

Convergence of a TPFA scheme for a diffusion-convection equation with a multiplicative stochastic noise

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April 4, 2023

Abstract

The aim of this paper is to address the convergence analysis of a finite-volume scheme for the approximation of a stochastic non-linear parabolic problem set in a bounded domain of \mathbb{R}^2 and under homogeneous Neumann boundary conditions. The considered discretization is semi-implicit in time and TPFA in space. By adapting well-known methods for the time-discretization of stochastic PDEs, one shows that the associated finite-volume approximation converges towards the unique variational solution of the continuous problem strongly in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$.

Keywords: Stochastic non-linear parabolic equation • Multiplicative Lipschitz noise • Finite-volume method • Upwind scheme • Diffusion-convection equation • Variational approach • Convergence analysis.

Mathematics Subject Classification (2020): 60H15 • 35K05 • 65M08.

1 Introduction

Let Λ be a bounded, open, connected, and polygonal set of \mathbb{R}^2 . Moreover let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space endowed with a right-continuous, complete filtration $(\mathcal{F}_t)_{t \geq 0}$ and let $(W(t))_{t \geq 0}$ be a standard, one-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t \geq 0}$ on $(\Omega, \mathcal{A}, \mathbb{P})$.

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For $T > 0$, we consider the following non-linear parabolic problem forced by a multiplicative stochastic noise:

$$\begin{aligned} du - \Delta u dt + \operatorname{div}_x(\mathbf{v}u) dt &= g(u) dW(t) + \beta(u) dt, & \text{in } \Omega \times (0, T) \times \Lambda; \\ u(0, \cdot) &= u_0, & \text{in } \Omega \times \Lambda; \\ \nabla u \cdot \mathbf{n} &= 0, & \text{on } \Omega \times (0, T) \times \partial\Lambda; \end{aligned} \quad (1.1)$$

where div_x is the divergence operator with respect to the space variable and \mathbf{n} denotes the unit normal vector to $\partial\Lambda$ outward to Λ . We assume the following hypotheses on the data:

\mathcal{A}_1 : $u_0 \in L^2(\Omega; H^1(\Lambda))$ is \mathcal{F}_0 -measurable.

\mathcal{A}_2 : $g : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz-continuous function.

\mathcal{A}_3 : $\beta : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz-continuous function with $\beta(0) = 0$.

\mathcal{A}_4 : $\mathbf{v} \in \mathcal{C}^1([0, T] \times \Lambda; \mathbb{R}^2)$, $\operatorname{div}_x(\mathbf{v}(t, x)) = 0$ for all $(t, x) \in [0, T] \times \Lambda$ and $\mathbf{v}(t, x) \cdot \mathbf{n}(x) = 0$ for all $(t, x) \in [0, T] \times \partial\Lambda$.

1.1 Notations

Let us introduce some notations and make precise the functional setting.

- $|x|$ denotes the euclidian norm of x in \mathbb{R}^2 and $x \cdot y$ the usual scalar product of x and y in \mathbb{R}^2 .
- For $p \in \{1, 2, 3\}$, $\|\cdot\|_\infty$ denotes the $L^\infty(\mathbb{R}^p)$ norm.
- $L_\beta \geq 0$ the Lipschitz constant of β .
- $L_g \geq 0$ the Lipschitz constant of g .
- $C_{L_g} \geq 0$ a constant only depending on L_g and $g(0)$, satisfying for all $r \in \mathbb{R}$

$$|g(r)|^2 \leq C_{L_g}(1 + |r|^2). \quad (1.2)$$

- $\mathbb{E}[\cdot]$ denotes the expectation, *i.e.* the integral over Ω with respect to the probability measure \mathbb{P} .
- For a given separable Banach space X , we denote by $L^2_{\mathcal{P}_T}(\Omega \times (0, T); X)$ the space of the predictable X -valued processes ([14] p.94 or [29] p.27). This space is the space $L^2(\Omega \times (0, T); X)$ for the product measure $d\mathbb{P} \otimes dt$ on the predictable σ -field \mathcal{P}_T (*i.e.* the σ -field generated by the sets $\mathcal{F}_0 \times \{0\}$ and the rectangles $A \times (s, t]$, for any $s, t \in [0, T]$ with $s \leq t$ and $A \in \mathcal{F}_s$).
For $X = L^2(\Lambda)$, one has $L^2_{\mathcal{P}_T}(\Omega \times (0, T); L^2(\Lambda)) \subset L^2(\Omega \times (0, T); L^2(\Lambda))$.

Remark 1.1. Note that the existence of the constant C_{L_g} is given by Assumption \mathcal{A}_2 . It allows us to apply our scheme for square integrable, additive noise with appropriate measurability assumptions.

1.2 Concept of solution and main result

The theoretical framework associated with Problem (1.1) is well established in the literature. Indeed, we can find many existence and uniqueness results for various concepts of solutions associated with this problem such as mild solutions, variational solutions, path-wise solutions and weak solutions, see, e.g., [14] and [25]. In the present paper we will be interested in the concept of solution as defined below, which we will call a variational solution:

Definition 1.2. *A stochastic process u in $L^2_{\mathcal{P}_T}(\Omega \times (0, T); L^2(\Lambda))$ is a variational solution to Problem (1.1) if it belongs to*

$$L^2(\Omega; \mathcal{C}([0, T]; L^2(\Lambda))) \cap L^2(\Omega; L^2(0, T; H^1(\Lambda)))$$

and satisfies, for all $t \in [0, T]$,

$$u(t) - u_0 - \int_0^t \Delta u(s) ds + \int_0^t \operatorname{div}_x(\mathbf{v}(s, \cdot)u(s)) ds = \int_0^t g(u(s)) dW(s) + \int_0^t \beta(u(s)) ds$$

in $L^2(\Lambda)$ and \mathbb{P} -a.s. in Ω .

Existence, uniqueness and regularity of this variational solution is well-known in the literature, see, e.g., [28],[24],[25]. The main result of this paper is to propose a finite-volume scheme for the approximation of such a variational solution and to show its stochastically strong convergence by passing to the limit with respect to the time and space discretization parameters.

Theorem 1.3. *Assume that hypotheses \mathcal{A}_1 to \mathcal{A}_4 hold. Let $(\mathcal{T}_m)_{m \in \mathbb{N}}$ be a sequence of admissible finite-volume meshes of Λ in the sense of Definition 2.1 such that the mesh size h_m tends to 0 and let $(N_m)_{m \in \mathbb{N}} \subset \mathbb{N}^*$ be a sequence of positive numbers which tends to infinity. For a fixed $m \in \mathbb{N}$, let u_{h_m, N_m}^r and u_{h_m, N_m}^l be respectively the right and left in time finite-volume approximations defined by (2.3), (2.5)-(2.6) with $\mathcal{T} = \mathcal{T}_m$ and $N = N_m$. Then $(u_{h_m, N_m}^r)_{m \in \mathbb{N}}$ and $(u_{h_m, N_m}^l)_{m \in \mathbb{N}}$ converge strongly in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ to the variational solution of Problem (1.1) in the sense of Definition 1.2.*

Remark 1.4. The proof of Theorem 1.3 will be done as follows: firstly, we will prove in Proposition 4.1 that the sequences $(u_{h_m, N_m}^r)_{m \in \mathbb{N}}$ and $(u_{h_m, N_m}^l)_{m \in \mathbb{N}}$ weakly converge in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ towards a common limit denoted by u . Secondly, thanks to Proposition 4.5 and Proposition 4.9 we will be able on the one hand to identify u as the unique variational solution of Problem (1.1) in the sense of Definition 1.2, and on the other hand to prove that the convergence finally holds strongly in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$.

1.3 State of the art

As mentioned in a previous paper [8] in collaboration with F. NABET, the study of numerical schemes for stochastic partial differential equations (SPDEs) has been a very

fashionable subject in recent decades and for this reason, an extensive literature on this topic is available. We refer the interested reader to [1], [15] and [27] for a general overview and associated references.

If we focus on the theoretical study of parabolic SPDEs with a non-linear first order operator, note that the variational techniques developed in [28], [24] and [25] can be applied whereas the semigroup approach is not available, and therefore the use of mild solutions is out of range. Note that for the theoretical study of (1.1) in \mathbb{R}^d ($d \geq 1$) instead of a bounded domain, we can refer to [31] or the appendix of [13].

Concerning the numerical analysis of these variational solutions, it is clear that in the past the use of finite-element methods has been favored and extensively employed (we refer to [10], [11] for a thorough exposition of existing papers). But we note that recently, for first order scalar conservation laws with multiplicative noise, finite-volume discretizations have been developed in [4], [5], [6], [22], [26], [16], [3], and [17] by adapting the deterministic framework. Then, in [7] and [8], convergence of finite-volume scheme for the particular case of stochastic heat equation (with respectively linear and non-linear multiplicative noise) has been investigated. As far as we know, stochastic equations of type (1.1) have not been studied yet from a numerical point of view.

1.4 Goal of the study and outline of the paper

In this work, our aim is to fill the gap left by the previous authors by proposing a convergence result for a both space and time-discretization of the non-linear diffusion-convection equation forced by a stochastic noise and under homogeneous Neumann boundary conditions (1.1). The added value comparing to existing results is threefold:

- Firstly, the taking into account of a convection term $\operatorname{div}_x(\mathbf{v}u)$ which is very interesting from a modeling point of view and opens the door to many extensions (Stefan's problem, porous medium equation...)
- Secondly, the fact that we avoid the use of technical tools from the stochastic framework for the passage to the limit in the non-linear terms (such as the theorem of Prokhorov, Skorokhod's representation theorem, concept of martingale solution, Gyöngy-Krylov argument...) as we did in a previous work [8] for the stochastic heat equation (which corresponds to (1.1) with $\mathbf{v} = \mathbf{0}$ and $\beta = 0$).
- Thirdly, the obtention of a strong convergence result in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ which is more satisfactory than in [8] where, in spite of the fact that $\mathbf{v} = \mathbf{0}$ and $\beta = 0$, we only proved convergence of the scheme in $L^p(\Omega; L^2(0, T; L^2(\Lambda)))$ for any $1 \leq p < 2$.

The main difficulty of the present study is to choose suitable tools of the finite-volume framework compatible with the stochastic one and the restrictions brought by the multiplicative noise. Particularly, we will see that the main challenge will be the identification

of weak limits coming from the discretization of the non-linear terms $g(u)$ and $\beta(u)$.

The paper is organized as follows. Section 2 is devoted to the introduction of the finite-volume framework: definition of the finite-volume mesh employed for the discretization of Λ , associated notations, definition of discrete norms and construction of the right and left finite-volume approximations denoted respectively by $(u_{h,N}^r)_{h,N}$ and $(u_{h,N}^l)_{h,N}$. In Section 3, we will derive stability estimates in suitable functional spaces satisfied by the sequences $(u_{h,N}^r)_{h,N}$, $(u_{h,N}^l)_{h,N}$, $(g(u_{h,N}^l))_{h,N}$ and $(\beta(u_{h,N}^r))_{h,N}$. These estimates will allow us in Section 4 to extract weakly converging subsequences. The remaining of the paper will be devoted to the passage to the limit in the numerical scheme: firstly we will prove the existence of an Itô stochastic process u , the weak limit (up to subsequences) of $(u_{h,N}^r)_{h,N}$ and $(u_{h,N}^l)_{h,N}$. Secondly, by taking advantage of the use of exponential weighted in time norm (which is a known method for the time-discretization of Stochastic PDE) and adapting it to our TPFA scheme, we will prove that this convergence finally holds strongly. Thirdly, using this last information, we will be able to identify weak limits coming from the non-linear terms $(g(u_{h,N}^l))_{h,N}$ and $(\beta(u_{h,N}^r))_{h,N}$. Fourthly, we will conclude that u is the unique variational solution of (1.1) in the sense of Definition 1.2 and fifthly that subsequences are not needed anymore.

2 The finite-volume framework

The following subsections 2.1, 2.2, 2.3 contain all the definitions and notations related to finite-volume framework and are the same as in our previous paper [8], but for a matter of self-containedness we choose to repeat them identically.

2.1 Admissible finite-volume meshes and notations

In order to perform a finite-volume approximation of the variational solution of Problem (1.1) on $[0, T] \times \Lambda$ we need first of all to set a choice for the temporal and spatial discretization. For the time-discretization, let $N \in \mathbb{N}^*$ be given. We define the fixed time step $\Delta t = \frac{T}{N}$ and divide the interval $[0, T]$ in $0 = t_0 < t_1 < \dots < t_N = T$ equidistantly with $t_n = n\Delta t$ for all $n \in \{0, \dots, N-1\}$. For the space discretization, we refer to [21] and consider finite-volume admissible meshes in the sense of the following definition.

Definition 2.1. (*Admissible finite-volume mesh*) An admissible finite-volume mesh \mathcal{T} of Λ (see Fig. 1) is given by a family of open, polygonal, and convex subsets K , called control volumes of \mathcal{T} , satisfying the following properties:

- $\bar{\Lambda} = \bigcup_{K \in \mathcal{T}} \bar{K}$.
- If $K, L \in \mathcal{T}$ with $K \neq L$ then $\text{int } K \cap \text{int } L = \emptyset$.
- If $K, L \in \mathcal{T}$, with $K \neq L$ then either the one-dimensional Lebesgue measure of $\bar{K} \cap \bar{L}$ is 0 or $\bar{K} \cap \bar{L}$ is the edge of the mesh denoted $\sigma = K|L$ separating the control volumes K and L .

- To each control volume $K \in \mathcal{T}$, we associate a point $x_K \in \overline{K}$ (called the center of K) such that: if $K, L \in \mathcal{T}$ are two neighbouring control volumes the straight line between the centers x_K and x_L is orthogonal to the edge $\sigma = K|L$.

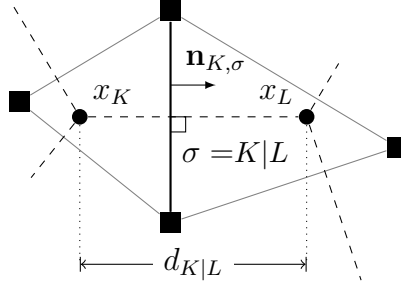


Figure 1: Notations of the mesh \mathcal{T} associated with Λ

Once an admissible finite-volume mesh \mathcal{T} of Λ is fixed, we will use the following notations.

Notations.

- $h = \text{size}(\mathcal{T}) = \sup\{\text{diam}(K) : K \in \mathcal{T}\}$ the mesh size.
- $d_h \in \mathbb{N}$ the number of control volumes $K \in \mathcal{T}$ with $h = \text{size}(\mathcal{T})$.
- λ_2 denotes the two-dimensional Lebesgue measure.
- \mathcal{E} is the set of the edges of the mesh \mathcal{T} and we define $\mathcal{E}_{\text{int}} := \{\sigma \in \mathcal{E} : \sigma \not\subseteq \partial\Lambda\}$, $\mathcal{E}_{\text{ext}} := \{\sigma \in \mathcal{E} : \sigma \subseteq \partial\Lambda\}$.
- For $K \in \mathcal{T}$, \mathcal{E}_K is the set of edges of K and $m_K := \lambda_2(K)$.
- For $K \in \mathcal{T}$, \mathbf{n}_K denotes the unit normal vector to ∂K outward to K and for $\sigma \in \mathcal{E}_K$, we denote the unit vector on the edge σ pointing out of K by $\mathbf{n}_{K,\sigma}$.
- Let $K, L \in \mathcal{T}$ be two neighbouring control volumes. For $\sigma = K|L \in \mathcal{E}_{\text{int}}$, let m_σ be the length of σ and $d_{K|L}$ the distance between x_K and x_L .
- For $\sigma = K|L \in \mathcal{E}_{\text{int}}$, the diamond D_σ (see Fig. 2) is the open quadrangle whose diagonals are the edge σ and the segment $[x_K, x_L]$. For $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K$, we define $D_\sigma := K$. Then, $\overline{\Lambda} = \bigcup_{\sigma \in \mathcal{E}} \overline{D_\sigma}$.
- $m_{D_\sigma} := \lambda_2(D_\sigma)$ is the two-dimensional Lebesgue measure of the diamond D_σ . Note that for $\sigma \in \mathcal{E}_{\text{int}}$, we have $m_{D_\sigma} = \frac{m_\sigma d_{K|L}}{2}$.

Using these notations, we introduce a positive number

$$\text{reg}(\mathcal{T}) = \max \left(\mathcal{N}, \max_{\substack{K \in \mathcal{T} \\ \sigma \in \mathcal{E}_K}} \frac{\text{diam}(K)}{d(x_K, \sigma)} \right) \quad (2.1)$$

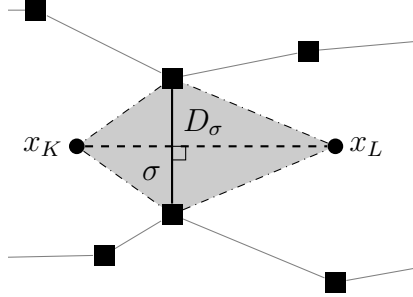


Figure 2: Notations on a diamond cell D_σ for $\sigma \in \mathcal{E}_{\text{int}}$

(where \mathcal{N} is the maximum of edges incident to any vertex) that measures the regularity of a given mesh and is useful to perform the convergence analysis of finite-volume schemes. This number should be uniformly bounded by a constant not depending on the mesh size h for the convergence results to hold. We have in particular $\forall K, L \in \mathcal{T}$,

$$\frac{h}{d_{K|L}} \leq \text{reg}(\mathcal{T}). \quad (2.2)$$

2.2 Discrete unknowns and piecewise constant functions

From now on and unless otherwise specified, we consider $N \in \mathbb{N}^*$, $\Delta t = \frac{T}{N}$ and \mathcal{T} an admissible finite-volume mesh of Λ in the sense of Definition 2.1 with a mesh size h . For $n \in \{0, \dots, N-1\}$ given, the idea of a finite-volume scheme for the approximation of Problem (1.1) is to associate to each control volume $K \in \mathcal{T}$ and time t_n a discrete unknown value denoted by $u_K^n \in \mathbb{R}$, expected to be an approximation of $u(t_n, x_K)$, where u is the variational solution of (1.1). Before presenting the numerical scheme satisfied by the discrete unknowns $\{u_K^n, K \in \mathcal{T}, n \in \{0, \dots, N-1\}\}$, let us introduce some general notations.

For any arbitrary vector $(w_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ we can define the piecewise constant function $w_h^n : \Lambda \rightarrow \mathbb{R}$ by

$$w_h^n(x) := \sum_{K \in \mathcal{T}} w_K^n \mathbf{1}_K(x), \quad \forall x \in \Lambda.$$

Note that since the mesh \mathcal{T} is fixed, by the continuous mapping defined from \mathbb{R}^{d_h} to $L^2(\Lambda)$ by

$$(w_K^n)_{K \in \mathcal{T}} \mapsto \sum_{K \in \mathcal{T}} \mathbf{1}_K w_K^n,$$

the space \mathbb{R}^{d_h} can be considered as a finite-dimensional subspace of $L^2(\Lambda)$ and we may naturally identify the function and the vector

$$w_h^n \equiv (w_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}.$$

Then, knowing for all $n \in \{0, \dots, N\}$ the function w_h^n , we can define the following piecewise constant functions in time and space $w_{h,N}^r, w_{h,N}^l : [0, T] \times \Lambda \rightarrow \mathbb{R}$ by

$$\begin{aligned} w_{h,N}^r(t, x) &:= \sum_{n=0}^{N-1} w_h^{n+1}(x) \mathbb{1}_{[t_n, t_{n+1})}(t) \text{ if } t \in [0, T) \text{ and } w_{h,N}^r(T, x) := w_h^N(x), \\ w_{h,N}^l(t, x) &:= \sum_{n=0}^{N-1} w_h^n(x) \mathbb{1}_{[t_n, t_{n+1})}(t) \text{ if } t \in (0, T] \text{ and } w_{h,N}^l(0, x) := w_h^0(x). \end{aligned} \quad (2.3)$$

Remark 2.2. The superscripts r and l in (2.3) do not refer to the continuity properties of the associated functions (which may be chosen either càdlàg or càglàd). The difference is that in our case, the finite-volume approximation $w_{h,N}^l$ will be adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$, whereas $w_{h,N}^r$ won't be.

As for the piecewise constant function in space, since \mathcal{T} and N are fixed, by the continuous mapping defined from $\mathbb{R}^{d_h \times N}$ to $L^2(0, T; L^2(\Lambda))$ by

$$(w_K^n)_{\substack{K \in \mathcal{T} \\ n \in \{0, \dots, N-1\}}} \mapsto \sum_{\substack{K \in \mathcal{T} \\ n \in \{0, \dots, N-1\}}} \mathbb{1}_K \mathbb{1}_{[t_n, t_{n+1})} w_K^n,$$

the space $\mathbb{R}^{d_h \times N}$ can be considered as a finite-dimensional subspace of $L^2(0, T; L^2(\Lambda))$ and we may naturally identify

$$\begin{aligned} w_{h,N}^l &\equiv (w_K^n)_{\substack{K \in \mathcal{T} \\ n \in \{0, \dots, N-1\}}} \in \mathbb{R}^{d_h \times N}, \\ w_{h,N}^r &\equiv (w_K^{n+1})_{\substack{K \in \mathcal{T} \\ n \in \{0, \dots, N-1\}}} \in \mathbb{R}^{d_h \times N}. \end{aligned}$$

Remark 2.3. Note that in the rest of the paper, when we will consider a time and space function $\alpha : [0, T] \times \Lambda \rightarrow \mathbb{R}$ on all the space Λ (respectively the time interval $[0, T]$) at a fixed time $t \in [0, T]$ (respectively at a fixed $x \in \Lambda$) we will omit the space (respectively time) variable in the notations and write $\alpha(t)$ (respectively $\alpha(x)$) instead of $\alpha(t, \cdot)$ (respectively $\alpha(\cdot, x)$).

2.3 Discrete norms and discrete gradient

Fix $n \in \{0, \dots, N-1\}$ and consider for the remainder of this subsection an arbitrary vector $(w_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ and use its natural identification with the piecewise constant function in space $w_h^n \equiv (w_K^n)_{K \in \mathcal{T}}$. We introduce in what follows the notions of discrete gradient and discrete norms for such a function w_h^n .

Definition 2.4 (Discrete L^2 -norm). *We define the L^2 -norm of $w_h^n \in \mathbb{R}^{d_h}$ as follows*

$$\|w_h^n\|_{L^2(\Lambda)} = \left(\sum_{K \in \mathcal{T}} m_K |w_K^n|^2 \right)^{\frac{1}{2}}.$$

Definition 2.5 (Discrete gradient). We define the gradient operator ∇^h that maps scalar fields $w_h^n \in \mathbb{R}^{d_h}$ into vector fields of $(\mathbb{R}^2)^{e_h}$ (where e_h is the number of edges in the mesh \mathcal{T}), we set $\nabla^h w_h^n = (\nabla_\sigma^h w_h^n)_{\sigma \in \mathcal{E}}$ with

$$\nabla_\sigma^h w_h^n := \begin{cases} 2 \frac{w_L^n - w_K^n}{d_{K|L}} \mathbf{n}_{K,\sigma}, & \text{if } \sigma = K|L \in \mathcal{E}_{\text{int}}; \\ 0, & \text{if } \sigma \in \mathcal{E}_{\text{ext}}. \end{cases}$$

We remark that $\nabla^h w_h^n$ is considered as a piecewise constant function, which is constant on the diamond D_σ .

Definition 2.6 (Discrete H^1 -seminorm). We define the H^1 -seminorm of $w_h^n \in \mathbb{R}^{d_h}$ as follows

$$|w_h^n|_{1,h} := \left(\sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} |w_K^n - w_L^n|^2 \right)^{\frac{1}{2}}.$$

Remark 2.7. Note that in particular,

$$\|\nabla^h w_h^n\|_{(L^2(\Lambda))^2}^2 = \sum_{\sigma \in \mathcal{E}_{\text{int}}} m_{D_\sigma} |\nabla_\sigma^h w_h^n|^2 = 2 \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} |w_K^n - w_L^n|^2 = 2 |w_h^n|_{1,h}^2$$

where the constant 2 corresponds to the space dimension 2.

Remark 2.8. If we consider another arbitrary vector $\tilde{w}_h^n \equiv (\tilde{w}_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$, by summing over the edges we may rearrange the sum on the left-hand side and get the following rule of "discrete partial integration"

$$\sum_{K \in \mathcal{T}} \sum_{\sigma=K|L \in \mathcal{E}_K \cap \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} (w_K^n - w_L^n) \tilde{w}_K^n = \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} (w_K^n - w_L^n) (\tilde{w}_K^n - \tilde{w}_L^n). \quad (2.4)$$

Now, we have all the necessary definitions and notations to present the finite-volume scheme studied in this paper. This is the aim of the next subsection.

2.4 The finite-volume scheme

Firstly, we define the vector $u_h^0 \equiv (u_K^0)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ by the discretization of the initial condition u_0 of Problem (1.1) over each control volume:

$$u_K^0 := \frac{1}{m_K} \int_K u_0(x) dx, \quad \forall K \in \mathcal{T}. \quad (2.5)$$

The finite-volume scheme we propose reads, for this given initial \mathcal{F}_0 -measurable random vector $u_h^0 \in \mathbb{R}^{d_h}$ as follows:

For any $n \in \{0, \dots, N-1\}$, knowing $u_h^n \equiv (u_K^n)_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ we search for $u_h^{n+1} \equiv$

$(u_K^{n+1})_{K \in \mathcal{T}} \in \mathbb{R}^{d_h}$ such that, for almost every $\omega \in \Omega$, the vector u_h^{n+1} is a solution to the following random equations

$$\begin{aligned} & \frac{m_K}{\Delta t} (u_K^{n+1} - u_K^n) + \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma}^{n+1} u_\sigma^{n+1} + \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \\ &= \frac{m_K}{\Delta t} g(u_K^n) (W^{n+1} - W^n) + m_K \beta(u_K^{n+1}), \quad \forall K \in \mathcal{T}, \end{aligned} \quad (2.6)$$

where, the $(d-1)$ -dimensional Lebesgue measure is denoted by γ ,

$$v_{K,\sigma}^{n+1} = \frac{1}{\Delta t m_\sigma} \int_{t_n}^{t_{n+1}} \int_\sigma \mathbf{v}(t, x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x) dt,$$

and u_σ^{n+1} denotes the upstream value at time t_{n+1} with respect to σ defined as follows: if $\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K$ is the interface between the control volumes K and L (i.e. $\sigma = K|L$), u_σ^{n+1} is equal to u_K^{n+1} if $v_{K,\sigma}^{n+1} \geq 0$ and to u_L^{n+1} if $v_{K,\sigma}^{n+1} < 0$. Note also that $W^{n+1} - W^n$ denotes the increments of the Brownian motion between t_{n+1} and t_n :

$$W^{n+1} - W^n = W(t_{n+1}) - W(t_n) \text{ for } n \in \{0, \dots, N-1\}.$$

Remark 2.9. Note that using the divergence-free property of \mathbf{v} (i.e. $\text{div}_x(\mathbf{v}(t, x)) = 0$ for all $(t, x) \in [0, T] \times \Lambda$), the scheme (2.6) can be rewritten in the following way:

$$\begin{aligned} & \frac{m_K}{\Delta t} (u_K^{n+1} - u_K^n) + \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma}^{n+1} (u_\sigma^{n+1} - u_K^{n+1}) + \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \\ &= \frac{m_K}{\Delta t} g(u_K^n) (W^{n+1} - W^n) + m_K \beta(u_K^{n+1}), \quad \forall K \in \mathcal{T}. \end{aligned} \quad (2.7)$$

Indeed,

$$\begin{aligned} \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma}^{n+1} u_K^{n+1} &= \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{u_K^{n+1}}{\Delta t} \int_{t_n}^{t_{n+1}} \int_\sigma \mathbf{v}(t, x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x) dt \\ &= \frac{u_K^{n+1}}{\Delta t} \int_{t_n}^{t_{n+1}} \int_{\partial K} \mathbf{v}(t, x) \cdot \mathbf{n}_K(x) d\gamma(x) dt \\ &= \frac{u_K^{n+1}}{\Delta t} \int_{t_n}^{t_{n+1}} \int_K \text{div}_x(\mathbf{v}(t, x)) dx dt \\ &= 0. \end{aligned}$$

Note that using (2.7) and the fact that* $v_{K,\sigma}^{n+1} = (v_{K,\sigma}^{n+1})^+ - (v_{K,\sigma}^{n+1})^-$, another equivalent formulation of the scheme (2.6) is given for any $K \in \mathcal{T}$ by

$$\begin{aligned} & \frac{m_K}{\Delta t} (u_K^{n+1} - u_K^n) + \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma (v_{K,\sigma}^{n+1})^- (u_K^{n+1} - u_L^{n+1}) \\ &+ \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) = \frac{m_K}{\Delta t} g(u_K^n) (W^{n+1} - W^n) + m_K \beta(u_K^{n+1}). \end{aligned} \quad (2.8)$$

*For any $a \in \mathbb{R}$, $a^+ = \max(a, 0)$ and $a^- = -\min(a, 0)$.

Proposition 2.10 (Existence of a discrete solution). *Assume that hypotheses \mathcal{A}_1 to \mathcal{A}_4 hold. Let \mathcal{T} be an admissible finite-volume mesh of Λ in the sense of Definition 2.1 with a mesh size h and $N \in \mathbb{N}^*$. Then, there exists a unique solution $(u_h^n)_{1 \leq n \leq N} \in (\mathbb{R}^{d_h})^N$ to Problem (2.6) associated with the initial vector u_h^0 defined by (2.5). Additionally, for any $n \in \{0, \dots, N\}$, u_h^n is a \mathcal{F}_{t_n} -measurable random vector.*

The solution $(u_h^n)_{1 \leq n \leq N} \in (\mathbb{R}^{d_h})^N$ of the scheme (2.5)-(2.6) is then used to build the right and left finite-volume approximations $u_{h,N}^r$ and $u_{h,N}^l$ defined by (2.3) for the variational solution u of Problem (1.1).

Proof. We refer to the preprint [9] exclusively dedicated to the proof of such an existence and uniqueness result. \square

3 Stability estimates

We will derive in this section several stability estimates satisfied by the discrete solution $(u_h^n)_{1 \leq n \leq N} \in (\mathbb{R}^{d_h})^N$ of the scheme (2.5)-(2.6) given by Proposition 2.10, and also by the associated right and left finite-volume approximations $(u_{h,N}^r)_{h,N}$ and $(u_{h,N}^l)_{h,N}$ defined by (2.3). We start by giving a bound on the discrete initial data.

Lemma 3.1. *Let u_0 be a given function satisfying assumption \mathcal{A}_1 . Then, the associated discrete initial data $u_h^0 \in \mathbb{R}^{d_h}$ defined by (2.5) satisfies \mathbb{P} -a.s. in Ω ,*

$$\|u_h^0\|_{L^2(\Lambda)} \leq \|u_0\|_{L^2(\Lambda)}.$$

The proof is a direct consequence of the definition of u_h^0 and the Cauchy-Schwarz inequality.

Now, we can give the bounds on the discrete solutions which is one of the key points of the proof of the convergence theorem.

Proposition 3.2 (Bounds on the discrete solutions). *There exists a constant $K_0 > 0$, depending only on u_0 , C_{L_g} , L_β , $|\Lambda|$ and T such that for any $N \in \mathbb{N}^*$ large enough and any $h \in \mathbb{R}_+^*$*

$$\mathbb{E} \left[\|u_h^n\|_{L^2(\Lambda)}^2 \right] + 2\mathbb{E} \left[\sum_{k=0}^{n-1} \|u_h^{k+1} - u_h^k\|_{L^2(\Lambda)}^2 \right] + 8\Delta t \sum_{k=0}^{n-1} \mathbb{E} [|u_h^{k+1}|_{1,h}^2] \leq K_0, \quad \forall n \in \{1, \dots, N\}.$$

Proof. Set $N \in \mathbb{N}^*$, $h \in \mathbb{R}_+^*$ and fix $n \in \{1, \dots, N\}$. For any $k \in \{0, \dots, n-1\}$, we multiply the numerical scheme (2.8) with u_K^{k+1} , take the expectation, and sum over

$K \in \mathcal{T}$ to obtain thanks to (2.4)

$$\begin{aligned}
& \sum_{K \in \mathcal{T}} \frac{m_K}{\Delta t} \mathbb{E} [(u_K^{k+1} - u_K^k) u_K^{k+1}] + \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} \mathbb{E} [|u_K^{k+1} - u_L^{k+1}|^2] \\
& + \sum_{K \in \mathcal{T}} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma (v_{K,\sigma}^{k+1})^- \mathbb{E} [(u_K^{k+1} - u_L^{k+1}) u_K^{k+1}] \\
& = \sum_{K \in \mathcal{T}} \frac{m_K}{\Delta t} \mathbb{E} [g(u_K^k) u_K^{k+1} (W^{k+1} - W^k)] + \sum_{K \in \mathcal{T}} m_K \mathbb{E} [\beta(u_K^{k+1}) u_K^{k+1}].
\end{aligned} \tag{3.1}$$

We consider the terms of (3.1) separately. Firstly note that

$$\sum_{K \in \mathcal{T}} \frac{m_K}{\Delta t} \mathbb{E} [(u_K^{k+1} - u_K^k) u_K^{k+1}] = \frac{1}{2} \sum_{K \in \mathcal{T}} \frac{m_K}{\Delta t} \mathbb{E} [|u_K^{k+1}|^2 - |u_K^k|^2 + |u_K^{k+1} - u_K^k|^2]. \tag{3.2}$$

Secondly, using the inequality $\forall a, b \in \mathbb{R}, b(b-a) \geq \frac{b^2}{2} - \frac{a^2}{2}$ with $b = u_K^{k+1}$ and $a = u_K^k$, one arrives at

$$(u_K^{k+1} - u_K^k) u_K^{k+1} \geq \frac{(u_K^{k+1})^2}{2} - \frac{(u_K^k)^2}{2}.$$

Note that $v_{L,\sigma}^{k+1} = -v_{K,\sigma}^{k+1}$, thus using the divergence-free property of \mathbf{v} one gets

$$\begin{aligned}
& \sum_{K \in \mathcal{T}} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma (v_{K,\sigma}^{k+1})^- \left(\frac{(u_L^{k+1})^2}{2} - \frac{(u_K^{k+1})^2}{2} \right) \\
& = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma}^{k+1} \frac{(u_K^{k+1})^2}{2} \\
& = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{(u_K^{k+1})^2}{2\Delta t} \int_{t_n}^{t_{n+1}} \int_\sigma \mathbf{v}(t, x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x) dt \\
& = \sum_{K \in \mathcal{T}} \frac{(u_K^{k+1})^2}{2\Delta t} \int_{t_n}^{t_{n+1}} \int_K \text{div}_x(\mathbf{v}(t, x)) dx dt \\
& = 0,
\end{aligned}$$

and this leads to

$$\sum_{K \in \mathcal{T}} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma (v_{K,\sigma}^{k+1})^- \mathbb{E} [(u_K^{k+1} - u_L^{k+1}) u_K^{k+1}] \geq 0. \tag{3.3}$$

Thirdly, since u_K^k and $(W^{k+1} - W^k)$ are independent one obtains

$$\sum_{K \in \mathcal{T}} \frac{m_K}{\Delta t} \mathbb{E} [g(u_K^k) u_K^k (W^{k+1} - W^k)] = 0,$$

and so by applying Young's inequality and using the Itô isometry one arrives at

$$\begin{aligned}
& \sum_{K \in \mathcal{T}} \frac{m_K}{\Delta t} \mathbb{E} [g(u_K^k) u_K^{k+1} (W^{k+1} - W^k)] \\
&= \sum_{K \in \mathcal{T}} \frac{m_K}{\Delta t} \mathbb{E} [g(u_K^k) (u_K^{k+1} - u_K^k) (W^{k+1} - W^k)] \\
&\leq \sum_{K \in \mathcal{T}} \frac{m_K}{\Delta t} \mathbb{E} [|g(u_K^k) (W^{k+1} - W^k)|^2] + \frac{1}{4} \sum_{K \in \mathcal{T}} \frac{m_K}{\Delta t} \mathbb{E} [|u_K^{k+1} - u_K^k|^2] \\
&\leq \Delta t \sum_{K \in \mathcal{T}} \frac{m_K}{\Delta t} \mathbb{E} [|g(u_K^k)|^2] + \frac{1}{4} \sum_{K \in \mathcal{T}} \frac{m_K}{\Delta t} \mathbb{E} [|u_K^{k+1} - u_K^k|^2].
\end{aligned} \tag{3.4}$$

Fourthly, using the Lipschitz property of β , the following holds

$$\sum_{K \in \mathcal{T}} m_K \mathbb{E} [\beta(u_K^{k+1}) u_K^{k+1}] \leq L_\beta \sum_{K \in \mathcal{T}} m_K \mathbb{E} [|u_K^{k+1}|^2]. \tag{3.5}$$

Combining (3.2)-(3.3)-(3.4) and (3.5) and multiplying the obtained inequality with $2\Delta t$, one gets

$$\begin{aligned}
& \sum_{K \in \mathcal{T}} m_K \mathbb{E} [|u_K^{k+1}|^2 - |u_K^k|^2 + |u_K^{k+1} - u_K^k|^2] + 2\Delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} \mathbb{E} [|u_K^{k+1} - u_L^{k+1}|^2] \\
&\leq 2\Delta t \sum_{K \in \mathcal{T}} m_K \mathbb{E} [|g(u_K^k)|^2] + \frac{1}{2} \sum_{K \in \mathcal{T}} m_K \mathbb{E} [|u_K^{k+1} - u_K^k|^2] + 2\Delta t L_\beta \sum_{K \in \mathcal{T}} m_K \mathbb{E} [|u_K^{k+1}|^2].
\end{aligned}$$

Then, from (1.2)

$$\begin{aligned}
& (1 - 2\Delta t L_\beta) \sum_{K \in \mathcal{T}} m_K \mathbb{E} [|u_K^{k+1}|^2 - |u_K^k|^2] + \frac{1}{2} \sum_{K \in \mathcal{T}} m_K \mathbb{E} [|u_K^{k+1} - u_K^k|^2] \\
&+ 2\Delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} \mathbb{E} [|u_K^{k+1} - u_L^{k+1}|^2] \leq 2C_{L_g} \Delta t |\Lambda| + 2\Delta t (C_{L_g} + L_\beta) \sum_{K \in \mathcal{T}} m_K \mathbb{E} [(u_K^k)^2].
\end{aligned}$$

For Δt small enough so that $1 - 2\Delta t L_\beta \geq \frac{1}{4}$, after summing over $k \in \{0, \dots, n-1\}$, one arrives at

$$\begin{aligned}
& \frac{1}{4} \mathbb{E} [\|u_h^n\|_{L^2(\Lambda)}^2 - \|u_h^0\|_{L^2(\Lambda)}^2] + \frac{1}{2} \sum_{k=0}^{n-1} \mathbb{E} [\|u_h^{k+1} - u_h^k\|_{L^2(\Lambda)}^2] + 2\Delta t \sum_{k=0}^{n-1} \mathbb{E} [\|u_h^{k+1}\|_{1,h}^2] \\
&\leq 2C_{L_g} T |\Lambda| + 2\Delta t (C_{L_g} + L_\beta) \sum_{k=0}^{n-1} \mathbb{E} [\|u_h^k\|_{L^2(\Lambda)}^2].
\end{aligned} \tag{3.6}$$

Then, it follows that

$$\mathbb{E} [\|u_h^n\|_{L^2(\Lambda)}^2] \leq \mathbb{E} [\|u_h^0\|_{L^2(\Lambda)}^2] + 8C_{L_g} |\Lambda| T + 8\Delta t (C_{L_g} + L_\beta) \sum_{k=0}^{n-1} \mathbb{E} [\|u_h^k\|_{L^2(\Lambda)}^2].$$

Applying the discrete Gronwall lemma yields

$$\mathbb{E} \left[\|u_h^n\|_{L^2(\Lambda)}^2 \right] \leq \left(\mathbb{E} \left[\|u_h^0\|_{L^2(\Lambda)}^2 \right] + 8C_{L_g}|\Lambda|T \right) e^{8T(C_{L_g}+L_\beta)}. \quad (3.7)$$

From (3.7) and Lemma 3.1 we may conclude that there exists a constant $\Upsilon > 0$ such that

$$\sup_{n \in \{1, \dots, N\}} \mathbb{E} \left[\|u_h^n\|_{L^2(\Lambda)}^2 \right] \leq \Upsilon. \quad (3.8)$$

Thanks to (3.8) and (1.2) one gets that for all $n \in \{1, \dots, N\}$

$$\Delta t \sum_{k=0}^{n-1} \mathbb{E} \left[\|g(u_h^k)\|_{L^2(\Lambda)}^2 \right] \leq C_{L_g}|\Lambda|T + C_{L_g}\Delta t \sum_{k=0}^{n-1} \mathbb{E} \left[\|u_h^k\|_{L^2(\Lambda)}^2 \right] \leq C_{L_g}T(\Upsilon + |\Lambda|). \quad (3.9)$$

From (3.6), Lemma 3.1 and (3.8) it now follows that for all $n \in \{1, \dots, N\}$

$$\begin{aligned} & \mathbb{E} \left[\|u_h^n\|_{L^2(\Lambda)}^2 \right] + 2 \sum_{k=0}^{n-1} \mathbb{E} \left[\|u_h^{k+1} - u_h^k\|_{L^2(\Lambda)}^2 \right] + 8\Delta t \sum_{k=0}^{n-1} \mathbb{E} \left[|u_h^{k+1}|_{1,h}^2 \right] \\ & \leq \mathbb{E} \left[\|u_0\|_{L^2(\Lambda)}^2 \right] + 8C_{L_g}T|\Lambda| + 8\Upsilon T(C_{L_g} + L_\beta). \end{aligned}$$

□

We are now interested in the bounds on the right and left finite-volume approximations defined by (2.3).

Lemma 3.3. *The sequences $(u_{h,N}^r)_{h,N}$ and $(u_{h,N}^l)_{h,N}$ are bounded in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ independently of the discretization parameters $N \in \mathbb{N}^*$ and $h \in \mathbb{R}_+^*$. Additionally, $(u_{h,N}^l)_{h,N}$ is bounded in $L_{\mathcal{P}_T}^2(\Omega \times (0, T); L^2(\Lambda))$.*

Proof. The boundedness of the sequences in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ is a direct consequence of Proposition 3.2. The predictability of $(u_{h,N}^l)_{h,N}$ with values in $L^2(\Lambda)$ is a consequence of the \mathcal{F}_{t_n} -measurability of u_K^n for all $n \in \{0, \dots, N\}$ and all $K \in \mathcal{T}$. Indeed, by construction, $(u_{h,N}^l)_{h,N}$ is then an elementary process adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ and so it is predictable. □

Thanks to Proposition 3.2 we can also obtain a $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ -bound on the discrete gradients of the finite-volume approximations.

Lemma 3.4. *There exist a constant $K_1 \geq 0$ depending only on u_0 , C_{L_g} , L_β , $|\Lambda|$, and T , and a constant $K_2 \geq 0$ additionally depending on the mesh regularity $\text{reg}(\mathcal{T})$ (defined by (2.1)), such that*

$$\int_0^T \mathbb{E} \left[|u_{h,N}^r(t)|_{1,h}^2 \right] dt \leq K_1 \quad (3.10)$$

and

$$\int_0^T \mathbb{E} \left[|u_{h,N}^l(t)|_{1,h}^2 \right] dt \leq K_2. \quad (3.11)$$

Proof. Firstly, note that

$$\int_0^T \mathbb{E} [|u_{h,N}^r(t)|_{1,h}^2] dt = \Delta t \sum_{k=0}^{N-1} \mathbb{E} [|u_h^{k+1}|_{1,h}^2]$$

and therefore (3.10) follows directly from Proposition 3.2. Secondly, using the definition of $u_{h,N}^l$ and (3.10), we get

$$\int_0^T \mathbb{E} [|u_{h,N}^l(t)|_{1,h}^2] dt \leq \Delta t \mathbb{E} [|u_h^0|_{1,h}^2] + \Delta t \sum_{k=0}^{N-1} \mathbb{E} [|u_h^{k+1}|_{1,h}^2] \leq \Delta t \mathbb{E} [|u_h^0|_{1,h}^2] + K_1.$$

Since u_0 is assumed to be in $L^2(\Omega; H^1(\Lambda))$, from [21, Lemma 9.4], it follows that there exists $C_\Lambda \geq 0$ depending on the mesh regularity $\text{reg}(\mathcal{T})$ such that

$$\mathbb{E} [|u_h^0|_{1,h}^2] \leq C_\Lambda \mathbb{E} [\|\nabla u_0\|_{L^2(\Lambda)}^2]$$

and therefore (3.11) follows. \square

Lemma 3.5. *The sequences $(g(u_{h,N}^r))_{h,N}$, $(g(u_{h,N}^l))_{h,N}$, $(\beta(u_{h,N}^r))_{h,N}$, and $(\beta(u_{h,N}^l))_{h,N}$ are bounded in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ independently of the discretization parameters $N \in \mathbb{N}^*$ and $h \in \mathbb{R}_+^*$.*

Proof. It is a direct consequence of the boundedness of the sequences $(u_{h,N}^r)_{h,N}$ and $(u_{h,N}^l)_{h,N}$ in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ given by Lemma 3.3 and of the Lipschitz nature of g and β . \square

Lemma 3.6. *There exists a constant $K_3 \geq 0$ depending only on u_0 , L_g , L_β , $|\Lambda|$, and T , and a constant $K_4 \geq 0$ additionally depending on the mesh regularity $\text{reg}(\mathcal{T})$ (defined by (2.1)), such that*

$$\int_0^T \mathbb{E} [|g(u_{h,N}^r(t))|_{1,h}^2] dt \leq K_3 \tag{3.12}$$

and

$$\int_0^T \mathbb{E} [|g(u_{h,N}^l(t))|_{1,h}^2] dt \leq K_4. \tag{3.13}$$

Proof. After noticing that:

$$\begin{aligned} \int_0^T \mathbb{E} [|g(u_{h,N}^r(t))|_{1,h}^2] dt &\leq L_g^2 \int_0^T \mathbb{E} [|u_{h,N}^r(t)|_{1,h}^2] dt \\ \text{and } \int_0^T \mathbb{E} [|g(u_{h,N}^l(t))|_{1,h}^2] dt &\leq L_g^2 \int_0^T \mathbb{E} [|u_{h,N}^l(t)|_{1,h}^2] dt, \end{aligned}$$

the result is immediate thanks to Lemma 3.4. \square

4 Convergence of the finite-volume scheme

Now, we have all the necessary material to pass to the limit in the numerical scheme. In the sequel, let $(\mathcal{T}_m)_{m \in \mathbb{N}}$ be a sequence of admissible meshes of Λ in the sense of Definition 2.1 such that the mesh size h_m tends to 0 when m tends to $+\infty$ and let $(N_m)_{m \in \mathbb{N}} \subset \mathbb{N}^*$ be a sequence with $\lim_{m \rightarrow +\infty} N_m = +\infty$, and $\Delta t_m := \frac{T}{N_m}$. For the sake of simplicity, for $m \in \mathbb{N}$, we shall use the notations $\mathcal{T} = \mathcal{T}_m$, $h = \text{size}(\mathcal{T}_m)$, $\Delta t = \Delta t_m$, and $N = N_m$ when the m -dependency is not useful for the understanding of the reader.

4.1 Weak convergence of finite-volume approximations

Firstly, thanks to the bounds on the discrete solutions, we obtain the following weak convergences.

Proposition 4.1. *There exist not relabeled subsequences of $(u_{h,N}^r)_m$ and of $(u_{h,N}^l)_m$ respectively and a process $u \in L^2(\Omega; L^2(0, T; H^1(\Lambda))) \cap L^2_{\mathcal{P}_T}(\Omega \times (0, T); L^2(\Lambda))$ such that*

$$u_{h,N}^l \rightarrow u \text{ and } u_{h,N}^r \rightarrow u \text{ weakly in } L^2(\Omega; L^2(0, T; L^2(\Lambda))) \text{ as } m \rightarrow +\infty.$$

Proof. From Lemma 3.3 it follows that the sequences $(u_{h,N}^r)_m$, $(u_{h,N}^l)_m$ respectively are bounded in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$, thus, up to a not relabeled subsequence, they are weakly convergent in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ towards possibly distinct elements u , \tilde{u} respectively. Moreover, from Lemma 3.4 and Remark 2.7, it follows that

$$\|\nabla^h u_{h,N}^r\|_{L^2(\Omega; L^2(0, T; L^2(\Lambda)))}^2 \leq 2K_1.$$

Consequently, there exists $\chi \in L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ such that, passing to a not relabeled subsequence if necessary, $\nabla^h u_{h,N}^r \rightarrow \chi$ weakly in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ for $m \rightarrow +\infty$. With similar arguments as in [20, Lemma 2] and [21, Theorem 14.3] we get the additional regularity $u \in L^2(\Omega; L^2(0, T; H^1(\Lambda)))$ and $\chi = \nabla u$. Since, by Proposition 3.2,

$$\mathbb{E} \left[\|u_{h,N}^r - u_{h,N}^l\|_{L^2(0, T; L^2(\Lambda))}^2 \right] = \Delta t \mathbb{E} \left[\sum_{n=0}^{N-1} \|u_h^{n+1} - u_h^n\|_{L^2(\Lambda)}^2 \right] \leq K_0 \Delta t, \quad (4.1)$$

it follows that $(u_{h,N}^r - u_{h,N}^l)_m$ converges strongly to 0 in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ for $m \rightarrow +\infty$, hence also weakly and therefore $u = \tilde{u}$. Note that the predictability property of u with values in $L^2(\Lambda)$ is inherited from $(u_{h,N}^l)_m$ at the limit. \square

Lemma 4.2. *There exist not relabeled subsequences of $(g(u_{h,N}^r))_m$ and $(g(u_{h,N}^l))_m$, and a process g_u in $L^2(\Omega; L^2(0, T; H^1(\Lambda))) \cap L^2_{\mathcal{P}_T}(\Omega \times (0, T); H^1(\Lambda))$ such that*

$$g(u_{h,N}^r) \rightarrow g_u \text{ and } g(u_{h,N}^l) \rightarrow g_u \text{ weakly in } L^2(\Omega; L^2(0, T; L^2(\Lambda))) \text{ as } m \rightarrow +\infty.$$

Proof. The existence of a common weak limit g_u in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ is a direct consequence of the boundedness results on $(g(u_{h,N}^r))_m$ and $(g(u_{h,N}^l))_m$ in the same space stated in Lemma 3.5, the Lipschitz property of g and Inequality (4.1). Moreover, from Lemma 3.6 and Remark 2.7, it follows that

$$\|\nabla^h g(u_{h,N}^l)\|_{L^2(\Omega; L^2(0, T; L^2(\Lambda)))}^2 \leq 2K_3.$$

Consequently, there exists $\tilde{\chi} \in L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ such that, passing to a not relabeled subsequence if necessary, $\nabla^h g(u_{h,N}^l) \rightarrow \tilde{\chi}$ weakly in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ as $m \rightarrow +\infty$. With similar arguments as in [20, Lemma 2] and [21, Theorem 14.3] we get the additional regularity $g_u \in L^2(\Omega; L^2(0, T; H^1(\Lambda)))$ and $\tilde{\chi} = \nabla g_u$. Note that the predictability property of g_u with values in $H^1(\Lambda)$ is inherited from $(g(u_{h,N}^l))_m$ at the limit. \square

Remark 4.3. Note that the information $g_u \in L^2_{\mathcal{P}_T}(\Omega \times (0, T); H^1(\Lambda))$ will be one of the keys points in the proof of Proposition 4.5 below.

Lemma 4.4. *There exist not relabeled subsequences of $(\beta(u_{h,N}^r))_m$ and $(\beta(u_{h,N}^l))_m$, and a process β_u in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ such that*

$$\beta(u_{h,N}^r) \rightarrow \beta_u \text{ and } \beta(u_{h,N}^l) \rightarrow \beta_u \text{ weakly in } L^2(\Omega; L^2(0, T; L^2(\Lambda))) \text{ as } m \rightarrow +\infty.$$

Proof. The weak convergence in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ of the sequences $(\beta(u_{h,N}^r))_m$ and $(\beta(u_{h,N}^l))_m$ is a direct consequence of their boundedness properties in such a space stated in Lemma 3.5. The fact that the weak limit β_u is common is due to Inequality (4.1) and again the Lipschitz property of β . \square

Proposition 4.5. *The weak limit u of our finite-volume scheme (2.5)-(2.6) introduced in Proposition 4.1 has \mathbb{P} -a.s. continuous paths with values in $L^2(\Lambda)$ and satisfies for all $t \in [0, T]$,*

$$u(t) = u_0 + \int_0^t \Delta u(s) ds - \int_0^t \operatorname{div}_x(\mathbf{v}(s, \cdot)u(s)) ds + \int_0^t g_u(s) dW(s) + \int_0^t \beta_u(s) ds,$$

in $L^2(\Lambda)$ and \mathbb{P} -a.s. in Ω , where g_u and β_u respectively are given by Lemmas 4.2 and 4.4.

Proof. Let $A \in \mathcal{A}$, $\xi \in \mathcal{D}(\mathbb{R})$ with $\xi(T) = 0$ and $\varphi \in \mathcal{D}(\mathbb{R}^2)$ with $\nabla \varphi \cdot \mathbf{n} = 0$ on $\partial\Lambda$, where we denote $\mathcal{D}(D) := \mathcal{C}_c^\infty(D)$ for any open subset $D \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$. We introduce the discrete function $\varphi_h : \Lambda \rightarrow \mathbb{R}$ defined by $\varphi_h(x) = \sum_{K \in \mathcal{T}} \mathbf{1}_K(x) \varphi(x_K)$ for any $x \in \Lambda$.

For $K \in \mathcal{T}$, $n \in \{0, \dots, N-1\}$ and $t \in [t_n, t_{n+1})$ we multiply (2.7) with $\mathbf{1}_A \xi(t) \varphi(x_K)$ to obtain

$$\begin{aligned} & \mathbf{1}_A \xi(t) \frac{m_K}{\Delta t} \left(u_K^{n+1} - u_K^n - g(u_K^n) (W^{n+1} - W^n) \right) \varphi(x_K) \\ & + \mathbf{1}_A \xi(t) \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \varphi(x_K) \\ & + \mathbf{1}_A \xi(t) \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma}^{n+1} (u_\sigma^{n+1} - u_K^{n+1}) \varphi(x_K) \\ & = \mathbf{1}_A \xi(t) m_K \beta(u_K^{n+1}) \varphi(x_K). \end{aligned} \tag{4.2}$$

Firstly, we sum (4.2) over each control volume $K \in \mathcal{T}$, we integrate over each time interval $[t_n, t_{n+1}]$, then we sum over $n = 0, \dots, N-1$, and finally we take the expectation to obtain

$$S_{1,m} + S_{2,m} + S_{3,m} + S_{4,m} = S_{5,m} \quad (4.3)$$

where

$$\begin{aligned} S_{1,m} &= \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} \mathbf{1}_A \xi(t) m_K \frac{u_K^{n+1} - u_K^n}{\Delta t} \varphi(x_K) dt \right] \\ S_{2,m} &= -\mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} \mathbf{1}_A \xi(t) m_K g(u_K^n) \frac{W^{n+1} - W^n}{\Delta t} \varphi(x_K) dt \right] \\ S_{3,m} &= \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} \mathbf{1}_A \xi(t) \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_\sigma}{d_{K|L}} (u_K^{n+1} - u_L^{n+1}) \varphi(x_K) dt \right] \\ S_{4,m} &= \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} \mathbf{1}_A \xi(t) \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma}^{n+1} (u_\sigma^{n+1} - u_K^{n+1}) \varphi(x_K) dt \right] \\ S_{5,m} &= \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} \mathbf{1}_A \xi(t) m_K \beta(u_K^{n+1}) \varphi(x_K) dt \right]. \end{aligned}$$

Let us study separately the limit as m goes to $+\infty$ of $S_{1,m}$, $S_{2,m}$, $S_{3,m}$, $S_{4,m}$ and $S_{5,m}$.

• Study of $S_{1,m}$: By using Proposition 4.1 and a discrete integration by parts formula, one shows that up to a subsequence denoted in the same way

$$S_{1,m} \xrightarrow{m \rightarrow +\infty} -\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_\Lambda u(t, x) \xi'(t) \varphi(x) dx dt \right] - \mathbb{E} \left[\mathbf{1}_A \int_\Lambda u_0(x) \xi(0) \varphi(x) dx \right].$$

Indeed, if we consider the following discrete integration by parts formula

$$\sum_{n=1}^N a_n (b_n - b_{n-1}) = a_N b_N - a_0 b_0 - \sum_{n=0}^{N-1} b_n (a_{n+1} - a_n)$$

applied to $a_n = u_K^n$ and $b_n = \xi(t_n)$, one has since $\xi(T) = 0$

$$\begin{aligned} & \mathbb{E} \left[\mathbf{1}_A \int_{\Delta t}^T \int_\Lambda u_{h,N}^l(t, x) \xi'(t - \Delta t) \varphi(x) dx dt \right] \\ &= \mathbb{E} \left[\mathbf{1}_A \sum_{n=1}^{N-1} \sum_{K \in \mathcal{T}} \int_{t_n}^{t_{n+1}} \int_K u_K^n \xi'(t - \Delta t) \varphi(x) dx dt \right] \\ &= \mathbb{E} \left[\mathbf{1}_A \sum_{n=1}^{N-1} \sum_{K \in \mathcal{T}} u_K^n (\xi(t_n) - \xi(t_{n-1})) \int_K \varphi(x) dx \right] \\ &= -\mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} u_K^0 \xi(0) \int_K \varphi(x) dx \right] - \mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} \sum_{n=0}^{N-1} (u_K^{n+1} - u_K^n) \xi(t_n) \int_K \varphi(x) dx \right]. \end{aligned}$$

Firstly, thanks to the weak convergence of $(u_{h,N}^l)_m$ towards u in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ given by Proposition 4.1, one gets

$$\mathbb{E} \left[\mathbf{1}_A \int_{\Delta t}^T \int_{\Lambda} u_{h,N}^l(t, x) \xi'(t - \Delta t) \varphi(x) dx dt \right] \xrightarrow{m \rightarrow +\infty} \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\Lambda} u(t, x) \xi'(t) \varphi(x) dx dt \right].$$

Secondly, we have

$$\begin{aligned} & \left| \mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} u_K^0 \xi(0) \int_K \varphi(x) dx \right] - \mathbb{E} \left[\mathbf{1}_A \int_{\Lambda} u_0(x) \xi(0) \varphi(x) dx \right] \right| \\ &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} \xi(0) \int_K (u_K^0 - u_0(x)) \varphi(x) dx \right] \right| \\ &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{K \in \mathcal{T}} \xi(0) \int_K \left(\frac{1}{m_K} \int_K u_0(y) - u_0(x) dy \right) \varphi(x) dx \right] \right| \\ &\leq \|\xi\|_{\infty} \|\varphi\|_{\infty} \mathbb{E} \left[\sum_{K \in \mathcal{T}} \int_K \left(\frac{1}{m_K} \int_K |u_0(y) - u_0(x)| dy \right) dx \right], \end{aligned}$$

and since u_0 belongs particularly to $L^1(\Omega; L^1(\Lambda))$, this last term tends to 0 as m goes to $+\infty$ thank's to Lebesgue's dominated convergence theorem.

• Study of $S_{2,m}$: Thanks to Lemma 4.2 and the properties of the stochastic integral, one shows that, up to a subsequence denoted in the same way,

$$S_{2,m} \xrightarrow{m \rightarrow +\infty} \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\Lambda} \int_0^t g_u(s, x) dW(s) \xi'(t) \varphi(x) dx dt \right].$$

To do so, we introduce the following terms

$$\begin{aligned} \tilde{S}_{2,m} &= -\mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} m_K \mathbf{1}_A \xi(t_n) g(u_K^n) (W^{n+1} - W^n) \varphi(x_K) \right] \\ \hat{S}_{2,m} &= -\mathbb{E} \left[\mathbf{1}_A \int_{\Lambda} \int_0^T \xi(t) g(u_{h,N}^l(t, x)) dW(t) \varphi(x) dx \right] \end{aligned}$$

and decompose $S_{2,m}$ as

$$S_{2,m} = (S_{2,m} - \tilde{S}_{2,m}) + (\tilde{S}_{2,m} - \hat{S}_{2,m}) + \hat{S}_{2,m}$$

and study separately each term of this decomposition.

Firstly, we have by applying successively Cauchy-Schwarz inequality on Ω and then on

the sum over $n \in \{0, \dots, N-1\}$ and $K \in \mathcal{T}$:

$$\begin{aligned}
& |S_{2,m} - \tilde{S}_{2,m}|^2 \\
&= \left| \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} m_K \mathbf{1}_A(\xi(t) - \xi(t_n)) g(u_K^n) \frac{W^{n+1} - W^n}{\Delta t} \varphi(x_K) dt \right] \right|^2 \\
&\leq \left(\mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \sum_{K \in \mathcal{T}} m_K \left| \mathbf{1}_A(\xi(t) - \xi(t_n)) g(u_K^n) \frac{W^{n+1} - W^n}{\Delta t} \varphi(x_K) \right| dt \right] \right)^2 \\
&\leq \frac{(\Delta t)^2}{(\Delta t)^2} \|\xi'\|_\infty^2 \|\varphi\|_\infty^2 \left(\sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} m_K \mathbb{E} \left[\left| \mathbf{1}_A g(u_K^n) (W^{n+1} - W^n) \right| \right] \right)^2 \\
&\leq \|\xi'\|_\infty^2 \|\varphi\|_\infty^2 \left(\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \Delta t m_K (\mathbb{E}[\mathbf{1}_A])^{\frac{1}{2}} \times \left\{ \mathbb{E} \left[g^2(u_K^n) (W^{n+1} - W^n)^2 \right] \right\}^{\frac{1}{2}} \right)^2 \\
&\leq \|\xi'\|_\infty^2 \|\varphi\|_\infty^2 \left(\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \Delta t m_K \right) \times \left(\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \Delta t m_K \mathbb{E} \left[g^2(u_K^n) (W^{n+1} - W^n)^2 \right] \right) \\
&\leq \|\xi'\|_\infty^2 \|\varphi\|_\infty^2 T |\Lambda| \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \Delta t m_K \mathbb{E} \left[g^2(u_K^n) \right] \times \mathbb{E} \left[(W^{n+1} - W^n)^2 \right] \\
&\leq \Delta t \|\xi'\|_\infty^2 \|\varphi\|_\infty^2 |\Lambda| T C_{L_g} (1 + \|u_{h,N}^l\|_{L^2(\Omega; L^2(0,T; L^2(\Lambda)))}^2),
\end{aligned}$$

which tends to 0 as m goes to $+\infty$.

Secondly, we start by rewriting $\tilde{S}_{2,m} - \hat{S}_{2,m}$ in the following manner

$$\begin{aligned}
-(\tilde{S}_{2,m} - \hat{S}_{2,m}) &= \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} m_K \int_{t_n}^{t_{n+1}} \xi(t_n) g(u_K^n) dW(t) \varphi(x_K) \right] \\
&\quad - \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} m_K \int_{t_n}^{t_{n+1}} \xi(t) g(u_K^n) dW(t) \varphi(x_K) \right] \\
&\quad + \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} m_K \int_{t_n}^{t_{n+1}} \xi(t) g(u_K^n) dW(t) \varphi(x_K) \right] \\
&\quad - \mathbb{E} \left[\mathbf{1}_A \int_\Lambda \int_0^T \xi(t) g(u_{h,N}^l(t, x)) dW(t) \varphi(x) dx \right].
\end{aligned} \tag{4.4}$$

Our aim is to analyse the difference between the two first terms (the two last terms, respectively) on the right hand side of (4.4). To do so, we involve the discrete function φ_h introduced at the beginning of the proof. Using successively Cauchy-Schwarz inequality

on $\Omega \times \Lambda$ and Itô isometry, one gets since $\mathbb{P}(A) \leq 1$

$$\begin{aligned}
& \left| \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} m_K \int_{t_n}^{t_{n+1}} (\xi(t_n) - \xi(t)) g(u_K^n) dW(t) \varphi(x_K) \right] \right| \\
& \leq \sum_{n=0}^{N-1} \left| \mathbb{E} \left[\mathbf{1}_A \int_{\Lambda} \int_{t_n}^{t_{n+1}} (\xi(t_n) - \xi(t)) g(u_{h,N}^l(t, x)) \varphi_h(x) dW(t) dx \right] \right| \\
& \leq \sqrt{|\Lambda|} \sum_{n=0}^{N-1} \left(\mathbb{E} \left[\int_{\Lambda} \left\{ \int_{t_n}^{t_{n+1}} (\xi(t_n) - \xi(t)) g(u_{h,N}^l(t, x)) \varphi_h(x) dW(t) \right\}^2 dx \right] \right)^{\frac{1}{2}} \\
& = \sqrt{|\Lambda|} \sum_{n=0}^{N-1} \left(\mathbb{E} \left[\int_{\Lambda} \int_{t_n}^{t_{n+1}} \left\{ (\xi(t_n) - \xi(t)) g(u_{h,N}^l(t, x)) \varphi_h(x) \right\}^2 dt dx \right] \right)^{\frac{1}{2}} \\
& \leq \sqrt{\Delta t} \sqrt{|\Lambda|} \|\xi'\|_{\infty} \|\varphi\|_{\infty} L_g \sum_{n=0}^{N-1} \Delta t \left(\sup_{t \in [0, T]} \mathbb{E} \left[\int_{\Lambda} |u_{h,N}^l(t, x)|^2 dx \right] \right)^{\frac{1}{2}}
\end{aligned}$$

which tends to 0 thanks to the bound given by Proposition 3.2. Let us now study the difference between the two last terms of the right hand side of (4.4). Using again Cauchy-Schwarz inequality on $\Omega \times \Lambda$ and Itô isometry, one arrives at

$$\begin{aligned}
& \left| \mathbb{E} \left[\mathbf{1}_A \int_{\Lambda} \int_0^T \xi(t) g(u_{h,N}^l(t, x)) (\varphi_h(x) - \varphi(x)) dW(t) dx \right] \right|^2 \\
& \leq |\Lambda| \times \mathbb{E} \left[\int_{\Lambda} \left\{ \int_0^T \xi(t) g(u_{h,N}^l(t, x)) (\varphi_h(x) - \varphi(x)) dW(t) \right\}^2 dx \right] \\
& = |\Lambda| \times \mathbb{E} \left[\int_{\Lambda} \int_0^T \left\{ \xi(t) g(u_{h,N}^l(t, x)) (\varphi_h(x) - \varphi(x)) \right\}^2 dt dx \right] \\
& \leq h^2 \|\xi\|_{\infty}^2 \|\nabla \varphi\|_{\infty}^2 |\Lambda| \mathbb{E} \left[\int_{\Lambda} \int_0^T C_{L_g} (1 + |u_{h,N}^l(t, x)|^2) dt dx \right],
\end{aligned}$$

which tends to 0 as m goes to $+\infty$ thanks to the control on $\mathbb{E} \left[\int_{\Lambda} \int_0^T |u_{h,N}^l(t, x)|^2 dt dx \right]$ given by Lemma 3.3. Finally, one can affirm that $\tilde{S}_{2,m} - \hat{S}_{2,m} \xrightarrow{m \rightarrow +\infty} 0$.

Thirdly, we show that the following convergence result holds:

$$\hat{S}_{2,m} \xrightarrow{m \rightarrow +\infty} -\mathbb{E} \left[\mathbf{1}_A \int_{\Lambda} \int_0^T \xi(s) g_u(s, x) dW(s) \varphi(x) dx \right].$$

To do so, recall that, thanks to Itô isometry, the linear application $I_T : L^2_{\mathcal{P}_T}(\Omega \times (0, T); L^2(\Lambda)) \rightarrow L^2(\Omega; L^2(\Lambda))$ defined by

$$I_T : X \mapsto \int_0^T X(\omega, t, x) dW(t)$$

is continuous. Set $(X_m)_m = (g(u_{h,N}^l)\xi)_m$, then thanks to Lemma 4.2, up to a subsequence denoted in the same way, $(X_m)_m$ converges weakly towards $g_u\xi$ in $L^2_{\mathcal{P}_T}(\Omega \times (0, T); L^2(\Lambda))$ and so since $\varphi\mathbb{1}_A \in L^2(\Omega; L^2(\Lambda))$ one gets

$$\begin{aligned}\hat{S}_{2,m} &= - \int_{\Omega} \int_{\Lambda} I_T(X_m)(x, \omega) \varphi(x) \mathbb{1}_A(\omega) dx d\mathbb{P}(\omega) \\ &\xrightarrow{m \rightarrow +\infty} - \int_{\Omega} \int_{\Lambda} I_T(g_u\xi)(\omega, x) \varphi(x) \mathbb{1}_A(\omega) dx d\mathbb{P}(\omega) \\ &= -\mathbb{E} \left[\mathbb{1}_A \int_{\Lambda} \int_0^T \xi(s) g_u(s, x) dW(s) \varphi(x) dx \right]\end{aligned}$$

Finally, thanks to Itô formula (see [14] Theorem 4.17 p.105) we may apply a rule of stochastic integration by parts to conclude

$$-\mathbb{E} \left[\mathbb{1}_A \int_{\Lambda} \int_0^T \xi(s) g_u(s, x) dW(s) \varphi(x) dx \right] = \mathbb{E} \left[\mathbb{1}_A \int_{\Lambda} \int_0^T \int_0^t g_u(s, x) dW(s) \xi'(t) \varphi(x) dt dx \right].$$

• Study of $S_{3,m}$: Following the arguments we developed in [8] (proof of Proposition 4.16), one shows that

$$S_{3,m} \xrightarrow{m \rightarrow +\infty} -\mathbb{E} \left[\mathbb{1}_A \int_0^T \int_{\Lambda} \xi(t) \Delta\varphi(x) u(t, x) dx dt \right].$$

Indeed, thanks to a discrete integration by part formula, $S_{3,m}$ can be written as

$$S_{3,m} = \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{1}_A \xi(t) \sum_{K \in \mathcal{T}} u_K^{n+1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_{\sigma} \left(\frac{\varphi(x_K) - \varphi(x_L)}{d_{K|L}} \right) dt \right].$$

Then, since $\nabla\varphi \cdot \mathbf{n} = 0$ on $\partial\Lambda$, thanks to the Green-Ostrogradski Theorem one has,

$$\int_K \Delta\varphi(x) dx = \int_{\partial K} \nabla\varphi(x) \cdot \mathbf{n}(x) d\gamma(x) = \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \int_{\sigma} \nabla\varphi(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x).$$

Thus, we have

$$\begin{aligned}S_{3,m} &= \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{1}_A \xi(t) \sum_{K \in \mathcal{T}} u_K^{n+1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_{\sigma} \left(\frac{\varphi(x_K) - \varphi(x_L)}{d_{K|L}} \right) dt \right] \\ &\quad - \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{1}_A \xi(t) \sum_{K \in \mathcal{T}} u_K^{n+1} \left(\int_K \Delta\varphi(x) dx - \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \int_{\sigma} \nabla\varphi(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x) \right) dt \right] \\ &= -\mathbb{E} \left[\int_0^T \mathbb{1}_A \xi(t) \int_{\Lambda} u_{h,N}^r(t, x) \Delta\varphi(x) dx dt \right] \\ &\quad + \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{1}_A \xi(t) \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} m_{\sigma} (u_K^{n+1} - u_L^{n+1}) R_{\sigma}^{\varphi} dt \right],\end{aligned}$$

with

$$R_\sigma^\varphi = \frac{1}{m_\sigma} \int_\sigma \nabla \varphi(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x) - \frac{\varphi(x_L) - \varphi(x_K)}{d_{K|L}}.$$

Using Proposition 4.1 and up to a subsequence denoted in the same way, one gets

$$-\mathbb{E} \left[\int_0^T \int_\Lambda \mathbf{1}_A \xi(t) u_{h,N}^r(t, x) \Delta \varphi(x) dx dt \right] \xrightarrow{m \rightarrow +\infty} -\mathbb{E} \left[\int_0^T \int_\Lambda \mathbf{1}_A \xi(t) u(t, x) \Delta \varphi(x) dx dt \right].$$

Note that one is able to control the deterministic rest R_σ^φ since for any $\sigma = K|L \in \mathcal{E}_{\text{int}}$, the orthogonality condition implies $x_L - x_K = d_{K|L} \mathbf{n}_{K,\sigma}$, thus thanks to the Taylor formula there exists a constant $C_\varphi > 0$ only depending on φ such that for any $\sigma \in \mathcal{E}_{\text{int}}$ one has

$$|R_\sigma^\varphi| \leq C_\varphi h.$$

Therefore, thanks to the Cauchy-Schwarz inequality and Lemma 3.4, one finally obtains

$$\begin{aligned} & \left| \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbf{1}_A \xi(t) \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} m_\sigma (u_K^{n+1} - u_L^{n+1}) R_\sigma^\varphi dt \right] \right| \\ & \leq C_\varphi h \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbf{1}_A |\xi(t)| \left(\sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} m_\sigma d_{K|L} \right)^{\frac{1}{2}} \left(\sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} m_\sigma \frac{|u_K^{n+1} - u_L^{n+1}|^2}{d_{K|L}} \right)^{\frac{1}{2}} dt \right] \\ & \leq \sqrt{2} C_\varphi |\Lambda|^{\frac{1}{2}} h \mathbb{E} \left[\int_0^T \mathbf{1}_A |\xi(t)| |u_{h,N}^r(t)|_{1,h} dt \right] \\ & \leq h \sqrt{2} C_\varphi |\Lambda|^{\frac{1}{2}} \|\xi \mathbf{1}_A\|_{L^2(\Omega \times (0,T))} \left(\mathbb{E} \left[\int_0^T |u_{h,N}^r(t)|_{1,h}^2 dt \right] \right)^{\frac{1}{2}} \xrightarrow{m \rightarrow +\infty} 0. \end{aligned}$$

• Study of $S_{4,m}$: Adapting to the stochastic case arguments exposed in [21] p.774-776 in the deterministic and elliptic case, one shows that

$$S_{4,m} \xrightarrow{m \rightarrow +\infty} -\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_\Lambda (\mathbf{v}(t, x) u(t, x)) \cdot \nabla \varphi(x) \xi(t) dx dt \right].$$

To do so, we decompose the term

$$S_{4,m} = \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbf{1}_A \xi(t) \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma v_{K,\sigma}^{n+1} (u_\sigma^{n+1} - u_K^{n+1}) \varphi(x_K) dt \right]$$

in the following manner: $S_{4,m} = S_{4,m} - \tilde{S}_{4,m} + \tilde{S}_{4,m} - \hat{S}_{4,m} + \hat{S}_{4,m}$ where

$$\tilde{S}_{4,m} = \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbf{1}_A \xi(t) \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} (u_\sigma^{n+1} - u_K^{n+1}) \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_\sigma \mathbf{v}(s, x) \cdot \mathbf{n}_{K,\sigma} \varphi(x) d\gamma(x) ds dt \right]$$

$$\hat{S}_{4,m} = -\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_\Lambda \xi(t) u_{h,N}^r(t, x) \operatorname{div}_x (\mathbf{v}(t, x) \varphi(x)) dx dt \right].$$

Firstly, one shows that $|S_{4,m} - \tilde{S}_{4,m}|$ tends to 0 as m goes to $+\infty$. Indeed, by noticing that

$$\left| \frac{1}{m_\sigma \Delta t} \int_{t_n}^{t_{n+1}} \int_\sigma \mathbf{v}(s, x) \cdot \mathbf{n}_{K,\sigma} (\varphi(x_K) - \varphi(x)) d\gamma(x) ds \right| \leq h \|\mathbf{v}\|_\infty \|\nabla \varphi\|_\infty$$

one obtains

$$\begin{aligned} |S_{4,m} - \tilde{S}_{4,m}| &= \left| \mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbf{1}_A \xi(t) \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_\sigma (u_\sigma^{n+1} - u_K^{n+1}) \right. \right. \\ &\quad \left. \left. \times \frac{1}{m_\sigma \Delta t} \int_{t_n}^{t_{n+1}} \int_\sigma \mathbf{v}(s, x) \cdot \mathbf{n}_{K,\sigma} (\varphi(x_K) - \varphi(x)) d\gamma(x) ds dt \right] \right| \\ &\leq \|\mathbf{v}\|_\infty \|\nabla \varphi\|_\infty \|\xi\|_\infty h \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \Delta t m_\sigma \mathbb{E} [|u_\sigma^{n+1} - u_K^{n+1}|] \\ &\leq \|\mathbf{v}\|_\infty \|\nabla \varphi\|_\infty \|\xi\|_\infty h \left(\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \Delta t m_\sigma d_{K|L} \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \Delta t \frac{m_\sigma}{d_{K|L}} \mathbb{E} [|u_K^{n+1} - u_L^{n+1}|^2] \right)^{\frac{1}{2}} \\ &\leq h \|\mathbf{v}\|_\infty \|\nabla \varphi\|_\infty \|\xi\|_\infty \sqrt{2T|\Lambda|} \left(\int_0^T \mathbb{E} [|u_{h,N}^r(t)|_{1,h}^2] dt \right)^{\frac{1}{2}} \xrightarrow{m \rightarrow +\infty} 0. \end{aligned}$$

Secondly, one shows that $|\tilde{S}_{4,m} - \hat{S}_{4,m}|$ tends to 0 as m goes to $+\infty$. Indeed, for any $\sigma = K|L \in \mathcal{E}_{\text{int}}$, since $\mathbf{n}_{K,\sigma} = -\mathbf{n}_{L,\sigma}$ then $v_{K,\sigma}^{n+1} = -v_{L,\sigma}^{n+1}$ and so one remarks that $\tilde{S}_{4,m}$ can be rewritten as

$$\tilde{S}_{4,m} = -\mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbf{1}_A \xi(t) \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} u_K^{n+1} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_\sigma \mathbf{v}(s, x) \cdot \mathbf{n}_{K,\sigma} \varphi(x) d\gamma(x) ds dt \right].$$

In this way, using the divergence-free property of \mathbf{v} , one gets

$$\begin{aligned} \tilde{S}_{4,m} &= -\mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbf{1}_A \xi(t) \sum_{K \in \mathcal{T}} u_K^{n+1} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_{\partial K} \mathbf{v}(s, x) \cdot \mathbf{n}_K(x) \varphi(x) d\gamma(x) ds dt \right] \\ &= -\mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbf{1}_A \xi(t) \sum_{K \in \mathcal{T}} u_K^{n+1} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_K \text{div}_x (\mathbf{v}(s, x) \varphi(x)) dx ds dt \right] \\ &= -\mathbb{E} \left[\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbf{1}_A \xi(t) \sum_{K \in \mathcal{T}} u_K^{n+1} \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \int_K \mathbf{v}(s, x) \cdot \nabla \varphi(x) dx ds dt \right]. \end{aligned}$$

In the same manner, we write $\hat{S}_{4,m}$ as

$$\hat{S}_{4,m} = -\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\Lambda} \xi(t) u_{h,N}^r(t, x) \mathbf{v}(t, x) \cdot \nabla \varphi(x) dx dt \right]$$

and then

$$\begin{aligned} & \left| \tilde{S}_{4,m} - \hat{S}_{4,m} \right| \\ &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} u_K^{n+1} \int_{t_n}^{t_{n+1}} \xi(t) \int_K \left\{ \mathbf{v}(t, x) \cdot \nabla \varphi(x) - \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \mathbf{v}(s, x) \cdot \nabla \varphi(x) ds \right\} dx dt \right] \right| \\ &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} u_K^{n+1} \int_{t_n}^{t_{n+1}} \xi(t) \int_K \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \left\{ \mathbf{v}(t, x) - \mathbf{v}(s, x) \right\} \cdot \nabla \varphi(x) ds dx dt \right] \right| \\ &\leq \Delta t \|\xi\|_{\infty} \|\partial_t \mathbf{v}\|_{\infty} \|\nabla \varphi\|_{\infty} \|u_{h,N}^r\|_{L^2(\Omega; L^2(0, T; L^2(\Lambda)))} \sqrt{T|\Lambda|} \xrightarrow{m \rightarrow +\infty} 0. \end{aligned}$$

Thirdly, owing to the weak convergence of $(u_{h,N}^r)_m$ towards u in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ given by Proposition 4.1, one gets directly that

$$\begin{aligned} \hat{S}_{4,m} &= -\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\Lambda} u_{h,N}^r(t, x) \operatorname{div}_x (\mathbf{v}(t, x) \varphi(x)) \xi(t) dx dt \right] \\ &\xrightarrow{m \rightarrow +\infty} -\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\Lambda} u(t, x) \operatorname{div}_x (\mathbf{v}(t, x) \varphi(x)) \xi(t) dx dt \right] \\ &= -\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\Lambda} (\mathbf{v}(t, x) u(t, x)) \cdot \nabla \varphi(x) \xi(t) dx dt \right] \end{aligned}$$

• Study of $S_{5,m}$: Thanks to Lemma 4.4, one shows that

$$S_{5,m} \xrightarrow{m \rightarrow +\infty} \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\Lambda} \beta_u(t, x) \varphi(x) \xi(t) dx dt \right].$$

Indeed, let us remark that $S_{5,m}$ can be decomposed as $S_{5,m} = S_{5,m} - \tilde{S}_{5,m} + \tilde{S}_{5,m}$ where

$$\tilde{S}_{5,m} = \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\Lambda} \xi(t) \beta(u_{h,N}^r) \varphi(x) dx dt \right] \xrightarrow{m \rightarrow +\infty} \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\Lambda} \beta_u(t, x) \varphi(x) \xi(t) dx dt \right]$$

and

$$\begin{aligned} |S_{5,m} - \tilde{S}_{5,m}| &= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{t_n}^{t_{n+1}} \int_K \xi(t) \beta(u_K^{n+1}) (\varphi(x_K) - \varphi(x)) dx dt \right] \right| \\ &\leq h \|\xi\|_{\infty} \|\nabla \varphi\|_{\infty} L_{\beta} \|u_{h,N}^r\|_{L^1(\Omega; L^1(0, T; L^1(\Lambda)))} \xrightarrow{m \rightarrow +\infty} 0. \end{aligned}$$

Gathering all the previous convergence results, we pass to the limit in (4.3) to get that P-a.s. in Ω , for all $\xi \in \{\phi \in \mathcal{D}(\mathbb{R}) : \phi(T) = 0\}$ and all $\varphi \in \{\psi \in \mathcal{D}(\mathbb{R}^2) : \nabla\psi \cdot \mathbf{n} = 0 \text{ on } \partial\Lambda\}$,

$$\begin{aligned}
& - \int_0^T \int_{\Lambda} \left(u(t, x) - \int_0^t g_u(s, x) dW(s) \right) \xi'(t) \varphi(x) dx dt - \int_{\Lambda} u_0(x) \xi(0) \varphi(x) dx \\
& - \int_0^T \int_{\Lambda} (\mathbf{v}(t, x) u(t, x)) \cdot \nabla \varphi(x) \xi(t) dx dt \\
& = \int_0^T \int_{\Lambda} u(t, x) \Delta \varphi(x) \xi(t) dx dt + \int_0^T \int_{\Lambda} \beta_u(t, x) \varphi(x) \xi(t) dx dt
\end{aligned} \tag{4.5}$$

which can be rewritten as

$$\begin{aligned}
& - \int_0^T \int_{\Lambda} \left(u(t, x) - \int_0^t g_u(s, x) dW(s) \right) \xi'(t) \varphi(x) dx dt - \int_{\Lambda} u_0(x) \xi(0) \varphi(x) dx \\
& + \int_0^T \int_{\Lambda} \operatorname{div}_x (\mathbf{v}(t, x) u(t, x)) \varphi(x) \xi(t) dx dt \\
& = - \int_0^T \int_{\Lambda} \nabla u(t, x) \cdot \nabla \varphi(x) \xi(t) dx dt + \int_0^T \int_{\Lambda} \beta_u(t, x) \varphi(x) \xi(t) dx dt,
\end{aligned} \tag{4.6}$$

since $\mathbf{v}(t, x) \cdot \mathbf{n}(x) = 0$ and $\nabla \varphi(x) \cdot \mathbf{n}(x) = 0$ for any $(t, x) \in [0, T] \times \partial\Lambda$. By [19, Theorem 1.1] the set $\{\psi \in \mathcal{D}(\mathbb{R}^2) \mid \nabla\psi \cdot \mathbf{n} = 0 \text{ on } \partial\Lambda\}$ is dense in $H^1(\Lambda)$ and therefore (4.6) applies to all $\varphi \in H^1(\Lambda)$. In the following, we denote the dual space of $H^1(\Lambda)$ by $H^1(\Lambda)^*$. Recall that we have the following continuous and dense embeddings

$$H^1(\Lambda) \hookrightarrow L^2(\Lambda) \hookrightarrow H^1(\Lambda)^*.$$

Let us denote the $H^1(\Lambda)$ - $H^1(\Lambda)^*$ duality bracket by $\langle \cdot, \cdot \rangle$ and the $L^2(\Lambda)$ scalar product by (\cdot, \cdot) . From Proposition 4.1, we know that the weak limit u belongs to $L^2(\Omega; L^2(0, T; H^1(\Lambda)))$ and thus it follows that

$$\Delta u \in L^2(\Omega; L^2(0, T; H^1(\Lambda)^*))$$

which yields

$$- \int_0^T \int_{\Lambda} \nabla u(t, x) \cdot \nabla \varphi(x) \xi(t) dx dt = \int_0^T \langle \Delta u(t, \cdot), \varphi \rangle \xi(t) dt \tag{4.7}$$

P-a.s. in Ω , for all $\xi \in \{\phi \in \mathcal{D}(\mathbb{R}) : \phi(T) = 0\}$ and all $\varphi \in H^1(\Lambda)$. Combining (4.6) with (4.7) and with the identity

$$- \int_{\Lambda} u_0(x) \varphi(x) \xi(0) dx = \int_0^T \int_{\Lambda} u_0(x) \varphi(x) \xi'(t) dx dt \tag{4.8}$$

(see, [30, Lemma 7.3]), from Fubini's theorem it follows that

$$\begin{aligned}
& \left\langle - \int_0^T \left(u(t) - \int_0^t g_u(s) dW(s) - u_0 \right) \xi'(t) dt, \varphi \right\rangle \\
& = \left\langle \int_0^T \left(\Delta u(t) - \operatorname{div}_x (\mathbf{v}(t, \cdot) u(t)) + \beta_u(t) \right) \xi(t) dt, \varphi \right\rangle,
\end{aligned}$$

\mathbb{P} -a.s. in Ω , for all $\xi \in \{\phi \in \mathcal{D}(\mathbb{R}) : \phi(T) = 0\}$ and all $\varphi \in H^1(\Lambda)$. Therefore

$$-\int_0^T \left(u(t) - \int_0^t g_u(s) dW(s) - u_0 \right) \xi'(t) dt = \int_0^T \left(\Delta u(t) - \operatorname{div}_x(\mathbf{v}(t, \cdot)u(t)) + \beta_u(t) \right) \xi(t) dt$$

in $H^1(\Lambda)^*$, for all $\xi \in \{\phi \in \mathcal{D}(\mathbb{R}) : \phi(T) = 0\}$, \mathbb{P} -a.s. in Ω since, by a separability argument, the exceptional set in Ω may be chosen independently of φ . Consequently, (see, e.g. [12, Proposition A6])

$$\left(u - \int_0^\cdot g_u dW - u_0 \right) \in W^{1,2}(0, T; H^1(\Lambda)^*) \quad \mathbb{P}\text{-a.s. in } \Omega$$

and so

$$\frac{d}{dt} \left(u(t) - \int_0^t g_u(s) dW(s) - u_0 \right) = \Delta u - \operatorname{div}_x(\mathbf{v}u) + \beta_u \text{ in } L^2(\Omega; L^2(0, T; H^1(\Lambda)^*)). \quad (4.9)$$

Since $g_u \in L^2_{\mathcal{P}_T}(\Omega \times (0, T); H^1(\Lambda))$, it follows thanks to the properties of the stochastic integral that (see [14] Proposition 4.15 p.104 or [29] Lemma 2.4.1 p.35)

$$\nabla \left(\int_0^\cdot g_u dW \right) = \int_0^\cdot \nabla g_u dW,$$

hence $u - \int_0^\cdot g_u dW$ is an element of $L^2(\Omega; L^2(0, T; H^1(\Lambda)))$. Note that the remaining of the proof is very similar to the one exposed in our previous paper [8], but for the sake of clarity, we decide to detail it again. From [30, Lemma 7.3] we obtain at first that $u \in L^2(\Omega; \mathcal{C}([0, T]; L^2(\Lambda)))$ and together with (4.9), the following rule of partial integration for all $0 \leq t \leq T$, \mathbb{P} -a.s. in Ω :

$$\begin{aligned} & \left(u(t) - \int_0^t g_u(\tau) dW(\tau) - u_0, \zeta(t) \right) - (u(0) - u_0, \zeta(0)) \\ &= \int_0^t \langle \Delta u(s) - \operatorname{div}_x(\mathbf{v}(s, \cdot)u(s)) + \beta_u(s), \zeta(s) \rangle ds \\ & \quad + \int_0^t \left\langle \partial_t \zeta(s), u(s) - \int_0^s g_u(\tau) dW(\tau) - u_0 \right\rangle ds \end{aligned} \quad (4.10)$$

for all $\zeta \in L^2(0, T; H^1(\Lambda))$ with $\partial_t \zeta \in L^2(0, T; H^1(\Lambda)^*)$. Choosing $\zeta : (t, x) \mapsto \xi(t)\varphi(x)$ with $\varphi \in H^1(\Lambda)$, $\xi \in \{\phi \in \mathcal{D}(\mathbb{R}) : \phi(T) = 0\}$ in (4.10), we get

$$\begin{aligned} & \left(u(t) - \int_0^t g_u(\tau) dW(\tau) - u_0, \varphi \right) \xi(t) - (u(0) - u_0, \varphi) \xi(0) \\ &= \int_0^t \xi(s) \langle \Delta u(s) - \operatorname{div}_x(\mathbf{v}(s, \cdot)u(s)) + \beta_u(s), \varphi \rangle ds \\ & \quad + \int_0^t \xi'(s) \left(u(s) - \int_0^s g_u(\tau) dW(\tau) - u_0, \varphi \right) ds \end{aligned} \quad (4.11)$$

\mathbb{P} -a.s. in Ω . The particular choice of $t = T$ and $\xi \in \mathcal{D}(\mathbb{R})$ with $\xi(T) = 0$ and $\xi(0) = 1$ in (4.11) combined with (4.5), (4.7) and (4.8) yields

$$(u(0) - u_0, \varphi) = 0 \quad \text{for all } \varphi \in H^1(\Lambda), \mathbb{P}\text{-a.s. in } \Omega$$

and therefore $u(0) = u_0$ \mathbb{P} -a.s. in Ω .

Now, we fix $t \in [0, T]$ and choose $\xi \in \mathcal{D}(\mathbb{R})$ with $\xi(T) = 0$ and $\xi(s) = 1$ for all $s \in [0, t]$. With this choice, from (4.11) we obtain

$$\left(u(t) - \int_0^t g_u(s) dW(s) - u(0), \varphi \right) = \int_0^t \langle \Delta u(s) - \operatorname{div}_x (\mathbf{v}(s, \cdot)u(s)) + \beta_u(s), \varphi \rangle ds \quad (4.12)$$

\mathbb{P} -a.s. in Ω for all $\varphi \in H^1(\Lambda)$. Since, for fixed $\varphi \in H^1(\Lambda)$,

$$t \mapsto \left(u(t) - \int_0^t g_u(s) dW(s) - u(0), \varphi \right)$$

$$\text{and } t \mapsto \int_0^t \langle \Delta u(s) - \operatorname{div}_x (\mathbf{v}(s, \cdot)u(s)) + \beta_u(s), \varphi \rangle ds$$

are continuous on $[0, T]$, \mathbb{P} -a.s. in Ω , the exceptional set in Ω in (4.12) may be chosen independently of $t \in [0, T]$ and (4.12) also holds for $t = T$. This yields

$$u(t) - u(0) - \int_0^t g_u(s) dW(s) + \int_0^t \operatorname{div}_x (\mathbf{v}(s, \cdot)u(s)) ds - \int_0^t \beta_u(s) ds = \int_0^t \Delta u(s) ds,$$

in $H^1(\Lambda)^*$ and \mathbb{P} -a.s. in Ω . To conclude, let us mention that since the left-hand side of the above equality is in $L^2(\Lambda)$, it also holds in $L^2(\Lambda)$. \square

Lemma 4.6. (*Stochastic energy equality*) For any $c > 0$, the stochastic process u introduced in Proposition 4.1 satisfies the following stochastic energy equality:

$$\begin{aligned} & e^{-ct} \mathbb{E} \left[\|u(t)\|_{L^2(\Lambda)}^2 \right] + 2 \int_0^t e^{-cs} \mathbb{E} \left[\|\nabla u(s)\|_{L^2(\Lambda)}^2 \right] ds \\ &= \mathbb{E} \left[\|u_0\|_{L^2(\Lambda)}^2 \right] - c \int_0^t e^{-cs} \mathbb{E} \left[\|u(s)\|_{L^2(\Lambda)}^2 \right] ds + \int_0^t e^{-cs} \mathbb{E} \left[\|g_u(s)\|_{L^2(\Lambda)}^2 \right] ds \\ &+ 2 \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta_u(s, x) u(s, x) dx \right] ds, \quad \forall t \in [0, T]. \end{aligned} \quad (4.13)$$

Proof. It is a direct application of Itô formula to the stochastic process u and the functional $\Psi : (t, v) \mapsto e^{-ct} \|v\|_{L^2(\Lambda)}^2$ defined on $[0, T] \times L^2(\Lambda)$. Let us precise that in the application of Itô formula, the following contribution of the flux term appears

$$2 \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} (\mathbf{v}(s, x)u(s, x)) \cdot \nabla u(s, x) dx \right] ds$$

and since \mathbf{v} is divergence-free and satisfies $\mathbf{v} \cdot \mathbf{n} = 0$ on $[0, T] \times \partial\Lambda$, one gets

$$\begin{aligned} & \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} (\mathbf{v}(s, x) u(s, x)) \cdot \nabla u(s, x) dx \right] ds \\ &= \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \mathbf{v}(s, x) \cdot \nabla \left(\int_0^{u(s, x)} z dz \right) dx \right] ds \\ &= 0. \end{aligned}$$

□

4.2 Identification of weak limits coming from the non-linear terms

The next result gives a lower bound on the inferior limit of the integral over the interval $[0, T]$ of an exponential weight in time norm of $\mathbb{E}[|u_{h_m, N_m}^r(\cdot)|_{1, h}^2]$. This result is needed for the identification of the weak limits g_u and β_u of respectively $(g(u_{h, N}^l))_m$ and $(\beta(u_{h, N}^r))_m$. Indeed, in such an identification procedure, we are led to make appear the term $\int_0^T \int_0^t e^{-cs} \mathbb{E} [\int_{\Lambda} |\nabla u(s, x)|^2 dx] ds dt$ (with a constant $c > 0$), and the following lemma is essential for that.

Lemma 4.7. *For any $c > 0$, the stochastic process u introduced in Proposition 4.1 satisfies the following inequality:*

$$\int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} |\nabla u(x, s)|^2 dx \right] ds dt \leq \liminf_{m \rightarrow +\infty} \int_0^T \int_0^t e^{-cs} \mathbb{E}[|u_{h_m, N_m}^r(s)|_{1, h_m}^2] ds dt. \quad (4.14)$$

Proof. The idea is to generalize the Lemma 2.2 of [23] to the evolutionary in time and stochastic case. Following their work, we start by introducing an approximation of u constructed from several density results. For this we will need the following functional spaces:

$$\begin{aligned} \mathcal{D}(\bar{\Lambda}) &= \{ \varphi|_{\Lambda}, \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^2) \} \\ \mathcal{V} &= \{ \varphi \in \mathcal{D}(\bar{\Lambda}) : \nabla \varphi \cdot \mathbf{n} = 0 \text{ on } \partial\Lambda \} \\ \mathcal{V} &= \left\{ \sum_{k \text{ finite}} \xi_k \varphi_k, \xi_k \in \mathcal{C}_c^\infty(0, T) \text{ and } \varphi_k \in \mathcal{V} \right\}. \end{aligned}$$

Firstly, note that from [2] (Chapter 6, Remark 6.2.1.p.223), \mathcal{V} is a dense part of $H^1(\Lambda)$, and secondly, from [18] (Corollary 1.3.1 p.13), \mathcal{V} is dense in $L^2(0, T; H^1(\Lambda))$. Now owing to [18] (Proposition 1.3.1 p.13), since u belongs to $L^2(\Omega; L^2(0, T; H^1(\Lambda)))$, then there exists a sequence $(u_p)_{p \in \mathbb{N}^*}$ of simple random variables $u_p = \sum_{i=1}^p \mathbb{1}_{A_i} \vartheta_i$, where for any i in $\{1, \dots, p\}$, $A_i \in \mathcal{A}$, $\vartheta_i \in L^2(0, T; H^1(\Lambda))$ and such that

$$u_p \xrightarrow{p \rightarrow +\infty} u \text{ in } L^2(\Omega; L^2(0, T; H^1(\Lambda))).$$

Set $p \in \mathbb{N}^*$, then for $i \in \{1, \dots, p\}$ since $\vartheta_i \in L^2(0, T; H^1(\Lambda))$ there exists a sequence $(\vartheta_{i, \epsilon_i})_{\epsilon_i > 0} \subset \mathcal{V}$ such that

$$\vartheta_{i, \epsilon_i} \xrightarrow{\epsilon_i \rightarrow 0} \vartheta_i \text{ in } L^2(0, T; H^1(\Lambda)).$$

Note that since p is finite, we can introduce a common $\epsilon > 0$ such that

$$\vartheta_{i, \epsilon} \xrightarrow{\epsilon \rightarrow 0} \vartheta_i \text{ in } L^2(0, T; H^1(\Lambda)).$$

Then we define $u_{p, \epsilon} = \sum_{i=1}^p \mathbb{1}_{A_i} \vartheta_{i, \epsilon}$ and we have for any fixed p ,

$$u_{p, \epsilon} \xrightarrow{\epsilon \rightarrow 0} u_p \text{ in } L^2(\Omega; L^2(0, T; H^1(\Lambda))).$$

Set $c > 0$, $p \in \mathbb{N}^*$ and $\epsilon > 0$. We denote $u_{p, \epsilon}^{n, K} = u_{p, \epsilon}(t_n, x_K)$ \mathbb{P} -a.s. in Ω , for any $n \in \{0, \dots, N-1\}$ and any $K \in \mathcal{T}$. Due to the weak convergence of the finite-volume approximation $(u_{h, N}^r)_m$ towards u in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$, we have

$$\begin{aligned} & \mathbb{E} \left[\int_0^T e^{-ct} \int_{\Lambda} \nabla u(t, x) \cdot \nabla u_{p, \epsilon}(t, x) dx dt \right] \\ &= - \mathbb{E} \left[\int_0^T e^{-ct} \int_{\Lambda} u(t, x) \operatorname{div}_x (\nabla u_{p, \epsilon}(t, x)) dx dt \right] \\ &= - \lim_{m \rightarrow +\infty} \mathbb{E} \left[\int_0^T e^{-ct} \int_{\Lambda} u_{h, N}^r(t, x) \operatorname{div}_x (\nabla u_{p, \epsilon}(t, x)) dx dt \right]. \end{aligned} \quad (4.15)$$

Note that

$$\begin{aligned} & \mathbb{E} \left[\int_0^T e^{-ct} \int_{\Lambda} u_{h, N}^r(t, x) \operatorname{div}_x (\nabla u_{p, \epsilon}(t, x)) dx dt \right] \\ &= \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{t_n}^{t_{n+1}} e^{-ct} \int_K u_K^{n+1} \operatorname{div}_x (\nabla u_{p, \epsilon}(t, x)) dx dt \right] \\ &= \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} u_K^{n+1} \int_{t_n}^{t_{n+1}} e^{-ct} \int_{\partial K} \nabla u_{p, \epsilon}(t, x) \cdot \mathbf{n}_K(x) d\gamma(x) dt \right] \\ &= \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} (u_K^{n+1} - u_L^{n+1}) \int_{t_n}^{t_{n+1}} e^{-ct} \int_{\sigma} \nabla u_{p, \epsilon}(t, x) \cdot \mathbf{n}_{K, L} d\gamma(x) dt \right] \\ &= T_{1, m} - T_{2, m} + T_{2, m} \end{aligned}$$

where

$$\begin{aligned} T_{1, m} &= \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} (u_K^{n+1} - u_L^{n+1}) \int_{t_n}^{t_{n+1}} e^{-ct} \int_{\sigma} \nabla u_{p, \epsilon}(t, x) \cdot \mathbf{n}_{K, L} d\gamma(x) dt \right] \\ \text{and } T_{2, m} &= \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} (u_K^{n+1} - u_L^{n+1}) \int_{t_n}^{t_{n+1}} e^{-ct} \frac{m_{\sigma}}{d_{K|L}} (u_{p, \epsilon}^{n+1, L} - u_{p, \epsilon}^{n+1, K}) dt \right]. \end{aligned}$$

Firstly, one shows that $|T_{1,m} - T_{2,m}| \xrightarrow{m \rightarrow +\infty} 0$. To show this, we rewrite $T_{1,m} - T_{2,m}$ as

$$T_{1,m} - T_{2,m} = \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \Delta t m_{\sigma} (u_K^{n+1} - u_L^{n+1}) \times R_{\sigma}^{n+1}(u_{p,\epsilon}) \right]$$

where

$$R_{\sigma}^{n+1}(u_{p,\epsilon}) = \frac{1}{\Delta t m_{\sigma}} \int_{t_n}^{t_{n+1}} e^{-ct} \int_{\sigma} \left(\nabla u_{p,\epsilon}(t, x) \cdot \mathbf{n}_{K,L} - \frac{u_{p,\epsilon}^{n+1,L} - u_{p,\epsilon}^{n+1,K}}{d_{K|L}} \right) d\gamma(x) dt \quad (4.16)$$

which can be rewritten as

$$\begin{aligned} R_{\sigma}^{n+1}(u_{p,\epsilon}) &= \frac{1}{\Delta t m_{\sigma}} \int_{t_n}^{t_{n+1}} e^{-ct} \int_{\sigma} \left(\nabla u_{p,\epsilon}(t, x) \cdot \mathbf{n}_{K,L} - \frac{u_{p,\epsilon}(t, x_L) - u_{p,\epsilon}(t, x_K)}{d_{K|L}} \right) d\gamma(x) dt \\ &\quad + \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} e^{-ct} \left(\frac{u_{p,\epsilon}(t, x_L) - u_{p,\epsilon}(t, x_K)}{d_{K|L}} - \frac{u_{p,\epsilon}^{n+1,L} - u_{p,\epsilon}^{n+1,K}}{d_{K|L}} \right) dt. \end{aligned}$$

Let us denote by $\mathcal{H}_{u_{p,\epsilon}}$ the Hessian matrix of $u_{p,\epsilon}$ defined for any $i, j \in \{1, \dots, d\}$ by $(\mathcal{H}_{u_{p,\epsilon}})_{ij} = \frac{\partial^2 u_{p,\epsilon}}{\partial x_i \partial x_j}$. Then, thanks to a Taylor's expansion, for any $t \in [0, T]$ and any $x \in \sigma$, we have

$$\begin{aligned} u_{p,\epsilon}(t, x_L) &= u_{p,\epsilon}(t, x) + (x_L - x) \cdot \nabla u_{p,\epsilon}(t, x) + \frac{1}{2} (x_L - x)^T \mathcal{H}_{u_{p,\epsilon}}(t, \bar{x}) (x_L - x) \\ u_{p,\epsilon}(t, x_K) &= u_{p,\epsilon}(t, x) + (x_K - x) \cdot \nabla u_{p,\epsilon}(t, x) + \frac{1}{2} (x_K - x)^T \mathcal{H}_{u_{p,\epsilon}}(t, \hat{x}) (x_K - x), \end{aligned}$$

where $\bar{x} = \lambda x_L + (1 - \lambda)x$ and $\hat{x} = \alpha x_K + (1 - \alpha)x$ for $\lambda, \alpha \in [0, 1]$. Remind that the orthogonality condition on the mesh implies that for any $\sigma = K|L \in \mathcal{E}_{\text{int}}$, $x_L - x_K = d_{K|L} \mathbf{n}_{K,\sigma}$ and so, using (2.2)

$$\begin{aligned} &\left| \nabla u_{p,\epsilon}(t, x) \cdot \mathbf{n}_{K,L} - \frac{u_{p,\epsilon}(t, x_L) - u_{p,\epsilon}(t, x_K)}{d_{K|L}} \right| \\ &= \left| \frac{1}{2} \frac{(x_K - x)^T \mathcal{H}_{u_{p,\epsilon}}(t, \hat{x}) (x_K - x) - (x_L - x)^T \mathcal{H}_{u_{p,\epsilon}}(t, \bar{x}) (x_L - x)}{d_{K|L}} \right| \quad (4.17) \\ &\leq h \operatorname{reg}(\mathcal{T}) \sup_{(t,y) \in [0,T] \times \Lambda} |\mathcal{H}_{u_{p,\epsilon}}(t, y)|. \end{aligned}$$

Additionally, using the following equalities for any t in $[t_n, t_{n+1}]$

$$u_{p,\epsilon}^{n+1,K} = u_{p,\epsilon}(t_{n+1}, x_K) = u_{p,\epsilon}(t, x_K) + \int_t^{t_{n+1}} \partial_t u_{p,\epsilon}(s, x_K) ds$$

and

$$u_{p,\epsilon}^{n+1,L} = u_{p,\epsilon}(t_{n+1}, x_L) = u_{p,\epsilon}(t, x_L) + \int_t^{t_{n+1}} \partial_t u_{p,\epsilon}(s, x_L) ds,$$

and the fact that for any s in $[0, T]$

$$\begin{aligned} |\partial_t u_{p,\epsilon}(s, x_K) - \partial_t u_{p,\epsilon}(s, x_L)| &= \left| \int_0^1 \left(\nabla(\partial_t u_{p,\epsilon})(s, \mu x_K + (1-\mu)x_L) \right) \cdot (x_K - x_L) d\mu \right| \\ &\leq 2h \sup_{(s,x) \in [0,T] \times \Lambda} |\nabla(\partial_t u_{p,\epsilon})(s, x)|, \end{aligned}$$

we obtain using again (2.2)

$$\begin{aligned} &\left| \frac{u_{p,\epsilon}(t, x_L) - u_{p,\epsilon}(t, x_K)}{d_{K|L}} - \frac{u_{p,\epsilon}^{n+1,L} - u_{p,\epsilon}^{n+1,K}}{d_{K|L}} \right| \\ &= \left| \int_t^{t_{n+1}} \frac{\partial_t u_{p,\epsilon}(s, x_K) - \partial_t u_{p,\epsilon}(s, x_L)}{d_{K|L}} ds \right| \\ &\leq \Delta t \operatorname{reg}(\mathcal{T}) \sup_{(s,x) \in [0,T] \times \Lambda} |\nabla(\partial_t u_{p,\epsilon})(s, x)|. \end{aligned} \tag{4.18}$$

Noticing that $\int_t^{t_{n+1}} e^{-ct} dt \leq \Delta t$ and combining (4.17) and (4.18), we obtain the existence of a constant $K_{p,\epsilon} > 0$ only depending on $u_{p,\epsilon}$ such that

$$\mathbb{E} [|R_\sigma^{n+1}(u_{p,\epsilon})|^2] \leq (K_{p,\epsilon}(\Delta t + h) \operatorname{reg}(\mathcal{T}))^2. \tag{4.19}$$

Applying Cauchy-Schwarz inequality yields

$$\begin{aligned} &|T_{1,m} - T_{2,m}|^2 \\ &= \left| \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \Delta t \frac{m_\sigma}{\sqrt{d_{K|L}}} (u_K^{n+1} - u_L^{n+1}) \times \sqrt{d_{K|L}} R_\sigma^{n+1}(u_{p,\epsilon}) \right] \right|^2 \\ &\leq \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \Delta t \frac{m_\sigma}{d_{K|L}} |u_K^{n+1} - u_L^{n+1}|^2 \right] \\ &\quad \times 2\mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \Delta t \frac{m_\sigma d_{K|L}}{2} |R_\sigma^{n+1}(u_{p,\epsilon})|^2 \right] \\ &\leq 2T|\Lambda| (K_{p,\epsilon}(\Delta t + h) \operatorname{reg}(\mathcal{T}))^2 \mathbb{E} \left[\int_0^T |u_{h,N}^r(t)|_{1,h}^2 dt \right], \end{aligned} \tag{4.20}$$

which tends to 0 as $m \rightarrow +\infty$. We can thus affirm that $T_{2,m}$ admits a limit which is known thanks to (4.15)

$$\lim_{m \rightarrow +\infty} T_{2,m} = \lim_{m \rightarrow +\infty} T_{1,m} = \mathbb{E} \left[\int_0^T e^{-ct} \int_\Lambda \nabla u(t, x) \cdot \nabla u_{p,\epsilon}(t, x) dx dt \right].$$

Secondly, thanks to Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|T_{2,m}|^2 &\leq \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \int_{t_n}^{t_{n+1}} e^{-ct} \frac{m_\sigma}{d_{K|L}} |u_K^{n+1} - u_L^{n+1}|^2 dt \right] \\
&\quad \times \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \int_{t_n}^{t_{n+1}} e^{-ct} \frac{m_\sigma}{d_{K|L}} |u_{p,\epsilon}^{n+1,K} - u_{p,\epsilon}^{n+1,L}|^2 dt \right] \quad (4.21) \\
&= \mathbb{E} \left[\int_0^T e^{-ct} |u_{h,N}^r(t)|_{1,h}^2 dt \right] \times \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \int_{t_n}^{t_{n+1}} e^{-ct} \frac{m_\sigma}{d_{K|L}} |u_{p,\epsilon}^{n+1,K} - u_{p,\epsilon}^{n+1,L}|^2 dt \right].
\end{aligned}$$

Now, due to the regularity of $u_{p,\epsilon}$, by adapting previous arguments applied to $u_{h,N}^r$, one can show that

$$\begin{aligned}
&\lim_{m \rightarrow +\infty} \mathbb{E} \left[\sum_{n=0}^{N-1} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \int_{t_n}^{t_{n+1}} e^{-ct} \frac{m_\sigma}{d_{K|L}} |u_{p,\epsilon}^{n+1,K} - u_{p,\epsilon}^{n+1,L}|^2 dt \right] \\
&= \mathbb{E} \left[\int_0^T e^{-ct} \int_\Lambda |\nabla u_{p,\epsilon}(t, x)|^2 dx dt \right].
\end{aligned}$$

Using this convergence result, one obtains by passing to the inferior limit in (4.21)

$$\begin{aligned}
&\left(\mathbb{E} \left[\int_0^T e^{-ct} \int_\Lambda \nabla u(t, x) \cdot \nabla u_{p,\epsilon}(t, x) dx dt \right] \right)^2 \\
&\leq \liminf_{m \rightarrow +\infty} \mathbb{E} \left[\int_0^T e^{-ct} |u_{h,N}^r(t)|_{1,h}^2 dt \right] \times \mathbb{E} \left[\int_0^T e^{-ct} \int_\Lambda |\nabla u_{p,\epsilon}(t, x)|^2 dx dt \right].
\end{aligned}$$

Then, using strong convergences in $L^2(\Omega; L^2(0, T; H^1(\Lambda)))$ of $(u_{p,\epsilon})_\epsilon$ towards u_p (for a fixed p) as ϵ goes to 0 and of $(u_p)_p$ towards u as p goes to $+\infty$, the following holds

$$\mathbb{E} \left[\int_0^T e^{-ct} \int_\Lambda |\nabla u(t, x)|^2 dx dt \right] \leq \liminf_{m \rightarrow +\infty} \mathbb{E} \left[\int_0^T e^{-ct} |u_{h,N}^r(t)|_{1,h}^2 dt \right].$$

Note that using the same reasoning we show that for any $t \in [0, T]$,

$$\mathbb{E} \left[\int_0^t e^{-cs} \int_\Lambda |\nabla u(s, x)|^2 dx ds \right] \leq \liminf_{m \rightarrow +\infty} \mathbb{E} \left[\int_0^t e^{-cs} |u_{h,N}^r(s)|_{1,h}^2 ds \right]$$

and thanks to Fatou's Lemma, we finally have

$$\begin{aligned}
\int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} |\nabla u(x, s)|^2 dx \right] ds dt &= \int_0^T \mathbb{E} \left[\int_0^t e^{-cs} \int_{\Lambda} |\nabla u(s, x)|^2 dx ds \right] dt \\
&\leq \int_0^T \liminf_{m \rightarrow +\infty} \mathbb{E} \left[\int_0^t e^{-cs} |u_{h,N}^r(s)|_{1,h}^2 ds \right] dt \\
&\leq \liminf_{m \rightarrow +\infty} \int_0^T \mathbb{E} \left[\int_0^t e^{-cs} |u_{h,N}^r(s)|_{1,h}^2 ds \right] dt \\
&= \liminf_{m \rightarrow +\infty} \int_0^T \int_0^t e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds dt,
\end{aligned}$$

and the proof is complete. \square

Remark 4.8. Let us detail here why we do consider such an approximation of u in the previous proof. Firstly, we need to have the following equality:

$$\mathbb{E} \left[\int_0^T e^{-ct} \int_{\Lambda} \nabla u(t, x) \cdot \nabla u_{p,\epsilon}(t, x) dx dt \right] = -\mathbb{E} \left[\int_0^T e^{-ct} \int_{\Lambda} u(t, x) \operatorname{div}_x (\nabla u_{p,\epsilon}(t, x)) dx dt \right],$$

and if $\nabla u_{p,\epsilon} \cdot \mathbf{n} \neq 0$ then the boundary term appears in the application of the Gauss-Green formula and our argumentation fails. For this reason, we choose the density of \mathcal{V} in $H^1(\Lambda)$. Secondly, we need a control of the term $\mathbb{E}[|R_{\sigma}^{n+1}(u_{p,\epsilon})|^2]$, and for this reason we choose a particular approximation $(u_{p,\epsilon})_{p,\epsilon}$ of elementary processes's type. Indeed, if we consider another approximation $(U_{p,\epsilon})_{p,\epsilon}$ with the only information $(U_{p,\epsilon})_{p,\epsilon} \subset L^2(\Omega; \mathcal{V})$, we don't know if we control or not the expectation of the following quantities:

$$\left(\sup_{(t,y) \in [0,T] \times \Lambda} |\mathcal{H}_{U_{p,\epsilon}}(t, y)| \right)^2 \text{ and } \left(\sup_{(s,x) \in [0,T] \times \Lambda} |\nabla(\partial_t U_{p,\epsilon})(s, x)| \right)^2.$$

Now, we have all the necessary tools on the one hand for the identification of g_u and β_u , and on the other hand for completing the proof of Theorem 1.3.

Proposition 4.9. *The sequences $(u_{h,N}^r)_m$ and $(u_{h,N}^l)_m$ converge strongly in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$ to the unique variational solution of Problem (1.1) in the sense of Definition 1.2.*

Proof. Let us fix $n \in \{0, \dots, N-1\}$, $K \in \mathcal{T}$, and multiply (2.8) by $\Delta t u_K^{n+1}$, use the formula $a(a-b) = \frac{1}{2}(a^2 - b^2 + (a-b)^2)$ with $a = u_K^{n+1}$ and $b = u_K^n$, take the expectation, and proceed as for the obtention of (3.4) to arrive at

$$\begin{aligned}
&\frac{m_K}{2} \mathbb{E} [(u_K^{n+1})^2 - (u_K^n)^2] + \frac{m_K}{2} \mathbb{E} [(u_K^{n+1} - u_K^n)^2] \\
&+ \Delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} m_{\sigma} (v_{K,\sigma}^{n+1})^{-} \mathbb{E} [(u_K^{n+1} - u_L^{n+1}) u_K^{n+1}] \\
&+ \Delta t \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{m_{\sigma}}{d_{K|L}} \mathbb{E} [(u_K^{n+1} - u_L^{n+1}) u_K^{n+1}] \\
&\leq \frac{m_K}{2} \mathbb{E} [(u_K^{n+1} - u_K^n)^2] + \frac{m_K \Delta t}{2} \mathbb{E} [g^2(u_K^n)] + \Delta t m_K \mathbb{E} [\beta(u_K^{n+1}) u_K^{n+1}].
\end{aligned}$$

Now, we multiply the last inequality by e^{-ct_n} for arbitrary $c > 0$. Then, summing over $K \in \mathcal{T}$ and $n \in \{0, \dots, k\}$ for $k \in \{0, \dots, N-1\}$, using (2.4) and reasoning as in the proof of (3.3) one gets

$$\begin{aligned} & \frac{1}{2} \sum_{n=0}^k \sum_{K \in \mathcal{T}} m_K e^{-ct_n} \mathbb{E} [(u_K^{n+1})^2 - (u_K^n)^2] + \Delta t \sum_{n=0}^k e^{-ct_n} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} \mathbb{E} [|u_K^{n+1} - u_L^{n+1}|^2] \\ & \leq \frac{\Delta t}{2} \sum_{n=0}^k \sum_{K \in \mathcal{T}} m_K e^{-ct_n} \mathbb{E} [g^2(u_K^n)] + \Delta t \sum_{n=0}^k \sum_{K \in \mathcal{T}} m_K e^{-ct_n} \mathbb{E} [\beta(u_K^{n+1})u_K^{n+1}]. \end{aligned}$$

Let us focus on each sum of this last inequality separately.

- Note that the general term of the first sum can be decomposed in the following way:

$$\begin{aligned} & e^{-ct_n} \mathbb{E} [(u_K^{n+1})^2 - (u_K^n)^2] \\ & = e^{-ct_n} \mathbb{E} [(u_K^{n+1})^2] - e^{-ct_{n-1}} \mathbb{E} [(u_K^n)^2] - \mathbb{E} [(u_K^n)^2] (e^{-ct_n} - e^{-ct_{n-1}}), \end{aligned}$$

where $t_{-1} := -\Delta t$. Firstly, we have

$$\begin{aligned} & \frac{1}{2} \sum_{n=0}^k \sum_{K \in \mathcal{T}} m_K (e^{-ct_n} \mathbb{E} [(u_K^{n+1})^2] - e^{-ct_{n-1}} \mathbb{E} [(u_K^n)^2]) \\ & = \frac{1}{2} \sum_{K \in \mathcal{T}} m_K e^{-ct_k} \mathbb{E} [(u_K^{k+1})^2] - \frac{1}{2} \sum_{K \in \mathcal{T}} m_K \mathbb{E} [(u_K^0)^2] e^{c\Delta t}. \end{aligned} \tag{4.22}$$

Secondly, since there exists $\xi \in (-ct_n, -ct_{n-1})$ such that

$$e^{-ct_n} - e^{-ct_{n-1}} = e^\xi (-ct_n + ct_{n-1}) = -c\Delta t e^\xi < -c\Delta t e^{-ct_n},$$

the following inequality holds

$$-\sum_{n=1}^k \sum_{K \in \mathcal{T}} m_K \mathbb{E} [(u_K^n)^2] (e^{-ct_n} - e^{-ct_{n-1}}) > c\Delta t \sum_{n=1}^k \sum_{K \in \mathcal{T}} m_K e^{-ct_n} \mathbb{E} [(u_K^n)^2].$$

In particular, we have since for any $s \in [t_n, t_{n+1}]$, $e^{-cs} \leq e^{-ct_n}$,

$$\begin{aligned} c\Delta t \sum_{n=1}^k \sum_{K \in \mathcal{T}} m_K e^{-ct_n} \mathbb{E} [(u_K^n)^2] & = c\Delta t \sum_{n=0}^{k-1} \sum_{K \in \mathcal{T}} m_K e^{-ct_{n+1}} \mathbb{E} [(u_K^{n+1})^2] \\ & = ce^{-c\Delta t} \sum_{n=0}^{k-1} \sum_{K \in \mathcal{T}} m_K \Delta t e^{-ct_n} \mathbb{E} [(u_K^{n+1})^2] \\ & \geq ce^{-c\Delta t} \sum_{n=0}^{k-1} \sum_{K \in \mathcal{T}} m_K \int_{t_n}^{t_{n+1}} e^{-cs} \mathbb{E} [(u_K^{n+1})^2] ds \\ & = ce^{-c\Delta t} \int_0^{t_k} e^{-cs} \mathbb{E} [\|u_{h,N}^r(s)\|_{L^2(\Lambda)}^2] ds. \end{aligned}$$

In this manner

$$\begin{aligned}
& -\frac{1}{2} \sum_{n=0}^k \sum_{K \in \mathcal{T}} m_K \mathbb{E} [(u_K^n)^2] (e^{-ct_n} - e^{-ct_{n-1}}) \\
& > -\frac{1}{2} \sum_{K \in \mathcal{T}} m_K \mathbb{E} [(u_K^0)^2] (1 - e^{c\Delta t}) + \frac{c}{2} e^{-c\Delta t} \int_0^{t_k} e^{-cs} \mathbb{E} [\|u_{h,N}^r(s)\|_{L^2(\Lambda)}^2] ds \\
& > \frac{c}{2} e^{-c\Delta t} \int_0^{t_k} e^{-cs} \mathbb{E} [\|u_{h,N}^r(s)\|_{L^2(\Lambda)}^2] ds.
\end{aligned} \tag{4.23}$$

• The second sum can be handled in the following manner

$$\begin{aligned}
\Delta t \sum_{n=0}^k e^{-ct_n} \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \frac{m_\sigma}{d_{K|L}} \mathbb{E} [|u_K^{n+1} - u_L^{n+1}|^2] &= \Delta t \sum_{n=0}^k e^{-ct_n} \mathbb{E} [|u_h^{n+1}|_{1,h}^2] \\
&\geq \int_0^{t_{k+1}} e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds.
\end{aligned} \tag{4.24}$$

Indeed, since for any $s \in [t_n, t_{n+1}]$, $e^{-cs} \leq e^{-ct_n}$ one gets

$$\begin{aligned}
\int_0^{t_{k+1}} e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds &= \sum_{n=0}^k \int_{t_n}^{t_{n+1}} e^{-cs} \mathbb{E} [|u_h^{n+1}|_{1,h}^2] ds \\
&\leq \sum_{n=0}^k \mathbb{E} [|u_h^{n+1}|_{1,h}^2] e^{-ct_n} \int_{t_n}^{t_{n+1}} 1 ds \\
&= \Delta t \sum_{n=0}^k e^{-ct_n} \mathbb{E} [|u_h^{n+1}|_{1,h}^2].
\end{aligned}$$

• We have the following majoration of the third sum:

$$\begin{aligned}
& \frac{\Delta t}{2} \sum_{n=0}^k \sum_{K \in \mathcal{T}} m_K e^{-ct_n} \mathbb{E} [g^2(u_K^n)] \\
& \leq \frac{\Delta t}{2} \sum_{K \in \mathcal{T}} m_K \mathbb{E} [g^2(u_K^0)] + \frac{1}{2} \int_0^{t_k} e^{-cs} \mathbb{E} [\|g(u_{h,N}^r)(s)\|_{L^2(\Lambda)}^2] ds.
\end{aligned} \tag{4.25}$$

Indeed,

$$\begin{aligned}
\frac{\Delta t}{2} \sum_{n=1}^k \sum_{K \in \mathcal{T}} m_K e^{-ct_n} \mathbb{E} [g^2(u_K^n)] &= \frac{\Delta t}{2} \sum_{n=0}^{k-1} \sum_{K \in \mathcal{T}} m_K e^{-ct_{n+1}} \mathbb{E} [g^2(u_K^{n+1})] \\
&\leq \frac{1}{2} \sum_{n=0}^{k-1} \sum_{K \in \mathcal{T}} m_K \int_{t_n}^{t_{n+1}} e^{-cs} \mathbb{E} [g^2(u_K^{n+1})] ds \\
&= \frac{1}{2} \int_0^{t_k} e^{-cs} \mathbb{E} [\|g(u_{h,N}^r)(s)\|_{L^2(\Lambda)}^2] ds.
\end{aligned}$$

- The last sum can be handled in the following manner:

$$\begin{aligned}
& \Delta t \sum_{n=0}^k \sum_{K \in \mathcal{T}} m_K e^{-ct_n} \mathbb{E} [\beta(u_K^{n+1}) u_K^{n+1}] \\
& \leq c \Delta t L_\beta \|u_{h,N}^r\|_{L^2(\Omega; L^2(0,T; L^2(\Lambda)))}^2 + \int_0^{t_{k+1}} e^{-cs} \mathbb{E} \left[\int_\Lambda \beta(u_{h,N}^r(s, x)) u_{h,N}^r(s, x) dx \right] ds.
\end{aligned} \tag{4.26}$$

Indeed,

$$\begin{aligned}
& \left| \Delta t \sum_{n=0}^k \sum_{K \in \mathcal{T}} m_K e^{-ct_n} \mathbb{E} [\beta(u_K^{n+1}) u_K^{n+1}] - \int_0^{t_{k+1}} e^{-cs} \mathbb{E} \left[\int_\Lambda \beta(u_{h,N}^r(s, x)) u_{h,N}^r(s, x) dx \right] ds \right| \\
& = \left| \sum_{n=0}^k \sum_{K \in \mathcal{T}} \int_{t_n}^{t_{n+1}} \int_K (e^{-ct_n} - e^{-cs}) \mathbb{E} [\beta(u_K^{n+1}) u_K^{n+1}] dx ds \right| \leq c \Delta t L_\beta \|u_{h,N}^r\|_{L^2(\Omega; L^2(0,T; L^2(\Lambda)))}^2.
\end{aligned}$$

Combining (4.22), (4.23), (4.24), (4.25) and (4.26) one gets

$$\begin{aligned}
& \sum_{K \in \mathcal{T}} m_K e^{-ct_k} \mathbb{E} [(u_K^{k+1})^2] + 2 \int_0^{t_{k+1}} e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds \\
& \leq e^{c\Delta t} \sum_{K \in \mathcal{T}} m_K \mathbb{E} [(u_K^0)^2] + \Delta t \sum_{K \in \mathcal{T}} m_K \mathbb{E} [g^2(u_K^0)] + \int_0^{t_k} e^{-cs} \mathbb{E} \left[\|g(u_{h,N}^r)(s)\|_{L^2(\Lambda)}^2 \right] ds \\
& \quad - ce^{-c\Delta t} \int_0^{t_k} e^{-cs} \mathbb{E} \left[\|u_{h,N}^r(s)\|_{L^2(\Lambda)}^2 \right] ds + 2c\Delta t L_\beta \|u_{h,N}^r\|_{L^2(\Omega; L^2(0,T; L^2(\Lambda)))}^2 \\
& \quad + 2 \int_0^{t_{k+1}} e^{-cs} \mathbb{E} \left[\int_\Lambda \beta(u_{h,N}^r(s, x)) u_{h,N}^r(s, x) dx \right] ds.
\end{aligned}$$

For $t \in [t_k, t_{k+1})$ since $e^{-ct} \leq e^{-ct_k}$ and $(t - \Delta t)^+ \leq t_k$, one gets

$$\begin{aligned}
& e^{-ct} \mathbb{E} \left[\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2 \right] + 2 \int_0^t e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds \\
& \leq e^{c\Delta t} \sum_{K \in \mathcal{T}} m_K \mathbb{E} [(u_K^0)^2] + \Delta t \sum_{K \in \mathcal{T}} m_K \mathbb{E} [g^2(u_K^0)] + \int_0^t e^{-cs} \mathbb{E} \left[\|g(u_{h,N}^r)(s)\|_{L^2(\Lambda)}^2 \right] ds \\
& \quad - ce^{-c\Delta t} \int_0^{(t-\Delta t)^+} e^{-cs} \mathbb{E} \left[\|u_{h,N}^r(s)\|_{L^2(\Lambda)}^2 \right] ds + 2c\Delta t L_\beta \|u_{h,N}^r\|_{L^2(\Omega; L^2(0,T; L^2(\Lambda)))}^2 \\
& \quad + 2 \int_0^t e^{-cs} \mathbb{E} \left[\int_\Lambda \beta(u_{h,N}^r(s, x)) u_{h,N}^r(s, x) dx \right] ds \\
& \quad + 2 \int_t^{t_{k+1}} e^{-cs} \mathbb{E} \left[\int_\Lambda \beta(u_{h,N}^r(s, x)) u_{h,N}^r(s, x) dx \right] ds
\end{aligned}$$

Using the fact that $(u_{h,N}^r)_m$ is bounded in $L^\infty(0, T; L^2(\Omega; L^2(\Lambda)))$ by Proposition 3.2 one

gets

$$\begin{aligned} & \int_{(t-\Delta t)^+}^t e^{-cs} \mathbb{E} \left[\|u_{h,N}^r(s)\|_{L^2(\Lambda)}^2 \right] ds + 2 \int_t^{t_{k+1}} e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta(u_{h,N}^r(s,x)) u_{h,N}^r(s,x) dx \right] ds \\ & \leq \Delta t (1 + 2L_{\beta}) \|u_{h,N}^r\|_{L^\infty(0,T;L^2(\Omega;L^2(\Lambda)))}^2 \end{aligned}$$

and so (since $-\int_0^{(t-\Delta t)^+} = -\int_0^t + \int_{(t-\Delta t)^+}^t$)

$$\begin{aligned} & e^{-ct} \mathbb{E} \left[\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2 \right] + 2 \int_0^t e^{-cs} \mathbb{E} [\|u_{h,N}^r(s)\|_{1,h}^2] ds \\ & \leq e^{c\Delta t} \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] + \Delta t L_g^2 \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] + \int_0^t e^{-cs} \mathbb{E} \left[\|g(u_{h,N}^r)(s)\|_{L^2(\Lambda)}^2 \right] ds \quad (4.27) \\ & \quad - ce^{-c\Delta t} \int_0^t e^{-cs} \mathbb{E} \left[\|u_{h,N}^r(s)\|_{L^2(\Lambda)}^2 \right] ds + 2c\Delta t L_{\beta} \|u_{h,N}^r\|_{L^2(\Omega;L^2(0,T;L^2(\Lambda)))}^2 \\ & \quad + \Delta t (1 + 2L_{\beta}) \|u_{h,N}^r\|_{L^\infty(0,T;L^2(\Omega;L^2(\Lambda)))}^2 + 2 \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta(u_{h,N}^r(s,x)) u_{h,N}^r(s,x) dx \right] ds. \end{aligned}$$

Moreover,

$$\begin{aligned} & \int_0^t e^{-cs} \mathbb{E} \left[\|g(u_{h,N}^r)(s)\|_{L^2(\Lambda)}^2 \right] ds = \int_0^t e^{-cs} \mathbb{E} \left[\|g(u_{h,N}^r)(s) - g(u)(s)\|_{L^2(\Lambda)}^2 \right] ds \\ & \quad + 2 \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} g(u_{h,N}^r)(s,x) g(u)(s,x) dx \right] ds - \int_0^t e^{-cs} \mathbb{E} \left[\|g(u)(s)\|_{L^2(\Lambda)}^2 \right] ds. \end{aligned} \quad (4.28)$$

In the same manner,

$$\begin{aligned} & -ce^{-c\Delta t} \int_0^t e^{-cs} \mathbb{E} \left[\|u_{h,N}^r(s)\|_{L^2(\Lambda)}^2 \right] ds = -ce^{-c\Delta t} \int_0^t e^{-cs} \mathbb{E} \left[\|u_{h,N}^r(s) - u(s)\|_{L^2(\Lambda)}^2 \right] ds \\ & \quad - 2ce^{-c\Delta t} \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} u_{h,N}^r(s,x) u(s,x) dx \right] ds + ce^{-c\Delta t} \int_0^t e^{-cs} \mathbb{E} \left[\|u(s)\|_{L^2(\Lambda)}^2 \right] ds. \end{aligned} \quad (4.29)$$

At last

$$\begin{aligned} & \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta(u_{h,N}^r(s,x)) u_{h,N}^r(s,x) dx \right] ds = \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta(u_{h,N}^r(s,x)) u(s,x) dx \right] ds \\ & \quad + \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} (\beta(u_{h,N}^r(s,x)) - \beta(u(s,x))) (u_{h,N}^r(s,x) - u(s,x)) dx \right] ds \quad (4.30) \\ & \quad + \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta(u(s,x)) (u_{h,N}^r(s,x) - u(s,x)) dx \right] ds. \end{aligned}$$

Note that there exists $c > 0$ depending only on L_g and L_{β} such that for any N big enough

$$\begin{aligned} & \int_0^t e^{-cs} \mathbb{E} \left[\|g(u_{h,N}^r)(s) - g(u(s))\|_{L^2(\Lambda)}^2 \right] ds - ce^{-c\Delta t} \int_0^t e^{-cs} \mathbb{E} \left[\|u_{h,N}^r(s) - u(s)\|_{L^2(\Lambda)}^2 \right] ds \\ & \quad + 2 \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} (\beta(u_{h,N}^r(s,x)) - \beta(u(s,x))) (u_{h,N}^r(s,x) - u(s,x)) dx \right] ds \leq 0. \end{aligned}$$

From now on, we will consider such a choice of c . After injecting (4.28), (4.29) and (4.30) in (4.27), we arrive at

$$\begin{aligned}
& e^{-ct} \mathbb{E} \left[\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2 \right] + 2 \int_0^t e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds \\
& \leq \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] + \Delta t L_g^2 \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] + 2 \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} g(u_{h,N}^r)(s,x) g(u)(s,x) dx \right] ds \\
& \quad - \int_0^t e^{-cs} \mathbb{E} \left[\|g(u)(s)\|_{L^2(\Lambda)}^2 \right] ds - 2ce^{-c\Delta t} \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} u_{h,N}^r(s,x) u(s,x) dx \right] ds \\
& \quad + ce^{-c\Delta t} \int_0^t e^{-cs} \mathbb{E} \left[\|u(s)\|_{L^2(\Lambda)}^2 \right] ds + 2c\Delta t L_{\beta} \|u_{h,N}^r\|_{L^2(\Omega;L^2(0,T);L^2(\Lambda))}^2 \\
& \quad + \Delta t (1 + 2L_{\beta}) \|u_{h,N}^r\|_{L^{\infty}(0,T;L^2(\Omega;L^2(\Lambda)))}^2 + (e^{c\Delta t} - 1) \mathbb{E} \left[\|u_0\|_{L^2(\Lambda)}^2 \right] \\
& \quad + 2 \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta(u_{h,N}^r(s,x)) u(s,x) dx \right] ds \\
& \quad + 2 \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta(u(s,x)) (u_{h,N}^r(s,x) - u(s,x)) dx \right] ds.
\end{aligned}$$

Then, integrating this last inequality from 0 to T the following holds

$$\begin{aligned}
& \int_0^T e^{-ct} \mathbb{E} \left[\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2 \right] dt + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds dt \\
& \leq \int_0^T \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] dt + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} g(u_{h,N}^r)(s,x) g(u)(s,x) dx \right] ds dt \\
& \quad - \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\|g(u)(s)\|_{L^2(\Lambda)}^2 \right] ds dt \\
& \quad - 2ce^{-c\Delta t} \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} u_{h,N}^r(s,x) u(s,x) dx \right] ds dt \\
& \quad + ce^{-c\Delta t} \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\|u(s)\|_{L^2(\Lambda)}^2 \right] ds dt + T (e^{c\Delta t} - 1) \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] \\
& \quad + T\Delta t \left((1 + 2L_{\beta}) \|u_{h,N}^r\|_{L^{\infty}(0,T;L^2(\Omega;L^2(\Lambda)))}^2 + L_g^2 \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] + 2cL_{\beta} \|u_{h,N}^r\|_{L^2(\Omega;L^2(0,T);L^2(\Lambda))}^2 \right) \\
& \quad + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta(u_{h,N}^r(s,x)) u(s,x) dx \right] ds dt \\
& \quad + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta(u(s,x)) (u_{h,N}^r(s,x) - u(s,x)) dx \right] ds dt.
\end{aligned}$$

Firstly, by passing to the superior limit one gets

$$\begin{aligned}
& \limsup_{m \rightarrow +\infty} \int_0^T e^{-ct} \mathbb{E} \left[\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2 \right] dt + 2 \liminf_{m \rightarrow +\infty} \int_0^T \int_0^t e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds dt \\
& \leq \int_0^T \mathbb{E} [\|u_0\|_{L^2(\Lambda)}^2] dt - c \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|u(s)\|_{L^2(\Lambda)}^2] ds dt \\
& + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} g_u(s,x) g(u)(s,x) dx \right] ds dt - \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|g(u(s))\|_{L^2(\Lambda)}^2] ds dt \\
& + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} \beta_u(s,x) u(s,x) dx \right] ds dt.
\end{aligned}$$

Secondly, thanks to the stochastic energy equality (4.13) one arrives at

$$\begin{aligned}
& \limsup_{m \rightarrow +\infty} \int_0^T e^{-ct} \mathbb{E} \left[\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2 \right] dt + 2 \liminf_{m \rightarrow +\infty} \int_0^T \int_0^t e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds dt \\
& \leq \int_0^T e^{-ct} \mathbb{E} [\|u(t)\|_{L^2(\Lambda)}^2] dt + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|\nabla u(s)\|_{L^2(\Lambda)}^2] ds dt \\
& - \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|g_u(s)\|_{L^2(\Lambda)}^2] ds dt + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} \left[\int_{\Lambda} g_u(s,x) g(u(s,x)) dx \right] ds dt \\
& - \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|g(u(s))\|_{L^2(\Lambda)}^2] ds dt,
\end{aligned}$$

which yields to

$$\begin{aligned}
& \limsup_{m \rightarrow +\infty} \int_0^T e^{-ct} \mathbb{E} \left[\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2 \right] dt + 2 \liminf_{m \rightarrow +\infty} \int_0^T \int_0^t e^{-cs} \mathbb{E} [|u_{h,N}^r(s)|_{1,h}^2] ds dt \\
& \leq \int_0^T e^{-ct} \mathbb{E} [\|u(t)\|_{L^2(\Lambda)}^2] dt + 2 \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|\nabla u(s)\|_{L^2(\Lambda)}^2] ds dt \\
& - \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|g(u(s)) - g_u(s)\|_{L^2(\Lambda)}^2] ds dt.
\end{aligned}$$

Thirdly, owing to (4.14) one obtains

$$\begin{aligned}
& \limsup_{m \rightarrow +\infty} \int_0^T e^{-ct} \mathbb{E} \left[\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2 \right] dt + \int_0^T \int_0^t e^{-cs} \mathbb{E} [\|g(u(s)) - g_u(s)\|_{L^2(\Lambda)}^2] ds dt \\
& \leq \int_0^T e^{-ct} \mathbb{E} [\|u(t)\|_{L^2(\Lambda)}^2] dt.
\end{aligned}$$

Note that by weak convergence of $(u_{h,N}^r)_m$ towards u in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$, the following inequality is always true

$$\int_0^T e^{-ct} \mathbb{E} [\|u(t)\|_{L^2(\Lambda)}^2] dt \leq \liminf_{m \rightarrow +\infty} \int_0^T e^{-ct} \mathbb{E} [\|u_{h,N}^r(t)\|_{L^2(\Lambda)}^2] dt,$$

and so this allows us to conclude firstly that $g_u = g(u)$, secondly that $(u_{h,N}^r)_m, (u_{h,N}^l)_m$ converges strongly to u in $L^2(\Omega; L^2(0, T; L^2(\Lambda)))$, that $\beta_u = \beta(u)$ and at last that u is the unique variational solution of Problem (1.1) in the sense of Definition 1.2. \square

Acknowledgments The authors would like to thank T. Gallouët for his valuable suggestions. This work has been supported by the German Research Foundation project (ZI 1542/3-1), the Institut de Mécanique et d’Ingenierie of Marseille and various Procope programs: Project-Related Personal Exchange France-Germany (49368YE), Procope Mobility Program (DEU-22-0004 LG1) and Procope Plus Project.

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