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Asymptotic normality for a modified quadratic variation of the Hermite process

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Abstract

We consider a modified quadratic variation of the Hermite process based on some well-chosen increments of this process. These special increments have the very useful property to be independent and identically distributed up to asymptotically negligible remainders. We prove that this modified quadratic variation satisfies a Central Limit Theorem and we derive its rate of convergence under the Wasserstein distance via Stein-Malliavin calculus. As a consequence, we construct, for the first time in the literature related to Hermite processes, a strongly consistent and asymptotically normal estimator for the Hurst parameter.

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Key words: Hermite process, fractional Brownian motion, parameter estimation, multiple Wiener-Itô integrals, Stein-Malliavin calculus, strong consistency, asymptotic normality, Ornstein-Uhlenbeck process, Hurst index estimation.

1 Introduction

Let an arbitrary integer $q \geq 1$. The Hermite process of order q is one of the most classical examples of a stochastic process belonging to the q th Wiener chaos. When $q = 1$ it reduces to the well-known Gaussian fractional Brownian motion. The Hermite process becomes non-Gaussian as soon as $q \geq 2$. Yet, similarly to fractional Brownian

motion, it is self-similar with stationary correlated increments possessing long memory property. We refer to the monographs [18] or [20] for a detailed presentation of Hermite processes and their applications and many other related topics.

Stochastic analysis of Hermite and related processes has been developed in the last decades. This is due to the fact that nice properties (non-Gaussianity, finiteness of all moments, self-similarity, long memory, and so on) of these processes make them natural candidates to model various phenomena. Also, stochastic analysis of Hermite processes is interesting in its own right since the classical Itô calculus fails to be applicable to them. It allows for a better understanding of properties of non-Gaussian chaotic stochastic processes, and it raises challenging mathematical problems involving Malliavin calculus and Stein method among many other powerful methodologies coming from probability theory and functional analysis. We refer to the monographs [14, 15, 17] for a detailed presentation of these topics.

One of the problems widely studied concerning the Hermite process is the statistical estimation of its Hurst parameter (or self-similarity index), usually denoted by H and which belongs to the open interval $(\frac{1}{2}, 1)$. This parameter determines many characteristics of the process (the scaling property, the moments, the intensity of the long memory, the regularity of sample paths, etc.) and therefore its estimation is of utmost importance for applications. A main approach to construct estimators for the Hurst parameter is based on the analysis of quadratic (or higher order) variations of Hermite process (see e.g. [7, 21, 11]), but other techniques have been also employed, such as wavelet analysis in [4, 9, 10], and least-squares-type estimators in [16]. While these approaches lead to consistent estimators for H , a general fact appearing in the references is that these estimators are not asymptotically normal, their limit distribution being in general a Rosenblatt random variable (the value at time 1 of the Hermite process of order $q = 2$). Such a non-Gaussian behavior in statistical estimation of Hurst parameter was even identified a long time ago in the very particular Gaussian case ($q = 1$) of fractional Brownian motion, when $H \in (3/4, 1)$ and its estimator is derived from usual quadratic variations (see for instance [19, 12]). Yet, it was shown in [13] that in this very special Gaussian case one can recover asymptotic normality for the latter estimator by replacing in quadratic variations the 1-order increments of the process by higher order increments. Unfortunately, such a strategy fails to work for all the non-Gaussian Hermite processes, that is as soon as $q \geq 2$ (see e.g. [8]).

The purpose of our present article is to propose a new strategy for overcoming the non-Gaussian behavior in statistical estimation of Hurst parameter of any Hermite process including fractional Brownian motion. To this end, we define a so-called modified quadratic variation of Hermite process, which is obtained only through some well-chosen 1-order increments of this process. The idea behind the definition of the modified quadratic variation comes from the recent paper [2] in which some well-chosen increments of the Hermite process, over consecutive dyadic numbers of the same level, were intro-

duced in order to bound from below local oscillations of the process. Roughly speaking, the crucial advantage in using these well-chosen increments is that their dominant parts are independent and identically distributed.

Thanks to the latter crucial advantage, by using techniques from the Stein-Malliavin calculus, we show that the modified quadratic variations sequence satisfies a Central Limit Theorem (CLT) and we determine the rate of convergence, under the Wasserstein distance, associated to this CLT. Then, by standard arguments, we define an estimator for the Hurst parameter of the Hermite process based on the modified quadratic variation and we prove its strong consistency and asymptotic normality. From the statistical point of view, our estimator needs less empirical data than those defined in e.g. [8] or [21], since it includes in its expression only a part of the increments of the Hermite process over consecutive dyadic numbers with the same level of the interval $[0, 1]$. We believe that our method has potential to be applied to other stochastic processes related to the class of Hermite processes. In order to illustrate this fact, we treat the case of the Ornstein-Uhlenbeck process associated to the Hermite process. We analyze the behavior of its modified quadratic variation and we discuss the estimation of its Hurst parameter.

The remaining of our article is organized as follows. In Section 2, we recall the definition of the Hermite process of any order $q \geq 1$ and its basic properties. In Section 3, we focus on some "nice" increments of it which are reminiscent of those previously introduced in [2]; we precisely show how these increments of Hermite process can be decomposed in dominant and negligible parts in such a way that the dominant parts be independent and identically distributed, then we determine the asymptotic behavior of the second and fourth moments of the dominant parts. Section 4 is the keystone of our article, we define in terms of the "nice" increments of Hermite process the modified quadratic variation V_N and its dominant part $V_{N,1}$ as well as its two negligible parts $V_{N,2}$ and $V_{N,3}$; then using techniques from the Stein-Malliavin calculus, we prove that the distribution of $V_{N,1}$, and consequently that of V_N , converges to a centered Gaussian distribution at a fast rate quantified in terms of the Wasserstein distance. The goal of Section 5 is to show that the estimator \hat{H}_N for the Hurst parameter, derived from V_N , is strongly consistent and asymptotically normal. In Section 6, it is shown that the method developed in the two previous sections can also be employed to obtain asymptotic normality of modified quadratic variation of the Hermite Ornstein-Uhlenbeck process (the solution to the Langevin equation with Hermite noise) as well as a strongly consistent and asymptotically normal estimator for its Hurst parameter. Finally, Section 7 is the Appendix where we included the notions related with Wiener chaos and Malliavin calculus needed in our work.

2 Preliminaries

In this preliminary part, we introduce the Hermite process and we recall its basic properties. Let an arbitrary integer $q \geq 1$ and a real number $H \in (\frac{1}{2}, 1)$. One denotes by $(Z_t^{H,q}, t \geq 0)$ the Hermite process of order q and with self-similarity index (or Hurst parameter) H . Using the convention that, for all $(x, a) \in \mathbb{R}^2$, one has $x_+^a = x^a$ when $x > 0$ and $x_+^a = 0$ otherwise, the process $(Z_t^{H,q}, t \geq 0)$ can be defined through multiple Wiener integral as

$$Z_t^{H,q} = c(H, q) \int_{\mathbb{R}^q} \left(\int_0^t (u - y_1)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \dots (u - y_q)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} du \right) dB(y_1) \dots dB(y_q), \quad (1)$$

where $(B(y), y \in \mathbb{R})$ is a two-sided Wiener process and $c(H, q)$ is a strictly positive normalizing constant chosen such that $\mathbf{E}(Z_t^{H,q})^2 = t^{2H}$. We can also express the random variable $Z_t^{H,q}$ as

$$Z_t^{H,q} = I_q(L_t^{H,q}),$$

where I_q stands for the multiple stochastic integral of order q with respect to the Brownian motion B and $L^{H,q}$ is the kernel of the Hermite process given by

$$L_t^{H,q}(y_1, \dots, y_q) = c(H, q) \int_0^t (u - y_1)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \dots (u - y_q)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} du, \quad (2)$$

for every $y_1, \dots, y_q \in \mathbb{R}$.

It is well-known that $L_t^{H,q}$ belongs to $L^2(\mathbb{R}^q)$ for every $t \geq 0$ and this ensures that the Hermite process $Z^{H,q}$ is well-defined. Moreover, the process $(Z_t^{H,q}, t \geq 0)$ is H -self-similar, it has stationary and correlated increments with long memory and its sample paths are, modulo a modification, Hölder continuous of any arbitrary order $\delta \in (0, H)$. Its covariance function reads

$$\mathbf{E}Z_t^{H,q}Z_s^{H,q} = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) \text{ for every } s, t \geq 0.$$

Moreover, the increments of the Hermite process satisfy, for every $s, t \geq 0$ and $p \geq 1$,

$$\mathbf{E} \left| Z_t^{H,q} - Z_s^{H,q} \right|^p = \mathbf{E} \left| Z_1^{H,q} \right|^p |t - s|^{Hp}. \quad (3)$$

Remark 1 *We notice that a different parametrization of the Hermite process is sometimes used in the literature, and in particular in the reference [2], whose results are used in the sequel. In this reference, the self-similarity index of the Hermite process (which is H in our work) is given by $q(H - 1) + 1$.*

3 Some nice increments of the Hermite process

In this part, we analyze some well-chosen increments of the Hermite process. Such increments have already been introduced in the reference [2]. Among all the increments of the Hermite process between two consecutive dyadic numbers of the same level N , we choose certain of them with nice properties. Actually, each chosen increment can be decomposed into the sum of a dominant part and of another term which is asymptotically negligible. Moreover, the dominant parts of these increments form a sequence of independent and identically distributed random variables.

Let us start with some notations. Let us fix $\beta \in (0, 1)$. For every $N \geq 1$, we consider the set

$$\mathcal{L}_N = \mathbb{N} \cap \left[1, \frac{2^N}{\lfloor 2^{N\beta} \rfloor} \right], \quad (4)$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$. Let us also introduce the following set

$$\mathcal{L}_{N,\gamma} = \mathcal{L}_N \cap [1, \lfloor 2^{N\gamma} \rfloor] \quad \text{with } \gamma < \beta. \quad (5)$$

It is clear that, for every $N \geq 1$, we have

$$\mathcal{L}_{N,\gamma} \subseteq \mathcal{L}_N$$

and

$$|\mathcal{L}_{N,\gamma}| \leq 2^{N\gamma}, \quad (6)$$

where $|\mathcal{L}_{N,\gamma}|$ denotes the cardinality of $\mathcal{L}_{N,\gamma}$. Moreover, for all N large enough, we have

$$|\mathcal{L}_{N,\gamma}| \geq 2^{N\gamma} - 1. \quad (7)$$

We mention in passing that the roles of the two parameters $\beta \in (0, 1)$ and $\gamma \in (0, \beta)$ is discussed in Remark 3 at the end of Section 5.

For $l \in \mathcal{L}_N$, we consider the following increment of length 2^{-N} of the Hermite process

$$\Delta Z_{l,N}^{H,q} = Z_{\frac{\lfloor 2^{N\beta} \rfloor + 1}{2^N}}^{H,q} - Z_{\frac{\lfloor 2^{N\beta} \rfloor}{2^N}}^{H,q} = Z_{e_{l,N,\beta} + 2^{-N}}^{H,q} - Z_{e_{l,N,\beta}}^{H,q}, \quad (8)$$

where we denoted, for $l \in \mathcal{L}_N$,

$$e_{l,N,\beta} = \frac{l \lfloor 2^{N\beta} \rfloor}{2^N}. \quad (9)$$

In view of (1) and (9), $\Delta Z_{l,N}^{H,q}$ can be expressed as

$$\begin{aligned}
\Delta Z_{l,N}^{H,q} &= c(H,q) \int_{\mathbb{R}^q} dB(y_1) \dots dB(y_q) 1_{\left(-\infty, \frac{l \lfloor 2^{N\beta} \rfloor + 1}{2^N}\right]^q}(y_1, \dots, y_q) \\
&\quad \int_{\frac{l \lfloor 2^{N\beta} \rfloor}{2^N}}^{\frac{l \lfloor 2^{N\beta} \rfloor + 1}{2^N}} (u - y_1)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \dots (u - y_q)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} du \\
&= c(H,q) \int_{\mathbb{R}^q} dB(y_1) \dots dB(y_q) 1_{(-\infty, e_{l,N,\beta} + 2^{-N}]^q}(y_1, \dots, y_q) \\
&\quad \int_{e_{l,N,\beta}}^{e_{l,N,\beta} + 2^{-N}} (u - y_1)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \dots (u - y_q)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} du.
\end{aligned}$$

Then, we decompose $\Delta Z_{l,N}^{H,q}$ as follows

$$\Delta Z_{l,N}^{H,q} = \tilde{\Delta} Z_{l,N}^{H,q} + \check{\Delta} Z_{l,N}^{H,q}.$$

Above we used the notation

$$\begin{aligned}
\tilde{\Delta} Z_{l,N}^{H,q} &= c(H,q) \int_{\mathbb{R}^q} dB(y_1) \dots dB(y_q) 1_{\left(\frac{(l-1) \lfloor 2^{N\beta} \rfloor + 1}{2^N}, \frac{l \lfloor 2^{N\beta} \rfloor + 1}{2^N}\right]^q}(y_1, \dots, y_q) \\
&\quad \int_{\frac{l \lfloor 2^{N\beta} \rfloor}{2^N}}^{\frac{l \lfloor 2^{N\beta} \rfloor + 1}{2^N}} (u - y_1)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \dots (u - y_q)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} du \\
&= c(H,q) \int_{\mathbb{R}^q} dB(y_1) \dots dB(y_q) 1_{(e_{l-1,N,\beta} + 2^{-N}, e_{l,N,\beta} + 2^{-N}]^q}(y_1, \dots, y_q) \\
&\quad \int_{e_{l,N,\beta}}^{e_{l,N,\beta} + 2^{-N}} (u - y_1)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \dots (u - y_q)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} du \tag{10}
\end{aligned}$$

and

$$\begin{aligned}
\check{\Delta} Z_{l,N}^{H,q} &= c(H,q) \int_{\mathbb{R}^q} dB(y_1) \dots dB(y_q) 1_{\left(\frac{(l-1) \lfloor 2^{N\beta} \rfloor + 1}{2^N}, \frac{l \lfloor 2^{N\beta} \rfloor + 1}{2^N}\right]^q}(y_1, \dots, y_q) \\
&\quad \int_{\frac{l \lfloor 2^{N\beta} \rfloor}{2^N}}^{\frac{l \lfloor 2^{N\beta} \rfloor + 1}{2^N}} (u - y_1)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \dots (u - y_q)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} du \\
&= c(H,q) \int_{\mathbb{R}^q} dB(y_1) \dots dB(y_q) 1_{(e_{l-1,N,\beta} + 2^{-N}, e_{l,N,\beta} + 2^{-N}]^q}(y_1, \dots, y_q) \\
&\quad \int_{e_{l,N,\beta}}^{e_{l,N,\beta} + 2^{-N}} (u - y_1)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \dots (u - y_q)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} du, \tag{11}
\end{aligned}$$

where

$$\begin{aligned} & \overline{\left(\frac{(l-1)[2^{N\beta}] + 1}{2^N}, \frac{l[2^{N\beta}] + 1}{2^N} \right)^q} \\ &= \left(-\infty, \frac{l[2^{N\beta}] + 1}{2^N} \right)^q \setminus \left(\frac{(l-1)[2^{N\beta}] + 1}{2^N}, \frac{l[2^{N\beta}] + 1}{2^N} \right)^q. \end{aligned}$$

We refer to $\tilde{\Delta}Z_{l,N}^{H,q}$ as the dominant part of the increment (8), while $\check{\Delta}Z_{l,N}^{H,q}$ can be viewed as its negligible part. The reason is given by the results in Propositions 1 and 2 below.

The following lemma plays an important role in the sequel. It shows that the dominant parts of the increments (8) are independent between them and identically distributed, while their negligible parts have all the same distribution.

Lemma 1 *For $N \geq 1$, we have*

1. *The random variables $\tilde{\Delta}Z_{l,N}^{H,q}, l \in \mathcal{L}_N$, given by (10), are independent and identically distributed.*
2. *The random variables $\check{\Delta}Z_{l,N}^{H,q}, l \in \mathcal{L}_N$, given by (11), are identically distributed.*

Proof: For point 1., the independence has been proven in [2]. To prove that $\tilde{\Delta}Z_{l,N}^{H,q}, l \in \mathcal{L}_N$, have the same distribution, we write (by $=^{(d)}$ we denote the equality in distribution), via (10) and the change of variables $\tilde{u} = u - e_{l,N,\beta}$ and then $\tilde{y}_i = y_i - e_{l,N,\beta}$ for $i = 1, \dots, q$,

$$\begin{aligned} \tilde{\Delta}Z_{l,N}^{H,q} &= c(H, q) \int_{\mathbb{R}^q} dB(y_1) \dots dB(y_q) 1_{\left(e_{l-1,N,\beta} + \frac{1}{2^N}, e_{l,N,\beta} + \frac{1}{2^N} \right)^q}(y_1, \dots, y_q) \\ &\quad \int_0^{2^{-N}} du (u + e_{l,N,\beta} - y_1)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \dots (u + e_{l,N,\beta} - y_q)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \\ &= c(H, q) \int_{\mathbb{R}^q} dB(y_1 + e_{l,N,\beta}) \dots dB(y_q + e_{l,N,\beta}) 1_{\left(\frac{-[2^{N\beta}]}{2^N} + 2^{-N}, 2^{-N} \right)^q}(y_1, \dots, y_q) \\ &\quad \int_0^{2^{-N}} du (u - y_1)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \dots (u - y_q)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \\ &\stackrel{(d)}{=} c(H, q) \int_{\mathbb{R}^q} dB(y_1) \dots dB(y_q) 1_{\left(\frac{-[2^{N\beta}]}{2^N} + 2^{-N}, 2^{-N} \right)^q}(y_1, \dots, y_q) \\ &\quad \int_0^{2^{-N}} du (u - y_1)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \dots (u - y_q)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)}. \end{aligned}$$

The last equality is a consequence of the fact that the Brownian motion B has stationary increments. A similar argument can be used to prove point 2.

We also need another auxiliary lemma.

Lemma 2 *Let φ and ψ be two arbitrary functions belonging to $L^2(\mathbb{R}^q)$, and let D be an arbitrary Borel subset of the real line \mathbb{R} . We denote by f and g the two functions of $L^2(\mathbb{R}^q)$ defined as: $f = \varphi 1_{D^q}$ and $g = \psi 1_{\overline{D^q}}$, where $\overline{D^q} = \mathbb{R}^q \setminus D^q$. Then, for every integer $a \geq 1$, we have*

$$\mathbf{E}I_q(f)^a I_q(g) = 0. \quad (12)$$

Proof: We can approximate g by simple functions belonging to $L^2(\mathbb{R}^q)$ with supports included in $\overline{D^q}$. Thus, there exists a sequence $(g_k, k \geq 1)$ of such functions for which we have $g_k \rightarrow_{k \rightarrow \infty} g$ in $L^2(\mathbb{R}^q)$. Moreover, for each $k \geq 1$, the simple function g_k can be chosen in such a way that it can be expressed, for some strictly positive integer M_k , as:

$$g_k = \sum_{j_1^{(k)}, \dots, j_q^{(k)}=1}^{M_k} a_{j_1^{(k)}, \dots, j_q^{(k)}}^{(k)} 1_{A_{j_1^{(k)}}^{(k)} \times \dots \times A_{j_q^{(k)}}^{(k)}}, \quad (13)$$

where:

- $A_1^{(k)}, \dots, A_{M_k}^{(k)}$ are disjoint bounded Borel subsets of \mathbb{R} such that each one of them is included in D , or included in $\overline{D} = \mathbb{R} \setminus D$, while $A_{j_1^{(k)}}^{(k)} \times \dots \times A_{j_q^{(k)}}^{(k)} \subseteq \overline{D^q} = \mathbb{R}^q \setminus D^q$;
- the coefficients $a_{j_1^{(k)}, \dots, j_q^{(k)}}^{(k)}, (j_1^{(k)}, \dots, j_q^{(k)}) \in \{1, \dots, M_k\}^q$, are real numbers such that any one of them vanishes as soon as at least two of its indices $j_1^{(k)}, \dots, j_q^{(k)}$ coincide i.e. when one has $j_m^{(k)} = j_n^{(k)}$ for some $m, n \in \{1, \dots, q\}$ with $m \neq n$.

Then, we can derive from (13) and an elementary property of multiple Wiener integral, that

$$I_q(g_k) = \sum_{j_1^{(k)}, \dots, j_q^{(k)}=1}^{M_k} a_{j_1^{(k)}, \dots, j_q^{(k)}}^{(k)} W(A_{j_1^{(k)}}^{(k)}) \dots W(A_{j_q^{(k)}}^{(k)}), \quad k \geq 1, \quad (14)$$

where W denotes the Brownian random measure associated with the Brownian motion B . Next, observe that, in view of the isometry property of multiple Wiener integral (see (58)) and Cauchy-Schwarz inequality, it turns out that, for deriving (12), it is enough to show that, for any integers $a \geq 1$ and $k \geq 1$, one has

$$\mathbf{E}I_q(f)^a I_q(g_k) = 0. \quad (15)$$

By (14) and linearity of expectation operator, to get (15) it suffices to prove that

$$\mathbf{E}I_q(f)^a W(A_1) \dots W(A_q) = 0, \quad (16)$$

where A_1, \dots, A_q are arbitrary disjoint bounded Borel subsets of \mathbb{R} such that each one of them is included in D , or included in \overline{D} , while we have

$$A_1 \times \dots \times A_q \subseteq \overline{D^q}. \quad (17)$$

Observe that (17) implies that there exists at least one index $j_0 \in \{1, 2, \dots, q\}$ such that $A_{j_0} \subseteq \overline{D}$. Let then J be the nonempty set of indices defined as $J = \{j \in \{1, 2, \dots, q\}, A_j \subseteq \overline{D}\}$, and $\overline{J} = \{1, 2, \dots, q\} \setminus J = \{j \in \{1, 2, \dots, q\}, A_j \subseteq D\}$ the complement of J . We denote by X_1 and X_2 the two random variables defined as $X_1 = I_q(f)^a \prod_{j \in \overline{J}} W(A_j)$ and $X_2 = \prod_{j \in J} W(A_j)$, with the convention that $X_1 = I_q(f)^a$ when \overline{J} is empty. We denote by $\mathcal{F}_{\overline{D}}$ (resp. \mathcal{F}_D) the sigma-algebra generated by all the random variables of the form $W(A)$, where A is an arbitrary bounded Borel subset of \overline{D} (resp. D). We know from a very classical property of the Brownian measure W and from the fact that the sets \overline{D} and D are disjoint, that the two sigma-algebras $\mathcal{F}_{\overline{D}}$ and \mathcal{F}_D are independent. Moreover, we know from the definitions of X_1 and f and from Lemma 1.2.5 in [17], that X_1 is \mathcal{F}_D measurable. Also, we know from the definition of X_2 that X_2 is $\mathcal{F}_{\overline{D}}$ -measurable. Then, we can derive from the independence of the two sigma-algebras $\mathcal{F}_{\overline{D}}$ and \mathcal{F}_D that X_1 and X_2 are two independent random variables. Thus, using their definitions we get that

$$\mathbf{E}I_q(f)^a W(A_1) \dots W(A_q) = \mathbf{E}(X_1 X_2) = \mathbf{E}(X_1) \mathbf{E}(X_2) = 0, \quad (18)$$

where the last equality is due to the fact that $\mathbf{E}(X_2) = 0$, which can be derived from the independence of the centered Gaussian random variables $W(A_j)$, $j \in J$, the sets A_j being disjoint.

Finally, it clearly follows from (18) that (16) is satisfied.

Let us now recall the following result from [2].

Proposition 1 *For every $N \geq 1$ large enough and $l \in \mathcal{L}_N$,*

$$\mathbf{E} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^2 \leq C 2^{-2NH} 2^{N\beta \frac{2H-2}{q}}. \quad (19)$$

Below, we deduce the asymptotic behavior of the second and fourth moments of the dominant parts denoted by $\tilde{\Delta} Z_{l,N}^{H,q}$.

Proposition 2 *For all $N \geq 1$ large enough and for every $l \in \mathcal{L}_N$, we have*

$$\left| 2^{2HN} \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - 1 \right| \leq C 2^{N\beta \frac{2H-2}{q}}, \quad (20)$$

which implies that

$$2^{2NH} \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \rightarrow_{N \rightarrow \infty} 1.$$

Also, we have

$$\left| 2^{4NH} \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^4 - \mathbf{E} |Z_1^{H,q}|^4 \right| \leq C 2^{N\beta \frac{2H-2}{q}}, \quad (21)$$

which entails that

$$2^{4NH} \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^4 \rightarrow_{N \rightarrow \infty} \mathbf{E} |Z_1^{H,q}|^4.$$

Proof: To prove (20), we notice that for $N \geq 1$ and $l \in \mathcal{L}_N$, by (3),

$$\mathbf{E} |\Delta Z_{l,N}^{H,q}|^2 = 2^{-2NH}$$

and by Lemma 2,

$$\begin{aligned} \mathbf{E} |\Delta Z_{l,N}^{H,q}|^2 &= \mathbf{E} |\tilde{\Delta} Z_{l,N}^{H,q}|^2 + \mathbf{E} |\check{\Delta} Z_{l,N}^{H,q}|^2 + 2\mathbf{E} \tilde{\Delta} Z_{l,N}^{H,q} \check{\Delta} Z_{l,N}^{H,q} \\ &= \mathbf{E} |\tilde{\Delta} Z_{l,N}^{H,q}|^2 + \mathbf{E} |\check{\Delta} Z_{l,N}^{H,q}|^2. \end{aligned} \quad (22)$$

On the other hand, by Proposition 1,

$$2^{2NH} \mathbf{E} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^2 \leq C 2^{N\beta \frac{2H-2}{q}} \rightarrow_{N \rightarrow \infty} 0.$$

Relation (22) implies

$$\begin{aligned} 2^{2HN} \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - 1 &= \left(2^{2HN} \mathbf{E} |\Delta Z_{l,N}^{H,q}|^2 - 1 \right) - 2^{2HN} \mathbf{E} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^2 \\ &= -2^{2HN} \mathbf{E} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^2 \end{aligned}$$

hence

$$\left| 2^{2HN} \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - 1 \right| \leq C 2^{N\beta \frac{2H-2}{q}}.$$

So, (20) is obtained. Let us now show (21).

By (3), we have

$$\mathbf{E} \left(\Delta Z_{l,N}^{H,q} \right)^4 = 2^{-4HN} \mathbf{E} |Z_1^{H,q}|^4,$$

for every $l \in \mathcal{L}_N$. On the other hand, by Lemma 2

$$\begin{aligned} \mathbf{E} \left(\Delta Z_{l,N}^{H,q} \right)^4 &= \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^4 + \mathbf{E} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^4 \\ &\quad + 6\mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \left(\check{\Delta} Z_{l,N}^{H,q} \right)^2 + 4\mathbf{E} \tilde{\Delta} Z_{l,N}^{H,q} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^3. \end{aligned}$$

Using the hypercontractivity property (61) and Proposition 1, we obtains that

$$\mathbf{E} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^4 \leq C \left(\mathbf{E} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^2 \right)^2 \leq C 2^{-4HN} 2^{N\beta \frac{4H-4}{q}}. \quad (23)$$

On the other hand, Cauchy-Schwarz's inequality, (61) and Proposition 1 imply that

$$\begin{aligned} \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \left(\check{\Delta} Z_{l,N}^{H,q} \right)^2 &\leq \left(\mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^4 \right)^{\frac{1}{2}} \left(\mathbf{E} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^4 \right)^{\frac{1}{2}} \\ &\leq C \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \mathbf{E} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^2 \leq C 2^{-4HN} 2^{N\beta \frac{2H-2}{q}} \end{aligned} \quad (24)$$

and

$$\begin{aligned} \mathbf{E} \tilde{\Delta} Z_{l,N}^{H,q} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^3 &\leq \left(\mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right)^{\frac{1}{2}} \left(\mathbf{E} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^6 \right)^{\frac{1}{2}} \\ &\leq C \left(\mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right)^{\frac{1}{2}} \left(\mathbf{E} \left(\check{\Delta} Z_{l,N}^{H,q} \right)^2 \right)^{\frac{3}{2}} \leq C 2^{-4HN} 2^{N\beta \frac{3H-3}{q}}. \end{aligned} \quad (25)$$

Finally, combining (23), (24) and (25), we get (21).

4 Modified quadratic variation of the Hermite process

We define the (centered) modified quadratic variation of the Hermite process, constructed by using not all the increments of the Hermite process over the consecutive dyadic numbers of arbitrary level N of the unit interval $[0, 1]$, but only the increments with nice properties introduced in Section 3. For any integer $N \geq 1$, let us set

$$\begin{aligned} V_N &= \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} \left[\left(Z_{e_{l,N,\beta}+2^{-N}}^{H,q} - Z_{e_{l,N,\beta}}^{H,q} \right)^2 - \mathbf{E} \left(Z_{e_{l,N,\beta}+2^{-N}}^{H,q} - Z_{e_{l,N,\beta}}^{H,q} \right)^2 \right] \\ &= \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} \left[(\Delta Z_{l,N}^{H,q})^2 - \mathbf{E} (\Delta Z_{l,N}^{H,q})^2 \right], \end{aligned} \quad (26)$$

where $|\mathcal{L}_{N,\gamma}|$ is the cardinality of the set $\mathcal{L}_{N,\gamma}$ defined in (5), and $\Delta Z_{l,N}^{H,q}$, $e_{l,N,\beta}$ are given by (8) and (9), respectively.

We will show that the sequence $(V_N, N \geq 1)$ satisfies a Central Limit Theorem (CLT) and we will deduce the rate of convergence under the Wasserstein distance. To do this, we decompose V_N as follows:

$$V_N = V_{N,1} + V_{N,2} + V_{N,3}, \quad (27)$$

where

$$V_{N,1} = \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} \left[(\tilde{\Delta} Z_{l,N}^{H,q})^2 - \mathbf{E}(\tilde{\Delta} Z_{l,N}^{H,q})^2 \right], \quad (28)$$

$$V_{N,2} = \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} \left[(\check{\Delta} Z_{l,N}^{H,q})^2 - \mathbf{E}(\check{\Delta} Z_{l,N}^{H,q})^2 \right], \quad (29)$$

and

$$V_{N,3} = 2 \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} \tilde{\Delta} Z_{l,N}^{H,q} \check{\Delta} Z_{l,N}^{H,q}. \quad (30)$$

The strategy to derive the CLT for V_N is the following: we first prove that the two last terms in the right-hand side of the equality (27), namely $V_{N,2}$ and $V_{N,3}$, converge to zero in $L^1(\Omega)$ as $N \rightarrow \infty$ (these two terms will be called the remainders), then using tools from the Stein-Malliavin calculus, we show that the sequence $V_{N,1}$ converges in distribution, as $N \rightarrow \infty$, to a Gaussian random variable. Let us first deal with the remainders.

4.1 The remainders

The goal of this subsection is to show that $V_{N,2}$ and $V_{N,3}$ converge to zero in $L^1(\Omega)$ as $N \rightarrow \infty$.

Proposition 3 *Consider the sequence $(V_{N,2}, N \geq 1)$ given by (29). Then, for N sufficiently large,*

$$\mathbf{E}|V_{N,2}| \leq C 2^{\frac{N\gamma}{2} + N\beta \frac{2H-2}{q}}.$$

In particular, $V_{N,2}$ converges to zero in $L^1(\Omega)$ as $N \rightarrow \infty$.

Proof: By using the inequalities (19) and (6), we obtain that

$$\begin{aligned} \mathbf{E}|V_{N,2}| &\leq C \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} \mathbf{E}(\check{\Delta} Z_{l,N}^{H,q})^2 \leq C \sqrt{|\mathcal{L}_{N,\gamma}|} 2^{N\beta \frac{2H-2}{q}} \\ &\leq C 2^{\frac{N\gamma}{2} + N\beta \frac{2H-2}{q}}. \end{aligned}$$

Since $\gamma < \beta$ (see the assumption (5)), we deduce that $\mathbf{E}|V_{N,2}| \rightarrow_{N \rightarrow \infty} 0$.

Regarding the summand $V_{N,3}$, we have the following result.

Proposition 4 *Consider the sequence $(V_{N,3}, N \geq 1)$ given by (30). Then, for N large enough,*

$$\mathbf{E}|V_{N,3}| \leq C 2^{\frac{N\gamma}{2} + N\beta \frac{H-1}{q}}.$$

In particular, $V_{N,3}$ converges to zero in $L^1(\Omega)$ as $N \rightarrow \infty$.

Proof: By (22), we clearly have for $l \in \mathcal{L}_N$,

$$\mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \leq 2^{-2HN}.$$

By the above inequality, (6), Cauchy-Schwarz's inequality and (19), we get

$$\begin{aligned} \mathbf{E}|V_{N,3}| &\leq \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} \left(\mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right)^{\frac{1}{2}} \left(\mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right)^{\frac{1}{2}} \\ &\leq C \sqrt{|\mathcal{L}_{N,\gamma}|} 2^{N\beta \frac{H-1}{q}} \leq C 2^{\frac{N\gamma}{2} + N\beta \frac{H-1}{q}}. \end{aligned}$$

Again, the assumption $\gamma < \beta$ (see (5)) ensures that $\mathbf{E}|V_{N,3}| \rightarrow_{N \rightarrow \infty} 0$.

4.2 The main term: Central Limit Theorem

The goal of this subsection is to show that the sequence $(V_{N,1}, N \geq 1)$, given by (28), converges in distribution to a centered Gaussian variable, and to estimate the rate of convergence for the Wasserstein distance via Stein-Malliavin calculus.

First, let us calculate the asymptotic variance of $V_{N,1}$. Notice that $\text{Var}(V_{N,1}) = \mathbf{E}|V_{N,1}|^2$ since $V_{N,1}$ is centered. We have, for every $N \geq 1$,

$$\begin{aligned} \mathbf{E}|V_{N,1}|^2 &= \frac{2^{4HN}}{|\mathcal{L}_{N,\gamma}|} \sum_{k,l \in \mathcal{L}_{N,\gamma}} \mathbf{E} \left[\left(\left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right) \left(\left(\tilde{\Delta} Z_{k,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{k,N}^{H,q} \right)^2 \right) \right] \\ &= \frac{2^{4HN}}{|\mathcal{L}_{N,\gamma}|} \sum_{l \in \mathcal{L}_{N,\gamma}} \mathbf{E} \left(\left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right)^2 \\ &= \frac{2^{4HN}}{|\mathcal{L}_{N,\gamma}|} \sum_{l \in \mathcal{L}_{N,\gamma}} \left[\mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^4 - \left(\mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right)^2 \right] \\ &= 2^{4HN} \left[\mathbf{E} \left(\tilde{\Delta} Z_{l_0,N}^{H,q} \right)^4 - \left(\mathbf{E} \left(\tilde{\Delta} Z_{l_0,N}^{H,q} \right)^2 \right)^2 \right], \end{aligned}$$

for some $l_0 \in \mathcal{L}_{N,\gamma}$ (for instance one may take $l_0 = 1$). Notice that, in our previous computations the second and the fourth equalities result from Lemma 1, point 1. It follows from these computations and Proposition 2 that for N large enough,

$$\left| \mathbf{E}|V_{N,1}|^2 - \left(\mathbf{E}|Z_1^{H,q}|^4 - 1 \right) \right| \leq C 2^{N\beta \frac{2H-2}{q}}, \quad (31)$$

and in particular,

$$\mathbf{E}|V_{N,1}|^2 \rightarrow_{N \rightarrow \infty} \mathbf{E}|Z_1^{H,q}|^4 - 1.$$

For later purpose, we need the following remark.

Remark 2 It follows from (27), Propositions 3 and 4, (31) and Cauchy-Schwarz's inequality that $\sup_{N \in \mathbb{N}} \mathbf{E}|V_N| < \infty$. Then using the fact that the random variable V_N belongs to a finite sum of Wiener chaoses of orders not more than $2q$ (the latter fact results from (26), (1) and the product formula (59)) and Theorem 5.10 in [14], we get that $\sup_{N \in \mathbb{N}} \mathbf{E}|V_N|^p < \infty$, for any fixed strictly positive real number p .

Now, we need to define some distances between the probability distributions of random variables. We refer to [15], Appendix C, for more details. Usually, the distance between the laws of two real-valued random variables F and G is defined as

$$d_W(F, G) = \sup_{h \in \mathcal{A}} |\mathbf{E}h(F) - \mathbf{E}h(G)|, \quad (32)$$

where \mathcal{A} is a class of functions satisfying $h(F), h(G) \in L^1(\Omega)$ for every $h \in \mathcal{A}$. When \mathcal{A} is the set of Lipschitz continuous functions $h : \mathbb{R} \rightarrow \mathbb{R}$ such that $\|h\|_{Lip} \leq 1$, where

$$\|h\|_{Lip} = \sup_{x, y \in \mathbb{R}, x \neq y} \frac{|h(x) - h(y)|}{|x - y|},$$

then (32) gives the Wasserstein distance. When \mathcal{A} is the set of indicator functions $\{1_{-\infty, z], z \in \mathbb{R}\}$ then (32) gives the Kolmogorov distance, for $\mathcal{A} = \{1_B, B \in \mathcal{B}(\mathbb{R})\}$ we have the total variation distance, while the choice of \mathcal{A} to be the class of functions h with $\|h\|_{Lip} + \|h\|_\infty < \infty$ leads to the Fortet-Mourier distance.

Let us recall a classical result from Stein-Malliavin calculus (see Theorem 5.1.3 and Remark 5.1.4 in [15]). Below, d could be any of the above distances (Kolmogorov, Total variation, Wasserstein or Fortet-Mourier). We refer to the Appendix for the definition of the Malliavin derivative D and of the Ornstein-Uhlenbeck operator L with respect to an isonormal Gaussian process $(B(\varphi), \varphi \in \mathcal{H})$, where \mathcal{H} is a real separable infinite-dimensional Hilbert space. Notice that in our present article, we have $\mathcal{H} = L^2(\mathbb{R})$.

Theorem 1 Let F be a random variable belonging to a finite sum of Wiener chaoses such that $\mathbf{E}F = 0$ and $\mathbf{E}F^2 = \sigma^2$. Let $\sigma' > 0$. Then

$$d(F, N(0, (\sigma')^2)) \leq C \left(\sqrt{\text{Var}(\langle DF, D(-L)^{-1}F \rangle_{\mathcal{H}})} + |\sigma^2 - (\sigma')^2| \right),$$

where $C > 0$ is a universal constant and $N(0, (\sigma')^2)$ the centered Gaussian (normal) distribution with standard deviation σ' .

By using the above result, we obtain the following limit theorem for the sequence $(V_{N,1}, N \geq 1)$, defined in (28). By " $\rightarrow^{(d)}$ " we denote the convergence in distribution and all the scalar products appearing in the sequel are in $L^2(\mathbb{R})$.

Proposition 5 Consider the sequence $(V_{N,1}, N \geq 1)$ given by (28). As $N \rightarrow \infty$,

$$V_{N,1} \xrightarrow{(d)} N \left(0, \left(\mathbf{E} |Z_1^{H,q}|^4 - 1 \right) \right),$$

and for N large enough,

$$d \left(V_{N,1}, N \left(0, \mathbf{E} |Z_1^{H,q}|^4 - 1 \right) \right) \leq C(q) 2^{-\frac{N\gamma}{2}}.$$

Proof: In order to use Theorem 1, let us first compute the quantity

$$\langle DV_{N,1}, D(-L)^{-1} V_{N,1} \rangle_{\mathcal{H}}.$$

By (10), we can write, for $l \in \mathcal{L}_N$,

$$\tilde{\Delta} Z_{l,N}^{H,q} = I_q(g_{l,N}) \quad (33)$$

with

$$\begin{aligned} g_{l,N}(y_1, \dots, y_q) &= c(H, q) \mathbf{1}_{(e_{l-1,N,\beta} + 2^{-N}, e_{l,N,\beta} + 2^{-N})^q}(y_1, \dots, y_q) \\ &\int_{e_{l,N,\beta}}^{e_{l,N,\beta} + 2^{-N}} (u - y_1)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} \dots (u - y_q)_+^{-\left(\frac{1}{2} + \frac{1-H}{q}\right)} du, \end{aligned}$$

for every $y_1, \dots, y_q \in \mathbb{R}$. By the product formula for multiple stochastic integrals (59) and (57),

$$\left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 = \sum_{r=0}^{q-1} r! \binom{q}{r}^2 I_{2q-2r}(g_{l,N} \otimes_r g_{l,N})$$

and therefore

$$D_* \left[\left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right] = \sum_{r=0}^{q-1} r! \binom{q}{r}^2 (2q - 2r) I_{2q-2r-1}(g_{l,N} \tilde{\otimes}_r g_{l,N}(\cdot, *)) \quad (34)$$

and

$$D_*(-L)^{-1} \left[\left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right] = \sum_{r=0}^{q-1} r! \binom{q}{r}^2 I_{2q-2r-1}(g_{l,N} \tilde{\otimes}_r g_{l,N}(\cdot, *)). \quad (35)$$

Above, ” \cdot ” stands for the $2q - 2r - 1$ variables associated with the integral $I_{2q-2r-1}$, and ” $*$ ” represents the remaining variable. The scalar product in $\mathcal{H} = L^2(\mathbb{R})$ considered

below is with respect to the variable " * ", Now,

$$\begin{aligned}
& \langle DV_{N,1}, D(-L)^{-1}V_{N,1} \rangle_{\mathcal{H}} \\
&= \frac{2^{4HN}}{|\mathcal{L}_{N,\gamma}|} \sum_{l,k \in \mathcal{L}_{N,\gamma}} \langle D_* \left[\left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right], D_*(-L)^{-1} \left[\left(\tilde{\Delta} Z_{k,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{k,N}^{H,q} \right)^2 \right] \rangle_{\mathcal{H}} \\
&= \frac{2^{4HN}}{|\mathcal{L}_{N,\gamma}|} \sum_{l \in \mathcal{L}_{N,\gamma}} \langle D_* \left[\left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right], D_*(-L)^{-1} \left[\left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right] \rangle_{\mathcal{H}}.
\end{aligned}$$

We used the fact that, since $g_{l,N}, g_{k,N}$ have disjoint supports coordinate by coordinate for $l \neq k$ (see also Lemma 4 in [5]), then

$$\begin{aligned}
& \langle D_* \left[\left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right], D_*(-L)^{-1} \left[\left(\tilde{\Delta} Z_{k,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{k,N}^{H,q} \right)^2 \right] \rangle_{\mathcal{H}} \\
&= \int_{\mathbb{R}} D_x \left[\left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right] D_x(-L)^{-1} \left[\left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 - \mathbf{E} \left(\tilde{\Delta} Z_{l,N}^{H,q} \right)^2 \right] dx = 0.
\end{aligned}$$

By (34), (35) and the definition of the contraction (60),

$$\begin{aligned}
& \langle DV_{N,1}, D(-L)^{-1}V_{N,1} \rangle_{\mathcal{H}} \\
&= \frac{2^{4HN}}{|\mathcal{L}_{N,\gamma}|} \sum_{l \in \mathcal{L}_{N,\gamma}} \sum_{r_1, r_2=0}^{q-1} r_1! r_2! \binom{q}{r_1}^2 \binom{q}{r_2}^2 (2q - 2r_1) \\
& \quad \int_{\mathbb{R}} dx I_{2q-2r_1-1} (g_{l,N} \tilde{\otimes}_{r_1} g_{l,N}(\cdot, x)) I_{2q-2r_2-1} (g_{l,N} \tilde{\otimes}_{r_2} g_{l,N}(\cdot, x)) \\
&= \frac{2^{4HN}}{|\mathcal{L}_{N,\gamma}|} \sum_{l \in \mathcal{L}_{N,\gamma}} \sum_{r_1, r_2=0}^{q-1} r_1! r_2! \binom{q}{r_1}^2 \binom{q}{r_2}^2 (2q - 2r_1) \\
& \quad \sum_{a=0}^{(2q-2r_1-1) \wedge (2q-2r_2-1)} a! \binom{2q-2r_1-1}{a} \binom{2q-2r_2-1}{a} \\
& \quad \times I_{4q-2r_1-2r_2-2a-2} ((g_{l,N} \tilde{\otimes}_{r_1} g_{l,N}) \otimes_{a+1} (g_{l,N} \tilde{\otimes}_{r_2} g_{l,N})),
\end{aligned}$$

where we used again the product formula (59). By separating the terms with $r_1 = r_2$

and $r_1 \neq r_2$ and by using (57),

$$\begin{aligned}
& \langle DV_{N,1}, D(-L)^{-1}V_{N,1} \rangle_{\mathcal{H}} = \mathbf{E} \langle DV_{N,1}, D(-L)^{-1}V_{N,1} \rangle_{\mathcal{H}} \quad (36) \\
& + \frac{2^{4HN}}{|\mathcal{L}_{N,\gamma}|} \sum_{l \in \mathcal{L}_{N,\gamma}} \sum_{r=0}^{q-1} r!^2 \binom{q}{r}^4 (2q-2r) \sum_{a=0}^{2q-2r-2} a! \binom{2q-2r-1}{a}^2 \\
& I_{4q-4r-2a-2} \left((g_{l,N} \tilde{\otimes}_r g_{l,N}) \otimes_{a+1} (g_{l,N} \tilde{\otimes}_r g_{l,N}) \right) \\
& + \frac{2^{4HN}}{|\mathcal{L}_{N,\gamma}|} \sum_{l \in \mathcal{L}_{N,\gamma}} \sum_{r_1, r_2=0; r_1 \neq r_2}^{q-1} r_1! r_2! \binom{q}{r_1}^2 \binom{q}{r_2}^2 (2q-2r_1) \\
& \sum_{a=0}^{(2q-2r_1-1) \wedge (2q-2r_2-1)} a! \binom{2q-2r_1-1}{a} \binom{2q-2r_2-1}{a} \\
& \times I_{4q-2r_1-2r_2-2a-2} \left((g_{l,N} \tilde{\otimes}_{r_1} g_{l,N}) \otimes_{a+1} (g_{l,N} \tilde{\otimes}_{r_2} g_{l,N}) \right).
\end{aligned}$$

In the sequel we denote by $\|\cdot\|$ the norm associated with a space $L^2(\mathbb{R}^s)$, for any arbitrary positive integer s . It follows from (33) and the isometry property of multiple stochastic integral (see (58)) that

$$q! \|g_{l,N}\|^2 = \mathbf{E}(\tilde{\Delta} Z_{l,N}^{H,q})^2 \leq 2^{-2HN}. \quad (37)$$

Recall that, if $f \in L^2(\mathbb{R}^m)$ and $g \in L^2(\mathbb{R}^n)$, then $\|f \otimes_r g\| \leq \|f\| \times \|g\|$ for every $0 \leq r \leq m \wedge n$. We then obtain the following estimates, for $0 \leq r \leq q-1$ and for $0 \leq a \leq 2q-2r-2$,

$$\begin{aligned}
& \mathbf{E} \left(I_{4q-4r-2a-2} \left((g_{l,N} \tilde{\otimes}_r g_{l,N}) \otimes_{a+1} (g_{l,N} \tilde{\otimes}_r g_{l,N}) \right) \right)^2 \\
& = C(q, r, a) \|(g_{l,N} \tilde{\otimes}_r g_{l,N}) \otimes_{a+1} (g_{l,N} \tilde{\otimes}_r g_{l,N})\|^2 \\
& \leq C(q, r, a) \|(g_{l,N} \tilde{\otimes}_r g_{l,N})\|^4 \leq C(q, r, a) \|(g_{l,N} \otimes_r g_{l,N})\|^4 \\
& \leq C(q, r, a) \|g_{l,N}\|^8 \leq C(q, r, a) 2^{-8HN},
\end{aligned}$$

due to (37). Also, for $0 \leq r_1 \neq r_2 \leq q-1$ and for $0 \leq a \leq (2q-2r_1-1) \wedge (2q-2r_2-1)$, one has

$$\begin{aligned}
& \mathbf{E} \left(I_{4q-2r_1-2r_2-2a-2} \left((g_{l,N} \tilde{\otimes}_{r_1} g_{l,N}) \otimes_{a+1} (g_{l,N} \tilde{\otimes}_{r_2} g_{l,N}) \right) \right)^2 \\
& = C(q, r_1, r_2, a) \|(g_{l,N} \tilde{\otimes}_{r_1} g_{l,N}) \otimes_{a+1} (g_{l,N} \tilde{\otimes}_{r_2} g_{l,N})\|^2 \\
& \leq C(q, r_1, r_2, a) \|(g_{l,N} \tilde{\otimes}_{r_1} g_{l,N})\|^2 \|(g_{l,N} \tilde{\otimes}_{r_2} g_{l,N})\|^2 \\
& \leq C(q, r_1, r_2, a) \|(g_{l,N} \otimes_{r_1} g_{l,N})\|^2 \|(g_{l,N} \otimes_{r_2} g_{l,N})\|^2 \\
& \leq C(q, r_1, r_2, a) \|g_{l,N}\|^8 \leq C(q, r_1, r_2, a) 2^{-8HN}.
\end{aligned}$$

By plugging these last two inequalities into (36), for all N large enough, we get (below we use again that $g_{l,N}$ and $g_{k,N}$ have disjoint supports, coordinate by coordinate, when

$k \neq l$, also we use (58))

$$\begin{aligned}
& \text{Var} (\langle DV_{N,1}, D(-L)^{-1}V_{N,1} \rangle_{\mathcal{H}}) \\
& \leq C(q) \frac{2^{8HN}}{|\mathcal{L}_{N,\gamma}|^2} \sum_{l \in \mathcal{L}_{N,\gamma}} \left[\sup_{r,a} \mathbf{E} (I_{4q-4r-2a-2} ((g_{l,N} \tilde{\otimes}_r g_{l,N}) \otimes_{a+1} (g_{l,N} \tilde{\otimes}_r g_{l,N})))^2 \right. \\
& \quad \left. + \sup_{r_1, r_2, a} \mathbf{E} (I_{4q-2r_1-2r_2-2a-2} ((g_{l,N} \tilde{\otimes}_{r_1} g_{l,N}) \otimes_{a+1} (g_{l,N} \tilde{\otimes}_{r_2} g_{l,N})))^2 \right] \\
& \leq C(q) \frac{1}{|\mathcal{L}_{N,\gamma}|} \leq C(q) 2^{-N\gamma}, \tag{38}
\end{aligned}$$

where the last inequality results from (7). Next, we derive from Theorem 1 and the estimates (31) and (38), that, for N sufficiently large,

$$d \left(V_{N,1}, N \left(0, \mathbf{E} |Z_1^{H,q}|^4 - 1 \right) \right) \leq C(q) \max \left(2^{N\beta \frac{2H-2}{q}}, 2^{-\frac{N\gamma}{2}} \right) \leq C(q) 2^{-\frac{N\gamma}{2}},$$

where we used the assumption $\beta > \gamma$.

Let us deduce the asymptotic behavior of the modified quadratic variation of the Hermite process. By d_W we denote the Wasserstein distance (see (32) for its definition).

Theorem 2 *Let $(V_N, N \geq 1)$ be the sequence defined in (26). Then, as $N \rightarrow \infty$,*

$$V_N \xrightarrow{(d)} N \left(0, \left(\mathbf{E} |Z_1^{H,q}|^4 - 1 \right) \right), \tag{39}$$

and for N large enough,

$$d_W \left(V_N, N \left(0, \mathbf{E} |Z_1^{H,q}|^4 - 1 \right) \right) \leq C(q) 2^{-\frac{N\gamma}{2}}. \tag{40}$$

Proof: The convergence in distribution follows immediately from the decomposition (27) and the results stated in Propositions 3, 4 and 5. To get the rate of convergence under the Wasserstein distance, we write, via the triangle inequality and the definition of the Wasserstein distance (32),

$$\begin{aligned}
& d_W \left(V_N, N \left(0, \mathbf{E} |Z_1^{H,q}|^4 - 1 \right) \right) \\
& \leq d_W \left(V_{N,1}, N \left(0, \mathbf{E} |Z_1^{H,q}|^4 - 1 \right) \right) + d_W(V_N, V_{N,1}) \\
& \leq d_W \left(V_{N,1}, N \left(0, \mathbf{E} |Z_1^{H,q}|^4 - 1 \right) \right) + \mathbf{E} |V_N - V_{N,1}| \\
& \leq d_W \left(V_{N,1}, N \left(0, \mathbf{E} |Z_1^{H,q}|^4 - 1 \right) \right) + \mathbf{E} |V_{N,2}| + \mathbf{E} |V_{N,3}|.
\end{aligned}$$

The estimates in Propositions 3, 4 and 5 imply the desired conclusion.

5 Hurst parameter estimation

In this section, the purpose is to estimate the Hurst parameter H of the Hermite process based on the discrete observation of this process. We only need to assume that the process $Z^{H,q}$ is observed at the times $(e_{l,N,\beta}, e_{l,N,\beta} + 2^{-N}, l \in \mathcal{L}_{N,\gamma})$ which is true, in particular, if $Z^{H,q}$ is observed on the interval $[0, 1]$ at the dyadic points $\frac{j}{2^N}$ for $j = 0, 1, \dots, 2^N$.

Consider the sequence

$$\begin{aligned} S_N &= \frac{1}{|\mathcal{L}_{N,\gamma}|} \sum_{l \in \mathcal{L}_{N,\gamma}} (\Delta Z_{l,N}^{H,q})^2 \\ &= \frac{1}{|\mathcal{L}_{N,\gamma}|} \sum_{l \in \mathcal{L}_{N,\gamma}} \left(Z_{e_{l,N,\beta}+2^{-N}}^{H,q} - Z_{e_{l,N,\beta}}^{H,q} \right)^2, \end{aligned} \quad (41)$$

with the notation (9). We clearly have, for every $N \geq 1$,

$$\mathbf{E}S_N = 2^{-2HN}. \quad (42)$$

In order to estimate H , the standard statistical procedure is to approximate $\mathbf{E}S_N$ by S_N in (41) and to take the (Napierian) logarithm in (42). Thus, we get the estimator

$$\hat{H}_N = \frac{-\log(S_N)}{2N \log 2}, \quad N \geq 1. \quad (43)$$

The purpose of the present section is to deduce, from the results in Section 4, the asymptotic properties of the above estimator. Let us start with an auxiliary result.

Proposition 6 *For $N \geq 1$, let V_N be given by (26) and consider the set $\mathcal{L}_{N,\gamma}$ defined by (5). We set*

$$U_N = \frac{V_N}{\sqrt{|\mathcal{L}_{N,\gamma}|}} = 2^{2HN} S_N - 1, \quad N \geq 1. \quad (44)$$

Then the sequence $(U_N, N \geq 1)$ converges almost surely to zero as $N \rightarrow \infty$ at the fast rate 2^{-N^a} , where $a \in (0, \gamma)$ is arbitrary and fixed.

Proof: Let G be a random variable with Gaussian distribution $N(0, \mathbf{E}|Z_1^{H,q}|^4 - 1)$. Using the fact that the absolute value on \mathbb{R} is a Lipschitz continuous function with Lipschitz semi-norm $\|\cdot\|_{Lip}$ equals to 1, we can derive from the definition of the Wasserstein distance d_W (see (32)) and from Theorem 2, that, for every N large enough, we have

$$\mathbf{E}|V_N| - \mathbf{E}|G| \leq |\mathbf{E}|V_N| - \mathbf{E}|G|| \leq d_W(V_N, G) \leq C(q)2^{-\frac{1}{2}N^\gamma}.$$

Then, combining (44) and (7), we get, for all $N \geq 1$,

$$\mathbf{E}|U_N| \leq C_1(q)2^{-2^{-1}N^\gamma}. \quad (45)$$

Next, let $a \in (0, \gamma)$ be arbitrary and fixed. We can derive from (45) and Markov's inequality that

$$\sum_{N \geq 1} P(|U_N| \geq 2^{-N^a}) \leq \sum_{N \geq 1} 2^{N^a} \mathbf{E}|U_N| \leq C \sum_{N \geq 1} 2^{(N^a - 2^{-1}N^\gamma)} < \infty.$$

Then, the almost sure convergence of the sequence $(U_N, N \geq 1)$ to zero at the fast rate 2^{-N^a} follows by Borel-Cantelli's lemma.

We have the following result.

Theorem 3 *The estimator (43) is strongly consistent, i.e. $\widehat{H}_N \rightarrow_{N \rightarrow \infty} H$ almost surely, and its almost sure convergence holds at the fast rate 2^{-N^a} , where $a \in (0, \gamma)$ is arbitrary and fixed. Moreover, it is asymptotically normal:*

$$2N(\log 2) \sqrt{|\mathcal{L}_{N,\gamma}|} (H - \widehat{H}_N) \xrightarrow{N \rightarrow \infty} N(0, \mathbf{E}|Z_1^{H,q}|^4 - 1). \quad (46)$$

Proof: Let us first precisely determine the relationship between \widehat{H}_N and the modified quadratic variation V_N . From (26), (41) and (44), we can write

$$V_N = \sqrt{|\mathcal{L}_{N,\gamma}|} (2^{2HN} S_N - 1),$$

so

$$\frac{V_N}{\sqrt{|\mathcal{L}_{N,\gamma}|}} + 1 = U_N + 1 = 2^{2HN} S_N,$$

or, equivalently,

$$2N \log 2 \sqrt{|\mathcal{L}_{N,\gamma}|} (H - \widehat{H}_N) = V_N + \sqrt{|\mathcal{L}_{N,\gamma}|} (\log(1 + U_N) - U_N). \quad (47)$$

Then, we can derive from (47) and the inequality

$$|\log(1 + x) - x| \leq x^2, \quad \text{for all } x \in [-1/2, 1/2], \quad (48)$$

and the fact that that $U_N = \frac{V_N}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \rightarrow_{N \rightarrow \infty} 0$ almost surely (see Proposition 6), that we have almost surely, for each N large enough,

$$|H - \widehat{H}_N| \leq (2 \log 2)^{-1} N^{-1} (|U_N| + U_N^2).$$

Thus, in view of Proposition 6, it turns out that the estimator \widehat{H}_N converges almost surely to the Hurst parameter H at the fast rate 2^{-N^a} , where $a \in (0, \gamma)$ is arbitrary and fixed.

Let us now show that (46) holds. In view of (47) and Theorem 2 it is enough to prove that

$$\sqrt{|\mathcal{L}_{N,\gamma}|} (\log(1 + U_N) - U_N) \rightarrow_{N \rightarrow \infty} 0 \text{ almost surely.} \quad (49)$$

It follows from Proposition 6, (48), and (44), that we have, almost surely, for all N large enough,

$$\sqrt{|\mathcal{L}_{N,\gamma}|} |\log(1 + U_N) - U_N| \leq \frac{V_N^2}{\sqrt{|\mathcal{L}_{N,\gamma}|}}.$$

Thus, (49) can be obtained by showing that

$$\frac{V_N^2}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \rightarrow_{N \rightarrow \infty} 0 \text{ almost surely.} \quad (50)$$

Observe that Remark 2 and (7) imply that

$$\mathbf{E} \left(\frac{V_N^2}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \right) = \mathcal{O} \left(2^{-2^{-1}N^\gamma} \right), \quad \text{for all } N \text{ large enough.} \quad (51)$$

Finally, we can derive from (51), Markov's inequality and Borel-Cantelli's lemma that (50) is satisfied.

Before ending this section let us discuss the roles of the two parameters $\beta \in (0, 1)$ and $\gamma \in (0, \beta)$ which were introduced in (4) and (5).

Remark 3 *As we have already emphasized, in our present work we use for statistical inference the modified quadratic variation V_N (see (26)), which is obtained through some well-chosen dyadic increments of the Hermite process $Z^{H,q}$ and not all its dyadic increments. The parameter $\beta \in (0, 1)$ is related with the selection procedure of these nice increments. Namely, among all the dyadic increments of the Hermite process $Z_{(k+1)/2^N}^{H,q} - Z_{k/2^N}^{H,q}$, $0 \leq k < 2^N$, we only select those for which the integer k is multiple of $[2^{N^\beta}]$. Thus, when β decreases and gets closer to zero the number of the selected increments increases; which somehow means that more data is available for statistical inference through our modified quadratic variation V_N . Yet, if we take in it all the selected increments for a given β (i.e. if we replace in (26) the set $\mathcal{L}_{N,\gamma}$ by the set \mathcal{L}_N), then our proof of the Central Limit Theorem for V_N (see Theorem 2) will no longer work. For this reason, we need to take in V_N about $[2^{N^\gamma}]$ of the selected increments, with $\gamma \in (0, \beta)$ being another parameter which should be chosen as close as possible to β , for improving as far as possible, for a given β , the rate of convergence toward normal distribution, provided by (40) in Theorem 2. It is an open difficult question to know what is the "best" choice of the parameter β . On one hand, it is tempting to seek to increase the value of β since the chosen value for γ can then be larger and thus the rate of convergence of V_N toward normal distribution becomes better; on the other hand increasing the value of β somehow means that less data is available for statistical inference, which may be a drawback in one way or the other.*

6 Quadratic variation and Hurst estimation for the Hermite-Ornstein-Uhlenbeck process

The results obtained in Sections 4 and 5 can be easily transferred to other Hermite-related stochastic processes. To illustrate this point, we study in the present section the case of the Ornstein-Uhlenbeck process associated to the Hermite process, which will be called the Hermite Ornstein-Uhlenbeck (HOU) process in the sequel.

The HOU process is defined as the unique solution of the Langevin equation

$$X_t = \xi - \int_0^t X_s ds + Z_t^{H,q}, \quad t \geq 0 \quad (52)$$

with initial condition $\xi \in L^2(\Omega)$, where $Z^{H,q}$ is a Hermite process of any order $q \geq 1$ with an arbitrary self-similarity index $H \in (\frac{1}{2}, 1)$. For simplicity, we take $\xi = 0$. We mention in passing that the solution of the Langevin equation (52) can be expressed as a Wiener-integral with respect to $Z^{H,q}$ (see [1] or [16]), but we will not need to use this integral representation in the present section. For more details on the HOU process, we refer to [1], [6] or [16]. For later purposes, we recall its following property: for every $p \geq 2$ and $T > 0$,

$$\sup_{t \in [0, T]} \mathbf{E}|X_t|^p < C, \quad (53)$$

where C is a positive finite constant depending only on p and T .

For every integer $N \geq 1$, the modified quadratic variation $V_N(X)$ is defined as

$$\begin{aligned} V_N(X) &= \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} \left[\left(X_{e_{l,N,\beta}+2^{-N}} - X_{e_{l,N,\beta}} \right)^2 - \mathbf{E} \left(Z_{e_{l,N,\beta}+2^{-N}}^{H,q} - Z_{e_{l,N,\beta}}^{H,q} \right)^2 \right] \\ &= \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} \left[(\Delta X_{l,N})^2 - \mathbf{E}(\Delta Z_{l,N}^{H,q})^2 \right], \end{aligned} \quad (54)$$

where $\mathcal{L}_{N,\gamma}$ is the same set as in (5) and $e_{l,N,\beta}$ is given by (9). Also, we mention that

$$\Delta X_{l,N} = X_{e_{l,N,\beta}+2^{-N}} - X_{e_{l,N,\beta}}.$$

As a consequence of Theorem 2, we deduce the following asymptotic behavior in distribution of $V_N(X)$:

Proposition 7 *Let $(V_N(X), N \geq 1)$ be given by (54). As $N \rightarrow \infty$,*

$$V_N(X) \rightarrow^{(d)} N \left(0, \left(\mathbf{E}|Z_1^{H,q}|^4 - 1 \right) \right),$$

and for N large enough,

$$d_W \left(V_N(X), N \left(0, \mathbf{E}|Z_1^{H,q}|^4 - 1 \right) \right) \leq C(q) 2^{-\frac{N\gamma}{2}}.$$

Proof: For $t \geq 0$, let

$$Y_t = - \int_0^t X_s ds,$$

where X is the solution to (52). Then we can derive from (52) and elementary computations that

$$V_N(X) = V_N + \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} (\Delta Y_{l,N})^2 + 2 \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} (\Delta Z_{l,N}^{H,q})(\Delta Y_{l,N}), \quad (55)$$

where V_N is given by (26) and $\Delta Y_{l,N} = Y_{e_{l,N,\beta}+2^{-N}} - Y_{e_{l,N,\beta}}$. Notice that, in view of Theorem 2, when N goes to ∞ , the first term in the right-hand side of (55) converges in distribution to the desired limit at the desired rate $2^{-\frac{N\gamma}{2}}$. Thus, in order to obtain the proposition, it is enough to show that, when N tends to ∞ , the other two terms, in the right-hand side of (55), converge to zero in $L^1(\Omega)$ at a faster rate than $2^{-\frac{N\gamma}{2}}$. This is true. Indeed, using Cauchy-Schwarz's inequality as well as the inequalities (53) and (6), we get that

$$\begin{aligned} \mathbf{E} \left| \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} (\Delta Y_{l,N})^2 \right| &= \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} \mathbf{E} \left(\int_{e_{l,N,\beta}}^{e_{l,N,\beta}+2^{-N}} X_s ds \right)^2 \\ &\leq \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} 2^{-N} \sum_{l \in \mathcal{L}_{N,\gamma}} \int_{e_{l,N,\beta}}^{e_{l,N,\beta}+2^{-N}} \mathbf{E} X_s^2 ds \leq C 2^{(2H-2)N} \sqrt{|\mathcal{L}_{N,\gamma}|} \\ &\leq C 2^{(2H-2)N + \frac{N\gamma}{2}} = o\left(2^{-(1-H)N}\right), \quad \text{for all } N \text{ large enough,} \end{aligned}$$

where the last equality results from the fact that $\gamma < 1$. Similar arguments allow us to derive that

$$\begin{aligned} \mathbf{E} \left| \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} (\Delta Z_{l,N}^{H,q})(\Delta Y_{l,N}) \right| &\leq \frac{2^{2HN}}{\sqrt{|\mathcal{L}_{N,\gamma}|}} \sum_{l \in \mathcal{L}_{N,\gamma}} \left(\mathbf{E} (\Delta Z_{l,N}^{H,q})^2 \right)^{\frac{1}{2}} \left(\mathbf{E} (\Delta Y_{l,N})^2 \right)^{\frac{1}{2}} \\ &\leq C 2^{(H-1)N + \frac{N\gamma}{2}} = o\left(2^{-(1-H)\frac{N}{2}}\right), \quad \text{for all } N \text{ large enough.} \end{aligned}$$

In view of Proposition 7, the same procedure as in Section 5 can be used in order to obtain a strongly consistent and asymptotically normal estimator for the Hurst parameter H in (52). More precisely, if X is the HOU process defined by (52), let

$$S_N(X) = \frac{1}{|\mathcal{L}_{N,\gamma}|} \sum_{l \in \mathcal{L}_{N,\gamma}} (\Delta X_{l,N})^2, \quad N \geq 1.$$

Similarly to (55), it can be shown that

$$2^{2HN} S_N(X) = 2^{2HN} S_N + \frac{2^{2HN}}{|\mathcal{L}_{N,\gamma}|} \sum_{l \in \mathcal{L}_{N,\gamma}} (\Delta Y_{l,N})^2 + 2 \frac{2^{2HN}}{|\mathcal{L}_{N,\gamma}|} \sum_{l \in \mathcal{L}_{N,\gamma}} (\Delta Z_{l,N}^{H,q})(\Delta Y_{l,N}),$$

where S_N is as in (41). Then, we can derive from Proposition 6, Borel-Cantelli's lemma and the estimates obtained in the proof of Proposition 7 that we have $S_N(X) \sim 2^{-2HN}$ (in the sense that $2^{2HN} S_N(X) - 1$ converges almost surely to zero as $N \rightarrow \infty$). Thus, it turns out that

$$\hat{H}_N(X) = \frac{-\log(S_N(X))}{2N \log 2}$$

is a strongly consistent estimator for the Hurst parameter H of the Hermite Ornstein-Uhlenbeck process. Moreover, by using Proposition 7 and by arguing as in the proof of Theorem 3, we can show that this estimator is asymptotically normal:

$$2N(\log 2) \sqrt{|\mathcal{L}_{N,\gamma}|} \left(H - \hat{H}_N(X) \right) \xrightarrow[N \rightarrow \infty]{(d)} N(0, \mathbf{E}|Z_1^{H,q}|^4 - 1).$$

7 Appendix: Multiple stochastic integrals and the Malliavin derivative

The basic tools from the analysis on Wiener space are presented in this section. We will focus on some elementary facts about multiple stochastic integrals. We refer to [17] or [15] for a complete review on the topic.

Consider \mathcal{H} a real separable infinite-dimensional Hilbert space with its associated inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, and $(B(\varphi), \varphi \in \mathcal{H})$ an isonormal Gaussian process on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, which is a centered Gaussian family of random variables such that $\mathbf{E}(B(\varphi)B(\psi)) = \langle \varphi, \psi \rangle_{\mathcal{H}}$ for every $\varphi, \psi \in \mathcal{H}$. Denote by $I_q (q \geq 1)$ the q th multiple stochastic integral with respect to B , which is an isometry between the Hilbert space $\mathcal{H}^{\odot q}$ (symmetric tensor product) equipped with the scaled norm $\sqrt{q!} \|\cdot\|_{\mathcal{H}^{\otimes q}}$ and the Wiener chaos of order q , which is defined as the closed linear span of the random variables $H_q(B(\varphi))$ where $\varphi \in \mathcal{H}$, $\|\varphi\|_{\mathcal{H}} = 1$ and H_q is the Hermite polynomial of degree $q \geq 1$ defined by :

$$H_q(x) = \frac{(-1)^q}{q!} \exp\left(\frac{x^2}{2}\right) \frac{d^q}{dx^q} \left(\exp\left(-\frac{x^2}{2}\right) \right), \quad x \in \mathbb{R}. \quad (56)$$

For $q = 0$,

$$\mathcal{H}_0 = \mathbb{R} \text{ and } I_0(x) = x \text{ for every } x \in \mathbb{R}. \quad (57)$$

The isometry property of multiple integrals can be written as follows : for $p, q \geq 0$, $f \in \mathcal{H}^{\otimes p}$ and $g \in \mathcal{H}^{\otimes q}$

$$\mathbf{E}\left(I_p(f)I_q(g)\right) = \begin{cases} q! \langle \tilde{f}, \tilde{g} \rangle_{\mathcal{H}^{\otimes q}} & \text{if } p = q, \\ 0 & \text{otherwise,} \end{cases} \quad (58)$$

where \tilde{f} stands for the symmetrization of f . When $\mathcal{H} = L^2(T)$, with T being an interval of \mathbb{R} , we have the following product formula: for $p, q \geq 0$, $f \in \mathcal{H}^{\odot p}$ and $g \in \mathcal{H}^{\odot q}$,

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} r! \binom{q}{r} \binom{p}{r} I_{p+q-2r}(f \otimes_r g), \quad (59)$$

where, for $r = 0, \dots, p \wedge q$, the contraction $f \otimes_r g$ is the function in $L^2(T^{p+q-2r})$ given by

$$(f \otimes_r g)(t_1, \dots, t_{p+q-2r}) = \int_{T^r} f(u_1, \dots, u_r, t_1, \dots, t_{p-r}) g(u_1, \dots, u_r, t_{p-r+1}, \dots, t_{p+q-2r}) du_1 \dots du_r. \quad (60)$$

An useful property of finite sums of multiple stochastic integrals is the hypercontractivity. Namely, for every fixed real number $p \geq 2$, there exists a universal deterministic finite constant C_p , such that, for any random variable F of the form $F = \sum_{k=0}^n I_k(f_k)$ with $f_k \in \mathcal{H}^{\otimes k}$, the following inequality holds:

$$\mathbf{E}|F|^p \leq C_p (\mathbf{E}F^2)^{\frac{p}{2}}. \quad (61)$$

We denote by D the Malliavin derivative operator that acts on cylindrical random variables of the form $F = g(B(\varphi_1), \dots, B(\varphi_n))$, where $n \geq 1$, $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function with compact support and $\varphi_i \in \mathcal{H}$, in the following way:

$$DF = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(B(\varphi_1), \dots, B(\varphi_n)) \varphi_i.$$

The operator D is closable and it can be extended to $\mathbb{D}^{1,2}$ which denotes the closure of the set of cylindrical random variables with respect to the norm $\|\cdot\|_{1,2}$ defined as

$$\|F\|_{1,2}^2 := \mathbf{E}|F|^2 + \mathbf{E}\|DF\|_{\mathcal{H}}^2.$$

If $F = I_p(f)$, where $f \in \mathcal{H}^{\odot p}$ with $\mathcal{H} = L^2(T)$ and $p \geq 1$, then

$$D_*F = pI_{p-1}(f(\cdot, *)),$$

where " \cdot " stands for $p-1$ variables.

The pseudo inverse $(-L)^{-1}$ of the Ornstein-Uhlenbeck operator L is defined, for $F = I_p(f)$ with $f \in \mathcal{H}^{\odot p}$ and $p \geq 1$, by

$$(-L)^{-1}F = \frac{1}{p}I_p(f).$$

At last notice that in our work, we have $\mathcal{H} = L^2(\mathbb{R})$ while the role of the isonormal process $(B(\varphi), \varphi \in \mathcal{H})$ is played by the usual Wiener integral on $L^2(\mathbb{R})$ associated with the Brownian motion $(B(y), y \in \mathbb{R})$.

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