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► **To cite this version:**

| Gilles Dowek. The physical Church thesis and the sensitivity to initial conditions. 2009. <hal-04078089>

HAL Id: hal-04078089

<https://hal.science/hal-04078089v1>

Preprint submitted on 25 Apr 2023

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The physical Church thesis and the sensitivity to initial conditions

Gilles Dowek

The *physical Church thesis* is a thesis about nature that expresses that all that can be computed by a physical system — a machine — is computable in the sense of computability theory. At a first look, this thesis seems contradictory with the existence, in nature, of chaotic dynamical systems, that is systems whose evolution cannot be “computed” because of their sensitivity to initial conditions.

The goal of this note is to show that there exist dynamical systems that are both computable and chaotic, and thus that the existence in nature of chaotic dynamical system is not, *per se*, a refutation of the physical Church thesis. Thus, chaos seems to be compatible with computability, in the same way as it is compatible with determinism.

1 The notion of computable function

1.1 Computability over the natural numbers

Several equivalent definitions of the notion of computable function may be given. One of the simplest is to define inductively a family $(C_p)_p$ where C_p is a set of partial functions from \mathbb{N}^p to \mathbb{N} by the six following rules

- the functions $x_1, \dots, x_p \mapsto x_i$ are elements of C_p ,
- the function $x_1, \dots, x_p \mapsto 0$ is an element of C_p ,
- the function $x \mapsto x + 1$ is an element of C_1 ,
- if f is an element of C_q and g_1, \dots, g_q are elements of C_p then the composition of f and g_1, \dots, g_q , that is the function $x_1, \dots, x_p \mapsto f(g_1(x_1, \dots, x_p), \dots, g_q(x_1, \dots, x_p))$ is an element of C_p ,
- if f is an element of C_p and g an element of C_{p+2} then the function h defined by induction as follows

$$h(x_1, \dots, x_p, 0) = f(x_1, \dots, x_p)$$

$$h(x_1, \dots, x_p, y + 1) = g(x_1, \dots, x_p, y, h(x_1, \dots, x_n, y))$$

is an element of C_{p+1} ,

- if f is an element of C_{p+1} then the function g defined by $g(x_1, \dots, x_n) = y$ if $f(x_1, \dots, x_n, y) = 0$ and for all $z < y$ $f(x_1, \dots, x_n, z) \neq 0$ is an element of C_p .

In this last case, the function f is not defined in x_1, \dots, x_p if the function $z \mapsto f(x_1, \dots, x_n, z)$ never takes the value 0.

1.2 Computability on rational numbers

The notion of computability extends easily to rational numbers. Let α be the injection from \mathbb{N}^2 to \mathbb{N} defined by $\alpha(n, p) = (n+p)(n+p+1)/2+p$. We define an injection π from \mathbb{Q} to \mathbb{N} by $\pi(r) = \alpha(\alpha(\text{sgn}(r), \text{num}(r)), \text{den}(r))$ where $\text{sgn}(r)$ is the sign of r — 0 or 1 —, $\text{num}(r)$ is the numerator of its reduced form and $\text{den}(r)$ its denominator. A partial function f from \mathbb{Q}^p to \mathbb{Q} is said to be computable if the partial function from \mathbb{N}^p to \mathbb{N} mapping $\pi(x_1), \dots, \pi(x_n)$ to $\pi(f(x_1, \dots, x_n))$ is computable.

At a first look, this definition depends of the chosen injection π from \mathbb{Q} to \mathbb{N} . In fact, it can easily be proved that this is not the case. If π' is another injection from \mathbb{Q} to \mathbb{N} such that the function that map $\pi'(x)$ and $\pi'(y)$ to $\pi'(x+y)$ and that maps $\pi'(x)$ and $\pi'(y)$ to $\pi'(xy)$ are computable, then the set of computable functions from \mathbb{Q}^p to \mathbb{Q} defined with the injection π' is the same as that defined with the injection π .

1.3 Computability on real numbers

The extension of the notion of computability to real numbers is less obvious, because the set of real numbers, unlike that of rational numbers, is not countable. A possible definition, that is equivalent to the classical definition of Grzegorzczuk and Lacombe [3], is that a function g from an interval I of \mathbb{R} to \mathbb{R} is computable if there exists a computable function G from $\mathbb{Q} \times \mathbb{Q}$ to \mathbb{Q} and a computable function η from the set of strictly positive rational numbers to itself, such that

$$|x - q| \leq \eta(\varepsilon) \Rightarrow |g(x) - G(\varepsilon, q)| \leq \varepsilon$$

In other words, to compute the value of $g(x)$ we must first choose an accuracy ε with which we want to know the value of $g(x)$. We must then apply the function η to ε to get the accuracy with which we are required to supply the argument x . Then, we apply the function G to ε and to a rational approximation q of x with an accuracy $\eta(\varepsilon)$ and we get the desired result. Repeating this process, we can obtain approximations of the value of $g(x)$ as accurate as we want, provided we can supply arbitrarily accurate approximations of x .

We can notice that a computable function from an interval I to \mathbb{R} is uniformly continuous on I .

1.4 Non deterministic algorithms

Let us finally mention another extension of the notion of computable function that we will need. A non necessarily functional binary relation R is said to

be computable if the function that maps x to the set of y 's such that $x R y$ is computable. This requires to define a notion of computable function taking values in a set of sets. Several solutions exist, depending on the cardinals of these sets and on the nature of their elements.

2 The physical Church thesis

Stating the physical Church thesis requires to consider a dynamical physical system, that is a system whose state evolves in time. Time can be discrete or continuous. The state of the system can be described by one or more discrete or continuous variables. It may be directly accessible to measurement or not. It may evolve in a deterministic way or not.

We then consider the evolution of this system during a time interval that starts at a predefined date and finishes either at another predefined date or when the system satisfies, for the first time, a given property, that is itself computable — for instance, when one of its discrete variables describing its state takes for the first time a given value.

We consider the evolution of the system during various time intervals defined as above and for which the initial states are different.

The physical Church thesis states that, whatever the considered physical system is, the function — when the system is deterministic — or the relation — when it is not — that relates the initial state to the final state of the system is computable.

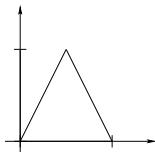
3 An example of chaotic and computable system

The existence, in nature, of chaotic dynamical systems, that is system whose evolution can not be “computed” because of its sensitivity to initial conditions seems to contradict the physical Church thesis.

In fact, as we shall see, there exist dynamical systems that are both chaotic and computable. To do so, we shall give a very simple example of such a system: *the baker's transformation*. It is a dynamical system with a discrete time, in which a point of coordinate x in $[0, 1]$ moves in one time step to the coordinate $b(x)$ where b is the function defined by

$$b(x) = 2x \text{ if } x \leq 1/2$$

$$b(x) = 2 - 2x \text{ if } x > 1/2$$



if the initial state of the system is x_0 , its state after n time units is thus $b^n(x_0)$.

It is not difficult to prove that this system is chaotic: for all η , if two initial positions have a difference less than η , after some time, the positions may differ of an arbitrary value. Let η be an arbitrary strictly positive real number and a and a' two arbitrary real numbers in $[0, 1]$. Let n be a natural number such that $1/2^n \leq \eta$. The initial conditions $x_0 = a/2^n$ and $x'_0 = a'/2^n$ are such that $|x_0 - x'_0| \leq \eta$. However, after n time units, $b^n(x_0) = a$ and $b^n(x'_0) = a'$.

A very small difference in the initial conditions can thus change completely the state of the system after a certain time and knowing an approximation of the initial position, by a measurement for instance, does not allow to predict the evolution of the system, even approximately.

It is nevertheless elementary to prove that the function that maps an initial position x_0 of the physical point and a date n to the position of the physical point at the date n , $F = x_0, n \mapsto b^n(x_0)$, is a computable function from $[0, 1] \times \mathbb{N}$ to $[0, 1]$.

What makes this system non predictable is not the fact that the function F is not computable, but the fact that to have a reasonable approximation, let us say with an accuracy of 0.001, of the state of the system at a date n , it would be necessary to have an approximation of the initial position with an accuracy $0.001/2^n$ and that for large enough values of n , it becomes impossible to know this initial value by a measurement. It is therefore not the absence of algorithm that makes this system non predictable, but the absence of argument to supply to this algorithm.

It can be noticed that the construction of such an algorithm to which no argument can be supplied is a common practice. All the consistency proofs of a theory T (for instance, hyperbolic geometry) relatively to that of a theory U (for instance, euclidean geometry) that proceed by constructing an algorithm that would transform a proof of a contradiction in the theory T into a proof of a contradiction in the theory U , use such an algorithm.

The fact that no argument can be supplied to an algorithm does not mean that this algorithm does not exist.

4 Two alternative descriptions

4.1 Really discrete positions

The classical description of the baker's transformation describes the position of a physical point in motion by a real number. This can be criticized, assuming that, in reality, the position of a physical point can take only a finite number of values in the interval $[0, 1]$ as it is the case in quantum physics, in some situations, for instance when the state space has a finite dimension, and as it is also the case, if we assume, like, for instance, Gandy [1], that information has a finite density in nature, that is that a system of finite size can have only a finite number of possible states.

Under such a description, the baker's transformation is not sensitive to initial

conditions, because if we call η the half of the minimal distance that separates two distinct possible states, the condition $|x_0 - x'_0| \leq \eta$ is equivalent to $x_0 = x'_0$.

In this case, the function b that maps the state of the dynamical system at a given time, to its state at the next time step is a function of finite domain and thus it is computable. The function that maps the initial position x_0 of the physical point and the date n to the position of the physical point at the date n , $F = x_0, n \mapsto b^n(x_0)$, is thus also computable.

4.2 Discretely perceived positions

Another way to criticize this description of the position of a physical point by a real number is to accept to give a meaning only to effectively measurable values. This leads to reject this notion of ideal position, that is not accessible to measurement, as metaphysical.

In this case, the baker's transformation becomes a non deterministic dynamical system, as the state of the system at a given time does not determine its state at the next step. For instance, if we use to measure the state of the system a device that provides three digits, the initial state 0.000 can be followed by 0.000 or by 0.001. As above, the system is not sensitive to initial conditions anymore and it is the fact that it is not deterministic — and not the fact that it is sensitive to initial conditions — that makes it non predictable.

In this case, as above, the number of possible measured values is finite and therefore the relation R that relates two measured values x and y if y is one of the measured values that may follow x is computable, because it is finite. The relation that relates a natural number n and two values x and y if y may follow x after n steps is thus also computable.

5 The limit state

In the case of the baker's transformation, the sequence $(F(x_0, n))_n$ is not always convergent. Thus, we cannot speak, in general, of the limit state of the system. In contrast, this becomes possible when we consider dissipative dynamical systems. We can, in this case, discuss the computability not only of the function F that maps an initial state x_0 of the system and a date n to the state of the system at this date, but also the function that maps an initial state x_0 of the system to its limit state, that is to the limit of the sequence $(F(x_0, n))_n$ when n goes to infinity. In this case, it is not difficult to find examples where this function is not computable. Any example where this function is not continuous works. Other examples of non computable properties in chaotic systems are given, for instance, in [2].

This observation however is not in contradiction with the physical Church thesis that is only concerned with the state of the system at a finite date n and says nothing about its limit state when n goes to infinity.

The fact that a dynamical system may be both computable and non predictable shows, once more, that the notion of computability largely exceeds the

practical notion of accessibility to computation.

Acknowledgments

To Olivier Bournez, Jean-Baptiste Joinet, Giuseppe Longo and Thierry Paul.

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