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# Active disturbance rejection control for a transport equation via a differentiator<sup>\*</sup>

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**Abstract:** This paper deals with the stabilization of a transport equation subject to a boundary disturbance. Our feedback design relies on the so-called strategy called active disturbance rejection control (ADRC). The unknown disturbance is estimated by Levant's differentiator and one of the feature of this differentiator is that it allows to estimate in finite-time the disturbance. We prove the existence of solutions of the closed-loop system and the global asymptotic stability of the closed-loop system. A numerical example is given to illustrate the efficiency of our strategy.

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## 1. INTRODUCTION

This paper deals with the stabilization of a transport equation subject to a boundary disturbance.

Transport equations have received much attention for many years due to the many physical phenomena they model: e.g. aeronomy (Schunk (1975)), crystallization (Mesbah et al. (2011); Omar and Rohani (2017)), biology (Perthame (2007)) and more specific on the concentration of polymers (Armiento et al. (2016)). A good overview of the actual research lines concerning this topic is provided in Bastin and Coron (2016) and Hayat (2021).

Stabilization of this kind of systems where the disturbances are rejected is not a new topic. The reader can refer to Terrand-Jeanne et al. (2019); Coron and Hayat (2019) which are based on PI controllers or Deutscher (2017a) and Deutscher (2017b) which are based on backstepping method and observer design. Note that, in these cases, the disturbances might not be located at the same position as the control. However, there are more constraints on the disturbance under consideration: either the disturbance is supposed to be constant or the dynamics of the latter is assumed to be known. The objective of the current paper is to propose a control strategy for a larger class of uncertainties/disturbances as it was the case in Liard et al. (2022); Tang and Krstic (2014); Balogoun et al. (2023) with sliding mode control.

As the sliding mode control method, the active disturbance rejection control (ADRC) is a powerful method to deal with disturbances. It was initially proposed in Han (2009) in the context of finite dimensional systems. The main idea of the ADRC is to build an observer to estimate a disturbance. Then, the disturbance is compensated in

a feedback law by its estimate. Recently, this approach has been successfully applied to systems described by one-dimensional partial differential equations (PDEs) (Liu and Wang (2015); Guo and Jin (2013); Zhang et al. (2019); Zhang and Wang (2021); Cai et al. (2022); Guo and Liu (2014)).

The contribution of this paper is to apply the ADRC strategy to design a feedback-law which allows to reject the disturbance in finite time and to ensure that the resulting closed-loop system is globally asymptotically stable. Our ADRC is based on the Levant differentiator (Levant (2003, 2005)) which allows to estimate the disturbance in finite-time. The PDE is reduced to an ordinary differential equation (ODE) (more specifically, an integrator chain) through the knowledge of  $n$ -th moment of the PDE's state, which is often possible in practice (Omar and Rohani (2017); Armiento et al. (2016)). Note that this is much better than what is normally done in the literature by reducing PDE thanks to test functions in ODE, where the full-state of PDE often have to be measurable, which is not possible in practice. In contrast with (Guo and Jin (2013); Guo and Liu (2014)), we do not derive the dynamics of the output from the weak formulation of the PDE, which, in general, leads to a scalar ODE. We rather compute the dynamics of all the moments up to a certain degree  $n$ , which leads to a much more complicated system. In our case, this system corresponds to a chain of integrators. We are, in general, closer to practical cases, since the output that is assumed to be known in Guo and Jin (2013); Guo and Liu (2014) does not correspond exactly to a moment.

This paper is organized as follows. Section 2 presents a linear hyperbolic equation. Section 3 presents the ADRC strategy and the main results of the paper. Section 4 contains the proofs of the main results. Section 5 illustrates via numerical simulations the efficiency of our ARDC

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strategy. Finally, Section 6 collects some remarks and introduces some future research lines to be followed.

Notation: The set of non-negative real numbers is denoted in this paper by  $\mathbb{R}_+$ . When a function  $f$  only depends on the time variable  $t$  (resp. on the space variable  $x$ ), its derivative is denoted by  $\dot{f}$  (resp.  $f'$ ). We define the function sign as the multivalued function defined on  $\mathbb{R}$  by  $\text{sign}(x) = \frac{x}{|x|}$  for  $x \neq 0$  and  $\text{sign}(0) = [-1, 1]$ . For every  $m \geq 0$  and  $x \in \mathbb{R}$ , we use  $|x|^m$  to denote  $|x|^m \text{sign}(x)$ . A function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is of class  $\mathcal{K}$ , if it is continuous, strictly increasing and satisfies  $\alpha(0) = 0$ . A function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is of class  $\mathcal{KL}$ , if for each fixed  $t \geq 0$ ,  $\beta(\cdot, t)$  is of class  $\mathcal{K}$ , and, for each fixed  $r \geq 0$ ,  $\beta(r, \cdot)$  is decreasing and satisfies  $\lim_{t \rightarrow \infty} \beta(r, t) = 0$ . For  $x \in \mathbb{R}^n$ , we use  $x_i$  with  $i = 0, 1, \dots, n-1$  to indicate the coordinates of  $x$ . Given  $L > 0$ , the set  $L^2(0, L)$  denotes the Hilbert space of measurable square-integrable function with values in  $\mathbb{R}$ . The set  $H^1(0, L)$  denotes the Sobolev space of absolutely continuous  $\mathbb{R}$ -valued functions whose derivatives are in  $L^2(0, L)$ .

## 2. PROBLEM STATEMENT

Let  $L > 0$  and  $\lambda > 0$ . We consider the following linear hyperbolic system

$$\begin{cases} y_t(t, x) + \lambda y_x(t, x) = 0, (t, x) \in \mathbb{R}_+ \times [0, L] \\ y(t, 0) = u(t) + d(t), t \in \mathbb{R}_+ \\ y(0, x) = y_0(x), x \in [0, L] \end{cases} \quad (1)$$

where  $y_0$  is real-valued, i.e. for all  $x \in [0, L]$ ,  $y_0(x) \in \mathbb{R}$ ,  $u(t) \in \mathbb{R}$  denotes the control and  $d(t) \in \mathbb{R}$  is an unknown disturbance. We assume that there exists a known positive constant  $C$  such that, for a.e  $t \in \mathbb{R}_+$ ,

$$|d(t)| \leq C. \quad (2)$$

When the system (1) is undisturbed ( $d = 0$ ), it is known that the feedback law

$$u(t) := ay(t, L),$$

allows to stabilize the system if  $|a| < 1$ , see (Bastin and Coron, 2016, Theorem 2.1.). Let  $n \in \mathbb{N}$  fixed. We assume that we measure  $y(t, L)$  and the  $n$ -th moment defined by  $\int_0^L x^n y(t, x) dx$  and we denote by  $\eta_i$  the  $i$ -th moment of  $y$ , where  $i = 0, \dots, n$  i.e.

$$\eta_i(t) := \int_0^L x^i y(t, x) dx. \quad (3)$$

Then, performing formal integrations by parts and using boundary conditions of (1), we prove that the moments satisfy the following chain of integrators:

$$\begin{cases} \dot{\eta}_n(t) = -\lambda L^n y(t, L) + n\lambda \eta_{n-1}(t), \\ \dot{\eta}_{n-1}(t) = -\lambda L^{n-1} y(t, L) + (n-1)\lambda \eta_{n-2}(t), \\ \vdots \\ \dot{\eta}_1(t) = -\lambda L y(t, L) + \lambda \eta_0(t), \\ \dot{\eta}_0(t) = -\lambda y(t, L) + \lambda(u(t) + d(t)). \end{cases} \quad (4)$$

In this paper, the first goal is to estimate the disturbance  $d$  in finite-time. Second, the second objective of this paper is to use the disturbance estimate to design feedback controller  $u$  in order to make the origin of (1) globally asymptotically stable.

## 3. MAIN RESULTS

To achieve the first goal of this paper, we design an observer for (4) using Levant's differentiator (Levant (2003)), of the form

$$\begin{cases} \dot{\hat{\eta}}_n(t) = -\lambda L^n y(t, L) + n\lambda \hat{\eta}_{n-1}(t) \\ \quad - k_n C^{\frac{1}{n+1}} [\hat{\eta}_n - \eta_n]^{\frac{1}{n+1}}, \\ \dot{\hat{\eta}}_{n-1}(t) = -\lambda L^{n-1} y(t, L) + (n-1)\lambda \hat{\eta}_{n-2}(t) \\ \quad - k_{n-1} C^{\frac{2}{n+1}} [\hat{\eta}_n - \eta_n]^{\frac{2}{n+1}}, \\ \vdots \\ \dot{\hat{\eta}}_1(t) = -\lambda L y(t, L) + \lambda \hat{\eta}_0(t) \\ \quad - k_1 C^{\frac{n}{n+1}} [\hat{\eta}_n - \eta_n]^{\frac{1}{n+1}}, \\ \dot{\hat{\eta}}_0(t) \in -\lambda y(t, L) + \lambda u(t) - k_0 C [\hat{\eta}_n - \eta_n]^0 \end{cases} \quad (5)$$

where  $k_i$  for all,  $i = 0, 1, \dots, n$ , are the output injection gains to be selected to ensure the convergence of the observer. We define the estimation error as  $e_i := \hat{\eta}_i - \eta_i$ . The error dynamics is given by

$$\begin{cases} \dot{e}_n(t) = n\lambda e_{n-1}(t) - k_n C^{\frac{1}{n+1}} [e_n]^{\frac{n}{n+1}}, \\ \dot{e}_{n-1}(t) = (n-1)\lambda e_{n-2}(t) - k_{n-1} C^{\frac{2}{n+1}} [e_n]^{\frac{n-1}{n+1}}, \\ \vdots \\ \dot{e}_1(t) = \lambda e_0(t) - k_1 C^{\frac{n}{n+1}} [e_n]^{\frac{1}{n+1}}, \\ \dot{e}_0(t) \in -\lambda d(t) - k_0 C [e_n]^0 \end{cases} \quad (6)$$

i.e

$$\begin{cases} \dot{e}_i(t) = i\lambda e_{i-1}(t) - k_i C^{\frac{n+1-i}{n+1}} [e_n]^{\frac{i}{n+1}}, \quad i = 1, \dots, n \\ \dot{e}_0(t) \in -\lambda C[-1, 1] - k_0 C [e_n]^0. \end{cases} \quad (7)$$

Using the following transformation

$$\begin{cases} w_i = \frac{(i+1)\lambda e_i}{k_{i+1} C}, \quad i = 1, \dots, n \\ w_0 = \frac{e_0}{k_1 C}, \\ k_{n+1} = 1 \end{cases} \quad (8)$$

we obtain

$$\begin{cases} \dot{w}_i(t) = -\tilde{k}_i \left( \left[ \frac{w_n}{\lambda(n+1)} \right]^{\frac{i}{n+1}} - w_{i-1}(t) \right), \quad i = 1, \dots, n \\ \dot{w}_0(t) \in -\tilde{k}_0 \left( \left[ \frac{w_n}{\lambda(n+1)} \right]^0 + \frac{\lambda}{k_0} [-1, 1] \right) \end{cases} \quad (9)$$

where

$$\tilde{k}_0 = \frac{k_0}{k_1} \quad \text{and} \quad \tilde{k}_i = \frac{\lambda(j+1)k_i}{k_{i+1}} \quad i = 1, \dots, n.$$

The solution of system (9) is understood in the sense of Filippov (Filippov (2013)) and the existence of the solution will be proved later. From (Cruz-Zavala and Moreno, 2018, Theorem 1 and Proposition 1), all trajectories of (9) converge to zero in finite-time. More precisely, we have the following Proposition.

**Proposition 1.** There exists a valid set of differentiator parameters  $k_i$ ,  $i = 1, \dots, n$  and a finite time  $t_r > 0$  such that  $w_i(t) = 0$ ,  $i = 1, \dots, n$  for any  $t > t_r$ .

Then, according to (8),  $w_i(t) = 0$ ,  $i = 1, \dots, n$  for all  $t \geq t_r$  implies that  $e_i(t) = 0$ ,  $i = 1, \dots, n$  for all  $t \geq t_r$ . Thus, according to the last line (6), the function

$$\tilde{d}(t) = -\frac{k_0 C}{\lambda} \text{sign}(e_n(t)), \quad t \geq 0 \quad (10)$$

is an estimation of  $d$  for all  $t > t_r$ , where  $t_r$  is the convergence time. But, for all  $t > t_r$ ,  $e_n(t) = 0$ . Thus,  $t > t_r$ ,  $\text{sign}(e_n(t)) = \text{sign}(0) = [-1, 1]$ . This is why a low-pass filter of the fast switching signal  $\text{sign}(0)$  is used for such an estimate.

Then, in order to reach the second goal of this paper, we design the feedback controller as follows:

$$u(t) = ay(t, L) + \frac{k_0 C}{\lambda} \text{sign}(e_n(t)), \quad |a| < 1. \quad (11)$$

Note that the first term in (11) is a usual control that makes the closed-loop system (1) exponentially stable without the disturbance (Bastin and Coron, 2016, Theorem 2.1). The second term is used to compensate the effect of the disturbance.

The closed-loop system (1)–(11) is finally given by

$$\begin{cases} y_t(t, x) + \lambda y_x(t, x) = 0, \\ y(t, 0) \in ay(t, L) + \frac{k_0 C}{\lambda} \text{sign}(\hat{\eta}_n(t) - \eta_n(t)) \\ \quad + d(t), \\ \dot{\hat{\eta}}_i(t) = -\lambda L^i y(t, L) + i\lambda \hat{\eta}_{i-1}(t) \\ \quad - k_i C^{\frac{n+1-i}{n+1}} [\hat{\eta}_n - \eta_n]^{\frac{i}{n+1}}, \quad i = 1, \dots, n \\ \dot{\hat{\eta}}_0(t) = \lambda y(t, L)(a-1) \\ y(0, x) = y_0(x), \\ \hat{\eta}(0) = \hat{\eta}^0 \in \mathbb{R}^{n+1}. \end{cases} \quad (12)$$

**Remark 1.** According to Proposition 1,  $e(t) = 0$  for all  $t > t_r$ . Thus,  $\dot{e}(t) = 0$  for all  $t > t_r$ . Then, the mild solution<sup>1</sup>  $y$  of (12) is a mild solution to

$$\begin{cases} y_t(t, x) + \lambda y_x(t, x) = 0, \\ y(t, 0) = ay(t, L) \end{cases} \quad (13)$$

for all  $t > t_r$ .

The main results of this paper can be formulated as follows:

**Theorem 1.** (Existence of solutions). Assume that (2) holds. Then, for all  $T > 0$  and for all  $(y_0, \hat{\eta}^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$ , the closed-loop system (12) admits a mild solution  $(y, \hat{\eta}) \in C(0, T; L^2(0, L) \times \mathbb{R}^{n+1})$ .

**Theorem 2.** (Global asymptotic stability). There exists a  $\mathcal{KL}$ -function  $\alpha$  such that, the following inequality

$$\|y(t, \cdot)\|_{L^2(0, L)} + |e(t)|_{\mathbb{R}^{n+1}} \leq \alpha(\|y_0\|_{L^2(0, L)} + |e^0|_{\mathbb{R}^{n+1}}, t) \quad (14)$$

is satisfied for any  $(y_0, \hat{\eta}^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$ , for any  $t \geq 0$  and for all solution  $(y, \hat{\eta})$  of (12), where  $e = \hat{\eta} - \eta$ ,  $e^0 = \hat{\eta}^0 - \eta^0$  and  $\eta_i^0 = \eta_i(0)$ .

## 4. PROOF OF THEOREM 1 AND THEOREM 2

### 4.1 Proof of Theorem 1

Let's consider the operator  $\mathcal{A} : \phi \in D(\mathcal{A}) \subset L^2(0, L) \mapsto \mathcal{A}\phi \in L^2(0, L)$  defined as

$$\begin{cases} \mathcal{A}\phi = -\lambda\phi', \\ D(\mathcal{A}) = \{\phi \in H^1(0, L) \mid \phi(0) = a\phi(L)\} \end{cases} \quad (15)$$

where  $\lambda$  is given in system (1). According to the proof of (Bastin and Coron, 2016, Theorem A.1), it generates a  $C_0$ -semigroup  $(\mathbb{T}(t))_{t \geq 0}$  of contractions in  $L^2(0, L)$ . Also, consider the operator  $\tilde{\mathcal{B}}$  defined as  $\lambda\langle \varphi, \mathcal{B}v \rangle_{D(\mathcal{A}^*), D(\mathcal{A}^*)}' =$

$\lambda\varphi(0)v$  for all  $v \in \mathbb{R}$  and  $\varphi \in D(\mathcal{A}^*)$  where  $\mathcal{A}^*$  is the adjoint operator of  $\mathcal{A}$  and  $\langle \cdot, \cdot \rangle_{D(\mathcal{A}^*), D(\mathcal{A}^*)}'$  is the dual product. Now, let's prove that  $\mathcal{B}$  is admissible<sup>2</sup> for  $(\mathbb{T}(t))_{t \geq 0}$ . To do so, consider the system

$$\begin{cases} \frac{d}{dt}z = \mathcal{A}^*z, \\ \gamma = \mathcal{B}^*z. \end{cases} \quad (16)$$

where  $\mathcal{A}^* : \varphi \in D(\mathcal{A}^*) \subset L^2(0, L) \mapsto \mathcal{A}^*\varphi \in L^2(0, L)$  and  $\mathcal{B}^* : \varphi \in D(\mathcal{A}^*) \mapsto \mathcal{B}^*\varphi \in \mathbb{R}$  are given by

$$\begin{cases} \mathcal{A}^*\varphi = \lambda\varphi', \\ D(\mathcal{A}^*) = \{\varphi \in H^1(0, L) \mid \varphi(L) = a\varphi(0)\}, \\ \mathcal{B}^*\varphi = \varphi \in D(\mathcal{A}^*) \mapsto \lambda\varphi(0). \end{cases} \quad (17)$$

For all  $z_0 \in D(\mathcal{A}^*)$ , the function

$$z(t) = \mathbb{T}^*(t)z_0 \quad (18)$$

defines the unique classical solution of (16) where  $\mathbb{T}^*(t)$ <sup>3</sup> is a  $C_0$ -semigroup with infinitesimal generator  $\mathcal{A}^*$  on  $L^2(0, L)$ . Now, consider the following function

$$E(t) = \int_0^L (z(t, x))^2 dx.$$

The time derivative of  $E$  along the trajectories of (16) reads as, for all  $t \geq 0$ ,

$$\begin{aligned} \dot{E}(t) &= 2 \int_0^L \partial_t z(t, x) z(t, x) dx \\ &= 2\lambda \int_0^L \partial_x z(t, x) z(t, x) dx \\ &= \lambda(|z(t, L)|^2 - |z(t, 0)|^2) \\ &= -\lambda|z(t, 0)|^2(1 - a^2) \\ &\leq 0. \end{aligned} \quad (19)$$

Then, from (19), one deduces that, for all  $T > 0$

$$\begin{aligned} \int_0^T |\gamma(t)|^2 dt &= \lambda^2 \int_0^T |z(t, 0)|^2 dt \\ &= \frac{\lambda}{a^2 - 1} \int_0^T \dot{E}(t) dt \\ &= \frac{\lambda}{1 - a^2} (E(0) - E(T)) \\ &\leq \frac{\lambda}{1 - a^2} E(0) = \frac{\lambda}{1 - a^2} \|z(0, \cdot)\|_{L^2(0, L)}^2 \end{aligned} \quad (20)$$

where  $\gamma$  come from (16). Then

$$\int_0^T |\gamma(t)|^2 dt \leq \frac{\lambda}{1 - a^2} \|z(0, \cdot)\|_{L^2(0, L)}^2. \quad (21)$$

Then according to (Tucsnak and Weiss, 2009, Definition 4.3.1 and Theorem 4.4.3.), this proves that  $\mathcal{B}$  is admissible for the  $C_0$ -semigroup  $(\mathbb{T}(t))_{t \geq 0}$ .

Now, let  $(y_0, \hat{\eta}^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$  such that  $\hat{\eta}^0 \neq \eta^0$ . Then, there exist  $t_0 > 0$  such that for all  $t \in [0, t_0]$ , we have  $\hat{\eta}(t) \neq \eta(t)$ . Therefore, for all  $t \in [0, t_0]$ ,  $\text{sign}(\hat{\eta}_n(t) - \eta_n(t)) = \pm 1$ . Thus, for all  $t \in [0, t_0]$ , the system (12) is equivalent to the following system

<sup>2</sup> See e.g (Tucsnak and Weiss, 2009, Definition 4.2.1)

<sup>3</sup> See e.g (Tucsnak and Weiss, 2009, Proposition 2.8.5)

<sup>1</sup> See e.g (Tucsnak and Weiss, 2009, Definition 4.1.5)

$$\begin{cases} y_t(t, x) + \lambda y_x(t, x) = 0, \\ y(t, 0) = ay(t, L) + d(t) \pm \frac{k_0 C}{\lambda}, \\ \dot{\hat{\eta}}_i(t) = -\lambda L^i y(t, L) + i\lambda \hat{\eta}_{i-1}(t) \\ \quad - k_i C \frac{n+1-i}{n+1} [\hat{\eta}_n - \eta_n]^{\frac{i}{n+1}}, \quad i = 1, \dots, n \\ \dot{\hat{\eta}}_0(t) = \lambda y(t, L)(a-1) \\ y(0, x) = y_0(x), \\ \hat{\eta}(0) = \hat{\eta}^0 \in \mathbb{R}^{n+1}. \end{cases} \quad (22)$$

Since the function  $\hat{d} : t \mapsto d(t) + \frac{k_0 C}{\lambda} \text{sign}(\hat{\eta}_n(t) - \eta_n(t))$  is bounded on  $[0, t_0]$  and  $\mathcal{B}$  is admissible for  $(\mathbb{T}(t))_{t \geq 0}$ , then, according to (Tucsnak and Weiss, 2009, Proposition 4.2.5), there exist a unique mild solution  $y \in C([0, t_0]; L^2(0, L))$  of (22). As a consequence, the functions  $y(\cdot, L)$  and  $\eta_n(\cdot)$  are continuous on  $[0, t_0]$ . Then, the right hand side of the system

$$\begin{cases} \dot{\hat{\eta}}_i(t) = -\lambda L^i y(t, L) + i\lambda \hat{\eta}_{i-1}(t) \\ \quad - k_i C \frac{n+1-i}{n+1} [\hat{\eta}_n - \eta_n]^{\frac{i}{n+1}}, \quad i = 1, \dots, n \\ \dot{\hat{\eta}}_0(t) = \lambda y(t, L)(a-1) \\ \hat{\eta}(0) = \hat{\eta}^0 \in \mathbb{R}^{n+1}. \end{cases} \quad (23)$$

is continuous on  $[0, t_0] \times \mathbb{R}^{n+1}$ . Thus, according to (Coddington and Levinson, 1955, Theorem 1.2), the system (23) admits a solution  $\hat{\eta} \in C^1([0, t_0])$ . As a consequence, the system (12) admits a mild solution on  $[0, t_0]$ . Moreover, for all  $t \in [0, t_0]$ ,  $y(t, \cdot) \in D(\mathcal{A})$  and satisfies the following equations in  $D(\mathcal{A}^*)'$

$$y(t, \cdot) - y_0 = -\lambda \int_0^t y_x(s, \cdot) ds + \int_0^t \mathcal{B} \hat{d}(s) ds, \forall t \in [0, t_0]. \quad (24)$$

Thus, for all  $t \in [0, t_0]$ ,  $y(t, \cdot) \in D(\mathcal{A})$  and for all  $i = 0, 1, \dots, n$ , for all  $(t, x) \in [0, t_0] \times [0, L]$ ,  $y$  satisfies

$$x^i y(t, x) - x^i y_0(x) = -\lambda \int_0^t x^i y_x(s, x) ds + \int_0^t x^i \mathcal{B} \hat{d}(s) ds. \quad (25)$$

As a consequence, we have for all  $i = 0, 1, \dots, n$ , and for all  $t \in [0, t_0]$

$$\begin{aligned} & \int_0^L x^i y(t, x) dx - \int_0^L x^i y_0(x) dx \\ &= -\lambda \int_0^t \int_0^L x^i y_x(s, x) dx ds + \int_0^t \int_0^L x^i \mathcal{B} dx \hat{d}(s) ds. \end{aligned} \quad (26)$$

Since  $\mathcal{B}$  is the product of the delta function at  $x = 0$  with  $\lambda$ , then for all  $i = 1, \dots, n$

$$\int_0^L x^i \mathcal{B} dx = 0 \quad (27)$$

and

$$\int_0^L \mathcal{B} dx = \lambda, \text{ because } 0 \in [0, L]. \quad (28)$$

Thus, for all  $i = 1, \dots, n$ , we have

$$\begin{aligned} & \int_0^L x^i y(t, x) dx - \int_0^L x^i y_0(x) dx \\ &= -\lambda \int_0^t \int_0^L x^i y_x(s, x) dx ds, \quad \forall t \in [0, t_0] \end{aligned} \quad (29)$$

and

$$\begin{aligned} & \int_0^L y(t, x) dx - \int_0^L y_0(x) dx \\ &= -\lambda \int_0^t \int_0^L y_x(s, x) dx ds + \lambda \int_0^t \hat{d}(s) ds, \quad \forall t \in [0, t_0]. \end{aligned} \quad (30)$$

Using an integration by parts, one immediately obtains

$$\begin{aligned} & \int_0^L y(t, x) dx - \int_0^L y_0(x) dx \\ &= -\lambda \int_0^t \left( y(s, L) - y(s, 0) - \hat{d}(s) \right) ds, \quad \forall t \in [0, t_0] \end{aligned} \quad (31)$$

and for all  $i = 1, \dots, n$ , for all  $t \in [0, t_0]$

$$\begin{aligned} & \int_0^L x^i y(t, x) dx - \int_0^L x^i y_0(x) dx \\ &= i\lambda \int_0^t \int_0^L x^{i-1} y(s, x) dx ds - \lambda \int_0^t L^i y(s, L) ds. \end{aligned} \quad (32)$$

Using (3) and the fact that, for all  $t \in [0, t_0]$ ,  $y(t, \cdot) \in D(\mathcal{A})$ , we obtain, for all  $t \in [0, t_0]$

$$\eta_0(t) - \eta_0(0) = -\lambda \int_0^t \left( y(s, L)(1-a) - \hat{d}(s) \right) ds \quad (33)$$

and for all  $i = 1, \dots, n$ , for all  $t \in [0, t_0]$

$$\eta_i(t) - \eta_i(0) = \int_0^t \left( i\lambda \eta_{i-1}(s) - \lambda L^i y(s, L) \right) ds. \quad (34)$$

This prove that, for all  $i = 0, 1, \dots, n$ , the  $i$ -th moment  $\eta_i$  of  $y$  is a Carathéodory solution to (4) on  $[0, t_0]$ .

Now, we assume without loss of generality that  $t_0 < t_r$ , where  $t_r$  is given in Proposition 1. Thus, according to Proposition 1,  $\hat{\eta}(t) \neq \eta(t)$  for all  $t \in [t_0, t_r[$ . As a result, the same reasoning at time interval  $[0, t_0]$  is step-by-step applied to time interval  $[t_0, t_r[$ , by considering  $y(t_0, \cdot)$  as the initial condition. As a consequence, the system (12) admits a solution on  $[0, t_r[$ .

Moreover, on one hand, according to Proposition 1,  $\hat{\eta}(t) = \eta(t)$  and  $d(t) - \hat{d}(t) = 0$  for all  $t \geq t_r$ . Thus, for all  $t \geq t_r$ , the system (12) is equivalent to the following system

$$\begin{cases} y_t(t, x) + \lambda y_x(t, x) = 0, \\ y(t, 0) = ay(t, L), \\ \dot{\hat{\eta}}_i(t) = -\lambda L^i y(t, L) + i\lambda \hat{\eta}_{i-1}(t), \quad i = 1, \dots, n \\ \dot{\hat{\eta}}_0(t) = \lambda y(t, L)(a-1) \\ y(0, x) = y_0(x), \\ \hat{\eta}(0) = \hat{\eta}^0 \in \mathbb{R}^{n+1}. \end{cases} \quad (35)$$

It can be proved as before that the system (35) admits a solution on  $[t_r, \infty[$ . Then, in conclusion, for all  $(y_0, \hat{\eta}^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$  such that  $\hat{\eta}^0 \neq \eta^0$ , the system (12) admits a solution on  $[0, \infty[$ .

Let  $(y_0, \hat{\eta}^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$  such that  $\hat{\eta}^0 = \eta^0$ . Then, there exist  $t_0 > 0$  such that, for all  $t \in [0, t_0[$ , we have  $\hat{\eta}(t) = \eta(t)$ . Then, there exists a measurable function  $c : [0, t_0[ \rightarrow [-1, 1]$  such that, for all  $t \in [0, t_0[$ ,  $y(t, 0) = ay(t, L) + d(t) + \frac{k_0 C}{\lambda} c(t)$ . Thus, for all,  $t \in [0, t_0[$  the system (12) is equivalent to the following system

$$\begin{cases} y_t(t, x) + \lambda y_x(t, x) = 0, \\ y(t, 0) = ay(t, L) + d(t) + \frac{k_0 C}{\lambda} c(t), \\ \dot{\hat{\eta}}_i(t) = -\lambda L^i y(t, L) + i\lambda \hat{\eta}_{i-1}(t), \quad i = 1, \dots, n \\ \dot{\hat{\eta}}_0(t) = \lambda y(t, L)(a-1) \\ y(0, x) = y_0(x), \\ \hat{\eta}(0) = \hat{\eta}^0 \in \mathbb{R}^{n+1}. \end{cases} \quad (36)$$

Since the function  $t \mapsto d(t) + \frac{k_0 C}{\lambda} c(t)$  is bounded on  $[0, t_0]$  then we conclude as before that the system (36) admits a solution on  $[0, t_0[$  and the  $i$ -th moment  $\eta_i$  of  $y$  is a Carathéodory solution to (4) on  $[0, t_0[$ .

However, since  $\hat{\eta}(t) = \eta(t)$  for all  $t \in [0, t_0[$ , then according to Proposition 1,  $t_0 > t_r$ . Thus, for all  $t \geq t_r$ , the system (12) is equivalent to system (35). As a consequence  $t_0 = \infty$ .

This concludes the proof of Theorem 1.

#### 4.2 Proof of Theorem 2

Let  $(y_0, \hat{\eta}^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$ . Using the error variable  $e$  defined in (6), we can write the equivalent system of (12) as follows:

$$\begin{cases} y_t(t, x) + \lambda y_x(t, x) = 0, \\ y(t, 0) = ay(t, L) - \frac{1}{\lambda} \dot{e}_0(t), \\ \dot{e}_i(t) = i\lambda e_{i-1}(t) - k_i C \frac{n+1-i}{n+1} |e_n|^{\frac{i}{n+1}}, \quad i = 1, \dots, n \\ \dot{e}_0(t) \in -\lambda d(t) - k_0 C \text{sign}(e_n(t)). \end{cases} \quad (37)$$

Then, according to Proposition 1 and Remark 1, there exists a finite time  $t_r$  such that, for all  $t > t_r$ , the solution  $y$  of (37) is equivalent to the system (13) and hence is globally exponentially stable in  $L^2(0, L)$  from (Bastin and Coron, 2016, Theorem 2.1). Therefore, to conclude the proof of Theorem 2, it is just necessary to prove that the system (37) depends continuously on initial conditions on the time interval  $[0, t_r]$ . It is stated in the following Lemma.

**Lemma 1.** There exists a  $\mathcal{K}$ -function  $\beta$  such that for all  $(y_0, e^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$ , for all  $t \in [0, t_r]$ ,

$$\|y(t, \cdot)\|_{L^2(0, L)} + |e|_{\mathbb{R}^{n+1}} \leq \beta(\|y_0\|_{L^2(0, L)} + |e^0|_{\mathbb{R}^{n+1}}). \quad (38)$$

for all solution  $(y, e)$  of (37).

**Proof.** Let  $(y_0, e^0) \in L^2(0, L) \times \mathbb{R}^{n+1}$  and we consider  $(y, e)$  a solution of (37) associated  $(y_0, e^0)$ . Then, according to (Tucsnak and Weiss, 2009, Proposition 2.1.2), there exists  $K_0 > 0$  such that, for all  $t \in [0, t_r]$ , we have

$$\begin{aligned} \|y(t, \cdot)\|_{L^2(0, L)} &\leq K_0 \|y_0(\cdot)\|_{L^2(0, L)} \\ &+ \left\| \frac{1}{\lambda} \int_0^t \mathbb{T}(t-s) \mathcal{B} \dot{e}_0(t) ds \right\|_{L^2(0, L)}. \end{aligned} \quad (39)$$

As a consequence, since  $(\mathbb{T}(t))_{t \geq 0}$  is exponentially stable and  $\mathcal{B}$  is admissible operator for  $(\mathbb{T}(t))_{t \geq 0}$ , then we have according to (Tucsnak and Weiss, 2009, Proposition 4.4.5), that there exists  $K_1 > 0$  independent of  $t_r$  such that

$$\|y(t, \cdot)\|_{L^2(0, L)} \leq K_1 \left( \|y_0(\cdot)\|_{L^2(0, L)} + \|\dot{e}_0(\cdot)\|_{L^2(0, t_r)} \right). \quad (40)$$

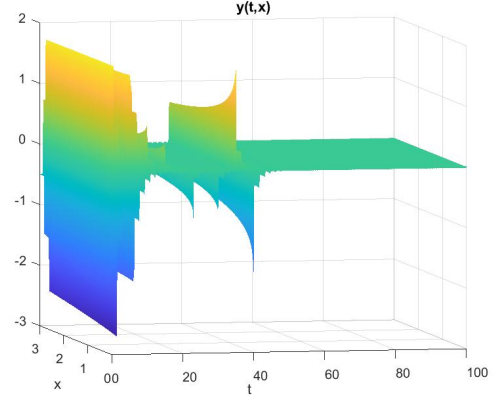


Fig. 1.  $y$  versus  $x$  and time.

Since  $d$  and the sign function are bounded then according to (6),  $\dot{e}_0$  is also bounded. Therefore, there exists  $K_2 > 0$  such that

$$\|\dot{e}_0\|_{L^2((0, t_r), \mathbb{R})}^2 \leq K_2 t_r. \quad (41)$$

Now, according to (Cruz-Zavala and Moreno, 2018, Theorem 1), there are positive constants  $K_3, K_4$  (dependent on the bound of  $d$ ) such that

$$\begin{cases} t_r < K_3 |e^0|_{\mathbb{R}^{n+1}}, \\ |e|_{\mathbb{R}^{n+1}} \leq K_4 |e^0|_{\mathbb{R}^{n+1}}. \end{cases} \quad (42)$$

As a consequence, according to (40), (41) and (42), there exists  $C_1 > 0$  (independent of  $t_r$ ) such that, for all  $t \in [0, t_r]$ ,

$$\|y(t, \cdot)\|_{L^2(0, L)} + |e|_{\mathbb{R}^{n+1}} \leq \beta(\|y_0\|_{L^2(0, L)} + |e^0|_{\mathbb{R}^{n+1}}). \quad (43)$$

where  $\beta$  is given by  $\beta : s \in \mathbb{R}_+ \mapsto C_1(s + \sqrt{s})$ . This concludes the proof of Lemma 1.

Since for all  $t > t_r$ , the solution  $y$  of system (37) is globally exponentially stable in  $L^2(0, L)$ , then according to Lemma 1 and Proposition 1, there exists a  $\mathcal{KL}$ -function  $\alpha$  such that, for any  $y_0 \in L^2(0, L)$ , for any  $\hat{\eta}^0 \in \mathbb{R}^{n+1}$  and for any  $t \geq 0$ :

$$\|y(t, \cdot)\|_{L^2(0, L)} + |e(t)|_{\mathbb{R}^{n+1}} \leq \alpha(\|y_0\|_{L^2(0, L)} + |e^0|_{\mathbb{R}^{n+1}}, t) \quad (44)$$

for all solution  $(y, e)$  of (37). This concludes the proof of Theorem 2.

## 5. NUMERICAL ILLUSTRATION

Let  $L = 3, \lambda = 2, n = 1, k_1 = k_0 = 2, a = 0.47, C = 1$ . The time-space step variation  $(\Delta t, \Delta x) = (0.0015, 0.0150)$  satisfies the CFL condition  $\frac{\lambda \Delta t}{\Delta x} < 1$ . We consider a step disturbance

$$\begin{aligned} d(t) &= \mathbb{1}_{(0,1)}(t) - \mathbb{1}_{(1,5)}(t) + 0.5\mathbb{1}_{(5,6)}(t) - 0.7\mathbb{1}_{(6,10)}(t) \\ &+ 0\mathbb{1}_{(10,15)}(t) + 0.6\mathbb{1}_{(15,23)}(t) - 0.2\mathbb{1}_{(23,30)}(t) \\ &- \mathbb{1}_{(30,35)}(t) + \mathbb{1}_{(35,40)}(t) - 0.8\mathbb{1}_{(40,+\infty)}(t). \end{aligned} \quad (45)$$

In Figure 1, the robust stabilization of  $y$  is illustrated. Figure 2 shows that,  $\tilde{d}$  estimates well  $d$ .

## 6. CONCLUSION

In this paper, we apply the ADRC method based on a differentiation to stabilization of a linear hyperbolic

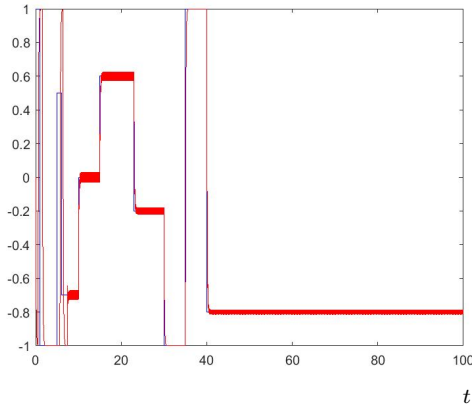


Fig. 2.  $\tilde{d}$  (---) and the disturbance  $d$  (—) versus time (sec).

system (transport equation to be more precise) subject to a boundary disturbance. The disturbance is supposed to have a known boundary. The Levant differentiator is used to estimate the disturbance in finite time. The existence of solutions of the closed-loop system is shown, and the global asymptotic stability of the closed-loop system is proven.

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