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An application of Sparre Andersen's fluctuation theorem for exchangeable and sign-invariant random variables

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Abstract

We revisit here a famous result by Sparre Andersen on persistence probabilities $\mathbf{P}(S_k \geq 0 \forall 0 \leq k \leq n)$ for symmetric random walks $(S_n)_{n \geq 0}$. We give a short proof of this result when considering sums of random variables that are only assumed *exchangeable* and *sign-invariant*. We then apply this result to the study of persistence probabilities of (symmetric) additive functionals of Markov chains, which can be seen as a natural generalization of integrated random walks.

Keywords: Randoms walks; exchangeability; fluctuation theory; persistence problem; additive functionals, Markov chains.

MSC2020 AMS classification: 60G50; 60J55; 60G09; 60J10.

1 Introduction

We discuss in this paper several results on persistence problems for symmetric random walks, in particular the well known (but nonetheless striking) fluctuation theorem due to Sparre Andersen [21]. To introduce the result, let ξ_1, \dots, ξ_n be real random variables and let us denote $S_0 = 0$, and $S_k = \sum_{i=1}^k \xi_i$ for $1 \leq k \leq n$. Then, one is interested in estimating the persistence probabilities

$$\mathbf{P}(S_k > 0 \text{ for all } 1 \leq k \leq n) \quad \text{or} \quad \mathbf{P}(S_k \geq 0 \text{ for all } 1 \leq k \leq n).$$

Sparre Andersen's result [21] states that if the $(\xi_i)_{1 \leq i \leq n}$ are i.i.d., symmetric and have no atom, then these probabilities do not depend on the law of ξ .

Theorem A. *If the $(\xi_i)_{1 \leq i \leq n}$ are i.i.d. and if the distribution of ξ_i is symmetric and has no atom, then we have*

$$\mathbf{P}(S_k > 0 \text{ for all } 1 \leq k \leq n) = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{4^n} \binom{2n}{n}.$$

In fact, [21] proves a number of results for exchangeable random variables, but the independence plays an important role in obtaining fluctuation's results, such as arcsine laws or persistence probabilities. Chapter XII in Feller's book [11] provides a streamlined approach to Sparre Andersen's result, using a duality argument together with a (purely combinatorial) cyclic lemma, see [11,

XII.6]: these two ingredients only require the law of (X_1, \dots, X_n) to be exchangeable. Independence is required in the last step of the proof, in order to obtain that the ladder epochs defined iteratively by $T_0^+ = 0$ and $T_k^+ = \min\{n \geq T_{k-1}^+, S_n > \max_{0 \leq j < n} \{S_j\}\}$ form a renewal sequence. The following is then deduced, see [11, XII.7, Thm. 1]:

Theorem B. *If the $(\xi_i)_{i \geq 1}$ are i.i.d., then $1 - \mathbf{E}[s^{T_1^+}] = \exp\left(-\sum_{n=1}^{\infty} \frac{s^n}{n} \mathbf{P}(S_n > 0)\right)$ for any $s \in [0, 1)$.*

Theorem B can be seen as a particular case of a Wiener–Hopf factorization, also known as Spitzer–Baxter formula, which gives the joint Laplace transform/characteristic function of the first ladder epoch and ladder height, see [11, XII.9]. Note that if the law of X_i is symmetric and has no atom, one has $\mathbf{P}(S_n > 0) = \frac{1}{2}$ for all n , so the generating function of T_1^+ is given by $\mathbf{E}[s^{T_1^+}] = 1 - \sqrt{1 - s}$.

Let us mention that another proof of Theorem A is presented in [9, Prop. 1.3]: the proof is remarkably simple and elegant and relies on a decomposition with respect to the first time that $(S_k)_{k \geq 0}$ hits its minimum $\min_{1 \leq j \leq n} \{S_j\}$, using the independence of the $(\xi_i)_{1 \leq i \leq n}$. The goal of our paper is to present a version of Theorem A valid for exchangeable and sign-invariant random variables $(\xi_i)_{1 \leq i \leq n}$, going beyond the independent and symmetric setting. Sparre Andersen was already aware of this result, see [21, Thm. 4] (its proof is however a bit laconic), but we give here a short and self-contained proof (with no combinatorial lemma), taking inspiration from [9]. We then present some application to persistence probabilities for symmetric additive functionals of birth-death chains, giving in particular a simple proof of Sinai’s result [20] on the integrated simple random walk.

2 The case of exchangeable and sign-invariant random variables

Below, we consider examples in which there is no independence: one is however still able to obtain the same conclusion as in Theorem A, using some weaker but natural assumption. Let us introduce some terminology: the law \mathbf{P} of (ξ_1, \dots, ξ_n) is said to be:

- (E) exchangeable if for any permutation $\sigma \in \mathfrak{S}_n$, $(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)})$ has the same distribution as (ξ_1, \dots, ξ_n) ;
- (S) sign-invariant if for any $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, $(\varepsilon_1 \xi_1, \dots, \varepsilon_n \xi_n)$ has the same distribution as (ξ_1, \dots, ξ_n) .

One can think about these two assumptions as an invariance of \mathbf{P} under the action of the groups: of permutations \mathfrak{S}_n for condition (E); of sign changes $\{-1, 1\}^n$ for condition (S).

The following result is actually already mentioned in [21, Thm. 4]: the present paper is meant as a way to put more emphasis on this result; we then provide some applications.

Theorem 2.1. *Assume that (ξ_1, \dots, ξ_n) satisfies conditions (E)-(S). Then, if \mathbf{P} has no atom,*

$$\mathbf{P}(S_k > 0 \text{ for all } 1 \leq k \leq n) = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{4^n} \binom{2n}{n}.$$

In general, if \mathbf{P} has atoms, we have

$$\mathbf{P}(S_k > 0 \text{ for all } 1 \leq k \leq n) \leq \frac{(2n-1)!!}{(2n)!!} = \frac{1}{4^n} \binom{2n}{n} \leq \mathbf{P}(S_k \geq 0 \text{ for all } 1 \leq k \leq n).$$

One can easily see that both conditions (E)-(S) are necessary to obtain the statement of Theorem 2.1, see for instance Section 4.2 below.

Remark 2.2 (About sign-invariance). *The sign-invariance of (ξ_1, \dots, ξ_n) can be seen as a strong form of symmetry for the law of (ξ_1, \dots, ξ_n) . Let us mention the article [6] which derives some properties of sign-invariant sequences: in particular, [6, Lem. 1.2] tells that sign-invariant (ξ_1, \dots, ξ_n) are independent conditionally on $(|\xi_1|, \dots, |\xi_n|)$. In other words, any exchangeable and sign-invariant law can be obtained by taking a random vector (Z_1, \dots, Z_n) (that can be constrained to have $0 \leq Z_1 \leq \dots \leq Z_n$) and by shuffling and changing the signs of its coordinates randomly, that is setting $\xi_i = \varepsilon_i Z_{\sigma(i)}$ for $1 \leq i \leq n$, where the $(\varepsilon_i)_{1 \leq i \leq n}$ are i.i.d. signs (i.e. such that $\mathbf{P}(\varepsilon_i = \pm 1) = \frac{1}{2}$) and σ is a random uniform permutation of $\{1, \dots, n\}$ (i.e. such that $\mathbf{P}(\sigma = \nu) = \frac{1}{n!}$ for all $\nu \in \mathfrak{S}_n$).*

Remark 2.3 (Finite vs. infinite sequences). *The law of an infinite sequence $(\xi_i)_{i \geq 1}$ of random variables is called exchangeable, resp. sign-invariant, if for any $n \geq 1$ the law of (ξ_1, \dots, ξ_n) is exchangeable, resp. sign-invariant. By de Finetti's theorem, one knows that the law of $\xi = (\xi_i)_{i \geq 1}$ is exchangeable if and only if it is conditionally i.i.d.: in other words, there exists a random probability distribution μ such that $\mathbf{P}(\xi \in \cdot \mid \mu) = \mu^{\otimes \infty}$, where $\mu^{\otimes \infty}$ denotes the law of an infinite sequence of i.i.d. random variables with law μ . In [5, Thm. 1], it is shown that an exchangeable sequence $(\xi_i)_{i \geq 1}$ is sign-invariant if and only if μ is almost surely symmetric. From this, it is easy to derive Theorem 2.1 for an exchangeable and sign-invariant sequence $(\xi_i)_{i \geq 1}$, simply by working conditionally on the realization of μ . Hence, the truly remarkable fact about Theorem 2.1 is that it holds for a finite exchangeable and sign-invariant (ξ_1, \dots, ξ_n) .*

3 Persistence probabilities of f -integrated birth-death chains

Let us consider $(X_n)_{n \geq 0}$ a birth and death Markov chain, starting from $X_0 = 0$: it is a Markov chain on \mathbb{Z} such that $|X_n - X_{n-1}| \leq 1$, with transition probabilities

$$p_x = p(x, x+1), \quad q_x = p(x, x-1), \quad r_x = p(x, x) = 1 - p_x - q_x \quad \text{for } x \geq 1.$$

We assume that $(X_n)_{n \geq 0}$ is symmetric, in the sense that $(-X_n)_{n \geq 0}$ has the same distribution as $(X_n)_{n \geq 0}$; in other words, the transition probabilities verify $p(x, y) = p(-x, -y)$ for any $x, y \in \mathbb{Z}$. For $x = 0$, we also set $p_0 = p(0, 1) \in (0, \frac{1}{2}]$, $q_0 = p(0, -1) = p_0$ and $r_0 = p(0, 0) = 1 - 2p_0$.

Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be any anti-symmetric function which preserves the sign of x , i.e. such that $f(x) > 0$ for $x > 0$ and $f(x) < 0$ for $x < 0$ (and naturally $f(0) = 0$). Then we define the f -integrated Markov chain, or *additive functional*, as follows: $\zeta_0 = 0$ and, for $n \geq 1$,

$$\zeta_n := \sum_{i=1}^n f(X_i). \tag{3.1}$$

We are now interested in the persistence (or survival) probabilities

$$\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \quad \text{or} \quad \mathbf{P}(\zeta_k \geq 0 \text{ for all } 1 \leq k \leq n).$$

A classical, well-studied example is when $(X_n)_{n \geq 0}$ is the simple symmetric random walk and f is the identity: then $(\zeta_n)_{n \geq 0}$ is the integrated random walk and the persistence probabilities are known to be of order $n^{-1/4}$, see [20]. We prove the following result.

Theorem 3.1. *Let $(X_n)_{n \geq 0}$ be a symmetric recurrent birth-death chain and let f be an anti-symmetric function which preserves the sign of x . Let $p := \mathbf{P}(X_1 \geq 0) = 1 - p_0$ and let $g(m) := \frac{1}{4^m} \binom{2m}{m}$, with by convention $g(m) = 0$ if $m < 0$. Then we have for all $n \geq 1$*

$$\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \leq \mathbf{E}[g(L_n)] \leq p^{-1} \mathbf{P}(\zeta_k \geq 0 \text{ for all } 1 \leq k \leq n),$$

where $L_n := \sum_{k=1}^n \mathbf{1}_{\{X_k=0\}}$ is the local time of the chain at 0 up to time n . We also have

$$\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \geq p(1-p) \mathbf{E}[g(L_n - 1) \mathbf{1}_{\{L_n \geq 1\}}].$$

Let us mention that our approach is based on an excursion decomposition of the process $(X_n)_{n \geq 0}$. Our results would still be valid for Markov chains that “cannot jump above 0”, *i.e.* such that $\mathbf{P}(X_n \leq 0 \mid X_{n-1} = x) = \mathbf{P}(X_n = 0 \mid X_{n-1} = x)$ for any $x \in \mathbb{N}$. We have chosen to stick with the birth-death setting since it is the most natural example of such chains (and already contains a wide class of behaviors).

In the Theorem 3.1 above, since we have $g(m) = \frac{1}{4^m} \binom{2m}{m} \sim (\pi m)^{-1/2}$ as $m \rightarrow \infty$ and since the chain is assumed to be recurrent (which implies that $L_n \rightarrow \infty$ a.s.), it is tempting to replace the term $\mathbf{E}[g(L_n)]$ by $(1 + o(1))\pi^{-1/2} \mathbf{E}[(1 + L_n)^{-1/2}]$. However, one needs a bit of caution: let us give a sufficient condition (often verified in practice) to obtain such a result¹.

Lemma 3.2. *Assume that:*

- (i) *there is a sequence $(b_n)_{n \geq 1}$, $\lim_{n \rightarrow \infty} b_n = +\infty$, such that $(\frac{1}{b_n} L_n)_{n \geq 1}$ converges in distribution to a random variable \mathcal{X} ;*
- (ii) *for all $t > 0$ we have $\sup_{n \geq 1} \mathbf{P}(\frac{1}{b_n}(1 + L_n) \leq t) \leq Ch(t)$, for some constant $C > 0$ and some non-decreasing function h that verifies $\int_0^\infty h(u)u^{-3/2} du < +\infty$.*

Then the random variable \mathcal{X} verifies $\mathbf{E}[\mathcal{X}^{-1/2}] < +\infty$ and, as $n \rightarrow \infty$,

$$\mathbf{E}[g(L_n - 1) \mathbf{1}_{\{L_n \geq 1\}}] \sim \mathbf{E}[g(L_n)] \sim \frac{1}{\sqrt{\pi}} \mathbf{E}[(1 + L_n)^{-1/2}] \sim \frac{1}{\sqrt{\pi b_n}} \mathbf{E}[\mathcal{X}^{-1/2}].$$

If $(X_n)_{n \geq 0}$ is positive recurrent, then $\mathbf{E}[g(L_n)] \sim \frac{\sqrt{\mathbf{E}[\tau_1]}}{\sqrt{\pi n}}$, where $\tau_1 := \inf\{n \geq 1, X_n = 0\}$.

Example 3.3. *We give below a class of example where Lemma 3.2 can be applied, but let us comment on the case of the simple symmetric random walk $(X_n)_{n \geq 0}$. In that case, Lemma 3.2 is verified with $b_n = \sqrt{n/2}$ and $\mathcal{X} = |Z|$, $Z \sim \mathcal{N}(0, 1)$; and with the function $h(u) = \min(u, 1)$. We then get that the persistence probabilities for the integrated random walk verify*

$$(1 + o(1)) \frac{1}{4} c_0 n^{-1/4} \leq \mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \leq (1 + o(1)) c_0 n^{-1/4}, \quad (3.2)$$

with $c_0 := \pi^{-1/2} 2^{1/4} \mathbf{E}[|Z|^{-1/2}] = \pi^{-1} \Gamma(1/4)$. This recovers a result by Sinai [20] (in the case $f(x) = x$) and makes the constants explicit.

Remark 3.4. *We stress here that the asymptotic bounds that we obtain combining Theorem 3.1 and Lemma 3.2 do not depend on the (anti-symmetric) function f in the definition (3.1) of ζ_n . In particular, the bounds (3.2) obtained in the case of the simple random walk (and in Proposition 3.5 below for Bessel-like random walks) are valid for any anti-symmetric function f .*

¹For non-asymptotic bounds, one could also use the following: for $m \geq 1$, $e^{-\frac{1}{6m}} \leq \sqrt{\pi m} g(m) \leq e^{-\frac{1}{24m}}$.

A class of examples: Bessel-like random walks

Let us give more precise result in the case of symmetric Bessel-like random walk, see [1, 13, 16]. This is a class of birth-death chains that includes the simple symmetric random walk. They have the following transition probabilities: for $x \geq 1$

$$p_x := p(x, x+1) = \frac{1}{2} \left(1 - \frac{\delta + \varepsilon_x}{2x} \right), \quad p(x, x-1) = 1 - p_x, \quad (3.3)$$

where $\delta \in \mathbb{R}$ and ε_x is such that $\lim_{x \rightarrow \infty} \varepsilon_x = 0$; we take $p(-x, -x-1) = p(x, x+1)$ and $p(0, 1) = p(0, -1) = \frac{1}{2}$, for symmetry reasons. We also assume uniform ellipticity, *i.e.* there is some $\eta > 0$ such that $p_x \in [\eta, 1 - \eta]$ for all $x \in \mathbb{Z}$. The parameter δ is called the *drift parameter* and we have the following behavior: the walk $(X_n)_{n \geq 0}$ is transient if $\delta < -1$, recurrent if $\delta > -1$, positive recurrent if $\delta > 1$; in the cases $\delta = -1$ and $\delta = 1$, the behavior depends on the function ε_x .

More precisely, letting $\lambda_x = \prod_{k=1}^x \frac{p_x}{1-p_x}$, the random walk $(X_n)_{n \geq 1}$ is recurrent if and only if $\sum_{x=1}^{\infty} \lambda_x = +\infty$. Moreover, with the notation (3.3), there is a constant $K_0 > 0$ such that

$$\lambda_x = \prod_{k=1}^x \frac{1-p_k}{p_k} \sim K_0 x^\delta \ell(x)^{-1} \quad \text{as } x \rightarrow \infty, \quad \text{where } \ell(x) := \exp \left(\sum_{k=1}^x \frac{\varepsilon_k}{k} \right). \quad (3.4)$$

Note that ℓ is a slowly varying function.

Then, in [1], the sharp tail of the distribution of $\tau_1 := \inf\{n \geq 1, X_n = 0\}$ is derived: we reported the result in Section 5.3 below, see (5.1). This enables us to verify that L_n satisfies the assumptions of Lemma 3.2 and to show the following result. We focus on the null-recurrent case, since the positive recurrent case is treated in Lemma 3.2.

Proposition 3.5. *Assume that $(X_n)_{n \geq 0}$ is null-recurrent (necessarily $\delta \in [-1, 1]$). Then let us set $\kappa := \frac{1+\delta}{2} \geq 0$ and define*

$$b_n := \begin{cases} n\mu(n)^{-1} & \text{if } \delta = 1, \\ \frac{K_0 \Gamma(\kappa)}{2^{1-\kappa} \Gamma(1-\kappa)} n^\kappa \ell(\sqrt{n})^{-1} & \text{if } \delta \in (-1, 1), \\ K_0^{-1} \nu(n)^{-1} & \text{if } \delta = -1, \end{cases} \quad \text{with} \quad \begin{aligned} \mu(n) &:= \sum_{k=1}^n \mathbf{P}(\tau_1 \geq k), \\ \nu(n) &:= \sum_{x \leq n, x \text{ even}} \frac{1}{x \ell(\sqrt{x})}. \end{aligned} \quad (3.5)$$

In the case $\delta = 1$, $\mu(n)$ is slowly varying (and $\lim_{n \rightarrow \infty} \mu(n) = \mathbf{E}[\tau_1]$); in the case $\delta = -1$, $\nu(n)$ is slowly varying and the recurrence is equivalent to having $\lim_{n \rightarrow \infty} \nu(n) = +\infty$.

We then have that

$$(1 + o(1)) \frac{1}{4} c_\kappa (b_n)^{-1/2} \leq \mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \leq (1 + o(1)) c_\kappa (b_n)^{-1/2},$$

with: $c_\kappa = 1$ if $\delta = 1$; $c_\kappa = \mathbf{E}[\mathcal{Z}^{\kappa/2}]$ if $\delta \in (-1, 1)$, where \mathcal{Z} is a one-sided κ -stable random variable of Laplace transform e^{-t^κ} ; $c_\kappa = \sqrt{\pi}$ if $\delta = -1$.

Further comments and comparison with the literature

Let us stress that there are several directions in which one could extend our results. First, one could consider more general underlying random walks or Markov chains, for instance random walks with non-necessarily symmetric increments or Markov chains that can “jump over 0”. One could also consider more general functions f , not necessarily symmetric. Another room for improvement is to obtain sharp asymptotics for the persistence probabilities, *i.e.* for instance finding the correct constant \widehat{c}_κ such that $\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \sim \widehat{c}_\kappa b_n^{-1/2}$ in Proposition 3.5.

A big part of the literature has considered the case of integrated random walks, that is considering the Markov chain $X_n = \sum_{k=1}^n \xi_k$ with $(\xi_k)_{k \geq 0}$ i.i.d. random variables and $\zeta_n = \sum_{i=1}^n X_i$ (*i.e.* taking $f(x) = x$), starting with the work of Sinai [20]. Under the condition that the ξ_k 's are centered with a finite second moment, the persistence probability $\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n)$ has been proven to be of order $n^{-1/4}$ in [9], and the sharp asymptotic $\sim c_1 n^{-1/4}$ in [10]. The case where ξ_k does not have a finite second moment remains mostly open, except in some specific (one-sided) cases, see e.g. [9, 22]. Our result can be seen as an extension to persistence problems for additive functionals of Markov chains $(X_n)_{n \geq 0}$; with the major restriction of symmetry and of the fact that the underlying Markov chain cannot jump above 0 (but with the advantage of having an elementary proof).

Another line of works considered persistence problems for additive functionals of *continuous-time* Markov processes $(X_t)_{t \geq 0}$. Let us mention [12] that considered the case of an integrated Brownian motion, and [14, 19] who considered the f -integral of a Brownian motion or a skew-Bessel process respectively, for the (possibly asymmetric) functional $f(x) = |x|^\gamma (c_+ \mathbf{1}_{\{x > 0\}} - c_- \mathbf{1}_{\{x < 0\}})$ for some $\gamma > -1$. More recently, in [4], we have pushed further the existing techniques (based on a Wiener–Hopf decomposition of a bi-variate Lévy process associated to the problem): we obtained the sharp asymptotics for persistence probabilities for a wide class of Markov processes (including one-dimensional generalized diffusions, see [15]) and of functions f . In particular, [4, Example 6] shows that the result applies to *continuous-time* birth-death processes, giving the existence of the constant \widehat{c}_κ mentioned above. The present article can be seen as a elementary approach to obtaining a sub-optimal result.

4 Exchangeable and symmetric sequences: proof of Theorem 2.1

Let us introduce some notation. For $n \in \mathbb{N}$ and for a fixed $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we define an exchangeable and symmetric vector $\xi = (\xi_1, \dots, \xi_n)$ with law denoted $\mathbf{P}^{(x)}$ by permuting the coordinates of (x_1, \dots, x_n) by a random uniform permutation and by changing the signs of the coordinates uniformly at random. More precisely, let $(\varepsilon_i)_{1 \leq i \leq n}$ be i.i.d. random variables with law $\mathbf{P}(\varepsilon_1 = 1) = \mathbf{P}(\varepsilon_1 = -1) = \frac{1}{2}$ and let σ be a random permutation of $\{1, \dots, n\}$ with uniform distribution $\mathbf{P}(\sigma = \nu) = \frac{1}{n!}$ for all $\nu \in \mathfrak{S}_n$, independent of $(\varepsilon_i)_{1 \leq i \leq n}$: we then define

$$(\xi_1, \dots, \xi_n) := (\varepsilon_1 x_{\sigma(1)}, \dots, \varepsilon_n x_{\sigma(n)}). \quad (4.1)$$

Note that we can restrict to the case where $x_i \in \mathbb{R}_+$ for all $1 \leq i \leq n$.

We then construct the random walk $S_k = \sum_{i=1}^k \xi_i$ for any $0 \leq k \leq n$, and we are interested in

the persistence probabilities

$$p_n(x) := \mathbf{P}^{(x)}(S_k > 0 \text{ for all } 1 \leq k \leq n) \quad \text{and} \quad \bar{p}_n(x) := \mathbf{P}^{(x)}(S_k \geq 0 \text{ for all } 1 \leq k \leq n).$$

We set by convention these probabilities equal to 1 for $n = 0$.

Proposition 4.1. *If $x = (x_1, \dots, x_n)$ is such that*

$$\sum_{i \in I} x_i \neq \sum_{j \in J} x_j \quad \text{for all } I, J \subset \{1, \dots, n\} \text{ with } I \neq J \quad (\text{H})$$

(by convention $\sum_{i \in \emptyset} x_i = 0$), then we have that

$$p_n(x) = \bar{p}_n(x) = \frac{1}{4^n} \binom{2n}{n} = \frac{(2n-1)!!}{(2n)!!} = \prod_{i=1}^n \left(1 - \frac{1}{2i}\right).$$

and in particular it does not depend on x . In general, we have $p_n(x) \leq \frac{(2n-1)!!}{(2n)!!} \leq \bar{p}_n(x)$.

Theorem 2.1 follows directly from Proposition 4.1, noting that the law \mathbf{P} of (ξ_1, \dots, ξ_n) , conditionally on $|\xi| = (|\xi_1|, \dots, |\xi_n|)$, is $\mathbf{P}^{(|\xi|)}$.

4.1 Proof of Proposition 4.1

Our proof is greatly inspired by that of [9, Prop. 1.3]. We start with the first statement: we are going to prove by recurrence on n that for any $x = (x_1, \dots, x_n)$ that verifies the assumption (H), the quantities $p_n(x) = \bar{p}_n(x)$ do not depend on x . The statement is trivial for $n = 1$ since we have $\mathbf{P}^{(x)}(\xi_1 > 0) = \mathbf{P}^{(x)}(\xi_1 \geq 0) = \frac{1}{2}$ if $x \neq 0$, so we directly proceed to the induction step.

Let us fix $n \geq 2$ and some $x = (x_1, \dots, x_n)$ that verifies assumption (H). We now apply a path decomposition used in [9]. Let $W = \min\{k, S_k = \max_{1 \leq i \leq n} S_i\}$: then, for any $\ell \in \{0, \dots, n\}$, we have

$$\begin{aligned} \{W = \ell\} &= \{\xi_\ell > 0, \xi_\ell + \xi_{\ell-1} > 0, \dots, \xi_\ell + \dots + \xi_1 > 0\} \\ &\quad \cap \{\xi_{\ell+1} \leq 0, \xi_{\ell+1} + \xi_{\ell+2} \leq 0, \dots, \xi_{\ell+1} + \dots + \xi_n \leq 0\}. \end{aligned}$$

Now, we have no independence at hand, but we can further decompose over permutations of (x_1, \dots, x_n) with a fixed image $I = \sigma(\{1, \dots, \ell\})$: we get

$$\begin{aligned} \mathbf{P}^{(x)}(W = \ell) &= \sum_{I \subset \{1, \dots, n\}, |I| = \ell} \sum_{\nu_1: \{1, \dots, \ell\} \leftrightarrow I} \sum_{\nu_2: \{\ell+1, \dots, n\} \leftrightarrow I^c} \\ &\frac{1}{n!} \mathbf{P} \left(\sum_{i=0}^j \varepsilon_{\ell-i} x_{\nu_1(\ell-i)} > 0 \text{ for all } 0 \leq j \leq \ell-1, \sum_{i=1}^j \varepsilon_{\ell+i} x_{\nu_2(\ell+i)} \leq 0 \text{ for all } 1 \leq j \leq n-\ell \right), \end{aligned}$$

where $\nu: A \leftrightarrow B$ means that ν is a bijection from A to B . By independence and symmetry of the $(\varepsilon_i)_{1 \leq i \leq n}$, then recombining the sums over the permutations and using the exchangeability, we get that $\mathbf{P}^{(x)}(W = \ell)$ is equal to

$$\sum_{I \subset \{1, \dots, n\}, |I| = \ell} \frac{\ell!(n-\ell)!}{n!} \mathbf{P}^{(x_I)}(S_k > 0 \text{ for all } 1 \leq k \leq \ell) \mathbf{P}^{(x_{I^c})}(S_k \geq 0 \text{ for all } 1 \leq k \leq n-\ell),$$

where we have defined $x_J = (x_i)_{i \in J}$ for any $J \subset \{1, \dots, n\}$. Since both x_I and x_{I^c} verify the assumption (H), we can apply the induction hypothesis: we have that $p_\ell(x_I), p_{n-\ell}(x_{I^c})$ does not depend on x_I, x_{I^c} , for $1 \leq \ell \leq n-1$. Hence, denoting $p_\ell := p_\ell(x_I), p_{n-\ell} := p_{n-\ell}(x_{I^c})$, we end up with $\mathbf{P}^{(x)}(W = \ell) = p_\ell p_{n-\ell}$, for any $\ell \in \{1, \dots, n-1\}$.

Since we have $\mathbf{P}^{(x)}(W = n) = \mathbf{P}^{(x)}(\xi_n > 0, \xi_n + \xi_{n-1} > 0, \dots, \xi_1 > 0) = p_n(x)$ by exchangeability and $\mathbf{P}^{(x)}(W = 0) = \mathbf{P}^{(x)}(S_k \leq 0 \text{ for all } 1 \leq k \leq n) = \bar{p}_n(x)$ by symmetry, we get

$$1 = \sum_{\ell=0}^n \mathbf{P}^{(x)}(W = \ell) = p_n(x) + \sum_{\ell=1}^{n-1} p_\ell p_{n-\ell} + \bar{p}_n(x).$$

Using that $\sum_{i \in I} x_i \neq \sum_{j \in J} x_j$ for all $I, J \subset \{1, \dots, n\}$ with $I \neq J$, we obtain that $p_n(x) = \bar{p}_n(x)$: the above identity shows that $p_n(x) = \bar{p}_n(x) =: p_n$ does not depend on x .

We can now determine the value of p_n , as done in [9]. From the above, $(p_n)_{n \geq 0}$ satisfies the recursive relation $1 = \sum_{\ell=0}^n p_\ell p_{n-\ell}$ for every $n \geq 0$ (recall that $p_0 = 1$). Constructing the generating function, we get that for any $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = \left(\sum_{\ell=0}^{\infty} p_\ell x^\ell \right)^2.$$

Therefore, the generating function of $(p_n)_{n \geq 0}$ is equal to $(1-x)^{-1/2}$, from which one deduces that $p_n = \frac{1}{4^n} \binom{2n}{n} = \frac{(2n-1)!!}{2n!!}$ for all $n \geq 0$.

For the general bounds, for any fixed x , for any $\delta > 0$ fixed, one can choose $y = y(x, \delta)$ such that $x + y$ verifies the assumption (H) and $\sum_{i=1}^n |y_i| \leq \delta$ (take e.g. a typical realization of i.i.d. random variables uniform in $[0, \delta/n]$). Then, we clearly have that

$$\mathbf{P}^{(x+y)}(S_k \geq 0 \text{ for all } 1 \leq k \leq n) \begin{aligned} &\geq \mathbf{P}^{(x)}(S_k \geq \delta \text{ for all } 1 \leq k \leq n), \\ &\leq \mathbf{P}^{(x)}(S_k \geq -\delta \text{ for all } 1 \leq k \leq n). \end{aligned}$$

Since $x + y$ verifies assumption (H), we get that the probability on the left-hand side does not depend on x, y (and is equal to $\frac{(2n-1)!!}{2n!!}$), so

$$\mathbf{P}^{(x)}(S_k \geq \delta \text{ for all } 1 \leq k \leq n) \leq \frac{(2n-1)!!}{2n!!} \leq \mathbf{P}^{(x)}(S_k \geq -\delta \text{ for all } 1 \leq k \leq n).$$

Since δ is arbitrary, letting $\delta \downarrow 0$ concludes the proof. \square

4.2 On the necessity of conditions (E)-(S)

Let us stress that in the construction (4.1) of the exchangeable and symmetric vector (ξ_1, \dots, ξ_n) , the two assumptions are essential:

(a) *The signs $(\varepsilon_i)_{1 \leq i \leq n}$ need to be independent.* As a counter-example, take $(\varepsilon_i)_{1 \leq i \leq n}$ uniform on $\{\omega \in \{-1, 1\}^n, \sum_{i=1}^n \omega_i \in \{2-n, n-2\}\}$, so that ξ is still symmetric. Let $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$ with $x_1 \geq x_2 \geq \dots \geq x_n$ (this is no restriction by definition of ξ). Then to have $S_k > 0$ for all $1 \leq k \leq n$, we need to have $\varepsilon_1 = +1$ (which happens with probability $1/2n$) and then, since all

other signs are $\varepsilon_i = -1$, we need to place x_1 in the first position (which happens with probability $1/n$): we get

$$p_n(x) = \mathbf{P}^{(x)}(S_k > 0 \text{ for all } 1 \leq k \leq n) = \begin{cases} \frac{1}{2n^2} & \text{if } x_1 > \sum_{i=2}^n x_i, \\ 0 & \text{if } x_1 \leq \sum_{i=2}^n x_i. \end{cases}$$

Therefore $p_n(x)$ depends on x .

(b) *The signs $(\varepsilon_i)_{1 \leq i \leq n}$ need to be independent from the permutation σ .* As a counter-example, take $n = 3$ and $x = (x_1, x_2, x_3) \in (\mathbb{R}_+)^3$ with $x_1 > x_2 > x_3 > 0$: then the probability $\mathbf{P}^{(x)}(S_1 > 0, S_2 > 0, S_3 > 0)$ is equal to (one needs to have $\varepsilon_1 = +1$)

$$\begin{aligned} & \mathbf{P}(\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1) + \mathbf{P}(\varepsilon_1 = \varepsilon_2 = 1, \varepsilon_3 = -1, x_{\sigma(3)} < x_{\sigma(1)} + x_{\sigma(2)}) \\ & + \mathbf{P}(\varepsilon_1 = \varepsilon_3 = 1, \varepsilon_2 = -1, x_{\sigma(2)} < x_{\sigma(1)}) + \mathbf{P}(\varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = -1, x_{\sigma(2)} + x_{\sigma(3)} < x_{\sigma(1)}). \end{aligned}$$

Now, if the joint distribution of (ε, σ) is such that $\varepsilon = (\varepsilon_i)_{1 \leq i \leq 3}$ is uniform on $\{-1, 1\}^3$ and $\mathbf{P}(\sigma(3) = 3 \mid \varepsilon_1 = \varepsilon_2 = +1, \varepsilon_3 = -1) = 1$, $\mathbf{P}(\sigma(2) = 3 \mid \varepsilon_1 = \varepsilon_2 = +1, \varepsilon_3 = -1) = 1$ and $\mathbf{P}(\sigma(1) = 1 \mid \varepsilon_1 = +1, \varepsilon_2 = \varepsilon_3 = -1) = 1$, since $x_1 > x_2 > x_3 > 0$, we get that

$$\mathbf{P}^{(x)}(S_1 > 0, S_2 > 0, S_3 > 0) = \frac{1}{4} + \frac{1}{8} \mathbf{1}_{\{x_1 > x_2 + x_3\}},$$

which depends on x . It simply remains to see that the above conditions on the joint distribution of (ε, σ) can be satisfied, which can be checked by hand.

5 Integrated birth and death chains

5.1 Persistence for integrated birth-death chains: Proof of Theorem 3.1

Let us define iteratively $\tau_0 = 0$ and, for $k \geq 1$, $\tau_k = \min\{n > \tau_{k-1}, X_n = 0\}$. Then, we define for $k \geq 1$, the random variable

$$\xi_k = \sum_{i=\tau_{k-1}+1}^{\tau_k} f(X_i),$$

i.e. the contribution of the k -th excursion of $(X_n)_{n \geq 0}$ to the f -integrated Markov chain. Note that by the Markov property, the $(\xi_k)_{k \geq 1}$ are i.i.d. We can therefore write $\zeta_n := \sum_{i=1}^n f(X_i)$ as

$$\zeta_n = \sum_{k=1}^{L_n} \xi_k + W_n, \quad W_n = \sum_{i=\tau_{L_n}+1}^n f(X_i)$$

where we recall that $L_n = \sum_{i=1}^n \mathbf{1}_{\{X_i=0\}}$ is the local time at 0. Since there is no change of sign during an excursion (recall that $(X_n)_{n \geq 0}$ is a birth-death chain), we have that $(\zeta)_{n \geq 0}$ is monotonous on each interval $(\tau_{k-1}, \tau_k]$.

We therefore get the following upper bound: removing the positivity condition on the last segment $(\tau_{L_n}, n]$, we get

$$\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \leq \mathbf{P}\left(\sum_{k=1}^{\ell} \xi_k > 0 \text{ for all } 1 \leq \ell \leq L_n\right).$$

The difficulty here is that the number of terms in the sum is random and that ξ_i and L_n are not independent. If we condition on L_n and on τ_{L_n} , we get that

$$\mathbf{P}\left(\sum_{k=1}^{\ell} \xi_k > 0 \text{ for all } 0 \leq \ell \leq L_n\right) = \mathbf{E}\left[\mathbf{P}\left(\sum_{k=1}^{\ell} \xi_k > 0 \text{ for all } 0 \leq \ell \leq L_n \mid L_n, \tau_{L_n}\right)\right].$$

Then, for any $0 \leq m \leq j \leq n$, conditionally on $\{L_n = m, \tau_m = j\}$, the random variables $(\xi_i)_{1 \leq i \leq m}$ are not independent, but they are easily seen to be exchangeable and sign-invariant (thanks to the Markov property and the fact that the chain $(X_n)_{n \geq 0}$ is symmetric). We can therefore apply Theorem 2.1 to obtain that

$$\mathbf{P}\left(\sum_{k=1}^{\ell} \xi_k > 0 \text{ for all } 0 \leq \ell \leq m \mid L_n = m, \tau_m = j\right) \leq \frac{1}{4^m} \binom{2m}{m} =: g(m).$$

This enables us to bound the conditional probability and we end up with

$$\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \leq \mathbf{E}[g(L_n)].$$

As far as the lower bound is concerned, imposing that the excursion straddling n is non-negative (with probability p) so that $W_n \geq 0$, and using that the sign of W_n is independent from the past by symmetry, we get

$$\mathbf{P}(\zeta_k \geq 0 \text{ for all } 1 \leq k \leq n) \geq p \mathbf{P}\left(\sum_{k=1}^{\ell} \xi_k \geq 0 \text{ for all } 1 \leq \ell \leq L_n\right).$$

As above, conditioning with respect to L_n, τ_{L_n} and using Theorem 2.1, we get that

$$\mathbf{P}(\zeta_k \geq 0 \text{ for all } 1 \leq k \leq n) \geq p \mathbf{E}[g(L_n)].$$

Let us now derive the last lower bound in Theorem 3.1. On the event $\{L_n \geq 1\}$, let us stress that $\xi_1 > 0$ if and only if $X_1 > 0$. Then, imposing on this event the first excursion to be positive and the last to be non-negative, we get that

$$\mathbf{P}(\zeta_k > 0 \text{ for all } 1 \leq k \leq n) \geq p(1-p) \mathbf{P}\left(\sum_{k=2}^{\ell} \xi_k > 0 \text{ for all } 1 \leq \ell \leq L_n, L_n \geq 1\right).$$

Then, conditioning with respect to $\{L_n = m, \tau_m = j\}$, and using the exchangeability and sign-invariance of $(\xi_i)_{2 \leq i \leq m}$, we get thanks to Theorem 2.1

$$\mathbf{P}\left(\sum_{k=2}^{\ell} \xi_k \geq 0 \text{ for all } 1 \leq \ell \leq L_n, L_n \geq 1 \mid L_n = m, \tau_m = j\right) \geq g(m-1) \mathbf{1}_{\{m \geq 1\}},$$

since there are only $m-1$ terms in the sum. This concludes the proof. \square

5.2 Proof of Lemma 3.2

Let us start by giving the asymptotic of $\mathbf{E}[(1 + L_n)^{-1/2}]$. First, we have that

$$\begin{aligned}\mathbf{E}[(1 + L_n)^{-1/2}] &= \int_0^\infty \mathbf{P}((1 + L_n)^{-1/2} \geq t) dt = \int_0^\infty \mathbf{P}(1 + L_n \leq t^{-2}) dt \\ &= b_n^{-1/2} \int_0^\infty \mathbf{P}(b_n^{-1}(1 + L_n) \leq u) \frac{1}{2} u^{-3/2} du.\end{aligned}$$

Now, using that $\mathbf{P}(b_n^{-1}(1 + L_n) \leq u) \leq Ch(u)$ with $\int_0^\infty h(u)u^{-3/2}du < +\infty$, together with the fact that $\frac{1}{b_n}(1 + L_n)$ converges in distribution to \mathcal{X} , we get by dominated convergence that the last integral converges to

$$\frac{1}{2} \int_0^\infty \mathbf{P}(\mathcal{X} \leq u) u^{-3/2} du = \int_0^\infty \mathbf{P}(\mathcal{X} \leq t^{-2}) dt = \mathbf{E}[\mathcal{X}^{-1/2}],$$

where the integral is convergent since $\mathbf{P}(\mathcal{X} \leq u) \leq Ch(u)$ with $\int_0^\infty h(u)u^{-3/2}du < +\infty$. This proves that $\mathbf{E}[(1 + L_n)^{-1/2}] = b_n^{-1/2} \mathbf{E}[\mathcal{X}^{-1/2}]$ as $n \rightarrow \infty$.

It remains to see that the first expectations are asymptotically equivalent. We simply need to show that the contribution to $\mathbf{E}[g(L_n)]$ is mostly concentrated on large values of L_n . Let us fix $K > 0$ arbitrarily large, and notice that since $g(m) \leq 1$,

$$\mathbf{E}[g(L_n) \mathbf{1}_{\{L_n \leq K\}}] \leq \mathbf{P}(L_n \leq K) = \mathbf{P}(b_n^{-1}(1 + L_n) \leq b_n^{-1}(1 + K)) \leq Ch((K + 1)b_n^{-1}).$$

Now, since h is non-decreasing with $\int_0^1 h(u)u^{-3/2}du < +\infty$, we get that $h(x)x^{-1/2}$ goes to 0 as $x \downarrow 0$. Hence, for any fixed $K > 0$, we have that $\mathbf{P}(L_n \leq K) = o(b_n^{-1/2})$. Choosing $K = K_n$ that goes to infinity slowly enough, this proves that

$$\mathbf{E}[g(L_n)] = \mathbf{E}[g(L_n) \mathbf{1}_{\{L_n > K_n\}}] + o(b_n^{-1/2}),$$

and similarly for $\mathbf{E}[(1 + L_n)^{-1/2}]$. Now, we clearly have that $g(m) = (1 + o(1))\pi^{-1/2}m^{-1/2}$ with the $o(1)$ uniform in $m \geq K_n$. We therefore end up with

$$\mathbf{E}[g(L_n)] \sim \pi^{-1/2} \mathbf{E}[(1 + L_n)^{-1/2}].$$

The expectation $\mathbf{E}[g(L_n - 1) \mathbf{1}_{\{L_n \geq 1\}}]$ is treated similarly.

We conclude by showing that the assumptions are verified in the case where $(X_n)_{n \geq 1}$ is positive recurrent. The ergodic theorem already shows that $\lim_{n \rightarrow \infty} \frac{1}{n} L_n = \frac{1}{\mathbf{E}[\tau]}$ a.s., so we can take $b_n = n$. We now show that

$$\mathbf{P}\left(\frac{1}{n}(1 + L_n) \leq t\right) = \mathbf{P}(1 + L_n \leq tn) \leq Ch(t) \quad \text{with } h(t) = \min(1, t^{3/4}).$$

Let us consider the successive return times $\tau_k = \min\{n > \tau_{k-1}, X_n = 0\}$, which form a renewal process $\tau = (\tau_i)_{i \geq 0}$. Then, our starting point is to write that, for any $t > 0$, we have

$$\mathbf{P}(1 + L_n \leq tn) = \mathbf{P}(\tau_{tn} > n)$$

(we omit the integer part in $\lfloor tn \rfloor$ for simplicity). If $t > \frac{1}{4\mathbf{E}[\tau_1]}$, we bound this probability by 1. If $t \leq \frac{1}{4\mathbf{E}[\tau_1]}$, then we use a standard bound: let $\delta(t) > 0$ to be chosen in a moment, and write

$$\begin{aligned} \mathbf{P}(\tau_{tn} > n) &\leq \mathbf{P}(\exists k \leq tn, \tau_k - \tau_{k-1} > \delta(t)n) + \mathbf{P}\left(\sum_{k=1}^{tn} (\tau_k - \tau_{k-1}) \mathbf{1}_{\{\tau_k - \tau_{k-1} \leq \delta(t)n\}} > n\right) \\ &\leq tn\mathbf{P}(\tau_1 > \delta(t)n) + e^{-un}\mathbf{E}\left[\exp(u\tau_1 \mathbf{1}_{\{\tau_1 \leq \delta(t)n\}})\right]^{tn}, \end{aligned}$$

with $u = \frac{1}{\delta(t)n}$. By Markov's inequality, the first term is bounded by $t\delta(t)^{-1}\mathbf{E}[\tau_1]$. For the second term, we use that $e^x \leq 1 + (e-1)x \leq 1 + 2x$ for all $x \in [0, 1]$: hence, for $u = \frac{1}{\delta(t)n}$, we have

$$e^{-un}\mathbf{E}\left[\exp(u\tau_1 \mathbf{1}_{\{\tau_1 \leq \delta(t)n\}})\right] \leq e^{-un}(1 + 2u\mathbf{E}[\tau_1])^{tn} \leq e^{-\frac{1}{2}un},$$

where we have also used that $(1 + 2u\mathbf{E}[\tau_1])^{tn} \leq e^{\frac{1}{2}un}$ since $2t\mathbf{E}[\tau_1] \leq \frac{1}{2}$. We therefore end up with $\mathbf{P}(1 + L_n \leq tn) \leq C\left(\frac{t}{\delta(t)} + e^{-\frac{1}{2\delta(t)}}\right)$ for $t \leq \frac{1}{4\mathbf{E}[\tau_1]}$. This concludes the proof, taking $\delta(t) = t^{1/4}$. \square

5.3 About Bessel-like random walks: proof of Proposition 3.5

The proof of Proposition 3.5 boils down to the proof of the following lemma, together with an application of Theorem 3.1 and Lemma 3.2. Recall the notation from Section 3, in particular (3.3) and (3.4). Recall also the definition (3.5) of the sequence $(b_n)_{n \geq 1}$.

Lemma 5.1. *In the null-recurrent case, $(\frac{1}{b_n}L_n)_{n \geq 1}$ converges in distribution to:*

(i) $\mathcal{X} = 1$ if $\delta = 1$; (ii) $\mathcal{X} = \mathcal{Z}^{-\kappa}$ if $\delta \in (-1, 1)$; (iii) $\mathcal{X} \sim \text{Exp}(1)$ if $\delta = -1$.

Additionally we have, uniformly in n , $\mathbf{P}(\frac{1}{b_n}(1 + L_n) \leq t) \leq C \min(t, 1)$ for all $t > 0$.

Proof. Let us report (part of) the statement of [1, Thm. 2.1]. Let $\delta \geq -1$ and set $\kappa := \frac{1+\delta}{2}$, then as $n \rightarrow \infty$

$$\begin{aligned} \text{if } \delta > -1 \ (\kappa > 0), \quad & \mathbf{P}(\tau_1 \geq x) \sim \frac{2^{1-\kappa}}{K_0\Gamma(\kappa)} x^{-\kappa} \ell(\sqrt{x}), \\ \text{if } \delta = -1 \ (\kappa = 0), \quad & \mathbf{P}(\tau_1 \geq x) \sim \frac{1}{K_0} \nu(x)^{-1}, \end{aligned} \tag{5.1}$$

where $\nu(x)$ is defined in (3.5); also, $(X_n)_{n \geq 0}$ is recurrent if and only if $\lim_{x \rightarrow \infty} \nu(x) = \infty$.

Since the tail distribution $\mathbf{P}(\tau_1 \geq n)$ is regularly varying with exponent $-\kappa$, we have the following convergences in distribution for $(\tau_n)_{n \geq 1}$; we can then transfer these convergences to L_n , writing again, for any $t > 0$, $\mathbf{P}(1 + L_n \leq tb_n) = \mathbf{P}(\tau_{tb_n} > n)$. Recall that we focus on the null-recurrent case, so $\delta \in [-1, 1]$, $\kappa \in [0, 1]$. Let $(a_n)_{n \geq 1}$ be a sequence such that $\mathbf{P}(\tau_1 \geq a_n) \sim n^{-1}$ as $n \rightarrow \infty$, then:

(i) *Case $\delta = 1$, $\kappa = 1$.* Then $a_n^{-1}(\tau_n - n\mu(a_n))$ converges in distribution to a 1 stable law, see [11, IX.8] (see Eq. (8.15) for the centering): in particular, $\frac{\tau_n}{n\mu(a_n)}$ converges in probability to 1 (recall that $\mu(\cdot)$ is defined in (3.5)).

Setting b_n such that $b_n\mu(a_{b_n}) \sim n$, then we have that $\frac{\tau_{tb_n}}{n}$ converges to t in probability, so $b_n^{-1}L_n$ converges in probability to 1. Then, we simply have to notice that if b_n is given by the above relation then we have $\mu(a_{b_n}) \sim \mu(n)$, see e.g. [3, Lem. 4.3], so $b_n \sim n/\mu(n)$ as defined in (3.5).

(ii) Case $\delta \in (-1, 1)$, $\kappa \in (0, 1)$. Setting $\tilde{a}_n = \Gamma(1 - \kappa)^{1/\kappa} a_n$ we have that $\frac{1}{\tilde{a}_n} \tau_n$ converges in distribution to a one-sided κ -stable random variable \mathcal{Z} with Laplace transform e^{-t^κ} , see [11, XIII.6].

Let us set b_n such that $\tilde{a}_{b_n} \sim n$; using that \tilde{a}_n is regularly varying with exponent $1/\kappa$, we get that $a_{tb_n} \sim t^{1/\kappa} a_{b_n} \sim t^{1/\kappa} n$. We therefore get that $\lim_{n \rightarrow \infty} \mathbf{P}(\tau_{tb_n} > n) = \mathbf{P}(\mathcal{Z} > t^{-1/\kappa})$, showing that $b_n^{-1}(1 + L_n)$ converges in distribution to \mathcal{Z}^κ . Now, recalling the definition of a_n and of \tilde{a}_n , we get that

$$\mathbf{P}(\tau_1 \geq n) \sim \mathbf{P}(\tau_1 \geq \tilde{a}_{b_n}) \sim \Gamma(1 - \kappa)^{-1} \mathbf{P}(\tau_1 \geq a_{b_n}) \sim \Gamma(1 - \kappa)^{-1} b_n^{-1},$$

which, combined with (5.1), gives the definition of b_n in (3.5).

(iii) Case $\delta = -1$, $\kappa = 0$. Then $\frac{n}{K_0 \nu(\tau_n)}$ converges in distribution to an $\text{Exp}(1)$ variable, see [8, Thm. 8].

Let $b_n = \frac{1}{K_0 \nu(n)}$. The slowly varying function ν is asymptotically equivalent to a strictly increasing and continuous function, so assume that ν is strictly increasing and continuous. Then, for any $t > 0$,

$$\mathbf{P}(1 + L_n \leq tb_n) = \mathbf{P}(\tau_{tb_n} > n) = \mathbf{P}\left(\frac{tb_n}{K_0 \nu(\tau_{tb_n})} < \frac{tb_n}{K_0 \nu(n)}\right) \xrightarrow{n \rightarrow \infty} 1 - e^{-t},$$

which proves the last claim.

It remains to show the uniform bound on the tail of L_n . Here again, we simply need a uniform bound on the tail of τ_n , of the type $\mathbf{P}(\tau_{tb_n} > n) \leq Ch(t)$ for some “good” function h . These estimates are standard and fall in the “big-jump” phenomenon: in all the cases we consider, there exists a constant $C > 0$ such that for any $t < 1$ (the event $\tau_{tb_n} > n$ is an upper large deviation) we have

$$\mathbf{P}(1 + L_n \leq tb_n) = \mathbf{P}(\tau_{tb_n} > n) \leq C \times tb_n \mathbf{P}(\tau_1 > n). \quad (5.2)$$

If $\delta = 1$, this is given by [17, Thm. 1.2]; note also that in this case $\lim_{n \rightarrow \infty} b_n \mathbf{P}(\tau_1 > n) = 0$, thanks to [7, Prop. 1.5.9.a.]. If $\delta \in (-1, 1)$, this is contained in [17, Thm. 1.1]; note that by definition of b_n we have that $b_n \mathbf{P}(\tau_1 > n)$ remains bounded in this case. In the case $\delta = -1$, this is due to [18] (see also [2]); and $b_n \mathbf{P}(\tau_1 > n)$ also remains bounded in this case, by definition of b_n . In all cases, bounding the probability by 1 if $t \geq 1$, we get that $\mathbf{P}(1 + L_n \leq tb_n) \leq C \min(t, 1)$ uniformly in n . \square

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