]$ -module, then  $M$  has the almost Ihara property.

**Applications:** the generalizations of the classical Ihara's lemma were introduced in [CHT08] to prove a non minimal  $R = \mathbb{T}$  theorem. The weaker statement  $R^{\mathrm{red}} = \mathbb{T}$  where  $R^{\mathrm{red}}$  is the reduced quotient of  $R$ , was later obtained unconditionally using Taylor's *Ihara avoidance* method, cf. [Tay08] which was enough to prove the Sato-Tate conjecture. However, the full  $R = \mathbb{T}$  theorem would have applications to special values of the adjoint  $L$ -function and would imply that  $R$  is a complete intersection. It should also be useful for generalizing the local-global compatibility results of [Eme].

In [Mos21], the author also proved that Ihara's property in the quasi-banal case is equivalent to the following result.

**Proposition 1.2.6.** (cf. [Mos21] corollary 9.5)

Let  $\mathfrak{m}$  be a non-Eisenstein maximal ideal of  $\mathbb{T}^S$  and  $f \in S_{\overline{G}}(\overline{K}^v \mathrm{GL}_d(\mathcal{O}_v), \overline{\mathbb{F}}_l)$ . Let  $\mathrm{Iw}_v$  be the Iwahori subgroup of  $\mathrm{GL}_d(\mathcal{O}_v)$ , then the  $\overline{\mathbb{F}}_l[\mathrm{Iw}_v \setminus \mathrm{GL}_d(F_v) / \mathrm{GL}_d(\mathcal{O}_v)]$ -submodule of  $S_{\overline{G}}(\overline{K}^v \mathrm{Iw}_v, \overline{\mathbb{F}}_l)$  generated by  $f$  is of dimension  $d!$ .

### 1.3 Main result

With the previous notations, let  $q_v$  be the order of the residue field of  $F_v$ . We fix some prime number  $l$  unramified in  $F^+$  and split in  $E$  and we place ourself in the limit case where  $q_v \equiv 1 \pmod{l}$  with  $l > d$ , which is, at least by base change, the crucial case to consider.

**Definition 1.3.1.** As in definition 2.5.1 of [CHT08], we say that a subgroup  $H \subseteq \mathrm{GL}_d(\overline{\mathbb{F}}_l)$  is big if :

- $H$  has no  $l$ -power order quotients;
- $H^i(H, \mathfrak{g}_d^0(\overline{\mathbb{F}}_l)) = (0)$  for  $i = 0, 1$  and where  $\mathfrak{g}_d := \mathrm{Lie} \mathrm{GL}_d$  and  $\mathfrak{g}_d^0$  is the trace zero subspace of  $\mathfrak{g}_d$ ;
- for all irreducible  $\overline{\mathbb{F}}_l[H]$ -submodules  $W$  of  $\mathfrak{g}_d(\overline{\mathbb{F}}_l)$ , we can find  $h \in H$  and  $\alpha \in \overline{\mathbb{F}}_l$  satisfying the following properties.
  - The  $\alpha$ -generalized eigenspace  $V(h, \alpha)$  of  $h$  on  $\overline{\mathbb{F}}_l^d$  is one dimensional.
  - Let  $\pi_{h, \alpha} : \overline{\mathbb{F}}_l^d \rightarrow V(h, \alpha)$  be the  $h$ -equivariant projection of  $\overline{\mathbb{F}}_l^d$  to  $V(h, \alpha)$  and let  $i_{h, \alpha} : V(h, \alpha) \hookrightarrow \overline{\mathbb{F}}_l^d$  be the  $h$ -equivariant injection of  $V(h, \alpha)$  into  $\overline{\mathbb{F}}_l^d$ . Then  $\pi_{h, \alpha} \circ W \circ i_{h, \alpha} \neq (0)$ .

**Theorem 1.3.2.** *In the limit case, suppose moreover that there exists a prime  $p_0 = u_0 \bar{u}_0$  split in  $E$  with a place  $v_0 | u_0$  of  $F$  such that  $\bar{B}_{v_0}$  is a division algebra. Consider  $\mathfrak{m}$  such that*

$$\bar{\rho}_{\mathfrak{m}} : G_F \longrightarrow \mathrm{GL}_n(\bar{\mathbb{F}}_l)$$

*is an irreducible representation which is unramified at all places of  $F$  lying above primes which do not split in  $E$  and which satisfies the following hypothesis:*

- *after semi-simplification  $\bar{\rho}_{\mathfrak{m},v}$  is a direct sum of characters;*
- *$\bar{F}^{\ker \mathrm{ad} \bar{\rho}}$  does not contain  $F(\zeta_l)$  where  $\zeta_l$  is any primitive  $l$ -root of 1;*
- *$\bar{\rho}(G_{F+(\zeta_l)})$  is big.*

*Then Ihara's lemma of the conjecture 1.2.2 is true, i.e. every irreducible  $\mathrm{GL}_d(F_v)$ -submodule of  $S_{\bar{G}}(\bar{K}^v, \bar{\mathbb{F}}_l)_{\mathfrak{m}}$  is generic.*

*Remark.* The last two hypothesis come from theorem 5.1.5 of [Gee11]. The first one corresponds to the weak form of Ihara's lemma of definition 1.2.3.

In [Boy20] we essentially proved conjecture 1.2.2 in the banal case under some restrictive hypothesis. The idea was mainly to transfer the property about irreducible subspaces to a similar one for the middle degree cohomology group of some KHT Shimura variety  $\mathrm{Sh}_{K^v(\infty)}$  associated to some similitude group  $G/\mathbb{Q}$  such that  $G(\mathbb{A}_{\mathbb{Q}}^{\infty,p}) \cong \bar{G}(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$ , cf. §2.3 for more details, and with level  $K^v(\infty) := \bar{K}^v$  meaning finite level outside  $v$  and infinite level at  $v$ .

The localization at  $\mathfrak{m}$  of the cohomology groups of  $\mathrm{Sh}_{K^v(\infty)}$  can be computed as the cohomology of the geometric special fiber  $\mathrm{Sh}_{K^v(\infty), \bar{s}_v}$  of  $\mathrm{Sh}_{K^v(\infty)}$ , with coefficient in the complex of nearby cycles  $\Psi_{K^v(\infty),v}$ .

The Newton stratification of  $\mathrm{Sh}_{K^v(\infty), \bar{s}_v}$  gives us a filtration of  $\Psi_{K^v(\infty),v}$  and so a filtration of  $H^{d-1}(\mathrm{Sh}_{K^v(\infty), \bar{\eta}_v}, \bar{\mathbb{Z}}_l)_{\mathfrak{m}}$  and the main point of [Boy20] is to prove that each graded part of this filtration verifies the Ihara property, i.e. each of their irreducible sub-space are generic. To realize this strategy we need first the cohomology groups of  $\mathrm{Sh}_{K^v(\infty)}$  to be torsion free: this point is now essentially settled by the main result of [Boy23a]. More crucially the previous filtration should be strict, i.e. its graded parts have to be torsion free, cf. theorem 2.4.3.

It appears that these graded parts are parabolically induced and in the limit case when the order  $q_v$  of the residue field is such that  $q_v \equiv 1 \pmod{l}$ , the socle of these parabolic induced representations are no more irreducible and do not fulfill the Ihara property. It then appears that we have:

- first to verify that the first non trivial graded part of our filtration verifies the genericity property of its irreducible submodule. For this we need a level raising statement given in [Gee11], cf. theorem 3.1.2.
- Then we have to understand that the extensions between the graded parts of our filtration, are non split.

The idea here is to work with the Galois representation  $\rho_{\mathfrak{m}}$  with values in the localized Hecke algebra  $\mathbb{T}(K^v(\infty))_{\mathfrak{m}}$  and more precisely with  $\rho_{\mathfrak{m}} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l$  which

should be as far as possible from being semi-simple, i.e. the graded parts of its socle filtration should be irreducible. Over  $\overline{\mathbb{Q}}_l$ ,  $\rho_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  is equipped with the action of the nilpotent monodromy operator  $N_v$  coming from a monodromy operator acting on  $\Psi_{K^v(\infty),v}$  constructed from the fact that any  $\overline{\mathbb{Q}}_l$ -representation with finite dimension is quasi-unipotent.

Over  $\overline{\mathbb{F}}_l$  such an arithmetic approach is hopeless because, up to consider a finite extension of  $F_v$ , such a  $\overline{\mathbb{F}}_l$ -representation has a trivial action of the inertia group. We are then looking for a geometric definition of  $N_v$  which exists in the semi-stable reduction case which corresponds to the case where the level at  $v$  of our Shimura variety is of Iwahori type which then corresponds to focus on automorphic representations  $\Pi$  such that the cuspidal support of  $\Pi_v$  is made of characters. In other words, we then tackle the weak form of Ihara's lemma of definition 1.2.3.

Our geometric monodromy operator then gives us a cohomological monodromy operator  $N_{v,\mathfrak{m}}^{coho}$  acting on  $H^0(\mathrm{Sh}_{K,\overline{s}_v}, \Psi_{K^v(\infty),v})_{\mathfrak{m}}$  as soon as the irreducible constituents of  $\overline{\rho}_{\mathfrak{m}}$  are characters. One of the main point, cf theorem 2.4.3, is that the graded parts of the filtration of  $H^0(\mathrm{Sh}_{K,\overline{s}_v}, \Psi_{K^v(\infty),v})_{\mathfrak{m}}$  induced by the nearby cycles spectral sequence, are all torsion free, so that in particular we perfectly understand the action of  $\overline{N}_{v,\mathfrak{m}}^{coho} := N_{v,\mathfrak{m}}^{coho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$  on  $H^0(\mathrm{Sh}_{K,\overline{s}_v}, \Psi_{K^v(\infty),v})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ , see *Main observation* at the end of §3.2.

Finally using typicity of  $H^0(\mathrm{Sh}_{K,\overline{s}_v}, \Psi_{K^v(\infty),v})_{\mathfrak{m}}$  in the sense of definition 3.3.2, we infer for every irreducible sub-quotient  $\varrho$  of  $\overline{\rho}_{\mathfrak{m}}$ , a monodromy operator  $N_{\mathfrak{m},\varrho}$  on  $\rho_{\mathfrak{m}}$ . We then consider the image of  $N_{\mathfrak{m},\varrho}^{r-1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$  where  $r$  is the multiplicity of  $\varrho$  in  $\overline{\rho}_{\mathfrak{m}}$ :

- From the theorem of Gee and the *main observation*, we know both that this image is non zero
- and it is associated to  $\varrho$ -generic representations.

Varying  $\varrho$  we infer genericness of irreducible quotients of  $H^0(\mathrm{Sh}_{K,\overline{s}_v}, \Psi_{K^v(\infty),v})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$  and we conclude about its irreducible subspaces through Grothendieck-Verdier duality as in proposition 3.7 (3) of [NT16].

## 2 Preliminaries

### 2.1 Representations of $\mathrm{GL}_d(L)$

Consider a finite extension  $L/\mathbb{Q}_p$  with residue field  $\mathbb{F}_q$ . We denote by  $|\cdot|$  its absolute value. For a representation  $\pi$  of  $\mathrm{GL}_d(L)$  and  $n \in \frac{1}{2}\mathbb{Z}$ , set

$$\pi\{n\} := \pi \otimes q^{-n \mathrm{val}_{\mathrm{det}}}.$$

**Notation 2.1.1.** For  $\pi_1$  and  $\pi_2$  representations of respectively  $\mathrm{GL}_{n_1}(L)$  and  $\mathrm{GL}_{n_2}(L)$ , we will denote by

$$\pi_1 \times \pi_2 := \mathrm{ind}_{P_{n_1, n_1+n_2}(L)}^{\mathrm{GL}_{n_1+n_2}(L)} \pi_1\left\{\frac{n_2}{2}\right\} \otimes \pi_2\left\{-\frac{n_1}{2}\right\},$$



the normalized parabolic induced representation where for any sequence  $\underline{r} = (0 < r_1 < r_2 < \dots < r_k = d)$ , we write  $P_{\underline{r}}$  for the standard parabolic subgroup of  $\mathrm{GL}_d$  with Levi

$$\mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2-r_1} \times \dots \times \mathrm{GL}_{r_k-r_{k-1}}.$$

Recall that a representation  $\rho$  of  $\mathrm{GL}_d(L)$  is called *cuspidal* (resp. *supercuspidal*) if it is not a subspace (resp. subquotient) of a proper parabolic induced representation. When the field of coefficients is of characteristic zero, these two notions coincides, but this is no more true over  $\overline{\mathbb{F}}_l$ .

**Definition 2.1.2.** (see [Zel80] §9 and [Boy10] §1.4) Let  $g$  be a divisor of  $d = sg$  and  $\pi$  an irreducible cuspidal  $\overline{\mathbb{Q}}_l$ -representation of  $\mathrm{GL}_g(L)$ . The induced representation

$$\pi\left\{\frac{1-s}{2}\right\} \times \pi\left\{\frac{3-s}{2}\right\} \times \dots \times \pi\left\{\frac{s-1}{2}\right\} \quad (1)$$

holds a unique irreducible quotient (resp. subspace) denoted  $\mathrm{st}_s(\pi)$  (resp.  $\mathrm{Speh}_s(\pi)$ ); it is a generalized Steinberg (resp. Speh) representation. Their cuspidal support is the Zelevinsky segment

$$[\pi\left\{\frac{1-s}{2}\right\}, \pi\left\{\frac{s-1}{2}\right\}] := \left\{ \pi\left\{\frac{1-s}{2}\right\}, \pi\left\{\frac{3-s}{2}\right\}, \dots, \pi\left\{\frac{s-1}{2}\right\} \right\}.$$

More generally the set of sub-quotients of the induced representation (1) is in bijection with the following set.

$$\mathrm{Dec}(s) = \{(t_1, \dots, t_r), \text{ such that } t_i \geq 1 \text{ and } \sum_{i=1}^r t_i = s\}.$$

For any  $\underline{s} \in \mathrm{Dec}(s)$ , we denote by  $\mathrm{st}_{\underline{s}}(\pi)$  the associated irreducible subquotient of (1). Following Zelevinsky, we fix this bijection such that  $\mathrm{Speh}_{\underline{s}}(\pi)$  corresponds to  $(s)$  and  $\mathrm{st}_{\underline{s}}(\pi)$  to  $(1, \dots, 1)$ . The Lubin-Tate representation

$LT_{h,s}(\pi)$  will also appear in the following, it corresponds with  $(\overbrace{1, \dots, 1}^h, s-h)$ .

**Proposition 2.1.3.** (cf. [Vig96] III.5.10) Let  $\pi$  be an irreducible cuspidal representation of  $\mathrm{GL}_g(K)$  with a stable  $\overline{\mathbb{Z}}$ -lattice<sup>1</sup>, then its modulo  $l$  reduction is irreducible and cuspidal (but not necessary supercuspidal).

We now suppose as explained in the introduction that

$$q \equiv 1 \pmod{l} \quad \text{and} \quad l > d$$

so the following facts are verified:

- the modulo  $l$  reduction of every irreducible cuspidal representation of  $\mathrm{GL}_g(L)$  for  $g \leq d$ , is supercuspidal.

---

<sup>1</sup>We say that  $\pi$  is integral.

- For a  $\overline{\mathbb{F}}_l$ -irreducible supercuspidal representation  $\varrho$  of  $\mathrm{GL}_g(L)$ , the parabolic induced representation  $\varrho \times \cdots \times \varrho$ , with  $s$  copies of  $\varrho$ , is semi-simple with irreducible constituents the modulo  $l$  reduction of  $\mathrm{st}_{\underline{s}}(\pi)$  for  $\underline{s} \in \mathrm{Dec}(s)$  and  $\pi$  any cuspidal representation whose modulo  $l$  reduction is isomorphic to  $\varrho$ .

Concerning the notion of genericity, consider the mirabolic subgroup  $M_d(L)$  of  $\mathrm{GL}_d(L)$  as the set of matrices with last row  $(0, \dots, 0, 1)$ : we denote by

$$V_d(L) = \{(m_{i,j}) \in M_d(L) : m_{i,j} = \delta_{i,j} \text{ for } j < d\}.$$

its unipotent radical. We fix a non trivial character  $\psi$  of  $L$  and let  $\theta$  be the character of  $V_d(L)$  defined by  $\theta((m_{i,j})) = \psi(m_{d-1,d})$ . For  $G = \mathrm{GL}_r(L)$  or  $M_r(L)$ , we denote by  $\mathrm{alg}(G)$  the abelian category of smooth representations of  $G$  and, following [BZ77], we introduce

$$\Psi^- : \mathrm{alg}(M_d(L)) \longrightarrow \mathrm{alg}(\mathrm{GL}_{d-1}(L)),$$

and

$$\Phi^- : \mathrm{alg}(M_d(L)) \longrightarrow \mathrm{alg}(M_{d-1}(L)),$$

defined by  $\Psi^- = r_{V_d,1}$  (resp.  $\Phi^- = r_{V_d,\theta}$ ) the functor of  $V_d$  coinvariants (resp.  $(V_d, \theta)$ -coinvariants), cf. [BZ77] 1.8. For  $\tau \in \mathrm{alg}(M_d(L))$ , the representation

$$\tau^{(k)} := \Psi^- \circ (\Phi^-)^{k-1}(\tau)$$

is called the  $k$ -th derivative of  $\tau$ . If  $\tau^{(k)} \neq 0$  and  $\tau^{(m)} = 0$  for all  $m > k$ , then  $\tau^{(k)}$  is called the highest derivative of  $\tau$ . In the particular case where  $k = d$ , there is a unique irreducible representation  $\tau_{nd}$  of  $M_d(L)$  with derivative of order  $d$ .

**Definition 2.1.4.** An irreducible representation  $\pi$  of  $\mathrm{GL}_d(L)$  is said generic, if its restriction to the mirabolic subgroup admits  $\tau_{nd}$  as a subquotient.

Let  $\pi$  be an irreducible generic representation of  $\mathrm{GL}_d(L)$  and consider any stable lattice which gives us by modulo  $l$  reduction a  $\overline{\mathbb{F}}_l$ -representation uniquely determined up to semi-simplification. Then this modulo  $l$  reduction admits a unique generic irreducible constituent.

## 2.2 Weil–Deligne inertial types

Recall that a Weil-Deligne representation of  $W_L$  is a pair  $(r, N)$  where

- $r : W_L \longrightarrow \mathrm{GL}(V)$  is a smooth<sup>2</sup> representation on a finite dimensional vector space  $V$ ; and

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<sup>2</sup>i.e. continuous for the discrete topology on  $V$

- $N \in \text{End}(V)$  is nilpotent such that

$$r(g)Nr(g)^{-1} = \|g\|N,$$

where  $\|\bullet\| : W_L \longrightarrow W_L/I_L \rightarrow q^{\mathbb{Z}}$  takes an arithmetic Frobenius element to  $q$ .

*Remark.* To a continuous<sup>3</sup> representation on a finite dimensional  $\mathbb{Q}_l$ -vector space  $V$ ,  $\rho : W_L \longrightarrow \text{GL}(V)$  is attached a Weil-Deligne representation denoted by  $\text{WD}(\rho)$ . A Weil representation of  $W_L$  is also said of Galois type if it comes from a representation of  $G_L$ .

*Main example:* let  $\rho : W_L \longrightarrow \text{GL}(V)$  be a smooth irreducible representation on a finite dimensional vector space  $V$ . For  $k \geq 1$  an integer, we then define a Weil-Deligne representation

$$\text{Sp}(\rho, k) := (V \oplus V(1) \oplus \cdots \oplus V(k-1), N),$$

where for  $0 \leq i \leq k-2$ , the isomorphism  $N : V(i) \cong V(i+1)$  is induced by some choice of a basis of  $\bar{L}(1)$  and  $N|_{V(k-1)}$  is zero. Then every Frobenius semi-simple Weil-Deligne representation of  $W_L$  is isomorphic to some  $\bigoplus_{i=1}^r \text{Sp}(\rho_i, k_i)$ , for smooth irreducible representations  $\rho_i : W_L \longrightarrow \text{GL}(V_i)$  and integers  $k_i \geq 1$ . Up to obvious reorderings, such a writing is unique.

Let now  $\rho$  be a continuous representation of  $W_L$ , or its Weil-Deligne representation  $\text{WD}(\rho)$ , and consider its restriction to  $I_L$ ,  $\tau := \rho|_{I_L}$ . Such an isomorphism class of a finite dimensional continuous representation of  $I_L$  is then called *an inertial type*.

**Notation 2.2.1.** Let  $\mathcal{I}_0$  the set of inertial types that extend to a continuous irreducible representation of  $G_L$ .

*Remark.*  $\tau \in \mathcal{I}_0$  might not be irreducible.

Let Part be the set of decreasing sequences of positive integers  $\underline{d} = (\underline{d}(1) \geq \underline{d}(2) \geq \cdots)$  viewed as a partition of  $\sum \underline{d} := \sum_i \underline{d}(i)$ .

**Notation 2.2.2.** Let  $f : \mathcal{I}_0 \longrightarrow \text{Part}$  with finite support. We then denote by  $\tau_f$  the restriction to  $I_L$  of

$$\bigoplus_{\tau_0 \in \mathcal{I}_0} \bigoplus_i \text{Sp}(\rho_{\tau_0}, f(\tau_0)(i)),$$

where  $\rho_{\tau_0}$  is a fixed extension of  $\tau_0$  to  $W_L$ .

*Remark.* By lemma 3.3 of [MS] the isomorphism class of  $\tau_f$  is independent of the choices of the  $\rho_{\tau_0}$ .

The map from  $\{f : \mathcal{I}_0 \longrightarrow \text{Part}\}$  to the set of inertial types given by  $f \mapsto \tau_f$ , is a bijection. The dominance order  $\preceq$  on Part induces a partial order on the set of inertial types.

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<sup>3</sup>relatively to the  $l$ -adic topology on  $V$

We let  $\text{rec}_L$  denote the local reciprocity map of [HT01, Theorem A]. Fix an isomorphism  $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ . We normalize the local reciprocity map  $\text{rec}$  of [HT01, Theorem A], defined on isomorphism classes of irreducible smooth representations of  $\text{GL}_n(L)$  over  $\mathbb{C}$  as follows: if  $\pi$  is the isomorphism class of an irreducible smooth representation of  $\text{GL}_n(L)$  over  $\overline{\mathbb{Q}}_\ell$ , then

$$\rho_\ell(\pi) \stackrel{\text{def}}{=} \iota^{-1} \circ \text{rec}_L \circ \iota(\pi \otimes_{\overline{\mathbb{Q}}_\ell} |\det|^{(1-n)/2}).$$

Then  $\rho_\ell(\pi)$  is the isomorphism class of an  $n$ -dimensional, Frobenius semisimple Weil–Deligne representation of  $W_L$  over  $\overline{\mathbb{Q}}_\ell$ , independent of the choice of  $\iota$ . Moreover, if  $\rho$  is an isomorphism class of an  $n$ -dimensional, Frobenius semisimple Weil–Deligne representation of  $W_L$  over  $M$ , then  $\rho_\ell^{-1}(\rho)$  is defined over  $M$  (cf. [CEG<sup>+</sup>16, §1.8]).

Recall the following compatibility of the Langlands correspondence.

**Lemma 2.2.3.** *If  $\pi$  and  $\pi'$  are irreducible generic representations of  $\text{GL}_d(L)$  such that  $\rho_\ell(\pi)|_{I_L} \cong \rho_\ell(\pi')|_{I_L}$  then*

$$\pi|_{\text{GL}_d(\mathcal{O}_L)} \cong \pi'|_{\text{GL}_d(\mathcal{O}_L)}.$$

## 2.3 Kottwitz–Harris–Taylor Shimura varieties

Let  $F = F^+E$  be a CM field where  $E/\mathbb{Q}$  is a quadratic imaginary extension and  $F^+/\mathbb{Q}$  is totally real. We fix a real embedding  $\tau : F^+ \hookrightarrow \mathbb{R}$ . For a place  $v$  of  $F$ , we will denote by

- $F_v$  the completion of  $F$  at  $v$ ,
- $\mathcal{O}_v$  the ring of integers of  $F_v$ ,
- $\varpi_v$  a uniformizer,
- $q_v$  the cardinal of the residue field  $\kappa(v) = \mathcal{O}_v/(\varpi_v)$ .

Let  $B$  be a division algebra with center  $F$ , of dimension  $d^2$  such that at every place  $v$  of  $F$ , either  $B_v$  is split or a local division algebra and suppose  $B$  provided with an involution of second kind  $*$  such that  $*|_F$  is the complex conjugation. For any  $\beta \in B^{*-1}$ , denote by  $\sharp_\beta$  the involution  $v \mapsto v^\sharp_\beta = \beta v^* \beta^{-1}$  and let  $G/\mathbb{Q}$  be the group of similitudes, denoted by  $G_\tau$  in [HT01], defined for every  $\mathbb{Q}$ -algebra  $R$  by

$$G(R) \cong \{(\lambda, g) \in R^\times \times (B^{op} \otimes_{\mathbb{Q}} R)^\times \text{ such that } gg^\sharp_\beta = \lambda\}$$

with  $B^{op} = B \otimes_{F,c} F$ . If  $x$  is a place of  $\mathbb{Q}$  split  $x = yy^c$  in  $E$  then

$$G(\mathbb{Q}_x) \cong (B_y^{op})^\times \times \mathbb{Q}_x^\times \cong \mathbb{Q}_x^\times \times \prod_{v_i^+} (B_{v_i^+}^{op})^\times, \quad (2)$$

where  $x = \prod_i v_i^+$  in  $F^+$  and we identify places of  $F^+$  over  $x$  with places of  $F$  over  $y$ .

**Convention 2.3.1.** For  $x = yy^c$  a place of  $\mathbb{Q}$  split in  $M$  and  $v$  a place of  $F$  over  $y$ , we shall make throughout the text the following abuse of notation: we denote  $G(F_v)$  the factor  $(B_{v|F^+}^{op})^\times$  in the formula (2) so that

$$G(\mathbb{A}_{\mathbb{Q}}^{\infty,v}) := G(\mathbb{A}_{\mathbb{Q}}^{\infty,p}) \times \left( \mathbb{Q}_p^\times \times \prod_{v_i^+ \neq v|F^+} (B_{v_i^+}^{op})^\times \right).$$

In [HT01], the authors justify the existence of some  $G$  like before such that moreover

- if  $x$  is a place of  $\mathbb{Q}$  non split in  $M$  then  $G(\mathbb{Q}_x)$  is quasi split;
- the invariants of  $G(\mathbb{R})$  are  $(1, d-1)$  for the embedding  $\tau$  and  $(0, d)$  for the others.

As in [HT01, page 90], a compact open subgroup  $K$  of  $G(\mathbb{A}_{\mathbb{Q}}^\infty)$  is said to be *sufficiently small* if there exists a place  $x$  of  $\mathbb{Q}$  such that the projection from  $K^x$  to  $G(\mathbb{Q}_x)$  does not contain any element of finite order except identity.

**Notation 2.3.2.** Denote by  $\mathcal{K}$  the set of sufficiently small compact open subgroups of  $G(\mathbb{A}^\infty)$ . For  $K \in \mathcal{K}$ , write  $\text{Sh}_{K,\eta} \rightarrow \text{Spec } F$  for the associated Shimura variety of Kottwitz-Harris-Taylor type.

**Definition 2.3.3.** Denote by  $\text{Spl}$  the set of places  $w$  of  $F$  such that  $p_w := w|_{\mathbb{Q}} \neq l$  is split in  $E$  and  $B_w^\times \cong \text{GL}_d(F_w)$ . For each  $K \in \mathcal{K}$ , we write  $\text{Spl}(K)$  for the subset of  $\text{Spl}$  of places such that  $K_v$  is the standard maximal compact of  $\text{GL}_d(F_v)$ .

In the sequel, we fix a place  $v$  of  $F$  in  $\text{Spl}$ . The scheme  $\text{Sh}_{K,\eta}$  has a projective model  $\text{Sh}_{K,v}$  over  $\text{Spec } \mathcal{O}_v$  with special geometric fiber  $\text{Sh}_{K,\bar{s}_v}$ . We have a projective system  $(\text{Sh}_{K,\bar{s}_v})_{K \in \mathcal{K}}$  which is naturally equipped with an action of  $G(\mathbb{A}_{\mathbb{Q}}^\infty) \times \mathbb{Z}$  such that any  $w_v \in W_{F_v}$  acts by  $-\deg(w_v) \in \mathbb{Z}$ , where  $\deg = \text{val} \circ \text{Art}_{F_v}^{-1}$  and  $\text{Art}_{F_v} : F_v^\times \xrightarrow{\sim} W_{F_v}^{ab}$ .

**Notation 2.3.4.** For  $K \in \mathcal{K}$ , the Newton stratification of the geometric special fiber  $\text{Sh}_{K,\bar{s}_v}$  is denoted by

$$\text{Sh}_{K,\bar{s}_v} =: \text{Sh}_{K,\bar{s}_v}^{\geq 1} \supset \text{Sh}_{K,\bar{s}_v}^{\geq 2} \supset \cdots \supset \text{Sh}_{K,\bar{s}_v}^{\geq d}$$

where  $\text{Sh}_{K,\bar{s}_v}^{\leq h} := \text{Sh}_{K,\bar{s}_v}^{\geq h} - \text{Sh}_{K,\bar{s}_v}^{\geq h+1}$  is an affine scheme, which is smooth and pure of dimension  $d-h$ . It is built up by the geometric points such that the connected part of the associated Barsotti–Tate group has rank  $h$ . For each  $1 \leq h < d$ , write

$$i_h : \text{Sh}_{K,\bar{s}_v}^{\geq h} \hookrightarrow \text{Sh}_{K,\bar{s}_v}^{\geq 1}, \quad j^{\geq h} : \text{Sh}_{K,\bar{s}_v}^{\leq h} \hookrightarrow \text{Sh}_{K,\bar{s}_v}^{\geq h},$$

and  $j^=h = i_h \circ j^{\geq h}$ .

For  $n \geq 1$ , with our previous abuse of notation, consider  $K^v(n) := K^v K_v(n)$  where

$$K_v(n) := \ker(\mathrm{GL}_d(\mathcal{O}_v) \rightarrow \mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n)).$$

Recall that  $\mathrm{Sh}_{I^v(n), \bar{s}_v}^{\bar{h}}$  is geometrically induced under the action of the parabolic subgroup  $P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$ , defined as the stabilizer of the first  $h$  vectors of the canonical basis of  $F_v^d$ . Concretely this means there exists a closed subscheme  $\mathrm{Sh}_{K^v(n), \bar{s}_v, 1}^{\bar{h}}$  stabilized by the Hecke action of  $P_{h,d}(F_v)$  and such that

$$\mathrm{Sh}_{K^v(n), \bar{s}_v}^{\bar{h}} = \mathrm{Sh}_{K^v(n), \bar{s}_v, 1}^{\bar{h}} \times_{P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)} \mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n),$$

meaning that  $\mathrm{Sh}_{K^v(n), \bar{s}_v}^{\bar{h}}$  is the disjoint union of copies of  $\mathrm{Sh}_{K^v(n), \bar{s}_v, 1}^{\bar{h}}$  indexed by  $\mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n)/P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$  and exchanged by the action of  $\mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n)$ . We will denote by  $\mathrm{Sh}_{K^v(n), \bar{s}_v, 1}^{\geq \bar{h}}$  the closure of  $\mathrm{Sh}_{K^v(n), \bar{s}_v, 1}^{\bar{h}}$  inside  $\mathrm{Sh}_{K^v(n), \bar{s}_v}$ .

**Notation 2.3.5.** Let  $1 \leq h \leq d$  and  $\Pi_h$  any representation of  $\mathrm{GL}_h(F_v)$ . For  $\chi_v$  a character of  $F_v^\times$ , we then denote by

$$\widetilde{HT}_1(\chi_v, \Pi_h) := \mathcal{L}(\chi_v, t)_1 \otimes \Pi_h^{K_v(n)} \otimes \Xi^{\frac{h-d}{2}}$$

the Harris-Taylor local system on the Newton stratum  $\mathrm{Sh}_{K^v(n), \bar{s}_v, 1}^{\bar{h}}$  where

- $\mathcal{L}(\chi_v, t)_1$  is the constant sheaf  $\overline{\mathbb{Z}}_l$  where the fundamental group acts through

$$\pi_1 \longrightarrow \mathcal{D}_{v,h}^\times \xrightarrow{\chi_v} \mathcal{O}_v^\times$$

where  $\mathcal{D}_{v,h}$  is the maximal order of the division algebra  $D_{v,h}/F_v$  with invariant  $1/h$ , and the first surjection is given by the Igusa varieties of [HT01];

- $\Xi : \frac{1}{2}\mathbb{Z} \rightarrow \overline{\mathbb{Z}}_l^\times$  is defined by  $\Xi(\frac{1}{2}) = q^{1/2}$ .

We also introduce the induced version

$$\widetilde{HT}(\chi_v, \Pi_h) := \left( \mathcal{L}(\chi_v, t)_1 \otimes \Pi_h^{K_v(n)} \otimes \Xi^{\frac{h-d}{2}} \right) \times_{P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)} \mathrm{GL}_d(\mathcal{O}_v/\mathcal{M}_v^n),$$

where the unipotent radical of  $P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n)$  acts trivially and the action of

$$(g^{\infty, v}, \begin{pmatrix} g_v^c & * \\ 0 & g_v^{et} \end{pmatrix}, \sigma_v) \in G(\mathbb{A}^{\infty, v}) \times P_{h,d}(\mathcal{O}_v/\mathcal{M}_v^n) \times W_v$$

is given

- by the action of  $g_v^c$  on  $\Pi_h^{K_v(n)}$  and  $\deg(\sigma_v) \in \mathbb{Z}$  on  $\Xi^{\frac{h-d}{2}}$ , and
- the action of  $(g^{\infty, v}, g_v^{et}, \mathrm{val}(\det g_v^c) - \deg \sigma_v) \in G(\mathbb{A}^{\infty, v}) \times \mathrm{GL}_{d-h}(\mathcal{O}_v/\mathcal{M}_v^n) \times \mathbb{Z}$  on  $\mathcal{L}_{\overline{\mathbb{Q}}_l}(\chi_v)_1 \otimes \Xi^{\frac{h-d}{2}}$ .

We also introduce

$$HT(\chi_v, \Pi_h)_1 := \widetilde{HT}(\chi_v, \Pi_h)_1[d - tg],$$

and the perverse sheaf

$$P(h, \chi_v)_1 := {}^p j_{1,!}^h HT(\chi_v, \text{St}_h(\chi_v))_1 \otimes \chi_v^{-1},$$

and their induced version,  $HT(\chi_v, \Pi_h)$  and  $P(h, \chi_v)$ .

Note that over  $\overline{\mathbb{Z}}_l$ , there are at least two notions of intermediate extension associated to the two classical  $t$ -structures  $p$  and  $p+$ . However in our situation they all coincide. Indeed as  $\text{Sh}_{K^v(n), \overline{s}_v, 1}^{\geq h}$  is smooth over  $\text{Spec } \overline{\mathbb{F}}_p$ , then  $HT(\chi_v, \Pi_h)_1$  is perverse for the two  $t$ -structures with

$$i_1^{h \leq +1, *} HT(\chi_v, \Pi_h)_1 \in {}^p \mathcal{D}^{<0} \text{ and } i_1^{h \leq +1, !} HT(\chi_v, \Pi_h)_1 \in {}^{p+} \mathcal{D}^{\geq 1}.$$

Let now denote by

$$\Psi_{K,v} := R\Psi_{\eta_v}(\overline{\mathbb{Z}}_l[d-1])\left(\frac{d-1}{2}\right)$$

the nearby cycles autodual free perverse sheaf on  $\text{Sh}_{K, \overline{s}_v}$ . Recall, cf. [Boy23b] proposition 3.1.3, that

$$\Psi_{K,v} \cong \bigoplus_{1 \leq g \leq d} \bigoplus_{\varrho \in \text{Scusp}(g)} \Psi_{K,\varrho}, \quad (3)$$

where

- $\text{Scusp}(g)$  is the set of equivalence classes of irreducible supercuspidal representations of  $\text{GL}_g(F_v)$ .
- The irreducible sub-quotients of  $\Psi_{K,\varrho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  are the Harris-Taylor perverse sheaves of  $\Psi_{K, \overline{\mathbb{Q}}_l}$  associated to irreducible cuspidal representations  $\pi_v$  with modulo  $l$  reduction having supercuspidal support a Zelevinsky segment associated to  $\varrho$ .

*Remark.* In the limit case when  $q_v \equiv 1 \pmod{l}$  and  $l > d$ , recall that we do not have to bother about cuspidal  $\overline{\mathbb{F}}_l$ -representation which are not supercuspidal. In particular in the previous formula we can

- replace  $\text{Scusp}(g)$  by the set  $\text{Cusp}(g)$  of equivalence classes of cuspidal representations,
- and the Harris-Taylor perverse sheaves of  $\Psi_{K,\varrho} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  are those associated to  $\pi_v$  such that its modulo  $l$  reduction is isomorphic to  $\varrho$ .

## 2.4 Filtrations of stratification

Using the Newton stratification and following the constructions of [Boy19], we can define a  $\overline{\mathbb{Z}}_l$ -filtration

$$\mathrm{Fil}_!^0(\Psi_{K,\varrho}) \hookrightarrow \cdots \hookrightarrow \mathrm{Fil}_!^d(\Psi_{K,\varrho}) = \Psi_{K,\varrho}$$

where each graded part  $\mathrm{gr}_!^k(\Psi_{K,\varrho})$  admits a filtration

$$\mathrm{Fil}^{-d}(\mathrm{gr}_!^k(\Psi_{K,\varrho})) \hookrightarrow \cdots \hookrightarrow \mathrm{Fil}^{-k}(\mathrm{gr}_!^k(\Psi_{K,\varrho})) = \mathrm{gr}_!^k(\Psi_{K,\varrho})$$

with

$$\mathrm{gr}^{-i}(\mathrm{gr}_!^k(\Psi_{K,\varrho})) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong \bigoplus_{\pi_v \in \mathrm{Cusp}(\varrho)} P(i, \pi_v) \left( \frac{i+1-2k}{2} \right),$$

where  $\mathrm{Cusp}(\varrho)$  is the set of equivalence classes of irreducible cuspidal representations with modulo  $l$  reduction isomorphic to  $\varrho$ . We then have spectral sequences

$$E_1^{p,q} = H^{p+q}(\mathrm{Sh}_{K,\overline{s}_v}, \mathrm{gr}^{-p}(\mathrm{gr}_!^k(\Psi_{K,\varrho}))) \Rightarrow H^{p+q}(\mathrm{Sh}_{K,\overline{s}_v}, \mathrm{gr}_!^k(\Psi_{K,\varrho})), \quad (4)$$

and

$$E_1^{p,q} = H^{p+q}(\mathrm{Sh}_{K,\overline{s}_v}, \mathrm{gr}_!^{-p}(\Psi_{K,\varrho})) \Rightarrow H^{p+q}(\mathrm{Sh}_{K,\overline{s}_v}, \Psi_{K,\varrho}). \quad (5)$$

**Definition 2.4.1.** For a finite set  $S$  of places of  $\mathbb{Q}$  containing the places where  $G$  is ramified, denote by  $\mathbb{T}_{abs}^S := \prod'_{x \notin S} \mathbb{T}_{x,abs}$  the abstract unramified Hecke algebra where  $\mathbb{T}_{x,abs} \cong \overline{\mathbb{Z}}_l[X^{un}(T_x)]^{W_x}$  for  $T_x$  a split torus,  $W_x$  the spherical Weyl group and  $X^{un}(T_x)$  the set of  $\overline{\mathbb{Z}}_l$ -unramified characters of  $T_x$ .

*Example.* For  $w \in \mathrm{Spl}$ , we have

$$\mathbb{T}_{v|\mathbb{Q},abs} = \overline{\mathbb{Z}}_l[T_{v',i} : i = 1, \dots, d, v'|(v|\mathbb{Q})]$$

where  $T_{v',i}$  is the characteristic function of

$$\mathrm{GL}_d(\mathcal{O}_{v'}) \mathrm{diag}(\overbrace{\varpi_{v'}, \dots, \varpi_{v'}}^i, \overbrace{1, \dots, 1}^{d-i}) \mathrm{GL}_d(\mathcal{O}_{v'}) \subseteq \mathrm{GL}_d(F_{v'}).$$

Recall that  $\mathbb{T}_{abs}^S$  acts through correspondances on each of the  $H^i(\mathrm{Sh}_{K,\overline{\eta}}, E)$  where  $K \in \mathcal{K}$  is maximal at each places outside  $S$ .

**Notation 2.4.2.** We denote by  $\mathbb{T}(K)$  the image of  $\mathbb{T}_{abs}^S$  inside  $\mathrm{End}_{\overline{\mathbb{Z}}_l}(H^{d-1}(\mathrm{Sh}_{K,\overline{\eta}}, \overline{\mathbb{Z}}_l))$ .

We also denote

$$H^{d-1}(\mathrm{Sh}_{K^v(\infty),\overline{\eta}}, \overline{\mathbb{Z}}_l) := \varinjlim_{K_v} H^{d-1}(\mathrm{Sh}_{K^v K_v, \overline{\eta}}, \overline{\mathbb{Z}}_l),$$

where  $K_v$  describe the set of open compact subgroup of  $\mathrm{GL}_d(\mathcal{O}_v)$ . We also use similar notation for others cohomology groups.



**Theorem 2.4.3.** *Let  $\mathfrak{m}$  be a maximal ideal of  $\mathbb{T}(K^v(\infty))$  such that  $\bar{\rho}_{\mathfrak{m}}$  is irreducible and the irreducible constituents of its restriction to the decomposition group at the place  $v$  are characters. Recall also that  $q_v \equiv 1 \pmod{l}$  and  $l > d$ . Then*

- (i)  $H^i(\mathrm{Sh}_{K,\bar{\eta}},\bar{\mathbb{Z}}_l)_{\mathfrak{m}}$  is zero if  $i \neq d-1$  and otherwise torsion free.
- (ii) Moreover the spectral sequences (4) and (5), localized at  $\mathfrak{m}$ , degenerate at  $E_1$  and the  $E_{1,\mathfrak{m}}^{p,q}$  are zero for  $p+q \neq 0$  and otherwise torsion free.

*Proof.* (i) It is the main theorem of [Boy23a].

(ii) From (3) we are led to study the initial terms of the spectral sequence given by the filtration of  $\Psi_{K,\varrho}$  for  $\varrho$  a character which is a constituent of  $\bar{\rho}_{\mathfrak{m},v}$ . Recall also, as we are in the limit case, that

- as there do not exist irreducible  $\bar{\mathbb{Q}}_l$ -cuspidal representation of  $\mathrm{GL}_g(F_v)$  for  $g \leq d$  with modulo  $l$  reduction not supercuspidal, the irreducible constituents of  $\Psi_{K,\varrho} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l$  are the Harris-Taylor perverse sheaves  $P(h, \chi_v)(\frac{h-1-2k}{2})$  where the modulo  $l$  reduction of  $\chi_v$  is isomorphic to  $\varrho$  and  $0 \leq k < h$ .
- Over  $\bar{\mathbb{Z}}_l$ , we do not have to worry about the difference between  $p$  and  $p+$  intermediate extensions.

From [Boy23b] §2.3, consider the following equivariant resolution

$$\begin{aligned} 0 \rightarrow j_!^{-d} HT(\chi_v, \Pi_h\{\frac{h-d}{2}\} \times \mathrm{Speh}_{d-h}(\chi_v\{h/2\})) \otimes \Xi^{\frac{d-h}{2}} \rightarrow \dots \\ \rightarrow j_!^{h+1} HT(\chi_v, \Pi_h\{-1/2\} \times \chi_v\{h/2\}) \otimes \Xi^{\frac{1}{2}} \rightarrow \\ j_!^h HT(\chi_v, \Pi_h) \rightarrow {}^p j_{!*}^h HT(\chi_v, \Pi_h) \rightarrow 0, \end{aligned} \quad (6)$$

where  $\Pi_h$  is any representation of  $\mathrm{GL}_h(F_v)$ , also called the infinitesimal part of the perverse sheaf  ${}^p j_{!*}^h HT(\chi_v, \Pi_h)$ .<sup>4</sup>

By adjunction property, for  $1 \leq \delta \leq d-h$ , the map

$$\begin{aligned} j_!^{h+\delta} HT(\chi_v, \Pi_h\{\frac{-\delta}{2}\} \times \mathrm{Speh}_{\delta}(\chi_v\{h/2\})) \otimes \Xi^{\delta/2} \\ \rightarrow j_!^{h+\delta-1} HT(\chi_v, \Pi_h\{\frac{1-\delta}{2}\} \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}} \end{aligned} \quad (7)$$

is given by

$$\begin{aligned} HT(\chi_v, \Pi_h\{\frac{-\delta}{2}\} \times \mathrm{Speh}_{\delta}(\chi_v\{h/2\})) \otimes \Xi^{\delta/2} \rightarrow \\ j_!^{h+\delta,*} ({}^p j_!^{h+\delta-1} HT(\chi_v, \Pi_h\{\frac{1-\delta}{2}\} \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}) \end{aligned} \quad (8)$$

To compute this last term we use the resolution (6) for  $h+\delta-1$ . Precisely denote by  $\mathcal{H} := HT(\chi_v, \mathrm{st}_h(\chi\{\frac{1-\delta}{2}\}) \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}}$ , and write the previous resolution for  $h+\delta-1$  as follows

$$0 \rightarrow K \rightarrow j_!^{h+\delta} \mathcal{H}' \rightarrow Q \rightarrow 0,$$

---

<sup>4</sup>In  $P(h, \chi_v)$  the infinitesimal part  $\Pi_h$  is  $\mathrm{st}_h(\chi_v)$ .

$$0 \rightarrow Q \rightarrow j_!^{=h+\delta-1} \mathcal{H} \rightarrow p j_{!*}^{=h+\delta-1} \mathcal{H} \rightarrow 0,$$

with

$$\mathcal{H}' := HT\left(\chi_v, \Pi_h \left\{ \frac{1-\delta}{2} \right\} \times (\text{Speh}_{\delta-1}(\chi_v \{-1/2\}) \times \chi_v \left\{ \frac{\delta-1}{2} \right\}) \{h/2\}\right) \otimes \Xi^{\delta/2}.$$

As the support of  $K$  is contained in  $\text{Sh}_{\bar{I}, \bar{s}_v}^{\geq h+\delta+1}$  then  $p_i^{h+\delta, !} K = K$  and  $j^{=h+\delta, *} (p_i^{h+\delta, !} K)$  is zero. Moreover  $p_i^{h+\delta, !} (p j_{!*}^{=h+\delta-1} \mathcal{H})$  is zero by construction of the intermediate extension. We then deduce that

$$\begin{aligned} j^{=h+\delta, *} (p_i^{h+\delta, !} (j_!^{=h+\delta-1} HT(\chi_v, \Pi_h \left\{ \frac{1-\delta}{2} \right\} \times \text{Speh}_{\delta-1}(\chi_v \{h/2\}))) \otimes \Xi^{\frac{\delta-1}{2}}) \\ \cong HT\left(\chi_v, \Pi_h \left\{ \frac{1-\delta}{2} \right\} \right. \\ \left. \times (\text{Speh}_{\delta-1}(\chi_v \{-1/2\}) \times \chi_v \left\{ \frac{\delta-1}{2} \right\}) \{h/2\}\right) \otimes \Xi^{\delta/2} \quad (9) \end{aligned}$$

In particular, up to homothety, the map (8), and so (7), is unique. Finally as the maps of (6) are strict, the given maps (7) are uniquely determined, that is, if we forget the infinitesimal parts, these maps are independent of the chosen  $h$  in (6), i.e. only depends on  $h + \delta$ .

For every  $1 \leq h \leq d$ , let denote by  $i(h)$  the smallest index  $i$  such that  $H^i(\text{Sh}_{K, \bar{s}_v}, p j_{!*}^{=h} HT(\chi_v, \Pi_h)_{\mathfrak{m}})$  has non trivial torsion: if it does not exist then we set  $i(h) = +\infty$  and note that it does not depend on the choice of the infinitesimal part  $\Pi_h$ . By duality, as  $p j_{!*} = p^+ j_{!*}$  for Harris-Taylor local systems associated to characters, note that when  $i(h)$  is finite then  $i(h) \leq 0$ . Suppose by absurdity there exists  $h$  with  $i(h)$  finite and denote  $h_0$  the biggest such  $h$ .

**Lemma 2.4.4.** *For  $1 \leq h \leq h_0$  then  $i(h) = h - h_0$ .*

Note that a similar result is proved in [Boy17] when the level is maximal at  $v$ .

*Proof.* a) We first prove that for every  $h_0 < h \leq d$ , the cohomology groups of  $j_!^{=h} HT(\chi_v, \Pi_h)$  are torsion free. Consider the following strict filtration in the category of free perverse sheaves

$$\begin{aligned} (0) = \text{Fil}^{-1-d}(\chi_v, h) \hookrightarrow \text{Fil}^{-d}(\chi_v, h) \hookrightarrow \dots \\ \hookrightarrow \text{Fil}^{-h}(\chi_v, h) = j_!^{=h} HT(\chi_v, \Pi_h) \quad (10) \end{aligned}$$

where the symbol  $\hookrightarrow$  means a strict monomorphism, with graded parts

$$\text{gr}^{-k}(\chi_v, h) \cong p j_{!*}^{=k} HT(\chi_v, \Pi_h \left\{ \frac{h-k}{2} \right\} \otimes \text{st}_{k-h}(\chi_v \{h/2\})) \left( \frac{h-k}{2} \right).$$

Over  $\bar{\mathbb{Q}}_l$ , the result is proved in [Boy09] §4.3. Over  $\bar{\mathbb{Z}}_l$ , the result follows from the general constructions of [Boy14] and the fact that the  $p$  and  $p^+$  intermediate extensions are isomorphic for Harris-Taylor perverse sheaves associated to characters. The associated spectral sequence localized at  $\mathfrak{m}$ , is then concentrated in middle degree and torsion free which gives the claim.

b) Before watching the cases  $h \leq h_0$ , note that the spectral sequence associated to (6) for  $h = h_0 + 1$ , has all its  $E_1$  terms torsion free and degenerates at its  $E_2$  terms. As by hypothesis the aims of this spectral sequence is free and equals to only one  $E_2$  terms, we deduce that all the maps

$$\begin{aligned} H^0(\mathrm{Sh}_{I, \bar{s}_v, j_1^{=h+\delta}} HT_\xi(\chi_v, \mathrm{st}_h(\chi_v\{\frac{-\delta}{2}\}) \times \mathrm{Speh}_\delta(\chi_v\{h/2\})) \otimes \Xi^{\delta/2})_{\mathfrak{m}} \\ \longrightarrow \\ H^0(\mathrm{Sh}_{I, \bar{s}_v, j_1^{=h+\delta-1}} HT_\xi(\chi_v, \mathrm{st}_h(\chi_v\{\frac{1-\delta}{2}\}) \\ \times \mathrm{Speh}_{\delta-1}(\chi_v\{h/2\})) \otimes \Xi^{\frac{\delta-1}{2}})_{\mathfrak{m}} \end{aligned} \quad (11)$$

are saturated, i.e. their cokernel are free  $\bar{\mathbb{Z}}_l$ -modules. Then from the previous fact stressed after (9), this property remains true when we consider the associated spectral sequence for  $1 \leq h' \leq h_0$ .

c) Consider now  $h = h_0$  and the spectral sequence associated to (6) where

$$E_2^{p,q} = H^{p+2q}(\mathrm{Sh}_{I, \bar{s}_v, j_1^{=h+q}} HT_\xi(\chi_v, \mathrm{st}_h(\chi_v\{-q/2\}) \times \mathrm{Speh}_q(\chi_v\{h/2\})) \otimes \Xi^{\frac{q}{2}})_{\mathfrak{m}} \quad (12)$$

By definition of  $h_0$ , we know that some of the  $E_\infty^{p,-p}$  should have a non trivial torsion subspace. We saw that

- the contributions from the deeper strata are torsion free and
- $H^i(\mathrm{Sh}_{I, \bar{s}_v, j_1^{=h_0}} HT_\xi(\chi_v, \Pi_{h_0}))_{\mathfrak{m}}$  are zero for  $i < 0$  and is torsion free for  $i = 0$ , whatever is  $\Pi_{h_0}$ .
- Then there should exist a non strict map  $d_1^{p,q}$ . But, we have just seen that it can not be maps between deeper strata.
- Finally, using the previous points, the only possibility is that the cokernel of

$$\begin{aligned} H^0(\mathrm{Sh}_{I, \bar{s}_v, j_1^{=h_0+1}} HT_\xi(\chi_v, \mathrm{st}_{h_0}(\chi_v\{\frac{-1}{2}\}) \times \chi_v\{h_0/2\})) \otimes \Xi^{1/2})_{\mathfrak{m}} \\ \longrightarrow \\ H^0(\mathrm{Sh}_{I, \bar{s}_v, j_1^{=h_0}} HT_\xi(\chi_v, \mathrm{st}_{h_0}(\chi_v)))_{\mathfrak{m}} \end{aligned} \quad (13)$$

has a non trivial torsion subspace.

In particular we have  $i(h_0) = 0$ .

d) Finally using the fact 2.18 and the previous points, for any  $1 \leq h \leq h_0$ , in the spectral sequence (12)

- by point a),  $E_2^{p,q}$  is torsion free for  $q \geq h_0 - h + 1$  and so it is zero if  $p + 2q \neq 0$ ;
- by affiness of the open strata, cf. [Boy17] theorem 1.8,  $E_2^{p,q}$  is zero for  $p + 2q < 0$  and torsion free for  $p + 2q = 0$ ;

- by point b), the maps  $d_2^{p,q}$  are saturated for  $q \geq h_0 - h + 2$ ;
- by point c),  $d_2^{-2(h_0-h+1), h_0-h+1}$  has a cokernel with a non trivial torsion subspace.
- Moreover, over  $\overline{\mathbb{Q}}_l$ , the spectral sequence degenerates at  $E_3$  and  $E_3^{p,q} = 0$  if  $(p, q) \neq (0, 0)$ .

We then deduce that  $H^i(\mathrm{Sh}_{K, \overline{s}_v}, {}^p j_{!*}^{-h} HT_\xi(\chi_v, \Pi_h))_{\mathfrak{m}}$  is zero for  $i < h - h_0$  and for  $i = h - h_0$  it has a non trivial torsion subspace.  $\square$

Consider now the filtration of stratification of  $\Psi_\rho$  constructed using the adjunction morphisms  $j_{!*}^{-h} j^{=h,*}$  as in [Boy14]

$$\mathrm{Fil}_!^1(\Psi_\rho) \hookrightarrow \mathrm{Fil}_!^2(\Psi_\rho) \hookrightarrow \cdots \hookrightarrow \mathrm{Fil}_!^d(\Psi_\rho) \quad (14)$$

where  $\mathrm{Fil}_!^h(\Psi_\rho)$  is the saturated image of  $j_{!*}^{-h} j^{=h,*} \Psi_\rho \rightarrow \Psi_\rho$ . For our fixed  $\chi_v$ , let denote  $\mathrm{Fil}_{!, \chi_v}^1(\Psi) \hookrightarrow \mathrm{Fil}_!^1(\Psi_\rho)$  such that  $\mathrm{Fil}_{!, \chi_v}^1(\Psi) \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l \cong \mathrm{Fil}_!^1(\Psi_{\chi_v})$  where  $\Psi_{\chi_v}$  is the direct factor of  $\Psi \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{Q}}_l$  associated to  $\chi_v$ , cf. [Boy14]. From [Boy23b] 3.3.5, we have the following resolution of  $\mathrm{gr}_{!, \chi_v}^h(\Psi)$

$$\begin{aligned} 0 \rightarrow j_{!*}^{-d} HT(\chi_v, LT_{h,d}(\chi_v)) \otimes \chi_v^* \left( \frac{d-h}{2} \right) \rightarrow \\ j_{!*}^{-d-1} HT(\chi_v, LT_{h,d-1}(\chi_v)) \otimes \chi_v^* \left( \frac{d-h-1}{2} \right) \rightarrow \\ \cdots \rightarrow j_{!*}^{-h} HT(\chi_v, st_h(\chi_v)) \otimes \chi_v^* \rightarrow \mathrm{gr}_{!, \chi_v}^h(\Psi) \rightarrow 0, \end{aligned} \quad (15)$$

where  $LT_{h,h+\delta}(\chi_v) \hookrightarrow st_h(\chi_v \{-\delta/2\}) \times \mathrm{Speh}_\delta(\chi_v \{h/2\})$ , is the only irreducible sub-space of this induced representation,

We can then apply the previous arguments a)-d) above: for  $h \leq h_0$  (resp.  $h > h_0$ ) the torsion of  $H^i(\mathrm{Sh}_{I, \overline{s}_v}, \mathrm{gr}_{!, \chi_v}^h(\Psi_{v, \xi}))_{\mathfrak{m}}$  is trivial for any  $i \leq h - h_0$  (resp. for all  $i$ ) and the free parts are concentrated for  $i = 0$ . Using the spectral sequence associated to the previous filtration, we can then conclude that  $H^{1-t_0}(\mathrm{Sh}_{I, \overline{s}_v}, \Psi_{v, \xi})_{\mathfrak{m}}$  would have non trivial torsion which is false as  $\mathfrak{m}$  is supposed to be KHT-free.  $\square$

In particular the previous spectral sequence gives us a filtration of  $H^{d-1}(\mathrm{Sh}_{K, \overline{\eta}_v}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$  whose graded parts are

$$H^0(\mathrm{Sh}_{K, \overline{s}_v}, \mathrm{gr}^{-p}(\mathrm{gr}_!^k(\Psi_{K, \rho})))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l,$$

for  $\rho$  describing the equivalence classes of irreducible  $\overline{\mathbb{F}}_l$ -supercuspidal representation of  $\mathrm{GL}_g(F_v)$  with  $1 \leq g \leq d$ , and then  $1 \leq k \leq p \leq \lfloor \frac{d}{g} \rfloor$ .

### 3 Genericity for KHT-Shimura varieties

As explained in the introduction, we follow the strategy of [Boy20] which consists to transfer the genericity property of Ihara's lemma concerning  $\overline{G}$  to the genericity of the cohomology of KHT-Shimura varieties.

Let  $\overline{G}$  be a similitude group as in the introduction such that moreover there exists a prime number  $p_0$  split in  $E$  and  $v_0^+$  a place of  $F^+$  above  $p_0$ , identified as before to a place  $v_0$  of  $F$ , such that  $\overline{B}_{v_0}$  is a division algebra: in particular  $v_0 \neq v$ . Consider then, with the usual abuse of notation,  $G/\mathbb{Q}$  such that  $G(\mathbb{A}_{\mathbb{Q}}^{\infty, v_0}) \cong \overline{G}(\mathbb{A}_{\mathbb{Q}}^{\infty, v_0})$  with  $G(F_{v_0}) \cong \mathrm{GL}_d(F_{v_0})$  and  $G(\mathbb{R})$  of signatures  $(1, n-1), (0, n)^r$ . The KHT Shimura variety  $\mathrm{Sh}_{K, v_0} \rightarrow \mathrm{spec} \mathcal{O}_{v_0}$  associated to  $G$  with level  $K$ , has a Newton stratification of its special fiber with

$$\mathrm{Sh}_{K, \overline{s}_{v_0}}^{\overline{d}} = \coprod_{i \in \ker^1(\mathbb{Q}, G)} \mathrm{Sh}_{K, \overline{s}_{v_0}, i}^{\overline{d}}.$$

For a equivariant sheaf  $\mathcal{F}_{K, i}$  on  $\mathrm{Sh}_{K^v(\infty), \overline{s}_{v_0}, i}^{\overline{d}}$  seen as compatible system of  $\mathrm{Sh}_{K^v K_v, \overline{s}_{v_0}, i}^{\overline{d}}$  for  $K_v$  describing the set of open compact subgroups of  $\mathrm{GL}_d(\mathcal{O}_v)$ , its fiber at a compatible system  $z_{K^v(\infty), i}$  of supersingular point  $z_{K^v K_v, i}$ , has an action of  $\overline{G}(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times \mathrm{GL}_d(F_v)^0$  where  $\mathrm{GL}_d(F_v)^0$  is the kernel of the valuation of the determinant so that, cf. [Boy09] proposition 5.1.1, as a  $\mathrm{GL}_d(F_v)$ -module, we have

$$H^0(\mathrm{Sh}_{K^v(\infty), \overline{s}_{v_0}, i}^{\overline{d}}, \mathcal{F}_{K^v(\infty), i}) \cong \left( \mathrm{ind}_{\overline{G}(\mathbb{Q})}^{\overline{G}(\mathbb{A}^{\infty, v}) \times \mathbb{Z}} z_{K^v(\infty), i}^* \mathcal{F}_{K^v(\infty), i} \right)^{K^v},$$

with  $\delta \in \overline{G}(\mathbb{Q}) \mapsto (\delta^{\infty, v_0}, \mathrm{val} \circ \mathrm{rn}(\delta_{v_0})) \in \overline{G}(\mathbb{A}^{\infty, v_0, v}) \times \mathbb{Z}$  and where the action of  $g_{v_0} \in \mathrm{GL}_d(F_{v_0})$  is given by those of  $(g_0^{-\mathrm{val} \det g_{v_0}} g_{v_0}, \mathrm{val} \det g_{v_0}) \in \mathrm{GL}_d(F_{v_0})^0 \times \mathbb{Z}$  where  $g_0 \in \mathrm{GL}_d(F_{v_0})$  is any fixed element with  $\mathrm{val} \det g_0 = 1$ . Moreover, cf. [Boy09] corollaire 5.1.2, if  $z_{K^v(\infty), i}^* \mathcal{F}_{K^v(\infty), i}$  is provided with an action of the kernel  $(D_{v_0, d}^{\times})^0$  of the valuation of the reduced norm, action compatible with those of  $\overline{G}(\mathbb{Q}) \hookrightarrow D_{v_0, d}^{\times}$ , then as a  $G(\mathbb{A}^{\infty})$ -module, we have

$$H^0(\mathrm{Sh}_{K^v(\infty), \overline{s}_{v_0}, i}^{\overline{d}}, \mathcal{F}_{K^v(\infty), i}) \cong \left( \mathcal{C}^{\infty}(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A}^{\infty}), \Lambda) \otimes_{D_{v_0, d}^{\times}} \mathrm{ind}_{(D_{v_0, d}^{\times})^0}^{D_{v_0, d}^{\times}} z_i^* \mathcal{F}_{\mathcal{I}, i} \right)^{K^v} \quad (16)$$

In particular, cf. lemma 2.3.1 of [Boy20], let  $\overline{\pi}$  be an irreducible sub- $\overline{\mathbb{F}}_l$ -representation of  $\mathcal{C}^{\infty}(\overline{G}(\mathbb{Q}) \backslash \overline{G}(\mathbb{A})/K^v, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$  for  $\mathfrak{m}$  such that  $\overline{\rho}_{\mathfrak{m}}$  is irreducible. Write its local component  $\overline{\pi}_{v_0} \cong \pi_{v_0}[s]_D$  with  $\pi_{v_0}$  an irreducible cuspidal representation of  $\mathrm{GL}_g(F_{v_0})$  with  $d = sg$ . Then  $(\overline{\pi}^{v_0})^{K^v}$  is a sub-representation of  $H^0(\mathrm{Sh}_{K^v(\infty), \overline{s}_{v_0}}^{\overline{d}}, HT(\pi_{v_0}^{\vee}, s))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$  and, cf. proposition 2.3.2 of [Boy20], a sub- $\overline{\mathbb{F}}_l$ -representation of  $H^{d-1}(\mathrm{Sh}_{K^v(\infty), \overline{\eta}_{v_0}}, \overline{\mathbb{F}}_l)_{\mathfrak{m}}$ . Indeed, cf. theorem 2.4.3,

- by the main result of [Boy23a], as  $l > d \geq 2$  and  $\overline{\rho}_{\mathfrak{m}}$  is irreducible, then  $\mathfrak{m}$  is KHT free so that hypothesis (H1) of [Boy20] is fulfilled.
- Theorem 2.4.3 gives us that the filtration of  $H^{d-1}(\mathrm{Sh}_{K^v(\infty), \overline{\eta}_{v_0}}, \overline{\mathbb{Z}}_l)_{\mathfrak{m}}$  induced by the filtration of the nearby cycles at  $v_0$  is strict, which was the reason to introduced hypothesis (H3) in [Boy20].

Finally if the analog of Ihara's lemma for  $H^{d-1}(\mathrm{Sh}_{K^v(\infty), \bar{\eta}}, \bar{\mathbb{F}}_l)_{\mathfrak{m}}$  is true for the action of  $\mathrm{GL}_d(F_v)$ , then this is also the case for  $\bar{G}$ . We now focus on the genericity of irreducible sub- $\mathrm{GL}_d(F_v)$ -modules of  $H^0(\mathrm{Sh}_{K^v(\infty), \bar{\eta}}, \bar{\mathbb{F}}_l)_{\mathfrak{m}}$  using the nearby cycles at the place  $v$ .

Using proposition 3.7 (3) of [NT16], through Verdier duality, this is also equivalent to prove the genericity of irreducible quotient of  $\mathrm{GL}_d(F_v)$ -modules of  $H^0(\mathrm{Sh}_{K^v(\infty), \bar{\eta}_v}, \bar{\mathbb{F}}_l)_{\mathfrak{m}}$ , which is the content of proposition 3.3.4 using 3.3.3.

### 3.1 Level raising

To a cohomological minimal prime ideal  $\tilde{\mathfrak{m}}$  of  $\mathbb{T}(K)$ , which corresponds to a maximal ideal of  $\mathbb{T}(K)[\frac{1}{l}]$ , is associated both a near equivalence class of  $\bar{\mathbb{Q}}_l$ -automorphic representation  $\Pi_{\tilde{\mathfrak{m}}}$  and a Galois representation

$$\rho_{\tilde{\mathfrak{m}}} : G_F := \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{GL}_d(\bar{\mathbb{Q}}_l)$$

such that the eigenvalues of the Frobenius morphism at an unramified place  $w$  are given by the Satake parameters of the local component  $\Pi_{\tilde{\mathfrak{m}}, w}$  of  $\Pi_{\tilde{\mathfrak{m}}}$ . The semi-simple class  $\bar{\rho}_{\mathfrak{m}}$  of the reduction modulo  $l$  of  $\rho_{\tilde{\mathfrak{m}}}$  depends only of the maximal ideal  $\mathfrak{m}$  of  $\mathbb{T}_K^S$  containing  $\tilde{\mathfrak{m}}$ .

We now allow infinite level at  $v$  and we denote by  $\mathbb{T}(K^v(\infty))$  the associated Hecke algebra. We fix a maximal ideal  $\mathfrak{m}$  in  $\mathbb{T}(K^v(\infty))$  such that

- the associated Galois representation  $\bar{\rho}_{\mathfrak{m}} : G_F \rightarrow \mathrm{GL}_d(\mathbb{F})$  is irreducible;
- $\bar{\rho}_{\mathfrak{m}}|_{W_{F_v}}$ , after semi-simplification, is a direct sum of characters.

*Remark.* For every minimal prime  $\tilde{\mathfrak{m}} \subseteq \mathfrak{m}$ , note that  $\Pi_{\tilde{\mathfrak{m}}, v}$  looks like  $\mathrm{st}_{s_1}(\chi_{v,1}) \times \cdots \times \mathrm{st}_{s_r}(\chi_{v,r})$  with  $s_1 + \cdots + s_r = d$ , so that it has non trivial invariant vectors under the Iwahori subgroup  $\mathrm{Iw}_v$  and, as a direct consequence of Cebotarev theorem, see for example lemma 3.2 of [DS74],

$$\mathbb{T}(K^v(\infty)) = \mathbb{T}(K^v \mathrm{Iw}_v). \quad (17)$$

Let  $\mathcal{S}_v(\mathfrak{m})$  be the supercuspidal support of the modulo  $l$  reduction of any  $\Pi_{\tilde{\mathfrak{m}}, v}$  in the near equivalence class associated to a minimal prime ideal  $\tilde{\mathfrak{m}} \subseteq \mathfrak{m}$ . Recall that  $\mathcal{S}_v(\mathfrak{m})$  is a multi-set, i.e. a set with multiplicities which only depends on  $\mathfrak{m}$ . We decompose it according to the set  $\mathcal{Z}$  of Zelevinsky lines: as we supposed  $q_v \equiv 1 \pmod{l}$  then every Zelevinsky line is reduced to a single equivalence class of an irreducible (super)cuspidal  $\bar{\mathbb{F}}_l$ -representations  $\varrho$  of some  $\mathrm{GL}_{g(\varrho)}(F_v)$  with  $1 \leq g(\varrho) \leq d$ . Moreover our second hypothesis tells us that we are only concerned with  $\varrho$  being a character:

$$\mathcal{S}_v(\mathfrak{m}) = \coprod_{\varrho \in \mathrm{Cusp}_{\bar{\mathbb{F}}_l}(1, v)} \mathcal{S}_{\varrho}(\mathfrak{m}),$$

where  $\mathrm{Cusp}_{\bar{\mathbb{F}}_l}(1, v)$  is the set of  $\bar{\mathbb{F}}_l$ -characters of  $F_v^\times$ .

**Notation 3.1.1.** We denote by  $l_{\varrho}(\mathfrak{m})$  the multiplicity of  $\mathcal{S}_{\varrho}(\mathfrak{m})$ .

For  $\tilde{\mathfrak{m}} \subseteq \mathfrak{m}$ , the local component  $\Pi_{\tilde{\mathfrak{m}},v}$  of  $\Pi_{\tilde{\mathfrak{m}}}$  can then be written as a full induced representation  $\prod_{\varrho \in \text{Cusp}_{\overline{\mathbb{F}}_l}(1,v)} \Pi_{\tilde{\mathfrak{m}},\varrho}$  where each  $\Pi_{\tilde{\mathfrak{m}},\varrho}$  is also a full induced representation

$$\Pi_{\tilde{\mathfrak{m}},\varrho} \cong \prod_{i=1}^{r(\varrho)} \text{St}_{l_{\varrho,i}(\tilde{\mathfrak{m}})}(\pi_{v,i})$$

where  $r_l(\pi_{v,i}) \cong \varrho$ ,  $l_{\varrho,1}(\varrho) \geq \dots \geq l_{\varrho,r(\varrho)}(\varrho)$  and  $\sum_{i=1}^r l_{\varrho,i}(\tilde{\mathfrak{m}}) = l_{\varrho}(\mathfrak{m})$ .

Suppose now that there exists  $\varrho \in \text{Cusp}_{\overline{\mathbb{F}}_l}(1,v)$  with  $r(\varrho) \geq 2$ , then

$$H^0(\text{Sh}_{K,\bar{s}_v}, \text{gr}^{-l_{\varrho,1}(\varrho)}(\text{gr}_l^1(\Psi_{K,\varrho})))_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$$

is a subspace of  $H^0(\text{Sh}_{K,\bar{s}_v}, \Psi_{K,v})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ . Moreover this subspace, as a  $\overline{\mathbb{F}}_l$ -representation of  $\text{GL}_d(F_v)$ , has a subspace of the following shape  $\text{st}_{l_1(\varrho)}(\varrho) \times \tau$  where the supercuspidal support of  $\tau$  contains  $\varrho$ . In particular as  $q_v \equiv 1 \pmod{l}$  and  $l > d$ , this induced representation has both a generic and a non generic subspace.

We can then conclude that for the genericity property to be true for KHT Shimura varieties, one needs a level raising property as in proposition 3.3.1 of [Boy20]. Hopefully such statements exist under some rather mild hypothesis as for example the following result of T. Gee.

**Theorem 3.1.2.** ([Gee11] theorem 5.1.5) *Let  $F = F^+E$  be a CM field where  $F^+$  is totally real and  $E$  is imaginary quadratic. Let  $d > 1$  and  $l > d$  be a prime which is unramified in  $F^+$  and split in  $E$ . Suppose that*

$$\bar{\rho} : G_F \longrightarrow \text{GL}_n(\overline{\mathbb{F}}_l)$$

*is an irreducible representation which is unramified at all places of  $F$  lying above primes which do not split in  $E$  and which satisfies the following properties.*

- $\bar{\rho}$  is automorphic of weight  $\underline{a}$ , where we assume that for all  $\tau \in (\mathbb{Z}^d)^{\text{hom}(F,\mathbb{C})}$  we have either

$$l - 1 - d \geq a_{\tau,1} \geq \dots \geq a_{\tau,d} \geq 0,$$

or

$$l - 1 - d \geq a_{c\tau,1} \geq \dots \geq a_{c\tau,d} \geq 0.$$

*Note in particular that these conditions imply  $\bar{\rho}^c \cong \bar{\rho}^\vee \epsilon^{1-d}$ .*

- $\overline{F}^{\ker \text{ad } \bar{\rho}}$  does not contain  $F(\zeta_l)$ .
- $\bar{\rho}(G_{F^+(\zeta_l)})$  is big.

*Let  $u$  be a finite place of  $F^+$  which split in  $F$  and not dividing  $l$ . Choose an inertial type  $\tau_v$  and a place  $v$  of  $F$  above  $u$ . Assume that  $\bar{\rho}|_{G_{F_v}}$  has a lift to characteristic zero of type  $\tau_v$ .*

*Then there is an automorphic representation  $\pi$  of  $\text{GL}_n(\mathbb{A}_F)$  of weight  $\underline{a}$  and level prime to  $l$  such that*

- $\bar{r}_{l,l}(\pi) \cong \bar{\rho}$ .

- $r_{l,l}(\pi)|_{G_{F_v}}$  has type  $\tau_v$ .
- $\pi$  is unramified at all places  $w \neq v$  of  $F$  at which  $\bar{\rho}$  is unramified.

*Remark.* In this text we focus only on the trivial coefficients  $\bar{\mathbb{Z}}_l$ , i.e. to the case  $a_{\tau,1} = \dots = a_{\tau,d} = a_{c\tau,1} = \dots = a_{c\tau,d} = 0$ , but we could also deal with other weights as in the previous theorem.

### 3.2 Local and global monodromy

The previous filtrations on  $\Psi_{K,\varrho}$  are compatible with the action of the monodromy operator  $N_v$  at  $v$ . More precisely, over  $\bar{\mathbb{Q}}_l$ , for  $i \geq 2$ , we have isomorphisms

$$N_v : \mathrm{gr}^{-i}(\mathrm{gr}_!^{k+1}(\Psi_{K,\varrho})) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l \longrightarrow \mathrm{gr}^{-i}(\mathrm{gr}_!^k(\Psi_{K,\varrho})) \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l.$$

As explained in the introduction, for a geometric definition of  $N_v$ , we now consider a level  $K$  such that  $K_v$  is the Iwahori subgroup  $\mathrm{Iw}_v$  of upper triangular matrices modulo  $q_v$ , so that  $\mathrm{Sh}_{K,\bar{s}_v}$  is semi-stable. Following Rapoport-Zink, we denote by  $N_v^{geo} : \Psi_{K^v \mathrm{Iw}_v, v} \longrightarrow \Psi_{K^v \mathrm{Iw}_v, v}(1)$  their monodromy operator.

Recall that for any  $\bar{\mathbb{F}}_l$ -character  $\varrho$  we have a direct factor  $\Psi_{K^v \mathrm{Iw}_v, \varrho}$  of  $\Psi_{K^v \mathrm{Iw}_v, v}$  on which our monodromy operator acts. For an open compact subgroup  $K$  where now  $K_v$  is any open compact subgroup of  $\mathrm{GL}_d(\mathcal{O}_v)$ , then  $\Psi_{K,\varrho}$  has a filtration with graded parts  $\bigoplus_{\chi_v \in \mathcal{C}(\varrho)} P(t, \chi_v)^{\binom{1-t+2k}{2}}$  where

- $\mathcal{C}(\varrho)$  is the set of character whose modulo  $l$  reduction is isomorphic to  $\varrho$ .<sup>5</sup>
- $1 \leq t \leq d$  and  $0 \leq k \leq t - 1$ .

Note that, up to homothety,  $P(t, \chi_v)$  admits a unique stable  $\bar{\mathbb{Z}}_l$ -lattice. There exists then an unique extension of our monodromy operator  $N_v^{geo}$  to  $\Psi_{K,\varrho}$  for any  $K$  as soon as  $\varrho$  is a  $\bar{\mathbb{F}}_l$ -character. Moreover it follows from [Ill94] 3.6.13 in the semi-stable case, that  $N_v^{geo}$  is as non trivial as possible, so that in our case, for  $1 \leq k < i \leq d$ ,

$$(N_v^{geo}) : \mathrm{gr}^{-i}(\mathrm{gr}_!^{k+1}(\Psi_{K,\varrho})) \longrightarrow \mathrm{gr}^{-i}(\mathrm{gr}_!^k(\Psi_{K,\varrho}))$$

is an isomorphism.

*Remark.* More generally let  $\varrho$  be an irreducible  $\bar{\mathbb{F}}_l$ -cuspidal representation of  $\mathrm{GL}_g(F_v)$  such that  $m(\varrho)$ <sup>6</sup> is strictly greater than  $\frac{d}{g}$ . Then, cf. [Boy11], for  $1 \leq s \leq \frac{d}{g}$  and for any lift  $\pi_v$  of  $\varrho$ , the representation  $\mathrm{st}_s(\pi_v)$  admits, up to homothety, a unique stable lattice. From the main results of [Boy23b] saying that, with the previous hypothesis on  $m(\varrho)$ , there exists a unique intermediate extension of Harris-Taylor local systems associated to the lifts of  $\varrho$ , we can then define a  $\bar{\mathbb{Z}}_l$ -version of the monodromy operator on  $\Psi_{K^v(\infty), \varrho}$  such that modulo  $l$  its order of nilpotency is maximal, equal to  $\lfloor \frac{d}{g} \rfloor$ .

<sup>5</sup>Recall that, as  $q_v \equiv 1 \pmod{l}$  with  $l > d$ , then there exists no cuspidal representation whose modulo  $l$  reduction has a supercuspidal support made of character except than characters.

<sup>6</sup>which is either equal to the order of the Zelevinsky line of  $\varrho$  when it is  $> 1$  otherwise  $m(\varrho) = l$



Let fix  $\mathfrak{m}$  a maximal ideal of  $\mathbb{T}^S(K^v(\infty))$  such that  $H^{d-1}(\mathrm{Sh}_{K^v \mathrm{Iw}_v, \bar{\eta}}, \bar{\mathbb{Z}}_l)_{\mathfrak{m}}$  is torsion free and non zero.

*Remark.* Note that  $H^\bullet(\mathrm{Sh}_{K^v(\infty), \bar{s}_v}, P(t, \varrho))_{\mathfrak{m}}$  is zero if  $\varrho$  is not a character that is a constituent of  $\bar{\rho}_{\mathfrak{m}, v}$ . We are then only concerned with the  $\Psi_{K^v(\infty), \varrho}$  for  $\varrho$  a character.

In [Boy19] over  $\bar{\mathbb{Q}}_l$ , we proved that the filtration  $\mathrm{Fil}_!^\bullet(\Psi_{K, v})$  coincides with the kernel filtration of monodromy on  $\Psi_{K, v}$ . The spectral sequence

$$E_{1, !, \mathfrak{m}}^{i, j} := H^{i+j}(\mathrm{Sh}_{K, \bar{s}_v}, \mathrm{gr}_!^{-i}(\Psi_{K, v}))_{\mathfrak{m}} \Rightarrow H^{i+j}(\mathrm{Sh}_{K, \bar{s}_v}, \Psi_{K, v})_{\mathfrak{m}}$$

is concentrated in  $i + j = 0$  and so degenerates in  $E_1$ . We denote by  $N_{\mathfrak{m}, \varrho}^{\mathrm{coho}}$  the monodromy operator associated to  $N_v^{\mathrm{geo}}$  on  $H^*(\mathrm{Sh}_{K^v(\infty), \bar{s}_v}, \Psi_{K^v(\infty), \varrho})_{\mathfrak{m}}$  as well as  $\bar{N}_{\mathfrak{m}, \varrho}^{\mathrm{coho}}$  for its modulo  $l$  version.

*Main observation:* Recall that over  $\bar{\mathbb{Q}}_l$ , the filtration on  $H^{i+j}(\mathrm{Sh}_{K, \bar{s}_v}, \Psi_{K, v})_{\mathfrak{m}}$  induced by  $\mathrm{Fil}_!^\bullet(\Psi_{K, \varrho})$  coincides with those of the iterated kernels of  $N_v$ . From the previous proposition and theorem 2.4.3, we then deduce that this fact remains true over  $\bar{\mathbb{F}}_l$ . In particular the image of  $(\bar{N}_{\mathfrak{m}, \varrho}^{\mathrm{coho}})^{l_\varrho(\mathfrak{m})-1}$  is, up to semi-simplification,

$$\bigoplus_{\chi_v \in \mathcal{C}(\varrho)} H^0(\mathrm{Sh}_{K^v(\infty), v}, P(\chi_v, l_\varrho(\mathfrak{m}))(\frac{l_\varrho(\mathfrak{m}) - 1}{2}))_{\mathfrak{m}} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l.$$

### 3.3 Typicity and monodromy

As explained in [HT01], the  $\bar{\mathbb{Q}}_l$ -cohomology of  $\mathrm{Sh}_{K, \bar{\eta}}$  can be written as

$$H^{d-1}(\mathrm{Sh}_{K, \bar{\eta}}, \bar{\mathbb{Q}}_l)_{\mathfrak{m}} \cong \bigoplus_{\pi \in \mathcal{A}_{\xi, K}(\mathfrak{m})} (\pi^\infty)^K \otimes V(\pi^\infty),$$

where

- $\mathcal{A}_K(\mathfrak{m})$  is the set of equivalence classes of automorphic representations of  $G(\mathbb{A})$  with non trivial  $K$ -invariants and such that its modulo  $l$  Satake's parameters outside  $S$  are prescribed by  $\mathfrak{m}$ ,
- and  $V(\pi^\infty)$  is a representation of  $\mathrm{Gal}_{F, S}$ .

As  $\bar{\rho}_{\mathfrak{m}}$  is supposed to be absolutely irreducible, then as explained in chapter VI of [HT01], if  $V(\pi^\infty)$  is non zero, then  $\pi$  is a weak transfer of a cohomological automorphic representation  $(\Pi, \psi)$  of  $\mathrm{GL}_d(\mathbb{A}_F) \times \mathbb{A}_F^\times$  with  $\Pi^\vee \cong \Pi^c$  where  $c$  is the complex conjugation. Attached to such a  $\Pi$  is a global Galois representation  $\rho_{\Pi, l} : \mathrm{Gal}_{F, S} \rightarrow \mathrm{GL}_d(\bar{\mathbb{Q}}_l)$  which is irreducible.

**Theorem 3.3.1.** (cf. [NF19] theorem 2.20)

If  $\rho_{\Pi, l}$  is strongly irreducible, meaning it remains irreducible when it is restricted to any finite index subgroup, then  $V(\pi^\infty)$  is a semi-simple representation of  $\mathrm{Gal}_{F, S}$ .

*Remark.* The Tate conjecture predicts that  $V(\pi^\infty)$  is always semi-simple.

**Definition 3.3.2.** (cf. [Sch18] §5) We say that  $\mathfrak{m}$  is KHT-typic for  $K$  if, as a  $\mathbb{T}(K)_{\mathfrak{m}}[\text{Gal}_{F,S}]$ -module,

$$H^{d-1}(\text{Sh}_{K,\bar{\eta}}, \bar{\mathbb{Z}}_l)_{\mathfrak{m}} \cong \sigma_{\mathfrak{m},K} \otimes_{\mathbb{T}(K)_{\mathfrak{m}}} \rho_{\mathfrak{m},K},$$

for some  $\mathbb{T}(K)_{\mathfrak{m}}$ -module  $\sigma_{\mathfrak{m},K}$  on which  $\text{Gal}_{F,S}$  acts trivially and

$$\rho_{\mathfrak{m},K} : \text{Gal}_{F,S} \longrightarrow \text{GL}_d(\mathbb{T}(K)_{\mathfrak{m}})$$

is the stable lattice of  $\bigoplus_{\tilde{\mathfrak{m}} \subseteq \mathfrak{m}} \rho_{\tilde{\mathfrak{m}}}$  introduced in the introduction.

**Proposition 3.3.3.** *We suppose that for all  $\pi \in \mathcal{A}_K(\mathfrak{m})$ , the Galois representation  $V(\pi^\infty)$  is semi-simple. Then  $\mathfrak{m}$  is KHT-typic for  $K$ .*

*Proof.* By proposition 5.4 of [Sch18] it suffices to deal with  $\bar{\mathbb{Q}}_l$ -coefficients. From [HT01] proposition VII.1.8 and the semi-simplicity hypothesis, then  $V(\pi^\infty) \cong \tilde{R}(\pi) \bigoplus^{n(\pi)}$  where  $\tilde{R}(\pi)$  is of dimension  $d$ . We then write

$$(\pi^\infty)^K \otimes_{\bar{\mathbb{Q}}_l} R(\pi) \cong (\pi^\infty)^K \otimes_{\mathbb{T}(K)_{\mathfrak{m},\bar{\mathbb{Q}}_l}} (\mathbb{T}(K)_{\mathfrak{m},\bar{\mathbb{Q}}_l})^d,$$

and  $(\pi^\infty)^K \otimes_{\bar{\mathbb{Q}}_l} V(\pi^\infty) \cong ((\pi^\infty)^K) \bigoplus^{n(\pi)} \otimes_{\mathbb{T}(K)_{\mathfrak{m},\bar{\mathbb{Q}}_l}} (\mathbb{T}(K)_{\mathfrak{m},\bar{\mathbb{Q}}_l})^d$  and finally

$$H^{d-1}(\text{Sh}_{K,\bar{\eta}}, \bar{\mathbb{Q}}_l)_{\mathfrak{m}} \cong \sigma_{\mathfrak{m},K,\bar{\mathbb{Q}}_l} \otimes_{\mathbb{T}(K)_{\mathfrak{m},\bar{\mathbb{Q}}_l}} (\mathbb{T}(K)_{\mathfrak{m},\bar{\mathbb{Q}}_l})^d,$$

with  $\sigma_{\mathfrak{m},K,\bar{\mathbb{Q}}_l} \cong \bigoplus_{\pi \in \mathcal{A}_K(\mathfrak{m})} ((\pi^\infty)^I) \bigoplus^{n(\pi)}$ . The result then follows from [HT01] theorem VII.1.9 which insures that  $R(\pi) \cong \rho_{\tilde{\mathfrak{m}}}$ , if  $\tilde{\mathfrak{m}}$  is the prime ideal associated to  $\pi$ ,  $\square$

Let  $\varrho$  be a  $\bar{\mathbb{F}}_l$ -character with  $l_\varrho(\mathfrak{m}) > 0$ . Then  $H^0(\text{Sh}_{K,\bar{\eta}_v}, \Psi_{K,\varrho})_{\mathfrak{m}}$  as a direct factor of  $H^{d-1}(\text{Sh}_{K,\bar{\eta}}, \bar{\mathbb{Q}}_l)_{\mathfrak{m}}$ , is also typic, i.e.

$$H^0(\text{Sh}_{K,\bar{\eta}_v}, \Psi_{K,\varrho})_{\mathfrak{m}} \cong \sigma_{\mathfrak{m},K,\varrho} \otimes_{\mathbb{T}(K)_{\mathfrak{m}}} \rho_{\mathfrak{m},K,\varrho}.$$

The monodromy operator  $N_{\mathfrak{m},\varrho}^{\text{cho}}$  acting on  $H^0(\text{Sh}_{K,\bar{\eta}_v}, \Psi_{K,\varrho})_{\mathfrak{m}}$  is such that

$$N_{\mathfrak{m},\varrho}^{\text{cho}} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{Q}}_l \cong \text{Id} \otimes N_{\mathfrak{m},\varrho,\bar{\mathbb{Q}}_l},$$

i.e. it acts trivially on the first factor  $\sigma_{\mathfrak{m},K,\varrho}$ . We then deduce that  $N_{\mathfrak{m},\varrho}^{\text{cho}}$  induces a nilpotent operator  $N_{\mathfrak{m},\varrho}$  (resp.  $\bar{N}_{\mathfrak{m},\varrho}$ ) on  $\rho_{\mathfrak{m}}$  (resp.  $\rho_{\mathfrak{m}} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l$ ).

*Remark.* For a  $K^v$  and  $\mathfrak{m}$  such that the irreducible constituent of  $\bar{\rho}_{\mathfrak{m},v}$  are characters, we notice that  $\rho_{\mathfrak{m},K}$  does not depend on the level  $K_v$  and we then denote it simply by  $\rho_{\mathfrak{m}}$ .

**Proposition 3.3.4.** *Let  $\tau_v$  be an irreducible  $\bar{\mathbb{F}}_l[\text{GL}_d(F_v)]$ -module which is a quotient of  $\sigma_{\mathfrak{m},K^v(\infty)} \otimes_{\bar{\mathbb{Z}}_l} \bar{\mathbb{F}}_l$ . Then  $\tau_v$  is generic.*

*Proof.* Recall that  $\tau_v$  can be written as a full parabolic induced representation

$$\bigtimes_{\varrho \in \text{Cusp}_{\mathbb{F}_l}(1,v)} \tau_{\varrho}$$

where the (super)cuspidal support of  $\tau_{\varrho}$  is a multiple of  $\varrho$ . We want to prove that  $\tau_{\varrho}$  is generic which, as  $q_v \equiv 1 \pmod{l}$  with  $l > d$ , means that  $\tau_{\varrho} \cong \text{st}_{l_{\varrho}(\mathfrak{m})}(\varrho)$ .

Write, cf (17),

$$\mathbb{T}(K^v \text{Iw}_v)_{\mathfrak{m}} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l \cong \mathbb{T}(K^v(\infty))_{\mathfrak{m}} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l \cong R/\mathfrak{N}$$

where  $R$  is a  $\overline{\mathbb{F}}_l$ -algebra and  $\sqrt{\mathfrak{N}}$  is maximal. We then consider the image of  $(N_{\mathfrak{m},\varrho}^{\text{coho}})^{l_{\varrho}(\mathfrak{m})-1} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l = \text{Id} \otimes \overline{N}_{\mathfrak{m},\varrho}^{l_{\varrho}(\mathfrak{m})-1}$

$$(\sigma_{\mathfrak{m},K^v(\infty)} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l) \otimes_{R/\mathfrak{N}} (\rho_{\mathfrak{m}} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l) \xrightarrow{\text{Id} \otimes \overline{N}_{\mathfrak{m},\varrho}^{l_{\varrho}(\mathfrak{m})-1}} (\sigma_{\mathfrak{m},K^v(\infty)} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l) \otimes_{R/\mathfrak{N}} \text{Im} \overline{N}_{\mathfrak{m},\varrho}^{l_{\varrho}(\mathfrak{m})-1}.$$

From theorem 3.1.2, we know there exists  $\tilde{\mathfrak{m}} \subseteq \mathfrak{m}$  such that  $\Pi_{\tilde{\mathfrak{m}},v} \cong \text{st}_{l_{\varrho}(\mathfrak{m})}(\chi_v) \times \tau_v$  where  $\chi_v$  is isomorphic to  $\varrho$  modulo  $l$ , and the cuspidal support of  $\tau_v$  modulo  $l$  does not contain  $\varrho$ . From the main observation at the end of §3.2, we then deduce that  $\text{Im} \overline{N}_{\mathfrak{m},\varrho}^{l_{\varrho}(\mathfrak{m})-1}$  is non zero.

Let denote by  $\text{Top}(\sigma_{\mathfrak{m},K^v(\infty)} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l)$ , the cosocle of  $\sigma_{\mathfrak{m},K^v(\infty)} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l$ , i.e. its largest semi-simple quotient.

**Lemma 3.3.5.** *The action of  $R/\mathfrak{N}$  on this cosocle, factorizes through its quotient  $R/\sqrt{\mathfrak{N}}$ .*

*Proof.* Let denote  $M = \text{Top}(\sigma_{\mathfrak{m},K^v(\infty)} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l)$ , and we have to prove that  $\sqrt{\mathfrak{N}}M = (0)$ . We argue by contradiction and let  $k \geq 1$  be maximal such that  $(\sqrt{\mathfrak{N}})^k M \neq (0)$ . The short exact sequence

$$0 \rightarrow (\sqrt{\mathfrak{N}})^k M \rightarrow M \rightarrow M/(\sqrt{\mathfrak{N}})^k M \rightarrow 0,$$

is split, as  $M$  is semi-simple so that  $M \cong (\sqrt{\mathfrak{N}})^k M \oplus M/(\sqrt{\mathfrak{N}})^k M$  is killed by  $(\sqrt{\mathfrak{N}})^k$ , which gives the expected contradiction.  $\square$

In particular this cosocle is a quotient of  $(\sigma_{\mathfrak{m},K^v(\infty)} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l)/\sqrt{\mathfrak{N}}$ , and

$$\text{Top}(\sigma_{\mathfrak{m},K^v(\infty)} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l) \otimes_{R/\sqrt{\mathfrak{N}}} \text{Im} \overline{N}_{\mathfrak{m},\varrho}^{l_{\varrho}(\mathfrak{m})-1} / \sqrt{\mathfrak{N}}$$

is a quotient of  $(\sigma_{\mathfrak{m},K^v(\infty)} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l) \otimes_{R/\mathfrak{N}} \text{Im} \overline{N}_{\mathfrak{m},\varrho}^{l_{\varrho}(\mathfrak{m})-1}$ . Write

$$\text{Top}(\sigma_{\mathfrak{m},K^v(\infty)} \otimes_{\mathbb{Z}_l} \overline{\mathbb{F}}_l) \cong \bigoplus_{i=1}^r \bar{\pi}_{v,i}$$

as a  $\overline{\mathbb{F}}_l[\mathrm{GL}_d(F_v)]$  semi-simple module so that

$$\mathrm{Top}\left(\sigma_{\mathfrak{m}, K^v(\infty)} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l\right) \otimes_{\overline{\mathbb{F}}_l} \left(\mathrm{Im} \overline{N}_{\mathfrak{m}, \varrho}^{l_\varrho(\mathfrak{m})-1} \otimes_{R/\mathfrak{M}} \overline{\mathbb{F}}_l\right) \cong \bigoplus_{i=1}^r \left(\overline{\pi}_{v,i} \otimes_{\overline{\mathbb{F}}_l} \left(\mathrm{Im} \overline{N}_{\mathfrak{m}, \varrho}^{l_\varrho(\mathfrak{m})-1} \otimes_{R/\mathfrak{M}} \overline{\mathbb{F}}_l\right)\right)$$

is a quotient of the image of  $(N_{\mathfrak{m}, \varrho}^{\mathrm{coh}\varrho})^{l_\varrho(\mathfrak{m})-1} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$  on  $H^0(\mathrm{Sh}_{K^v(\infty), \overline{s}_v}, \Psi_{K^v(\infty), \varrho})_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l$ . But we saw that this image, up to semi-simplification, is

$$H^0(\mathrm{Sh}_{K^v(\infty), \overline{s}_v}, P(\varrho, l_\varrho(\mathfrak{m})))_{\mathfrak{m}} \cong \left(\bigoplus_i \overline{\phi}_{v,i}\right) \otimes \varrho,$$

where each  $\overline{\phi}_{v,i}$  are of the following shape  $\mathrm{st}_{l_\varrho(\mathfrak{m})}(\varrho) \times \overline{\phi}_{v,i}^\varrho$  where  $\varrho$  does not belong to the (super)cuspidal support of  $\overline{\phi}_{v,i}^\varrho$ . We then deduce that the  $\varrho$ -part of any of the irreducible constituents  $\tau_v \cong \bigtimes_{\varrho \in \mathrm{Cusp}(1,v)} \tau_\varrho$  of  $\mathrm{Top}\left(\sigma_{\mathfrak{m}, K^v(\infty)} \otimes_{\overline{\mathbb{Z}}_l} \overline{\mathbb{F}}_l\right)$  are such that  $\tau_\varrho$  is generic. As it is true for all  $\varrho \in \mathrm{Cusp}(1, v)$ , then  $\tau_v$  is generic as stated.  $\square$

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