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Invoking Fractional Calculus to extend the classic Riemann Zeta-Bernoulli relation

Shubhayan Ghosal

Department of Mathematics, Jadavpur University

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Email: shubhayang.math.ug@jadavpuruniversity.in

Abstract

In this paper, we explore and extend the classical definition of the zeta function and the Riemann-Liouville Fractional calculus on the entire complex plane, except the origin where it has a simple pole and residue 1, through the analytic continuation of Bernoulli Numbers, as a fractional differintegral.

1 Some less noticed properties of a Bernoulli Polynomial

1.1 Defining Appell Sequence

A sequence $s_n(x)$ is said to be Appell[5] for a function $g(t)$, provided that it is not identically 0, if the following holds

$$g(t)e^{xt} = \sum_{k=0}^{\infty} s_k(x) \cdot \frac{t^k}{k!}$$

where $y \in \mathbb{C}$ with field characteristic 0 and which upon Maclaurian expansion, Cauchy product of series and comparison of L.H.S and R.H.S gives us

$$s_k(x) = \sum_{k=0}^n \binom{n}{k} g^k(0) \cdot x^{n-k}$$

where $g^k(0)$ is the k^{th} derivative of g at $t = 0$. Basically, it is a Sheffer sequence in $(g(t), t)$. Another property which is often used to describe the Appell sequence is

$$s'_n(x) = n s_{n-1}(x)$$

which is also one of the properties inherited by Bernoulli Polynomials. A well-known property of Appell sequence can be derived from the Binomial theorem for natural index

$$s_n(x+y) = \sum_{r=0}^n \binom{n}{r} s_r(x) \cdot y^{n-r} \quad (1)$$

1.2 Bernoulli Polynomials as a Special case of Appell

A Bernoulli Polynomial[3] is an Appell sequence with $g(t) = \frac{e^t-1}{t}$ and the coefficients hence derived are called Bernoulli Numbers. The Bernoulli Polynomial is given as

$$\frac{t}{e^t-1} e^{xt} = \sum_{k=0}^{\infty} B_k(x) \cdot \frac{t^k}{k!}$$

and explicitly, as before

$$B_n(x) = \sum_{r=0}^n \binom{n}{r} B_r \cdot x^{n-r}$$

Here, the coefficients of the function in the Maclaurian expansion have been termed as the Bernoulli Numbers.

The Bernoulli Numbers can thus be defined as $\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$, thus yielding

$$B_n = \lim_{x \rightarrow 0} D_x^n \left(\frac{x}{e^x-1} \right)$$

We are now to extend the Bernoulli Numbers as a new function $\mathcal{B}(x)$ with $x \in \mathbb{R}$ [6]. The function will return the standard Bernoulli numbers at positive integer arguments. Note that, to extend the function, we resort to the standard fractional calculus definition of Riemann-Liouville. The Riemann-Liouville fractional integral is defined on the complex plane as

$$I^s f(x) = \frac{1}{\Gamma(s)} \int_a^x (x-t)^{s-1} \cdot f(t) dt$$

We intend to show the relation $\zeta(-n) = -\frac{B_{n+1}}{n+1}$ [4] by proving this relation $\forall s \in \mathbb{C}$ on the half planes $Re(s) \leq 0$, $Re(s) \geq 0$, $s \neq 0$.

Our approach would be fractional calculus. First we define a few terms related to fractional calculus. The fractional calculus concept mainly deals with viewing the differentiation and integration as operators and inverses of each other, with the ability to extend the number of times the operator acting to any non-integer order, perhaps the complex numbers too! However, in this paper, we would be mostly dealing with the Riemann-Liouville concepts and some preliminary ideas like

$$I^\alpha I^\beta f = I^{\alpha+\beta} f$$

and

$$[1]D^s f = D^{\lfloor \text{Re}(s) \rfloor} I^{-(\{\text{Re}(s)\} + i \cdot \text{Im}(s))} f, \quad \text{Re}(s) > 0. \quad (2)$$

Indeed, we intend to present the proof of how a generic operator can be extended on the complex plane, thanks to S.C. Woon.[7]

2 Analytical Continuation of an Operator Using Nested Series Expansion

Thinking of continuing an operator some natural number ‘ n ’ times is quite usual as we think. But to continue it to some complex ‘ z ’ times is what is somewhat off path. With reference to [6], we assume a generic operator is allowed to continue n times, denoting it as A^n . Then we have

$$\begin{aligned} D^z &= (w\mathbf{1} - (w\mathbf{1} - D))^z \\ &= w^z \left(\mathbf{1} - \left(\mathbf{1} - \frac{D}{w} \right) \right)^z \\ &= w^z \left[\mathbf{1} + \sum_{n=0}^{\infty} (-1)^n \binom{z}{n} \left(\mathbf{1} - \frac{D}{w} \right)^n \right] \\ &= w^z \left[\mathbf{1} + \sum_{n=0}^{\infty} (-1)^n \binom{z}{n} \left(\mathbf{1} + \sum_{k=0}^n \left(\frac{-1}{w} \right)^k \binom{n}{k} D^k \right) \right] \end{aligned} \quad (3)$$

where $z \in \mathbb{C}$, $w \in \mathbb{R}^+$, $\text{Re}(z) > \frac{1}{w}$ and $\mathbf{1}$ is the identity operator. Also,
 $\binom{x}{n} = \frac{1}{n!} \cdot \prod_{k=0}^{n-1} (x - k) \quad \forall x \in \mathbb{R}.$

3 Analytical Continuation of Bernoulli Numbers as a Fractional Derivative and the proof

Intutively speaking, the integral representation $\zeta(s) \cdot \Gamma(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$ gives us some hint of the connection of the Riemann-Liouville integral, but not rigorously. Let us start with a function $\mathcal{B}(s) = \lim_{x \rightarrow 0} D_x^s \left(\frac{x}{e^x - 1} \right)$, $s \in \mathbb{C}$, $\text{Re}(s) < 0$. In the subsequent steps we change $t \mapsto x - t$ and then $nt = k$, also we set $a \rightarrow -\infty$.

$$\begin{aligned}
\mathcal{B}(s) &= \lim_{x \rightarrow 0} D_x^s \left(\frac{x}{e^x - 1} \right) \\
&= - \sum_{n=0}^{\infty} \lim_{x \rightarrow 0} D_x^s (x e^{nx}) \\
&= - \frac{1}{\Gamma(-s)} \sum_{n=0}^{\infty} \lim_{x \rightarrow 0} \int_a^x (x-t)^{-s-1} \cdot t e^{nt} dt \\
&= \frac{1}{\Gamma(-s)} \sum_{n=0}^{\infty} \lim_{x \rightarrow 0} \int_{x-a}^0 t^{-s-1} \cdot (x-t) e^{n(x-t)} dt \\
&= \frac{1}{\Gamma(-s)} \left(\lim_{x \rightarrow 0} \int_{\infty}^0 t^{-s-1} \cdot (x-t) dt + \sum_{n=1}^{\infty} \lim_{x \rightarrow 0} \int_{\infty}^0 t^{-s-1} (x-t) e^{n(x-t)} dt \right) \\
&= \frac{1}{\Gamma(-s)} \sum_{n=1}^{\infty} \lim_{x \rightarrow 0} \left(x \int_{\infty}^0 t^{-s-1} e^{-nt} dt - \int_{\infty}^0 t^{-s} e^{-nt} dt \right) \\
&= \frac{1}{\Gamma(-s)} \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} \left(\int_0^{\infty} k^{-s} e^k dk \right) \\
&= \frac{1}{\Gamma(-s)} \sum_{n=1}^{\infty} \frac{1}{n^{s-1}} \cdot \Gamma(-s+1) \\
&= \zeta(1-s) \cdot (-s)
\end{aligned}$$

$$\Rightarrow \boxed{\zeta(1-s) = -\frac{\mathcal{B}(s)}{s}} \quad (4)$$

$$\forall \operatorname{Re}(s) \leq 0, \quad s \in \mathbb{C}, \quad s \neq 0$$

And now, given that $\operatorname{Re}(s) > 0$, we use (5) to complete the rest of the proof. Let $m = \lfloor \operatorname{Re}(s) \rfloor$, $s' = \{\operatorname{Re}(s)\} + i \cdot \operatorname{Im}(s)$, then $s = m + s'$. On the 6th line, the quantity $C = \int_{\infty}^0 t^{-s'-1} dt$ is already 0 if $m = 1$ or else $D_x^m(Cx) = 0$.

$$\begin{aligned}
\mathcal{B}(s) &= \lim_{x \rightarrow 0} D_x^s \left(\frac{x}{e^x - 1} \right) \\
&= \lim_{x \rightarrow 0} D_x^{(m+s')} \left(\frac{x}{e^x - 1} \right) \\
&= -\frac{1}{\Gamma(-s')} \sum_{n=0}^{\infty} \lim_{x \rightarrow 0} D_x^m \left(\int_a^x (x-t)^{-s'-1} t e^{nt} dt \right) \\
&= \frac{1}{\Gamma(-s')} \sum_{n=0}^{\infty} \lim_{x \rightarrow 0} D_x^m \left(\int_{x-a}^0 t^{-s'-1} (x-t) e^{n(x-t)} dt \right) \\
&= \frac{1}{\Gamma(-s')} \lim_{x \rightarrow 0} D_x^m \left(\int_{-\infty}^0 t^{-s'-1} (x-t) dt + \sum_{n=1}^{\infty} \int_{-\infty}^0 t^{-s'-1} (x-t) e^{n(x-t)} dt \right) \\
&= \frac{1}{\Gamma(-s')} \sum_{n=1}^{\infty} \lim_{x \rightarrow 0} D_x^m \left(\int_{-\infty}^0 t^{-s'-1} (x-t) e^{n(x-t)} dt \right) \\
&= \frac{1}{\Gamma(-s')} \sum_{n=1}^{\infty} \lim_{x \rightarrow 0} D_x^m e^{nx} \left(\int_{-\infty}^0 t^{-s'-1} (x-t) e^{-nt} dt \right) \\
&= \frac{1}{\Gamma(-s')} \sum_{n=1}^{\infty} \lim_{x \rightarrow 0} \left[D_x^m x e^{nx} \left(\int_{-\infty}^0 t^{-s'-1} e^{-nt} dt \right) - D_x^m e^{nx} \left(\int_{-\infty}^0 t^{-s'} e^{-nt} dt \right) \right] \\
&= \frac{1}{\Gamma(s')} \sum_{n=1}^{\infty} \left[m n^{m-1} \left(\int_{-\infty}^0 t^{-s'-1} e^{-nt} dt \right) - n^m \left(\int_{-\infty}^0 t^{-s'} e^{-nt} dt \right) \right] \\
&= -\frac{1}{\Gamma(-s')} \sum_{n=1}^{\infty} n^{(m+s'-1)} (m \Gamma(-s') - \Gamma(-s' + 1)) \\
&= \zeta(1-s) \cdot (-s)
\end{aligned} \tag{5}$$

Hence, the result $\boxed{\zeta(1-s) = -\frac{\mathcal{B}(s)}{s}}$ holds $\forall s \in \mathbb{C} - \{0\}$. [2] But then,

$$\zeta(-n) = -\frac{\mathcal{B}(n+1)}{n+1} = -\frac{\lim_{x \rightarrow 0} D_x^{n+1} \left(\frac{x}{e^x - 1} \right)}{n+1} = -\frac{B_{n+1}}{n+1}$$

which are the standard Bernoulli numbers. Hence $\mathcal{B}(s)$ is indeed the analytical continuation of Bernoulli numbers, more precisely Bernoulli Polynomials at argument 0 i.e., $B_n(0)$.

It is not very difficult to realise the reflection formula of the Bernoulli numbers from the analytic relation of the Riemann-zeta function.

$$\mathcal{B}(1-s) = \left(\frac{1-s}{s} \right) 2^s \cdot \pi^{s-1} \sin \left(\frac{\pi s}{2} \right) \cdot \Gamma(1-s) \cdot \mathcal{B}(s) \tag{6}$$

Conclusion

It can be noted that not only we have extended the relation, we made an extension to a particular class of polynomials and invoked a new method involving fractional calculus to derive a classical formula. May be this method turns out to somehow throw some light on the age old Riemann hypothesis, owing to its unique representation as a fractional (complex!) differintegral. In the next paper, we will explore the intriguing properties of these extended numbers and their applications in various areas of mathematics, including Hurwitz and Lerch Zeta Functions and Distribution of primes.

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