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# An Existence, stability and convergence of global weak solutions result for a new class of sediment transport models based on viscous nonhomogeneous distortion shallow water equations

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## Abstract

In this paper, a new sediment transport model in viscous nonhomogeneous distortion shallow water equations with Reynolds dissipation is formulated, and then, an existence theorem of global weak solutions is proposed. The model extends several sediment transport models based on shallow water equations that do not account for the turbulence created by the acceleration of the water flow near the mobile sediment bed. The model is inspired from recent theories developed in [Sediment transport models in Generalized shear shallow water flow equations. CARI 2022, Oct 2022, Dschang, Cameroon. <https://hal.archives-ouvertes.fr/hal-03735893>.] and [A sediment transport theory based on distortion-free-boundary nonhomogeneous fluid flows. *Application and Engineering Science*, 100148. <https://doi.org/10.1016/j.apples.2023.100148>]. To prove the existence theorem of global weak solutions some energy and entropy inequalities are derived and investigated. Additionally, a convergence of weak solutions is also rigorously proved. The derived results can be used or adapted for a class of distortion shallow water-based models. Extensions of these results for two-dimensional nonhomogeneous and homogeneous distortion shallow water based equations cases are in investigation. The obtained results generalize and improve some recent mathematical analysis results from shallow water based equations.

*Keywords:* Sediment transport model (STM); Viscous nonhomogeneous distortion shallow water equations; existence theorems of global weak solutions; convergence of weak solutions.

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## 1 Introduction

Several Sediment transport models in coastal environments based on shallow water type models have been developed in the literature. Due to the presence of an abrupt moving topography quickly spatially variable, the flow becomes turbulent. The mathematical models widely used to describe the Shallow Water phenomena do not account for the distortion effects (see Ngatcha [12]). One can regard distortion as a combination of strain and rotation. The effect of the rotational component is to weaken the effect of the strain somewhat. The anisotropy produced is such that the streamwise component of velocity has the largest fluctuations and the normal component has the smallest fluctuations. Indeed, it is well-known that the Shallow Water models are derived from a first-order approximation of long wave theory. In this work, a nonlinear analysis of a new one-dimensional sediment transport model is performed. The model is derived from the second-order approximation of long wave theory and is able to describe the distortion-free-boundary nonhomogeneous fluids viscous flow. We consider a one-dimensional mixing distortion flow between two hyperplans  $H_1$  and  $H_2$ :

$$H_1 = \{x \in \Omega, z = \eta(x, t), \forall 0 < t \leq T\} \quad (1)$$

$$H_2 = \{x \in \Omega, z = Z_b^*(x, t), \forall 0 < t \leq T\} \quad (2)$$

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The problem analyzed in this work consists to study all the unknown variables of a sediment transport model defined *p.p.* in  $Q_T$  with  $Q_T = \Omega \times ]0, T]$  as:

$$\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (h\bar{u}) &= E_w |\hat{u}| + \frac{\mathcal{E} - \mathcal{D}}{1 - \phi_s}, \\
\frac{\partial}{\partial t} (h\bar{u}) + \frac{\partial}{\partial x} (2h\bar{\kappa}) + gh \frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho} \frac{\partial \rho}{\partial x} &= h\overline{\mathcal{F}}_x + \mu \frac{\partial}{\partial x} (hD(\bar{u})) + E_w |\hat{u}| u_\xi - \frac{\mathcal{E} - \mathcal{D}}{1 - \phi_s} u_{Z_b^*}, \\
\frac{\partial}{\partial t} (h\bar{\kappa}) + \frac{\partial}{\partial x} (h\bar{u}(\bar{\kappa} + \mathcal{P})) + gh\bar{u} \frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho} \bar{u} \frac{\partial \rho}{\partial x} &= h\bar{u}\overline{\mathcal{F}}_x + E_w |\hat{u}| \kappa_\xi - \frac{\mathcal{E} - \mathcal{D}}{1 - \phi_s} \kappa_{Z_b^*} \\
+ \mu \bar{u} \frac{\partial}{\partial x} (hD(\bar{u})) + \frac{\partial}{\partial x} \left( 2\kappa \hat{u} \frac{\partial \hat{u}}{\partial x} \right), \\
\frac{\partial (h\bar{\alpha}_s)}{\partial t} + \frac{\partial}{\partial x} (h\bar{u} \bar{\alpha}_s) &= (\mathcal{E} - \mathcal{D}) + \frac{\partial}{\partial x} \left( \gamma h \frac{\partial \bar{\alpha}_s}{\partial x} \right), \\
\frac{\partial Z_b^*}{\partial t} + u_{Z_b^*} \frac{\partial Z_b^*}{\partial x} &= \frac{\mathcal{D} - \mathcal{E}}{1 - \phi_s} + \frac{2A_g}{1 - \phi_s} \frac{\partial}{\partial x} \left( 2\kappa \hat{u} \frac{\partial \hat{u}}{\partial x} \right) + w(Z_b^*), \\
h(0, x) &= h_0, \quad (hu)(0, x) = (hu)_0, \quad Z_b^*(0, x) = (Z_b^*)_0, \quad (h\bar{\kappa})(0, x) = (h\bar{\kappa})_0, \quad h\bar{\alpha}_s = (h\bar{\alpha}_s)_0, \quad \forall x \in \Omega \\
h &= h|_{\partial\Omega}, \quad hu = hu|_{\partial\Omega}, \quad \bar{\kappa} = \bar{\kappa}|_{\partial\Omega}, \quad h\bar{\alpha}_s = h\bar{\alpha}_s|_{\partial\Omega} \quad \text{on } \partial\Omega, \quad t \in ]0, T]
\end{aligned} \tag{3}$$

where  $h[m]$  is the water depth,  $h\bar{u}[m^2/s]$  is the water discharge ( $\bar{u}$  is the averaged velocity),  $\bar{\kappa}[m^2/s^2]$  is the kinetic energy  $\mathcal{P}[m^2/s^2]$  is the shear stress,  $C[m^3/s^3]$  is the sediment concentration and  $Z_b^*[m]$  is the bed level. The friction source term is  $\overline{\mathcal{F}}_x$  and  $\bar{\kappa} = \frac{\overline{uu} + \overline{u'u'}}{2}$  is the turbulent kinetic energy.  $\kappa_\xi = (uu)(\eta)$ ,  $\kappa_{Z_b^*} = (uu)(Z_b^*)$ .

In the system (3), we have:

$$\phi_f + \phi_s = 1, \quad \xi = Z_b^* + h, \quad \rho = \bar{\alpha}_s \rho_s + (1 - \bar{\alpha}_s) \rho_f = \rho_f + \delta \rho \bar{\alpha}_s \quad \text{and} \quad \delta \rho = \rho_s - \rho_f.$$

We have also  $\mu = \frac{\nu}{\rho}$ , where  $\nu$  is the viscosity of the clear water. In this context,  $u(t, x, Z_b^*)$  is the characteristic velocity at the bed-load should be defined as functions of the averaged quantities characterizing the water flow. Here,  $\partial\Omega$  is Lipschitz (i.e. is continuous). We have accounted the term  $\mathcal{P} = \overline{u'u'}$  that measures the distortion velocity in the vertical direction [10]. The derivation of this model is available in appendix.

Here; the first equation of the above model is the sediment/water mass conservation, the second is the mixture sediment/water momentum conservation equation, the third equation is the averaged kinetic energy conservation equation, the fourth equation is the sediment concentration evolution equation and the last equation is the bottom evolution equation. The above system of equation has been derived first by Ngatcha et al., [8] and improved by Ngatcha et al., [11]. The general theory for two-dimensional case has been introduced by Ngatcha and Nkonga [10]. All these models describe the morphodynamic and the sediment transport in a turbulent flow particularly in a context where the distortion effect is accounted. In nonhomogeneous distortion shallow water equations the distortion combines rotation and deformation and appears when the water moves near an abrupt topography. This effect is widely ignored in several sediment transport used in coastal environment. Therefore we present a new sediment transport model that accounts the distortion of horizontal profile velocity in the vertical direction. Moreover, it differentiates the fluid velocity from sediment velocity (phase-lag). The proposed model is one of most general found in the literature and extends several averaged sediment transport models based on shallow water equations available in the literature (see for instance [5], [6], [7], [16]). The model also extends the one developed in [8] and has been solved by a variant of path-conservative central-upwind AENO (Averaging Essentially Non Oscillatory) methods (see Ngatcha and Njifenjou [9] and Ngatcha [12]). When  $\mathcal{P} = 0$ , we retrieve a turbidity current model developed in shallow

water context (see [12]). A steady states solution of the proposed system of equation satisfy:

$$\begin{aligned}
u\partial_x h &= 0, \\
\partial_x(u^2) + \partial_x \mathcal{P} + \frac{\mathcal{P}}{h}\partial_x h + gh\partial_x \eta + gh\frac{\delta\rho}{2\rho}\partial_x C &= 0, \\
u\partial_x u^2 + 3u\mathcal{P} + \frac{\mathcal{P}}{h}u\partial_x h + gu\partial_x h + gu\partial_x Z_b + gh\frac{\delta\rho}{2\rho}\partial_x C &= 0, \\
u\partial_x C &= 0, \\
uZ_b^* \partial_x Z_b &= 0,
\end{aligned} \tag{4}$$

From these equations two particular families stationary solutions can be found. In absence of water entrainment source term, the steady state at rest for wet-cells reads:

$$h + Z_b^* = cste, \quad h\bar{u} = 0, \quad \mathcal{D} - \mathcal{E} = 0, \quad \mathcal{P} = 0, \quad \rho = cste. \tag{5}$$

The steady state for dry-cells reads:

$$h = 0, \quad \bar{u} = 0, \quad \mathcal{D} - \mathcal{E} = 0, \quad \mathcal{P} = 0. \tag{6}$$

An admissible space solution reads:

$$\mathbb{W} = \{(h, h\bar{u}, h\bar{K}, hC, Z_b) \in \mathbb{R}^5, h > 0, \mathcal{P} > 0, C > 0\} \tag{7}$$

The model is proved hyperbolic in  $\mathbb{W}$  and the eigenvalues of Jacobian matrix  $\mathcal{A}$  are given:

$$\lambda_1 = \bar{u} - \sqrt{gh + 3\mathcal{P}}, \quad \lambda_{2,3} = \bar{u}, \quad \lambda_3 = uZ_b^*, \quad \lambda_5 = \bar{u} + \sqrt{gh + 3\mathcal{P}} \tag{8}$$

The one-sided local speeds of propagation  $a^\pm$  are upper/lower bounds on the largest/smallest eigenvalues of Jacobian matrix of the system

$$a^+ = \min\{\lambda_1, \lambda_1, 0\}, \quad a^- = \max\{\lambda_5, \lambda_5, 0\}. \tag{9}$$

Solutions of the model (3) may develop discontinuities in finite time. The discontinuous solutions are weak solutions that are not always admissible. Thus we consider that they satisfy the following entropy inequality:

$$\partial_t E + \partial_x G \leq 0, \tag{10}$$

where the entropy-flux couple  $(E, G)$  are given in distortion shallow water context through the text (see below). For the particular choice of the family of segments  $\Phi(s, W_L, W_R) = W_L + s(W_R - W_L)$ . For the proposed model, a discontinuous solution  $W = W_L + (W_R - W_L)\mathbb{1}_{x>\sigma t}$  ( $\mathbb{1}_{x>\sigma t}$  is the Heaviside function) satisfies the jump condition or R-H relations given by:

$$\begin{aligned}
[[h\bar{u}]] &= \sigma[[h]], \\
[[2h\bar{K} + \frac{1}{2}gh^2]] + g\{\{h\}\}[[Z_b^*]] + g\frac{\rho_s - \rho_w}{2\bar{\rho}}[\{\{h\}\}[[h\bar{\alpha}_s]] - \{\{h\bar{\alpha}_s\}\}[[h]]] &= \sigma[[h\bar{u}]], \\
[[h\bar{K}\bar{u} + \frac{1}{2}gh^2\bar{u} + \mathcal{P}\bar{u}]] + g\{\{h\bar{u}\}\}[[Z_b^*]] + g\frac{\rho_s - \rho_w}{2\bar{\rho}}\left[\{\{h\bar{u}\}\}[[h\bar{\alpha}_s]] - \frac{\{h\bar{u}\}\{h\bar{\alpha}_s\}}{\{h\}}[[h]]\right] &= \sigma[[h\bar{K}]], \\
[[h\bar{u}\bar{\alpha}_s]] &= \sigma[[h\bar{\alpha}_s]], \\
\{\{u_b^*\}\}[[Z_b^*]] &= \sigma[[Z_b^*]].
\end{aligned} \tag{11}$$

Here,  $\sigma$  is a shock speed propagation and  $\bar{\rho} = \frac{\rho_R + \rho_L}{2}$ . Such condition facilitates the numerical and physical studies of waves solution of the model (see [8], [11]).

In this study, we are interested in a nonlinear analysis of a problem governing sediment transport equations in a mixing turbulent viscous flow and this is a novelty in the literature. The viscous effects are included in

the model via the term  $\mu \frac{\partial(hD(\bar{u}))}{\partial x}$ . The account of this term allows to improve the regularity of the water depth. However, another viscous parametrization is possible. We can use for example  $\frac{\partial(hD(\bar{u}))}{\partial x}$  instead of  $\mu \frac{\partial(hD(\bar{u}))}{\partial x}$  without extra difficulties but the problem of consistency will appear. While the diffusion term  $\mu \frac{\partial(hD(\bar{u}))}{\partial x}$  improves the regularity of the  $h$  its also produces energetically consistence solution.

Particularly, we prove the existence of theorems of a global weak solution. Such a study is difficult to meet in modern literature. The literature offers among other things, numerous results related to the existence theorem global weak solutions for viscous shallow water-based equations (or viscous sedimentation in shallow water equations). [1], [2], [16]. An existence of global weak solution has been also proved for Boussinesq-type model by [3] and for an equation describing the motion of waves at the free surface of shallow water under the influence of gravity (see [4]). Many above results have are limited inspired us in this work, but however, many modifications have been proven necessary to maintain the physical and mathematical properties of the model. In this paper energy and entropy inequalities are proposed to the first time to show the stability of these weak solutions. The solution functions of the problem are located in vectorial spaces and are defined p.p in these spaces. However, should be noted that functions in vector spaces are defined better than almost anywhere. The variational capacity makes it possible function more finely than the Lebesgue measure. We show here that the proposed STM is energetically consistent without the use of any restricted variable or data as in [14] and some other works.

The rest of the paper is organized as follows. In section 2, we propose a new sediment transport model without Reynolds dissipation and we present some notations and essential preliminary results in this work. We expose also the variational formulation of the model. In section 3, entropy and energy inequalities are given and we propose a existence theorem of global weak solution. In section 4, a convergence proof of global weak solutions is proposed. The derivation of the model and some functional analysis results and mathematical tools are available in appendices [AppendixA](#), [AppendixB](#) and [AppendixC](#).

## 2 The model without dissipation effect: Mathematical tools and preliminaries results

In this section, we recall the model without Reynolds dissipation effect and some important preliminary results. These results we will permit us to give some new mathematical and mechanical justifications of the proposed sediment transport model in nonhomogeneous distortion shallow water equations. The convection-diffusion STM without Reynolds dissipation effect  $\forall(t, x) \in ]0, T] \times \Omega \subset \mathbb{R}_+^* \times \mathbb{R}$ , reads:

$$\begin{aligned}
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) &= E_w|\hat{u}| + \frac{\mathcal{E} - \mathcal{D}}{1 - \phi_s}, \\
\frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}(2h\bar{K}) + gh\frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho}\frac{\partial \rho}{\partial x} &= h\bar{\mathcal{F}}_x + \mu\frac{\partial}{\partial x}(hD(\bar{u})) + E_w|\hat{u}|u_\xi - \frac{\mathcal{E} - \mathcal{D}}{1 - \phi_s}u_{z_b^*}, \\
\frac{\partial}{\partial t}(h\bar{K}) + \frac{\partial}{\partial x}(h\bar{u}(\bar{K} + \mathcal{P})) + gh\bar{u}\frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho}\bar{u}\frac{\partial \rho}{\partial x} &= h\bar{u}\bar{\mathcal{F}}_x + E_w|\hat{u}|K_\xi - \frac{\mathcal{E} - \mathcal{D}}{1 - \phi_s}K_{z_b^*} + \mu\bar{u}\frac{\partial}{\partial x}(hD(\bar{u})), \\
\frac{\partial(h\bar{\alpha}_s)}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}\bar{\alpha}_s) &= (\mathcal{E} - \mathcal{D}) + \frac{\partial}{\partial x}\left(\gamma h\frac{\partial \bar{\alpha}_s}{\partial x}\right), \\
\frac{\partial Z_b^*}{\partial t} + u_{z_b^*}\frac{\partial Z_b^*}{\partial x} &= \frac{\mathcal{D} - \mathcal{E}}{1 - \phi_s},
\end{aligned} \tag{12}$$

The derivation of the above model can be seen in appendix. We joint to these equations the initial and boundary conditions.

### Initial and boundary conditions

The initial conditions are:

$$h(x, 0) = h^0(x), h\bar{u}(x, 0) = h\bar{u}^0(x), h\bar{\alpha}_s(x, 0) = h\bar{\alpha}_s^0(x), (h\bar{K})(x, 0) = (h\bar{K})^0(x), Z_b^*(x, 0) = (Z_b^*)^0(x). \tag{13}$$

The Neumann boundary conditions associated to the model are given by:

$$h.n = 0, (h\bar{u}).n = 0, (h\bar{\alpha}_s).n = 0, \mathcal{P}.n = 0, (h\bar{K}).n = 0, Z_b^*.n = 0. \quad (14)$$

### 2.1 Preliminary results

We propose to show that a solution of the model problem consisting to resolve (3) is not unique. We start by re-writing the new model in compact form following:

$$\begin{cases} \partial_t W + \partial_x F(W) + \sum_i B_i(gh)\partial_x W = S(W), & x \in \Omega, \quad t \in (0, T], \\ W(x, 0) = W^0, & x \in \Omega \\ \text{Neumann condition,} \end{cases} \quad (15)$$

where  $W$  is a vector to  $j$  unknowns (including for example the water depth  $h$ , the water discharge  $h\bar{u}$ , the energy  $h\bar{K}$ , the sediment concentration  $h\bar{\alpha}_s$ , the topography  $Z_b^*$ ),  $F(W)$  is the physical flux,  $B_i(gh)$  is the nonconservative terms associated to loss potential energy and depends on wave celerity  $v^2 = gh$ ,  $S(W)$  is a source term containing the bed friction and diffusion effects.

### Results of non-uniqueness of a solution of the model problem

We suppose here that  $T > 0$ . If we assume that  $\partial_t W \in (C^0(\bar{\Omega} \times ]0, T])^j \forall x \in \bar{\Omega}$ ,  $\partial_x F(W) \in (C^0(\bar{\Omega} \times ]0, T])^j$ , then, so that the source term  $S(W(x, t)) \in (C^0(\bar{\Omega} \times ]0, T])^j$ , we would tempted to say that  $B_i(h)\partial_x W \in (C^0(\bar{\Omega} \times ]0, T])^j$ . In this case, it is necessary that  $B_i$  either includes  $(C^0(\bar{\Omega} \times ]0, T])^j$ ,  $\forall i$  since  $\partial_x W \in (C^0(\bar{\Omega} \times ]0, T])^j$ . Say  $B_i(h) \in (C^0(\bar{\Omega} \times ]0, T])^j$  would not be true since  $W \in (C^1(\bar{\Omega} \times ]0, T])^j$ . It would be true that  $B_i(h) \in (C^1(\bar{\Omega} \times ]0, T])^j$ , and in this situation the product  $\sum_i B_i \partial_x W$  is not still includes  $(C^0(\bar{\Omega} \times ]0, T])^j$ .

All the above comments are resumed in the following non-uniqueness result of the model problem.

**Proposition 2.1** (Non-uniqueness of a smooth solution). *Let us the problem given by (15) with  $T > 0$ . We assume that  $\partial_t W \in (C^1(\bar{\Omega} \times ]0, T])^j, \forall x \in \bar{\Omega}$ , and  $S(W) \in (C^1(\bar{\Omega} \times ]0, T])^j$ . Therefore, the model problem admits a solution possibly discontinuous. Particularly it admits at least a solution.*

### Non uniqueness of a solution in $L^2(\Omega)$ .

The fact that the unknown vector  $W(x, t)$  is described in  $(L^2(]0, T], \Omega))^j$  and that there exists an initial solution, we incite to say that  $W(\cdot, 0) \in (L^2(\Omega))^j$ . Moreover, we can rewrite  $\lim_{t \rightarrow +\infty} W(t, \bullet) = W^0 \in (L^2(\Omega))^j$ , but we cannot still that affirm  $\lim_{t \rightarrow +\infty} B_i(W(t, \bullet))\partial_x W(t, \bullet) = B_i(W(x, 0))\partial_x W(x, 0) \in (L^2(\Omega))^j$ .

Now we assume that  $W^0 \in (L^2(\Omega))^j$  and  $S(W) \in (C^0(]0, T], L^2(\Omega))^j$  then it is necessary that  $B_i(W(t, x))\partial_x W \in (C^0(]0, T], L^2(\Omega))^j$  to ensure the existence of a unique solution. In fact, this implies that  $W \in (C^1(]0, T], L^2(\Omega))^j$  and thus  $W \in L^p(0, T, L^2(\Omega))$  since  $C(]0, T], L^2(\Omega))$  is dense  $W \in L^p(0, T, L^2(\Omega))$ ,  $1 \leq p < \infty$  which leads to say that  $B_i(W(t, x)) \in (C^1(]0, T], L^2(\Omega))^j$ , i.e. to know that the product  $B_i(W(t, \bullet))\partial_x W$  is possibly discontinuous.

One has the following

**Proposition 2.2** (Non uniqueness in  $(L^2(\Omega))^j$ ). *We assume that  $W^0 \in (L^2(\Omega))^j$  and  $S(W) \in (C^0(]0, T], L^2(\Omega))^j$ . Then, the product  $B_i(W(t, \bullet))\partial_x W$  is possibly discontinuous. Therefore, the nonconservative problem admits a non unique solution in  $(L^2(]0, T], \Omega))^j$ .*

## 2.2 Variational formulation

We have the following basis hypothesis:

$$\begin{aligned}
h^0 &\in L^2(\Omega), \sqrt{h^0} \in L^2(\Omega), \partial h^0 \in L^2(\Omega), h^0 \bar{u}^0 \bar{u}^0 \in L^1(\Omega) \\
h^0 \bar{u}^0 &\in L^1(\Omega), h^0 \bar{\alpha}_s^0 \in L^2(\Omega), \bar{\alpha}_s^0 = \frac{h^0 \bar{\alpha}_s^0}{h^0} \in L^2(\Omega), \mathcal{P}^0 \in L^1(\Omega), \\
h^0 \bar{K}^0 &= h^0 \bar{u}^0 \bar{u}^0 + \mathcal{P}^0 \in L^1(\Omega), \frac{(\widehat{\mathcal{P}}^2)^0}{h} \in L^1(\Omega) \text{ with } \widehat{\mathcal{P}} = h\sqrt{\mathcal{P}}, (Z_b^*)^0 \in H^1(\Omega) \hookrightarrow C^0(\bar{\Omega}).
\end{aligned} \tag{16}$$

**Définition 2.1.** We say that  $(h, \bar{u}, \bar{K}, \bar{\alpha}_s, Z_b^*) \in \mathbb{W}$  is a weak solution of (3), in  $\mathcal{D}(Q_T) = C^\infty(Q_T)$  with initial data given in (16) verifying the entropy inequality (27) of all the smooth test functions  $\varphi(x, t)$  ( $\varphi$  with compact support in  $R \times R^+$ , such that  $\varphi(x, t)$  and  $\varphi(x, 0) = \varphi^0$ ). We have:

$$\begin{aligned}
& - \int_{\Omega} h(x, 0)\varphi(x, 0) - \int_{Q_T} h \partial_t \varphi + \int_{Q_T} (h\bar{u}) \cdot \partial_x \varphi = \int_{Q_T} P_1(u, h)\varphi; \\
& - (h\bar{u})(x, 0)\varphi(x, 0) + \int_{Q_T} h\bar{u} \partial_t \varphi + \int_{Q_T} h\bar{u}^2 \partial_x \varphi + \int_{Q_T} (\partial_x \mathcal{P})\varphi + \int_{Q_T} g \partial_x (1/2h^2)\varphi + \int_{Q_T} \left(\frac{gh^2}{2\rho} \frac{\partial \rho}{\partial x}\right)\varphi \\
& + \int_{Q_T} g(h\partial_x Z_b^*)\varphi + \int_{\Omega} \mu h D(\bar{u}) \frac{\partial \varphi}{\partial x} = \int_{Q_T} P_2(\bar{u}, h)\varphi; \\
& - \int_{\Omega} (h\bar{K})(x, 0)\varphi(x, 0) + \int_{Q_T} h\bar{u}^3 \partial_x \varphi + \frac{3}{2} \int_{Q_T} h\bar{u}(\partial_x \mathcal{P})\varphi + \frac{3}{2} \int_{Q_T} \mathcal{P}\bar{u}(\partial_x h)\varphi + \int_{Q_T} \frac{gh^2}{2\rho} \bar{u} \frac{\partial \rho}{\partial x} \varphi \\
& + \int_{Q_T} g(h\bar{u} \partial_x Z_b^*)\varphi - \int_{\Omega} \mu |\sqrt{h} D(u)|^2 \varphi = \int_{Q_T} P_3(\bar{u}, h)\varphi; \\
& - \int_{\Omega} (h^0 \bar{\alpha}_s^0) \varphi(x, 0) - \int_{Q_T} (h\bar{\alpha}_s) \partial_t \varphi + \int_{Q_T} ((h\bar{u}) \partial_x \bar{\alpha}_s)\varphi + \int_{Q_T} h \partial_x \bar{\alpha}_s \partial_x \varphi = \int_{Q_T} P_4(u, h)\varphi; \\
& - \int_{\Omega} (Z_b^*)^0 \varphi(x, 0) - \int_{Q_T} Z_b^* \partial_t \varphi + \int_{Q_T} (u(t, x, Z_b^*) \partial_x Z_b^*)\varphi = \int_{Q_T} P_5(\bar{u}, h)\varphi;
\end{aligned} \tag{17}$$

where  $P_1(\bar{u}, h) = E_w |\widehat{u}| + \frac{\mathcal{E} - \mathcal{D}}{1 - \phi_s}$ ;  $P_2(\bar{u}, h) = h\overline{\mathcal{F}}_x + E_w |\widehat{u}| u_\xi - \frac{\mathcal{E} - \mathcal{D}}{1 - \phi_s} u_{Z_b^*}$ ;  $P_3(\bar{u}, h) = h\bar{u}\overline{\mathcal{F}}_x + E_w |\widehat{u}|_{K_\xi} - \frac{\mathcal{E} - \mathcal{D}}{1 - \phi_s} K_{Z_b^*}$ ;  $P_4(\bar{u}, h) = \mathcal{E} - \mathcal{D}$ ;  $P_5(\bar{u}, h) = -\frac{\mathcal{E} - \mathcal{D}}{1 - \phi_s}$ .

This definition authorizes the function at least of class  $C^1$  and even the discontinuous solution to be weak solution since a hyperbolic nature of the model problem. Reciprocally, all the weak solution that satisfy (17) is also solution of the problem model given by (3). The time derivative of  $h$ ,  $h\bar{u}$ ,  $h\bar{K}$ ,  $h\bar{\alpha}_s$ ,  $Z_b^*$  are understood in the usual sense of distributions on  $\Omega \times ]0, T]$ . Let us

$$\chi = \left\{ \sum_{i=1}^n \varphi_i(t) Y_i(x), n \in \mathbb{N}, \varphi_i \in C_c^\infty(]0, T]), Y_i \in C_c^\infty(\Omega) \right\} \tag{18}$$

the set of tensorial functions in  $C^\infty(\Omega \times ]0, T])$ . The fact that  $\chi$  is dense in  $L^p(0, T; H^1(\Omega))$  ensures that the distribution derivative  $\partial_t$  belongs to  $(L^p(0, T; H^1(\Omega)))'$  and we have the following continuous relations for the norm  $L^1(0, T; H^1(\Omega))$

$$\varphi \in \chi \rightarrow \langle \partial_t W^{(k)}, \varphi \rangle = \varphi(T) W^{(k)}(T) - \int_0^T \int_{\Omega} W^{(k)}(x, t) \partial_t \varphi$$

**Proposition 2.3.** Let  $t \mapsto \sigma(t)$  be a regular curve. Let  $h$ ,  $h\bar{u}$ ,  $h\bar{K}$ ,  $h\bar{\alpha}_s$ ,  $Z_b^*$  functions of class  $C^1(\Omega)$ , bounded and its derivatives in  $\Omega_- = \{(t, x), x < \sigma(t)\}$  and of class  $C^1$  in  $\Omega_+ = \{(t, x), x > \sigma(t)\}$  with  $\Omega = \Omega_- \cap \Omega_+$ . Then,  $W^-$  and  $W^+$  are weak solution of the variational given by (17).

**Remark 1.** *The above relations or variational formulation of the problem model are essential to prove some results presented below. Particularly we are going to prove an existence theorem of global weak solution  $(h, h\bar{u}, h\bar{K}, hC, Z_b$  that satisfying energy and inequalities relation to be establish and where the initial condition given by (13) satisfy the system (16).*

### 3 Energy and entropy inequalities

In this section, we give some energy and entropy inequalities associated to the proposed model. We show how the presence of the turbulence modifies and improves several inequalities often obtained in shallow water context. We will recall the inequalities obtained in shallow water context. The entropy and energy relations are essential to establish several estimates used to prove consistency and stability of the model. Thus we prove in this section that the proposed model admits a global weak solution in a sense where it verify the energy and entropy inequalities developed here.

#### 3.1 Energy and entropy inequalities without distortion and viscous effects

We start by recalling the energy and entropy equations verified for a smooth solution of the model obtained when the turbulence effect  $\mathcal{P} = \hat{u}^2$  and the viscous term  $\mu \frac{\partial}{\partial x}(hD(\bar{u}))$  are removed. In this situation, one has the following

**Lemma 1.** *Let  $(h, h\bar{u}, h\bar{\alpha}_s, Z_b^*)^T$  be a solution of the system:*

$$\begin{aligned} \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\bar{u}) &= \frac{dF_\xi}{dt} - \frac{dF_b}{dt} \\ \frac{\partial}{\partial t}(h\bar{u}) + \frac{\partial}{\partial x}(h\bar{u}^2) + gh \frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho} \frac{\partial \rho}{\partial x} &= h\mathcal{F}_x + \frac{dF_\xi}{dt} u_\xi - \frac{dF_b}{dt} u_{Z_b^*} \\ \frac{\partial}{\partial t}(h\bar{\alpha}_s) + \frac{\partial}{\partial x}(h\bar{u}\bar{\alpha}_s) &= -\phi_f \frac{dF_b}{dt} + \frac{\partial}{\partial x} \left( \gamma h \frac{\partial \bar{\alpha}_s}{\partial x} \right) \\ \frac{\partial Z_b^*}{\partial t} + u(t, x, Z_b^*) \frac{\partial Z_b^*}{\partial x} &= \frac{dF_b}{dt}. \end{aligned} \quad (19)$$

Then the following energy equation holds:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} h\bar{u}^2 + \frac{3}{2} \frac{d}{dt} \int_{\Omega} \frac{1}{2} gh^2 + \int_{\Omega} gZ_b^* \partial_t h & \\ = \int_{\Omega} \bar{u} P_2(\bar{u}, h) + \int_{\Omega} \frac{gh}{2} P_1(\bar{u}, h) \int_{\Omega} g(h + Z_b^*) P_1(\bar{u}, h). & \end{aligned} \quad (20)$$

Eq. (20) is substantially different to that obtain with homogeneous fluid. Therefore the presence of sediment in the fluid modifies the mechanical energy of the system. When all the source terms are removed ( $P_i(\bar{u}, h) = 0, i = 1, 2$ ), Eq.(20) becomes:

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} h\bar{u}^2 + \frac{3}{2} \frac{d}{dt} \int_{\Omega} \frac{1}{2} gh^2 + \int_{\Omega} gZ_b^* \partial_t h = 0. \quad (21)$$

*Proof.* To obtain Eq.(20), we multiply the momentum equation of (19) by  $\bar{u}$  and we integrate over  $\Omega$ . In some integrations, we use the Green formula to achieve many simplifications (see below). A similar result can be obtained by integrating the kinetic energy equation (given by the third equation of (3)) without distortion effect where  $\kappa = \frac{1}{2} h\bar{u}^2$ .  $\square$

In the domain where the weak solution is regular, the following equality is satisfied by the functions  $(h, h\bar{u}, h\bar{K}, h\bar{\alpha}_s, Z_b^*)^T$  and ones has:

$$\partial_t \xi + \bar{u} \partial_x (\xi + h\mathcal{P} + gh^2/2) + gh\bar{u} \partial_x Z_b^* + \frac{gh^2 \bar{u} \delta \rho}{2\rho} \partial_x \bar{\alpha}_s = P_3(\bar{u}, h) + ghP_1(\bar{u}, h) + \bar{u} \partial_x (h\nu \partial_x \bar{u}) \quad (22)$$



where the mechanical energy is given by  $\xi = h\bar{\kappa} + gh^2/2$ . The equation is obtained by multiplying  $h$  equation by  $gh$  and by adding with  $h\bar{\kappa}$  equation. After integration, we obtain:

$$\int_{\Omega} \frac{d}{dt} \left( \xi + \frac{gh^2}{2} + ghZ_b^* \right) d\Omega + \int_{\Omega} \bar{u} \partial_x h \mathcal{P} + \int_{\Omega} \frac{gh^2 \bar{u} \delta \rho}{2\rho} \partial_x \bar{\alpha}_s = \int_{\Omega} P_3(\bar{u}, h) + \int_{\Omega} ghP_1(\bar{u}, h) - \frac{\nu}{2} \int_{\Omega} h |\partial_x \bar{u}|^2 \quad (23)$$

Moreover, solutions of the model (19) may develop discontinuities in finite time. These discontinuous solutions are weak solutions that are not always admissible. Thus we consider that they satisfy the following entropy inequality:

$$\partial_t \xi + \partial_x G \leq 0, \quad (24)$$

where the energy written as:

$$\xi = \frac{1}{2} gh^2 + h\mathcal{P} + ghZ_b^* + \frac{h|\bar{u}|^2}{2}, \quad (25)$$

and where the energy flux reads

$$G = \left( g(Z_b^* + h) + \frac{3}{2} \mathcal{P} + \frac{gh}{2\rho} \delta \rho \bar{\alpha}_s + \frac{\bar{u}^2}{2} \right) h\bar{u}. \quad (26)$$

The function  $\xi$  acts as mathematical entropy. From one shock to another, the energy decreases.

At a point of discontinuity, the steady state solutions should verify the dissipation entropy:

$$\partial_x \left( g(Z_b^* + h) + \frac{3}{2} \mathcal{P} + \frac{gh}{2\rho} \delta \rho \bar{\alpha}_s + \frac{\bar{u}^2}{2} \right) h\bar{u} \leq 0, \quad (27)$$

and this allows to conclude that  $h\bar{u}$  is constant in whole the domain. When  $\rho = \rho_w$  we have:

$$\partial_x \left( g(Z_b^* + \frac{h}{2}) + \frac{3}{2} \mathcal{P} + \frac{\bar{u}^2}{2} \right) h\bar{u} \leq 0, \quad (28)$$

**Lemma 2.** *If the functions  $(h, h\bar{u}, h\bar{\kappa}, h\bar{\alpha}_s, Z_b^*)^T$  are solution of the system (3) with initial data  $(u^0, h^0, \mathcal{P}^0, C^0, Z_b^0)$  given by (13) and satisfying (16) then we have the following entropy equality: With a test function  $\varphi \in D(Q_T)$  we have for:*

$$\int_{\Omega} \partial_t \xi(u^0, h^0, \mathcal{P}^0, C^0, (Z_b^*)^0) \varphi(x, 0) dx + \int_{\Omega} \int_0^T \xi(h, u, \mathcal{P}, C, Z_b^*) \partial_t \varphi dx dt + \int_{\Omega} \int_0^T \partial_x \varphi G(h, u, \mathcal{P}, C, Z_b^*) dx dt \leq 0, \quad (29)$$

**Remark 2.** *The relation does not ensures the existence of non-unique entropy solution in sense of Kruzkov of the problem model (3). Although every wave solution of the model propagate at the finite speed  $a^{\pm}$  given in (9).*

### 3.2 Energy and entropy inequalities with distortion and viscous effects

We will derive some energy and entropy inequalities necessary to ensure the stability of the model in energy sense.

#### Integration of $h$ equation

We derive the  $h$  equation of (3) with respect to  $x$  and we multiply by  $\partial_x h$ . We get after integration on  $\Omega$  the following result:

$$\frac{d}{dt} \int_{\Omega} \partial_x |h|^2 \partial_x^2 \bar{u} + \int_{\Omega} \frac{1}{2} |\partial_x h|^2 \bar{u} = \int_{\Omega} \partial_x h \partial_x P_1(\bar{u}, h). \quad (30)$$

### Integration of $h\bar{u}$ equation and reformulation

We multiply the  $h\bar{u}$  equation by  $\frac{\partial_x h}{h}$  and we integrate over  $\Omega$ . We get:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \bar{u} \cdot \partial_x h + \int_{\Omega} \partial_x h \partial_x \mathcal{P} + g \int_{\Omega} \partial_x h \partial_x Z_b^* + \int_{\Omega} g(\rho) |\partial_x h|^2 + g \int_{\Omega} |\partial_x h|^2 + \int_{\Omega} g(\rho) \partial_x h \partial_x h \bar{\alpha}_s \\ = \int_{\Omega} \mu \partial_x (hD(u)) \frac{\partial_x h}{h} + \int_{\Omega} \frac{\partial_x h}{h} P_2(\bar{u}, h). \end{aligned} \quad (31)$$

We have about the diffusion term:

$$\int_{\Omega} \mu \partial_x (hD(\bar{u})) \frac{\partial_x h}{h} = \int_{\Omega} \mu \frac{(\partial_x h)^2}{h} D(\bar{u}) + \int_{\Omega} \mu \partial_x h \partial_x^2 \bar{u},$$

where we can also note that the diffusion improves the regularity of the water depth  $h$  by the following relation:

$$\int_{\Omega} \mu \partial_x h \partial_x^2 \bar{u} = - \int_{\Omega} \partial_x \bar{u} \cdot \partial_x^2 h.$$

Using this simplification relation, the equation (31) becomes:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \bar{u} \cdot \partial_x h + \int_{\Omega} \partial_x h \partial_x \mathcal{P} + g \int_{\Omega} \partial_x h \partial_x Z_b^* + \int_{\Omega} g(\rho) |\partial_x h|^2 + g \int_{\Omega} |\partial_x h|^2 + \int_{\Omega} g(\rho) \partial_x h \partial_x h \bar{\alpha}_s \\ + \int_{\Omega} \mu \frac{(\partial_x h)^2}{h} D(\bar{u}) - \int_{\Omega} \partial_x \bar{u} \cdot \partial_x^2 h = \int_{\Omega} \frac{\partial_x h}{h} P_2(\bar{u}, h). \end{aligned} \quad (32)$$

If we remove all the source terms in mass conservation equation given by the first equation of (3) we can have:

$$\int_{\Omega} \bar{u} \partial_x h = - \int_{\Omega} \log h (\text{div}(h\bar{u})) = \int_{\Omega} \log(h \partial_t h) = \frac{d}{dt} \int_{\Omega} (h \log h - h) \quad (33)$$

And in this case Eq.(32) can write as:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} (h \log h - h) + \int_{\Omega} \partial_x h \partial_x \mathcal{P} + g \int_{\Omega} \partial_x h \partial_x Z_b^* + \int_{\Omega} g(\rho) |\partial_x h|^2 + g \int_{\Omega} |\partial_x h|^2 + \int_{\Omega} g(\rho) \partial_x h \partial_x h \bar{\alpha}_s \\ + \int_{\Omega} \mu \frac{(\partial_x h)^2}{h} D(\bar{u}) - \int_{\Omega} \partial_x \bar{u} \cdot \partial_x^2 h = \int_{\Omega} \frac{\partial_x h}{h} P_2(\bar{u}, h). \end{aligned} \quad (34)$$

### Integration of $h\bar{K}$ equation

We integrate over  $\Omega$  the equation  $h\bar{K}$  and we get:

$$\begin{aligned} \frac{d}{dt} \left[ \int_{\Omega} h\bar{K} + \int_{\Omega} - \int_{\Omega} \frac{3}{4} gh^2 \right] + \int_{\Omega} g Z_b^* \partial_t h + \int_{\Omega} \bar{u} \cdot \partial_x h \mathcal{P} + \int_{\Omega} \bar{u} \cdot \partial_x \frac{1}{2} gh^2 + \int_{\Omega} \mu |\sqrt{h} D(\bar{u})|^2 \\ = \int_{\Omega} g(h + Z_b^*) P_1(\bar{u}, h) + \int_{\Omega} P_3(\bar{u}, h). \end{aligned} \quad (35)$$

Here, we have used the Green formula and some simplifications:

$$\begin{aligned} \int_{\Omega} \mu \partial_x (hD(\bar{u})) \bar{u} &= \int_{\Omega} \mu h \partial_x \bar{u} \partial_x \bar{u} = - \int_{\Omega} \mu |\sqrt{h} D(\bar{u})|^2, \\ \int_{\Omega} gh^2 \bar{u} \partial_x \rho &= \frac{d}{dt} \int_{\Omega} \frac{3}{4} gh^2, \\ \int_{\Omega} gh \bar{u} \partial_x (h + Z_b^*) &= \frac{d}{dt} \int_{\Omega} \frac{1}{2} gh^2 + \int_{\Omega} g Z_b^* \partial_t h. \end{aligned}$$

**Lemma 3.** *If the functions  $(h, h\bar{u}, h\bar{K}, h\bar{\alpha}_s, Z_b^*)^T$  are solutions of the system (3), then we have the following energy equality:*

$$\begin{aligned} & \frac{d}{dt} \left[ \int_{\Omega} h\bar{K} + \int_{\Omega} \frac{3}{4}gh^2 + \int_{\Omega} \bar{u} \cdot \partial_x h \right] + \int_{\Omega} (g(\rho) + g) |\partial_x h|^2 + \int_{\Omega} \partial_x |h|^2 \partial_x^2 \bar{u} + \int_{\Omega} gZ_b^* \partial_t h \\ & + \int_{\Omega} g(\rho) \partial_x h \partial_x h \bar{\alpha}_s + \int_{\Omega} \mu |\sqrt{h}D(\bar{u})|^2 + \int_{\Omega} \partial_x h \partial_x Z_b^* + \int_{\Omega} \bar{u} \cdot \partial_x h \mathcal{P} + \int_{\Omega} \mu \frac{(\partial_x h)^2}{h} D(u) \\ & - \int_{\Omega} \partial_x u \cdot \partial_x^2 h + \int_{\Omega} \bar{u} \cdot \partial_x \left( \frac{1}{2}gh^2 \right) + \int_{\Omega} \partial_x h \partial_x \mathcal{P} = G(u, h), \end{aligned} \quad (36)$$

where the term  $G(\bar{u}, h)$  reads:

$$G(\bar{u}, h) = \int_{\Omega} g(h + Z_b^*)P_1(\bar{u}, h) + \int_{\Omega} P_3(\bar{u}, h) + \int_{\Omega} \frac{\partial_x h}{h} P_2(\bar{u}, h) + \int_{\Omega} \partial_x h \partial_x P_1(\bar{u}, h). \quad (37)$$

The inequality can be obtained by using the fact that  $ab \leq \frac{1}{2}(a^2 + b^2)$ , for all  $a, b \in \mathbb{R}$ . We can also use the Holder inequality:

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad a, b \in \mathbb{R}, \varepsilon > 0. \quad (38)$$

The relation (27) and (36) are keys to prove several original results about the new model (3) (that describes the sediment transport and morphodynamic in the context of accelerated shallow water flow). It is expected that the entropy and energy relations obtained using distortion shallow water equations context generalize those obtained using shallow water equations [2], [14], [15], [16]. These relations ensure the existence of a global weak solution of the model.

### 3.3 Existence theorem of a global weak solution of the developed 1D model

Here, we proposed a existence theorem of a global weak solution satisfying certain condition exposed above. We start by writing the following result.

**Theorem 3.1.** *Let us the initial values of the model given above. We assume that the unknowns of the model satisfy the variational formulation given by Eq. (17). There exists a global weak solution  $(h, h\bar{u}, h\bar{K}, h\bar{\alpha}_s, Z_b^*)^T$  of (3) satisfying the energy and entropy inequalities (27)-(29) and (36) respectively.*

We have also the following

**Proposition 3.1.** *According to the above relations (27) and (36), we have the following estimates:*

$$\begin{aligned} \|P_1(\bar{u}, h)\|_{L^2(0,T;H^{-1}(\Omega))} &\leq c \\ \|\partial_x \sqrt{h}\|_{L^2(0,T;H^1(\Omega))} &\leq c \\ \|\partial_x h\|_{L^\infty(0,T;H^1(\Omega))} &\leq c \\ \|\bar{u}\sqrt{h}\|_{L^\infty(0,T;L^2(\Omega))} &\leq c \\ \|\partial_x Z_b^*\|_{L^\infty(0,T;H^1(\Omega))} &\leq c \\ \|\partial_x h\bar{\alpha}_s\|_{L^2(0,T;L^2(\Omega))} &\leq c \\ \|Z_b^*\|_{L^\infty(0,T;L^2(\Omega))} &\leq c \\ \|\partial_x \mathcal{P}\|_{L^2(0,T;L^2(\Omega))} &\leq c \\ \|\partial_x \sqrt{h}\|_{L^2(0,T;H^1(\Omega))} &\leq c \\ \|\partial_x \sqrt{h}D(u)\|_{L^2(0,T;L^2(\Omega))} &\leq c \end{aligned} \quad (39)$$

**Proposition 3.2.** *According to the definition of the energy-space  $\mathbf{W}(0, T)$ , the above relations (27)-(36) and the estimate relations given by (39), the following relations hold (for  $c$  constant):*

$$\begin{aligned} \|\sqrt{h}\|_{\mathbf{W}(0,T)} &\leq c, & \|h\|_{\mathbf{W}(0,T)} &\leq c, & \|Z_b^*\|_{\mathbf{W}(0,T)} &\leq c, \\ \|h\bar{\alpha}_s\|_{\mathbf{W}(0,T)} &\leq c, & \|h\bar{u}\|_{\mathbf{W}(0,T)} &\leq c, & \|\hat{u}\|_{\mathbf{W}(0,T)} &\leq c, \end{aligned} \quad (40)$$

The above proposition 3.2 can help to prove that a global weak solution of the model problem is bounded in  $\mathbf{W}(0, T)$ . The fact to prove that the solution is bounded in  $\mathbf{W}(0, T)$  solidify our stability result proposed in this work.

#### 4 Convergence result of the model

We consider here a sequence of approximate global weak solution  $(h_n, (h\bar{u})_n, (h\bar{K})_n, (h\bar{\alpha}_s)_n, (Z_b^*)_n)^T$  satisfying the above model given by (3). A sequence of approximate weak solution satisfies also the variational problem given (17).

We assume that its initial values satisfy (for  $c$  constant):

$$\begin{aligned} \bar{\alpha}_{s0}^n &\rightarrow \bar{\alpha}_{s0} \text{ strongly in } L^2(\Omega); & (Z_b^*)_0^n &\rightarrow (Z_b^*)_0 \text{ strongly in } L^2(\Omega); & h_0^n &\rightarrow h_0 \text{ strongly in } L^2(\Omega); \\ \partial h_0^n &\rightarrow \partial h_0 \text{ strongly in } L^2(\Omega); & \frac{(q_0^2)^n}{h_0^n} + \mathcal{P}_0^n &\rightarrow \frac{(q_0^2)}{h_0} + \mathcal{P}_0 \text{ strongly in } L^1(\Omega); & \mathcal{P}_0^n &\rightarrow \mathcal{P}_0 \text{ strongly in } L^1(\Omega); \\ \partial h_0^n \bar{\alpha}_0^n &\rightarrow \partial h_0 \bar{\alpha}_0 \text{ strongly in } L^1(\Omega). \end{aligned}$$

This initial solution verifies also the relation given by:

$$\|\partial_x \sqrt{h_0}\|_{L^2(\Omega)} \leq c; \quad \|h_0 \bar{u}_0 \bar{\alpha}_{s0}\|_{L^2(\Omega)} = \left\| \frac{m_0}{h_0} \right\|_{L^2(\Omega)} \leq c. \quad (41)$$

And in thus we have the following estimation:

$$\int_{\Omega} \left( \frac{1}{2} g h_0^2 + h_0 \mathcal{P}_0 + g h_0 (Z_b^*)_0 + \frac{h_0 \mathcal{P}_0}{2} + \frac{h_0 |\bar{u}_0|^2}{2} \right) \leq c \quad (42)$$

The inequality (42) allows us to show that the kinetic energy equation with distortion nature of the non-homogeneous flow is stable and bounded. However, more other relations can be used to prove it and this require some investigations. The proposed paper give a brief introduction of the mathematical analysis of a new class of sediment transport equations recently put in place by myself and my collaborators.

**Theorem 4.1** (Convergence). *There exists a global weak sequence solution  $(h_n, h_n \bar{u}_n, h_n \bar{K}_n, h_n \bar{\alpha}_{sn}, (Z_b^*)_n)^T$  of (3) satisfying the energy and entropy inequalities (27) and (36) respectively.*

**Proposition 4.1.** *A sequence of weak solution satisfy also the equation (27) and (36). According to the result given in (3.1) and the we have the following estimates:*

$$\begin{aligned} \|P_1(\bar{u}_n, h_n)\|_{L^2(0,T;H^{-1}(\Omega))} &\leq c \\ \|\partial_x \sqrt{h_n}\|_{L^2(0,T;H^1(\Omega))} &\leq c \\ \|\partial_x h_n\|_{L^\infty(0,T;H^1(\Omega))} &\leq c \\ \|\bar{u}_n\|_{L^\infty(0,T;L^2(\Omega))} &\leq c \\ \|\partial_x (Z_b^*)_n\|_{L^\infty(0,T;L^2(\Omega))} &\leq c \\ \|\partial_x h_n \bar{\alpha}_{sn}\|_{L^2(0,T;L^2(\Omega))} &\leq c \\ \|(Z_b^*)_n\|_{L^\infty(0,T;L^2(\Omega))} &\leq c \\ \|\partial_x \mathcal{P}_n\|_{L^2(0,T;L^2(\Omega))} &\leq c \\ \|\partial_x \sqrt{h_n}\|_{L^2(0,T;H^1(\Omega))} &\leq c \\ \|\partial_x \sqrt{h_n} D(u_n)\|_{L^2(0,T;L^2(\Omega))} &\leq c \end{aligned} \quad (43)$$

Moreover, according to the Lemma 4 one has:

$$\|\sqrt{h_n}\bar{u}_n\|_{L^2(0,T;L^2(\Omega))} \leq c \quad \|\sqrt{h_n}\hat{u}_n\|_{L^2(0,T;L^2(\Omega))} \leq c \quad (44)$$

We have also the following

**Proposition 4.2.** *According to the definition of  $\mathbf{W}(0, T)$ , the relations (27)-(36) and the estimation relations given by (43), the following relations hold:*

$$\|\sqrt{h_n}\|_{\mathbf{W}(0,T)} \leq c, \quad \|h_n\|_{\mathbf{W}(0,T)} \leq c, \quad \|(Z_b^*)_n\|_{\mathbf{W}(0,T)} \leq c, \quad \|\sqrt{h}\sqrt{\mathcal{P}}\|_{\mathbf{W}(0,T)} \leq c, \quad \|\sqrt{h\bar{u}}\|_{\mathbf{W}(0,T)} \leq c. \quad (45)$$

The following result is due to proposition AppendixB.4:

**Proposition 4.3.** *Let  $\Omega$  be a bounded in  $\mathbb{R}$ . Any bounded sequence  $(h_n)_n, (\sqrt{h_n})_n, ((Z_b^*)_n)_n$  in  $W(0, T)$  has a subsequence that converges strongly in  $L^2(0, T; L^2(\Omega))$ .*

## 5 Proof of the convergence result of the model

This section is devoted to prove that the proposed model converge when we pass to the limit each equation of the model (mass conservation equation, momentum equation, kinetic energy equation, sediment concentration equation and bed evolution equation). We will prove that each of these equation is satisfied by a sequence of global weak solution and admit a limit when  $n$  tends to  $\infty$ .

### 5.1 Pass to the limit in the mass conservation

Let us the mass equation of the above model given by the first equation of the studied model above. An sequence of global weak solution of the model satisfies also this model due to proposition 4.1. We prove here that the mass equations satisfied by a sequence of global weak solutions given by:

$$\frac{\partial h_n}{\partial t} + \frac{\partial}{\partial x} (h_n \bar{u}_n) = P_1(\bar{u}_n, h_n) = P_1^n(u, h), \quad (46)$$

and by

$$\frac{\partial \sqrt{h_n}}{\partial t} + \frac{\partial}{\partial x} (\sqrt{h_n} \sqrt{h_n} \bar{u}_n) = P_1(\bar{u}_n, \sqrt{h_n}), \quad (47)$$

admit a limit when  $n$  tends to  $\infty$ .

From the equation (46), we have  $\frac{\partial h_n}{\partial t} = -\frac{\partial}{\partial x} (h_n \bar{u}_n) + P_1(\bar{u}_n, h_n)$  that implies that  $\frac{\partial h_n}{\partial t} \rightarrow \frac{\partial h}{\partial t}$  in  $L^2(0, T; H^{-1}(\Omega))$ . Since  $\|h\|_{L^\infty(0,T;H^1(\Omega))} \cap L^2(0,T;H^1(\Omega)) \leq c$ , and according to embedding given by (B.2), we can extract a limit of a subsequence  $h \in L^2(0, T; H^1(\Omega))$  that  $\partial_x h$  belong to  $L^\infty(0, T; L^2(\Omega))$  and  $h_n \rightarrow h$  in  $L^2(0, T; L^2(\Omega))$ . Now, since  $\|\sqrt{h_n}\|_{L^\infty(0,T;L^2(\Omega))}$ , we have  $\sqrt{h_n} \bar{u}_n \rightarrow \sqrt{h\bar{u}}$  in  $L^2(0, T; L^2(\Omega))$ . We prove that  $h_n \bar{u}_n \rightarrow h\bar{u}$ . As  $h_n \bar{u}_n = \sqrt{h_n}(\sqrt{h_n} \bar{u}_n)$ , we can write:

$$\|h_n \bar{u}_n\|_{L^p(0,T;L^p(\Omega))} = \|\sqrt{h_n}\|_{L^2(0,T;L^2(\Omega))} \|\sqrt{h_n} \bar{u}_n\|_{L^p(0,T;L^p(\Omega))}. \quad (48)$$

Notice that  $\sqrt{h_n} \bar{u}_n \rightarrow \sqrt{h\bar{u}}$  in  $L^2(0, T; L^2(\Omega))$ , and so, we can prove that  $\sqrt{h_n} \rightarrow \sqrt{h}$  in  $L^p(0, T; L^p(\Omega))$ . According to Eq. (47), we have  $\partial_t \sqrt{h_n}$  is bounded in  $L^2(0, T; H^{-1}(\Omega))$ . Previously, we have showed that  $\|\partial_x \sqrt{h_n}\|_{L^2(0,T;H^1(\Omega))} \leq c$  and  $\|\sqrt{h_n} \bar{u}\|_{L^2(0,T;L^2(\Omega))} \leq c$ . We have therefore,  $\sqrt{h_n} \rightarrow \sqrt{h}$  in  $L^p(0, T; L^p(\Omega))$ .

Really,  $h_n u_n \rightarrow hu$  is not easy to prove since  $h_n u_n$  equation is associated to the momentum equation we prefer use the fact that  $\partial_x h_n u_n = u_n \partial_x h_n + h_n \partial_x u_n$ . We have  $\partial_x h_n \rightarrow \partial_x h$  strongly in  $L^2(0, T; L^2(\Omega))$  and  $u_n \rightarrow u$  weakly in  $L^2(0, T; L^2(\Omega))$ . Moreover  $\partial_x u_n$  is bounded in  $L^2(0, T; L^2(\Omega))$  and  $\partial_x u_n L^2(0, T; L^2(\Omega)) \in$ . We can deduce that  $h_n \bar{u}_n \rightarrow h\bar{u}$  in  $L^2(0, T; L^2(\Omega))$ . We can also prove that  $h_n u_n \rightarrow hu$  by remarking that  $\int_{Q_T} h_n u_n dt dx = \langle \sqrt{h_n} u_n, \sqrt{h_n} \rangle_\Omega$  i.e. by proving the  $\sqrt{h_n} u_n \rightarrow \sqrt{h} u$  in  $L^2(0, T; L^2(\Omega))$  strongly and using the fact that  $\sqrt{h_n} \rightarrow \sqrt{h}$ , we conclude.

## 5.2 Pass to the limit in the momentum conservation equation

According to the proposition 4.1, we have:

$$\begin{aligned} & \frac{\partial}{\partial t} (h_n \bar{u}_n) + \frac{\partial}{\partial x} (2h_n \bar{\kappa}_n) + gh_n \frac{\partial(h_n + (Z_b^*)_n)}{\partial x} + \frac{gh_n^2}{2\rho_n} \frac{\partial \rho_n}{\partial x} \\ & - \int_{\Omega} \mu \frac{(\partial_x h_n)^2}{h_n} D(u_n) + \int_{\Omega} \partial_x u_n \cdot \partial_x^2 h_n = P_2(u_n, h_n). \end{aligned} \quad (49)$$

From the equation (49), we have:

$$\frac{\partial}{\partial t} (h_n \bar{u}_n) = -\frac{\partial}{\partial x} (2h_n \bar{\kappa}_n) - gh_n \frac{\partial(h_n + (Z_b^*)_n)}{\partial x} - \frac{gh_n^2}{2\rho_n} \frac{\partial \rho_n}{\partial x} + P_2(h_n, \bar{u}_n), \quad (50)$$

where  $h_n \bar{\kappa}_n = \frac{(h_n \bar{u}_n) \bar{u}_n}{2} + \frac{h_n \mathcal{P}_n}{2}$ .

We have  $\partial_t h_n \bar{u}_n, \partial_t h_n \bar{\kappa}_n \in L^2(0, T; H^{-s}(\Omega))$ , s great. We have  $\bar{u}_n, \partial_x \bar{u}_n$  bounded in  $L^2(0, T; H^1(\Omega))$  as  $\bar{u}_n \in \text{since } L^2(0, T; L^2(\Omega))$ . As  $\partial_t \bar{u}_n \in L^2(0, T; L^2(\Omega))$  we have  $\bar{u}_n \in L^2(0, T; H^1(\Omega))$  and we conclude that  $\bar{u}_n \rightarrow \bar{u}$  in  $L^2(0, T; H^1(\Omega))$ . Using the motion conservation equation  $\partial_t h_n \bar{u}_n \in L^2(0, T; L^2(\Omega))$  therefore  $h_n \bar{u}_n \rightarrow h \bar{u}$  strongly in  $L^2(0, T; H^{-1}(\Omega))$ . Remarking that  $\int_{Q_T} h_n \bar{u}_n \bar{u}_n dt dx = \langle h_n \bar{u}_n, \bar{u}_n \rangle_{L^2(0, T; H^{-1}(\Omega)), L^2(0, T; H^{-1}(\Omega))}$  The Aubin-Lions lemma ensures that  $h_n \bar{u}_n \rightarrow h \bar{u}$  strongly in  $L^2(0, T; H^{-1}(\Omega))$  and we obtain:

$$\int_{Q_T} h_n \bar{u}_n \bar{u}_n dt dx \rightarrow \int_{Q_T} h |\bar{u}|^2 dx dt \quad (51)$$

We conclude that  $h_n \bar{u}_n \rightarrow h \bar{u}$  strongly in  $L^2(0, T; L^2(\Omega))$  thanks to embedding compact (see appendix).

In the same sense we have proved that  $h_n \mathcal{P}_n = (\sqrt{h_n} \sqrt{\mathcal{P}_n})^2$  is bounded in  $L^2(0, T; H^{-1}(\Omega))$ . Remarking that  $\int_{Q_T} h_n \mathcal{P}_n dx dt = \int_{Q_T} |\sqrt{h_n} \sqrt{\mathcal{P}_n}|^2 dx dt = \langle \sqrt{h_n} \sqrt{\mathcal{P}_n}, \sqrt{h_n} \sqrt{\mathcal{P}_n} \rangle_{L^2(0, T; H^{-1}(\Omega)), L^2(0, T; H^{-1}(\Omega))}$

Since  $\sqrt{h_n} \sqrt{\mathcal{P}_n}$  is bounded in  $L^2(0, T; L^2(\Omega))$  and  $\sqrt{h_n} \sqrt{\mathcal{P}_n} \in L^2(0, T; L^2(\Omega))$ . The compactness Aubin-Lions Lemma ensure that  $\sqrt{h_n} \sqrt{\mathcal{P}_n} \rightarrow \sqrt{h} \sqrt{\mathcal{P}}$  strongly in  $L^2(0, T; L^2(\Omega))$  and we obtain:

$$\langle \sqrt{h_n} \sqrt{\mathcal{P}_n}, \sqrt{h_n} \sqrt{\mathcal{P}_n} \rangle \rightarrow \langle \sqrt{h} \sqrt{\mathcal{P}}, \sqrt{h} \sqrt{\mathcal{P}} \rangle \quad (52)$$

i.e.

$$\int_{Q_T} |\sqrt{h_n} \sqrt{\mathcal{P}_n}|^2 dx dt \rightarrow \int_{Q_T} |\sqrt{h} \sqrt{\mathcal{P}}|^2 dx dt \quad (53)$$

Therefore  $\sqrt{h_n} \sqrt{\mathcal{P}_n} \rightarrow \sqrt{h} \sqrt{\mathcal{P}}$  and  $h_n \mathcal{P}_n \rightarrow h \mathcal{P}$  strongly in  $L^2(0, T; L^2(\Omega))$ . Using (51) and (53) we have  $\int_{Q_T} h_n \bar{\kappa}_n dx dt \rightarrow \int_{Q_T} h \bar{\kappa} dx dt$ . As  $\partial_t h_n \bar{u}_n$  is bounded in  $L^2(0, T; H^{-1}(\Omega))$  then  $\partial_x h_n \bar{\kappa}_n$  is bounded in  $L^2(0, T; H^{-1}(\Omega)) \subset L^2(0, T; L^2(\Omega))$ . To pass to limit on  $h_n \bar{\kappa}_n$  becomes easy.

To prove that, we will show that the product  $(\sqrt{h_n} \sqrt{\mathcal{P}_n})^2$  and  $\sqrt{h_n} \bar{u}_n$  are bounded in  $L^2(0, T; L^2(\Omega))$ .

We know that  $\left( \frac{\hat{\mathcal{P}}_n}{\sqrt{h_n}} \right)_n$  and  $\frac{q_n}{\sqrt{h_n}}$  are bounded in  $L^2(0, T; L^2(\Omega))$ , with  $h_n > 0, n \in \mathbb{N}$ . We can apply the Fatou Lemma and this leads to:

$$\sup_{t \in [0, T]} \int_{\Omega} \liminf \left( \frac{\hat{\mathcal{P}}_n}{\sqrt{h_n}} \right) \left( \frac{\hat{\mathcal{P}}_n}{\sqrt{h_n}} \right) \leq \sup \liminf \int_{\Omega} \left( \frac{\hat{\mathcal{P}}_n}{\sqrt{h_n}} \right) \left( \frac{\hat{\mathcal{P}}_n}{\sqrt{h_n}} \right) \quad (54)$$

$$\sup_{t \in [0, T]} \int_{\Omega} \liminf \left( \frac{q_n}{\sqrt{h_n}} \right) \left( \frac{q_n}{\sqrt{h_n}} \right) \leq \sup \liminf \int_{\Omega} \left( \frac{q_n}{\sqrt{h_n}} \right) \left( \frac{q_n}{\sqrt{h_n}} \right) \quad (55)$$

We can define  $\sqrt{\mathcal{P}}$  and  $\bar{u}$  as follows:

$$\sqrt{\mathcal{P}} = \begin{cases} \frac{\hat{\mathcal{P}}}{h} & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

$$\bar{u} = \begin{cases} \frac{q}{h} & \text{if } h \neq 0 \\ 0 & \end{cases}$$

Such that  $q = h\bar{u}$  and  $\widehat{\mathcal{P}}$  following that

$$\sup_{t \in [0, T]} \int_{\Omega} \frac{q^2}{h} = \sup_{t \in [0, T]} \int_{\Omega} |\sqrt{h}\bar{u}|^2 < +\infty \quad (56)$$

$$\sup_{t \in [0, T]} \int_{\Omega} \frac{\widehat{\mathcal{P}}^2}{h} = \sup_{t \in [0, T]} \int_{\Omega} |\sqrt{h}\sqrt{\widehat{\mathcal{P}}}|^2 < +\infty \quad (57)$$

According to equations (54), (55), (56) and (57), we have:

$$\begin{aligned} \left( \sqrt{h_n} \bar{u} \right)_n &= \left( \frac{q_n}{\sqrt{h_n}} \right)_n \rightarrow \sqrt{h} \bar{u} = \frac{q}{\sqrt{h}}, \quad h \neq 0, \quad \forall x \\ \left( \sqrt{h_n} \sqrt{\mathcal{P}_n} \right)_n &= \left( \frac{\widehat{\mathcal{P}}_n}{\sqrt{h_n}} \right)_n \rightarrow \sqrt{h} \sqrt{\widehat{\mathcal{P}}} = \frac{\widehat{\mathcal{P}}}{\sqrt{h}}, \quad h \neq 0, \quad \forall x \end{aligned} \quad (58)$$

Therefore, we can write  $\frac{\sqrt{h_n} \sqrt{\mathcal{P}_n} \sqrt{h_n} \sqrt{\mathcal{P}_n}}{2} \in L^2(0, T; L^2(\Omega))$  and  $\left( \frac{\widehat{\mathcal{P}}_n}{\sqrt{h_n}} \right) \left( \frac{\widehat{\mathcal{P}}_n}{\sqrt{h_n}} \right) \in L^2(0, T; L^2(\Omega))$ , since one has:

$$\int_{\Omega} |\sqrt{h_n} \sqrt{\mathcal{P}_n} - \sqrt{h} \sqrt{\mathcal{P}}|^2 \leq \int_{\Omega} |\sqrt{h_n} \sqrt{\mathcal{P}_n} - \sqrt{h} \sqrt{\mathcal{P}}| |\sqrt{h_n} \sqrt{\mathcal{P}_n}| + \int_{\Omega} |\sqrt{h_n} \sqrt{\mathcal{P}_n} - \sqrt{h} \sqrt{\mathcal{P}}| |\sqrt{h} \sqrt{\mathcal{P}}|. \quad (59)$$

The Holder inequality permit to obtain:

$$\begin{aligned} \int_{\Omega} |\sqrt{h_n} \sqrt{\mathcal{P}_n} - \sqrt{h} \sqrt{\mathcal{P}}|^2 &\leq \|\sqrt{h_n} \sqrt{\mathcal{P}_n} - \sqrt{h} \sqrt{\mathcal{P}}\|_{L^2(0, T; L^2(\Omega))} \|\sqrt{h_n} \sqrt{\mathcal{P}_n}\|_{L^2(0, T; L^2(\Omega))} \\ &+ \|\sqrt{h_n} \sqrt{\mathcal{P}_n} - \sqrt{h} \sqrt{\mathcal{P}}\|_{L^2(0, T; L^2(\Omega))} \|\sqrt{h} \sqrt{\mathcal{P}}\|_{L^2(0, T; L^2(\Omega))}. \end{aligned} \quad (60)$$

According to error estimates given by (44), we have:

$$\int_{\Omega} |\sqrt{h_n} \sqrt{\mathcal{P}_n} - \sqrt{h} \sqrt{\mathcal{P}}|^2 \leq +\infty. \quad (61)$$

Following the same procedure, we obtain also

$$\int_{\Omega} |\sqrt{h_n} \bar{u}_n - \sqrt{h} \bar{u}|^2 \leq +\infty \quad (62)$$

Therefore,  $\sqrt{h_n} \bar{u}_n \rightarrow \sqrt{h} \bar{u}$  in  $L^2(0, T; L^2(\Omega))$  and  $\sqrt{h_n} \sqrt{\mathcal{P}_n} \rightarrow \sqrt{h} \sqrt{\mathcal{P}}$  in  $L^2(0, T; L^2(\Omega))$ .

The function  $P_2(h_n, \bar{u}_n) \in L^2(0, T; H^{-1}(\Omega))$ . We put  $gh_n \frac{\partial h_n}{\partial x} = g(h_n) \left( \frac{\partial h_n}{\partial x} \right)$ . Since  $h_n \in L^\infty(0, T; L^2(\Omega))$  and  $\frac{\partial h_n}{\partial x} \in L^2(0, T; L^2(\Omega))$ , we can write  $gh_n \frac{\partial h_n}{\partial x} \in L^2(0, T; L^2(\Omega))$ . Therefore,  $\frac{1}{2} gh_n \frac{\partial h_n}{\partial x}$  is bounded  $L^2(0, T; L^2(\Omega))$ .

$gh_n \frac{\partial (Z_b^*)_n}{\partial x}$  is bounded in  $L^\infty(0, T; H^{-1}(\Omega))$  thank's to compact inclusion (B.2).  $gh_n \bar{\alpha}_{s_n} \frac{\partial h_n}{\partial x} \in L^2(0, T; L^2(\Omega))$  since we have  $h_n \bar{\alpha}_{s_n} \in L^\infty(0, T; L^2(\Omega))$  and  $\partial_x h_n \in L^2(0, T; L^2(\Omega))$  to thank's the compact inclusion (B.2). We also prove that  $h_n \partial_x h_n \bar{\alpha}_{s_n} \rightarrow h \partial_x h \bar{\alpha}_s$  strongly in  $L^2(0, T; L^2(\Omega))$ . Indeed, we have  $h_n \partial_x h_n \bar{\alpha}_{s_n} = h_n^2 \partial_x \bar{\alpha}_{s_n} + h_n \bar{\alpha}_{s_n} \partial_x h_n$ . We also have  $\partial_x h_n \in L^2(0, T; L^2(\Omega))$  and  $\partial_x \bar{\alpha}_{s_n} \in L^2(0, T; L^2(\Omega))$ . About the diffusion term, we have  $\frac{(\partial_x h_n)^2}{h_n} D(u_n) \rightarrow \frac{(\partial_x h)^2}{h} D(u)$  strongly in  $L^2(0, T; L^2(\Omega))$ . We also have:

$$- \int_{\Omega} \partial_x u_n \cdot \partial_x^2 h_n = \int_{\Omega} \mu \partial_x h_n \partial_x^2 u_n$$

$\partial_x^2 u_n$  is bounded in  $L^2(0, T; H^{-1}(\Omega))$ , since  $\partial_x u_n$  is bounded in  $L^2(0, T; L^2(\Omega))$ .

### 5.3 Pass to the limit in the kinetic evolution equation

According to the proposition 4.1, we have:

$$\begin{aligned} \frac{\partial}{\partial t} (h_n \bar{\kappa}_n) &+ \frac{\partial}{\partial x} (h_n \bar{\kappa}_n \bar{u}_n + h_n \mathcal{P}_n \bar{u}_n) + gh_n \bar{u}_n \frac{\partial(h_n + (Z_b^*)_n)}{\partial x} \\ &+ \frac{gh_n^2 \bar{u}_n}{2\rho_n} \frac{\partial \rho_n}{\partial x} + \mu \frac{\partial}{\partial x} \left( h_n \frac{\partial \bar{u}_n}{\partial x} \right) + P_3(\bar{u}_n, h_n) \end{aligned} \quad (63)$$

From the Eq. (63), we have:

$$\begin{aligned} \frac{\partial}{\partial t} (h_n \bar{\kappa}_n) &= -\frac{\partial}{\partial x} (h_n \bar{\kappa}_n \bar{u}_n + h_n \mathcal{P}_n \bar{u}_n) - gh_n \bar{u}_n \frac{\partial(h_n + (Z_b^*)_n)}{\partial x} \\ &- \frac{gh_n^2 \bar{u}_n}{2\rho_n} \frac{\partial \rho_n}{\partial x} + P_3(\bar{u}_n, h_n) + \mu \frac{\partial}{\partial x} \left( h_n \frac{\partial \bar{u}_n}{\partial x} \right) \end{aligned} \quad (64)$$

Using Eq.(64), we have  $\frac{\partial}{\partial t} (h_n \bar{\kappa}_n) \rightarrow \frac{\partial}{\partial t} (h \bar{\kappa})$  weakly in  $L^2(0, T; H^{-1}(\Omega))$  since  $P_3(\bar{u}_n, h_n) \in L^2(0, T; H^{-1}(\Omega))$ . Thus, we can remark that:

$$\int_{Q_T} h_n \bar{\kappa}_n \bar{u}_n dt dx = \langle h_n \bar{\kappa}_n, \bar{u}_n \rangle_{(L_T^2 H^{-1}, L_T^2 H^1)}$$

As proved previously,  $\partial_t(h_n \bar{\kappa}_n)$  is bounded in  $L^2(0, T; H^{-1}(\Omega))$  and  $\bar{u}_n \in L^2(0, T; H^{-1}(\Omega))$ . Therefore  $h_n \bar{\kappa}_n \rightarrow h \bar{\kappa}$  strongly in  $L^2(0, T; H^{-1}(\Omega))$ .

We obtain

$$\int_{Q_T} h_n \bar{\kappa}_n \bar{u}_n dt dx \rightarrow \int_{Q_T} h \bar{\kappa} \bar{u} dt dx$$

and that  $h_n \bar{\kappa}_n \rightarrow h \bar{\kappa}$  in  $L^2(0, T; L^2(\Omega))$ . We prove easily that  $h_n \bar{\kappa}_n \bar{u}_n \rightarrow h \bar{\kappa} \bar{u}$  using the fact  $h_n \bar{\kappa}_n \rightarrow h \bar{\kappa}$  in  $L^2(0, T; L^2(\Omega))$  and  $\bar{u}_n \rightarrow \bar{u}$  weakly in  $L^2(0, T; L^2(\Omega))$ .

In the same sense, one has  $\partial_t(h_n \bar{\kappa}_n)$  is bounded in  $L^2(0, T; H^{-1}(\Omega))$  and we can write

$$\int_{Q_T} h_n \mathcal{P}_n \bar{u}_n dt dx = \langle h_n \mathcal{P}_n, \bar{u}_n \rangle_{(L_T^2 H^{-1}, L_T^2 H^1)}$$

As proved previously and using the embedding inequalities, we write  $|h_n \mathcal{P}_n| = |\sqrt{h_n} \sqrt{\mathcal{P}_n}|^2 \rightarrow |\sqrt{h} \sqrt{\mathcal{P}}|^2$  strongly in  $L^2(0, T; L^2(\Omega))$ .

Finally, we prove that:

$$\int_{Q_T} h_n \mathcal{P}_n \bar{u}_n dt dx = \langle h_n \mathcal{P}_n, \bar{u}_n \rangle_{(L_T^2 H^{-1}, L_T^2 H^1)} \rightarrow \langle h \mathcal{P}, \bar{u} \rangle_{(L_T^2 L_T^2, L_T^2 L_T^2)}$$

by using the fact that  $h_n \mathcal{P}_n \rightarrow h \mathcal{P}$  strongly in  $L^2(0, T; L^2(\Omega))$  and  $\bar{u}_n \rightarrow \bar{u}$  weakly in  $L^2(0, T; L^2(\Omega))$ .

We have finally that

$$\int_{Q_T} (h_n \bar{\kappa}_n \bar{u}_n + h_n \mathcal{P}_n \bar{u}_n) dt dx \rightarrow \int_{Q_T} (h \bar{\kappa} \bar{u} + h \mathcal{P} \bar{u}) dt dx \quad (65)$$

and that  $(h_n \bar{\kappa}_n + h_n \mathcal{P}_n) \bar{u}_n \rightarrow (h \bar{\kappa} + h \mathcal{P}) \bar{u}$  strongly in  $L^2(0, T; L^2(\Omega))$ .

According to Poincaré inequality and the sequence of kinetic conservation equations we have the estimate follows:

$$\begin{aligned} \left| \int_{Q_T} \partial_x (h_n \bar{\kappa}_n \bar{u}_n + h_n \mathcal{P}_n \bar{u}_n) dt dx \right| &\leq \int_{Q_T} |\partial_x (h_n \bar{\kappa}_n \bar{u}_n + h_n \mathcal{P}_n \bar{u}_n)| dt dx \\ &\leq \int_{Q_T} |\partial_x (h_n \bar{\kappa}_n \bar{u}_n)| + \int_{Q_T} |\partial_x (h_n \mathcal{P}_n \bar{u}_n)| dt dx \\ &\leq \tilde{C}_1 \|\bar{u}_n\|_{L_T^2 H^1}^2 + \tilde{C}_2 \|h_n \mathcal{P}_n\|_{L_T^2 H^1}^2 + \tilde{C}_1 \|h_n \bar{u}_n^2\|_{L_T^2 H^1}^2 \end{aligned} \quad (66)$$



We have  $\partial_x (h_n \bar{\kappa}_n + h_n \mathcal{P}_n) \bar{u}_n \rightarrow \partial_x (h \bar{\kappa} + h \mathcal{P}) \bar{u}$  strongly in  $L_T^2 H^{-1} \subset L_T^2 L_T^2$  and as  $\partial_x (h_n \bar{\kappa}_n + h_n \mathcal{P}_n) \bar{u}_n$  is bounded in  $L_T^2 L_T^2$ .

We have  $gh_n \bar{u}_n \frac{\partial (Z_b^*)_n}{\partial x} = g(h_n \bar{u}_n) \frac{\partial (Z_b^*)_n}{\partial x}$ . Since  $h_n \bar{u}_n \in L^2(0, T; L^2(\Omega))$ , and  $\frac{\partial (Z_b^*)_n}{\partial x}$  bounded in  $L^2(0, T; L^2(\Omega))$ ,  $gh_n \bar{u}_n \frac{\partial (Z_b^*)_n}{\partial x} \in L^2(0, T; L^2(\Omega))$ . Therefore, we can write that  $gh_n \bar{u}_n \frac{\partial (Z_b^*)_n}{\partial x} \rightarrow gh \bar{u} \frac{\partial Z_b^*}{\partial x}$  weakly in  $L^2(0, T; H^{-1}(\Omega))$ .

In the same sense, we have  $gh_n \bar{u}_n \frac{\partial h_n}{\partial x} \rightarrow gh \bar{u} \frac{\partial h}{\partial x}$  weakly in  $L^2(0, T; L^2(\Omega))$ .

We have from (64),  $\frac{\partial}{\partial x} (h_n \bar{\kappa}_n \bar{u}_n + h_n \mathcal{P}_n \bar{u}_n) \in L^2(0, T; H^{-1}(\Omega))$ . Indeed, we put  $h_n \bar{\kappa}_n \bar{u}_n + h_n \mathcal{P}_n \bar{u}_n = h_n \bar{u}_n \left( \frac{\bar{u}_n^2}{2} + \frac{3}{2} \mathcal{P}_n \right)$ . On the one hand  $h_n \bar{u}_n$  is bounded in  $L^2(0, T; L^2(\Omega))$ , and on the other hand  $\frac{\bar{u}_n^2}{2} + \frac{3}{2} \mathcal{P}_n$  is bounded in  $L^2(0, T; L^2(\Omega))$ . Following the same demonstration above, we prove that  $\frac{gh_n^2 \bar{u}_n}{2 \rho_n} \frac{\partial \rho_n}{\partial x} \rightarrow \frac{gh^2 \bar{u}}{2 \rho} \frac{\partial \rho}{\partial x}$  strongly in  $L^2(0, T; L^2(\Omega))$ . The source term  $P_4(\bar{u}_n, h_n) \rightarrow P_4(\bar{u}_n, h) \in L^2(0, T; H^{-1}(\Omega))$ . For the term  $|\sqrt{h_n} D(\bar{u}_n)|^2$ , we have  $|\sqrt{h_n} D(\bar{u}_n)|^2 \rightarrow |\sqrt{h} D(\bar{u}_n)|^2$  in  $L^2(0, T; L^2(\Omega))$ .

#### 5.4 Pass to the limit in the sediment concentration equation

According to the proposition 4.1, we have:

$$\frac{\partial}{\partial t} (h_n \bar{\alpha}_{sn}) + \frac{\partial}{\partial x} (h_n \bar{u}_n \bar{\alpha}_{sn}) = P_4(h_n, \bar{u}_n) + \frac{\partial}{\partial x} \left( \gamma h_n \frac{\partial \bar{\alpha}_{sn}}{\partial x} \right) \quad (67)$$

From the equation (67), we have

$$\frac{\partial}{\partial t} (h_n \bar{\alpha}_{sn}) = -\frac{\partial}{\partial x} (h_n \bar{u}_n \bar{\alpha}_{sn}) + P_4(h_n, \bar{u}_n) + \frac{\partial}{\partial x} \left( \gamma h_n \frac{\partial \bar{\alpha}_{sn}}{\partial x} \right) \quad (68)$$

We mention that  $\frac{\partial}{\partial t} (h_n \bar{\alpha}_{sn}) \in L^2(0, T; H^{-1}(\Omega))$ . By a simple reformulation, one has:

$$\frac{\partial}{\partial x} (h_n \bar{u}_n \bar{\alpha}_{sn}) = \bar{\alpha}_{sn} \frac{\partial}{\partial x} (h_n \bar{u}_n) + h_n \bar{u}_n \frac{\partial}{\partial x} \bar{\alpha}_{sn}$$

Previously, we have proved that  $\frac{\partial}{\partial x} (h_n \bar{u}_n) \in L^2(0, T; H^{-1}(\Omega))$  due to fact that  $\frac{\partial h_n}{\partial t} = -\frac{\partial}{\partial x} (h_n \bar{u}_n) + P_1(\bar{u}_n, h_n) \in L^2(0, T; H^{-1}(\Omega))$ . We can write  $h_n \bar{u}_n \frac{\partial}{\partial x} \bar{\alpha}_{sn} \in L^2(0, T; H^{-1}(\Omega))$  because  $h_n \bar{u}_n \rightarrow h \bar{u}$  in  $L^2(0, T; L^2(\Omega))$ . Note that  $\frac{\partial}{\partial x} \bar{\alpha}_{sn}$  is bounded in  $L^2(0, T; L^2(\Omega))$ . We can write

$$\frac{\partial}{\partial x} \left( \gamma h_n \frac{\partial \bar{\alpha}_{sn}}{\partial x} \right) = \gamma h_n \partial_x^2 \bar{\alpha}_{sn} + \gamma \frac{\partial \bar{\alpha}_{sn}}{\partial x} \frac{\partial h_n}{\partial x}$$

With that, one write that  $\frac{\partial}{\partial x} \left( \gamma h_n \frac{\partial \bar{\alpha}_{sn}}{\partial x} \right)$  is bounded in  $L^2(0, T; L^2(\Omega))$ . On the other hand  $\frac{\partial \bar{\alpha}_{sn}}{\partial x}$  is bounded in  $L^2(0, T; L^2(\Omega))$ , this leads us to say that  $\partial_x^2 \bar{\alpha}_{sn}$  is bounded in  $L^2(0, T; H^{-1}(\Omega))$ . We also write that  $(\bar{\alpha}_{sn})_n \rightarrow \bar{\alpha}_s$  strongly in  $L^2(0, T; H^{-1}(\Omega))$ .

#### 5.5 Pass to the limit in the bed evolution equation

According to the proposition 4.1, we have:

$$\begin{cases} \frac{\partial Z_b^*}{\partial t} + u_b \frac{\partial Z_b^*}{\partial x} = \frac{\mathcal{E} - \mathcal{D}}{1 - \phi_s}, & \text{p.p. in } Q_T \\ Z_b^*(0, \cdot) = (Z_b^*)^0, & \text{in } \Omega \\ Z_b^* = Z_b^*|_{\partial \Omega} & \text{on } \partial \Omega \end{cases} \quad (69)$$

where we have noted that  $u_b = u(t, x, Z_b^*)$ . We assume that the initial topography  $(Z_b^*)^0$  is in  $H^1(\Omega)$ , then  $Z_b^*|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$  and the operator  $Z_b^* \mapsto Z_b^*|_{\partial\Omega}$  is surjective from  $H^1(\Omega)$  to  $H^{1/2}(\partial\Omega)$ . We can also have  $(\frac{\partial Z_b^*}{\partial x}) \cdot n_{Z_b}$  for  $Z_b^* \in H^2(\Omega)$  ( $(\frac{\partial Z_b^*}{\partial x}) \cdot n_{Z_b} = \frac{\partial Z_b^*}{\partial n_{Z_b}}$ ). In this case the operator  $(Z_b^*|_{\partial\Omega}, \frac{\partial Z_b^*}{\partial n_{Z_b}})$  is linear continuous and surjective from  $H^2(\Omega)$  on  $H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ .

The function  $(Z_b^*)^0$  thank to Sobolev injection therefore  $(Z_b^*)^0 \in C^0(\bar{\Omega})$  and  $(Z_b^*)^0$  is differentiable in  $L^1$  sense p.p in  $\Omega$ .

We have  $Z_b^* \in \{H^1(Q_T) \cap L^\infty(0, T; H^1(\Omega))\}$ . Moreover, there exists  $(Z_b^*)^k \in H^1(\Omega)$  such that

$$\frac{\partial(Z_b^*)_n}{\partial t} + u_{b,n} \frac{\partial Z_b^*}{\partial x} = w_{b,n} + P_5(\bar{u}_n, h_n), \quad \text{p.p. in } Q_T \quad (70)$$

with

$$\inf_{\Omega} \text{ess}(Z_b^*)_n^0 \leq \inf_{\Omega} \text{ess}(Z_b^*)^{n-1} \leq (Z_b^*)^{n-1} \leq \sup_{\Omega} \text{ess}(Z_b^*)^{n-1} \leq \sup_{\Omega} \text{ess}(Z_b^*)_n^0 \quad (71)$$

From (70), we have:

$$\frac{\partial(Z_b^*)_n}{\partial t} = -u_{b,n} \frac{\partial Z_b^*}{\partial x} + P_5(\bar{u}_n, h_n). \quad (72)$$

We can write that  $\frac{\partial(Z_b^*)_n}{\partial t} \in L^2(0, T; H^{-1}(\Omega))$ . The fact that  $u_b < \bar{u}$  (see [? ]), leads to write:

$$\|u_b\|_{L^p(0, T; L^p(\Omega))} < \|\bar{u}\|_{L^2(0, T; L^2(\Omega))} \leq c. \quad (73)$$

We have  $u_n \rightarrow u$  weakly in  $L^2(0, T; L^2(\Omega))$ . From this estimate, we have therefore  $u_{b,n}$  bounded in  $L^p(0, T; L^p(\Omega))$ . This can leads us to say that  $u_{b,n}$  bounded in  $L^2(0, T; H^{-1}(\Omega))$ .

### Convergence in the case of the presence of resonance phenomenon

Let us  $u_{b,n}$  a sequence of function of  $L^1(\Omega)$  we assume that the resonance phenomenon appears. A resonance condition in sediment transport in absence of shear effect has been proposed in [7]. In a such condition, we can suppose:

$$u_{b,n} \rightarrow \hat{u}, \forall t, x \in \Omega$$

We can write according the fact that the fluid velocity is greater than the characteristic velocity of the bottom  $|u_b| < \bar{u} p.p$ ,  $\bar{u} \in L^1(\Omega)$ . Then,  $\hat{u} \in L^1(\Omega)$  and  $\|u_{b,n} - \hat{u}\|_{L^1} \rightarrow 0$  therefore  $u_{b,n}(Z_b^*)_n \rightarrow \hat{u}Z_b^*$  in  $L^1(0, T; L^1(\Omega))$ . Moreover, in case of resonance, we have  $u_{b,n}\partial_x(Z_b^*)_n \approx \partial_x(\hat{u}_n(Z_b^*)_n)$  and therefore,  $\partial_x(\hat{u}_n(Z_b^*)_n)$  converges to  $\partial_x(\hat{u}_n(Z_b^*)_n)$  in  $\mathcal{D}'(Q_T)$ .

### General case:

It is well-known that  $(Z_b^*)_n \in L^\infty(0, T; H^1(\Omega))$  and  $u_{b,n} \in L^2(0, T; L^2(\Omega))$  We have thanks to 4:

$$\|u_{b,n}\partial_x(Z_b^*)_n\|_{L^2(0, T; L^2(\Omega))} \leq \|u_{b,n}\|_{L^2(0, T; L^2(\Omega))} \|\partial_x(Z_b^*)_n\|_{L^\infty(0, T; L^2(\Omega))}. \quad (74)$$

Therefore, we can say that  $u_{b,n}\partial_x((Z_b^*)_n)$  is bounded in  $L^2(0, T; L^2(\Omega))$ . Because  $u_b\partial_x Z_b^* \in L^2(0, T; L^2(\Omega))$ , we conclude that  $u_{b,n}\partial_x((Z_b^*)_n) \rightarrow u_b\partial_x Z_b^*$  strongly in  $L^2(0, T; L^2(\Omega))$ . To pass to the limit in bed evolution is justified.

### 5.6 Pass to the limit in friction terms

The friction terms in momentum and kinetic equations is oriented by  $\frac{-q}{|q|}$  and its expression depends on nature of the flow. In some application this term writes:

$$\mathcal{F}_x = \frac{\varrho|q|q}{h^\beta}, \quad \beta = \begin{cases} 2 & \text{Chezy} \\ 7/3 & \text{Manning-Stricker} \end{cases} \quad (75)$$

where  $\varrho$  is a friction coefficient. In this work we have used the case where  $\beta = 2$ .

$$\begin{aligned} |\mathcal{F}_{x,n} - \mathcal{F}_{x,n}| &= \varrho|\bar{u}_n^2 - \bar{u}^2| \leq \varrho[|\bar{u}_n||\bar{u}_n - \bar{u}| + |\bar{u}||\bar{u}_n - \bar{u}|] \\ \int_{Q_T} |\mathcal{F}_{x,n} - \mathcal{F}_{x,n}| &\leq \varrho|\bar{u}_n^2 - \bar{u}^2| \leq \varrho\left[\int_{Q_T} |\bar{u}_n||\bar{u}_n - \bar{u}| + \int_{Q_T} |\bar{u}||\bar{u}_n - \bar{u}|\right] \\ &\leq \varrho\left[\int_0^T \left(\int_\Omega |\bar{u}_n|^2\right)^{1/2} \int_0^T \left(\int_\Omega |\bar{u}_n - \bar{u}|^2\right)^{1/2} \right. \\ &\quad \left. + \int_0^T \left(\int_\Omega |\bar{u}|^2\right)^{1/2} \int_0^T \left(\int_\Omega |\bar{u}_n - \bar{u}|^2\right)^{1/2} \right] \\ &\leq \|\bar{u}_n - \bar{u}\|_{L^2(0,T;L^2(\Omega))} (\|\bar{u}_n\|_{L^2(0,T;L^2(\Omega))} + \|\bar{u}\|_{L^2(0,T;L^2(\Omega))}) \end{aligned} \quad (76)$$

We have  $\|\bar{u}_n\|$  and  $\|\bar{u}\|$  are bounded in  $L^2(0, T; L^2(\Omega))$  and  $\bar{u}_n \rightarrow \bar{u}$  in  $L^2(0, T; L^2(\Omega))$ . Finally, we get the strong convergence of  $\mathcal{F}_{x,n}$  to  $\mathcal{F}_x$  in  $L^p(0, T; L^p(\Omega))$ . Moreover, we have the strong convergence of  $\bar{u}_n \mathcal{F}_{x,n}$  to  $u \mathcal{F}_x$  in  $L^2(0, T; L^2(\Omega))$ . These convergence permit to pass to the limit in friction terms. More generally to pass to the limit in the proposed model.

## 6 Conclusion

In this paper, we have presented a new sediment transport model in viscous shear shallow water mixing flow equations. This model is inspired by that recently developed by the same author and solved by the path-conservative central-upwind methods. The mathematical properties (energy and entropy inequalities) of the model have been proposed. It's expected that these properties are considerably different from those admitted by shallow water type models. The shear effects influence these inequalities by modifying the kinetic energy. Moreover, the presence of sediment increases the potential energy of the system. We proposed an existence theorem of global weak solutions of this model. Moreover, the convergence of the solution of this model is obtained in  $W(0, T) - space$ . As a reminder, we used a better energy-norm space more refined than  $L^2(\Omega)$  to study the convergence of a sequence of global weak solutions. The proposed results in this work can be improved for better literature. However, it can serve as the base for the development of future existence and non-uniqueness results. An existence theorem of a global weak solution in a two-dimensional case remains an open problem.

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## AppendixA Mathematical tools and Notations

Here, some tools used in this work are exposed. We first expose some vectorial spaces as Sobolev spaces and its related spaces.

### Some vectorial spaces

For a function  $h : \Omega \rightarrow \mathbb{R}$  the essential supremum is:

$$ess - \sup(h) = \inf \{a \in \mathbb{R}; \{h > a\} \text{ has measure zero}\} \quad (\text{A.1})$$

For  $f : \Omega \rightarrow \mathbb{C}$  we define  $\|f\|_\infty := ess - \sup |f|$ . The normed vector space  $L^\infty(\Omega)$  consists of function which are essentially 'bounded'

$$L^\infty(\Omega) = \{f : \Omega \rightarrow \mathbb{C}; \|f\|_\infty < \infty\} \quad (\text{A.2})$$

Here, the standard norm reads  $\|f\|_{L^p} = \inf \sup |f(x, t)|$ . We note that  $L^p = L^p(\Omega)$  ( $1 \leq p < +\infty$ ) is the space of measurable function  $f$  such that  $\|f\|_{L^p} = \int_\Omega |f(x, t)|^p dx < +\infty$ .

The space  $C^\infty(\Omega) = \bigcap_{k \in \mathbb{N}} C^k$  is the set of function indefinitely derivable. We associated the norm  $\|\cdot\|_\infty$ . We define the set of continuous function with compact support as:

$$C_c^\infty(\Omega) = \{f \in C^\infty(\Omega) : f(x) = 0, \forall x \in \Omega \setminus K, K \text{ compact } \subset \Omega\} \quad (\text{A.3})$$

We resume that:

$$C_c^\infty(\Omega) = C^\infty \cap C_c \quad (\text{A.4})$$

### Some Sobolev spaces and related spaces

Suppose that  $E$  is the Hilbert space  $L^p(0, T, E)$  denotes the space of Lebesgue measurable functions  $f : ]0, T[ \rightarrow E$  and such that  $\left(\int_0^T \|f(t)\|_E^p dt\right)^{\frac{1}{p}} < +\infty$ , with  $1 \leq p < \infty$ . Moreover,  $L^\infty(0, T, E)$  denotes the space of Lebesgue measurable functions  $f : ]0, T[ \rightarrow E$  and such that  $\exists C > 0, \|f(t)\|_E \leq C$ , *p.p.t.* Note that  $C([0, T], E)$  is dense in  $L^p(0, T, E)$ .

If let us  $X = L^2(0, T, E)$ . Then, it follows that the dual of  $X$  is  $X' = L^2(0, T, E')$ .

For positive integers  $\kappa_x, \kappa_t$  we defined the Sobolev space

$$H^{\kappa_x, \kappa_t}(\Omega) = \{f \in L^2(\Omega), \partial^{|\alpha|} f \in L^2(\Omega) \forall \alpha \text{ with } 0 \leq |\alpha| \leq \kappa_1, \partial_t^i f \in L^2(\Omega), i = 0, \dots, \kappa_2\} \quad (\text{A.5})$$

particularly we have:

$$H^{1,1}(\Omega) = \{f \in L^2(\Omega \times (0, T)), \nabla f \in (L^2(\Omega)), \partial_t f \in L^2(\Omega), \text{Neumann condition}\} \quad (\text{A.6})$$

The Sobolev space  $H^{\kappa_x, \kappa_t}(\Omega)$  is the Hilbert space with the following inner product:

$$\langle f, g \rangle = \int_0^T (f, g)_\Omega dt + \sum_{|\alpha|=0}^{\kappa_1} \int_0^T (\partial^{|\alpha|} f, \partial^{|\alpha|} g)_\Omega + \sum_{i=0}^{\kappa_2} \int_0^T (\partial_t^i f, \partial_t^i g)_\Omega.$$

We note  $H^r(0, T; H^s)$  for nonnegative integers  $r$  and  $s$  us the completion of  $C^\infty(0, T; H^s(\Omega))$  in the norm:

$$\|f\|_{H^r(0, T; H^s)} = \sum_{j=0}^r \int_0^T \left\| \left( \frac{\partial}{\partial t} \right)^j f(\bullet, t) \right\|_s^2 dt$$

where  $C^\infty(0, T; H^s(\Omega))$  denotes the set to indefinitely differentiable from  $[0, T]$  into  $H^s(\Omega)$  for which all derivatives have continuous extensions to  $[0, T]$ .

We verify by the Fubini theorem that the space  $L^2(\Omega \times (0, T))$  identify with the space  $L^2(0, T, L^2(\Omega))$ . The hypotheses  $L^2(0, T; L^2(\Omega))$  coincide with the hypotheses  $f \in L^2(\Omega \times ]0, T])$ .

We define also the energy-space  $\mathbf{W}(0, T)$  by:

$$\mathbf{W}(0, T) = \{f \in L^2(0, T; H^{1,0}(\Omega)), \partial_t f \in L^2(0, T; H^{1,0}(\Omega))\}$$

The energy-norm is given by:

$$\|f\|_{\mathbf{W}(0, T)} = \|f\|_{L^2(0, T; H^{1,0}(\Omega))} + \|\partial_t f\|_{L^2(0, T; (H^{1,0}(\Omega)))}.$$

## AppendixB Functional analysis: Preliminary results

### A convergence result and inequalities

The following results are essential to prove some estimate errors and convergence results in the next. We have the following compact inclusion:

$$H^{s+1}(\Omega) \subset H^s(\Omega) \subset H^{-1}(\Omega), \quad s \in [0, 1]. \quad (\text{B.1})$$

Moreover,  $\forall s \in [0, 1]$  we have the following embedding following [13]:

$$H^{s+1,1}(\Omega) \hookrightarrow H^{s,1}(\Omega); \mathbf{W}(0, T) \hookrightarrow L^2(0, T; H^{1,1}(\Omega)); L^2(0, T; H^{1,1}(\Omega)) \hookrightarrow L^2(0, T; L^2(\Omega)) \quad (\text{B.2})$$

According to these relations, we have:

**Proposition AppendixB.1.** *Let  $\Omega$  be a bounded in  $\mathbb{R}$ . Any bounded sequence  $(f_n)_n$  in  $\mathbf{W}(0, T)$  has a subsequence that converges strongly in  $L^2(0, T, L^2(\Omega))$ .*

**Lemma 4.** *For any  $f_1 \in L^2(0, T, L^2(\Omega))$  and  $f_2 \in L^\infty(0, T, L^2(\Omega))$ , we have:*

$$\|f_1 f_2\|_{L^2(0, T, L^2(\Omega))} \leq \|f_1\|_{L^2(0, T, L^2(\Omega))} \|f_2\|_{L^\infty(0, T, L^2(\Omega))} \quad (\text{B.3})$$

### Weak convergence and consequence

**Définition AppendixB.1.** *Let  $X$  a Banach space and  $X'$  its dual. We say that  $(f_n)_n \subset X'$  converges to  $f \in X'$  in the weak- $*$  topology and we denote it by  $f_n \rightharpoonup f$  if*

$$\langle f_n, v \rangle_{X' \times X} \rightarrow \langle f, v \rangle_{X' \times X}, \quad \forall v \in X \quad (\text{B.4})$$

**Theorem AppendixB.1.** *Let  $X$  a separable space and  $\{f_n\}_n \subset X'$  a bounded sequence, then there exists a subsequence  $\{f_{n_k}\}_k$  that converges in the weak- $*$  to some  $f \in X'$ .*

**Theorem AppendixB.2.** *Let  $1 \leq p \leq +\infty$ . If  $\{f_n\}_n \subset L^p(Q_T)$  and  $f \in L^p(Q_T)$  such that  $\|f_n - f\|_{L^p(Q_T)} \rightarrow 0$ , then there exists a subsequence  $\{f_{n_k}\}_k$  such that converges to  $f$  almost everywhere in  $Q_T$ .*

**Proposition AppendixB.2.** *Let  $(v_n)_n$  a sequence of  $\mathbf{H}$  and  $v \in \mathbf{H}$ .  $v_n \rightarrow v \Rightarrow (v_n)_{n \in \mathbb{N}}$  is bounded and  $\|v\| \leq \liminf \|v_n\|$ .*

**Proposition AppendixB.3.** *Let  $(v_n)_n$  and  $(u_n)_n$  two sequences of a Hilbert space  $\mathbf{H}$  such that  $v_n \rightarrow v$  and such that  $u_n \rightharpoonup u$ , then, we have  $\langle v_n, u_n \rangle \rightarrow \langle v, u \rangle$  when  $n \rightarrow \infty$*

**Theorem AppendixB.3.** *If  $\{f_n\}_n \subset X'$  converges to  $f \in X'$ , in the then there exists a subsequence  $\{v_n\}_n \subset X$ ,  $v \in X$  such that  $\|v_n - v\|_{L^p(Q_T)} \rightarrow 0$ , then*

$$\langle f_n, v_n \rangle_{X' \times X} \rightarrow \langle f, v \rangle_{X' \times X}. \quad (\text{B.5})$$

## Compact embedding

**Lemma 5.** *Let  $X \subset B \subset Y$  Banach spaces such that the inclusion  $X \subset B$  is a compact embedding. Then for any  $1 < p < \infty$ ,  $1 \leq q \leq \infty$ , the space  $\left\{ f : f \in L^p(0, T; X) \text{ and } \frac{df}{dt} \in L^q(0, T; Y) \right\}$  is compact embedded in  $L^p(0, T; B)$ .*

**Remark AppendixB.1.** *Particularly, if  $q = p = 2$ ,  $X = L^2(\Omega)$ ,  $Y = X'$  and  $B = L^2()$  it follows that  $W(0, T) \hookrightarrow L^2(0, T; L^2(\Omega))$ . Moreover if there exists a Hilbert space  $\mathbf{H}$  such that  $L^2(\Omega)$  dense on  $\mathbf{H}$  then we have*

$$L^2(\Omega) \subset \mathbf{H} \subset L^2(\Omega) \text{ and } W(0, T) \subset C^0([0, T]; \mathbf{H}).$$

**Proposition AppendixB.4.** *Let  $\Omega$  be a bounded in  $\mathbb{R}$ . Any bounded sequence  $(f_n)_n$  in  $\mathbf{W}(0, T)$  has a subsequence that converges strongly in  $L^2(0, T, L^2(\Omega))$ .*

## AppendixC Derivation of the model

### Assumptions

The following assumptions are used:

(i) Long waves propagating assumption  $\varepsilon = \frac{\mathbf{H}}{\mathbf{L}} \ll 1$ , where  $\mathbf{H}$  and  $\mathbf{L}$  are two scale length characteristics. Therefore  $h = (\varepsilon)$ . (ii) The fluid is viscous and incompressible, no heat transfer (the horizontal gradient temperature is zero). (iii) The suspension is assumed to be sufficiently dilute to justify the use of the Boussinesq approximation. (iv) The sediment diameters  $d_{50}$  are uniform. (v) The pressure is assumed hydrostatic. i.e.  $\frac{\partial p}{\partial z} = \mathcal{O}(\varepsilon)$ . (vi) The vertical velocity at the bottom is neglected i.e.,  $w(t, x, Z_b^*) = 0$ . (vii) The averaged velocity  $u = u' + \bar{u}$  where  $\bar{u} = \frac{1}{h} \int_I u dz$ , and where the fluctuation  $u' = u - \bar{u}$  such that  $\frac{1}{h} \int_I u' dz = 0$ . (viii) We consider that  $|u - \bar{u}| = \mathcal{O}(\varepsilon^2)$ . (ix) The assumption of the smallness of the horizontal vorticity (hypothesis of weakly sheared flows) allows us to keep the second order depth averaged correlations in the governing equations without neglecting the third order correlations, and thus close the governing system. (x) We have  $\frac{1}{h} \int_I u' u' dz = \overline{u' u'} = \hat{u}^2$ , where  $\hat{u}$  is the distortion velocity. (xi) We have  $|\hat{u}| = \mathcal{O}(\varepsilon^\beta) \gg \mathcal{O}(\varepsilon)$ ,  $\beta < 1$ . (xii) The terms  $\left| \left( \frac{1}{h} \int_I u' u' u' dz \right) \right| = \mathcal{O}(\varepsilon^{3\beta})$  with  $\varepsilon^{3\beta} \ll \varepsilon^{2\beta}$  for  $\beta < 1$ , are considered (assumption of weak dissipation).

### AppendixC.1 Basic equations

The starting point is to consider the two-phase equations based on continuum scale approach. In two-phase models, both the fluid phase (water) and the dispersed sediment particle phase are respectively considered as a continuum, with each phase having its own continuity and momentum equations. The governing two-phase can be summed due to small distance between each phase compared to their size. From summed system it is possible to derive a single mixture system of equations that reads:

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0, \\ \frac{\partial}{\partial t} (u) + \frac{\partial}{\partial x} \left( uu + \frac{p}{\rho} \right) + \frac{\partial}{\partial z} (uw) &= \mathcal{F}_x, \\ \frac{\partial}{\partial z} (p) &= \mathcal{F}_z, \end{aligned} \tag{C.1}$$

where  $u, w$  are the horizontal and vertical velocity,  $\mathcal{F}$  is the source term,  $p$  is the pressure term,  $\rho$  is the mixture density influenced by the sediment concentration  $c$ :

$$\rho = \rho_w(1 - c) + \rho_s c, \quad (\text{C.2})$$

where  $\rho_s$  and  $\rho_w$  are sediment and water densities respectively assumed constant and  $\rho_s \neq \rho_w$ . With (C.2) ones compute the water-sediment mixture velocity  $\mathbf{u}^m$  (noting that  $\mathbf{u} = (u, w)$ ) as follows:

$$\mathbf{u}^m = (1 - c)\mathbf{u}^f + c\mathbf{u}^s,$$

where  $\mathbf{u}^s$  and  $\mathbf{u}^f$  are the velocity of solid and fluid respectively. Assuming that  $\mathbf{u}^f = \mathbf{u}^s$  in clear water layer, one has  $\mathbf{u}^m = \mathbf{u}$ . The first equation of (C.1) is the mass conservation equation. The second equation of (C.1) is the momentum-balance for the fluid-sediment. We have obtained the third equation of the system given by (C.1) using the hydrostatic assumption (that consists to neglect the vertical acceleration of the fluid). Note that when the volume concentrations of sediment is small enough i.e.  $\rho_w(1 - c) \gg \rho_s c$ , the mixture density is almost constant.

We shall assume that the volumetric sediment concentration  $c$  satisfies the equation:

$$\frac{\partial c}{\partial t} + \frac{\partial(cu)}{\partial x} + \frac{\partial(cw)}{\partial z} = 0. \quad (\text{C.3})$$

To finish, we consider the bed evolution equation, based on the mass balanced equation on a arbitrary control volume  $\Omega$ . Here we assume that there no vertical discontinuity of the bed and we denote by  $m$  the mass of sediment within the control volume  $\Omega$ . The equation for the conservation of mass  $m$  present on the bed is given by:

$$\frac{dm}{dt} = 0 \longrightarrow \int_{z=b}^{z=Z_b} \int_S (\rho_s(1 - \phi_s)) dS = 0, \quad (\text{C.4})$$

where  $\phi_s$  is the bed porosity,  $\rho_s$  is the sediment density.

**Remark 3.** *The partial differential nonhomogeneous Navier-Stokes equations (C.1)-(C.3) describe the dynamics of the flow by the variables  $u, w$ , and  $\alpha_s$  in a domain defined by two moving surfaces parameterized by  $z = \xi(t, x)$  for the upper surface of the flow and by  $z = b(x) + Z_b(t, x)$  for the bedload surface.*

### Appendix C.2 Boundary and kinematic conditions

The model developed here is thus a 1D sediment transport model. Here, the region  $\mathbf{D}_t$  occupied by the flow is given by:

$$\mathbf{D}_t = \{(x, z), x \in \mathbb{R}, Z_b^*(t, x) \leq z \leq Z_b^*(t, x) + h(t, x), t \in \mathbb{R}_+^*\}, \quad (\text{C.5})$$

where  $Z_b^*(t, x) = Z_b(t, x) + b(x)$  and  $h(x, t)$  are respectively the water depth and the bed level. The free surface is  $\xi(t, x) = Z_b(t, x) + b(x) + h(t, x)$  and its perturbations due to the turbulence are neglected. We assume that the perturbations of the free surface are due to the movements of sediment bed form. Using these assumptions some conditions should be imposed on the free surface and the bed interface parameterized respectively as:

$$\{(x, z(t)), x \in \mathbb{R}, z = \xi(t, x), t \in \mathbb{R}_+^*\}, \quad (\text{C.6})$$

and

$$\{(x, z(t)), x \in \mathbb{R}, z = b(x) + Z_b(t, x), t \in \mathbb{R}_+^*\}, \quad (\text{C.7})$$

where  $\xi(t, x)$  and  $b(x)$  are the vertical positions of respectively the air-fluid interface and the non-erodible bedload. We denote by  $I = [Z_b^*, \eta]$  the water depth integration.

## Kinematic conditions

A position of a fluid particle in a domain  $\Omega \subset \mathbb{R}$  at the time  $t$  is defined by:

$$\mathcal{X} = (x, z(t))^T = \{(x, z(t)), x \in \mathbb{R}, Z_b^*(x, t) \leq z \leq \eta(x, t), t \in \mathbb{R}_+\}. \quad (\text{C.8})$$

Note that this position is located on curves generated by the free surface and the bottom interface given respectively by (C.6)-(C.7). Taking into account the net water volume rate exchange per unit of time denoted  $F_\xi(t) = z(t) - \eta(x, t)$ , one has:

$$\frac{\partial \xi}{\partial t} + u(t, x, \xi) \frac{\partial \xi}{\partial x} - w(t, x, \xi) = \frac{dF_\xi}{dt} \quad (\text{C.9})$$

The material derivative  $\frac{dF_\xi}{dt}$  describes the entrainment process and is responsible to mixing between the suspension zone, the clear water zone and the bedload surface. The fluid flow is hydrostatic and with distortion. We will use a additional variable  $\hat{u}$  to describe the distortion in the flow with  $\frac{dF_\xi}{dt} = f(\hat{u})$ . In the same way, at the bedload interface, using the fact that the material derivative of  $b(x(t))$  is zero, we obtain with  $F_b(t) = z(t) - Z_b^*(t, x(t))$ , the following condition

$$\frac{\partial Z_b^*}{\partial t} + u(t, x, Z_b^*) \frac{\partial Z_b^*}{\partial x} - w(t, x, Z_b^*) = \frac{dF_b}{dt}, \quad (\text{C.10})$$

where  $Z_b^*(t, x(t)) = b(x(t)) + Z_b(t, x(t))$  is the elevation of the bedload surface.

## Sediment flux at the bottom and bedload equation

The sediment flux near the bed is function of sediment entrainment/deposition exchange:

$$W_s c + \sigma_z \frac{\partial c}{\partial z} = \mathcal{D} - \mathcal{E}, \quad \text{at } z = Z_b^* \quad (\text{C.11})$$

From the equation (C.11), we can have:

$$-\sigma_z \frac{\partial c}{\partial z} = \mathcal{E}, \quad \text{at } z = Z_b^* \quad (\text{C.12})$$

Given a bed porosity  $\phi_s$ , the density  $\rho_s$  and a body sedimentary  $Z_b^*(x, t)$  we can define a sediment mass  $m(x, t)$  per length unit  $dx$  and its time derivative by:

$$m = (1 - \phi_s) \rho_s Z_b^* dx, \quad \frac{\partial m}{\partial t} = (1 - \phi_s) \rho_s \frac{\partial Z_b^*}{\partial t} dx \quad (\text{C.13})$$

The sediment flux density of mass  $Q_m [m^3 \cdot s^{-1} \cdot m^{-1}]$  by:

$$\oint_{\Omega} Q_m \cdot n_{Z_b^*} ds = \rho_s \frac{\partial Q_b}{\partial x} dx + \rho_s \frac{dF_b}{dt} dx, \quad (\text{C.14})$$

$n_{Z_b^*}$  is the normal to bedload surface. This equation states that the sediment flux density of mass is the sum of horizontal and vertical sediment transport fluxes. In this case, the sediment mass conservation reads:

$$\frac{\partial m}{\partial t} + \oint_{\Omega} Q_m \cdot n_{Z_b^*} ds = 0, \quad (\text{C.15})$$



### Appendix C.3 Depth averaged equations

We define the depth average  $\bar{\phi} = \bar{\phi}(t, x)$  for any quantity  $\phi(t, x, z)$  by

$$\bar{\phi} = \frac{1}{h} \int_{Z_b^*}^{\xi} \phi dz \quad \text{where} \quad Z_b^* = b + Z_b, \quad h(t, x) = \xi(t, x) - Z_b^*(t, x) \quad (\text{C.16})$$

The fluctuation with respect to the average value is  $\phi' = \phi - \bar{\phi}$  and clearly we have

$$\overline{\phi'} = 0$$

Moreover, we have the following identities:

$$\int_{Z_b^*}^{\xi} \frac{\partial \phi}{\partial x} dz = \frac{\partial}{\partial x} (h\bar{\phi}) - \frac{\partial \xi}{\partial x} \phi(t, x, \xi) + \frac{\partial Z_b^*}{\partial x} \phi(t, x, Z_b^*) \quad (*)$$

and

$$\int_{Z_b^*}^{\xi} \frac{\partial \phi}{\partial t} dz = \frac{\partial}{\partial t} (h\bar{\phi}) - \frac{\partial \xi}{\partial t} \phi(t, x, \xi) + \frac{\partial Z_b^*}{\partial t} \phi(t, x, Z_b^*) \quad (**)$$

Integrating the first equation of the system given by (C.1) (also named divergence-free equation) over the depth of water and using relations (\*) and (\*\*) for  $\phi \equiv u$  yields

$$\frac{\partial}{\partial x} (h\bar{u}) - \frac{\partial \xi}{\partial x} u(t, x, \xi) + \frac{\partial Z_b^*}{\partial x} u(t, x, Z_b^*) + w(t, x, \xi) - w(t, x, Z_b^*) = 0$$

Then, using the kinematic condition (C.9) and (C.10), we derive that

$$-\frac{\partial \xi}{\partial x} u(t, x, \xi) + \frac{\partial Z_b^*}{\partial x} u(t, x, Z_b^*) + w(t, x, \xi) - w(t, x, Z_b^*) = \frac{\partial h}{\partial t} - \frac{DF_{\xi}}{Dt} - \frac{DF_b}{Dt}$$

The evolution of the water depth finally writes as

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} (h\bar{u}) = \frac{DF_{\xi}}{dt} + \frac{DF_b}{dt} \quad (\text{C.17})$$

The mass conservation equation (C.17) describes the variation of the water depth is influenced by the sediment exchange between the accelerated current and bedload surface and by water entrained and mixed by the turbulence. Sediment exchange involves two distinct mechanisms, i.e., sediment entrainment due to turbulence and sediment deposition due to gravitational action.

In the context of hydrostatic approximation (long-wave approximation), the pressure is given by  $p = p_0 - \rho g(z - \xi)$ . where  $p_0$  is the atmospheric pressure at the free surface, and where  $\rho$  is the density varying in the vertical direction. Assuming that  $p_0$  is constant in space, the gradient of the pressure is defined by

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \xi}{\partial x} + (\xi - z) g \frac{\partial \rho}{\partial x} \quad \text{and} \quad \int_{Z_b^*}^{\xi} \frac{1}{\rho} \frac{\partial p}{\partial x} dz = gh \frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho} \frac{\partial \rho}{\partial x}$$

Using the conservation of the mixture mass principle the first equation of the momentum writes as :

$$\frac{\partial u}{\partial t} + 2\frac{\partial K}{\partial x} + \frac{\partial}{\partial z} (uw) + \frac{1}{\rho} \frac{\partial p}{\partial x} = \mathcal{F}_x \quad \text{where} \quad \mathcal{F}_x = \frac{1}{\rho} \mathbf{F}_x + \frac{\mu}{\rho} \frac{\partial \tau_{xx}}{\partial x} + \frac{\mu}{\rho} \frac{\partial \tau_{xz}}{\partial z} \quad \text{and} \quad K = \frac{uu}{2}$$

Averaging this equation in the vertical direction yields :

$$\begin{aligned} \frac{\partial}{\partial t} (h\bar{u}) + \frac{\partial}{\partial x} (2h\bar{K}) + gh \frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho} \frac{\partial \rho}{\partial x} &= h\bar{\mathcal{F}}_x + \left( \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} u - w \right) u(t, x, \xi) \\ &\quad - \left( \frac{\partial Z_b^*}{\partial t} + \frac{\partial Z_b^*}{\partial x} u - w \right) u(t, x, Z_b^*) \end{aligned}$$

Thanks to the kinematic conditions we get that

$$\frac{\partial}{\partial t} (h\bar{u}) + \frac{\partial}{\partial x} (2h\bar{\kappa}) + gh \frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho} \frac{\partial \rho}{\partial x} = h\bar{\mathcal{F}}_x + \frac{D_{F\xi}}{Dt} u(t, x, \xi) - \frac{D_{Fb}}{Dt} u(t, x, Z_b^*) \quad (\text{C.18})$$

where

$$\bar{\kappa} = \frac{\overline{u\bar{u}}}{2} + \frac{\mathcal{P}}{2} \quad \text{with} \quad \mathcal{P} = \overline{u'u'}.$$

In the above equation given by (C.18), it is important to precise the term  $\bar{\mathcal{F}}_x$  that contains the friction term and the diffusion term. We have:

$$\mathcal{F}_x = \frac{1}{\rho} \mathbf{F}_x + \frac{\mu}{\rho} \frac{\partial \tau_{xx}}{\partial x} + \frac{\mu}{\rho} \frac{\partial \tau_{xz}}{\partial z}, \quad (\text{C.19})$$

where in RHS the first term is the friction term and the two last terms are the diffusion terms. According to [5] the friction term reads:

We can see that the average  $\bar{\kappa}$  is not entirely defined for us, as we still need to deal with non-averaged components of the velocity. The classical shallow water model is only valid under the assumption that  $\overline{u'u'}$  is negligible. To take into account some amount of vertical shear, we will now derive an equation for the average  $\bar{\kappa}$ . Starting from the momentum equation, we can derive the following set of equations,

$$\frac{\partial \kappa}{\partial t} + u \frac{\partial}{\partial x} (u\kappa) + u \frac{\partial}{\partial z} (u\kappa) + \frac{u}{\rho} \frac{\partial p}{\partial x} = u\mathcal{F}_x$$

As the velocity is a divergence-free function and under hydrostatic assumptions the previous set of equations becomes

$$\frac{\partial \kappa}{\partial t} + \frac{\partial}{\partial x} (u\kappa) + \frac{\partial}{\partial z} (w\kappa) + gu \frac{\partial \xi}{\partial x} + \frac{\xi - z}{\rho} gu \frac{\partial \rho}{\partial x} = u\mathcal{F}_x$$

Averaging this equation in the vertical direction gives

$$\begin{aligned} \frac{\partial}{\partial t} (h\bar{\kappa}) + \frac{\partial}{\partial x} (h\overline{\kappa u}) + gh\bar{u} \frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho} \bar{u} \frac{\partial \rho}{\partial x} &= -2g \frac{\partial \rho}{\partial x} \overline{(\eta - x_3)u'} + h\overline{u\mathcal{F}_x} + \frac{d_{F\xi}}{dt} \bar{\kappa}(t, x, \xi) \\ &\quad - \frac{D_{Fb}}{Dt} \bar{\kappa}(t, x, Z_b^*) \end{aligned}$$

where

$$2\overline{\kappa u} = 2\bar{\kappa}\bar{u} + 2\mathcal{P}\bar{u} + \overline{u'u'u'}$$

The third-order fluctuations is here assumed to be smaller than the second order fluctuations. So we can formulate the third-order fluctuations as a gradient, to produce dissipation of the depth average kinetic energy:

$$\overline{u'u'u'} \simeq -2\kappa \frac{\partial \mathcal{P}}{\partial x} \quad \implies \quad \bar{\kappa u} = \bar{\kappa}\bar{u} + \mathcal{P}\bar{u} - 2\kappa \hat{u} \frac{\partial \hat{u}}{\partial x}. \quad (\text{C.20})$$

This term shows how the kinetic energy is transported by the fluctuating motion since it can also write as  $\frac{1}{2} \overline{u'u'u'} = \overline{u'\hat{u}^2}$ .

We assume that  $\overline{u\mathcal{F}_x} \simeq \bar{u}\bar{\mathcal{F}}_x$  and then, the evolution of the averaged kinetic energy writes as

$$\begin{aligned} \frac{\partial}{\partial t} (h\bar{\kappa}) + \frac{\partial}{\partial x} (h\bar{u}(\bar{\kappa} + \mathcal{P})) + gh\bar{u} \frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho} \bar{u} \frac{\partial \rho}{\partial x} &= -2g \frac{\partial \rho}{\partial x} \overline{(\eta - x_3)u'} + h\bar{u}\bar{\mathcal{F}}_x + \frac{\partial}{\partial x} \left( 2\kappa \hat{u} \frac{\partial \hat{u}}{\partial x} \right) \\ &\quad + \frac{D_{F\xi}}{Dt} \bar{\kappa}_\xi - \frac{D_{Fb}}{Dt} \bar{\kappa}_{Z_b^*}, \end{aligned} \quad (\text{C.21})$$

We can now use the relation (C.20) reformulating the bed-load interface evolution (C.10) as:

$$\frac{\partial Z_b^*}{\partial t} + \frac{2A_g}{1 - \phi_s} \frac{\partial}{\partial x} (\bar{\kappa}\bar{u} + \mathcal{P}\bar{u}) = \frac{\mathcal{D} - \mathcal{E}}{1 - \phi_s} + \frac{2A_g}{1 - \phi_s} \frac{\partial}{\partial x} \left( \kappa \hat{u} \frac{\partial \hat{u}}{\partial x} \right) \quad (\text{C.22})$$

The equation for averaged volume concentration reads as:

$$\frac{\partial h\overline{\alpha_s}}{\partial t} + \frac{\partial}{\partial x} (h\overline{u\alpha_s}) = \left( \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} u - w \right) \alpha_s(t, x, \xi) - \left( \frac{\partial Z_b^*}{\partial t} + \frac{\partial Z_b^*}{\partial x} u - w \right) \alpha_s(t, x, Z_b^*).$$

We have assumed that  $\alpha_s(t, x, \xi) = 0$  and  $\alpha_s(t, x, Z_b^*) = 1 - \phi_s$ . To compute the average  $\overline{u\alpha_s}$ , we use a Fick's law like-approximation written as

$$\overline{u\alpha_s} \simeq \bar{u} \overline{\alpha_s} - \gamma \frac{\partial \overline{\alpha_s}}{\partial x},$$

where  $\gamma$  is a nonnegative coefficient. The final averaged sediment concentration equation writes:

$$\frac{\partial}{\partial t} (h\overline{\alpha_s}) + \frac{\partial}{\partial x} (h\overline{\alpha_s} \bar{u}) = \frac{\partial}{\partial x} \left( \gamma h \frac{\partial \overline{\alpha_s}}{\partial x} \right) - (\mathcal{D} - \mathcal{E}). \quad (\text{C.23})$$

The proposed model is thus obtained by coupling (C.17), (C.18), (C.21), (C.23) and (C.22).

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