Numerical Analysis of Quasi-Static Unilateral Contact Problems with Local Friction

Rémi Rocca, Marius Cocou

NUMERICAL ANALYSIS OF QUASI-STATIC UNILATERAL CONTACT PROBLEMS WITH LOCAL FRICTION*

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Abstract. A mixed finite element method is adopted to approximate the quasi-static unilateral contact problem with local Coulomb friction. The existence of a solution with bounds independent of the discretization parameter is obtained by an incremental scheme. This approach enables us to select a sequence of discrete solutions which converges strongly towards a solution of the quasi-static unilateral contact problem with Coulomb friction law.

Key words. unilateral contact, local friction, Coulomb law, quasi-static problems, finite elements, mixed formulations, backward scheme, convergence results

AMS subject classifications. 35K85, 49J40, 65N22

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1. Introduction. The aim of this paper is to study the approximation of the unilateral quasi-static contact problem with local Coulomb friction by a mixed finite element method.

The first result concerning the unilateral contact problem with local friction was obtained by Nečas, Jarušek, and Haslinger [1] for the static case by using a shifting technique and a fixed point argument. This result was extended by Jarušek [2] for domains in which the contact zone is smooth enough. Haslinger [3] proved the existence of solutions for the discrete problem obtained by a mixed finite element method and the existence of solutions converging towards a solution of the static unilateral contact problem with local friction when the discretization parameters tend to zero. To obtain this result he assumed that the normal component of the stress vector, the solution of the static unilateral contact problem, is sufficiently regular at the boundary of the contact zone.

The quasi-static unilateral contact problem has been recently studied by Andersson [4] and Rocca and Cocou [5, 6] who proved that there exists a solution if the friction coefficient is sufficiently small and smooth. The proof is based on a regularization of the contact functional and a shifting technique.

This paper is organized as follows. In section 2, we present the continuous and discrete variational formulations for the initial problem and its approximation. In section 3, we obtain a regularity result for the normal component of the stress vector for an auxiliary problem where the threshold of sliding is given. In section 4 we prove the existence of a saddle point for the incremental regularized discrete problem. Using some error estimates and the result of section 3 we prove that the Lagrangian multiplier is bounded in $H^{-1/2+\alpha}$ independently of the spatial discretization parameter, provided that the friction coefficient is small enough in an L^∞ norm. Section 5 deals with convergence results.

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2. Variational formulation and approximation. A linear elastic body occupies a polygonal set Ω of $\mathbb{R}^d$, $d = 2$ or 3, and its boundary $\partial\Omega$ is denoted by Γ. Let Γ_1, Γ_2, and Γ_3 be three open disjoint parts of Γ such that $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, $\overline{\Gamma}_1 \cap \overline{\Gamma}_3 = \emptyset$, and $mes(\Gamma_1) > 0$. We assume for the sake of simplicity that Γ_3 is a segment for $d = 2$ and a rectangle for $d = 3$. We denote by $\mathbf{u} = (u_1, \ldots, u_d)$ the displacement field, $\boldsymbol{\epsilon} = (\epsilon_{ij}(\mathbf{u})) = (\frac{1}{2}(u_{i,j} + u_{j,i}))$ the strain tensor, and $\boldsymbol{\sigma} = (\sigma_{ij}(\mathbf{u})) = (a_{ijkl}\,\epsilon_{kl}(\mathbf{u}))$ the stress tensor with the usual summation convention, where $i, j, k, l = 1, \ldots, d$. We adopt the following notation for the normal and tangential components of the displacement vector and stress vector: $u_N = u_i n_i$, $\mathbf{u}_T = \mathbf{u} - u_N \mathbf{n}$, $\sigma_N = \sigma_{ij} n_i n_j$, $(\boldsymbol{\sigma}_T)_i = \sigma_{ij} n_j - \sigma_N n_i$, where $\mathbf{n} = (n_i)$ is the outward unit normal vector to $\partial\Omega$.

Let us denote by $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ the densities of the body forces and traction forces, respectively. The initial displacement of the body is denoted by $\mathbf{u}^0$, and we assume that $a_{ijkl} \in L^\infty(\Omega)$, $1 \le i, j, k, l \le d$, with the usual conditions of symmetry and ellipticity, that is,

$$a_{ijkl} = a_{jikl} = a_{klij}, 1 \le i, j, k, l \le d,$$
$$\exists\, m_0 > 0\ \forall\, \zeta = (\zeta_{ij}) \in \mathbb{R}^{d^2},\ \zeta_{ij} = \zeta_{ji},\ 1 \le i, j \le d,\ a_{ijkl}\zeta_{ij}\zeta_{kl} \ge m_0\,|\zeta|^2.$$

The classical problem is as follows: Find $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ such that $\mathbf{u}(0, \cdot) = \mathbf{u}^0(\cdot)$ in Ω and for all $t \in [0, T]$

$$\text{(2.1)} \quad \text{div}\,\boldsymbol{\sigma}(\mathbf{u}) = -\boldsymbol{\phi} \text{ in } \Omega,$$

$$\text{(2.2)} \quad \sigma_{ij}(\mathbf{u}) = a_{ijkl}\epsilon_{kl}(\mathbf{u}) \text{ in } \Omega,$$

$$\text{(2.3)} \quad \mathbf{u} = \mathbf{U}(t, \mathbf{x}) \text{ on } \Gamma_1,$$

$$\text{(2.4)} \quad \boldsymbol{\sigma}\mathbf{n} = \boldsymbol{\psi} \text{ on } \Gamma_2,$$

$$\text{(2.5)} \quad u_N \le 0,\ \sigma_N(\mathbf{u}) \le 0,\ u_N\,\sigma_N(\mathbf{u}) = 0 \text{ on } \Gamma_3,$$

$$\text{(2.6)} \quad |\boldsymbol{\sigma}_T| \le -\mu\,\sigma_N \text{ on } \Gamma_3 \text{ and } \begin{cases} |\boldsymbol{\sigma}_T| < -\mu\,\sigma_N \Rightarrow \dot{\mathbf{u}}_T = \mathbf{0}, \\ |\boldsymbol{\sigma}_T| = -\mu\,\sigma_N \Rightarrow \exists \lambda \ge 0, \dot{\mathbf{u}}_T = -\lambda\,\boldsymbol{\sigma}_T. \end{cases}$$

The prescribed displacement $\mathbf{U}$ on Γ_1 is such that there exists $\overline{\mathbf{U}} \in W^{1,2}(0, T; [H^{3/2}(\Omega)]^d)$ satisfying $\overline{\mathbf{U}} = \mathbf{U}$ a.e. on Γ_1 and $\overline{\mathbf{U}}(t, \mathbf{x}) = \mathbf{0}$ in a neighborhood of Γ_3 for all $t \in [0, T]$. Therefore, by a translation we are able to choose $\mathbf{U}(t, \mathbf{x}) = \mathbf{0}$, which we shall assume in what follows.

We require also that a_{ijkl} are of class $C^{0,1/2+\iota}$ with $1/2 > \iota > 0$ in Ω.

The friction coefficient μ is assumed to belong to $L^\infty(\Gamma)$ and to the set of the multipliers of $H^{1/2}(\Gamma)$ (see [7]) denoted by $\mathcal{M}(H^{1/2}(\Gamma))$. Therefore, the mapping

$$H^{1/2}(\Gamma) \ni v \to \mu v \in H^{1/2}(\Gamma)$$

is bounded by the norm $\|\mu\|_{\mathcal{M}(H^{1/2}(\Gamma))}$.

We define V and K by $V = \{\mathbf{v} \in [H^1(\Omega)]^d, \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_1\}$, $K = \{\mathbf{v} \in V; v_N \le 0 \text{ a.e. on } \Gamma_3\}$. The duality pairings on $H^{1/2}(\Gamma)$, $H^{-1/2}(\Gamma)$ and on $[H^{1/2}(\Gamma)]^d$, $[H^{-1/2}(\Gamma)]^d$ are denoted by $\langle\ ,\ \rangle$. Let us introduce $\mathcal{B}(\Gamma)$, the space of bounded measures h on Γ with respect to the norm given by

$$\sup_{\substack{\varphi \in C(\Gamma) \\ \|\varphi\|_{L^\infty(\Gamma)} \le 1}} \int_\Gamma \varphi(x) dh(x),$$

where $C(\Gamma)$ is the space of continuous functions on Γ. Let $\mathcal{B}_-(\Gamma)$ be the set of bounded negative measures on Γ.

In the following, we will denote $f(t,.)$ by $f(t)$ for any mapping $f = f(t,\mathbf{x})$.

We suppose that $\mathbf{u}^0 \in K, \boldsymbol{\phi} \in W^{1,2}(0,T;[L^2(\Omega)]^d), \boldsymbol{\psi} \in W^{1,2}(0,T;[L^2(\Gamma)]^d)$ with $\mathrm{supp}(\boldsymbol{\psi}(t)) \subset \Gamma_2$ for all $t \in [0,T]$. In what follows we shall use the notation

$$\langle\ s\ \sigma_N(\mathbf{u}), w_N\rangle = \langle \sigma_N(\mathbf{u}), s\ w_N\rangle\ \forall s \in \mathcal{M}(H^{1/2}(\Gamma)).$$

By Green's formula, one can show a variational formulation of (2.1)–(2.6) as the following problem: *Find a mapping* $t \to \mathbf{u}(t)$ *such that* $\mathbf{u} \in W^{1,2}(0,T;V),\ \mathbf{u}(0) = \mathbf{u}^0$, *and for almost all* $t \in]0,T[,\ \mathbf{u}(t) \in K$, *and*

$$\begin{aligned} (2.7)\qquad & a(\mathbf{u}, \mathbf{v} - \dot{\mathbf{u}}) - \langle \mu\theta\sigma_N(\mathbf{u}), |\mathbf{v}_T| - |\dot{\mathbf{u}}_T|\rangle \geq (\boldsymbol{\phi}, \mathbf{v} - \dot{\mathbf{u}}) + \langle \boldsymbol{\psi}, \mathbf{v} - \dot{\mathbf{u}}\rangle \\ & \qquad\qquad + \langle\ \theta\sigma_N(\mathbf{u}), v_N - \dot{u}_N\ \rangle \qquad \forall\ \mathbf{v} \in V, \\ (2.8)\qquad & \langle\ \theta\sigma_N(\mathbf{u}), z_N - u_N\ \rangle \geq 0 \qquad \forall\ \mathbf{z} \in K, \end{aligned}$$

where $a(\mathbf{u},\mathbf{v}) = \int_\Omega a_{ijkl}\epsilon_{ij}(\mathbf{u})\epsilon_{kl}(\mathbf{v})\ dx$. We denote by m, M, respectively, the constants of coercivity and continuity of the bilinear form $a(.,.)$.

The cut-off function θ has the property that $\theta = 1$ on $\overline{\Gamma}_3$ and $\theta = 0$ on $\overline{\Gamma}_2^0$ with Γ_2^0 an open subset such that for all $t \in [0,T]$, supp $\psi(t) \subset \Gamma_2^0 \subset \overline{\Gamma}_2^0 \subset \Gamma_2$. For all $\mathbf{w} \in V$, such that $\mathbf{w}_T = \mathbf{0}$ a.e. on Γ, we have

$$(2.9)\qquad \langle \theta\sigma_N(\mathbf{u}), w_N\rangle = a(\mathbf{u},\mathbf{w}) - (\boldsymbol{\phi}, \mathbf{w}).$$

Using Green's formula, it is straightforward to verify that (2.7), (2.8) is equivalent to the following mixed variational formulation.

Problem P. Find two mappings $t \to \mathbf{u}(t)$ and $t \to \lambda(t)$ such that $\mathbf{u} \in W^{1,2}(0,T;V)$, $\lambda \in W^{1,2}(0,T;H^{-1/2}(\Gamma)),\ \mathbf{u}(0) = \mathbf{u}^0,\ \lambda(0) = \lambda^0 = \theta\sigma_N(\mathbf{u}^0),\ \lambda(t) \in C^{*-}$ for almost all $t \in]0,T[$, and

$$\begin{aligned} (2.10)\qquad & a(\mathbf{u}, \mathbf{v} - \dot{\mathbf{u}}) + j(\lambda, \mathbf{v}) - j(\lambda, \dot{\mathbf{u}}) \geq (\boldsymbol{\phi}, \mathbf{v} - \dot{\mathbf{u}}) + \langle \boldsymbol{\psi}, \mathbf{v} - \dot{\mathbf{u}}\rangle \\ & \qquad\qquad + \langle\ \lambda, v_N - \dot{u}_N\ \rangle \qquad \forall\ \mathbf{v} \in V, \\ (2.11)\qquad & \langle\ \pi - \lambda, u_N\ \rangle \geq 0 \qquad \forall\ \pi \in C^{*-}, \end{aligned}$$

where $C^{*-} = \{\pi \in H^{-1/2}(\Gamma);\ \mathrm{supp}(\pi) \subset \overline{\Gamma}_3\} \cap \mathcal{B}_-(\Gamma),\ j(\lambda,\mathbf{v}) = -\langle \mu\lambda, |\mathbf{v}_T|\rangle$.

The Lagrange multiplier λ satisfies $\lambda = \theta\sigma_N(\mathbf{u})$.

We assume that the initial displacement field $\mathbf{u}^0$ satisfies the compatibility condition

$$\begin{aligned} (2.12)\qquad & a(\mathbf{u}^0, \mathbf{v} - \mathbf{u}^0) + j(\lambda^0, \mathbf{v}) - j(\lambda^0, \mathbf{u}^0) \geq (\boldsymbol{\phi}(0), \mathbf{v} - \mathbf{u}^0) \\ & \qquad\qquad + \langle \boldsymbol{\psi}(0), \mathbf{v} - \mathbf{u}^0\rangle\ \ \forall \mathbf{v} \in K. \end{aligned}$$

In order to prove the existence of discrete solutions converging towards $\mathbf{u}$, we consider the following finite element approximations.

Let $(\mathcal{T}_h)_h$ be a set of regular triangulations of $\overline{\Omega}$ (see [8]). Each triangulation is a collection of elements Ω_i such that $\overline{\Omega} = \bigcup_{i\in\mathcal{I}_h} \overline{\Omega}_i$ with $\Omega_k \cap \Omega_l = \emptyset$ for all $k, l \in \mathcal{I}_h,\ k \neq l$. We assume that each triangulation is compatible with the partition of Γ; that is, each point where the boundary condition changes is a node of a set Ω_i.

Let $(\mathcal{T}_H^{'})_H$ be a set of triangulations of $\overline{\Gamma}_3 = \bigcup_{i\in\mathcal{J}_H} \Gamma_{3,i}$. We suppose that the elements $\Gamma_{3,i},\ i \in \mathcal{J}_H$, are segments for $d = 2$ and rectangles for $d = 3$. We set $H_i = \mathrm{length}(\Gamma_{3,i})$ for $d = 2$, and $H_{1,i},\ H_{2,i}$ are the lengths of the edges of $\Gamma_{3,i}$ for

$d = 3$. We assume that the triangulations are regular; that is, there exists a constant $\gamma > 0$ such that

$$\min_{i\in\mathcal{J}_H}(H_i)/H \geq \gamma, \quad \text{where } H = \max_{i\in\mathcal{J}_H}(H_i) \text{ for } d = 2,$$
$$\min_{\substack{i\in\mathcal{J}_H\\ j=1,2}}(H_{j,i})/H \geq \gamma, \text{ where } H = \max_{\substack{i\in\mathcal{J}_H\\ j=1,2}}(H_{j,i}) \text{ for } d = 3.$$

We consider the sets

$$V_h = \Big\{\mathbf{v}_h \in [C(\overline{\Omega})]^d; \mathbf{v}_{h|\Omega_i} \in [P_1(\Omega_i)]^d \ \ \forall i \in \mathcal{I}_h, \ \mathbf{v}_h = \mathbf{0} \text{ on } \Gamma_1\Big\},$$
$$\tilde{V}_h = \Big\{\mathbf{v}_h \in V_h; \mathbf{v}_{hT} = \mathbf{0} \text{ on } \Gamma, \ v_{hN} = 0 \text{ on } \Gamma\backslash\Gamma_3\Big\},$$
$$L_H = \Big\{\pi_H \in L^2(\Gamma), \ \pi_H = 0 \text{ on } \Gamma\backslash\overline{\Gamma}_3, \ \pi_{H|\Gamma_{3,i}} \in P_0(\Gamma_{3,i}) \ \ \forall i \in \mathcal{J}_H\Big\},$$
$$\Pi_H = \Big\{\pi_H \in L_H, \ \ \pi_H \leq 0 \text{ on } \Gamma_{3,i} \ \forall i \in \mathcal{J}_H\Big\},$$
$$K_{hH} = \Big\{\mathbf{v}_h \in V_h; \ \langle \pi_H, v_{hN}\rangle \geq 0 \ \ \forall \pi_H \in \Pi_H\Big\},$$

where $P_k(\omega)$ denotes the space of polynomials of degree lower or equal to k on ω.

We introduce r_h and R_H, the orthogonal projection operators from V on V_h and from $L^2(\Gamma)$ on L_H. Therefore the following properties hold:

$$\forall \mathbf{v} \in V, \ r_h(\mathbf{v}) \to \mathbf{v} \text{ in } [H^1(\Omega)]^d, \ h \to 0^+, \tag{2.13}$$
$$\forall \pi \in L^2(\Gamma), \ \pi = 0 \ \text{ a.e. on } \Gamma\backslash\overline{\Gamma}_3, \ R_H(\pi) \to \pi \text{ in } L^2(\Gamma), \ H \to 0^+, \tag{2.14}$$
$$\forall \alpha \in]0,1[, \ \forall \mathbf{v} \in [H^{1+\alpha}(\Omega)]^d, \ \|r_h(\mathbf{v}) - \mathbf{v}\|_{[H^1(\Omega)]^d} \leq Ch^\alpha \|\mathbf{v}\|_{[H^{1+\alpha}(\Omega)]^d} \ \forall h, \tag{2.15}$$
$$\forall \alpha \in]0,1/2], \ \forall \pi \in L^2(\Gamma) \ \pi = 0 \ \text{ a.e. on } \Gamma\backslash\overline{\Gamma}_3, \tag{2.16}$$
$$\|R_H(\pi) - \pi\|_{H^{-1/2}(\Gamma)} \leq CH^\alpha \|\pi\|_{H^{-1/2+\alpha}(\Gamma)} \ \forall H$$

with C a constant which depends on α.

As r_H is the orthogonal projection operator from V on V_h, we have that

$$\|r_H(\mathbf{v}) - \mathbf{v}\|_{[H^1(\Omega)]^d} \leq \|\mathbf{w}_h - \mathbf{v}\|_{[H^1(\Omega)]^d} \ \ \forall \mathbf{w}_h \in V_h.$$

For $\mathbf{v} \in [H^2(\Omega)]^d$ we can choose the Lagrange interpolate as $\mathbf{w}_h$, so that we have

$$\|\mathbf{w}_h - \mathbf{v}\|_{[H^1(\Omega)]^d} \leq Ch^\alpha \|\mathbf{v}\|_{[H^{1+\alpha}(\Omega)]^d}.$$

We obtain (2.15) by a density argument; see, e.g., [3] to obtain relation (2.16).

We consider the following semidiscrete variational problem.

Problem P_{hH}. Find two mappings $t \to \mathbf{u}_h(t)$ and $t \to \lambda_H(t)$ such that $\mathbf{u}_h(0) = \mathbf{u}_h^0, \ \lambda_H(0) = \lambda_H^0, \ \mathbf{u}_h \in W^{1,2}(0,T;V_h), \ \lambda_H \in W^{1,2}(0,T;L_H), \ \lambda_H(t) \in \Pi_H$, and

$$a(\mathbf{u}_h, \mathbf{v}_h - \dot{\mathbf{u}}_h) + j(\lambda_H, \mathbf{v}_h) - j(\lambda_H, \dot{\mathbf{u}}_h) \geq (\boldsymbol{\phi}, \mathbf{v}_h - \dot{\mathbf{u}}_h) + \langle \boldsymbol{\psi}, \mathbf{v}_h - \dot{\mathbf{u}}_h\rangle \tag{2.17}$$
$$+\langle \ \lambda_H, v_{hN} - \dot{u}_{hN} \ \rangle \qquad \forall \ \mathbf{v}_h \in V_h,$$
$$\langle \ \pi_H - \lambda_H, u_{hN} \ \rangle \geq 0 \qquad \forall \ \pi_H \in \Pi_H. \tag{2.18}$$

We assume that $\mathbf{u}^0$ is such that there exist two sequences $(\mathbf{u}_h^0)_h, \ (\lambda_H^0)_H$ which satisfy the following properties:

$(\mathbf{u}_h^0,\ \lambda_H^0)$ is a solution of the discrete problem

$$
\begin{aligned}
(2.19)\quad & a(\mathbf{u}_h^0, \mathbf{v}_h - \mathbf{u}_h^0) + j(\lambda_H^0, \mathbf{v}_h) - j(\lambda_H^0, \mathbf{u}_h^0) \geq (\boldsymbol{\phi}(0), \mathbf{v}_h - \mathbf{u}^0) \\
& \qquad + \langle \boldsymbol{\psi}(0), \mathbf{v}_h - \mathbf{u}_h^0 \rangle + \langle \lambda_H^0, v_{hN} - u_{hN}^0 \rangle \ \ \forall \mathbf{v}_h \in V_h, \\
(2.20)\quad & \langle \pi_H - \lambda_H^0, u_{hN}^0 \rangle \geq 0 \qquad \forall \pi_H \in \Pi_H.
\end{aligned}
$$

$$
\mathbf{u}_h^0 \to \mathbf{u}^0 \text{in} [H^1(\Omega)]^d,\ \lambda_H^0 \to \lambda^0 = \theta\sigma_N(\mathbf{u}^0) \text{in} H^{-1/2}(\Gamma).
$$

We remark that we obtain (2.12) by passing to the limit with respect to h, H in (2.19) and (2.20).

We adopt the following time discretization. For all $n \in \mathbb{N}^*$ we use the notation $\boldsymbol{\phi}^i = \boldsymbol{\phi}(i\Delta t)$ and $\boldsymbol{\psi}^i = \boldsymbol{\psi}(i\Delta t)$ for $i = 0, \ldots, n$, where $\Delta t = T/n$. We consider the following incremental problem for $i = 0, \ldots, n-1$.

Problem P_{hH}^i. For $\mathbf{u}_h^i \in K_{hH}$, find $\mathbf{u}_h^{i+1} \in K_{hH}, \lambda_H^{i+1} \in \Pi_H$ such that

$$
\begin{aligned}
& a(\mathbf{u}_h^{i+1}, \mathbf{v}_h - \mathbf{u}_h^{i+1}) + j(\lambda_H^{i+1}, \mathbf{v}_h - \mathbf{u}_h^i) - j(\lambda_H^{i+1}, \mathbf{u}_h^{i+1} - \mathbf{u}_h^i) \geq (\boldsymbol{\phi}^{i+1}, \mathbf{v}_h - \mathbf{u}_h^{i+1}) \\
(2.21)\quad & \qquad + \langle \boldsymbol{\psi}^{i+1}, \mathbf{v}_h - \mathbf{u}_h^{i+1} \rangle + \langle\ \lambda_H^{i+1}, v_{hN} - u_{hN}^{i+1} \rangle \quad \forall\ \mathbf{v}_h \in V_h, \\
(2.22)\quad & \langle\ \pi_H - \lambda_H^{i+1}, u_{hN}^{i+1}\ \rangle \geq 0 \qquad \forall\ \pi_H \in \Pi_H.
\end{aligned}
$$

We also consider the following regularized problem: For $\mathbf{u}_h^i \in K_{hH}$, find $\mathbf{u}_{h\nu} \in K_{hH}, \lambda_{H\nu} \in \Pi_H$ such that

$$
\begin{aligned}
(2.23)\quad & a(\mathbf{u}_{h\nu}, \mathbf{v}_h - \mathbf{u}_{h\nu}) + j_\nu(\lambda_{H\nu}, \mathbf{v}_h - \mathbf{u}_h^i) - j_\nu(\lambda_{H\nu}, \mathbf{u}_{h\nu} - \mathbf{u}_h^i) \geq (\boldsymbol{\phi}^{i+1}, \mathbf{v}_h - \mathbf{u}_{h\nu}) \\
& \qquad + \langle \boldsymbol{\psi}^{i+1}, \mathbf{v}_h - \mathbf{u}_{h\nu} \rangle + \langle\ \lambda_{H\nu}, v_{hN} - u_{h\nu N} \rangle \quad \forall\ \mathbf{v}_h \in V_h, \\
(2.24)\quad & \langle\ \pi_H - \lambda_{H\nu}, u_{h\nu N}\ \rangle \geq 0 \qquad \forall\ \pi_H \in \Pi_H,
\end{aligned}
$$

where $j_\nu(\lambda, \mathbf{v}) = -\langle \mu\lambda, \eta_\nu(|\mathbf{v}_T|) \rangle$ with $(\eta_\nu)_\nu$ a family of functions such that for all $\nu > 0$, η_ν is convex, $\eta_\nu \in C^1(\mathbb{R})$, $0 \leq \eta_\nu^{'} \leq 1$, $\eta_\nu(0) = \eta_\nu^{'}(0) = 0$, and $0 \leq s - \eta_\nu(s) \leq \nu$ for all $s \in \mathbb{R}$. A family of functions satisfying the previous properties is given, for example, by $\eta_\nu(x) = |x| - \nu/2$ if $|x| > \nu$ and $\eta_\nu(x) = x^2/2\nu$ if $|x| \leq \nu$.

3. A regularity result. In order to obtain a convergence result, we need to prove a regularity result for the following problem, where the threshold of sliding is given: For $g \in C^{*-}$, find $\mathbf{u}_g = \mathbf{u}(g) \in K$ such that

$$
\begin{aligned}
(3.1)\quad & a(\mathbf{u}_g, \mathbf{v} - \mathbf{u}_g) - \langle \mu g, \eta_\nu(|\mathbf{v}_T - \mathbf{u}_{hT}^i|) - \eta_\nu(|\mathbf{u}_{gT} - \mathbf{u}_{hT}^i|) \rangle \geq (\boldsymbol{\phi}^{i+1}, \mathbf{v} - \mathbf{u}_g) \\
& \qquad + \langle \boldsymbol{\psi}^{i+1}, \mathbf{v} - \mathbf{u}_g \rangle \ \ \forall \mathbf{v} \in K.
\end{aligned}
$$

We assume, only in this section, that Ω belongs to the class of domains such that $\overline{\Gamma}_3$ is of class $C^{1,1/2+\iota}$. This means that every point of $\overline{\Gamma}_3$ has a neighborhood where Γ can be locally represented by $x_d = \Psi(x_1, \ldots, x_{d-1})$, $\mathbf{x} = (\mathbf{x}^{'}, x_d) \in \mathbb{R}^d$, and Ψ is of class $C^{1,1/2+\iota}$ in its neighborhood. The domain considered in section 2 is a particular case of it.

In the next section we will use the notation $\lambda(g) = \theta\sigma_N(\mathbf{u}_g)$.

THEOREM 3.1. *If μ is sufficiently small and, if under the assumptions of section 2 the following relation holds: For all $\alpha \in [0, 1/2[$,*

$$
\begin{aligned}
(3.2)\quad & \|\theta\sigma_N(\mathbf{u}_g)\|_{H^{-1/2+\alpha}(\Gamma)} \leq C_1 \|\mu\|_{L^\infty(\Gamma)} \|g\|_{H^{-1/2+\alpha}(\Gamma)} \\
& \qquad + C_2 \Big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[H^{-1/2}(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d} \Big),
\end{aligned}
$$

then we equally have

$$\|\theta\sigma_N(\mathbf{u}_g)\|_{L^2(\Gamma)} \le C_3\ \|\mu\|_{L^\infty(\Gamma)}\|g\|_{L^2(\Gamma)} + C_4\Big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\Big). \tag{3.3}$$

Proof. First, by an argument on variational inequalities of the second kind, we note that problem (3.1) has a unique solution. Then we consider the following auxiliary problem: Find $\mathbf{u}_\epsilon \in V$ such that

$$a(\mathbf{u}_\epsilon, \mathbf{v}-\mathbf{u}_\epsilon) - \langle \mu g, \eta_\nu(|\mathbf{v}_T - \mathbf{u}_{hT}^i|) - \eta_\nu(|\mathbf{u}_{\epsilon T} - \mathbf{u}_{hT}^i|)\rangle \ge (\boldsymbol{\phi}^{i+1}, \mathbf{v}-\mathbf{u}_\epsilon) + \langle \boldsymbol{\psi}^{i+1}, \mathbf{v}-\mathbf{u}_\epsilon\rangle - \frac{1}{\epsilon}\int_{\Gamma_3}[u_{\epsilon N}]_+(v_N - u_{\epsilon N})ds \ \ \forall \mathbf{v}\in V.$$

We consider a covering $(U_j)_{j\in I}$ of $\overline{\Gamma}_3$ and $(\rho_j)_{j\in I}$ a subordinate C^∞ partition of unity. In order to obtain estimate (3.2) we use a shifting technique which is valid because the test functions are in V and $\overline{\Gamma}_1 \cap \overline{\Gamma}_3 = \emptyset$. As supp $(\theta\sigma_N(\mathbf{u}_\epsilon)) \subset \overline{\Gamma}_3$, we obtain as in [6, 9] the relation

$$\|\rho_j\ \theta\sigma_N(\mathbf{u}_\epsilon)\|_{H^{-1/2+\alpha}(\Gamma)} \le C_5\|\mu\|_{L^\infty(\Gamma)}\|\rho_j\ g\|_{H^{-1/2+\alpha}(\Gamma)} + C_6\Big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[H^{-1/2}(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\Big).$$

By the equivalence of $\|.\|_{H^{-1/2+\alpha}(\Gamma)}$ and the norm induced by the partition of unity $(\rho_j)_{j\in I}$ we obtain

$$\|\theta\sigma_N(\mathbf{u}_\epsilon)\|_{H^{-1/2+\alpha}(\Gamma)} \le C_1\ \|\mu\|_{L^\infty(\Gamma)}\|g\|_{H^{-1/2+\alpha}(\Gamma)} + C_2\Big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[H^{-1/2}(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\Big).$$

We can pass to the limit with respect to the parameter ϵ so that we obtain (3.2). Relation (3.3) is a consequence of the following trace lemma (see [10]). □

LEMMA 3.2. *There exists a constant $c>0$ such that for all $u \in H^{1,1/2}(\mathbb{R}^{d-1}\times]0,r[)$, $r>0$, such that $u(x_1,\dots,x_{d-1},r)=0$, the following relation holds:*

$$\|u(x_1,\dots,x_{d-1},0)\|_{H^1(\mathbb{R}^{d-1})} \le c\ \|u\|_{H^{1,1/2}(\mathbb{R}^{d-1}\times]0,r[)},$$

where $H^{1,\alpha}(\mathbb{R}^{d-1}\times]0,r[), 0<\alpha<1$, is the subset of elements w of $H^1(\mathbb{R}^{d-1}\times]0,r[)$ which satisfy

$$\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}^{d-1}\times]0,r[}\left(\left(\frac{\partial w}{\partial x_i}\right)_{-\mathbf{t}} - \frac{\partial w}{\partial x_i}\right)^2 |\mathbf{t}|^{-d+1-2\alpha}dxdt < +\infty$$

with $w_{-\mathbf{t}}(\mathbf{x}) = w(x_1+t_1,\dots,x_{d-1}+t_{d-1},x_d)$.

4. Existence of incremental solution. In order to prove the existence of a solution for problem (2.21), (2.22) we consider the following auxiliary problem associated with (2.23), (2.24): For $g_H \in \Pi_H$, find $\mathbf{u}_h = \mathbf{u}_h(g_H) \in K_{hH}, \lambda_H \in \Pi_H$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) - \langle \mu g_H, \eta_\nu(|\mathbf{v}_{hT} - \mathbf{u}_{hT}^i|) - \eta_\nu(|\mathbf{u}_{hT} - \mathbf{u}_{hT}^i|)\rangle \ge (\boldsymbol{\phi}^{i+1}, \mathbf{v}_h - \mathbf{u}_h)$$

$$+\langle \boldsymbol{\psi}^{i+1}, \mathbf{v}_h - \mathbf{u}_h\rangle + \langle\ \lambda_H, v_{hN} - u_{hN}\rangle \quad \forall\ \mathbf{v}_h \in V_h, \tag{4.1}$$

$$\langle \pi_H - \lambda_H, u_{hN}\ \rangle \ge 0 \quad \forall\ \pi_H \in \Pi_H. \tag{4.2}$$

THEOREM 4.1. *For all h,H there exists a solution* $(\mathbf{u}_h, \lambda_H)$ *for the problem* (4.1), (4.2). *Moreover,* $\mathbf{u}_h$ *is unique and, if we assume that the following condition holds:*

$$\langle \lambda_H, v_{hN} \rangle = 0 \ \ \forall\, \mathbf{v}_h \in \tilde{V}_h \Longrightarrow \lambda_H = 0, \tag{4.3}$$

then λ_H *is unique.*

Proof. Let $J(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - \langle \mu g_H, \eta_\nu(|\mathbf{v}_T - \mathbf{u}^i_{hT}|) \rangle - (\boldsymbol{\phi}^{i+1}, \mathbf{v}) - \langle \boldsymbol{\psi}^{i+1}, \mathbf{v} \rangle$ and $\mathcal{L}(\mathbf{v}, \pi_H) = J(\mathbf{v}) - \langle \pi_H, v_{hN} \rangle$. The Lagrangian functional $\mathcal{L}(.,.)$ has a saddle point $(\mathbf{u}_h, \lambda_H)$ on $V_h \times \Pi_H$ (see [11, Prop. 2.2, p. 161]).

From the coercivity of a(.,.) and relation (4.1) it follows that $\mathbf{u}_h$ is unique. The properties of functions η_ν imply that inequality (4.1) is equivalent to the following: For all $\mathbf{v}_h \in V_h$

$$a(\mathbf{u}_h, \mathbf{v}_h) - \left\langle \mu g_H, \eta_\nu'(|\mathbf{u}_{hT} - \mathbf{u}^i_{hT}|)\ \mathbf{v}_{hT}. \frac{\mathbf{u}_{hT} - \mathbf{u}^i_{hT}}{|\mathbf{u}_{hT} - \mathbf{u}^i_{hT}|} \right\rangle = (\boldsymbol{\phi}^{i+1}, \mathbf{v}_h) + \langle \boldsymbol{\psi}^{i+1}, \mathbf{v}_h \rangle + \langle \lambda_H, v_{hN} \rangle.$$

From this inequality we have that for all $\mathbf{v}_h \in \tilde{V}_h$

$$\langle \lambda_H, v_{hN} \rangle = a(\mathbf{u}_h, \mathbf{v}_h) - (\boldsymbol{\phi}^{i+1}, \mathbf{v}_h). \tag{4.4}$$

By condition (4.3), we obtain that λ_H is also unique. □

Condition (4.3) is fulfilled if $2h \leq H$, and we will assume that this relation holds in the following. Theorem 4.1 enables us to define the mapping Φ_H as follows:

$$\Pi_H \ni g_H \rightarrow \Phi_H(g_H) = \lambda_H \in \Pi_H,$$

where λ_H is the solution of (4.1), (4.2). The end of this section is devoted to proving that Φ_H has a fixed point with an estimate independent of h, H for the Lagrange multiplier λ_H. In order to prove this estimate, we remark that there exists a constant β (see [12]) independent of h and H such that

$$\beta \, \|\pi_H\|_{H^{-1/2}(\Gamma)} \leq \|\pi_H\|_{h, H^{-1/2}(\Gamma)} \quad \forall \pi_H \in \Pi_H \tag{4.5}$$

when the ratio h/H is sufficiently small. We will assume in what follows that this condition holds. We denote

$$\|\pi_H\|_{h, H^{-1/2}(\Gamma)} = \sup_{\mathbf{v}_h \in \tilde{V}_h,\ \mathbf{v}_h \neq 0} \frac{\langle \pi_H, v_{hN} \rangle}{\|v_{hN}\|_{H^{1/2}(\Gamma)}},$$

which is a norm by (4.3).

LEMMA 4.2. *The mapping* Φ_H *is continuous on* $H^{-1/2}(\Gamma)$.

Proof. Let $(\mathbf{u}_h, \lambda_H)$ and $(\tilde{\mathbf{u}}_h, \tilde{\lambda}_H)$ be the solutions of problem (4.1), (4.2) for g_H and $\tilde{g}_H$, respectively. By setting $\mathbf{v}_h = \tilde{\mathbf{u}}_h$ for g_H and $\mathbf{v}_h = \mathbf{u}_h$ for $\tilde{g}_H$, we get

$$a(\mathbf{u}_h - \tilde{\mathbf{u}}_h, \mathbf{u}_h - \tilde{\mathbf{u}}_h) \leq \ \langle \mu(g_H - \tilde{g}_H), \eta_\nu(|\mathbf{u}_{hT} - \mathbf{u}^i_{hT}|) - \eta_\nu(|\tilde{\mathbf{u}}_{hT} - \mathbf{u}^i_{hT}|) \rangle.$$

By the coercivity of a(.,.) we obtain

$$m \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|^2_{[H^1(\Omega)]^d} \leq \|\mu\|_{L^\infty(\Gamma)} \|g_H - \tilde{g}_H\|_{L^2(\Gamma)} \|\eta_\nu(|\mathbf{u}_{hT} - \mathbf{u}^i_{hT}|) - \eta_\nu(|\tilde{\mathbf{u}}_{hT} - \mathbf{u}^i_{hT}|)\|_{L^2(\Gamma)}$$

and finally

$$\|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{[H^1(\Omega)]^d} \leq C_7 \, \|g_H - \tilde{g}_H\|_{L^2(\Gamma)}.$$

Thus by (4.4) we obtain that

$$\langle \lambda_H - \tilde{\lambda}_H, v_{hN} \rangle \leq M \, \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{[H^1(\Omega)]^d} \, \|v_{hN}\|_{H^{1/2}(\Gamma)}$$

so that there exists a constant $C_8 > 0$ such that

$$\|\lambda_H - \tilde{\lambda}_H\|_{h,H^{-1/2}(\Gamma)} \leq C_8 \, \|g_H - \tilde{g}_H\|_{L^2(\Gamma)}. \qquad \square$$

THEOREM 4.3. *Under the above hypotheses there exists a solution of problem* (2.23), (2.24) *for any value of the friction coefficient.*

Proof. By introducing $\mathbf{v}_h = \mathbf{u}_h^i$ in (4.1) and as $\langle \lambda_H, u_{hN} \rangle = 0$, one obtains that

$$a(\mathbf{u}_h, \mathbf{u}_h^i - \mathbf{u}_h) + \underbrace{\langle \mu g_H, \eta_\nu(|\mathbf{u}_{hT} - \mathbf{u}_{hT}^i|) \rangle}_{\leq 0} \geq (\boldsymbol{\phi}^{i+1}, \mathbf{u}_h^i - \mathbf{u}_h) + \langle \boldsymbol{\psi}^{i+1}, \mathbf{u}_h^i - \mathbf{u}_h \rangle + \underbrace{\langle \lambda_H, u_{hN}^i \rangle}_{\geq 0}.$$

There exists a constant $C_9 > 0$ such that

$$\|\mathbf{u}_h\|_{[H^1(\Omega)]^d} \leq C_9 \, \left(\|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d} + \|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[H^{-1/2}(\Gamma)]^d} \right).$$

By relation (4.4) one obtains

$$\|\lambda_H\|_{h,H^{-1/2}(\Gamma)} \leq C_{10} \, \left(\|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d} + \|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[H^{-1/2}(\Gamma)]^d} \right).$$

We conclude by (4.5) and by Schauder's second fixed point theorem (see [13]) that the mapping Φ_H has a fixed point which is a solution of (2.23), (2.24). $\square$

LEMMA 4.4. *There exist two constants* $C_1^{'} > 0$, $C_2^{'} > 0$ *such that for all* $(\mathbf{v}_h, \pi_H) \in V_h \times \Pi_H$ *we have*

$$\|\lambda(g_H) - \lambda_H\|_{H^{-1/2}(\Gamma)} \leq C_1^{'} \Big(\|\mathbf{u}(g_H) - \mathbf{u}_h\|_{[H^1(\Omega)]^d} + \|\lambda(g_H) - \pi_H\|_{H^{-1/2}(\Gamma)} \Big), \tag{4.6}$$

$$\begin{aligned} \|\mathbf{u}(g_H) - \mathbf{u}_h\|^2_{[H^1(\Omega)]^d} \leq C_2^{'} \Big(& \|\mathbf{u}(g_H) - \mathbf{v}_h\|^2_{[H^1(\Omega)]^d} + \|\lambda(g_H) - \pi_H\|^2_{H^{-1/2}(\Gamma)} \\ & + \langle \theta \boldsymbol{\sigma}_{\boldsymbol{T}}, \mathbf{v}_{hT} - \mathbf{u}_T(g_H) \rangle + \langle \pi_H - \lambda(g_H), u_N(g_H) \rangle \\ & - \langle \mu g_H, \eta_\nu(|\mathbf{v}_{hT} - \mathbf{u}_{hT}^i|) - \eta_\nu(|\mathbf{u}_T(g_H) - \mathbf{u}_{hT}^i|) \rangle \Big). \end{aligned} \tag{4.7}$$

Proof. By (4.4) and (4.5), it follows that

$$\begin{aligned} \langle \pi_H - \lambda_H, v_{hN} \rangle &= \langle \pi_H, v_{hN} \rangle - a(\mathbf{u}_h, \mathbf{v}_h) + (\boldsymbol{\phi}^{i+1}, \mathbf{v}_h) \\ &= a(\mathbf{u}(g_H) - \mathbf{u}_h, \mathbf{v}_h) + \langle \pi_H - \lambda(g_H), v_{hN} \rangle \\ &\leq C_3^{'} (\|\mathbf{u}(g_H) - \mathbf{u}_h\|_{[H^1(\Omega)]^d} + \|\pi_H - \lambda(g_H)\|_{H^{-1/2}(\Gamma)}) \, \|v_{hN}\|_{H^{1/2}(\Gamma)}. \end{aligned}$$

We conclude by the triangle inequality.

In order to prove (4.7) we introduce the notation $\mathcal{U} = (\mathbf{u}(g_H), \ \lambda(g_H)), \mathcal{V} = (\mathbf{v}, \pi), \ \mathcal{U}_{\mathcal{H}} = (\mathbf{u}_h, \ \lambda_H), \mathcal{V}_{\mathcal{H}} = (\mathbf{v}_h, \pi_H)$ with $\mathcal{H} = (h, H)$. We also set $\mathcal{A}(\mathcal{U}, \mathcal{V}) = a(\mathbf{u}(g_H), \mathbf{v}) - \langle \lambda(g_H), v_N \rangle + \langle \pi, u_N(g_H) \rangle$, $j(\mathcal{V}) = -\langle \mu g_H, \eta_\nu(|\mathbf{v}_T - \mathbf{u}_{hT}^i|) \rangle$, and $(\Phi, \mathcal{V}) = (\boldsymbol{\phi}^{i+1}, \mathbf{v}) + \langle \boldsymbol{\psi}^{i+1}, \mathbf{v} \rangle$. Problems (3.1) and (4.1), (4.2) can be written as follows: Find $\mathcal{U}$ such that for all $\mathcal{V} \in V \times C^{*-}$

$$\mathcal{A}(\mathcal{U}, \mathcal{V} - \mathcal{U}) + j(\mathcal{V}) - j(\mathcal{U}) \geq (\Phi, \mathcal{V} - \mathcal{U}),$$

and find $\mathcal{U}_{\mathcal{H}}$ such that for all $\mathcal{V}_h \in V_h \times \Pi_H$

$$\mathcal{A}(\mathcal{U}_{\mathcal{H}}, \mathcal{V}_{\mathcal{H}} - \mathcal{U}_{\mathcal{H}}) + j(\mathcal{V}_{\mathcal{H}}) - j(\mathcal{U}_{\mathcal{H}}) \geq (\Phi, \mathcal{V}_{\mathcal{H}} - \mathcal{U}_{\mathcal{H}}).$$

Therefore, we have (see, e.g., [14])

$$\begin{aligned}
\mathcal{A}(\mathcal{U} - \mathcal{U}_{\mathcal{H}}, \mathcal{U} - \mathcal{U}_{\mathcal{H}}) &= \mathcal{A}(\mathcal{U} - \mathcal{U}_{\mathcal{H}}, \mathcal{U} - \mathcal{V}_{\mathcal{H}}) + \mathcal{A}(\mathcal{U}, \mathcal{V}_{\mathcal{H}} - \mathcal{U}_{\mathcal{H}}) - \mathcal{A}(\mathcal{U}_{\mathcal{H}}, \mathcal{V}_{\mathcal{H}} - \mathcal{U}_{\mathcal{H}}) \\
&\leq \mathcal{A}(\mathcal{U} - \mathcal{U}_{\mathcal{H}}, \mathcal{U} - \mathcal{V}_{\mathcal{H}}) + \mathcal{A}(\mathcal{U}, \mathcal{V}_{\mathcal{H}} - \mathcal{U}) + \mathcal{A}(\mathcal{U}, \mathcal{U} - \mathcal{U}_{\mathcal{H}}) \\
&+ j(\mathcal{V}_{\mathcal{H}}) - j(\mathcal{U}_{\mathcal{H}}) + (\Phi, \mathcal{U}_h - \mathcal{V}_h) \\
&\leq \mathcal{A}(\mathcal{U} - \mathcal{U}_{\mathcal{H}}, \mathcal{U} - \mathcal{V}_{\mathcal{H}}) + \mathcal{A}(\mathcal{U}, \mathcal{V}_{\mathcal{H}} - \mathcal{U}) + j(\mathcal{V}_{\mathcal{H}}) - j(\mathcal{U}) \\
&+ (\Phi, \mathcal{U} - \mathcal{V}_{\mathcal{H}}).
\end{aligned}$$

The previous relation can be written as follows:

$$\begin{aligned}
a(\mathbf{u}(g_H) - \mathbf{u}_h, \mathbf{u}(g_H) - \mathbf{u}_h) &\leq a(\mathbf{u}(g_H) - \mathbf{u}_h, \mathbf{u}(g_H) - \mathbf{v}_h) + (\boldsymbol{\phi}^{i+1}, \mathbf{u}(g_H) - \mathbf{v}_h) \\
&+ \langle \boldsymbol{\psi}^{i+1}, \mathbf{u}(g_H) - \mathbf{v}_h \rangle + a(\mathbf{u}(g_H), \mathbf{v}_h - \mathbf{u}(g_H)) + \langle \lambda(g_H) - \lambda_H, u_{hN}(g_H) - v_{hN} \rangle \\
&+ \langle \lambda(g_H), v_{hN} - u_N(g_H) \rangle - \langle \mu g_H, \eta_\nu(|\mathbf{v}_{hT} - \mathbf{u}_{hT}^i|) - (|\mathbf{u}_T(g_H) - \mathbf{u}_{hT}^i|) \rangle \\
&+ \langle \lambda(g_H) - \pi_H, u_{hN}(g_H) - u_{hN} \rangle - \langle \lambda(g_H) - \pi_H, u_{hN}(g_H) \rangle.
\end{aligned}$$

By Green's formula, one obtains

$$\begin{aligned}
a(\mathbf{u}(g_H), \mathbf{v}_h - \mathbf{u}(g_H)) &= (\boldsymbol{\phi}^{i+1}, \mathbf{v}_h - \mathbf{u}(g_H)) + \langle \boldsymbol{\psi}^{i+1}, \mathbf{v}_h - \mathbf{u}(g_H) \rangle \\
&+ \langle \theta \boldsymbol{\sigma}_T(\mathbf{u}(g_H)), \mathbf{v}_{hT} - \mathbf{u}_T(g_H) \rangle + \langle \theta \sigma_N(\mathbf{u}(g_H)), v_{hN} - u_N(g_H) \rangle.
\end{aligned}$$

Thus we have

$$\begin{aligned}
m \|\mathbf{u}(g_H) - \mathbf{u}_h\|^2_{[H^1(\Omega)]^d} &\leq a(\mathbf{u}(g_H) - \mathbf{u}_h, \mathbf{u}(g_H) - \mathbf{v}_h) + \langle \theta \boldsymbol{\sigma}_T(\mathbf{u}(g_H)), \mathbf{v}_{hT} - \mathbf{u}_T(g_H) \rangle \\
&+ \langle \lambda(g_H) - \lambda_H, u_{hN}(g_H) - v_{hN} \rangle + \langle \lambda(g_H) - \pi_H, u_{hN}(g_H) - u_{hN} \rangle \\
&+ \langle \pi_H - \lambda(g_H), u_N(g_H) \rangle - \langle \mu g_H, \eta_\nu(|\mathbf{v}_{hT} - \mathbf{u}_{hT}^i|) - \eta_\nu(|\mathbf{u}_T(g_H) - \mathbf{u}_{hT}^i|) \rangle.
\end{aligned}$$

By Young's inequality we finally get

$$\begin{aligned}
C_4^{'} \|\mathbf{u}(g_H) - \mathbf{u}_h\|^2_{[H^1(\Omega)]^d} &\leq C_5^{'} \|\lambda(g_H) - \lambda_H\|^2_{H^{-1/2}(\Gamma)} + C_6^{'} \|\lambda(g_H) - \pi_H\|^2_{H^{-1/2}(\Gamma)} \\
&+ C_7^{'} \|\mathbf{u}(g_H) - \mathbf{v}_h\|^2_{[H^1(\Omega)]^d} + \langle \pi_H - \lambda(g_H), u_N(g_H) \rangle \\
&- \langle \mu g_H, \eta_\nu(|\mathbf{v}_{hT} - \mathbf{u}_{hT}^i|) - \eta_\nu(|\mathbf{u}_T(g_H) - \mathbf{u}_{hT}^i|) \rangle \\
&+ \langle \theta \boldsymbol{\sigma}_T(\mathbf{u}(g_H)), \mathbf{v}_{hT} - \mathbf{u}_T(g_H) \rangle.
\end{aligned}$$

We conclude by (4.6). □

LEMMA 4.5. *Let $\mathcal{T}_h$ and $\mathcal{T}_H^{'}$ be such that there exist $\tau_1 > 0, \tau_2 > 0$ independent of h and H with $\tau_1 \leq h/H \leq \tau_2$. Then there exist positive constants D_1, D_2, D_3, D_4 such that for $0 < \alpha < 1/2$ the following estimates hold:*

$$\begin{aligned}
(4.8) \quad \|\lambda(g_H) - \lambda_H\|_{H^{-1/2}(\Gamma)} &\leq D_1 \|\mu\|_{L^\infty(\Gamma)} h^\alpha \|g_H\|_{H^{-1/2+\alpha}(\Gamma)} \\
&+ D_2 h^\alpha \left(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d} \right), \\
(4.9) \quad \|\mathbf{u}(g_H) - \mathbf{u}_h\|_{[H^1(\Omega)]^d} &\leq D_3 \|\mu\|_{L^\infty(\Gamma)} h^\alpha \|g_H\|_{H^{-1/2+\alpha}(\Gamma)} \\
&+ D_4 h^\alpha \left(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d} \right).
\end{aligned}$$

Proof. From the regularity result of section 3, it follows that

$$\|\mathbf{u}(g_H)\|_{[H^{1+\alpha}(\Omega)]^d} \leq D_4\ \|\mu\|_{L^\infty(\Gamma)}\ \|g_H\|_{H^{-1/2+\alpha}(\Gamma)} + D_5\Big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\Big). \tag{4.10}$$

We also have

$$-\langle \mu g_H, \eta_\nu(|\mathbf{v}_{hT} - \mathbf{u}_{hT}^i|) - \eta_\nu(|\mathbf{u}_T(g_H) - \mathbf{u}_{hT}^i|)\rangle \leq \|\mu\|_{L^\infty(\Gamma)}\ \|g_H\|_{L^2(\Gamma)} \times \|\mathbf{v}_{hT} - \mathbf{u}_T(g_H)\|_{[L^2(\Gamma)]^d}$$

and

$$\langle \theta \boldsymbol{\sigma}_T(\mathbf{u}(g_H)), \mathbf{v}_{hT} - \mathbf{u}_T(g_H)\rangle \leq -\int_{\Gamma_3} \mu g_H |\mathbf{v}_{hT} - \mathbf{u}_T(g_H)|\, ds.$$

We introduce $\mathbf{v}_h = r_h(\mathbf{u}(g_h))$ in (4.7) and $\pi_H = R_H(\lambda(g_H))$ in (4.6), (4.7). Using (2.16), (4.10), and the fact that $\|\pi_H\|_{L^2(\Gamma)} \leq c_1\ H^{-1/2+\alpha}\ \|\pi_H\|_{H^{-1/2+\alpha}(\Gamma)}$ for all $\pi_H \in L_H$, we obtain

$$\begin{aligned}
\|g_H\|_{L^2(\Gamma)}\|\mathbf{v}_{hT} - \mathbf{u}_T(g_H)\|_{[L^2(\Gamma)]^d} &\leq c_2 H^{-1/2+\alpha}\|g_H\|_{H^{-1/2+\alpha}(\Gamma)} h^{1/2}\|\mathbf{v}_h - \mathbf{u}(g_H)\|_{[H^1(\Omega)]^d} \\
&\leq c_3\ h^\alpha\ H^\alpha\ \|g_H\|_{H^{-1/2+\alpha}(\Gamma)}\ \|\mathbf{u}(g_H)\|_{[H^{1+\alpha}(\Omega)]^d} \\
&\leq c_4\ h^{2\alpha}\ \|g_H\|^2_{H^{-1/2+\alpha}(\Gamma)} + c_5\ h^{2\alpha}\big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} \\
&\qquad + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\big).
\end{aligned}$$

As $0 < \alpha < 1/2$, we equally have by (3.3) and (4.10) (see, e.g., [3])

$$\begin{aligned}
\langle \pi_H - \lambda(g_H), u_N(g_H)\rangle &\leq \tilde{c}_1\ h^{1/2+\alpha}\ \|\lambda(g_H)\|_{L^2(\Gamma)}\|\mathbf{u}(g_H)\|_{[H^{1/2+\alpha}(\Gamma)]^d} \\
&\leq h^{1/2+\alpha}\Big(\tilde{c}_2\ \|\mu\|_{L^\infty(\Gamma)}\ \|g_H\|_{L^2(\Gamma)} + \tilde{c}_3\big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} \\
&\qquad + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\big)\Big)\|\mathbf{u}(g_H)\|_{[H^{1+\alpha}(\Omega)]^d} \\
&\leq h^{1/2+\alpha}\Big(\tilde{c}_4\ \|\mu\|_{L^\infty(\Gamma)}\ H^{-1/2+\alpha}\ \|g_H\|_{H^{-1/2+\alpha}(\Gamma)} \\
&\qquad + \tilde{c}_3\big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\big)\Big) \\
&\qquad \times \Big(\tilde{c}_5\ \|\mu\|_{L^\infty(\Gamma)}\ \|g_H\|_{H^{-1/2+\alpha}(\Gamma)} + \tilde{c}_6\big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} \\
&\qquad + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\big)\Big) \\
&\leq \Big(\tilde{c}_7\ \|\mu\|_{L^\infty(\Gamma)}\ h^{2\alpha}\ \|g_H\|_{H^{-1/2+\alpha}(\Gamma)} \\
&+ \tilde{c}_8\ h^{2\alpha}\ \big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\big)\Big) \\
&\qquad \times\Big(\tilde{c}_5\ \|\mu\|_{L^\infty(\Gamma)}\ \|g_H\|_{H^{-1/2+\alpha}(\Gamma)} + \tilde{c}_6\big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} \\
&\qquad + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\big)\Big) \\
&\leq \tilde{c}_9\ \|\mu\|^2_{L^\infty(\Gamma)}\ h^{2\alpha}\ \|g_H\|^2_{H^{-1/2+\alpha}(\Gamma)} + \tilde{c}_{10}\ h^{2\alpha}\ \big(\|\boldsymbol{\phi}^{i+1}\|^2_{[L^2(\Omega)]^d} \\
&\qquad + \|\boldsymbol{\psi}^{i+1}\|^2_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\big).
\end{aligned}$$

By the previous relations, we obtain relation (4.9). Inequalities (2.16), (3.2), (4.6), and (4.10) enable us to obtain relation (4.8). ∎

THEOREM 4.6. *Let $\mathcal{T}_h$, $\mathcal{T}_H^{'}$, and α satisfy the same conditions as in Lemma* 4.5. *Then there exists a value $\mu^* > 0$ such that the following relation holds:*

$$\begin{aligned}(4.11)\qquad \|\lambda_H\|_{H^{-1/2+\alpha}(\Gamma)} &\leq \|\mu\|_{L^\infty(\Gamma)}/\mu^*\ \|g_H\|_{H^{-1/2+\alpha}(\Gamma)}\\ &+\mathcal{C}_1\Big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\Big).\end{aligned}$$

Thus for $\mu \in \mathcal{M}(H^{1/2}(\Gamma))$ such that $\|\mu\|_{L^\infty(\Gamma)} < \mu^*$, there exists $r_0 > 0$ such that the mapping Φ_H has a fixed point in $\{\pi_H \in \Pi_H; \|\pi_H\|_{H^{-1/2+\alpha}(\Gamma)} \leq r_0\}$. Consequently, problem (2.23), (2.24) *has a solution.*

Proof. By the triangle inequality, (2.16), (4.8), and by taking π_H as in Lemma 4.5, one obtains

$$\begin{aligned}\|\lambda(g_H) - \lambda_H\|_{H^{-1/2+\alpha}(\Gamma)} &\leq \|\lambda(g_H) - \pi_H\|_{H^{-1/2+\alpha}(\Gamma)} + \|\pi_H - \lambda_H\|_{H^{-1/2+\alpha}(\Gamma)}\\ &\leq \hat{c}_1\ H^{1/2-\alpha}\ \|\lambda(g_H)\|_{L^2(\Gamma)} + \hat{c}_2\ H^{-\alpha}\ \|\pi_H - \lambda_H\|_{H^{-1/2}(\Gamma)}\\ &\leq \hat{c}_1\ H^{1/2-\alpha}\ \|\lambda(g_H)\|_{L^2(\Gamma)} + \hat{c}_2\ H^{-\alpha}\ \Big(\|\pi_H - \lambda(g_H)\|_{H^{-1/2}(\Gamma)}\\ &\qquad +\|\lambda(g_H) - \lambda_H\|_{H^{-1/2}(\Gamma)}\Big)\\ &\leq \hat{c}_3\ H^{1/2-\alpha}\ \|\lambda(g_H)\|_{L^2(\Gamma)} + \hat{c}_2\ H^{-\alpha}\ \|\lambda(g_H) - \lambda_H\|_{H^{-1/2}(\Gamma)}\\ \leq \hat{c}_4\ \|\mu\|_{L^\infty(\Gamma)}\ \|g_H\|_{H^{-1/2+\alpha}(\Gamma)} &+ \hat{c}_5\Big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\Big).\end{aligned}$$

Using (3.2), we deduce that

$$\begin{aligned}\|\lambda_H\|_{H^{-1/2+\alpha}(\Gamma)} &\leq \|\lambda_H - \lambda(g_H)\|_{H^{-1/2+\alpha}(\Gamma)} + \|\lambda(g_H)\|_{H^{-1/2+\alpha}(\Gamma)}\\ &\leq \|\mu\|_{L^\infty(\Gamma)}/\mu^*\ \|g_H\|_{H^{-1/2+\alpha}(\Gamma)}\\ &\quad + \mathcal{C}_1\Big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d} + \|\mathbf{u}_h^i\|_{[H^1(\Omega)]^d}\Big).\end{aligned}$$

We conclude by Schauder's fixed point theorem. ∎

THEOREM 4.7. *There exists a solution $(\mathbf{u}_h^{i+1}, \lambda_H^{i+1})$ to problem* (2.21), (2.22) *satisfying the following estimate:*

$$\begin{aligned}(4.12)\qquad &\|\lambda_H^{i+1}\|_{H^{-1/2+\alpha}(\Gamma)}\\ &\leq \mathcal{C}_2\Big(\|\boldsymbol{\psi}^{i+1}\|_{[L^2(\Gamma)]^d}, \|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d}, \|\boldsymbol{\psi}^{i}\|_{[L^2(\Gamma)]^d}, \|\boldsymbol{\phi}^{i}\|_{[L^2(\Omega)]^d}\Big).\end{aligned}$$

Moreover, $(\mathbf{u}_h^{i+1}, \lambda_H^{i+1})$ satisfies the following relation for all $\mathbf{v}_h \in \tilde{V}_h$:

$$(4.13)\qquad \langle \lambda_H^{i+1}, v_{hN}\rangle = a(\mathbf{u}_h^{i+1}, \mathbf{v}_h) - (\boldsymbol{\phi}^{i+1}, \mathbf{v}_h).$$

Proof. The solution $\mathbf{u}_{h\nu}$ satisfies the following relation:

$$\begin{aligned}a(\mathbf{u}_{h\nu}, \mathbf{u}_{h\nu}) - \left\langle \mu\lambda_{H\nu}, \eta_\nu^{'}(|\mathbf{u}_{h\nu T} - \mathbf{u}_{hT}^i|)\ \mathbf{u}_{h\nu T}\cdot\frac{\mathbf{u}_{h\nu T} - \mathbf{u}_{hT}^i}{|\mathbf{u}_{h\nu T} - \mathbf{u}_{hT}^i|}\right\rangle &= (\boldsymbol{\phi}^{i+1}, \mathbf{u}_{h\nu})\\ &+\langle \boldsymbol{\psi}^{i+1}, \mathbf{u}_{h\nu}\rangle + \langle \lambda_H, u_{h\nu N}\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}m\|\mathbf{u}_{h\nu}\|^2_{[H^1(\Omega)]^d} \leq -\langle \mu\lambda_{H\nu}, |\mathbf{u}_{h\nu T}|\rangle &+ |(\boldsymbol{\phi}^{i+1}, \mathbf{u}_{h\nu})|\\ &+|\langle \boldsymbol{\psi}^{i+1}, \mathbf{u}_{h\nu}\rangle| + |\langle \lambda_H, u_{h\nu N}\rangle|.\end{aligned}$$

As $\langle \lambda_H, u_{h\nu N}\rangle = 0$, we obtain that

$$m\|\mathbf{u}_{h\nu}\|_{[H^1(\Omega)]^d} \leq \|\mu\|_{L^\infty(\Gamma)} C_{tr} \|\lambda_{H\nu}\|_{H^{-1/2}(\Gamma)} + \|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[H^{-1/2}(\Gamma)]^d},$$

and relations (4.4) and (4.5) enable us to prove that

$$\|\lambda_{H\nu}\|_{H^{-1/2}(\Gamma)} \leq (Mc_{tr})/\beta \, \|\mathbf{u}_{h\nu}\|_{[H^1(\Omega)]^d} + \, c_{tr}/\beta \, \|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d},$$

where C_{tr}, c_{tr} are two constants depending on the trace operators. If $\|\mu\|_{L^\infty(\Gamma)} < \tilde{\mu} = m/(MC_{tr}c_{tr})$, then there exists a constant $\mathcal{C}_3 > 0$ such that

$$\|\mathbf{u}_{h\nu}\|_{[H^1(\Omega)]^d} \leq \mathcal{C}_3 \Big(\|\boldsymbol{\phi}^{i+1}\|_{[L^2(\Omega)]^d} + \|\boldsymbol{\psi}^{i+1}\|_{[H^{-1/2}(\Gamma)]^d} \Big). \tag{4.14}$$

This enables us to select a sequence still denoted by (ν) such that

$$\begin{array}{rcll} \nu & \to & 0^+, & \\ \mathbf{u}_{h\nu} & \rightharpoonup & \mathbf{u}_h^{i+1} & \text{in} \quad [H^1(\Omega)]^d, \\ \lambda_{H\nu} & \rightharpoonup & \lambda_H^{i+1} & \text{in} \quad H^{-1/2+\alpha}(\Gamma). \end{array}$$

It follows that

$$\begin{aligned} a(\mathbf{u}_h^{i+1}, \mathbf{u}_h^{i+1}) &\leq \liminf_{\nu \to 0^+} a(\mathbf{u}_{h\nu}, \mathbf{u}_{h\nu}), \\ (\boldsymbol{\phi}^{i+1}, \mathbf{v}_h - \mathbf{u}_h^{i+1}) &= \lim_{\nu \to 0^+} (\boldsymbol{\phi}^{i+1}, \mathbf{v}_h - \mathbf{u}_{h\nu}), \\ \langle \boldsymbol{\psi}^{i+1}, \mathbf{v}_h - \mathbf{u}_h^{i+1} \rangle &= \lim_{\nu \to 0^+} \langle \boldsymbol{\psi}^{i+1}, \mathbf{v}_h - \mathbf{u}_{h\nu} \rangle. \end{aligned}$$

From the relation

$$\Big| \eta_\nu(|\mathbf{v}_{hT} - \mathbf{u}_{hT}^i|) - |\mathbf{v}_{hT} - \mathbf{u}_{hT}^i| \, \Big| \; \leq \; \nu$$

one obtains

$$j(\lambda_H^{i+1}, \mathbf{v}_h - \mathbf{u}_h^i) = \lim_{\nu \to 0^+} j_\nu(\lambda_{H\nu}, \mathbf{v}_h - \mathbf{u}_h^i). \tag{4.15}$$

We also have by the compact imbedding of $H^{-1/2+\alpha}(\Gamma)$ in $H^{-1/2}(\Gamma)$ that

$$\begin{aligned} \lim_{\nu \to 0^+} j(\lambda_{H\nu}, \mathbf{u}_{h\nu} - \mathbf{u}_h^i) &= \lim_{\nu \to 0^+} j(\lambda_{H\nu} - \lambda_H^{i+1}, \mathbf{u}_{h\nu} - \mathbf{u}_h^i) + \lim_{\nu \to 0^+} j(\lambda_H^{i+1}, \mathbf{u}_{h\nu} - \mathbf{u}_h^i) \\ &= j(\lambda_H^{i+1}, \mathbf{u}_h^{i+1} - \mathbf{u}_h^i). \end{aligned}$$

By the same manner as for (4.15), we get

$$\liminf_{\nu \to 0^+} j_\nu(\lambda_{H\nu}, \mathbf{u}_{h\nu} - \mathbf{u}_h^i) \geq \liminf_{\nu \to 0^+} j(\lambda_{H\nu}, \mathbf{u}_{h\nu} - \mathbf{u}_h^i).$$

Therefore,

$$\liminf_{\nu \to 0^+} j_\nu(\lambda_{H\nu}, \mathbf{u}_{h\nu} - \mathbf{u}_h^i) \geq j(\lambda_H^{i+1}, \mathbf{u}_h^{i+1} - \mathbf{u}_h^i).$$

The following relations are also valid:

$$\begin{aligned} \lim_{\nu \to 0^+} \langle \lambda_{H\nu}, u_{h\nu N} \rangle \geq \lim_{\nu \to 0^+} & \langle \lambda_{H\nu} - \lambda_H^{i+1}, u_{h\nu N} \rangle \\ & + \lim_{\nu \to 0^+} \langle \lambda_H^{i+1}, u_{h\nu N} \rangle = \langle \lambda_H^{i+1}, u_{hN} \rangle \end{aligned}$$

and

$$\lim_{\nu \to 0^+} \langle \pi_H, u_{h\nu N} \rangle = \langle \pi_H, u_{hN}^{i+1} \rangle.$$

All the previous relations concerning the limits with respect to the parameter ν enable

us to pass to the limit in (2.23), (2.24) and to obtain (2.21), (2.22).

Relation (4.13) is obtained by passing to the limit in (4.4). In order to get (4.12) we used (4.11) and (4.14) for index i. □

5. Convergence results. The aim of this section is to prove that there exists a sequence of discrete solutions of problem (2.21), (2.22) which converges towards the $(\mathbf{u}, \lambda)$ solution of problem (2.10), (2.11).

THEOREM 5.1. *There exists a solution to problem* (2.17), (2.18) *such that*

$$\|\lambda_H(t)\|_{H^{-1/2+\alpha}(\Gamma)} \leq \mathcal{C}_4 \left(\sup_{s\in[0,T]} \|\boldsymbol{\psi}(s)\|_{[L^2(\Gamma)]^d}, \sup_{s\in[0,T]} \|\boldsymbol{\phi}(s)\|_{[L^2(\Omega)]^d} \right) \text{ on }]0,T[. \tag{5.1}$$

Proof. We introduce $\mathbf{v}_h = \mathbf{u}_h^i$ in (2.21). It follows that

$$a(\mathbf{u}_h^{i+1}, \mathbf{u}_h^i - \mathbf{u}_h^{i+1}) - j(\lambda_H^{i+1}, \mathbf{u}_h^{i+1} - \mathbf{u}_h^i) \geq (\boldsymbol{\phi}^{i+1}, \mathbf{u}_h^i - \mathbf{u}_h^{i+1}) + \langle \boldsymbol{\psi}^{i+1}, \mathbf{u}_h^i - \mathbf{u}_h^{i+1} \rangle \tag{5.2}$$

due to the fact that (2.22) implies that

$$\langle \lambda_H^{i+1}, u_{hN}^{i+1} \rangle = 0,$$

and as $\mathbf{u}_h^i \in K_{hH}$ we have

$$\langle \lambda_H^{i+1}, u_{hN}^i \rangle \geq 0.$$

Similarly, we set $\mathbf{v}_h = \mathbf{u}_h^{i+1}$ in relation (2.21) corresponding to $i-1$ and $\mathbf{v}_h = \mathbf{u}_h^1$ in (2.19) if $i = 0$. Thus one obtains that

$$\begin{aligned} a(\mathbf{u}_h^i, \mathbf{u}_h^{i+1} - \mathbf{u}_h^i) + j(\lambda_H^i, \mathbf{u}_h^{i+1} - \mathbf{u}_h^{i-1}) - j(\lambda_H^i, \mathbf{u}_h^i - \mathbf{u}_h^{i-1}) \geq (\boldsymbol{\phi}^i, \mathbf{u}_h^{i+1} - \mathbf{u}_h^i) \\ + \langle \boldsymbol{\psi}^i, \mathbf{u}_h^{i+1} - \mathbf{u}_h^i \rangle. \end{aligned} \tag{5.3}$$

It is straightforward to verify that

$$j(\lambda_H^i, \mathbf{u}_h^{i+1} - \mathbf{u}_h^{i-1}) - j(\lambda_H^i, \mathbf{u}_h^i - \mathbf{u}_h^{i-1}) \leq j(\lambda_H^i, \mathbf{u}_h^{i+1} - \mathbf{u}_h^i).$$

From (5.2), (5.3), and by setting $\Delta f^i = f^{i+1} - f^i$, we have that

$$a(\Delta \mathbf{u}_h^i, \Delta \mathbf{u}_h^i) \leq -j(\Delta \lambda_H^i, \Delta \mathbf{u}_h^i) + (\Delta \boldsymbol{\phi}^i, \Delta \mathbf{u}_h^i) + \langle \Delta \boldsymbol{\psi}^i, \Delta \mathbf{u}_h^i \rangle$$

and

$$j(\Delta \lambda_H^i, \Delta \mathbf{u}_h^i) \leq \|\mu\|_{\mathcal{M}(H^{1/2}(\Gamma))} \, \|\Delta \lambda_H^i\|_{H^{-1/2}(\Gamma)} \, \|\Delta \mathbf{u}_h^i\|_{H^{1/2}(\Gamma)}.$$

Relations (4.13) and (4.5) enable us to prove that

$$\beta \|\Delta \lambda_H^i\|_{H^{-1/2}(\Gamma)} \leq M c_{tr} \, \|\Delta \mathbf{u}_h^i\|_{[H^1(\Omega)]^d} + c_{tr} \, \|\Delta \boldsymbol{\phi}^i\|_{[L^2(\Omega)]^d}.$$

If $\|\mu\|_{\mathcal{M}(H^{1/2}(\Gamma))} < \tilde{\mu}$, then there exist two constants $\mathcal{C}_5 > 0$, $\mathcal{C}_6 > 0$ such that

$$\begin{aligned} \|\Delta \mathbf{u}_h^i\|_{[H^1(\Omega)]^d} &\leq \mathcal{C}_5 \Big(\|\Delta \boldsymbol{\phi}^i\|_{[H^1(\Omega)]^d} + \|\Delta \boldsymbol{\psi}^i\|_{[H^{-1/2}(\Gamma)]^d} \Big), \\ \|\Delta \lambda_H^i\|_{H^{-1/2}(\Gamma)} &\leq \mathcal{C}_6 \Big(\|\Delta \boldsymbol{\phi}^i\|_{[H^1(\Omega)]^d} + \|\Delta \boldsymbol{\psi}^i\|_{[H^{-1/2}(\Gamma)]^d} \Big). \end{aligned}$$

These two estimates allow us to construct some sequences of discrete solutions for the displacement fields and the Lagrange multipliers which converge towards some absolutely continuous mappings with respect to time, as follows.

For $i = 0, \ldots, n$, we set

$$\begin{aligned}
&\mathbf{u}_{hn}(t) = \mathbf{u}_h^{i+1},\\
&\lambda_{Hn}(t) = \lambda_H^{i+1},\\
&\tilde{\mathbf{u}}_{hn}(t) = \mathbf{u}_h^i + (t - t_i)(\mathbf{u}_h^{i+1} - \mathbf{u}_h^i)/\Delta t,\\
&\tilde{\lambda}_{Hn}(t) = \lambda_H^i + (t - t_i)(\lambda_H^{i+1} - \lambda_H^i)/\Delta t,\\
&\mathbf{u}_{hn}(0) = \tilde{\mathbf{u}}_{hn}(0) = \mathbf{u}_h^0,\ \lambda_{Hn}(0) = \tilde{\lambda}_{Hn}(0) = \lambda_H^0
\end{aligned}$$

with $t \in]t_i, t_{i+1}]$, $t_i = i\Delta t$. There exist two elements $\mathbf{u}_h, \lambda_H$ and a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\begin{aligned}
\tilde{\mathbf{u}}_{hn_k} &\rightharpoonup \mathbf{u}_h && \text{in} && W^{1,2}(0,T;[H^1(\Omega)]^d),\\
\tilde{\lambda}_{Hn_k} &\rightharpoonup \lambda_H && \text{in} && W^{1,2}(0,T;H^{-1/2}(\Gamma)),\\
\mathbf{u}_{hn_k}(t) &\rightharpoonup \mathbf{u}_h(t) && \text{in} && [H^1(\Omega)]^d \ \ \forall t \in [0,T],\\
\lambda_{Hn_k}(t) &\to \lambda_H(t) && \text{in} && H^{-1/2}(\Gamma) \ \ \forall t \in [0,T].
\end{aligned}$$

The proof is similar to the one given in [6, 9]. In the following we still denote by $(\tilde{\mathbf{u}}_{hn})$, $(\tilde{\lambda}_{Hn})$, $(\mathbf{u}_{hn})$, and (λ_{Hn}) the previous subsequences.

By the weak convergence of $(\mathbf{u}_{hn}(t))$ and $(\lambda_{Hn}(t))$ it follows that

$$\lim_{n\to 0}\int_0^T a(\mathbf{u}_{hn}(t), \mathbf{v}_h(t))dt = \int_0^T a(\mathbf{u}_h(t), \mathbf{v}_h(t))dt, \tag{5.4}$$

$$\lim_{n\to 0}\int_0^T j(\lambda_{Hn}(t), \mathbf{v}_h(t))dt = \int_0^T j(\lambda_H(t), \mathbf{v}_h(t))dt, \tag{5.5}$$

$$\lim_{n\to 0}\int_0^T \langle\lambda_{Hn}(t), v_{hN}(t)\rangle dt = \int_0^T \langle\lambda_H(t), v_{hN}(t)\rangle dt. \tag{5.6}$$

We also have, by setting $\boldsymbol{\phi}_n(t) = \boldsymbol{\phi}^{i+1}$, $\boldsymbol{\psi}_n(t) = \boldsymbol{\psi}^{i+1}$ for $t \in]t_i, t_{i+1}]$, the following results:

$$\lim_{n\to 0}\int_0^T \left(\boldsymbol{\phi}_n(t), \mathbf{v}_h(t) - \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(t)\right) dt = \int_0^T \left(\boldsymbol{\phi}(t), \mathbf{v}_h(t) - \frac{d\tilde{\mathbf{u}}_h}{dt}(t)\right) dt, \tag{5.7}$$

$$\lim_{n\to 0}\int_0^T \left\langle\boldsymbol{\psi}_n(t), \mathbf{v}_h(t) - \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(t)\right\rangle dt = \int_0^T \left\langle\boldsymbol{\psi}(t), \mathbf{v}_h(t) - \frac{d\tilde{\mathbf{u}}_h}{dt}(t)\right\rangle dt. \tag{5.8}$$

Next we have

$$\begin{aligned}
\liminf_{n\to 0}\int_0^T j\left(\lambda_{Hn}(t), \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(t)\right) dt \geq \liminf_{n\to 0}\int_0^T j\left(\lambda_{Hn}(t) - \lambda_H(t), \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(t)\right) dt\\
+ \liminf_{n\to 0}\int_0^T j\left(\lambda_H(t), \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(t)\right) dt.
\end{aligned}$$

We have $\lim_{n\to\infty}\|\lambda_{Hn}(t) - \lambda_H(t)\|_{H^{-1/2}(\Gamma)} = 0$ $forall$ $t \in [0,T]$, and by the Cauchy–Schwarz inequality we obtain that

$$\left\| \int_0^T j\left(\lambda_{Hn}(t)-\lambda_H(t), \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(t)\right) dt \right\| \leq \left(\int_0^T \|\mu(\lambda_{Hn}(t)-\lambda_H(t))\|^2_{H^{-1/2}(\Gamma)} dt \right)^{1/2} \times \left(\int_0^T \left\| \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(t) \right\|^2_{[H^1(\Omega)]^d} dt \right)^{1/2}.$$

We conclude by Lebesgue's theorem that

$$\lim_{n\to+\infty} \int_0^T j\left(\lambda_{Hn}(t)-\lambda_H(t), \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(t)\right) dt = 0,$$

and by the convexity of $j(\lambda_H, .)$ we obtain that

$$\liminf_{n\to+\infty} \int_0^T j\left(\lambda_{Hn}(t), \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(t)\right) dt \geq \int_0^T j\left(\lambda_H(t), \frac{d\tilde{\mathbf{u}}_{h}}{dt}(t)\right) dt. \tag{5.9}$$

As previously, we have

$$\lim_{n\to+\infty} \int_0^T \left\langle \lambda_{Hn}(t), \frac{d\tilde{u}_{hNn}}{dt}(t) \right\rangle dt = \lim_{n\to+\infty} \int_0^T \left\langle \lambda_{Hn}(t)-\lambda_H(t), \frac{d\tilde{u}_{hNn}}{dt}(t) \right\rangle dt + \lim_{n\to+\infty} \int_0^T \left\langle \lambda_H(t), \frac{d\tilde{u}_{hNn}}{dt}(t) \right\rangle dt,$$

$$\lim_{n\to+\infty} \int_0^T \langle \pi_H - \lambda_{Hn}(t), u_{hNn}(t)\rangle dt = \lim_{n\to+\infty} \int_0^T \langle \lambda_H(t) - \lambda_{Hn}(t), u_{hNn}(t)\rangle dt + \lim_{n\to+\infty} \int_0^T \langle \pi_H - \lambda_H(t), u_{hNn}(t)\rangle dt.$$

We finally obtain that

$$\lim_{n\to+\infty} \int_0^T \left\langle \lambda_{Hn}(t), \frac{d\tilde{u}_{hNn}}{dt}(t) \right\rangle dt = \int_0^T \left\langle \lambda_H(t), \frac{d\tilde{u}_{hN}}{dt}(t) \right\rangle dt, \tag{5.10}$$

$$\lim_{n\to+\infty} \int_0^T \langle \pi_H - \lambda_{Hn}(t), u_{hNn}(t)\rangle dt = \int_0^T \langle \pi_H - \lambda_H(t), u_{hN}(t)\rangle dt. \tag{5.11}$$

By setting $\mathbf{v}_h = \Delta t\, \mathbf{w}_h + \mathbf{u}_h^i$ in (2.21), we have

$$a\left(\mathbf{u}_{hn}, \mathbf{w}_h - \frac{d\tilde{\mathbf{u}}_{hn}}{dt}\right) + j(\lambda_{Hn}, \mathbf{w}_h) - j\left(\lambda_{Hn}, \frac{d\tilde{\mathbf{u}}_{hn}}{dt}\right) \geq \left(\boldsymbol{\phi}_n, \mathbf{w}_h - \frac{d\tilde{\mathbf{u}}_{hn}}{dt}\right) + \left\langle \boldsymbol{\psi}_n, \mathbf{w}_h - \frac{d\tilde{\mathbf{u}}_{hn}}{dt} \right\rangle + \left\langle \lambda_{Hn}, w_{hN} - \frac{d\tilde{u}_{hNn}}{dt} \right\rangle \quad \forall \mathbf{w}_h \in V_h,$$

$$\langle \pi_H - \lambda_{Hn}, u_{hNn} \rangle \geq 0 \quad \forall \pi_H \in \Pi_H.$$

Therefore, we get

$$\int_0^T \left(a\left(\mathbf{u}_{hn}(s), \mathbf{w}_h(s) - \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(s)\right) + j(\lambda_{Hn}(s), \mathbf{w}_h(s)) - j\left(\lambda_{Hn}(s), \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(s)\right)\right) ds$$
$$\geq \int_0^T \left(\left(\boldsymbol{\phi}_n(s), \mathbf{w}_h(s) - \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(s)\right) + \left\langle \boldsymbol{\psi}_n(s), \mathbf{w}_h(s) - \frac{d\tilde{\mathbf{u}}_{hn}}{dt}(s)\right\rangle \right) ds$$
$$+ \int_0^T \left\langle \lambda_{Hn}(s), w_{hN}(s) - \frac{d\tilde{u}_{hNn}}{dt}(s)\right\rangle ds \quad \forall \mathbf{w}_h \in L^2(0,T;V_h),$$
$$\int_0^T \langle \pi_H(s) - \lambda_{Hn}(s), u_{hNn}(s)\rangle \, ds \geq 0 \ \ \forall \pi_H \in L^2(0,T;L_H)$$
$$\text{with } \pi_H \in \Pi_H \text{ a.e. on }]0,T[.$$

In order to pass to the limit by using the relations (5.4)–(5.10), we introduce $\mathbf{w}_h(s) = \mathbf{v}_h$ for $s \in [t, t+\tau]$ and $\mathbf{w}_h(s) = \dot{\mathbf{u}}(s)$ otherwise. Then, using Lebesgue's theorem, one obtains (2.17). In order to prove relation (2.18), we first use (5.11), and we set $\pi_H(s) = \pi_H$ for $s \in [t, t+\tau]$ and $\pi_H(s) = \lambda_H$ elsewhere.

Relation (5.1) is a consequence of (4.12). □

THEOREM 5.2. *Let $(\mathcal{T}_{h_i})_{i\in\mathbb{N}}$ and $(\mathcal{T}'_{H_i})_{i\in\mathbb{N}}$ be such that for all $i \in \mathbb{N}$ we have $\tau_1 \leq h_i/H_i \leq \tau_2$. Let $\mu \in \mathcal{M}(H^{1/2}(\Gamma))$ be such that $\|\mu\|_{L^\infty(\Gamma)} < \mu^*$ and $\|\mu\|_{\mathcal{M}(H^{1/2}(\Gamma))} < \tilde{\mu}$. Then there exists a subsequence $(i_k)_{k\in\mathbb{N}}$ such that $(\mathbf{u}_{h_{i_k}}, \lambda_{H_{i_k}})_{k\in\mathbb{N}}$ converges weakly towards the $(\mathbf{u}, \lambda = \theta\sigma_N(\mathbf{u}))$ solution of (2.10), (2.11). Moreover, we have*

$$\|\mathbf{u}_{h_{i_k}} - \mathbf{u}\|_{[H^1(\Omega)]^d} \to 0, \tag{5.12}$$
$$\|\lambda_{H_{i_k}} - \lambda\|_{H^{-1/2}(\Gamma)} \to 0. \tag{5.13}$$

Proof. For all $i \in \mathbb{N}$ we have

$$\|\mathbf{u}_{h_i}\|_{W^{1,2}(0,T;[H^1(\Omega)]^d)} \leq \mathcal{C}_7\Big(\|\boldsymbol{\phi}\|_{W^{1,2}(0,T;[L^2(\Omega)]^d)} + \|\boldsymbol{\psi}\|_{W^{1,2}(0,T;[H^{-1/2}(\Gamma)]^d)}\Big),$$
$$\|\lambda_{H_i}\|_{W^{1,2}(0,T;H^{-1/2}(\Gamma))} \leq \mathcal{C}_7\Big(\|\boldsymbol{\phi}\|_{W^{1,2}(0,T;[L^2(\Omega)]^d)} + \|\boldsymbol{\psi}\|_{W^{1,2}(0,T;[H^{-1/2}(\Gamma)]^d)}\Big).$$

It follows that there exists a constant $\mathcal{C}_8 > 0$ such that

$$\|\mathbf{u}_{h_i}(t)\|_{[H^1(\Omega)]^d} \leq \mathcal{C}_8 \ \ \forall t \in [0,T],$$
$$\|\lambda_{H_i}(t)\|_{H^{-1/2}(\Gamma)} \leq \mathcal{C}_8 \ \ \forall t \in [0,T].$$

Then there exist $(\mathbf{u}, \lambda) \in W^{1,2}(0,T;[H^1(\Omega)]^d) \times W^{1,2}(0,T;H^{-1/2}(\Gamma))$ and a subsequence $(h_{i_k})_{k\in\mathbb{N}}$ such that

$$\begin{array}{rcll} \mathbf{u}_{h_{i_k}} & \rightharpoonup & \mathbf{u} & \text{in } [H^1(\Omega)]^d \ \forall t \in [0,T], \\ \lambda_{H_{i_k}} & \to & \lambda & \text{in } H^{-1/2}(\Gamma) \ \forall t \in [0,T], \\ \mathbf{u}_{h_{i_k}} & \rightharpoonup & \mathbf{u} & \text{in } W^{1,2}(0,T;[H^1(\Omega)]^d), \\ \lambda_{H_{i_k}} & \rightharpoonup & \lambda & \text{in } W^{1,2}(0,T;H^{-1/2}(\Gamma)). \end{array}$$

For the sake of simplicity, we still denote by $(\mathbf{u}_{h_i}, \lambda_{H_i})_{i\in\mathbb{N}}$ these subsequences. Then the following relations hold:

$$\liminf_{i\to+\infty} \int_0^T a(\mathbf{u}_{h_i}(s), \dot{\mathbf{u}}_{h_i}(s)) ds \geq \frac{1}{2}\Big(\liminf_{i\to+\infty} a(\mathbf{u}_{h_i}(T), \mathbf{u}_{h_i}(T)) - \lim_{i\to+\infty} a(\mathbf{u}_{h_i}(0), \mathbf{u}_{h_i}(0))\Big)$$
$$\geq \frac{1}{2}\Big(a(\mathbf{u}(T), \mathbf{u}(T)) - a(\mathbf{u}(0), \mathbf{u}(0))\Big)$$
$$= \int_0^T a(\mathbf{u}(s), \dot{\mathbf{u}}(s)) ds,$$

$$\liminf_{i\to+\infty}\int_0^T j(\lambda_{H_i},\dot{\mathbf{u}}_{h_i}(s))ds \geq \liminf_{i\to+\infty}\int_0^T j(\lambda_{H_i}-\lambda,\dot{\mathbf{u}}_{h_i}(s))ds$$
$$+\liminf_{i\to+\infty}\int_0^T j(\lambda,\dot{\mathbf{u}}_{h_i}(s))ds \geq \int_0^T j(\lambda,\dot{\mathbf{u}}(s))ds.$$

If we set $\mathbf{v}_{h_i}=r_{h_i}(\mathbf{v})$ with $\mathbf{v}\in L^2(0,T;V)$, it follows that

$$\lim_{i\to+\infty}\int_0^T j(\lambda_{H_i},\mathbf{v}_{h_i}(s))ds=\int_0^T j(\lambda,\mathbf{v}(s))ds,$$
$$\lim_{i\to+\infty}\int_0^T a(\mathbf{u}_{h_i}(s),\mathbf{v}_{h_i}(s))ds\geq\int_0^T a(\mathbf{u}(s),\mathbf{v}(s))ds,$$
$$\lim_{i\to+\infty}\int_0^T (\boldsymbol{\phi},\mathbf{v}_{h_i}(s)-\dot{\mathbf{u}}_{h_i}(s))ds=\int_0^T(\boldsymbol{\phi},\mathbf{v}(s)-\dot{\mathbf{u}}(s))ds,$$
$$\lim_{i\to+\infty}\int_0^T \langle\boldsymbol{\psi},\mathbf{v}_{h_i}(s)-\dot{\mathbf{u}}_{h_i}(s)\rangle ds=\int_0^T\langle\boldsymbol{\psi},\mathbf{v}(s)-\dot{\mathbf{u}}(s)\rangle ds.$$

If we set $\pi_{H_i}=R_{H_i}(\overline{\pi})$ with $\overline{\pi}\in L^2(0,T;H^{-1/2}(\Gamma))$ such that $\overline{\pi}\in C^{*-}$ a.e. on $]0,T[$, then we have

$$\lim_{i\to+\infty}\int_0^T\langle\pi_{H_i}-\lambda_{H_i},u_{h_iN}(s)\rangle ds=\lim_{i\to+\infty}\int_0^T\langle\pi_{H_i}-\lambda,u_{h_iN}(s)\rangle ds$$
$$+\lim_{i\to+\infty}\int_0^T\langle\lambda-\lambda_{H_i},u_{h_iN}(s)\rangle ds=\int_0^T\langle\pi-\lambda,u_N(s)\rangle ds.$$

Therefore, we obtain

$$\int_0^T\left(a\left(\mathbf{u}(s),\mathbf{v}(s)-\frac{d\tilde{\mathbf{u}}}{dt}(s)\right)+j(\lambda,\mathbf{v}(s))-j\left(\lambda,\frac{d\tilde{\mathbf{u}}}{dt}(s)\right)\right)ds$$
$$\geq\int_0^T\left(\left(\boldsymbol{\phi}(s),\mathbf{v}(s)-\frac{d\tilde{\mathbf{u}}}{dt}(s)\right)+\left\langle\boldsymbol{\psi}(s),\mathbf{v}(s)-\frac{d\tilde{u}}{dt}(s)\right\rangle\right)ds$$
$$+\int_0^T\left\langle\lambda,v_N(s)-\frac{d\tilde{u}_N}{dt}(s)\right\rangle ds\quad\forall\,\mathbf{v}\in L^2(0,T;V),$$
$$\int_0^T\langle\overline{\pi}-\lambda,u_N(s)\rangle ds\ \geq 0\ \ \forall\overline{\pi}\in L^2(0,T;H^{-1/2}(\Gamma))\text{ such that }\overline{\pi}\in C^{*-}\text{ a.e. on }]0,T[.$$

Next we set $\mathbf{v}(s)=\mathbf{w}$ for $s\in[t,t+\tau]$ and $\mathbf{w}(s)=\dot{\mathbf{u}}(s)$ otherwise, $\overline{\pi}(s)=\pi$ for $s\in[t,t+\tau]$, and $\pi(s)=\lambda$ elsewhere. We pass to the limit with respect to τ using Lebesgue's theorem so that we obtain (2.10), (2.11), which is equivalent to (2.7), (2.8). Thus we have $\lambda=\theta\sigma_N(\mathbf{u})\in C^{*-}$.

In order to prove (5.12) we proceed as follows. We set $\mathbf{v}=\mathbf{0}$ and $\mathbf{v}=2\dot{\mathbf{u}}$ in (2.10), which implies that

$$a(\mathbf{u},\dot{\mathbf{u}})+j(\mathbf{u},\dot{\mathbf{u}})=(\boldsymbol{\phi},\dot{\mathbf{u}})+\langle\boldsymbol{\psi},\dot{\mathbf{u}}\rangle+\langle\lambda,\dot{u}_N\rangle.$$

Then we set $\mathbf{v}_h=r_{h_i}(\dot{\mathbf{u}})$ in (2.17) corresponding to h_i,H_i so that we obtain

$$a(\mathbf{u}_{h_i},\dot{\mathbf{u}}_{h_i})\leq a(\mathbf{u}_{h_i},r_{h_i}(\dot{\mathbf{u}}))+j(\lambda_{H_i},r_{h_i}(\dot{\mathbf{u}}))-j(\lambda_{H_i},\dot{\mathbf{u}}_{h_i})+(\boldsymbol{\phi},r_{h_i}(\dot{\mathbf{u}})-\dot{\mathbf{u}}_{h_i})$$
$$+\langle\boldsymbol{\psi},r_{h_i}(\dot{\mathbf{u}})-\dot{\mathbf{u}}_{h_i}\rangle+\langle\lambda_{H_i},(r_{h_i}(\dot{\mathbf{u}}))_N-\dot{u}_{h_iN}\rangle.$$

Thus for all $t \in]0,T[$ we get

$$\limsup_{i\to+\infty} \int_0^t a(\mathbf{u}_{h_i}(s), \dot{\mathbf{u}}_{h_i}(s))ds \leq \int_0^t a(\mathbf{u}(s), \dot{\mathbf{u}}(s))ds.$$

We conclude that

$$\lim_{i\to+\infty} \int_0^t a(\mathbf{u}_{h_i}(s), \dot{\mathbf{u}}_{h_i}(s))ds = \int_0^t a(\mathbf{u}(s), \dot{\mathbf{u}}(s))ds,$$

which implies

$$\lim_{i\to+\infty} a(\mathbf{u}_{h_i}(t), \mathbf{u}_{h_i}(t)) = a(\mathbf{u}(t), \mathbf{u}(t)).$$

Finally, we have that for all $t \in [0,T]$

$$\begin{aligned}\|\mathbf{u}_{h_i}(t) - \mathbf{u}(t)\|^2_{[H^1(\Omega)]^d} &\leq \frac{1}{m}\Big(a(\mathbf{u}_{h_i}(t) - \mathbf{u}(t), \mathbf{u}_{h_i}(t) - \mathbf{u}(t))\Big) \\ &\leq \frac{1}{m}\Big(a(\mathbf{u}_{h_i}(t), \mathbf{u}_{h_i}(t)) + a(\mathbf{u}, \mathbf{u}) - 2a(\mathbf{u}_{h_i}(t), \mathbf{u}(t))\Big).\end{aligned}$$

Passing to the limit, one obtains (5.12) and the proof is complete. □

6. Conclusions. In this paper we have considered an implicit Euler scheme with respect to time and a mixed finite element method for the space discretization. The fully discrete problem is solved by a fixed point approach, and a regularity result on the whole contact zone is established for the normal component of the stress vector. This result and some estimates independent of the discretization parameters enable us to pass to the limit with respect to mesh size and time. To our knowledge, this paper presents the first convergence results for quasi-static unilateral contact problems with local Coulomb friction.

It would be interesting to consider the numerical analysis of unilateral contact problems with local friction for curved contact zone.

We have used an error estimate between the solutions of the fully discrete problem and a kind of "static problem" with given threshold of sliding. The problem of obtaining rates of convergence for the quasi-static contact problem remains open.

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