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# Uniqueness and regularity of weak solutions of a fluid-rigid body interaction system under the Prodi-Serrin condition

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## Abstract

In this article, we study the weak uniqueness and the regularity of the weak solutions of a fluid-structure interaction system. More precisely, we consider the motion of a rigid ball in a viscous incompressible fluid and we assume that the fluid-rigid body system fills the entire space  $\mathbb{R}^3$ . We prove that the corresponding weak solutions that additionally satisfy a classical Prodi-Serrin condition, including a critical one, are unique. We also show that the weak solutions are regular under the Prodi-Serrin conditions, with a smallness condition in the critical case.

**Key words.** Navier-Stokes system, fluid-structure interaction, weak solution

**AMS subject classifications.** 35Q35, 76D03, 76D05, 74F10.

## 1 Introduction

### 1.1 Presentation of the model

We consider a fluid-structure interaction system composed by a rigid body immersed into a viscous incompressible fluid. We suppose that the rigid body is a homogeneous ball of radius 1 and that the fluid fills the exterior of the rigid body in  $\mathbb{R}^3$ . In particular, the structure domain  $\mathcal{S}(t)$  and the fluid domain  $\mathcal{F}(t)$  at time  $t \geq 0$  are given by

$$\mathcal{S}(t) = B(h(t), 1), \quad \mathcal{F}(t) = \mathbb{R}^3 \setminus \mathcal{S}(t), \quad (1.1)$$

where  $B(c, r)$  is the closed ball of  $\mathbb{R}^3$  of center  $a$  and of radius  $r$ .

The velocity and pressure fields in the fluid are denoted by  $u = u(t, x) \in \mathbb{R}^3$  and  $p = p(t, x) \in \mathbb{R}$  whereas the linear and angular velocities of the rigid ball are denoted by  $\ell = \ell(t) \in \mathbb{R}^3$  and  $\omega = \omega(t) \in \mathbb{R}^3$ . The system of equations governing the motion of this fluid-structure interaction system can be obtained by combining the Navier-Stokes system and the Newton laws:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \mu \Delta u + \nabla p = 0 & (t > 0, x \in \mathcal{F}(t)), \\ \operatorname{div} u = 0 & (t > 0, x \in \mathcal{F}(t)), \\ u(t, x) = \ell(t) + \omega(t) \times (x - h(t)) & (t > 0, x \in \partial \mathcal{S}(t)), \end{cases} \quad (1.2)$$

$$\begin{cases} m\ell'(t) = - \int_{\partial \mathcal{S}(t)} \mathbb{T}(u, p)\nu \, d\gamma_x & (t > 0), \\ J\omega'(t) = - \int_{\partial \mathcal{S}(t)} (x - h(t)) \times \mathbb{T}(u, p)\nu \, d\gamma_x & (t > 0), \end{cases} \quad (1.3)$$

$$h' = \ell \quad (t > 0), \quad (1.4)$$

$$u(0, \cdot) = a \quad \text{in } \mathcal{F}, \quad h(0) = 0, \quad \ell(0) = \ell_a, \quad \omega(0) = \omega_a. \quad (1.5)$$

Without loss of generality, we can assume as above that the initial position of the center of the rigid body is at the origin. To simplify the study, we suppose that the viscosity  $\mu$  of the fluid and the densities of fluid and of the structure are positive constants. We can assume that the fluid density is equal to 1 and if we denote by  $\rho_S > 0$  the density of the rigid body, its mass and its inertia tensor are given by the standard formula:

$$m = \frac{4}{3}\pi\rho_S, \quad J = \frac{2m}{5}\mathbb{I}_3.$$

We have denoted by  $\nu$  the unit vector field normal to  $\partial\mathcal{S}(t)$  and directed towards the interior of  $\mathcal{S}(t)$ . Finally, the Cauchy stress tensor field in the fluid is given by the constitutive law

$$\mathbb{T}(u, p) = -p\mathbb{I}_3 + 2\mu\mathbb{D}u, \quad \mathbb{D}u = \frac{1}{2}(\nabla u + \nabla u^\top),$$

and where  $M^\top$  denotes the transpose of a matrix  $M$ .

The initial and boundary value problem (1.2)–(1.5) and its generalization in the case of a rigid body of general shape has been studied for many years. Let us mention some related works from the literature: existence of Leray-Hopf weak solutions ([29, 9, 30]), existence and uniqueness of strong solutions ([31, 11, 8, 3, 12, 17, 23]), existence of steady states ([15, 16]), existence of time periodic solutions ([13, 14]), large time behaviour ([4, 10, 24]), etc.

Concerning the weak solutions of (1.2)–(1.5), as in the case without structure, it is an open problem to know if the weak solutions are unique or smooth. In the case where the fluid–rigid body system fills a bounded cavity  $\Omega$  (instead of the whole space  $\mathbb{R}^3$ ), the uniqueness of weak solutions in dimension 2 of space has been obtained only recently, see [18, 2]. Note that these uniqueness results are valid up to a contact between rigid bodies or between a rigid body and the boundary of  $\Omega$ , as it is not clear how the solutions can be continued after the contact (see [6, 28]). In dimension 3 and for a bounded cavity, the uniqueness and regularity of weak solutions under the classical Prodi-Serrin condition was proved in [27] and [26] respectively. However, the critical case of Prodi-Serrin condition was not included in these two articles, and in [26] regularity of weak solutions was proved under additional assumptions on the rigid velocities besides the Prodi-Serrin condition (see Remark 1.7 and Remark 1.9 below).

Our aim in this article is to prove the uniqueness of weak solutions of the system (1.2)–(1.5), under the Prodi-Serrin condition including the critical case (see Theorem 1.4). Note that in the case without structure (that is for the standard Navier-Stokes system), the uniqueness of weak solutions under the Prodi-Serrin condition in the critical case was obtained by [21] by using a first result of [25]. We are going to follow the same approach in our case and one of main arguments consists in showing the existence of mild solutions for initial conditions in  $L^2 \cap L^3$ . We are also going to show the regularity of weak solutions if we assume the Prodi-Serrin condition without the critical case (Theorem 1.8). In the critical case, we obtain the regularity property with an additional smallness assumption (Theorem 1.10). Note that this condition may be technical since for the incompressible Navier-Stokes system (without any structure), the regularity result is also true in the critical case without the smallness assumption (see [5]).

## 1.2 Statement of the main results

First, one can perform a change of variables in order to write the above system using the fixed spatial domains

$$\mathcal{S} := B(0, 1), \quad \mathcal{F} := \mathbb{R}^3 \setminus \mathcal{S}. \quad (1.6)$$

Setting

$$v(t, y) := u(t, y + h(t)), \quad q(t, y) := p(t, y + h(t)),$$

1 the system (1.2)–(1.5) can be transformed in the following system

$$2 \quad \begin{cases} \partial_t v + ((v - \ell) \cdot \nabla)v - \mu \Delta v + \nabla q = 0 & (t > 0, y \in \mathcal{F}), \\ \operatorname{div} v = 0 & (t > 0, y \in \mathcal{F}), \\ v(t, y) = \ell(t) + \omega(t) \times y & (t > 0, y \in \partial \mathcal{S}), \end{cases} \quad (1.7)$$

$$3 \quad \begin{cases} m\ell'(t) = - \int_{\partial \mathcal{S}} \mathbb{T}(v, q) \nu \, d\gamma_y & (t > 0), \\ J\omega'(t) = - \int_{\partial \mathcal{S}} y \times \mathbb{T}(v, q) \nu \, d\gamma_y & (t > 0), \end{cases} \quad (1.8)$$

$$4 \quad v(0, \cdot) = a \quad \text{in } \mathcal{F}, \quad \ell(0) = \ell_a, \quad \omega(0) = \omega_a, \quad (1.9)$$

$$5 \quad h' = \ell \quad (t > 0), \quad h(0) = 0. \quad (1.10)$$

6 Note that in the above system, the equations for the velocities (1.7)–(1.9) are decoupled from the equation  
7 (1.10) for the position of the rigid ball. In particular, the uniqueness and the regularity properties can be stated  
8 only for the system (1.7)–(1.9) and will induce the same properties for the whole system.

In what follows, we extend the fluid velocity by the rigid velocity in  $\mathcal{S}$ , that is

$$v(t, y) := \ell(t) + \omega(t) \times y \quad (t > 0, y \in \mathcal{S})$$

and we do the same for the initial velocity  $a$ . We recall that (see, for instance, [32, Lemma 1.1]),  $\mathbb{D}\varphi = 0$  in  $\mathcal{S}$   
if and only if there exist  $\ell_\varphi, \omega_\varphi \in \mathbb{R}^3$  such that

$$\varphi(y) := \ell_\varphi + \omega_\varphi \times y \quad (y \in \mathcal{S}).$$

8 With this notation, we have in (1.7)–(1.9)  $\ell = \ell_v$  and  $\omega = \omega_v$ .

9 We now introduce several function spaces in order to state our main results. Let  $G$  be a domain in  $\mathbb{R}^3$ . For  
10 any  $q \in [1, \infty]$ , and  $s \geq 0$ , we denote by  $L^q(G)$  and  $W^{s,q}(G)$  the usual Lebesgue and Sobolev-Slobodeckij spaces.  
11 We also use the notation  $H^s(G) = W^{s,2}(G)$ . We also define the following spaces, associated with our problem:  
12 for any  $q \in [1, \infty]$  and  $s \geq 0$

$$13 \quad \mathcal{L}^q = \{ \varphi \in L^q(\mathbb{R}^3) \mid \operatorname{div} \varphi = 0 \text{ in } \mathbb{R}^3, \quad \mathbb{D}(\varphi) = 0 \text{ in } \mathcal{S} \}, \quad (1.11)$$

$$14 \quad \mathcal{H}^s = \{ \varphi \in H^s(\mathbb{R}^3) \mid \operatorname{div} \varphi = 0 \text{ in } \mathbb{R}^3, \quad \mathbb{D}(\varphi) = 0 \text{ in } \mathcal{S} \}, \quad (1.12)$$

$$\mathcal{W}^{s,q} = \{ \varphi \in W^{s,q}(\mathbb{R}^3) \mid \operatorname{div} \varphi = 0 \text{ in } \mathbb{R}^3, \quad \mathbb{D}(\varphi) = 0 \text{ in } \mathcal{S} \}.$$

We define a global density

$$\rho := \begin{cases} 1 & \text{in } \mathcal{F}, \\ \rho_{\mathcal{S}} & \text{in } \mathcal{S}, \end{cases}$$

15 and the scalar product in  $\mathcal{L}^2(\mathbb{R}^3)$

$$\langle f, g \rangle_{\mathcal{L}^2} := \int_{\mathbb{R}^3} \rho f \cdot g \, dy. \quad (1.14)$$

Note that if  $f, g \in \mathcal{L}^2$ , then

$$\langle f, g \rangle_{\mathcal{L}^2} = \int_{\mathcal{F}} f \cdot g \, dy + m\ell_f \cdot \ell_g + J\omega_f \cdot \omega_g.$$

16 The above definitions and notations allow us to define the notion of weak solutions for the system (1.7)–(1.9):

17 **Definition 1.1.** *Assume  $a \in \mathcal{L}^2$  and  $T > 0$ . We say that  $v$  is a weak solution of (1.7)–(1.9) if*

$$v \in L^\infty(0, T; \mathcal{L}^2) \cap L^2(0, T; \mathcal{H}^1), \quad (1.15)$$

18 and if for any  $\varphi \in H^1(0, T; \mathcal{H}^1)$  such that  $\varphi(T, \cdot) = 0$ ,

$$\begin{aligned} - \int_0^T \langle v, \partial_t \varphi \rangle_{\mathcal{L}^2} \, dt + \iint_{(0,T) \times \mathbb{R}^3} 2\mu \mathbb{D}(v) : \mathbb{D}(\varphi) \, dy \, d\tau + \iint_{(0,T) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla)v \cdot \varphi \, dy \, d\tau \\ = \langle a, \varphi(0, \cdot) \rangle_{\mathcal{L}^2}. \end{aligned} \quad (1.16)$$

**Remark 1.2.** Note that if  $v$  satisfies (1.15), then all the terms in (1.16) are well-defined. Indeed, by the Sobolev embedding  $H^1(\mathcal{F}) \hookrightarrow L^6(\mathcal{F})$ , we have

$$v \cdot \nabla v \in L^1(0, T; L^{3/2}(\mathcal{F})), \quad \ell_v \cdot \nabla v \in L^2(0, T; L^2(\mathcal{F})),$$

whereas  $\varphi \in H^1(0, T; \mathcal{H}^1)$  yields  $\varphi \in L^\infty(0, T; L^3(\mathcal{F}))$  and  $\varphi \in L^2(0, T; L^2(\mathcal{F}))$ . Therefore, in view of Hölder's inequality all the terms in (1.16) are well-defined.

As mentioned in the introduction, the well-posedness of system (1.7)–(1.9) has been studied by several authors (including if the rigid body has a general shape). In particular, we have the following result (see, for instance, [30]):

**Theorem 1.3.** *There exists a weak solution  $v$  of (1.7)–(1.9) in the sense of Definition 1.1, and it satisfies the following energy inequality*

$$\frac{1}{2} \|v(t, \cdot)\|_{\mathcal{L}^2}^2 + 2\mu \int_0^t \int_{\mathbb{R}^3} |\mathbb{D}(v)|^2 \, dy \, d\tau \leq \frac{1}{2} \|a\|_{\mathcal{L}^2}^2 \quad (t \in [0, T]). \quad (1.17)$$

Our first main result corresponds to the Prodi-Serrin condition for the uniqueness of weak solutions:

**Theorem 1.4.** *Let  $v^{(1)}, v^{(2)}$  be two weak solutions of (1.7)–(1.9) associated with the same initial condition  $a \in \mathcal{L}^2$  in the sense of Definition 1.1. Assume  $v^{(1)}$  satisfies the energy inequality (1.17) and that  $v^{(2)} \in L^r(0, T; L^q(\mathcal{F}))^3$  with*

$$\frac{2}{r} + \frac{3}{q} = 1, \quad r \in [2, \infty], \quad q \in [3, \infty]. \quad (1.18)$$

Then

$$v^{(1)} = v^{(2)}.$$

**Remark 1.5.** *Let us remark that Theorem 1.4 holds if we replace (1.18) by the weaker assumption that  $v^{(2)} \in L^r(0, T; L^q(\mathcal{F}))^3$  with*

$$\frac{2}{r} + \frac{3}{q} \leq 1, \quad r \in [2, \infty], \quad q \in [3, \infty]. \quad (1.19)$$

Indeed, if  $q > 3$  and (1.19) holds, there exists  $r' \leq r$  such that  $\frac{2}{r'} + \frac{3}{q} = 1$  and  $v^{(2)} \in L^{r'}(0, T; L^q(\mathcal{F}))$  so that we can apply Theorem 1.4 and deduce the result.

To prove Theorem 1.4, we are going to consider two cases:

- the noncritical case :  $q \in (3, \infty]$  (or  $r \in [2, \infty)$ );
- the critical case :  $r = \infty$  and  $q = 3$ .

The proof in the critical case combines arguments of [25] and [21]. In particular, we first show a weaker result: we replace the hypothesis  $v^{(2)} \in L^\infty(0, T; L^3(\mathcal{F}))^3$  by the stronger hypothesis  $v^{(2)} \in C_{\mathbb{R}}^0([0, T]; L^3(\mathcal{F}))^3$  where  $C_{\mathbb{R}}^0$  denotes the set of right-continuous functions. The corresponding result states as follows:

**Theorem 1.6.** *Let  $v^{(1)}, v^{(2)}$  be two weak solutions of (1.7)–(1.9) associated with the same initial condition  $a \in \mathcal{L}^2$  in the sense of Definition 1.1. Assume  $v^{(1)}$  satisfies the energy inequality (1.17) and that*

$$v^{(2)} \in C_{\mathbb{R}}^0([0, T]; L^3(\mathcal{F}))^3 \cap L^\infty(0, T; L^3(\mathcal{F}))^3.$$

Then  $v^{(1)} = v^{(2)}$ .

Finally, using Theorem 1.6 and the method of [21] we prove Theorem 1.4 in the critical case, see Section 5.

**Remark 1.7.** *Let us remark that in the case where the fluid-rigid body fills a bounded cavity, the corresponding result was obtained in [27] in the noncritical case  $q \in (3, \infty)$ .*

Our second main result corresponds to the regularity of weak solutions under the Prodi-Serrin condition:

**Theorem 1.8.** *Let  $v$  be a weak solution of (1.7)–(1.9) associated with the initial condition  $a \in \mathcal{L}^2$  in the sense of Definition 1.1. Assume  $v \in L^r(0, T; L^q(\mathcal{F}))^3$  with*

$$\frac{2}{r} + \frac{3}{q} = 1, \quad r \in [2, \infty), \quad q \in (3, \infty]. \quad (1.20)$$

Then

$$\ell_v, \omega_v \in C^\infty((0, T]; \mathbb{R}^3), \quad v \in C^\infty((0, T] \times \overline{\mathcal{F}})^3.$$

**Remark 1.9.** *Let us remark that in the case where the fluid-rigid body fills a bounded cavity, a similar result was obtained in [26] in the noncritical case  $q \in (3, \infty)$ , and with the additional assumptions that  $\ell'_v, \omega'_v \in L^\infty(0, T)^3$ .*

In the critical case, i.e.  $q = 3$  and  $r = \infty$ , we need a smallness condition:

**Theorem 1.10.** *There exists  $\varepsilon > 0$  such that if  $v$  is a weak solution of (1.7)–(1.9) associated with the initial condition  $a \in \mathcal{L}^2$  in the sense of Definition 1.1 and if  $v \in L^\infty(0, T; L^3(\mathcal{F}))^3$  with*

$$\|v\|_{L^\infty(0, T; L^3(\mathcal{F}))} \leq \varepsilon \quad (1.21)$$

then

$$\ell_v, \omega_v \in C^\infty((0, T]; \mathbb{R}^3), \quad v \in C^\infty((0, T] \times \overline{\mathcal{F}})^3.$$

The outline of this article is as follows: first in the next section, we define the “fluid-structure” operator associated with the system (1.7)–(1.9) and its properties. We also give some useful inequalities and define some mollifiers. Then in Section 3, we show some important properties of the weak solutions of (1.7)–(1.9). In order to prove Theorem 1.4 in the critical case, we need to show the existence of mild solutions for (1.7)–(1.9) if the initial condition satisfies  $a \in \mathcal{L}^2 \cap \mathcal{L}^3$ . This is the aim of Section 4. Section 5 is devoted to the proof of the uniqueness of weak solutions, that is to the proof of Theorem 1.4. Finally, in Section 6, we show Theorem 1.8 and Theorem 1.10 for the smoothness of the weak solutions of (1.7)–(1.9).

**Notation 1.11.** *In the whole paper, we use  $C$  as a generic positive constant that does not depend on the other terms of the inequality. The value of the constant  $C$  may change from one appearance to another. We also use the notation  $X \lesssim_p Y$  if there exists a constant  $C > 0$  depending only on the parameter  $p$  such that we have the inequality  $X \leq CY$ .*

## 2 Preliminaries

In this section, we introduce the “fluid-structure” operator associated with the system (1.7)–(1.9), and we present some useful properties of this operator. We also recall some basic inequalities and properties of the mollifiers.

### 2.1 The “fluid-structure” operator.

We start by the following Helmholtz type decomposition of the space  $\mathcal{L}^q$  defined in (1.11). From [33], we have the following decomposition

$$L^q(\mathbb{R}^3)^3 = \mathcal{L}^q \oplus \mathcal{G}^q \quad (2.1)$$

where

$$\mathcal{G}^q := \left\{ \varphi \in L^q(\mathbb{R}^3)^3 \mid \exists \pi \in L^1_{loc}(\mathcal{F}), \quad \varphi = \nabla \pi \quad \text{in } \mathcal{F}, \right. \\ \left. \int_S \rho_S \varphi \, dy + \int_{\partial S} \pi \nu \, d\gamma_x = 0, \quad \int_S \rho_S \varphi \times x \, dx + \int_{\partial S} \pi \nu \times x \, d\gamma_x = 0 \right\}.$$

1 In the particular case  $q = 2$ , note that  $\mathcal{L}^2$  and  $\mathcal{G}^2$  are orthogonal for the scalar product in  $\langle \cdot, \cdot \rangle_{\mathcal{L}^2}$  (defined by  
2 (1.14)). Note also that  $\varphi \in \mathcal{G}^q$  and  $\mathbb{D}\varphi = 0$  in  $\mathcal{S}$  if and only if there exists  $\pi \in L^1_{loc}(\mathcal{F})$ ,  $\nabla\pi \in L^q(\mathcal{F})^3$  such that

$$\varphi = \nabla\pi \quad \text{in } \mathcal{F}, \quad \ell_\varphi = -m^{-1} \int_{\partial\mathcal{S}} \pi \nu \, d\gamma_x, \quad \omega_\varphi = -J^{-1} \int_{\partial\mathcal{S}} \pi \nu \times x \, d\gamma_x. \quad (2.2)$$

We denote by  $\mathbb{P}_q : L^q(\mathbb{R}^3)^3 \rightarrow \mathcal{L}^q$  the projection onto  $\mathcal{L}^q$  along  $\mathcal{G}^q$ . Then, we can define the operator  $\mathbb{A}_q : \mathcal{D}(\mathbb{A}_q) \rightarrow \mathcal{L}^q$  as follows

$$\mathcal{D}(\mathbb{A}_q) := \left\{ \varphi \in \mathcal{W}^{1,q} \mid \varphi|_{\mathcal{F}} \in W^{2,q}(\mathcal{F})^3 \right\}, \quad \mathbb{A}_q \varphi := \mathbb{P}_q \tilde{\mathbb{A}}_q \varphi,$$

where

$$\tilde{\mathbb{A}}_q \varphi := \begin{cases} \mu \Delta \varphi & (y \in \mathcal{F}), \\ -m^{-1} \left( \int_{\partial\mathcal{S}} 2\mathbb{D}(\varphi) \nu \, d\gamma_x \right) - J^{-1} \left( \int_{\partial\mathcal{S}} x \times 2\mathbb{D}(\varphi) \nu \, d\gamma_x \right) \times y & (y \in \mathcal{S}). \end{cases}$$

We also define

$$\mathbb{F}(v) := \begin{cases} -v \otimes (v - \ell_v) & \text{in } \mathcal{F}, \\ 0 & \text{in } \mathcal{S}, \end{cases}$$

and using that  $(v \otimes (v - \ell_v)) \nu = 0$  on  $\partial\mathcal{S}$ , we have

$$\operatorname{div} \mathbb{F}(v) := \begin{cases} -((v - \ell_v) \cdot \nabla) v & \text{in } \mathcal{F}, \\ 0 & \text{in } \mathcal{S}. \end{cases}$$

3 This allows us to write (1.7)–(1.9) in the following abstract way:

$$v' = \mathbb{A}_q v + \mathbb{P}_q \operatorname{div} \mathbb{F}(v) \quad (t > 0), \quad v(0) = a. \quad (2.3)$$

4 We now recall some properties of the “fluid-structure” operator  $\mathbb{A}_q$  from [4]. First we have (see [4, Proposition  
5 7.3])

6 **Proposition 2.1.** *Let  $1 < q < \infty$ . Assume that  $\varphi \in \mathcal{D}(\mathbb{A}_q)$  and  $\mathbb{A}_q \varphi = \mathbb{P}_q(F|_{\mathcal{F}})$  with  $F \in W^{m,q}(\mathcal{F})^3$  for some  
7  $m \in \mathbb{N}$ . Then  $\varphi|_{\mathcal{F}} \in W^{m+2,q}(\mathcal{F})^3$ .*

8 Second, we have (see [4, Theorem 6.1] or [24, Theorem 4.3])

**Proposition 2.2.** *Let  $1 < p, q < \infty$ . Then  $\mathbb{A}_q$  is the infinitesimal generator of a bounded analytic semigroup  
on  $\mathcal{L}^q$ . Moreover,  $\mathbb{A}_q$  satisfies the maximal  $L^p$ -regularity property: for any  $T > 0$  and for any  $\varphi \in L^p(0, T; \mathcal{L}^q)$ ,  
there exists a unique solution  $v \in W^{1,p}(0, T; \mathcal{L}^q) \cap L^p(0, T; \mathcal{D}(\mathbb{A}_q))$  of*

$$v' = \mathbb{A}_q v + \varphi \quad \text{in } (0, T), \quad v(0) = 0.$$

9 We also recall the  $L^q - L^r$  decay estimates of the “fluid-structure” semigroup obtained in [4, Theorem 7.1]:

**Theorem 2.3.** *Assume  $q \in (1, \infty)$  and*

$$q \leq r < \infty \quad \text{or} \quad r = \infty.$$

10 *Then*

$$\|e^{t\mathbb{A}_q} \varphi\|_{\mathcal{L}^r} \lesssim_{q,r} t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{r})} \|\varphi\|_{\mathcal{L}^q} \quad (t > 0, \varphi \in \mathcal{L}^q). \quad (2.4)$$

11 *Assume  $1 < q \leq r \leq 3$ . Then*

$$\|\nabla e^{t\mathbb{A}_q} \varphi\|_{L^r(\mathcal{F})^3} \lesssim_{q,r} t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{r}) - \frac{1}{2}} \|\varphi\|_{\mathcal{L}^q} \quad (t > 0, \varphi \in \mathcal{L}^q). \quad (2.5)$$

12 Finally, we state a useful lemma ([4, Lemma 8.1]) to deal with the estimate of the nonlinear terms:

**Lemma 2.4.** *Assume*

$$\frac{3}{2} \leq q \leq r < \infty.$$

*There for any  $F \in L^q(\mathbb{R}^3)^9$ ,  $F = 0$  in  $\mathcal{S}$ ,  $\operatorname{div} F \in L^r(\mathbb{R}^3)^3$ ,*

$$\|e^{t\mathbb{A}_r} \mathbb{P}_r \operatorname{div} F\|_{\mathcal{L}^r} \lesssim_{q,r} C t^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{r}) - \frac{1}{2}} \|F\|_{L^q(\mathcal{F})^9}.$$

## 2.2 Some inequalities and lemmas

Several of the results below are adapted from arguments of Navier-Stokes system without any rigid body, see for instance [21], [25], [7], etc.

First, we recall that a consequence of the Sobolev-Galiardo-Nirenberg inequality is the following standard estimate: for any  $w \in H^1(\mathbb{R}^3)$  and for any  $r \in [2, 6]$ ,

$$\|u\|_{L^r(\mathbb{R}^3)} \lesssim_r \|\nabla u\|_{L^2(\mathbb{R}^3)}^{\alpha_r} \|u\|_{L^2(\mathbb{R}^3)}^{1-\alpha_r}, \quad \text{where } \alpha_r := 3 \left( \frac{1}{2} - \frac{1}{r} \right) \in [0, 1]. \quad (2.6)$$

Next, let us show a standard property for the nonlinear terms of our system:

**Lemma 2.5.** *Assume  $v \in \mathcal{H}^1$  and  $w \in H^1(\mathbb{R}^3)^3$ . Then*

$$\int_{\mathcal{F}} ((v - \ell_v) \cdot \nabla) w \cdot w \, dy = 0.$$

In particular, if  $v \in \mathcal{H}^1$  and  $w^{(1)}, w^{(2)} \in H^1(\mathbb{R}^3)^3$ , then

$$\int_{\mathcal{F}} ((v - \ell_v) \cdot \nabla) w^{(1)} \cdot w^{(2)} \, dy = - \int_{\mathcal{F}} ((v - \ell_v) \cdot \nabla) w^{(2)} \cdot w^{(1)} \, dy.$$

*Proof.* Let us consider a sequence  $(w_k)_k \subset C_c^\infty(\mathbb{R}^3)^3$ ,  $w_k \rightarrow w$  in  $H^1(\mathbb{R}^3)^3$ . Then an integration by parts and the definition (1.12) of  $\mathcal{H}^1$  imply

$$\int_{\mathcal{F}} ((v - \ell_v) \cdot \nabla) w_k \cdot w_k \, dy = - \int_{\mathcal{F}} \operatorname{div}(v - \ell_v) \frac{|w_k|^2}{2} \, dy + \int_{\partial \mathcal{S}} (v - \ell_v) \cdot \nu \frac{|w_k|^2}{2} \, d\gamma = 0.$$

Now, we can write

$$\begin{aligned} & \int_{\mathcal{F}} ((v - \ell_v) \cdot \nabla) w \cdot w \, dy - \int_{\mathcal{F}} ((v - \ell_v) \cdot \nabla) w_k \cdot w_k \, dy \\ &= \int_{\mathcal{F}} ((v - \ell_v) \cdot \nabla) (w - w_k) \cdot w \, dy + \int_{\mathcal{F}} ((v - \ell_v) \cdot \nabla) w_k \cdot (w - w_k) \, dy \end{aligned}$$

and using (2.6), we have  $w \in L^3(\mathcal{F})^3$  and  $w_k \rightarrow w$  in  $L^3(\mathcal{F})^3$ . This proves the lemma.  $\square$

We now show some estimates for the nonlinear terms by using the Prodi-Serrin condition:

**Lemma 2.6.** *Assume  $w \in L^2(0, T; H^1(\mathcal{F})^3)$ ,  $v \in L^2(0, T; H^1(\mathcal{F})^3) \cap L^\infty(0, T; L^2(\mathcal{F})^3)$  and  $u \in L^r(0, T; L^q(\mathcal{F}))$  with*

$$\frac{2}{r} + \frac{3}{q} = 1, \quad q \in [3, \infty], \quad r \in [2, \infty].$$

If  $q \in (3, \infty)$ ,

$$\begin{aligned} & \left| \iint_{(0,t) \times \mathcal{F}} (v \cdot \nabla w) \cdot u \, dy \, ds \right| \\ & \lesssim \|\nabla w\|_{L^2(0,t;L^2(\mathcal{F})^9)} \|\nabla v\|_{L^2(0,t;L^2(\mathcal{F})^9)}^{3/q} \left( \int_0^t \|v(s, \cdot)\|_{L^2(\mathcal{F})^3}^2 \|u(s, \cdot)\|_{L^q(\mathcal{F})^3}^r \, ds \right)^{1/r}. \quad (2.7) \end{aligned}$$

If  $q = \infty$ , then

$$\left| \iint_{(0,t) \times \mathcal{F}} (v \cdot \nabla w) \cdot u \, dy \, ds \right| \lesssim \|\nabla w\|_{L^2(0,t;L^2(\mathcal{F})^9)} \left( \int_0^t \|v(s, \cdot)\|_{L^2(\mathcal{F})^3}^2 \|u(s, \cdot)\|_{L^\infty(\mathcal{F})^3}^2 \, ds \right)^{1/2}. \quad (2.8)$$



1 If  $q = 3$ , then

$$\left| \iint_{(0,t) \times \mathcal{F}} (v \cdot \nabla w) \cdot u \, dy \, ds \right| \lesssim \|\nabla w\|_{L^2(0,t;L^2(\mathcal{F})^9)} \|\nabla v\|_{L^2(0,t;L^2(\mathcal{F})^9)} \|u\|_{L^\infty(0,t;L^3(\mathcal{F})^3)}. \quad (2.9)$$

*Proof.* First we assume  $q \in (3, \infty)$ . Using the Sobolev-Galiardo-Nirenberg inequality

$$\|v\|_{L^6(\mathcal{F})^3} \lesssim \|\nabla v\|_{L^2(\mathcal{F})^9},$$

2 and the Hölder inequality, we deduce that

$$\left| \int_{\mathcal{F}} (v \cdot \nabla w) \cdot u \, dy \right| \lesssim \|\nabla w\|_{L^2(\mathcal{F})^9} \|u\|_{L^q(\mathcal{F})^3} \|\nabla v\|_{L^2(\mathcal{F})^9}^{3/q} \|v\|_{L^2(\mathcal{F})^3}^{1-3/q}. \quad (2.10)$$

3 If  $q = \infty$ , that is if  $u \in L^2(0, T; L^\infty(\Omega)^3)$ , then

$$\left| \int_{\mathcal{F}} (v \cdot \nabla w) \cdot u \, dy \right| \lesssim \|\nabla w\|_{L^2(\mathcal{F})^9} \|u\|_{L^\infty(\mathcal{F})^3} \|v\|_{L^2(\mathcal{F})^3}. \quad (2.11)$$

4 If  $q = 3$ , that is if  $u \in L^\infty(0, T; L^3(\Omega)^3)$ , then

$$\left| \int_{\mathcal{F}} (v \cdot \nabla w) \cdot u \, dy \right| \lesssim \|\nabla w\|_{L^2(\mathcal{F})^9} \|u\|_{L^3(\mathcal{F})^3} \|\nabla v\|_{L^2(\mathcal{F})^3}. \quad (2.12)$$

5 Integrating (2.10), (2.10) and (2.10) in  $(0, t)$  and using the Hölder inequality, we obtain (2.7), (2.8) and (2.9).  $\square$

6 In the critical case  $((r, q) = (\infty, 3))$ , the above lemma is not sufficient to prove the uniqueness result. We  
7 need the two following results obtained in [25]:

**Lemma 2.7.** Assume  $f \in L^1(0, T)$ ,  $f \geq 0$  and for any  $t \in (0, T]$ ,  $\int_0^t f(s) \, ds > 0$ . Assume  $(g_k)_k$  is sequence in  $L^\infty(0, T)$  with  $g_k \geq 0$ , right-continuous at  $t = 0$  and such that for a.e.  $t \in [0, T]$ , the sequence  $(g_k(t))_k$  is non increasing and converging to 0. Then

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall k \geq N, \forall t \in [0, T], \int_0^t f g_k \, ds \leq \varepsilon \int_0^t f \, ds.$$

**Lemma 2.8.** Assume  $w \in L^2(t_1, t_2; H^1(\mathcal{F})^3)$  and  $u \in L^\infty(t_1, t_2; L^3(\mathcal{F})^3)$ ,  $u$  right-continuous at  $t = t_1$  in the norm  $L^3(\mathcal{F})$ . Suppose that

$$\forall t \in (t_1, t_2), \int_{t_1}^t \|\nabla w\|_{L^2(\mathcal{F})^9}^2 \, ds > 0.$$

8 Then for any  $\eta > 0$ , there exists  $C_\eta > 0$ , independent of  $t$ , such that

$$\left| \iint_{(t_1, t) \times \mathcal{F}} (w \cdot \nabla w) \cdot u \, dy \, ds \right| \leq \eta \|\nabla w\|_{L^2(t_1, t; L^2(\mathcal{F})^9)}^2 + C_\eta \|w\|_{L^2(t_1, t; L^2(\mathcal{F})^3)}^2, \quad t_1 \leq t \leq t_2. \quad (2.13)$$

### 9 2.3 Definition and properties of the mollifiers

Let us introduce some standard mollifiers. We consider  $j \in C^\infty(\mathbb{R})$ , odd, nonnegative, with support in  $(-1, 1)$  and such that  $\int_{\mathbb{R}} j(t) \, dt = 1$ . Then we set for  $\varepsilon > 0$ ,

$$j_\varepsilon(t) := \frac{1}{\varepsilon} j\left(\frac{t}{\varepsilon}\right).$$

10 Then, for  $T \in (0, \infty)$ ,  $r \in [1, \infty)$ , and  $\mathcal{X}$  a Banach space, we consider the following approximation of  $w \in$   
11  $L^r(0, T; \mathcal{X})$ :

$$w_\varepsilon(t, \cdot) := \int_0^T j_\varepsilon(t - \tau) w(\tau, \cdot) \, d\tau. \quad (2.14)$$

12 It has the following standard properties

**Lemma 2.9.** Assume  $T \in (0, \infty)$ . For any  $w \in L^r(0, T; \mathcal{X})$  with  $r \in [1, \infty)$ ,

$$w_\varepsilon \in C^\infty([0, T]; \mathcal{X}), \quad \lim_{\varepsilon \rightarrow 0} \|w - w_\varepsilon\|_{L^r(0, T; \mathcal{X})} = 0.$$

For any  $w \in C^0([0, T]; \mathcal{X})$ ,

$$\lim_{\varepsilon \rightarrow 0} w_\varepsilon(0, \cdot) = \frac{1}{2}w(0, \cdot), \quad \lim_{\varepsilon \rightarrow 0} w_\varepsilon(T, \cdot) = \frac{1}{2}w(T, \cdot).$$

Assume that  $\mathcal{X}$  is a Hilbert space. Then for any  $w^{(1)}, w^{(2)} \in L^r(0, T; \mathcal{X})$

$$\int_0^T \langle \partial_t w_\varepsilon^{(1)}, w^{(1)} \rangle_{\mathcal{X}} dt = 0, \quad \int_0^T \langle \partial_t w_\varepsilon^{(1)}, w^{(2)} \rangle_{\mathcal{X}} dt = - \int_0^T \langle \partial_t w_\varepsilon^{(2)}, w^{(1)} \rangle_{\mathcal{X}} dt.$$

1 The proof of the above lemma is standard and we skip it.

## 2 3 Properties of the weak solutions

3 In this section, we derive some properties of the weak solution to (1.7)–(1.9) (see Definition 1.1). This results  
4 are in the spirit of Navier-Stokes equation without any rigid body.

5 **Proposition 3.1.** Assume that  $v$  is a weak solution of (1.7)–(1.9). Then  $t \mapsto v(t, \cdot)$  is continuous in  $[0, T]$  for  
6 the weak topology of  $L^2(\mathbb{R}^3)^3$  and for any  $0 \leq s < t \leq T$  and for any  $\varphi \in H^1(s, t; \mathcal{H}^1)$ ,

$$\begin{aligned} - \int_s^t \langle v, \partial_t \varphi \rangle_{\mathcal{L}^2} d\tau + 2\mu \iint_{(s, t) \times \mathbb{R}^3} \mathbb{D}(v) : \mathbb{D}(\varphi) dy d\tau + \iint_{(s, t) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla) v \cdot \varphi dy d\tau \\ = \langle v(s, \cdot), \varphi(s, \cdot) \rangle_{\mathcal{L}^2} - \langle v(t, \cdot), \varphi(t, \cdot) \rangle_{\mathcal{L}^2}. \end{aligned} \quad (3.1)$$

*Proof.* Assume  $t \in [0, T)$  and  $\varphi \in H^1(0, t; \mathcal{H}^1)$ . We consider  $\theta \in C^1(\mathbb{R}; [0, 1])$ ,  $\theta \equiv 1$  in  $(-\infty, 1)$ , and  $\theta \equiv 0$  in  $(2, \infty)$ . We set for  $\alpha > 0$  small enough,

$$\theta_\alpha(\tau) := \theta \left( \frac{\tau - t + \alpha}{\alpha} \right)$$

7 and we use in (1.16) the test function  $\theta_\alpha \varphi \in H^1(0, T; \mathcal{H}^1)$  that cancels at  $t = T$  for  $\alpha$  small enough. Passing to  
8 the limit as  $\alpha \rightarrow 0$ , we obtain (3.1) for  $s = 0$  and for a.e.  $t \in (0, T)$ . In particular, if  $\varphi \in \mathcal{H}^1$  (independent of  
9 time), this yields

$$\langle v(t, \cdot), \varphi \rangle_{\mathcal{L}^2} = \langle a, \varphi \rangle_{\mathcal{L}^2} - 2\mu \int_0^t \int_{\mathbb{R}^3} \mathbb{D}(v) : \mathbb{D}(\varphi) dy d\tau - \int_0^t \int_{\mathcal{F}} ((v - \ell_v) \cdot \nabla) v \cdot \varphi dy d\tau. \quad (3.2)$$

10 This shows the continuity of  $t \mapsto \langle v(t, \cdot), \varphi \rangle_{\mathcal{L}^2}$  for any  $\varphi \in \mathcal{H}^1$ , and by density for any  $\varphi \in \mathcal{L}^2$ . Then, we use the  
11 orthogonal decomposition (2.1) for  $q = 2$  to conclude.  $\square$

12 We now show that under Prodi-Serrin condition (1.18), weak solutions of (1.7)–(1.9) satisfy the energy  
13 equality instead of the energy inequality (1.17):

14 **Proposition 3.2.** If  $v$  is a weak solution of (1.7)–(1.9) such that

$$v \in L^4(0, T; L^4(\mathcal{F})^3), \quad (3.3)$$

15 then

$$\frac{1}{2} \|v(t, \cdot)\|_{\mathcal{L}^2}^2 + 2\mu \iint_{(0, t) \times \mathbb{R}^3} |\mathbb{D}(v)|^2 dy d\tau = \frac{1}{2} \|a\|_{\mathcal{L}^2}^2 \quad (t \in [0, T]). \quad (3.4)$$

16 In particular, if  $v$  satisfies the Prodi-Serrin condition (1.18), then it satisfies (3.4).

*Proof.* First, if  $v$  satisfies the Prodi-Serrin condition (1.18), then it satisfies (3.3). Indeed, if  $q \geq 4$ , then we can combine (1.18) with  $v \in L^\infty(0, T; L^2(\mathcal{F})^3)$  to deduce

$$v \in L^{4\frac{q-2}{q-3}}(0, T; L^4(\mathcal{F})^3) \subset L^4(0, T; L^4(\mathcal{F})^3).$$

If  $3 \leq q < 4$ , then we can combine (1.18) with  $v \in L^2(0, T; L^6(\mathcal{F})^3)$  to deduce

$$v \in L^{4\frac{6-q}{q-2q}}(0, T; L^4(\mathcal{F})^3) \subset L^4(0, T; L^4(\mathcal{F})^3).$$

1 Assume now that  $v$  is a weak solution of (1.7)–(1.9) satisfying (3.3). We consider the approximation  $v_\varepsilon \in$   
 2  $H^1(0, t; \mathcal{H}^1)$  given by (2.14) and we use it as the test function in (3.1) for  $s = 0$ . We obtain

$$\begin{aligned} - \int_0^t \langle v, \partial_t v_\varepsilon \rangle_{\mathcal{L}^2} d\tau + 2\mu \iint_{(0,t) \times \mathbb{R}^3} \mathbb{D}(v) : \mathbb{D}(v_\varepsilon) dy d\tau + \iint_{(0,t) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla) v \cdot v_\varepsilon dy d\tau \\ = \langle a, v_\varepsilon(0, \cdot) \rangle_{\mathcal{L}^2} - \langle v(t, \cdot), v_\varepsilon(t, \cdot) \rangle_{\mathcal{L}^2}. \end{aligned} \quad (3.5)$$

From Lemma 2.5, we have

$$\iint_{(0,t) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla) v \cdot v_\varepsilon dy d\tau = - \iint_{(0,t) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla) v_\varepsilon \cdot v dy d\tau$$

3 and thus

$$\begin{aligned} \left| \iint_{(0,t) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla) v \cdot v dy d\tau - \iint_{(0,t) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla) v_\varepsilon \cdot v dy d\tau \right| \\ \lesssim \left( \|v\|_{L^4(0,T;L^4(\mathcal{F})^3)}^2 + \|v\|_{L^\infty(0,T;L^2)}^2 \right) \|v - v_\varepsilon\|_{L^2(0,t;\mathcal{H}^1)}. \end{aligned}$$

4 Using Lemma 2.9 and Lemma 2.5, we deduce

$$\lim_{\varepsilon \rightarrow 0} \iint_{(0,t) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla) v_\varepsilon \cdot v dy d\tau = \iint_{(0,t) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla) v \cdot v dy d\tau = 0.$$

5 Moreover, Lemma 2.9 and the continuity of  $v$  in the weak topology of  $L^2(\mathbb{R}^3)^3$  imply

$$\lim_{\varepsilon \rightarrow 0} \iint_{(0,t) \times \mathbb{R}^3} \mathbb{D}(v) : \mathbb{D}(v_\varepsilon) dy d\tau = \iint_{(0,t) \times \mathbb{R}^3} |\mathbb{D}(v)|^2 dy d\tau,$$

$$\int_0^t \langle v, \partial_t v_\varepsilon \rangle_{\mathcal{L}^2} d\tau = 0, \quad \lim_{\varepsilon \rightarrow 0} \langle v(t, \cdot), v_\varepsilon(t, \cdot) \rangle_{\mathcal{L}^2} = \frac{1}{2} \|v(t, \cdot)\|_{\mathcal{L}^2}^2, \quad \lim_{\varepsilon \rightarrow 0} \langle a, v_\varepsilon(0, \cdot) \rangle_{\mathcal{L}^2} = \frac{1}{2} \|a\|_{\mathcal{L}^2}^2.$$

7 Combining the above relations together with (3.5), we complete the proof of the proposition.  $\square$

## 8 4 Existence and uniqueness of mild solutions

9 In this section, we consider mild solutions for the system (1.7)–(1.9). More precisely, writing (1.7)–(1.9) as (2.3),  
 10 with the help of the fluid-structure operator  $\mathbb{A}_q$ , the function  $\mathbb{F}$  and the projection operator  $\mathbb{P}_q$  introduced in  
 11 Section 2.1, we say that  $v$  is a mild solution of (1.7)–(1.9) or of (2.3), if  $v$  satisfies the following integral equation

$$v(t) = e^{t\mathbb{A}} a + \int_0^t e^{(t-s)\mathbb{A}} \mathbb{P} \operatorname{div} \mathbb{F}(v)(s) ds. \quad (4.1)$$

12 Here and in what follows, we can drop the indices  $q$  for  $\mathbb{A}_q$  and  $\mathbb{P}_q$  to simplify the writing. There is a strong  
 13 connection between mild solutions and weak solutions even if these notions are usually not equivalent. Here, we  
 14 consider initial data in  $\mathcal{L}^2 \cap \mathcal{L}^3$  and we prove the local in time existence and uniqueness of mild solutions that  
 15 are also weak solutions in the sense of Definition 1.1. The precise statement is as follows:

1 **Theorem 4.1.** *Let us assume that  $a \in \mathcal{L}^2 \cap \mathcal{L}^3$ . Then there exist  $T_* \in (0, T]$  and a unique mild solution*  
 2  *$v \in C^0([0, T_*]; \mathcal{L}^3)$  of (1.7)–(1.9) on  $(0, T_*)$  with*

$$t^{1/4}v \in C^0([0, T_*]; \mathcal{L}^6), \quad t^{1/2}v \in C^0([0, T_*]; \mathcal{L}^\infty), \quad t^{1/2}\nabla v \in C^0([0, T_*]; L^3(\mathcal{F})^9), \quad (4.2)$$

3 *and*

$$\lim_{t \rightarrow 0} \left\| t^{1/4}v(t) \right\|_{\mathcal{L}^6} + \left\| t^{1/2}v(t) \right\|_{\mathcal{L}^\infty} + \left\| t^{1/2}\nabla v(t) \right\|_{L^3(\mathcal{F})^9} = 0. \quad (4.3)$$

4 *Furthermore,  $v$  is a weak solution of (1.7)–(1.9) on  $(0, T_*)$  and it satisfies the energy equality (3.4) and*

$$t^{3/2(1/3-1/q)}v \in C^0([0, T_*]; \mathcal{L}^q) \quad (q \in [3, \infty]). \quad (4.4)$$

5 In [4, Theorem 8.2], it was already proved that if  $a \in \mathcal{L}^3$ , there exist  $T_* \in (0, T]$  and a unique mild solution  
 6  $v \in C^0([0, T_*]; \mathcal{L}^3)$  of (1.7)–(1.9) on  $(0, T_*)$  with (4.2) and (4.3). It is also shown in [4, Theorem 8.2] that this  
 7 solution satisfies (4.4). The proof of the existence of a mild solution is based on the Banach fixed point in the  
 8 Banach space

$$\mathcal{C}_T = \left\{ v \mid t^{1/4}v \in C^0([0, T]; \mathcal{L}^6), \quad t^{1/2}v \in C^0([0, T]; \mathcal{L}^\infty), \quad t^{1/2}\nabla v \in C^0([0, T]; L^3(\mathcal{F})^9) \right\}, \quad T > 0,$$

9 equipped with the norm

$$\|v\|_{\mathcal{C}_T} := \left\| t^{1/4}v(t) \right\|_{L^\infty(0, T; \mathcal{L}^6)} + \left\| t^{1/2}v(t) \right\|_{L^\infty(0, T; \mathcal{L}^\infty)} + \left\| t^{1/2}\nabla v(t) \right\|_{L^\infty(0, T; L^3(\mathcal{F})^9)}. \quad (4.5)$$

10 More precisely, let us denote by

$$\mathcal{C}_{T, R} := \{v \in \mathcal{C}_T \mid \|v\|_{\mathcal{C}_T} \leq R\}, \quad (4.6)$$

11 the closed ball of radius  $R > 0$ , and by  $\mathcal{N}$  the map defined by

$$\mathcal{N}v(t) := e^{t\mathbb{A}}a + \int_0^t e^{(t-s)\mathbb{A}}\mathbb{P} \operatorname{div} \mathbb{F}(v)(s) \, ds. \quad (4.7)$$

Then it is shown in [4] that for  $R > 0$  small enough, there exists  $T_R > 0$  such that  $\mathcal{C}_{T_R, R}$  is invariant by  $\mathcal{N}$  and that on this ball,  $\mathcal{N}$  is a strict contraction. It thus has a fixed point that satisfies (4.1). Furthermore,  $v$  is the limit of the sequence

$$v_1 = 0, \quad v_{n+1} = \mathcal{N}v_n \quad (n \in \mathbb{N}),$$

12 for which we have  $v_n \in \mathcal{C}_{T_R, R}$  for all  $n \in \mathbb{N}$ . It is then proved in [4] that

$$\left\| t^{3/2(1/3-1/q)}v_n \right\|_{L^\infty(0, T_R; \mathcal{L}^q)} \lesssim_q R \quad (q \in [3, \infty]). \quad (4.8)$$

13 We are now in a position to prove Theorem 4.1:

14 *Proof of Theorem 4.1.* First we can show that by taking  $R > 0$  possibly smaller, we have

$$v \in C^0([0, T_R]; \mathcal{L}^2), \quad t^{1/2}\nabla v \in C^0([0, T_R]; L^2(\mathcal{F})^9). \quad (4.9)$$

15 Using (4.7), (4.8), Theorem 2.3 and Lemma 2.4, we find for  $t \in [0, T_R]$

$$\begin{aligned} \|v_{n+1}(t)\|_{\mathcal{L}^2} &= \|\mathcal{N}v_n(t)\|_{\mathcal{L}^2} \lesssim \|a\|_{\mathcal{L}^2} + \left\| t^{1/2}v_n \right\|_{L^\infty(0, T_R; \mathcal{L}^\infty)} \|v_n\|_{L^\infty(0, T_R; \mathcal{L}^2)} \int_0^t (t-s)^{-1/2} s^{-1/2} \, ds \\ &\lesssim \|a\|_{\mathcal{L}^2} + R \|v_n\|_{L^\infty(0, T_R; \mathcal{L}^2)}, \end{aligned}$$

16 and

$$\begin{aligned}
\left\| t^{1/2} \nabla v_{n+1}(t) \right\|_{L^2(\mathcal{F})^9} &\lesssim \|a\|_{\mathcal{L}^2} + t^{1/2} \|\ell_{v_n}\|_{L^\infty(0, T_R)} \left\| t^{1/2} \nabla v_n \right\|_{L^\infty(0, T_R; L^2(\mathcal{F})^9)} \int_0^t (t-s)^{-1/2} s^{-1/2} \, ds \\
&+ t^{1/2} \left\| t^{1/8} v_n \right\|_{L^\infty(0, T_R; \mathcal{L}^4)} \left\| t^{1/2} \nabla v_n \right\|_{L^\infty(0, T_R; L^2(\mathcal{F})^9)} \int_0^t (t-s)^{-7/8} s^{-1/8} s^{-1/2} \, ds \\
&\lesssim \|a\|_{\mathcal{L}^2} + R \left\| t^{1/2} \nabla v_n \right\|_{L^\infty(0, T_R; L^2(\mathcal{F})^9)}.
\end{aligned}$$

Thus taking  $R$  small enough, the sequences  $(v_n)_n$  and  $(t^{1/2} \nabla v_n)_n$  are bounded in  $L^\infty(0, T_R; \mathcal{L}^2(\mathcal{F}))$  and  $L^\infty(0, T_R; L^2(\mathcal{F})^9)$  respectively. Passing to the limit, we deduce

$$v \in L^\infty(0, T_R; \mathcal{L}^2) \quad \text{and} \quad t^{1/2} \nabla v \in L^\infty(0, T_R; L^2(\mathcal{F})^9).$$

1 The arguments to deduce the continuity in time of  $v$  and  $\nabla v$  from the above relations are classical, see, for  
2 instance [20] or [1, p.118]. We take  $T_* = T_R$  from now on.

We deduce from (4.2) and (4.9) that

$$t\mathbb{P} \operatorname{div} \mathbb{F}(v) \in L^2(0, T_*; \mathcal{L}^2),$$

3 so that  $\tilde{v}(t) := tv(t)$  is the strong solution of

$$\tilde{v}' = \mathbb{A}_2 \tilde{v} + t\mathbb{P} \operatorname{div} \mathbb{F}(v) + \tilde{v} \quad (t \in (0, T_*)), \quad \tilde{v}(0) = 0.$$

4 We thus deduce from Proposition 2.2 that  $\tilde{v} \in L^2(0, T_*; \mathcal{D}(\mathbb{A}_2)) \cap H^1(0, T_*; \mathcal{L}^2)$  and that  $v \in L^2(\varepsilon, T_*; \mathcal{D}(\mathbb{A}_2)) \cap$   
5  $H^1(\varepsilon, T_*; \mathcal{L}^2)$  satisfies

$$v' = \mathbb{A}_2 v + \mathbb{P}_2 \operatorname{div} \mathbb{F}(v) \tag{4.10}$$

6 in a strong sense in  $(\varepsilon, T_*)$  for any  $\varepsilon > 0$ . Multiplying the above equation by  $v$ , integrating over the interval  
7  $(\varepsilon, t)$  and using Lemma 2.5, we obtain

$$\frac{1}{2} \|v(t, \cdot)\|_{\mathcal{L}^2}^2 + 2\mu \int_\varepsilon^t \int_{\mathcal{F}} |\mathbb{D}(v)|^2 \, dy \, d\tau = \frac{1}{2} \|v(\varepsilon)\|_{\mathcal{L}^2}^2 \quad (0 < \varepsilon \leq t \leq T_*).$$

8 Letting  $\varepsilon \rightarrow 0$ , in the above identity we obtain the energy equality. From the Korn and the Poincaré inequalities,  
9 we also deduce that  $v \in L^2(0, T_*; \mathcal{H}^1)$ . Assume  $\varphi \in H^1(0, T_*; \mathcal{H}^1)$  with  $\varphi(T_*, \cdot) = 0$ . Multiplying (4.10) by  $\varphi$ ,  
10 integrating over the interval  $(\varepsilon, t)$  and integrating by parts, we find

$$\begin{aligned}
& - \int_\varepsilon^{T_*} \langle v, \partial_t \varphi \rangle_{\mathcal{L}^2} \, dt + \iint_{(\varepsilon, T_*) \times \mathbb{R}^3} 2\mu \mathbb{D}(v) : \mathbb{D}(\varphi) \, dy \, d\tau + \iint_{(\varepsilon, T_*) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla) v \cdot \varphi \, dy \, d\tau \\
& \qquad \qquad \qquad = \langle v(\varepsilon, \cdot), \varphi(\varepsilon, \cdot) \rangle_{\mathcal{L}^2}.
\end{aligned}$$

11 By letting  $\varepsilon \rightarrow 0$ , we recover (1.16) and we deduce that  $v$  is a weak solution of (1.7)–(1.9).  $\square$

## 12 5 Uniqueness of weak solutions

13 In this section, we show Theorem 1.4 in the non critical case and Theorem 1.6 by proceeding as in [25] and then  
14 we show Theorem 1.4 in the critical case by following the method of [21].

15 Let  $v_\varepsilon^{(1)}$  and  $v_\varepsilon^{(2)}$  be the mollifiers of  $v^{(1)}$  and  $v^{(2)}$  respectively. We apply Proposition 3.1 with  $\varphi = v_\varepsilon^{(1)}$  and  
16  $\varphi = v_\varepsilon^{(2)}$ . This leads to

$$\begin{aligned}
& - \int_0^t \langle v^{(1)}, \partial_t v_\varepsilon^{(2)} \rangle_{\mathcal{L}^2} \, d\tau + 2\mu \iint_{(0, t) \times \mathbb{R}^3} \mathbb{D}(v^{(1)}) : \mathbb{D}(v_\varepsilon^{(2)}) \, dy \, d\tau + \iint_{(0, t) \times \mathcal{F}} ((v^{(1)} - \ell_{v^{(1)}}) \cdot \nabla) v^{(1)} \cdot v_\varepsilon^{(2)} \, dy \, d\tau \\
& \qquad \qquad \qquad = \langle a, v_\varepsilon^{(2)}(0, \cdot) \rangle_{\mathcal{L}^2} - \langle v^{(1)}(t, \cdot), v_\varepsilon^{(2)}(t, \cdot) \rangle_{\mathcal{L}^2}, \tag{5.1}
\end{aligned}$$

1 and

$$\begin{aligned}
& - \int_0^t \langle v^{(2)}, \partial_t v_\varepsilon^{(1)} \rangle_{\mathcal{L}^2} d\tau + 2\mu \iint_{(0,t) \times \mathbb{R}^3} \mathbb{D}(v^{(2)}) : \mathbb{D}(v_\varepsilon^{(1)}) dy d\tau + \iint_{(0,t) \times \mathcal{F}} ((v^{(2)} - \ell_{v^{(2)}}) \cdot \nabla) v^{(2)} \cdot v_\varepsilon^{(1)} dy d\tau \\
& = \langle a, v_\varepsilon^{(1)}(0, \cdot) \rangle_{\mathcal{L}^2} - \langle v^{(2)}(t, \cdot), v_\varepsilon^{(1)}(t, \cdot) \rangle_{\mathcal{L}^2}. \quad (5.2)
\end{aligned}$$

2 Then using Lemma 2.9, we have

$$- \int_0^t \langle v^{(1)}, \partial_t v_\varepsilon^{(2)} \rangle_{\mathcal{L}^2} ds - \int_0^t \langle v^{(2)}, \partial_t v_\varepsilon^{(1)} \rangle_{\mathcal{L}^2} ds = 0, \quad (5.3)$$

3

$$\langle a, v_\varepsilon^{(2)}(0, \cdot) \rangle_{\mathcal{L}^2} - \langle v^{(1)}(t, \cdot), v_\varepsilon^{(2)}(t, \cdot) \rangle_{\mathcal{L}^2} \rightarrow \frac{1}{2} \langle a, a \rangle_{\mathcal{L}^2} - \frac{1}{2} \langle v^{(1)}(t, \cdot), v^{(2)}(t, \cdot) \rangle_{\mathcal{L}^2}, \quad (5.4)$$

4

$$\langle a, v_\varepsilon^{(1)}(0, \cdot) \rangle_{\mathcal{L}^2} - \langle v^{(2)}(t, \cdot), v_\varepsilon^{(1)}(t, \cdot) \rangle_{\mathcal{L}^2} \rightarrow \frac{1}{2} \langle a, a \rangle_{\mathcal{L}^2} - \frac{1}{2} \langle v^{(1)}(t, \cdot), v^{(2)}(t, \cdot) \rangle_{\mathcal{L}^2}. \quad (5.5)$$

5 Using these relations, summing (5.1) and (5.2) and passing to the limit  $\varepsilon \rightarrow 0$ , we deduce

$$\begin{aligned}
& \langle v^{(1)}(t, \cdot), v^{(2)}(t, \cdot) \rangle_{\mathcal{L}^2} + 4\mu \iint_{(0,t) \times \mathbb{R}^3} \mathbb{D}(v^{(1)}) : \mathbb{D}(v^{(2)}) dy d\tau \\
& + \iint_{(0,t) \times \mathcal{F}} ((v^{(1)} - \ell_{v^{(1)}}) \cdot \nabla) v^{(1)} \cdot v^{(2)} dy d\tau + \iint_{(0,t) \times \mathcal{F}} ((v^{(2)} - \ell_{v^{(2)}}) \cdot \nabla) v^{(2)} \cdot v^{(1)} dy d\tau = \langle a, a \rangle_{\mathcal{L}^2}, \quad (5.6)
\end{aligned}$$

Applying Proposition 3.2 and using that  $v^{(1)}$  satisfies the energy inequality (1.17), we can also write

$$\frac{1}{2} \|v^{(1)}(t, \cdot)\|_{\mathcal{L}^2}^2 + 2\mu \iint_{(0,t) \times \mathbb{R}^3} |\mathbb{D}(v^{(1)})|^2 dy d\tau \leq \frac{1}{2} \|a\|_{\mathcal{L}^2}^2 \quad (t \in [0, T]),$$

$$\frac{1}{2} \|v^{(2)}(t, \cdot)\|_{\mathcal{L}^2}^2 + 2\mu \iint_{(0,t) \times \mathbb{R}^3} |\mathbb{D}(v^{(2)})|^2 dy d\tau = \frac{1}{2} \|a\|_{\mathcal{L}^2}^2 \quad (t \in [0, T]).$$

6 Combining the two above estimates with (5.6), we deduce

$$\begin{aligned}
& \frac{1}{2} \|(v^{(2)} - v^{(1)})(t, \cdot)\|_{\mathcal{L}^2}^2 + 2\mu \iint_{(0,t) \times \mathbb{R}^3} |\mathbb{D}(v^{(2)} - v^{(1)})|^2 dy d\tau \\
& \leq \iint_{(0,t) \times \mathcal{F}} ((v^{(2)} - \ell_{v^{(2)}}) \cdot \nabla) v^{(2)} \cdot v^{(1)} dy d\tau \\
& + \iint_{(0,t) \times \mathcal{F}} ((v^{(1)} - \ell_{v^{(1)}}) \cdot \nabla) v^{(1)} \cdot v^{(2)} dy d\tau, \quad (t \in [0, T]). \quad (5.7)
\end{aligned}$$

Set

$$w := v^{(1)} - v^{(2)}.$$

7 Using Lemma 2.5, we obtain

$$\begin{aligned}
& \int_{\mathcal{F}} ((v^{(2)} - \ell_{v^{(2)}}) \cdot \nabla) v^{(2)} \cdot v^{(1)} dy + \int_{\mathcal{F}} ((v^{(1)} - \ell_{v^{(1)}}) \cdot \nabla) v^{(1)} \cdot v^{(2)} dy \\
& = \int_{\mathcal{F}} ((w - \ell_w) \cdot \nabla) v^{(1)} \cdot v^{(2)} dy = \int_{\mathcal{F}} ((w - \ell_w) \cdot \nabla) w \cdot v^{(2)} dy.
\end{aligned}$$

8 Combining this with (5.7), we finally arrive to

$$\frac{1}{2} \|w(t, \cdot)\|_{\mathcal{L}^2}^2 + 2\mu \iint_{(0,t) \times \mathbb{R}^3} |\mathbb{D}(w)|^2 dy d\tau \leq \iint_{(0,t) \times \mathcal{F}} ((w - \ell_w) \cdot \nabla) w \cdot v^{(2)} dy d\tau \quad (t \in [0, T]). \quad (5.8)$$

9 We are now in a position to prove the uniqueness results. We start by proving Theorem 1.4 in the non-critical  
10 case:

1 *Proof of Theorem 1.4 in the noncritical case :  $q > 3$ .* We apply the estimate (2.7) on (5.8) to obtain

$$\begin{aligned} \frac{1}{2} \|w(t, \cdot)\|_{\mathcal{L}^2}^2 + 2\mu \iint_{(0,t) \times \mathbb{R}^3} |\mathbb{D}(w)|^2 \, dy \, d\tau \\ \lesssim \|\nabla w\|_{L^2(0,t;L^2(\mathcal{F})^9)}^{1+3/q} \left( \int_0^t \|w(s, \cdot)\|_{L^2(\mathcal{F})^3}^2 \|v^{(2)}(s, \cdot)\|_{L^q(\mathcal{F})^3}^r \, ds \right)^{1/r} \\ + \|v^{(2)}\|_{L^\infty(0,T;L^2)} \|\nabla w\|_{L^2(0,t;L^2(\mathcal{F})^9)} \left( \int_0^t |\ell_w(s)|^2 \, ds \right)^{1/2}. \end{aligned}$$

2 Therefore by Young's inequality, for any  $\eta > 0$  there exists  $C_\eta > 0$  such that

$$\begin{aligned} \frac{1}{2} \|w(t, \cdot)\|_{\mathcal{L}^2}^2 + 2\mu \iint_{(0,t) \times \mathbb{R}^3} |\mathbb{D}(w)|^2 \, dy \, d\tau \\ \leq \eta \|\nabla w\|_{L^2(0,t;L^2(\mathcal{F})^9)}^2 + C_\eta \int_0^t \|w(s, \cdot)\|_{L^2(\mathcal{F})^3}^2 \|v^{(2)}(s, \cdot)\|_{L^q(\mathcal{F})^3}^r \, ds + C_\eta \|v^{(2)}\|_{L^\infty(0,T;L^2)}^2 \int_0^t |\ell_w(s)|^2 \, ds. \end{aligned} \quad (5.9)$$

3 In particular, taking  $\eta$  small enough, we deduce

$$\|w(t, \cdot)\|_{\mathcal{L}^2}^2 \lesssim \int_0^t \left( \|v^{(2)}(s, \cdot)\|_{L^q(\mathcal{F})^3}^r + \|v^{(2)}\|_{L^\infty(0,T;L^2)}^2 \right) \|w(s, \cdot)\|_{\mathcal{L}^2}^2 \, ds \quad (t \in [0, T]), \quad (5.10)$$

4 and from the Grönwall lemma,  $w = 0$  in  $[0, T]$ .  $\square$

5 *Proof of Theorem 1.6.* We claim that there is a  $t_0 > 0$  such that  $v^{(1)} = v^{(2)}$  on  $[0, t_0]$ . Else, for any  $t \in (0, T]$ ,  
6  $w := v^{(1)} - v^{(2)}$  satisfies

$$\int_0^t \|\nabla w(s, \cdot)\|_{L^2(\mathcal{F})^9}^2 \, ds > 0. \quad (5.11)$$

7 In that case, since  $v^{(2)} \in L^\infty(0, T; L^3(\mathcal{F})^3)$  and since  $v^{(2)}$  right-continuous at  $t = 0$  in the norm of  $L^3$ , we can  
8 apply Lemma 2.8 on (5.8): for any  $\eta > 0$ , there exists  $C_\eta > 0$  such that

$$\begin{aligned} \frac{1}{2} \|w(t, \cdot)\|_{\mathcal{L}^2}^2 + 2\mu \iint_{(0,t) \times \mathbb{R}^3} |\mathbb{D}(w)|^2 \, dy \, d\tau \leq \eta \|\nabla w\|_{L^2(0,t;L^2(\mathcal{F})^9)}^2 \\ + C_\eta \int_0^t \|w(s, \cdot)\|_{L^2(\mathcal{F})^3}^2 \, ds + C_\eta \|v^{(2)}\|_{L^\infty(0,T;L^2)}^2 \int_0^t |\ell_w(s)|^2 \, ds \quad (t \in [0, T]). \end{aligned} \quad (5.12)$$

9 Taking  $\eta$  small enough and using the Grönwall lemma, we conclude that  $w \equiv 0$  which contradicts (5.11).

10 In a similar manner, if  $v^{(1)}(t, \cdot) = v^{(2)}(t, \cdot)$  for  $t \in (0, T)$ , then there exists  $t_0 > 0$  such that  $v^{(1)} = v^{(2)}$  on  
11  $[t, t + t_0]$ . Using this property and the weak continuity of  $v^{(1)}$  and  $v^{(2)}$  (see Proposition 3.1, we deduce the  
12 result.  $\square$

13 It remains to show the uniqueness in the critical case, and we can obtain it by following the method of [21]  
14 and by using Theorem 1.6.

15 *Proof of Theorem 1.4 in the critical case.* We show that  $v^{(2)} \in C_{\mathbb{R}}^0([0, T]; L^3(\mathcal{F})^3)$  so that we can apply Theo-  
16 rem 1.6 and deduce that  $v^{(2)} = v^{(1)}$ .

Assume  $t \in [0, T)$  and let  $(t_k)$  be a sequence in  $(0, T)$  such that  $t_k \rightarrow t$  and such that

$$\|v^{(2)}(t_k, \cdot)\|_{L^3(\mathcal{F})^3} \leq \|v^{(2)}\|_{L^\infty(0,T;L^3(\mathcal{F})^3)}.$$

Then, using Proposition 3.1, we deduce that up to a subsequence,

$$v^{(2)}(t_k, \cdot) \rightharpoonup v^{(2)}(t, \cdot) \quad \text{weakly in } L^2(\mathcal{F})^3 \cap L^3(\mathcal{F})^3.$$

1 In particular, we deduce that  $v^{(2)}(t, \cdot) \in \mathcal{L}^2 \cap \mathcal{L}^3$ . From Theorem 4.1, there exist  $T_* > 0$  and a weak solution  
2  $\tilde{v}^{(2)} \in C^0([t, T_*]; \mathcal{L}^3)$  of (1.7)–(1.8) in  $[t, T_*]$  with initial condition  $\tilde{v}^{(2)}(t, \cdot) = v^{(2)}(t, \cdot)$  (instead of (1.9)). Applying  
3 Proposition 3.2,  $v^{(2)}$  satisfies the energy inequality (1.17) and thus from Theorem 1.6, we deduce that  $\tilde{v}^{(2)} = v^{(2)}$   
4 in  $[t, T_*]$ . Consequently,  $v^{(2)} \in C^0([t, T_*]; \mathcal{L}^3)$  and is right-continuous at  $t$ . We deduce  $v^{(2)} \in C_{\mathbb{R}}^0([0, T]; L^3(\mathcal{F}))^3$   
5 and this concludes the proof of Theorem 1.4 in the critical case.  $\square$

## 6 Regularity of weak solutions

7 In this section, we prove Theorem 1.8 and Theorem 1.10. Assume  $v$  is a weak solution of (1.7)–(1.9) in the  
8 sense of Definition 1.1, and satisfies the hypothesis of either Theorem 1.8 or Theorem 1.10:

$$v \in L^r(0, T; L^q(\mathcal{F}))^3 \quad \text{for some } q \in (3, \infty) \text{ and } \frac{2}{r} + \frac{3}{q} = 1, \quad (6.1)$$

9 or for  $\varepsilon > 0$

$$v \in L^\infty(0, T; L^3(\mathcal{F}))^3 \text{ with } \|v\|_{L^\infty(0, T; \mathcal{L}^3)} \leq \varepsilon. \quad (6.2)$$

10 In order to show the regularity property in Theorem 1.8, we notice that  $v$  satisfies (in a weak sense and for  
11  $f = 0$ ) the following linear problem

$$\left\{ \begin{array}{l} \partial_t w - \mu \Delta w + \nabla \pi + ((v - \ell_v) \cdot \nabla) w = f \quad (t \in (0, T), y \in \mathcal{F}), \\ \operatorname{div} w = 0 \quad (t \in (0, T), y \in \mathcal{F}), \\ w(t, y) = \ell_w(t) + \omega_w(t) \times y \quad (t \in (0, T), y \in \partial \mathcal{S}), \\ m \ell'_w(t) = - \int_{\partial \mathcal{S}} \mathbb{T}(w, \pi) \nu \, d\gamma_y + m \ell_f \quad (t \in (0, T)), \\ J \omega'_w(t) = - \int_{\partial \mathcal{S}} y \times \mathbb{T}(w, \pi) \nu \, d\gamma_y + J \omega_f \quad (t \in (0, T)), \\ w(0, \cdot) = a \quad \text{in } \mathcal{F}, \quad \ell_w(0) = \ell_a, \quad \omega_w(0) = \omega_a. \end{array} \right. \quad (6.3)$$

12 Let us introduce the bilinear operator

$$\mathbb{B}(v, w) := -\mathbb{P}(((v - \ell_v) \cdot \nabla) w \mathbf{1}_{\mathcal{F}}), \quad (6.4)$$

13 where  $\mathbb{P}$  is the projection operator defined in Section 2.1. Using the operator  $\mathbb{A}$ , also introduced in Section 2.1,  
14 we can write the above system in the following form:

$$\partial_t w = \mathbb{A}w + \mathbb{B}(v, w) + \mathbb{P}f \quad \text{in } (0, T), \quad w(0) = a. \quad (6.5)$$

15 The proofs of Theorem 1.8 and Theorem 1.10 rely on the regularity of this linear system. Indeed, in what  
16 follows, we are going to show that  $v = w$  a.e. in  $[0, T] \times \mathcal{F}$ , and that  $w$  has more regularity than that initially  
17 assumed on  $v$ .

### 6.1 Strong solutions for the linear system

18 Here, we consider strong solutions of (6.3) or equivalently of (6.5), that is solutions in a space

$$\mathcal{X}^q(0, T) = L^q(0, T; \mathcal{D}(\mathbb{A}_q)) \cap W^{1, q}(0, T; \mathcal{L}^q),$$

for some  $q > 1$ . In the case  $q = 2$ , we recall that

$$\mathcal{X}^2(0, T) \subset C^0([0, T]; \mathcal{H}^1),$$



1 and we can thus equip  $\mathcal{X}^2(0, T)$  with the norm

$$\|w\|_{\mathcal{X}^2(0, T)} := \|w\|_{L^2(0, T; H^2(\mathcal{F})^3)} + \|w\|_{H^1(0, T; \mathcal{L}^2)} + \|w\|_{C^0([0, T]; \mathcal{H}^1)}.$$

2 We can show the following result:

3 **Proposition 6.1.**

- 4 • Assume  $v$  is a weak solution of (1.7)–(1.9) and that  $v$  satisfies (6.1). Assume also that  $\mathbb{P}_2 f \in L^2(0, T; \mathcal{L}^2)$   
5 and  $a \in \mathcal{H}^1$ . Then there exists a unique strong solution  $w \in \mathcal{X}^2(0, T)$  of (6.5).  
6 • There exists  $\varepsilon > 0$  such that for any  $v$  weak solution of (1.7)–(1.9) satisfying (6.2) and for any  $\mathbb{P}_2 f \in$   
7  $L^2(0, T; \mathcal{L}^2)$ ,  $a \in \mathcal{H}^1$ , there exists a unique strong solution  $w \in \mathcal{X}^2(0, T)$  of (6.5).

8 *Proof.* We consider for  $\tau \in (0, T]$  the mapping

$$\mathcal{N} : L^2(0, \tau; \mathcal{L}^2) \rightarrow L^2(0, \tau; \mathcal{L}^2), \quad g \mapsto \mathbb{B}(v, w) + \mathbb{P}f, \quad (6.6)$$

9 where  $w \in \mathcal{X}^2(0, \tau)$  is the solution (see Proposition 2.2) of

$$\partial_t w = \mathbb{A}w + g \quad \text{in } (0, \tau), \quad w(0) = a. \quad (6.7)$$

We have the estimate

$$\|w\|_{\mathcal{X}^2(0, \tau)} \lesssim \|g\|_{L^2(0, \tau; \mathcal{L}^2)} + \|a\|_{\mathcal{H}^1}.$$

10 We can see that the corresponding constant does not depend on  $\tau$  by extending  $g$  by 0 on  $(\tau, \infty)$ .

11 We first consider the case when  $v$  satisfies (6.1). If  $q \in (3, \infty)$ , using interpolation and Sobolev embedding  
12 yield that for a.e.  $t \in (0, \tau)$ ,

$$\|\nabla w(t)\|_{L^{2q/(q-2)}(\mathcal{F})^9} \lesssim \|\nabla w(t)\|_{H^{3/q}(\mathcal{F})^9} \lesssim \|w(t)\|_{H^2(\mathcal{F})^3}^{3/q} \|w(t)\|_{H^1(\mathcal{F})^3}^{1-3/q},$$

13 and thus

$$\|\nabla w\|_{L^{2r/(r-2)}(0, \tau; L^{2q/(q-2)}(\mathcal{F})^9)} \lesssim \|w\|_{L^\infty(0, \tau; H^1(\mathcal{F})^3)}^{2/r} \|w\|_{L^2(0, \tau; H^2(\mathcal{F})^3)}^{1-2/r} \lesssim \|w\|_{\mathcal{X}^2(0, \tau)}.$$

14 From the Hölder inequality, we deduce

$$\|\mathbb{B}(v, w)\|_{L^2(0, \tau; \mathcal{L}^2)} \lesssim \left( \|v\|_{L^r(0, \tau; L^q(\mathcal{F})^3)} + \|\ell_v\|_{L^2(0, \tau; \mathbb{R}^3)} \right) \|w\|_{\mathcal{X}^2(0, \tau)}, \quad (6.8)$$

15 and thus  $\mathcal{N}$  is well-defined. If  $q = \infty$ , then we have

$$\begin{aligned} \|\mathbb{B}(v, w)\|_{L^2(0, \tau; \mathcal{L}^2)} &\lesssim \left( \|v\|_{L^2(0, \tau; L^\infty(\mathcal{F})^3)} + \|\ell_v\|_{L^2(0, \tau; \mathbb{R}^3)} \right) \|\nabla w\|_{L^\infty(0, \tau; L^2(\mathcal{F})^9)} \\ &\lesssim \left( \|v\|_{L^2(0, \tau; L^\infty(\mathcal{F})^3)} + \|\ell_v\|_{L^2(0, \tau; \mathbb{R}^3)} \right) \|w\|_{\mathcal{X}^2(0, \tau)}. \end{aligned} \quad (6.9)$$

Therefore, in this case also  $\mathcal{N}$  is well-defined. Moreover, we deduce from the above estimates the existence of a constant  $C_0 > 0$  such that

$$\left\| \mathcal{N}g^{(1)} - \mathcal{N}g^{(2)} \right\|_{L^2(0, \tau; \mathcal{L}^2)} \leq C_0 \left( \|v\|_{L^r(0, \tau; L^q(\mathcal{F})^3)} + \|\ell_v\|_{L^2(0, \tau; \mathbb{R}^3)} \right) \|g^{(1)} - g^{(2)}\|_{L^2(0, \tau; \mathcal{L}^2)}$$

and there exists  $\tau_1 \in (0, T]$  such that

$$C_0 \left( \|v\|_{L^r(0, \tau_1; L^q(\mathcal{F})^3)} + \|\ell_v\|_{L^2(0, \tau_1; \mathbb{R}^3)} \right) \leq \frac{1}{2}.$$

For such a  $\tau_1$ ,  $\mathcal{N}$  admits a unique fixed point  $g \in L^2(0, \tau_1; \mathcal{L}^2)$  and the corresponding solution  $w$  of (6.7) is a strong solution of (6.5) in  $(0, \tau_1)$ . By considering an increasing sequence  $\tau_k > 0$ ,  $k = 1, \dots, N$  with  $\tau_N = T$  and such that

$$C_0 \left( \|v\|_{L^r(\tau_k, \tau_{k+1}; L^q(\mathcal{F})^3)} + \|\ell_v\|_{L^2(\tau_k, \tau_{k+1}; \mathbb{R}^3)} \right) \leq \frac{1}{2}, \quad (k = 1, \dots, N-1),$$

1 we can extend this solution to a solution of (6.5) in  $(0, T)$ .

2 It remains to consider the case where  $v$  satisfies (6.2). In this case, we take  $\tau = T$ , and using Sobolev  
3 embedding, we infer that

$$\begin{aligned} \|\mathbb{B}(v, w)\|_{L^2(0, T; \mathcal{L}^2)} &\lesssim \|v\|_{L^\infty(0, T; L^3(\mathcal{F}))^3} \|\nabla w\|_{L^2(0, T; L^6(\mathcal{F}))^9} + \|\ell_v\|_{L^\infty(0, T; \mathbb{R}^3)} \|\nabla w\|_{L^2(0, T; L^2(\mathcal{F}))^9} \\ &\lesssim \left( \|v\|_{L^\infty(0, T; L^3(\mathcal{F}))^3} + \|\ell_v\|_{L^\infty(0, T; \mathbb{R}^3)} \right) \|w\|_{\mathcal{X}^2(0, T)}, \end{aligned} \quad (6.10)$$

and thus  $\mathcal{N}$  is well-defined and there exists a constant  $C_1 > 0$  such that

$$\left\| \mathcal{N}g^{(1)} - \mathcal{N}g^{(2)} \right\|_{L^2(0, T; \mathcal{L}^2)} \leq C_1 \left( \|v\|_{L^\infty(0, T; L^3(\mathcal{F}))^3} + \|\ell_v\|_{L^\infty(0, T; \mathbb{R}^3)} \right) \|g^{(1)} - g^{(2)}\|_{L^2(0, T; \mathcal{L}^2)}.$$

4 Therefore, by choosing  $\varepsilon < \frac{1}{2C_1}$  in (6.2), we obtain that  $\mathcal{N}$  admits a unique fixed point  $g \in L^2(0, T; \mathcal{L}^2)$  and  
5 the corresponding solution  $w$  of (6.7) is a strong solution of (6.5) in  $(0, T)$ . This completes the proof of the  
6 proposition.  $\square$

7 Let us consider the adjoint system of (6.5):

$$-\partial_t \varphi = \mathbb{A}\varphi - \mathbb{B}(v, \varphi) + \mathbb{P}f \quad \text{in } (0, T), \quad \varphi(T) = a. \quad (6.11)$$

8 We can show the following result with a proof similar to to the proof of Proposition 6.1:

9 **Proposition 6.2.** *Assume  $v$  is a weak solution of (1.7)–(1.9) and that  $v$  satisfies (6.1) or (6.2) with  $\varepsilon$  given in  
10 Proposition 6.1. Assume also that  $\mathbb{P}_2 f \in L^2(0, T; \mathcal{L}^2)$  and  $a \in \mathcal{H}^1$ . Then there exists a unique strong solution  
11  $\varphi \in \mathcal{X}^2(0, T)$  of (6.11).*

## 12 6.2 Weak solutions for the linear system

13 Let us introduce two different notions of weak solutions for (6.5):

14 **Definition 6.3.** *Assume  $\mathbb{P}_2 f \in L^2(0, T; \mathcal{L}^2)$  and  $a \in \mathcal{L}^2$ .*

15 • *We say that  $w \in L^\infty(0, T; \mathcal{L}^2) \cap L^2(0, T; \mathcal{H}^1)$  is a weak solution of (6.5) if for any  $\varphi \in H^1(0, T; \mathcal{H}^1)$  such  
16 that  $\varphi(T, \cdot) = 0$ ,*

$$\begin{aligned} - \int_0^T \langle w, \partial_t \varphi \rangle_{\mathcal{L}^2} dt + \iint_{(0, T) \times \mathbb{R}^3} 2\mu \mathbb{D}(w) : \mathbb{D}(\varphi) dy d\tau + \iint_{(0, T) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla) w \cdot \varphi dy d\tau \\ = \int_0^T \langle \mathbb{P}_2 f, \varphi \rangle_{\mathcal{L}^2} dt + \langle a, \varphi(0, \cdot) \rangle_{\mathcal{L}^2}. \end{aligned} \quad (6.12)$$

17 • *We say that  $w \in L^2(0, T; \mathcal{L}^2)$  is a very weak solution of (6.5) if for any  $\varphi \in H^1(0, T; \mathcal{D}(\mathcal{A}))$  such that  
18  $\varphi(T, \cdot) = 0$ ,*

$$\begin{aligned} - \int_0^T \langle w, \partial_t \varphi \rangle_{\mathcal{L}^2} dt - \int_0^T \langle w, \mathbb{A}\varphi \rangle_{\mathcal{L}^2} dt - \iint_{(0, T) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla) \varphi \cdot w dy d\tau \\ = \int_0^T \langle \mathbb{P}_2 f, \varphi \rangle_{\mathcal{L}^2} dt + \langle a, \varphi(0, \cdot) \rangle_{\mathcal{L}^2}. \end{aligned} \quad (6.13)$$

19 From the above definition, one can check that a strong solution of (6.5) is a weak solution of (6.5), and that  
20 a weak solution of (6.5) is a very weak solution of (6.5).

21 **Lemma 6.4.** *Assume  $v$  is a weak solution of (1.7)–(1.9) and that  $v$  satisfies (6.1) or (6.2) with  $\varepsilon$  given in  
22 Proposition 6.1. Then there exists at most one very weak solution of (6.5).*

1 *Proof.* Let us consider  $w^{(1)}, w^{(2)} \in L^2(0, T; \mathcal{L}^2)$  two very weak solution of (6.5). Then for any  $\varphi \in H^1(0, T; \mathcal{D}(\mathcal{A}))$   
 2 such that  $\varphi(T, \cdot) = 0$ , we have

$$\begin{aligned}
 & - \int_0^T \langle w^{(1)} - w^{(2)}, \partial_t \varphi \rangle_{\mathcal{L}^2} dt - \int_0^T \langle w^{(1)} - w^{(2)}, \mathbb{A}\varphi \rangle_{\mathcal{L}^2} dt \\
 & \quad - \iint_{(0, T) \times \mathcal{F}} ((v - \ell_v) \cdot \nabla) \varphi \cdot (w^{(1)} - w^{(2)}) dy d\tau = 0. \quad (6.14)
 \end{aligned}$$

Then, using Proposition 6.2, there exists a strong solution  $\varphi \in \mathcal{X}^2(0, T)$  of

$$-\partial_t \varphi = \mathbb{A}\varphi - \mathbb{B}(v, \varphi) + w^{(1)} - w^{(2)} \quad \text{in } (0, T), \quad \varphi(T) = 0.$$

3 We can use the approximations  $\varphi_\varepsilon \in H^1(0, T; \mathcal{D}(\mathcal{A}))$  of  $\varphi$  given by (2.14) as test functions in (6.14). Using  
 4 Lemma 2.9 and (6.8), we can pass to the limit  $\varepsilon \rightarrow 0$  in (6.14) and we deduce  $w^{(1)} = w^{(2)}$ .  $\square$

**Remark 6.5.** *In particular, if  $v$  is a weak solution of (1.7)–(1.9) and if  $v$  satisfies (6.1) or (6.2) with  $\varepsilon$  given  
 in Proposition 6.1, then a very weak solution  $w$  of (6.5) with  $f = 0$  and with the same initial condition  $a \in \mathcal{L}^2$   
 as in (1.9), satisfies*

$$w = v.$$

5 *We use this property in what follows to deduce properties of  $v$  from the same properties for  $w$ .*

6 **Proposition 6.6.** *Assume  $v$  is a weak solution of (1.7)–(1.9) and that  $v$  satisfies (6.1) or (6.2) with  $\varepsilon$  given  
 7 in Proposition 6.1. Assume also  $\mathbb{P}_2 f \in L^2(0, T; \mathcal{L}^2)$  and  $a \in \mathcal{L}^2$ . Then there exists a unique weak solution  $w$  of  
 8 (6.5). Moreover  $w \in C^0([0, T]; \mathcal{L}^2)$ .*

9 *Proof.* The uniqueness comes from Lemma 6.4. For the existence, we consider a sequence  $a^{(k)} \in \mathcal{H}^1$  such that  
 10  $a^{(k)} \in \mathcal{H}^1$  and  $a^{(k)} \rightarrow a$  in  $\mathcal{L}^2$ . Then for any  $k \geq 0$ , there exists a unique strong solution  $w^{(k)} \in \mathcal{X}^2(0, T)$  of  
 11 (6.5) where  $a$  is replaced by  $a^{(k)}$ . We have in particular the relations

$$\begin{aligned}
 \partial_t (w^{(k)} - w^{(k')}) &= \mathbb{A} (w^{(k)} - w^{(k')}) + \mathbb{B} \left( v, (w^{(k)} - w^{(k')}) \right) \quad \text{in } (0, T), \\
 (w^{(k)} - w^{(k')})(0) &= (a^{(k)} - a^{(k')}).
 \end{aligned} \quad (6.15)$$

Multiplying the above first relation with  $(w^{(k)} - w^{(k')})$  and using Lemma 2.5, we deduce that

$$\left\| w^{(k)} - w^{(k')} \right\|_{L^\infty(0, T; \mathcal{L}^2)} + \left\| w^{(k)} - w^{(k')} \right\|_{L^2(0, T; \mathcal{H}^1)} \lesssim \left\| a^{(k)} - a^{(k')} \right\|_{\mathcal{L}^2},$$

12 so that  $(w^{(k)})$  is a Cauchy sequence in  $C^0([0, T]; \mathcal{L}^2) \cap L^2(0, T; \mathcal{H}^1)$  and its limit  $w$  is a weak solution of (6.5).  $\square$

### 13 6.3 Regularity of the weak solutions of the linear system

14 **Lemma 6.7.** *Assume  $v$  is a weak solution of (1.7)–(1.9) and that  $v$  satisfies (6.1) or (6.2) with  $\varepsilon$  given in  
 15 Proposition 6.1. Then  $tv \in \mathcal{X}^2(0, T)$ .*

16 *Proof.* Let us set

$$w(t, \cdot) = tv(t, \cdot) \quad (t \in (0, T)). \quad (6.16)$$

17 Then we deduce from (1.16) that  $w$  is a weak solution of

$$\partial_t w = \mathbb{A}_2 w + \mathbb{B}(v, w) + v \quad \text{in } (0, T), \quad w(0) = 0.$$

18 Since  $v \in L^2(0, T; \mathcal{L}^2)$ , we deduce from Proposition 6.1 that the above equation admits a unique strong solution  
 19 in  $\mathcal{X}^2(0, T)$ . By using, Lemma 6.4, we conclude that  $w \in \mathcal{X}^2(0, T)$ .  $\square$

1 **Lemma 6.8.** Assume  $v$  is a weak solution of (1.7)–(1.9) and that  $v$  satisfies (6.1) or (6.2) with  $\varepsilon$  given in  
 2 Proposition 6.1. Then  $t^4v \in \mathcal{X}^4(0, T)$ .

3 *Proof.* Let us set

$$w(t, \cdot) = t^2v(t, \cdot) \quad (t \in (0, T)). \quad (6.17)$$

4 Then we deduce from Lemma 6.7 that  $w$  is a strong solution of

$$\partial_t w = \mathbb{A}w + \mathbb{B}(tv, tv) + 2tv \quad \text{in } (0, T), \quad w(0) = 0,$$

5 with  $tv \in \mathcal{X}^2(0, T)$ .

On the other hand, from Sobolev embeddings (see, for instance, [22, Lemma 3.3, p.80], [17, Proposition 4.3]), we have

$$L^2(0, T; H^2(\mathcal{F})) \cap H^1(0, T; L^2(\mathcal{F})) \hookrightarrow L^{10}(0, T; L^{10}(\mathcal{F})) \cap L^{10/3}(0, T; W^{1,10/3}(\mathcal{F})).$$

We thus deduce that

$$\mathbb{B}(tv, tv) + 2tv \in L^{5/2}(0, T; \mathcal{L}^{5/2})$$

6 and applying Proposition 2.2, we obtain  $w = t^2v \in \mathcal{X}^{5/2}(0, T)$ .

7 We can now repeat the argument: we set

$$w(t, \cdot) = t^4v(t, \cdot) \quad (t \in (0, T)) \quad (6.18)$$

8 which is the strong solution of

$$\partial_t w = \mathbb{A}w + \mathbb{B}(t^2v, t^2v) + 4t^3v \quad \text{in } (0, T), \quad w(0) = 0,$$

with  $t^2v \in \mathcal{X}^{5/2}(0, T)$ . From Sobolev embeddings (see, for instance, [22, Lemma 3.3, p.80], [17, Proposition 4.3]), we have

$$L^{5/2}(0, T; W^{2,5/2}(\mathcal{F})) \cap W^{1,5/2}(0, T; L^{5/2}(\mathcal{F})) \hookrightarrow L^r(0, T; L^r(\mathcal{F})) \cap L^5(0, T; W^{1,5}(\mathcal{F}))$$

for any  $r > 5/2$ . We thus deduce that

$$\mathbb{B}(t^2v, t^2v) + 4t^3v \in L^4(0, T; \mathcal{L}^4)$$

9 and applying Proposition 2.2, we obtain  $w = t^4v \in \mathcal{X}^4(0, T)$ .

10

**Lemma 6.9.** Assume  $v$  is a weak solution of (1.7)–(1.9) and that  $v$  satisfies (6.1) or (6.2) with  $\varepsilon$  given in Proposition 6.1. Then

$$t^{16}v \in H^2(0, T; \mathcal{L}^2) \cap H^1(0, T; \mathcal{D}(\mathbb{A}_2)) \cap L^2(0, T; H^4(\mathcal{F})^3).$$

11 *Proof.* Let us set

$$w(t, \cdot) = t^8\partial_tv(t, \cdot) \quad (t \in (0, T)). \quad (6.19)$$

12 Then we deduce from Lemma 6.7 that  $w$  is a very weak solution of

$$\partial_t w = \mathbb{A}w + \mathbb{B}(v, w) + \mathbb{B}(t^4\partial_tv, t^4v) + 8t^7\partial_tv \quad \text{in } (0, T), \quad w(0) = 0.$$

Since  $t^4v \in \mathcal{X}^4(0, T)$  and  $tv \in \mathcal{X}^2(0, T)$ ,

$$\mathbb{B}(t^4\partial_tv, t^4v) + 8t^7\partial_tv \in L^2(0, T; \mathcal{L}^2).$$

13 Combining Proposition 6.1 and Lemma 6.4, we deduce that  $w = t^8\partial_tv \in \mathcal{X}^2(0, T)$ . Note in particular that

$$t^8v \in H^1(0, T; H^2(\mathcal{F})^3) \hookrightarrow L^\infty(0, T; L^\infty(\mathcal{F})^3). \quad (6.20)$$

From (1.7), we have

$$t^8 \mathbb{A}_2 v = -t^8 \partial_t v - t^8 \mathbb{B}(v, v). \quad (6.21)$$

Using that  $t^4 v \in \mathcal{X}^4(0, T)$  and  $tv \in \mathcal{X}^2(0, T)$ , we deduce that

$$t^8 ((v - \ell) \cdot \nabla) v \in L^2(0, T; H^1(\mathcal{F}))^3.$$

Then, using Proposition 2.1, we deduce that  $t^8 v \in L^2(0, T; H^3(\mathcal{F}))^3$ . Using this relation and (6.20), we deduce

$$t^{16} ((v - \ell) \cdot \nabla) v \in L^2(0, T; H^2(\mathcal{F}))^3,$$

and repeating the above argument on (6.21) allows us to obtain  $t^{16} v \in L^2(0, T; H^4(\mathcal{F}))^3$ .  $\square$

We are now in a position to prove Theorem 1.8 and Theorem 1.10.

*Proof of Theorem 1.8.* We show by induction that for any  $k \geq 2$ , there exists  $m \geq 1$  such that

$$t^m \partial_t^j v \in L^2(0, T; H^{2(k-j)}(\mathcal{F})^3) \quad (0 \leq j \leq k). \quad (6.22)$$

We can check that  $w = t^{2m+1} \partial_t^k v$  is a very weak solution of

$$\partial_t w = \mathbb{A}w + \mathbb{B}(v, w) + t^{2m+1} \sum_{j=1}^k \binom{k}{j} \mathbb{B}(\partial_t^j v, \partial_t^{k-j} v) + (2m+1)t^{2m} v \quad \text{in } (0, T), \quad w(0) = 0. \quad (6.23)$$

We deduce from (6.22) that for  $1 \leq j \leq k$ ,

$$t^{m+1} \partial_t^{k-j} v \in L^2(0, T; H^{2j}(\mathcal{F})^3) \cap H^1(0, T; H^{2(j-1)}(\mathcal{F})^3)$$

and thus,

$$\nabla (t^{m+1} \partial_t^{k-j} v) \in L^\infty(0, T; H^{2j-2}(\mathcal{F})^9).$$

For  $j \geq 2$ , we deduce from a Sobolev embedding that  $t^{m+1} \partial_t^{k-j} \nabla v \in L^\infty(0, T; L^\infty(\mathcal{F})^9)$  and thus

$$t^{2m+1} \mathbb{B}(\partial_t^j v, \partial_t^{k-j} v) \in L^2(0, T; \mathcal{L}^2).$$

For  $j = 1$ , we deduce from a Sobolev embedding that

$$t^{m+1} \partial_t^{k-1} \nabla v \in L^\infty(0, T; L^2(\mathcal{F})^9), \quad t^m \partial_t v \in L^2(0, T; H^{2k-2}(\mathcal{F})^3) \hookrightarrow L^2(0, T; L^\infty(\mathcal{F})^3)$$

since  $k \geq 2$ . We thus deduce from Proposition 6.1 and Lemma 6.4, applied on (6.23), that  $w \in \mathcal{X}^2(0, T)$ .

It remains to show that for  $j = 0, \dots, k-1$ , there exists  $m_j \geq 1$  such that

$$t^{m_j} \partial_t^j v \in L^2(0, T; H^{2(k-j+1)}(\mathcal{F})^3). \quad (6.24)$$

We have already obtained the above relation for  $j = k+1$  and  $j = k$ , and we proceed by backwards induction.

Let us assume that (6.24) holds from  $k+1$  to  $j$  with  $1 \leq j \leq k-1$ . We can assume that  $m_j \geq m+1$  where  $m$  is the exponent in (6.22) and we want to show (6.24) for  $j-1$ . From the equation satisfied by  $v$ , we deduce

$$t^{2m_j} \mathbb{A} \partial_t^{j-1} v = t^{2m_j} \partial_t^j v - t^{2m_j} \sum_{i=0}^{j-1} \binom{j-1}{i} \mathbb{B}(\partial_t^i v, \partial_t^{j-1-i} v). \quad (6.25)$$

Using (6.22), we have

$$t^{m_j} \partial_t^i v \in L^2(0, T; H^{2(k-j+1)}(\mathcal{F})^3), \quad t^{m_j} \nabla \partial_t^{j-1-i} v \in L^\infty(0, T; H^{2(k-j+1)}(\mathcal{F})^9), \quad (i \in \{1, \dots, j-1\}),$$

1 and for  $i = 0$ ,

$$t^{m_j} v \in L^\infty(0, T; H^{2k-1}(\mathcal{F})^3), \quad t^{m_j} \nabla \partial_t^{j-1} v \in L^2(0, T; H^{2k-2j+1}(\mathcal{F})^9). \quad (6.26)$$

Thus, using that  $2k - 2j + 1 \geq 3$  so that  $H^{2k-2j+1}(\mathcal{F})$  is an algebra, we deduce from (6.25) and Proposition 2.1 that

$$t^{2m_j} \partial_t^{j-1} v \in L^2(0, T; H^{2k-2j+3}(\mathcal{F})^3).$$

2 Using this, we can improve (6.26) and obtain for  $j \geq 2$

$$t^{2m_j} v \in L^\infty(0, T; H^{2k-1}(\mathcal{F})^3), \quad t^{2m_j} \nabla \partial_t^{j-1} v \in L^2(0, T; H^{2(k-j+1)}(\mathcal{F})^9) \quad (6.27)$$

3 and for  $j = 1$

$$t^{2m_j} v \in L^\infty(0, T; H^{2k}(\mathcal{F})^3), \quad t^{2m_j} \nabla v \in L^2(0, T; H^{2k}(\mathcal{F})^9). \quad (6.28)$$

In both cases, we can use Proposition 2.1 and we find

$$t^{4m_j} \partial_t^{j-1} v \in L^2(0, T; H^{2(k-j+2)}(\mathcal{F})^3)$$

and this concludes our inductions in  $k$  and  $j$ . We thus deduce (6.22) and thus that for any  $j, k \in \mathbb{N}$ , there exists  $m \geq 1$  such that

$$t^m v \in H^j(0, T; H^{2k}(\mathcal{F})^3)$$

4 and this ends the proof of the theorem. □

## 5 References

- 6 [1] Jacob Bedrossian and Vlad Vicol. *The mathematical analysis of the incompressible Euler and Navier-Stokes*  
7 *equations. An introduction*, volume 225 of *Grad. Stud. Math.* Providence, RI: American Mathematical  
8 Society (AMS), 2022.
- 9 [2] Marco Bravin. Energy equality and uniqueness of weak solutions of a “viscous incompressible fluid + rigid  
10 body” system with Navier slip-with-friction conditions in a 2D bounded domain. *J. Math. Fluid Mech.*,  
11 21(2):Paper No. 23, 31, 2019.
- 12 [3] Patricio Cumsille and Takéo Takahashi. Wellposedness for the system modelling the motion of a rigid body  
13 of arbitrary form in an incompressible viscous fluid. *Czechoslovak Math. J.*, 58(133)(4):961–992, 2008.
- 14 [4] Sylvain Ervedoza, Debayan Maity, and Marius Tucsnak. Large time behaviour for the motion of a solid in  
15 a viscous incompressible fluid. *Math. Ann.*, 385(1-2):631–691, 2023.
- 16 [5] Luis Escauriaza, Gregory Seregin, and Vladimír Šverák. Backward uniqueness for the heat operator in a  
17 half-space. *St. Petersburg. Math. J.*, 15(1):139–148, 2004.
- 18 [6] Eduard Feireisl. On the motion of rigid bodies in a viscous incompressible fluid. *J. Evol. Equ.*, 3(3):419–441,  
19 2003.
- 20 [7] Giovanni P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*,  
21 volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994. Linearized steady  
22 problems.
- 23 [8] Giovanni P. Galdi. Slow motion of a body in a viscous incompressible fluid with application to particle  
24 sedimentation. In *Recent developments in partial differential equations*, pages 1–35. Rome: Aracne, 1998.
- 25 [9] Giovanni P. Galdi. On the motion of a rigid body in a viscous liquid: a mathematical analysis with ap-  
26 plications. In *Handbook of mathematical fluid dynamics. Vol. 1*, pages 653–791. Amsterdam: Elsevier,  
27 2002.

- 1 [10] Giovanni P. Galdi. Large-time behavior of a rigid body of arbitrary shape in a viscous fluid under the action  
2 of prescribed forces and torques. *arXiv preprint arXiv:2212.11909*, 2022.
- 3 [11] Giovanni P. Galdi and Ana L. Silvestre. Strong solutions to the problem of motion of a rigid body in a  
4 Navier-Stokes liquid under the action of prescribed forces and torques. In *Nonlinear problems in mathe-*  
5 *matical physics and related topics I. In honor of Professor O. A. Ladyzhenskaya*, pages 121–144. New York,  
6 NY: Kluwer Academic/Plenum Publishers, 2002.
- 7 [12] Giovanni P. Galdi and Ana L. Silvestre. Strong solutions to the Navier-Stokes equations around a rotating  
8 obstacle. *Arch. Ration. Mech. Anal.*, 176(3):331–350, 2005.
- 9 [13] Giovanni P. Galdi and Ana L. Silvestre. Existence of time-periodic solutions to the Navier-Stokes equations  
10 around a moving body. *Pac. J. Math.*, 223(2):251–267, 2006.
- 11 [14] Giovanni P. Galdi and Ana L. Silvestre. Further results on steady-state flow of a Navier-Stokes liquid  
12 around a rigid body. Existence of the wake. *RIMS Kôkyûroku Bessatsu*, B1:127–143, 2007.
- 13 [15] Giovanni P. Galdi and Ana L. Silvestre. The steady motion of a Navier-Stokes liquid around a rigid body.  
14 *Arch. Ration. Mech. Anal.*, 184(3):371–400, 2007.
- 15 [16] Giovanni P. Galdi and Ana L. Silvestre. On the motion of a rigid body in a Navier-Stokes liquid under the  
16 action of a time-periodic force. *Indiana Univ. Math. J.*, 58(6):2805–2842, 2009.
- 17 [17] Matthias Geissert, Karoline Götze, and Matthias Hieber.  $L^p$ -theory for strong solutions to fluid-rigid body  
18 interaction in Newtonian and generalized Newtonian fluids. *Trans. Amer. Math. Soc.*, 365(3):1393–1439,  
19 2013.
- 20 [18] Olivier Glass and Franck Sueur. Uniqueness results for weak solutions of two-dimensional fluid-solid systems.  
21 *Arch. Ration. Mech. Anal.*, 218(2):907–944, 2015.
- 22 [19] Hirokazu Iwashita.  $L_q$ - $L_r$  estimates for the solutions of the nonstationary Stokes equations in an exterior  
23 domain and the Navier-Stokes initial value problems in  $L_q$  spaces. *Math. Ann.*, 285:265–288, 1989.
- 24 [20] Tosio Kato. Strong  $L^p$ -solutions of the Navier-Stokes equation in  $R^m$ , with applications to weak solutions.  
25 *Math. Z.*, 187:471–480, 1984.
- 26 [21] Hideo Kozono and Hermann Sohr. Remark on uniqueness of weak solutions to the Navier-Stokes equations.  
27 *Analysis*, 16(3):255–271, 1996.
- 28 [22] Olga A. Ladyženskaja, Vsevolod A. Solonnikov, and Nina N. Uralceva. *Linear and quasilinear equations*  
29 *of parabolic type*. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society,  
30 Providence, R.I., 1968. Translated from the Russian by S. Smith.
- 31 [23] Debayan Maity and Marius Tucsnak.  $L^p$ - $L^q$  maximal regularity for some operators associated with lin-  
32 earized incompressible fluid-rigid body problems. In *Mathematical analysis in fluid mechanics—selected*  
33 *recent results*, volume 710 of *Contemp. Math.*, pages 175–201. Amer. Math. Soc., Providence, RI, 2018.
- 34 [24] Debayan Maity and Marius Tucsnak. Motion of rigid bodies of arbitrary shape in a viscous incompressible  
35 fluid: wellposedness and large time behaviour. *hal-03992154*, February 2023.
- 36 [25] Kyūya Masuda. Weak solutions of Navier-Stokes equations. *Tohoku Math. J. (2)*, 36(4):623–646, 1984.
- 37 [26] Boris Muha, Šárka Nečasová, and Ana Radošević. On the regularity of weak solutions to the fluid-rigid  
38 body interaction problem. *arXiv preprint arXiv:2211.03080*, 2022.
- 39 [27] Boris Muha, Šárka Nečasová, and Ana Radošević. A uniqueness result for 3D incompressible fluid-rigid  
40 body interaction problem. *J. Math. Fluid Mech.*, 23(1):Paper No. 1, 39, 2021.

- 1 [28] Jorge San Martín, Victor Starovoitov, and Marius Tucsnak. Global weak solutions for the two-dimensional  
2 motion of several rigid bodies in an incompressible viscous fluid. *Arch. Ration. Mech. Anal.*, 161(2):113–147,  
3 2002.
- 4 [29] Denis Serre. Chute libre d’un solide dans un fluide visqueux incompressible. Existence. (Free falling body  
5 in a viscous incompressible fluid. Existence). *Japan J. Appl. Math.*, 4:99–110, 1987.
- 6 [30] Ana Leonor Silvestre. On the self-propelled motion of a rigid body in a viscous liquid and on the attainability  
7 of steady symmetric self-propelled motions. *J. Math. Fluid Mech.*, 4(4):285–326, 2002.
- 8 [31] Takéo Takahashi and Marius Tucsnak. Global strong solutions for the two-dimensional motion of an infinite  
9 cylinder in a viscous fluid. *J. Math. Fluid Mech.*, 6(1):53–77, 2004.
- 10 [32] Roger Temam. *Problèmes mathématiques en plasticité*, volume 12 of *Méthodes Mathématiques de*  
11 *l’Informatique [Mathematical Methods of Information Science]*. Gauthier-Villars, Montrouge, 1983.
- 12 [33] Yun Wang and Zhouping Xin. Analyticity of the semigroup associated with the fluid-rigid body problem  
13 and local existence of strong solutions. *J. Funct. Anal.*, 261(9):2587–2616, 2011.