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# THE BOHR COMPACTIFICATION OF AN ARITHMETIC GROUP

BACHIR BEKKA

ABSTRACT. Given a group  $\Gamma$ , its Bohr compactification  $\text{Bohr}(\Gamma)$  and its profinite completion  $\text{Prof}(\Gamma)$  are compact groups naturally associated to  $\Gamma$ ; moreover,  $\text{Prof}(\Gamma)$  can be identified with the quotient of  $\text{Bohr}(\Gamma)$  by its connected component  $\text{Bohr}(\Gamma)_0$ . We study the structure of  $\text{Bohr}(\Gamma)$  for an arithmetic subgroup  $\Gamma$  of an algebraic group  $\mathbf{G}$  over  $\mathbf{Q}$ . When  $\mathbf{G}$  is unipotent, we show that  $\text{Bohr}(\Gamma)$  can be identified with the direct product  $\text{Bohr}(\Gamma^{\text{Ab}})_0 \times \text{Prof}(\Gamma)$ , where  $\Gamma^{\text{Ab}} = \Gamma/[\Gamma, \Gamma]$  is the abelianization of  $\Gamma$ . In the general case, using a Levi decomposition  $\mathbf{G} = \mathbf{U} \rtimes \mathbf{H}$  (where  $\mathbf{U}$  is unipotent and  $\mathbf{H}$  is reductive), we show that  $\text{Bohr}(\Gamma)$  can be described as the semi-direct product of a certain quotient of  $\text{Bohr}(\Gamma \cap \mathbf{U})$  with  $\text{Bohr}(\Gamma \cap \mathbf{H})$ . When  $\mathbf{G}$  is simple and has higher  $\mathbf{R}$ -rank,  $\text{Bohr}(\Gamma)$  is isomorphic, up to a finite group, to the product  $K \times \text{Prof}(\Gamma)$ , where  $K$  is the maximal compact factor of  $\mathbf{G}(\mathbf{R})$ .

## 1. INTRODUCTION

Given a topological group  $G$ , the **Bohr compactification** of  $G$  is a pair  $(\text{Bohr}(G), \beta)$  consisting of a compact (Hausdorff) group  $\text{Bohr}(G)$  and a continuous homomorphism  $\beta : G \rightarrow \text{Bohr}(G)$  with dense image, satisfying the following universal property: for every compact group  $K$  and every continuous homomorphism  $\alpha : G \rightarrow K$ , there exists a continuous homomorphism  $\alpha' : \text{Bohr}(G) \rightarrow K$  such that the diagram

$$\begin{array}{ccc}
 & & \text{Bohr}(G) \\
 & \nearrow \beta & \downarrow \alpha' \\
 G & \xrightarrow{\alpha} & K
 \end{array}$$

commutes. The pair  $(\text{Bohr}(G), \beta)$  is unique in the following sense: if  $(K', \beta')$  is a pair consisting of a compact group  $K'$  and a continuous homomorphism  $\beta' : G \rightarrow K'$  with dense image satisfying the same universal property (such a pair will be called a Bohr compactification of

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$G$ ), then there exists an isomorphism  $\alpha : \text{Bohr}(G) \rightarrow K'$  of topological groups such that  $\beta' = \alpha \circ \beta$ .

The compact group  $\text{Bohr}(G)$  was first introduced by A. Weil ([Wei40, Chap.VII]) as a tool for the study of almost periodic functions on  $G$ , a subject initiated by H. Bohr ([Boh25a], [Boh25b]) in the case  $G = \mathbf{R}$  and generalized to other groups by J. von Neumann ([vN34]) among others. For more on this subject, see [Dix77, §16] or [BH, 4.C]).

The group  $\text{Bohr}(\Gamma)$  has been determined for only very few non abelian *discrete* groups  $\Gamma$  (for some general results, see [HK01] and [Hol64]; for the well-known case of abelian groups, see [AK43] and Section 11).

In contrast, there is a second much more studied completion of  $\Gamma$ , namely the **profinite completion** of  $\Gamma$ , which is a pair  $(\text{Prof}(\Gamma), \alpha)$  consisting of a profinite group (that is, a projective limit of finite groups)  $\text{Prof}(\Gamma)$  satisfying a similar universal property with respect to such groups, together with a homomorphism with  $\alpha : \Gamma \rightarrow \text{Prof}(\Gamma)$  with dense image. The group  $\text{Prof}(\Gamma)$  can be realized as the projective limit  $\varprojlim \Gamma/H$ , where  $H$  runs over the family of the normal subgroups of finite index of  $\Gamma$ . For all this, see [RZ00].

The universal property of  $\text{Bohr}(\Gamma)$  gives rise to a continuous epimorphism  $\alpha' : \text{Bohr}(\Gamma) \rightarrow \text{Prof}(\Gamma)$ . It is easy to see (see Proposition 7 below) that the kernel of  $\alpha'$  is  $\text{Bohr}(\Gamma)_0$ , the connected component of  $\text{Bohr}(\Gamma)$ ; so, we have a short exact sequence

$$1 \longrightarrow \text{Bohr}(\Gamma)_0 \longrightarrow \text{Bohr}(\Gamma) \longrightarrow \text{Prof}(\Gamma) \longrightarrow 1.$$

In this paper, we will deal with the case where  $\Gamma$  is an arithmetic subgroup in a linear algebraic group. The setting is as follows. Let  $\mathbf{G}$  be a connected linear algebraic group over  $\mathbf{Q}$  with a fixed faithful representation  $\rho : \mathbf{G} \rightarrow GL_m$ . We consider the subgroup  $\mathbf{G}(\mathbf{Z})$  of the group  $\mathbf{G}(\mathbf{Q})$  of  $\mathbf{Q}$ -points of  $\mathbf{G}$ , that is,

$$\mathbf{G}(\mathbf{Z}) = \rho^{-1}(\rho(\mathbf{G}) \cap GL_m(\mathbf{Z})).$$

A subgroup  $\Gamma$  of  $\mathbf{G}(\mathbf{Q})$  is called an **arithmetic subgroup** if  $\Gamma$  is commensurable to  $\mathbf{G}(\mathbf{Z})$ , that is,  $\Gamma \cap \mathbf{G}(\mathbf{Z})$  has finite index in both  $\Gamma$  and  $\mathbf{G}(\mathbf{Z})$ . Observe that  $\Gamma$  is a discrete subgroup of the real Lie group  $\mathbf{G}(\mathbf{R})$ .

We first deal with the case where  $\mathbf{G}$  is unipotent. More generally, we describe the Bohr compactification of any finitely generated nilpotent group. Observe that an arithmetic subgroup in a unipotent algebraic  $\mathbf{Q}$ -group is finitely generated (see Corollary 2 of Theorem 2.10 in [Rag72]).

For two topological groups  $H$  and  $L$ , we write  $H \cong L$  if  $H$  and  $L$  are topologically isomorphic. We observe that, when  $\Delta$  is a finitely

generated abelian group,  $\text{Bohr}(\Delta)$  splits as a direct sum  $\text{Bohr}(\Delta) = \text{Bohr}(\Delta)_0 \oplus \text{Prof}(\Delta)$ ; see Proposition 11.

**Theorem 1.** *Let  $\Gamma$  be a finitely generated nilpotent group. We have a direct product decomposition*

$$\text{Bohr}(\Gamma) \cong \text{Bohr}(\Gamma^{\text{Ab}})_0 \times \text{Prof}(\Gamma),$$

where  $\Gamma^{\text{Ab}} = \Gamma/[\Gamma, \Gamma]$  is the abelianization of  $\Gamma$ . This isomorphism is induced by the natural maps  $\Gamma \rightarrow \text{Bohr}(\Gamma^{\text{Ab}})$  and  $\Gamma \rightarrow \text{Prof}(\Gamma)$ , together with the projection  $\text{Bohr}(\Gamma^{\text{Ab}}) \rightarrow \text{Bohr}(\Gamma^{\text{Ab}})_0$ .

A crucial tool in the proof of Theorem 1 is the fact that elements in the commutator subgroup  $[\Gamma, \Gamma]$  of a nilpotent group  $\Gamma$  are distorted (see Proposition 15).

We now turn to the case of a general algebraic group  $\mathbf{G}$  over  $\mathbf{Q}$ . Let  $\mathbf{U}$  be the unipotent radical of  $\mathbf{G}$ . Then  $\mathbf{U}$  is defined over  $\mathbf{Q}$  and there exists a connected reductive  $\mathbf{Q}$ -subgroup  $\mathbf{H}$  such that we have a **Levi decomposition** as semi-direct product  $\mathbf{G} = \mathbf{U} \rtimes \mathbf{H}$  (see [Mos56]).

The group  $\Lambda = \mathbf{H}(\mathbf{Z})$  acts by automorphisms on  $\Delta = \mathbf{U}(\mathbf{Z})$  and hence on  $\text{Bohr}(\Delta)$ , by the universal property of  $\text{Bohr}(\Delta)$ . In general, this action does not extend to an action of  $\text{Bohr}(\Lambda)$  on  $\text{Bohr}(\Delta)$ . However, as we will see below (proof of Theorem 2),  $\text{Bohr}(\Lambda)$  acts naturally by automorphisms on an appropriate quotient of  $\text{Bohr}(\Delta)$ .

Observe that (see [BHC62, Corollary 4.6]) every arithmetic subgroup of  $\mathbf{G}(\mathbf{Q})$  is commensurable to  $\Delta(\mathbf{Z}) \rtimes \mathbf{H}(\mathbf{Z})$ . Recall that two topological groups  $G_1$  and  $G_2$  are (abstractly) commensurable if there exist finite index subgroups  $H_1$  and  $H_2$  of  $G_1$  and  $G_2$  such that  $H_1$  is topologically isomorphic to  $H_2$ . If this is the case, then  $\text{Bohr}(G_1)$  and  $\text{Bohr}(G_2)$  are commensurable; in fact, each one of the groups  $\text{Bohr}(G_1)$  or  $\text{Bohr}(G_2)$  can be described in terms of the other (see Propositions 8 and 9). For this reason, we will often deal with only one chosen representative of the commensurability class of an arithmetic group.

**Theorem 2.** *Let  $\mathbf{G}$  be a connected linear algebraic group over  $\mathbf{Q}$ , with Levi decomposition  $\mathbf{G} = \mathbf{U} \rtimes \mathbf{H}$ . Set  $\Lambda := \mathbf{H}(\mathbf{Z})$ ,  $\Delta := \mathbf{U}(\mathbf{Z})$ , and  $\Gamma := \Delta \rtimes \Lambda$ . Let  $\widehat{\Delta^{\text{Ab}}}_{\Lambda\text{-fin}}$  be the subgroup of the dual group  $\widehat{\Delta^{\text{Ab}}}$  of  $\Delta^{\text{Ab}}$  consisting of the characters with finite  $\Lambda$ -orbit. We have a semi-direct decomposition*

$$\text{Bohr}(\Gamma) \cong (Q \times \text{Prof}(\Delta)) \rtimes \text{Bohr}(\Lambda),$$

where  $Q$  is the connected component of  $\text{Bohr}(\Delta^{\text{Ab}})/N$  and  $N$  is the annihilator of  $\widehat{\Delta^{\text{Ab}}}_{\Lambda\text{-fin}}$  in  $\text{Bohr}(\Delta^{\text{Ab}})$ . This isomorphism is induced by the natural homomorphisms  $\Delta \rightarrow \text{Bohr}(\Delta^{\text{Ab}})/N$  and  $\Lambda \rightarrow \text{Bohr}(\Lambda)$ .

Theorems 1 and 2 reduce the determination of  $\text{Bohr}(\Gamma)$  for an arithmetic group  $\Gamma$  in  $\mathbf{G}$  to the case where  $\mathbf{G}$  is reductive. We have a further reduction to the case where  $\mathbf{G}$  is simply connected and almost simple. Indeed, recall that a group  $L$  is the **almost direct product** of subgroups  $L_1, \dots, L_n$  if the product map  $L_1 \times \dots \times L_n \rightarrow L$  is a surjective homomorphism with finite kernel.

Let  $\mathbf{G}$  be a connected reductive algebraic group over  $\mathbf{Q}$ . The commutator subgroup  $\mathbf{L} := [\mathbf{G}, \mathbf{G}]$  of  $\mathbf{G}$  is a connected semi-simple  $\mathbf{Q}$ -group and  $\mathbf{G}$  is an almost direct product  $\mathbf{G} = \mathbf{T}\mathbf{L}$  for a central  $\mathbf{Q}$ -torus  $\mathbf{T}$  (see (14.2) and (18.2) in [Bor91]) Moreover,  $\mathbf{L}$  is an almost direct product  $\mathbf{L} = \mathbf{L}_1 \cdots \mathbf{L}_n$  of connected almost  $\mathbf{Q}$ -simple  $\mathbf{Q}$ -subgroups  $\mathbf{L}_i$ , called the almost  $\mathbf{Q}$ -simple factors of  $\mathbf{L}$  (see [Bor91, (22.10)]). For every  $i \in \{1, \dots, n\}$ , let  $\tilde{\mathbf{L}}_i$  be the simply connected covering group  $\mathbf{L}_i$ . Set  $\tilde{\mathbf{G}} = \mathbf{T} \times \tilde{\mathbf{L}}_1 \times \dots \times \tilde{\mathbf{L}}_n$ . Let  $\tilde{\Gamma}$  be the arithmetic subgroup  $\mathbf{T}(\mathbf{Z}) \times \tilde{\mathbf{L}}_1(\mathbf{Z}) \times \dots \times \tilde{\mathbf{L}}_n(\mathbf{Z})$  in  $\tilde{\mathbf{G}}(\mathbf{Q})$ . The image  $\Gamma$  of  $\tilde{\Gamma}$  under the isogeny  $p : \tilde{\mathbf{G}} \rightarrow \mathbf{G}$  is an arithmetic subgroup of  $\mathbf{G}(\mathbf{Q})$  (see Corollaries 6.4 and 6.11 in [BHC62]). The map  $p : \tilde{\Gamma} \rightarrow \Gamma$  induces an isomorphism  $\text{Bohr}(\Gamma) \cong \text{Bohr}(\tilde{\Gamma})/F$ , where  $F$  is the finite normal subgroup  $F = \tilde{\beta}(\ker p)$  and  $\tilde{\beta} : \tilde{\Gamma} \rightarrow \text{Bohr}(\tilde{\Gamma})$  is the natural map (see Proposition 10).

As an easy consequence of Margulis' superrigidity results, we give a description of the Bohr compactification of an arithmetic lattice in a simple algebraic  $\mathbf{Q}$ -group  $\mathbf{G}$  under a higher rank assumption. Such a description does not seem possible for arbitrary  $\mathbf{G}$ . For instance, the free non abelian group  $F_2$  on two generators is an arithmetic lattice in  $SL_2(\mathbf{Q})$ , but we know of no simple description of  $\text{Bohr}(F_2)$ .

**Theorem 3.** *Let  $\mathbf{G}$  be a connected, simply connected, and almost simple  $\mathbf{Q}$ -group. Assume that the real semisimple Lie group  $\mathbf{G}(\mathbf{R})$  is not locally isomorphic to any group of the form  $SO(m, 1) \times K$  or  $SU(m, 1) \times K$  for a compact Lie group  $K$ . Let  $\mathbf{G}_{\text{nc}}$  be the product of the almost  $\mathbf{R}$ -simple factors  $\mathbf{G}_i$  of  $\mathbf{G}$  for which  $\mathbf{G}_i(\mathbf{R})$  is non compact. Let  $\Gamma \subset \mathbf{G}(\mathbf{Q})$  be an arithmetic subgroup. We have a direct product decomposition*

$$\text{Bohr}(\Gamma) \cong \text{Bohr}(\Gamma)_0 \times \text{Prof}(\Gamma)$$

and an isomorphism

$$\text{Bohr}(\Gamma)_0 \cong \mathbf{G}(\mathbf{R})/\mathbf{G}_{\text{nc}}(\mathbf{R}),$$

induced by the natural maps  $\Gamma \rightarrow \mathbf{G}(\mathbf{R})/\mathbf{G}_{\text{nc}}(\mathbf{R})$  and  $\Gamma \rightarrow \text{Prof}(\Gamma)$ .

A group  $\Gamma$  as in Theorem 3 is an irreducible lattice in the Lie group  $G = \mathbf{G}(\mathbf{R})$ , that is, the homogeneous space  $G/\Gamma$  carries a  $G$ -invariant

probability measure; moreover,  $\Gamma$  is cocompact in  $G$  if and only if  $\mathbf{G}$  is anisotropic over  $\mathbf{Q}$  (for all this, see [BHC62, (7.8), (11.6)]). The following corollary is a direct consequence of Theorem 3 and of the fact that a non cocompact arithmetic lattice in a semisimple Lie group has nontrivial unipotent elements (see [Mor15, (5.5.14)]).

**Corollary 4.** *With the notation as in Theorem 3, assume that  $\mathbf{G}$  is isotropic over  $\mathbf{Q}$ . For every arithmetic subgroup  $\Gamma$  of  $\mathbf{G}(\mathbf{Q})$ , the natural map  $\text{Bohr}(\Gamma) \rightarrow \text{Prof}(\Gamma)$  is an isomorphism.*

As shown in Section 6, it may happen that  $\text{Bohr}(\mathbf{G}(\mathbf{Z})) \cong \text{Prof}(\mathbf{G}(\mathbf{Z}))$ , even when  $\mathbf{G}(\mathbf{Z})$  is cocompact in  $\mathbf{G}(\mathbf{R})$ .

A general arithmetic lattice  $\Gamma$  has a third completion: the **congruence completion**  $\text{Cong}(\Gamma)$  of  $\Gamma$  is the projective limit  $\varprojlim \Gamma/H$ , where  $H$  runs over the family of the congruence subgroups of  $\Gamma$ ; recall that a normal subgroup of  $\Gamma$  is a congruence subgroup if it contains the kernel of the map  $\mathbf{G}(\mathbf{Z}) \rightarrow \mathbf{G}(\mathbf{Z}/N\mathbf{Z})$  of the reduction modulo  $N$ , for some integer  $N \geq 1$ . There is a natural surjective homomorphism  $\pi : \text{Prof}(\Gamma) \rightarrow \text{Cong}(\Gamma)$ . The so-called **congruence subgroup problem** asks whether  $\pi$  is injective and hence an isomorphism of topological groups; more generally, one can ask for a description of the kernel of  $\pi$ . This problem has been extensively studied for arithmetic subgroups (and, more generally, for  $S$ -arithmetic subgroups) in various algebraic groups; for instance, it is known that  $\pi$  is an isomorphism when  $\Gamma = SL_n(\mathbf{Z})$  for  $n \geq 3$  or  $\Gamma = Sp_{2n}(\mathbf{Z})$  for  $n \geq 2$  (see [BMS67]); moreover, the same conclusion is true when  $\Gamma = \mathbf{T}(\mathbf{Z})$  for a torus  $\mathbf{T}$  (see [Che51]) and when  $\Gamma = \mathbf{U}(\mathbf{Z})$  for a unipotent group  $\mathbf{U}$  (see Proposition 16 below). For more on the congruence subgroup problem, see for instance [Rag76] or [PR94, §9.5].

This paper is organized as follows. In Section 2, we establish some general facts about the Bohr compactifications of commensurable groups and the relationship between Bohr compactifications and unitary representations; we also give an explicit description of the Bohr compactification for a finitely generated abelian group. In Section 3, we give the proof of Theorem 1. Section 4 contains the proof of Theorem 2 and Section 5 the proof of Theorem 3. Section 6 is devoted to the explicit computation of the Bohr compactification for various examples of arithmetic groups.

## 2. SOME PRELIMINARIES

**2.1. Bohr compactifications and unitary representations.** Given a topological group  $G$ , we will consider finite dimensional unitary representations of  $G$ , that is, continuous homomorphisms  $G \rightarrow U(n)$ . Two

such representations are equivalent if they are conjugate by a unitary matrix. A representation  $\pi$  is irreducible if  $\mathbf{C}^n$  and  $\{0\}$  there are only  $\pi(G)$ -invariant subspaces of  $\mathbf{C}^n$ . We denote by  $\text{Rep}_{\text{fd}}(G)$  the set of equivalence classes of finite dimensional unitary representations of  $G$  and by  $\widehat{G}_{\text{fd}}$  the subset of irreducible ones. Every  $\pi \in \text{Rep}_{\text{fd}}(G)$  is a direct sum of representations from  $\widehat{G}_{\text{fd}}$

When  $K$  is a compact group, every irreducible unitary representation of  $K$  is finite dimensional and  $\widehat{K}_{\text{fd}} = \widehat{K}$  is the unitary dual space of  $K$ . By the Peter-Weyl theorem,  $\widehat{K}$  separates the points of  $K$ .

Let  $\beta : G \rightarrow H$  be a continuous homomorphism of topological groups  $G$  and  $H$  with dense image; then  $\beta$  induces *injective* maps

$$\widehat{\beta} : \text{Rep}_{\text{fd}}(H) \rightarrow \text{Rep}_{\text{fd}}(G) \quad \text{and} \quad \widehat{\beta} : \widehat{H}_{\text{fd}} \rightarrow \widehat{G}_{\text{fd}},$$

given by  $\widehat{\beta}(\pi) = \pi \circ \beta$  for  $\pi \in \text{Rep}_{\text{fd}}(H)$ . The following proposition, which may be considered as well-known, is a useful tool for identifying the Bohr compactification of a group.

**Proposition 5.** *Let  $G$  be a topological group,  $K$  a compact group, and  $\beta : G \rightarrow K$  a continuous homomorphism with dense image. The following properties are equivalent:*

- (i)  $(K, \beta)$  is a Bohr compactification of  $G$ ;
- (ii) the induced map  $\widehat{\beta} : \widehat{K} \rightarrow \widehat{G}_{\text{fd}}$  is surjective;
- (iii) the induced map  $\widehat{\beta} : \text{Rep}_{\text{fd}}(K) \rightarrow \text{Rep}_{\text{fd}}(G)$  is surjective.

*Proof.* Assume that (i) holds and let  $\pi : G \rightarrow U(n)$  be an irreducible representation of  $G$ ; by the universal property of the Bohr compactification, there exists a continuous homomorphism  $\pi' : K \rightarrow U(n)$  such that  $\pi = \widehat{\beta}(\pi')$  and (ii) follows.

Conversely, assume that (ii) holds. Let  $L$  be a compact group and  $\alpha : G \rightarrow L$  a continuous homomorphism with dense image. Choose a family  $\pi_i : L \rightarrow U(n_i)$  of representatives of  $\widehat{L}$ . By the Peter-Weyl theorem, we may identify  $L$  with its image in  $\prod_i U(n_i)$  under the map  $x \mapsto \bigoplus_i \pi_i(x)$ . For every  $i$ , we have  $\pi_i \circ \alpha \in \widehat{G}_{\text{fd}}$  and hence  $\pi_i \circ \alpha = \widehat{\beta}(\pi'_i) = \pi'_i \circ \beta$  for some representation  $\pi'_i : K \rightarrow U(n_i)$  of  $K$ . Define a continuous homomorphism

$$\alpha' : K \rightarrow \prod_i U(n_i) \quad x \mapsto \bigoplus_i \pi'_i(x).$$

We have  $\alpha' \circ \beta = \alpha$  and hence

$$\alpha'(K) = \alpha' \left( \overline{\beta(G)} \right) \subset \overline{\alpha(G)} = L.$$

So, (i) and (ii) are equivalent. It is obvious that (ii) is equivalent to (iii). □

The profinite completion  $(\text{Prof}(G), \alpha)$  of  $G$  may be similarly characterized in terms of certain unitary representations of  $G$ . Recall first that  $(\text{Prof}(G), \alpha)$  is a pair consisting of a profinite group  $\text{Prof}(G)$  and a continuous homomorphism  $\alpha : G \rightarrow \text{Prof}(G)$  with dense image, satisfying the following universal property: for every profinite group  $K$  and every continuous homomorphism  $f : G \rightarrow K$ , there exists a continuous homomorphism  $f' : \text{Prof}(G) \rightarrow K$  such that the diagram

$$\begin{array}{ccc}
 & & \text{Prof}(G) \\
 & \nearrow \alpha & \downarrow f' \\
 G & \xrightarrow{f} & K
 \end{array}$$

commutes. Recall that the class of profinite groups coincides with the class of totally disconnected compact groups (see [BH, Proposition 4.C.10]).

Denote by  $\text{Rep}_{\text{finite}}(G)$  the set of equivalence classes of finite dimensional unitary representations  $\pi$  of  $G$  for which  $\pi(G)$  is finite; let  $\widehat{G}_{\text{finite}}$  be the subset of irreducible representations from  $\text{Rep}_{\text{finite}}(G)$ .

If  $\alpha : G \rightarrow H$  is a continuous homomorphism of topological groups  $G$  and  $H$  with dense image, then  $\alpha$  induces *injective* maps

$$\widehat{\alpha} : \text{Rep}_{\text{finite}}(H) \rightarrow \text{Rep}_{\text{finite}}(G) \quad \text{and} \quad \widehat{\alpha} : \widehat{H}_{\text{finite}} \rightarrow \widehat{G}_{\text{finite}}.$$

Observe that  $\widehat{K} = \widehat{K}_{\text{finite}}$  if  $K$  is a profinite group. (Conversely, it follows from Peter-Weyl theorem that, if  $K$  is a compact group with  $\widehat{K} = \widehat{K}_{\text{finite}}$ , then  $K$  is profinite.) The proof of the following proposition is similar to the proof of Proposition 5 and will be omitted.

**Proposition 6.** *Let  $K$  be a totally disconnected compact group and  $\alpha : G \rightarrow K$  a continuous homomorphism with dense image. The following properties are equivalent:*

- (i)  $(K, \alpha)$  is a profinite completion of  $G$ ;
- (ii) the induced map  $\widehat{\alpha} : \widehat{K} \rightarrow \widehat{G}_{\text{finite}}$  is surjective;
- (ii) the induced map  $\widehat{\beta} : \text{Rep}_{\text{finite}}(K) \rightarrow \text{Rep}_{\text{finite}}(G)$  is surjective.



The universal property of  $\text{Bohr}(G)$  implies that there is a continuous epimorphism  $\alpha' : \text{Bohr}(G) \rightarrow \text{Prof}(G)$  such that the diagram

$$\begin{array}{ccc} & & \text{Bohr}(G) \\ & \nearrow \beta & \downarrow \alpha' \\ G & \xrightarrow{\alpha} & \text{Prof}(G) \end{array}$$

commutes. We record the following elementary but basic fact mentioned in the introduction.

**Proposition 7.** *The kernel of  $\alpha' : \text{Bohr}(G) \rightarrow \text{Prof}(G)$  coincides with the connected component  $\text{Bohr}(G)_0$  of  $\text{Bohr}(G)$ .*

*Proof.* Since  $\text{Bohr}(G)_0$  is connected and  $\text{Prof}(G)$  is totally disconnected,  $\text{Bohr}(G)_0$  is contained in  $\text{Ker}\alpha'$ . So,  $\alpha'$  factorizes to a continuous epimorphism  $\alpha'' : K \rightarrow \text{Prof}(G)$ , where  $K := \text{Bohr}(G)/\text{Bohr}(G)_0$  and we have a commutative diagram

$$\begin{array}{ccc} & & K \\ & \nearrow p \circ \beta & \downarrow \alpha'' \\ G & \xrightarrow{\alpha} & \text{Prof}(G). \end{array}$$

where  $p : \text{Bohr}(G) \rightarrow K$  is the canonical epimorphism. Since  $K$  is a totally disconnected compact group, there exists a continuous epimorphism  $f : \text{Prof}(G) \rightarrow K$  and we have a commutative diagram

$$\begin{array}{ccc} & & K \\ & \nearrow p \circ \beta & \uparrow f \\ G & \xrightarrow{\alpha} & \text{Prof}(G). \end{array}$$

For every  $g \in G$ , we have

$$f(\alpha''(p \circ \beta(g))) = f(\alpha(g)) = p \circ \beta(g);$$

since  $p \circ \beta(G)$  is dense in  $K$ , it follows that  $f \circ \alpha''$  is the identity on  $K$ . This implies that  $\alpha''$  is injective and hence an isomorphism.  $\square$

**2.2. Bohr compactifications of commensurable groups.** Let  $G$  be a topological group and  $H$  be a closed subgroup of finite index in  $G$ . We first determine  $\text{Bohr}(H)$  in terms of  $\text{Bohr}(G)$ .

**Proposition 8.** *Let  $(\text{Bohr}(G), \beta)$  be the Bohr compactification of  $G$ . Set  $K := \overline{\beta(H)}$ .*

- (i)  $K$  is a subgroup of finite index of  $\text{Bohr}(G)$ .
- (ii)  $(K, \beta|_H)$  is a Bohr compactification of  $H$ .

- (iii)  $K$  and  $\text{Bohr}(G)$  have the same connected component of the identity.

*Proof.* Item (i) is obvious and Item (iii) follows from Item (i). To show Item (ii), let  $\pi$  be a unitary representation of  $H$  on  $\mathbf{C}^n$ . Since  $H$  has finite index in  $G$ , the induced representation  $\rho := \text{Ind}_H^G \pi$ , which is a unitary representation of  $G$ , is finite dimensional. Hence, there exists  $\rho' \in \text{Rep}_{\text{fd}}(\text{Bohr}(G))$  such that  $\rho = \rho' \circ \beta$ . Now,  $\pi$  is equivalent to a subrepresentation of the restriction of  $\rho$  to  $H$  (see [BH, 1.F]); so, we may identify  $\pi$  with the representation of  $H$  defined by a  $\rho(H)$ -invariant subspace  $W$  of the space of  $\rho$ . Then  $W$  is  $\rho'(K)$ -invariant and defines therefore a representation  $\pi'$  of  $K$ . We have  $\pi = \pi' \circ (\beta|_H)$  and Proposition 5 shows that Item (ii) holds.  $\square$

Next, we want to determine  $\text{Bohr}(G)$  in terms of  $\text{Bohr}(H)$ .

Given a compact group  $K$  and a finite set  $X$ , we define another compact group, we call the **induced group** of  $(K, X)$ , as

$$\text{Ind}(K, X) := K^X \rtimes \text{Sym}(X),$$

where the group  $\text{Sym}(X)$  of bijections of  $X$  acts by permutations of indices on  $K^X$ :

$$\sigma((g_x)_{x \in X}) = (g_{\sigma^{-1}(x)})_{x \in X} \quad \text{for all } \sigma \in \text{Sym}(X), (g_x)_{x \in X} \in K^X$$

Observe that, if  $\pi : K \rightarrow U(n)$  is a representation of  $K$  on  $V = \mathbf{C}^n$ , then a unitary representation  $\text{Ind}(\pi)$  of  $\text{Ind}(K, X)$  on  $V^X$  is defined by

$$\text{Ind}(\pi)((g_x)_{x \in X}, \sigma)(v_x)_{x \in X} = (\pi(g_x)v_{\sigma^{-1}(x)})_{x \in X},$$

for  $((g_x)_{x \in X}, \sigma) \in \text{Ind}(K, X)$  and  $(v_x)_{x \in X} \in V^X$ .

Coming back to our setting, where  $H$  is a closed subgroup of finite index in  $G$ , we fix a transversal  $X$  for the right cosets of  $H$ ; so, we have a disjoint union  $G = \bigsqcup_{x \in X} Hx$ . For every  $g \in G$  and  $x \in X$ , let  $x \cdot g$  and  $c(x, g)$  be the unique elements in  $X$  and  $H$  such that  $xg = c(x, g)(x \cdot g)$ . Observe that

$$X \times G \rightarrow X, \quad (x, g) \mapsto x \cdot g$$

is an action of  $G$  on  $X$  (on the right), which is equivalent to the natural action of  $G$  on  $H \backslash G$  given by right multiplication. In particular, for every  $g \in G$ , the map  $\sigma(g) : x \mapsto x \cdot g^{-1}$  belongs to  $\text{Sym}(X)$  and we have a homomorphism

$$G \mapsto \text{Sym}(X), \quad g \mapsto \sigma(g).$$

**Proposition 9.** *Let  $(\text{Bohr}(H), \beta)$  be the Bohr compactification of  $H$ . Let  $\text{Ind}(\text{Bohr}(H), X)$  be the compact group defined as above. Consider the map  $\tilde{\beta} : G \rightarrow \text{Ind}(\text{Bohr}(H), X)$  defined by*

$$\tilde{\beta}(g) = (\beta(c(x, g)))_{x \in X}, \sigma(g) \quad \text{for all } g \in G.$$

*The closure of  $\tilde{\beta}(G)$  in  $\text{Ind}(\text{Bohr}(H), X)$ , together with the map  $\tilde{\beta}$ , is a Bohr compactification of  $G$ .*

*Proof.* It is readily checked that  $\tilde{\beta} : G \rightarrow \text{Ind}(\text{Bohr}(H), X)$  is a continuous homomorphism. Let  $\rho : G \rightarrow U(n)$  be a finite dimensional unitary representation of  $G$ . Set  $\pi := \rho|_H \in \text{Rep}_{\text{fd}}(H)$ . There exists  $\pi' \in \text{Rep}_{\text{fd}}(\text{Bohr}(H))$  such that  $\pi = \pi' \circ \beta$ . Let  $\tilde{\pi} := \text{Ind}_H^G \pi$ . As is well-known (see [BH, 1.F]),  $\tilde{\pi}$  can be realized on  $V^X$  for  $V := \mathbf{C}^n$  by the formula

$$\tilde{\pi}(g)(v_x)_{x \in X} = (\pi(c(x, g))v_{x \cdot g})_{x \in X} = (\pi(c(x, g))v_{\sigma(g^{-1})x})_{x \in X},$$

for all  $g \in G$  and  $(v_x)_{x \in X} \in V^X$ . With the unitary representation  $\text{Ind}(\pi')$  of  $\text{Ind}(\text{Bohr}(H), X)$  defined as above, we have therefore

$$(*) \quad \tilde{\pi}(g) = \text{Ind}(\pi')(\tilde{\beta}(g)) \quad \text{for all } g \in G,$$

that is,  $\tilde{\pi} = \text{Ind}(\pi') \circ \tilde{\beta}$ . Now,

$$\tilde{\pi} = \text{Ind}_H^G \pi = \text{Ind}_H^G(\rho|_H)$$

is equivalent to the tensor product representation  $\rho \otimes \lambda_{G/H}$ , where  $\lambda_{G/H}$  is the regular representation of  $G/H$  (see [BHV08, E.2.5]). Since  $\lambda_{G/H}$  contains the trivial representation of  $G$ , it follows that  $\rho$  is equivalent to a subrepresentation of  $\tilde{\pi}$ ; so, we can identify  $\rho$  with the representation of  $G$  defined by a  $\tilde{\pi}(G)$ -invariant subspace  $W$  of  $V^X$ . Denoting by  $L$  the closure of  $\tilde{\beta}(G)$ , it follows from  $(*)$  that  $W$  is invariant under  $\text{Ind}(\pi')(L)$  and so defines a representation  $\rho'$  of  $L$ . Then  $\rho = \rho' \circ \tilde{\beta}$  and the claim follows from Proposition 5.  $\square$

We will also need the following well-known (see [HK01, Lemma 2.2]) description of the Bohr compactification of a quotient of  $G$  in terms of the Bohr compactification of  $G$ .

**Proposition 10.** *Let  $(\text{Bohr}(G), \beta)$  be the Bohr compactification of the topological group  $G$  and let  $N$  be a closed normal subgroup of  $G$ . Let  $K_N$  be the closure of  $\beta(N)$  in  $\text{Bohr}(G)$*

- (i)  $K_N$  is a normal subgroup of  $\text{Bohr}(G)$  and  $\beta$  induces a continuous homomorphism  $\bar{\alpha} : G/N \rightarrow \text{Bohr}(G)/K_N$
- (ii)  $(\text{Bohr}(G)/K_N, \bar{\alpha})$  is a Bohr compactification of  $G/N$ .

*Proof.* Let  $(\text{Bohr}(G/N), \bar{\beta})$  be the Bohr compactification of  $G/N$ . The canonical homomorphism  $\alpha : G \rightarrow G/N$  induces a continuous homomorphism  $\alpha' : \text{Bohr}(G) \rightarrow \text{Bohr}(G/N)$  such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\beta} & \text{Bohr}(G) \\ \downarrow \alpha & & \downarrow \alpha' \\ G/N & \xrightarrow{\bar{\beta}} & \text{Bohr}(G/N) \end{array}$$

commutes. It follows that  $\beta(N)$  and hence  $K_N$  is contained in  $\text{Ker } \alpha'$ . So, we have induced homomorphisms  $\bar{\alpha} : G/N \rightarrow \text{Bohr}(G)/K_N$  and  $\bar{\alpha}' : \text{Bohr}(G)/K_N \rightarrow \text{Bohr}(G/N)$ , giving rise to a commutative diagram

$$\begin{array}{ccc} & \text{Bohr}(G)/K_N & \\ & \nearrow \bar{\alpha} & \downarrow \bar{\alpha}' \\ G/N & \xrightarrow{\bar{\beta}} & \text{Bohr}(G/N). \end{array}$$

It follows that  $(\text{Bohr}(G)/K_N, \bar{\alpha})$  has the same universal property for  $G/N$  as  $(\text{Bohr}(G/N), \bar{\beta})$ . Since  $\bar{\alpha}$  has dense image,  $(\text{Bohr}(G)/K_N, \bar{\alpha})$  is therefore a Bohr compactification of  $G/N$ .  $\square$

### 2.3. Bohr compactification of finitely generated abelian groups.

Let  $G$  be a locally compact abelian group. Its dual group  $\widehat{G}$  consists of the continuous homomorphism from  $G$  to the circle group  $\mathbf{S}^1$ ; equipped with the topology of uniform convergence on compact subsets,  $\widehat{G}$  is again a locally compact abelian group. Let  $\widehat{G}_{\text{disc}}$  be the group  $\widehat{G}$  equipped with the discrete topology. It is well-known (see e.g. [BH, Proposition 4.C.4]) that the Bohr compactification of  $G$  coincides with the dual group  $K$  of  $\widehat{G}_{\text{disc}}$ , together with the embedding  $i : G \rightarrow K$  given by  $i(g)(\chi) = \chi(g)$  for all  $g \in G$  and  $\chi \in \widehat{G}$ . Notice that this implies that, by Pontrjagin duality, the dual group of  $\text{Bohr}(G)$  coincides with  $\widehat{G}_{\text{disc}}$ .

A more precise information on the structure of the Bohr compactification is available in the case of a (discrete) finitely generated abelian group. As is well-known, such a group  $\Gamma$  splits a direct sum  $\Gamma = F \oplus A$  of a finite group  $F$  (which is its torsion subgroup) and a free abelian group  $A$  of finite rank  $k \geq 0$ , called the rank of  $\Gamma$ . Recall that  $\mathbf{Z}_p$  denotes the ring of  $p$ -adic integers for a prime  $p$  and  $\mathbf{A}$  the ring of adèles over  $\mathbf{Q}$ .

**Proposition 11.** *Let  $\Gamma$  be a finitely generated abelian group of rank  $k$ .*

(i) *We have a direct sum decomposition*

$$\text{Bohr}(\Gamma) \cong \text{Bohr}(\Gamma)_0 \oplus \text{Prof}(\Gamma).$$

(ii) *We have*

$$\text{Prof}(\Gamma) \cong F \oplus \prod_{p \text{ prime}} \mathbf{Z}_p^k,$$

where  $F$  is a finite group.

(iii) *We have*

$$\text{Bohr}(\Gamma)_0 \cong \prod_{\omega \in \mathfrak{c}} \mathbf{A}^k / \mathbf{Q}^k,$$

a product of uncountably many copies of the adelic solenoid  $\mathbf{A}^k / \mathbf{Q}^k$ .

*Proof.* We have  $\Gamma \cong F \oplus \mathbf{Z}^k$  for a finite group  $F$  and  $\text{Bohr}(\mathbf{Z}^k) = \text{Bohr}(\mathbf{Z})^k$ . So, it suffices to determine  $\text{Bohr}(\mathbf{Z})$ . As mentioned above,  $\text{Bohr}(\mathbf{Z})$  can be identified with the dual group of the circle  $\mathbf{S}^1$  viewed as discrete group. Choose a linear basis  $\{1\} \cup \{x_\omega \mid \omega \in \mathfrak{c}\}$  of  $\mathbf{R}$  over  $\mathbf{Q}$ . Then  $\mathbf{S}^1 \cong \mathbf{R}/\mathbf{Z}$  is isomorphic to the abelian group

$$(\mathbf{Q}/\mathbf{Z}) \oplus \bigoplus_{\omega \in \mathfrak{c}} \mathbf{Q}.$$

Hence,

$$\text{Bohr}(\mathbf{Z}) \cong \widehat{\mathbf{Q}/\mathbf{Z}} \oplus \prod_{\omega \in \mathfrak{c}} \widehat{\mathbf{Q}}.$$

Now,

$$\mathbf{Q}/\mathbf{Z} = \bigoplus_{p \text{ prime}} Z(p^\infty),$$

with  $Z(p^\infty) = \varinjlim_k \mathbf{Z}/p^k \mathbf{Z}$  the  $p$ -primary component of  $\mathbf{Q}/\mathbf{Z}$ . Hence,

$$\widehat{Z(p^\infty)} \cong \varprojlim_k \mathbf{Z}/p^k \mathbf{Z} = \mathbf{Z}_p.$$

On the other hand,  $\widehat{\mathbf{Q}}$  can be identified with the solenoid  $\mathbf{A}/\mathbf{Q}$  (see e.g. [HR79, (25.4)]). It follows that

$$\text{Bohr}(\Gamma) \cong \prod_{p \text{ prime}} \mathbf{Z}_p \oplus \prod_{\omega \in \mathfrak{c}} \mathbf{A}/\mathbf{Q}.$$

□

**2.4. Restrictions of representations to normal subgroups.** Let  $\Gamma$  be a group and  $N$  a normal subgroup of  $\Gamma$ . Recall that  $\Gamma$  acts on  $\widehat{N}_{\text{fd}}$ : for  $\sigma \in \widehat{N}_{\text{fd}}$  and  $\gamma \in \Gamma$ , the conjugate representation  $\sigma^\gamma \in \widehat{N}_{\text{fd}}$  is defined by

$$\sigma^\gamma(n) = \sigma(\gamma^{-1}n\gamma), \quad \text{for all } n \in N.$$

The stabilizer  $\Gamma_\sigma$  of  $\sigma$  is the subgroup consisting of all  $\gamma \in \Gamma$  for which  $\sigma^\gamma$  is equivalent to  $\sigma$ ; observe that  $\Gamma_\sigma$  contains  $N$ .

Given a unitary representation  $\rho$  of  $N$  on a finite dimensional vector space  $V$  and  $\sigma \in \widehat{N}_{\text{fd}}$ , we denote by  $V^\sigma$  the  $\sigma$ -isotypical component of  $\rho$ , that is, the sum of all  $\rho$ -invariant subspaces  $W$  for which the restriction of  $\rho$  to  $W$  is equivalent to  $\sigma$ . Observe that  $V$  decomposes as direct sum  $V = \bigoplus_{\sigma \in \Sigma_\rho} V^\sigma$ , where  $\Sigma_\rho$  is the finite set of  $\sigma \in \widehat{N}_{\text{fd}}$  with  $V^\sigma \neq \{0\}$ .

**Proposition 12.** *Let  $\pi$  be an irreducible unitary representation of  $\Gamma$  on a finite dimensional vector space  $V$ . Let  $V = \bigoplus_{\sigma \in \Sigma_{\pi|_N}} V^\sigma$  be the decomposition of the restriction  $\pi|_N$  of  $\pi$  to  $N$  into isotypical components. Then  $\Sigma_{\pi|_N}$  coincides with a  $\Gamma$ -orbit: there exists  $\sigma \in \widehat{N}_{\text{fd}}$  such that  $\Sigma_{\pi|_N} = \{\sigma^\gamma : \gamma \in \Gamma\}$ ; in particular,  $\Gamma_\sigma$  has finite index in  $\Gamma$ .*

*Proof.* Let  $\sigma \in \Sigma_{\pi|_N}$  and fix a transversal  $T$  for the left cosets of  $\Gamma_\sigma$  with  $e \in T$ . Then  $V^{\sigma^t} = \pi(t)V^\sigma$  for all  $t \in T$ . Since  $\pi$  is irreducible and  $\sum_{t \in T} \pi(t)V^\sigma$  is  $\pi(\Gamma)$ -invariant, it follows that  $\Sigma_{\pi|_N}$  is a  $\Gamma$ -orbit.  $\square$

### 3. PROOF OF THEOREM 1

**3.1. Distortion and Bohr compactification.** Let  $\Gamma$  be a finitely generated group with a finite set  $S$  of generators. For  $\gamma \in \Gamma$ , denote by  $\ell_S(\gamma)$  the word length of  $\gamma$  with respect to  $S \cup S^{-1}$  and set

$$t(\gamma) = \liminf_{n \rightarrow \infty} \frac{\ell_S(\gamma^n)}{n}.$$

The number  $t(\gamma)$  is called the *translation number* of  $\gamma$  in [GS91]

**Definition 13.** An element  $\gamma \in \Gamma$  is said to be **distorted** if  $t(\gamma) = 0$ .

In fact, since the sequence  $n \mapsto \ell_S(\gamma^n)$  is subadditive, we have, by Fekete's lemma,

$$t(\gamma) = \lim_{n \rightarrow \infty} \frac{\ell_S(\gamma^n)}{n} = \inf \left\{ \frac{\ell_S(\gamma^n)}{n} : n \in \mathbf{N}^* \right\}$$

The property of being distorted is independent of the choice of the set of generators. Distorted elements are called *algebraically parabolic* in [BGS85, (7.5), p.90], but we prefer to use the terminology from [FH06]. The relevance of distortion to the Bohr compactification lies

in the following proposition; for a related result with a similar proof, see [LMR00, (2.4)].

**Proposition 14.** *Let  $\Gamma$  be a finitely generated group and  $\gamma \in \Gamma$  a distorted element. Then, for every finite dimensional unitary representation  $\pi : \Gamma \rightarrow U(N)$  of  $\Gamma$ , the matrix  $\pi(\gamma) \in U(N)$  has finite order.*

*Proof.* It suffices to show that all eigenvalues of the unitary matrix  $\pi(\gamma)$  are roots of unity. Assume, by contradiction, that  $\pi$  has an eigenvalue  $\lambda \in \mathbf{S}^1$  of infinite order.

Let  $S$  be a finite set of generators of  $\Gamma$  with  $S = S^{-1}$ . The group  $\pi(\Gamma)$  is generated by the set  $\{\pi(s) \mid s \in S\}$ . Hence,  $\pi(G)$  is contained in  $GL_N(L)$ , where  $L$  is the subfield of  $\mathbf{C}$  generated by the matrix coefficients of the  $\pi(s)$ 's. It follows that  $\lambda$  is contained in a finitely generated extension  $\ell$  of  $L$ . By a lemma of Tits ([Tit72, Lemma 4.1]), there exists a locally compact field  $k$  endowed with an absolute value  $|\cdot|$  and a field embedding  $\sigma : \ell \rightarrow k$  such that  $|\sigma(\lambda)| \neq 1$ . Upon replacing  $\gamma$  by  $\gamma^{-1}$ , we may assume that  $|\sigma(\lambda)| > 1$ .

Define a function (“norm”)  $\xi \mapsto \|\xi\|$  on  $k^N$  by

$$\|\xi\| = \max\{|\xi_1|, \dots, |\xi_N|\} \quad \text{for all } \xi = (\xi_1, \dots, \xi_N) \in k^N.$$

For a matrix  $A \in GL_N(k)$ , set  $\|A\| = \sup_{\xi \neq 0} \|A\xi\|/\|\xi\|$ . It is obvious that  $\|A\xi\| \leq \|A\|\|\xi\|$  for all  $\xi \in k^N$  and hence

$$(*) \quad \|AB\| \leq \|A\|\|B\| \quad \text{for all } A, B \in GL_N(k).$$

In particular, we have  $\|A^n\| \leq \|A\|^n$  for all  $A \in GL_N(k)$  and  $n \in \mathbf{N}$ .

For a matrix  $w \in GL_n(\ell)$ , denote by  $\sigma(w)$  the matrix in  $GL_n(k)$  obtained by applying  $\sigma$  to the entries of  $w$ . Set  $A_s = \sigma(\pi(s))$  for  $s \in S$  and  $A := \sigma(\pi(\gamma))$ . With

$$C := \max\{\|A_s\| : s \in S\},$$

it is clear that Inequality (\*) implies that

$$(**) \quad \|A^n\| = \|\sigma(\pi(\gamma^n))\| \leq C^{\ell_S(\gamma^n)} \quad \text{for all } n \in \mathbf{N}.$$

On the other hand,  $\sigma(\lambda)$  is an eigenvalue of  $A$ ; so, there exists  $\xi \in k^N \setminus \{0\}$  such that  $A\xi = \sigma(\lambda)\xi$  and hence  $A^n\xi = \sigma(\lambda)^n\xi$  for all  $n \in \mathbf{N}$ . So, for every  $n \in \mathbf{N}$ , we have

$$\|A^n\xi\| = |\sigma(\lambda)|^n \|\xi\|$$

and this implies that

$$\|A^n\| \geq |\sigma(\lambda)|^n.$$

In view of (\*\*), we obtain therefore

$$\frac{\ell_S(\gamma^n) \log C}{n} \geq \log |\sigma(\lambda)| \quad \text{for all } n \in \mathbf{N}.$$

Since  $|\sigma(\lambda)| > 1$ , this contradicts the fact that  $\liminf_{n \rightarrow \infty} \frac{\ell_S(\gamma^n)}{n} = 0$ .  $\square$

**3.2. Distorted elements in nilpotent groups.** Let  $\Gamma$  be a finitely generated nilpotent subgroup. For subsets  $A, B$  in  $\Gamma$ , we let  $[A, B]$  denote the subgroup of  $\Gamma$  generated by all commutators  $[a, b] = aba^{-1}b^{-1}$ , for  $a \in A$  and  $b \in B$ . Let

$$\Gamma^{(0)} \supset \Gamma^{(1)} \supset \dots \supset \Gamma^{(d-1)} \supset \Gamma^{(d)} = \{e\}$$

be the lower central series of  $\Gamma$ , defined inductively by  $\Gamma^{(0)} = \Gamma$  and  $\Gamma^{(k+1)} = [\Gamma^{(k)}, \Gamma]$ . The step of nilpotency of  $\Gamma$  is the smallest  $d \geq 1$  such that  $\Gamma^{(d-1)} \neq \{e\}$  and  $\Gamma^{(d)} = \{e\}$ .

**Proposition 15.** *Let  $\Gamma$  be a finitely generated nilpotent subgroup. Every  $\gamma \in \Gamma^{(1)} = [\Gamma, \Gamma]$  is distorted.*

*Proof.* Let  $S$  be a finite set of generators of  $\Gamma$  with  $S = S^{-1}$ . Let  $d \geq 1$  be the step of nilpotency of  $\Gamma$ . The case  $d = 1$  being trivial, we will assume that  $d \geq 2$ . We will show by induction on  $i \in \{1, \dots, d-1\}$  that every  $\gamma \in \Gamma^{(d-i)}$  is distorted.

- *First step.* Assume that  $i = 1$ . It is well-known that every element  $\gamma$  in  $\Gamma^{(d-1)}$  is distorted (see for instance [BGS85, (7.6), p. 91]); in fact, more precise estimates are available: for every  $\gamma \in \Gamma^{(d-1)}$ , we have  $\ell_S(\gamma^n) = O(n^{1/d})$  as  $n \rightarrow \infty$  (see [Tit81, 2.3 Lemme] or [DK18, Lemma 14.15]).

- *Second step.* Assume that, for every finitely generated nilpotent subgroup  $\Lambda$  of step  $d' \geq 2$ , every element  $\delta \in \Lambda^{(d'-i)}$  is distorted for  $i \in \{1, \dots, d'-2\}$ . Let  $\gamma \in \Gamma^{(d-i-1)}$  and fix  $\varepsilon > 0$ .

The quotient group  $\bar{\Gamma} = \Gamma/\Gamma^{(d-1)}$  is nilpotent of step  $d' = d-1$  and  $p(\gamma) \in \bar{\Gamma}^{(d'-i)}$ , where  $p : \Gamma \rightarrow \bar{\Gamma}$  is the quotient map. By induction hypothesis,  $p(\gamma)$  is distorted in  $\bar{\Gamma}$  with respect to the generating set  $\bar{S} := p(S)$ . So, we have  $\lim_{n \rightarrow \infty} \frac{\ell_{\bar{S}}(p(\gamma)^n)}{n} = 0$ ; hence, we can find an integer  $N \geq 1$  such that

$$(*) \quad \forall n \geq N, \exists \delta_n \in \Gamma^{(d-1)} : \frac{\ell_S(\gamma^n \delta_n)}{n} \leq \varepsilon.$$

By the first step, we have  $\lim_{k \rightarrow \infty} \frac{\ell_S(\delta_N^k)}{k} = 0$ , since  $\delta_N \in \Gamma^{(d-1)}$ ; so, there exists  $K \geq 1$  such that

$$(**) \quad \forall k \geq K : \frac{\ell_S(\delta_N^k)}{k} \leq \varepsilon.$$



Let  $k \geq K$ . We have

$$(***) \quad \frac{\ell_S(\gamma^{Nk})}{Nk} = \frac{\ell_S((\gamma^{Nk}\delta_N^k)(\delta_N^{-1})^k)}{Nk} \leq \frac{\ell_S(\gamma^{Nk}\delta_N^k)}{Nk} + \frac{\ell_S(\delta_N^k)}{Nk}.$$

Now, since  $\Gamma^{(d-1)}$  is contained in the center of  $\Gamma$ , the elements  $\delta_N$  and  $\gamma_N$  commute and hence, by (\*), we have

$$\frac{\ell_S(\gamma^{Nk}\delta_N^k)}{Nk} = \frac{\ell_S((\gamma^N\delta_N)^k)}{Nk} \leq k \frac{\ell_S(\gamma^N\delta_N)}{Nk} = \frac{\ell_S(\gamma^N\delta_N)}{N} \leq \varepsilon.$$

So, together with (\*\*\*) and (\*\*), we obtain

$$\forall k \geq K : \frac{\ell_S(\gamma^{Nk})}{Nk} \leq 2\varepsilon.$$

This shows that  $t(\gamma) = 0$ .  $\square$

**3.3. Congruence subgroups in unipotent groups.** The following result, which shows that the congruence subgroup problem has a positive solution for unipotent groups, is well-known (see the sketch in [Rag76, p.108]); for the convenience of the reader, we reproduce its short proof.

**Proposition 16.** *Let  $\mathbf{U}$  be a unipotent algebraic group over  $\mathbf{Q}$ . Let  $\Gamma$  be an arithmetic subgroup of  $\mathbf{U}(\mathbf{Q})$ . Then every finite index subgroup of  $\Gamma$  is a congruence subgroup.*

*Proof.* We can find a sequence

$$\mathbf{U} = \mathbf{U}_0 \supset \mathbf{U}_1 \supset \cdots \supset \mathbf{U}_{d-1} \supset \mathbf{U}_d = \{e\}$$

of normal  $\mathbf{Q}$ -subgroups of  $\mathbf{U}$  such that the factor groups  $\mathbf{U}_i/\mathbf{U}_{i+1}$  are  $\mathbf{Q}$ -isomorphic to  $\mathbf{G}_a$ , the additive group of dimension 1 (see [Bor63, (15.5)]).

We proceed by induction on  $d \geq 1$ . If  $d = 1$ , then  $\Gamma$  is commensurable to  $\mathbf{Z}$  and the claim is obvious true. Assume that  $d \geq 2$ . Then  $\mathbf{U}$  can be written as semi-direct product  $\mathbf{U} = \mathbf{U}_1 \rtimes \mathbf{G}_a$ . By [BHC62, Corollary 4.6],  $\Gamma$  is commensurable to  $\mathbf{U}_1(\mathbf{Z}) \rtimes \mathbf{Z}$ . Let  $H$  a subgroup of finite index in  $\Gamma$ . Then  $H \cap \mathbf{U}_1(\mathbf{Z})$  has finite index in  $\mathbf{U}_1(\mathbf{Z})$  and hence, by induction hypothesis, contains the kernel of the reduction  $\mathbf{U}_1(\mathbf{Z}) \rightarrow \mathbf{U}_1(\mathbf{Z}/N_1\mathbf{Z})$  modulo some  $N_1 \geq 1$ . Moreover,  $H \cap \mathbf{Z} = N_2\mathbf{Z}$  for some  $N_2 \geq 1$ . Hence,  $H$  contains the kernel of the reduction  $\mathbf{U}(\mathbf{Z}) \rightarrow \mathbf{U}(\mathbf{Z}/N_1N_2\mathbf{Z})$  modulo  $N_1N_2$ .  $\square$

**3.4. Proof of Theorem 1.** Let  $\Gamma$  be a finitely generated nilpotent group and  $\alpha : \Gamma \rightarrow \text{Prof}(\Gamma)$  the canonical homomorphism. Recall (see Proposition 11) that the Bohr compactification of  $\Gamma^{\text{Ab}} = \Gamma/[\Gamma, \Gamma]$  splits as a direct sum

$$\text{Bohr}(\Gamma^{\text{Ab}}) = \text{Bohr}(\Gamma^{\text{Ab}})_0 \oplus B_1,$$

for a closed subgroup  $B_1 \cong \text{Prof}(\Gamma^{\text{Ab}})$ . Let  $p : \text{Bohr}(\Gamma^{\text{Ab}}) \rightarrow \text{Bohr}(\Gamma^{\text{Ab}})_0$  be the corresponding projection. Denote by  $\beta_0 : \Gamma \rightarrow \text{Bohr}(\Gamma^{\text{Ab}})$  the map induced by the quotient homomorphism  $\Gamma \rightarrow \Gamma^{\text{Ab}}$ . Set

$$K := \text{Bohr}(\Gamma^{\text{Ab}})_0 \times \text{Prof}(\Gamma),$$

and let  $\beta : \Gamma \rightarrow K$  be the homomorphism  $\gamma \mapsto (p \circ \beta_0(\gamma), \alpha(\gamma))$ . We claim that  $(K, \beta)$  is a Bohr compactification for  $\Gamma$ .

- *First step.* We claim that  $\beta(\Gamma)$  is dense in  $K$ . Indeed, let  $L$  be the closure of  $\beta(\Gamma)$  in  $K$  and  $L_0$  its connected component. Since  $\text{Prof}(\Gamma)$  is totally disconnected, the projection of  $L_0$  on  $\text{Prof}(\Gamma)$  is trivial; hence  $L_0 = K_0 \times \{1\}$  for a connected closed subgroup  $K_0$  of  $\text{Bohr}(\Gamma^{\text{Ab}})_0$ . The projection of  $L$  on  $\text{Bohr}(\Gamma^{\text{Ab}})_0$  induces then a continuous homomorphism

$$f : L/L_0 \rightarrow \text{Bohr}(\Gamma^{\text{Ab}})_0/K_0.$$

Observe that  $f$  has dense image, since  $p \circ \beta_0(\Gamma)$  is dense in  $\text{Bohr}(\Gamma^{\text{Ab}})_0$ ; so,  $f$  is surjective by compactness of  $L/L_0$ . It follows, by compactness again, that  $\text{Bohr}(\Gamma^{\text{Ab}})_0/K_0$  is topologically isomorphic to a quotient of  $L/L_0$ . As  $L/L_0$  is totally disconnected, this implies (see [Bou71, Chap. 3, §4, Corollaire 3]) that  $\text{Bohr}(\Gamma^{\text{Ab}})_0/K_0$  is also totally disconnected and hence that  $K_0 = \text{Bohr}(\Gamma^{\text{Ab}})_0$ . So,  $\text{Bohr}(\Gamma^{\text{Ab}})_0 \times \{1\}$  is contained in  $L$ . It follows that  $L$  is the product of  $\text{Bohr}(\Gamma^{\text{Ab}})_0$  with a subgroup of  $\text{Prof}(\Gamma)$ . Since  $\alpha(\Gamma)$  is dense in  $\text{Prof}(\Gamma)$ , this subgroup coincides with  $\text{Prof}(\Gamma)$ , that is,  $L = K$  and the claim is proved.

- *Third step.* We claim that every irreducible unitary representation  $\pi : \Gamma \rightarrow U(N)$  of  $\Gamma$  is of the form  $\chi \otimes \rho$  for some  $\chi \in \widehat{\Gamma^{\text{Ab}}}$  and  $\rho \in \widehat{\Gamma}_{\text{finite}}$ .

Indeed, Propositions 14 and 15, imply that  $\pi([\Gamma, \Gamma])$  is a periodic subgroup of  $U(N)$ . Since  $\Gamma$  is finitely generated,  $[\Gamma, \Gamma]$  is finitely generated (in fact, every subgroup of  $\Gamma$  is finitely generated; see [Rag72, 2.7 Theorem]). Hence, by Schur's theorem (see [Weh73, 4.9 Corollary]),  $\pi([\Gamma, \Gamma])$  is finite. It follows that there exists a finite index normal subgroup  $H$  of  $[\Gamma, \Gamma]$  so that  $\pi|_H$  is the trivial representation of  $H$ .

Next, we claim that there exists a normal subgroup  $\Delta$  of finite index in  $\Gamma$  such that  $\Delta \cap [\Gamma, \Gamma] = H$ . Indeed, since  $\Gamma/[\Gamma, \Gamma]$  is abelian and finitely generated, we have  $\Gamma/[\Gamma, \Gamma] \cong \mathbf{Z}^k \oplus F$  for some finite subgroup  $F$  and some integer  $k \geq 0$ . Let  $\Gamma_1$  be the inverse image in  $\Gamma$  of the copy

of  $\mathbf{Z}^k$  in  $\Gamma/[\Gamma, \Gamma]$ . Then  $\Gamma_1$  is a normal subgroup of finite index of  $\Gamma$ . Moreover,  $\Gamma_1$  can be written as iterated semi-direct product

$$\Gamma_1 = (\dots (([\Gamma, \Gamma] \rtimes \mathbf{Z}) \rtimes \mathbf{Z}) \rtimes \mathbf{Z}).$$

Set

$$\Delta := (\dots ((H \rtimes \mathbf{Z}) \rtimes \mathbf{Z}) \rtimes \mathbf{Z}).$$

Then  $\Delta$  is a normal subgroup of finite index of  $\Gamma$  with  $\Delta \cap [\Gamma, \Gamma] = H$ .

Since  $\pi|_H$  is trivial on  $H$  and since  $[\Delta, \Delta] \subset H$ , the restriction  $\pi|_\Delta$  of  $\pi$  to  $\Delta$  factorizes through  $\Delta^{\text{Ab}}$ . So, by Proposition 12, there exists a finite  $\Gamma$ -orbit  $\mathcal{O}$  in  $\widehat{\Delta^{\text{Ab}}}$  such that we have a direct sum decomposition  $V = \bigoplus_{\chi \in \mathcal{O}} V^\chi$ , where  $V^\chi$  is the  $\chi$ -isotypical component of  $\pi|_\Delta$ .

Fix  $\chi \in \mathcal{O}$ . Since  $\chi$  is trivial on  $H$  and since  $\Delta \cap [\Gamma, \Gamma] = H$ , we can view  $\chi$  as a unitary character of the subgroup  $\Delta/(\Delta \cap [\Gamma, \Gamma])$  of  $\Gamma^{\text{Ab}}$ . Hence,  $\chi$  extends to a character  $\tilde{\chi} \in \widehat{\Gamma^{\text{Ab}}}$  (see, e.g. [HR79, (24.12)]). This implies that  $\Gamma_\chi = \Gamma$ ; indeed,

$$\chi^\gamma(\delta) = \tilde{\chi}(\gamma^{-1}\delta\gamma) = \tilde{\chi}(\delta) = \chi(\delta)$$

for every  $\gamma \in \Gamma$  and  $\delta \in \Delta$ . This shows that  $\mathcal{O}$  is a singleton and so  $V = V^\chi$ . We write

$$\pi = \tilde{\chi} \otimes (\overline{\tilde{\chi}} \otimes \pi).$$

Then  $\rho := \overline{\tilde{\chi}} \otimes \pi$  is an irreducible unitary representation of  $\Gamma$  which is trivial on  $\Delta$ ; so,  $\rho$  has finite image and  $\pi = \tilde{\chi} \otimes \rho$ .

• *Third step.* Let  $\pi \in \widehat{\Gamma}_{\text{fd}}$ . We claim that there exists a representation  $\pi' \in \widehat{K}$  such that  $\pi = \pi' \circ \beta$ . Once proved, Proposition 5 will imply that  $(K, \beta)$  is a Bohr compactification for  $\Gamma$ .

By the second step, we can write  $\pi = \chi \otimes \rho$  for some  $\chi \in \widehat{\Gamma^{\text{Ab}}}$  and  $\rho \in \widehat{\Gamma}_{\text{finite}}$ . On the one hand, we can write  $\rho = \rho' \circ \alpha$  for some  $\rho' \in \widehat{\text{Prof}(\Gamma)}$ , by the universal property of  $\widehat{\text{Prof}(\Gamma)}$ . On the other hand, we can decompose  $\chi$  as  $\chi = \chi_0 \chi_1$  with  $\chi_0 \in \widehat{\Gamma^{\text{Ab}}}$  of infinite order and  $\chi_1 \in \widehat{\Gamma^{\text{Ab}}}$  of finite order. We have  $\chi_0 = \chi'_0 \circ (p \circ \beta_0)$  and  $\chi_1 = \chi'_1 \circ \alpha$  for unitary characters  $\chi'_0$  of  $\text{Bohr}(\Gamma^{\text{Ab}})_0$  and  $\chi'_1$  of  $\text{Prof}(\Gamma^{\text{Ab}})$ . For  $\pi' = \chi_0 \otimes (\chi'_1 \otimes \rho')$ , we have  $\pi' \in \widehat{K}$  and  $\pi = \pi' \circ \beta$ .

#### 4. PROOF OF THEOREMS 2

Let  $\mathbf{G} = \mathbf{U} \rtimes \mathbf{H}$  be a Levi decomposition of  $\mathbf{G}$  and set

$$\Lambda = \mathbf{H}(\mathbf{Z}), \quad \Delta = \mathbf{U}(\mathbf{Z}), \quad \text{and} \quad \Gamma = \Delta \rtimes \Lambda.$$

Denote by  $\beta_\Delta : \Delta \rightarrow \text{Bohr}(\Delta)$  and  $\beta_\Lambda : \Lambda \rightarrow \text{Bohr}(\Lambda)$  the natural homomorphisms. Observe that, by the universal property of  $\text{Bohr}(\Delta)$ , every element  $\lambda \in \Lambda$  defines a continuous automorphism  $\theta_b(\lambda)$  of  $\text{Bohr}(\Delta)$

such that

$$\theta_b(\lambda)(\delta) = \beta_\Delta(\lambda\delta\lambda^{-1}) \quad \text{for all } \delta \in \Delta.$$

The corresponding homomorphism  $\theta_b : \Lambda \rightarrow \text{Aut}(\text{Bohr}(\Delta))$  defines an action of  $\Lambda$  on  $\text{Bohr}(\Delta)$ . By Theorem 1, we have

$$\text{Bohr}(\Delta) = \text{Bohr}(\Delta^{\text{Ab}})_0 \times \text{Prof}(\Delta).$$

The group  $\Lambda$  acts naturally on  $\Delta^{\text{Ab}}$  and, by duality, on  $\widehat{\Delta^{\text{Ab}}}$ . Let

$$H := \widehat{\Delta^{\text{Ab}}}_{\Lambda\text{-fin}} \subset \widehat{\Delta^{\text{Ab}}}$$

be the subgroup of characters of  $\Delta^{\text{Ab}}$  with finite  $\Lambda$ -orbits. Observe that  $H$  contains the torsion subgroup of  $\widehat{\Delta^{\text{Ab}}}$ .

Let

$$\alpha : \Lambda \rightarrow \text{Aut}(H)$$

be the homomorphism given by the action of  $\Lambda$  on  $H$ .

For a locally compact group  $G$ , the group  $\text{Aut}(G)$  of continuous automorphisms of  $G$  will be endowed with the compact-open topology for which it is also a (not necessarily locally compact) topological group (see [HR79, (26.3)]).

• *First step.* We claim that the closure of  $\alpha(\Lambda)$  in  $\text{Aut}(H)$  is compact. Indeed, let us identify  $\text{Aut}(H)$  with a subset of the product space  $H^H$ . The topology of  $\text{Aut}(H)$  coincides with the topology induced by the product topology on  $H^H$ . Viewed this way,  $\alpha(\Lambda)$  is a subspace of the product  $\prod_{\chi \in H} \chi^\Lambda$  of the finite  $\Lambda$ -orbits  $\chi^\Lambda$ . Since  $\prod_{\chi \in H} \chi^\Lambda$  is compact and hence closed, the claim is proved.

Next, let  $N$  be the annihilator of  $H$  in  $\text{Bohr}(\Delta^{\text{Ab}})$ . Then  $N$  is  $\Lambda$ -invariant and the induced action of  $\Lambda$  on  $\text{Bohr}(\Delta^{\text{Ab}})/N$  is a quotient of the action given by  $\theta_b$ .

Let  $C$  be the connected component of  $\text{Bohr}(\Delta^{\text{Ab}})/N$ . Then  $C$  coincides with the image of  $\text{Bohr}(\Delta^{\text{Ab}})_0$  in  $\text{Bohr}(\Delta^{\text{Ab}})/N$  (see [Bou71, Chap. 3, §4, Corollaire 3]) and so

$$C \cong \text{Bohr}(\Delta^{\text{Ab}})_0 / (N \cap \text{Bohr}(\Delta^{\text{Ab}})_0).$$

Since  $C$  is invariant under  $\Lambda$ , we obtain an action of  $\Lambda$  on  $C$ ; let

$$\widehat{\alpha} : \Lambda \rightarrow \text{Aut}(C)$$

be the corresponding homomorphism.

• *Second step.* We claim that the action  $\widehat{\alpha}$  of  $\Lambda$  on  $C$  extends to an action of  $\text{Bohr}(\Lambda)$ ; more precisely, there exists a continuous homomorphism

$$\widehat{\alpha}' : \text{Bohr}(\Lambda) \rightarrow \text{Aut}(C)$$

such that the diagram

$$\begin{array}{ccc} & & \text{Bohr}(\Lambda) \\ & \nearrow \beta_\Lambda & \downarrow \widehat{\alpha}' \\ \Lambda & \xrightarrow{\widehat{\alpha}} & \text{Aut}(C) \end{array}$$

commutes. Indeed, by the first step, the closure  $K$  of  $\alpha(\Lambda)$  in  $\text{Aut}(H)$  is a compact group. Hence, by the universal property of  $\text{Bohr}(\Lambda)$ , there exists a continuous homomorphism

$$\alpha' : \text{Bohr}(\Lambda) \rightarrow K \subset \text{Aut}(H)$$

such that the diagram

$$\begin{array}{ccc} & & \text{Bohr}(\Lambda) \\ & \nearrow \beta_\Lambda & \downarrow \alpha' \\ \Lambda & \xrightarrow{\alpha} & \text{Aut}(H) \end{array}$$

commutes. Since  $\widehat{H} = \text{Bohr}(\Delta^{\text{Ab}})/N$ , we obtain by duality a continuous homomorphism  $\widehat{\alpha}' : \text{Bohr}(\Lambda) \rightarrow \text{Aut}(\text{Bohr}(\Delta^{\text{Ab}})/N)$ . The connected component  $C$  of  $\text{Bohr}(\Delta^{\text{Ab}})/N$  is invariant under  $\text{Bohr}(\Lambda)$ . This proves the existence of the map  $\widehat{\alpha}' : \text{Bohr}(\Lambda) \rightarrow \text{Aut}(C)$  with the claimed property.

Next, observe that, by the universal property of  $\text{Prof}(\Delta)$ , every element  $\lambda \in \Lambda$  defines a continuous automorphism  $\theta_p(\lambda)$  of  $\text{Prof}(\Delta)$  such that

$$\theta_p(\lambda)(\delta) = \beta_\Delta(\lambda\delta\lambda^{-1}) \quad \text{for all } \delta \in \Delta.$$

The corresponding homomorphism  $\theta_p : \Lambda \rightarrow \text{Aut}(\text{Prof}(\Delta))$  defines an action of  $\Lambda$  on  $\text{Prof}(\Delta)$ .

• *Third step.* We claim that the action  $\theta_p$  of  $\Lambda$  on  $\text{Prof}(\Delta)$  extends to an action of  $\text{Bohr}(\Lambda)$ ; more precisely, there exists a homomorphism  $\theta' : \text{Bohr}(\Lambda) \rightarrow \text{Aut}(\text{Prof}(\Delta))$  such that the diagram

$$\begin{array}{ccc} & & \text{Prof}(\Delta) \\ & \nearrow \beta_\Lambda & \downarrow \theta' \\ \Lambda & \xrightarrow{\theta_p} & \text{Aut}(\text{Prof}(\Delta)) \end{array}$$

commutes. Indeed, since  $\Delta$  is finitely generated and since its image in  $\text{Bohr}(\Delta)$  is dense, the profinite group  $\text{Bohr}(\Delta)$  is finitely generated (that is, there exists a finite subset of  $\text{Bohr}(\Delta)$  which generates a dense subgroup). This implies that  $\text{Aut}(\text{Bohr}(\Delta))$  is a profinite

group (see [RZ00, Corollary 4.4.4]) and so there exists a homomorphism  $\theta'_p : \text{Prof}(\Lambda) \rightarrow \text{Aut}(\text{Prof}(\Delta))$  such that  $\theta'_p \circ \alpha_\Lambda = \theta_p$ . We then lift  $\theta'_p$  to a homomorphism  $\theta' : \text{Bohr}(\Lambda) \rightarrow \text{Aut}(\text{Prof}(\Delta))$ .

We set

$$Q := \text{Bohr}(\Delta)/(N \cap \text{Bohr}(\Delta^{\text{Ab}})_0) = C \times \text{Prof}(\Delta);$$

we have an action of  $\Lambda$  on  $Q$  given by the homomorphism

$$\widehat{\alpha} \oplus \theta_p : \Lambda \rightarrow \text{Aut}(C) \times \text{Aut}(\text{Prof}(\Delta)) \subset \text{Aut}(Q)$$

and, by the second and third step, an action of  $\text{Bohr}(\Lambda)$  on  $Q$  given by

$$\widehat{\alpha}' \oplus \theta' : \text{Bohr}(\Lambda) \rightarrow \text{Aut}(C) \times \text{Aut}(\text{Prof}(\Delta))$$

such that the diagram

$$\begin{array}{ccc} & & \text{Bohr}(\Lambda) \\ & \nearrow \beta_\Lambda & \downarrow \widehat{\alpha}' \oplus \theta' \\ \Lambda & \xrightarrow{\widehat{\alpha} \oplus \theta_p} & \text{Aut}(C) \times \text{Aut}(\text{Prof}(\Delta)) \end{array}$$

commutes.

Let

$$B := (C \times \text{Prof}(\Delta)) \rtimes \text{Bohr}(\Lambda)$$

be the semi-direct product defined by  $\widehat{\alpha}' \oplus \theta'$ . Let

$$p : \text{Bohr}(\Delta) \rightarrow C = \text{Bohr}(\Delta^{\text{Ab}})_0/(N \cap \text{Bohr}(\Delta^{\text{Ab}})_0)$$

be the quotient epimorphism.

• *Fourth step.* We claim that  $B$ , together with the map  $\beta : \Gamma \rightarrow B$ , given by

$$\beta(\delta, \lambda) = (p(\beta_\Delta(\delta)), \beta_\Lambda(\lambda)) \quad \text{for all } (\delta, \lambda) \in \Gamma,$$

is a Bohr compactification for  $\Gamma = \Delta \rtimes \Lambda$ .

First, we have to check that  $\beta$  is a homomorphism with dense image. Since  $p \circ \beta_\Delta$  and  $\beta_\Lambda$  are homomorphisms with dense image, it suffices to show that

$$\beta(\lambda\delta\lambda^{-1}, e) = ((\widehat{\alpha}' \oplus \theta')(\beta_\Lambda(\lambda))(p(\beta_\Delta(\delta)), e) \quad \text{for all } (\delta, \lambda) \in \Gamma.$$

This is indeed the case: since  $p$  is equivariant for the  $\Lambda$ -actions, we have

$$p(\beta_\Delta(\lambda\delta\lambda^{-1})) = p(\theta_b(\lambda)\beta_\Delta(\delta)) = (\widehat{\alpha}' \oplus \theta')(\beta_\Lambda(\lambda))p(\beta_\Delta(\delta)).$$

Next, let  $\pi$  be a unitary representation of  $\Gamma$  on a finite dimensional vector space  $V$ . By Proposition 5, we have to show that there exists a unitary representation  $\widetilde{\pi}$  of  $B$  on  $V$  such that  $\pi = \widetilde{\pi} \circ \beta$ .

Consider a decomposition of  $V = V_1 \oplus \cdots \oplus V_s$  into irreducible  $\pi(\Delta)$ -invariant subspaces  $V_i$ ; denote by  $\sigma_1, \dots, \sigma_s$  the corresponding irreducible representations of  $\Delta$ . By Theorem 1, every  $\sigma_i$  is of the form  $\sigma_i = \chi_i \otimes \rho_i$  for some  $\chi_i \in \widehat{\Delta^{\text{Ab}}}$  and  $\rho_i \in \widehat{\Delta}_{\text{finite}}$ .

We decompose every  $\chi_i$  as a product  $\chi_i = \chi'_i \chi''_i$  with  $\chi'_i \in \widehat{\Delta^{\text{Ab}}}$  of finite order and  $\chi''_i \in \widehat{\Delta^{\text{Ab}}}$  of infinite order. Since  $\chi'_i$  has finite image, upon replacing  $\rho_i$  by  $\chi'_i \otimes \rho_i$ , we may and will assume that every non trivial  $\chi_i$  has infinite order.

Fix  $i \in \{1, \dots, s\}$ . We can extend  $\chi_i$  and  $\rho_i$  to unitary representations of  $\text{Bohr}(\Delta)$ , that is, we can find representations  $\tilde{\chi}_i$  and  $\tilde{\rho}_i$  of  $\text{Bohr}(\Delta)$  on  $V_i$  such that  $\chi_i = \tilde{\chi}_i \circ \beta_\Delta$  and  $\rho_i = \tilde{\rho}_i \circ \beta_\Delta$ . By Proposition 12, the stabilizer  $\Gamma_{\sigma_i}$  of  $\sigma_i$  has finite index in  $\Gamma$ . It follows that the  $\Lambda$ -orbit of  $\sigma_i$  is finite, and this implies that  $\chi_i \in H$ ; hence,  $\tilde{\chi}_i$  factorizes through

$$C = \text{Bohr}(\Delta^{\text{Ab}})_0 / (N \cap \text{Bohr}(\Delta^{\text{Ab}})_0)$$

and we have  $\chi_i = \tilde{\chi}_i \circ (p \circ \beta_\Delta)$ . Since  $\rho_i$  has finite image,  $\tilde{\rho}_i$  factorizes through  $\text{Prof}(\Delta)$ . So,  $\tilde{\sigma}_i := \tilde{\chi}_i \otimes \tilde{\rho}_i$  is a unitary representation of  $C \times \text{Prof}(\Delta)$  on  $V_i$ . Set

$$\tilde{\pi}_\Delta := \bigoplus_{i=1}^s \tilde{\sigma}_i.$$

Then  $\tilde{\pi}_\Delta$  is a unitary representation of  $C \times \text{Prof}(\Delta)$  on  $V$  such that  $\pi|_\Delta = \tilde{\pi}_\Delta \circ (\beta|_\Delta)$ .

On the other hand, since  $\pi|_\Lambda$  is a finite dimensional representation of  $\Lambda$ , we can find a representation  $\tilde{\pi}_\Lambda$  of  $\text{Bohr}(\Lambda)$  on  $V$  such that  $\pi|_\Lambda = \tilde{\pi}_\Lambda \circ (\beta|_\Lambda)$ . For  $\lambda \in \Lambda$  and  $\delta \in \Delta$ , we have

$$\begin{aligned} \tilde{\pi}_\Delta(\beta(\lambda)\beta(\delta)\beta(\lambda)^{-1}) &= \tilde{\pi}_\Delta(\beta(\lambda\delta\lambda)^{-1}) \\ &= \pi(\lambda\delta\lambda)^{-1} \\ &= \pi(\lambda)\pi(\delta)\pi(\lambda)^{-1} \\ &= \tilde{\pi}_\Lambda(\beta(\lambda))\tilde{\pi}_\Delta(\beta(\delta))\tilde{\pi}_\Lambda(\beta(\lambda))^{-1}. \end{aligned}$$

Since  $\beta$  has dense image in  $B$ , it follows that

$$\tilde{\pi}_\Delta(bab^{-1}) = \tilde{\pi}_\Lambda(b)\tilde{\pi}_\Delta(a)\tilde{\pi}_\Lambda(b)^{-1} \quad \text{for all } (a, b) \in B$$

and therefore the formula

$$\tilde{\pi}(a, b) = \tilde{\pi}_\Delta(a)\tilde{\pi}_\Lambda(b) \quad \text{for all } (a, b) \in B$$

defines a unitary representation of  $B$  on  $V$  such that  $\pi = \tilde{\pi} \circ \beta$ .

## 5. PROOF OF THEOREM 3

Recall that we are assuming that  $\mathbf{G}$  is a connected, simply-connected and almost  $\mathbf{Q}$ -simple algebraic group. The group  $\mathbf{G}$  can be obtained from an absolutely simple algebraic group  $\mathbf{H}$  by the so-called restriction of scalars; more precisely (see [BT65, 6.21, (ii)]), there exists a number field  $K$  and an absolutely simple algebraic group  $\mathbf{H}$  over  $K$  which is absolutely simple with the following property:  $\mathbf{G}$  can be written as (more precisely, is  $\mathbf{Q}$ -isomorphic to) the  $\mathbf{Q}$ -group  $\mathbf{H}^{\sigma_1} \times \cdots \times \mathbf{H}^{\sigma_s}$ , where the  $\sigma_i$ 's are the different (non conjugate) embeddings of  $K$  in  $\mathbf{C}$ . Assuming that  $\sigma_1, \dots, \sigma_{r_1}$  are the embeddings such that  $\sigma_i(K) \subset \mathbf{R}$ , we can identify  $\mathbf{G}(\mathbf{R})$  with

$$\mathbf{H}^{\sigma_1}(\mathbf{R}) \times \cdots \times \mathbf{H}^{\sigma_{r_1}}(\mathbf{R}) \times \mathbf{H}^{\sigma_{r_1+1}}(\mathbf{C}) \times \cdots \times \mathbf{H}^{\sigma_{r_s}}(\mathbf{C}).$$

Let  $\mathbf{G}_c$  be the product of the  $\mathbf{H}^{\sigma_i}$ 's for which  $\mathbf{H}^{\sigma_i}(\mathbf{R})$  is compact.

We assume now that the real semisimple Lie group  $\mathbf{G}(\mathbf{R})$  is not locally isomorphic to a group of the form  $SO(m, 1) \times L$  or  $SU(m, 1) \times L$  for a compact Lie group  $L$ . Let  $\Gamma \subset \mathbf{G}(\mathbf{Q})$  be an arithmetic subgroup.

Set  $K := \mathbf{G}_c(\mathbf{R}) \times \text{Prof}(\Gamma)$  and let  $\beta : \Gamma \rightarrow K$  be defined by  $\beta(\gamma) = (p(\gamma), \alpha(\gamma))$ , where  $p : \mathbf{G}(\mathbf{R}) \rightarrow \mathbf{G}_c(\mathbf{R})$  is the canonical projection and  $\alpha : \Gamma \rightarrow \text{Prof}(\Gamma)$  the map associated to  $\text{Prof}(\Gamma)$ . We claim that  $(K, \beta)$  is a Bohr compactification of  $\Gamma$ ,

First, we show that  $\beta(\Gamma)$  has dense image. Observe that  $\mathbf{G}_c(\mathbf{R})$  is connected (see [Bor91, (24.6.c)]). By the Strong Approximation Theorem (see [PR94, Theorem 7.12]),  $p(\mathbf{G}(\mathbf{Z}))$  is dense in  $\mathbf{G}_c(\mathbf{R})$ . Since  $\mathbf{G}_c(\mathbf{R})$  is connected and since  $\Gamma$  is commensurable to  $\mathbf{G}(\mathbf{Z})$ , it follows that  $p(\Gamma)$  is dense in  $\mathbf{G}_c(\mathbf{R})$ . Now,  $\alpha(\Gamma)$  is dense in  $\text{Prof}(\Gamma)$  and  $\text{Prof}(\Gamma)$  is totally disconnected. As in the first step of the proof of Theorem 1, we conclude that  $\beta(\Gamma)$  is dense in  $K$ .

Let  $\pi : \Gamma \rightarrow U(n)$  be a finite dimensional unitary representation of  $\Gamma$ . Then, by Margulis' superrigidity theorem (see [Mar91, Chap. VIII, Theorem B]), [Mor15, Corollary 16.4.1]), there exists a continuous homomorphism  $\rho_1 : \mathbf{G}(\mathbf{R}) \rightarrow U(n)$  and a homomorphism  $\rho_2 : \Gamma \rightarrow U(n)$  such that

- (i)  $\rho_2(\Gamma)$  is finite;
- (ii)  $\rho_1(g)\rho_2(\gamma) = \rho_2(\gamma)\rho_1(g)$  for all  $g \in \mathbf{G}(\mathbf{R})$  and  $\gamma \in \Gamma$ ;
- (iii)  $\pi(\gamma) = \rho_1(\gamma)\rho_2(\gamma)$  for all  $\gamma \in \Gamma$ .

By a classical result of Segal and von Neumann [SvN50],  $\rho_1$  factorizes through  $\mathbf{G}_c(\mathbf{R})$ , that is,  $\rho_1 = \rho'_1 \circ p$  for a unitary representation  $\rho'_1$  of  $\mathbf{G}_c(\mathbf{R})$ . It follows from (i) that  $\rho_2 = \rho'_2 \circ \alpha$  for a unitary representation  $\rho'_2$  of  $\text{Prof}(\Gamma)$ . Moreover, (ii) and (iii) show that  $\pi = (\rho_1|_{\Gamma}) \otimes \rho_2$ . Hence,



$\pi = (\rho'_1 \otimes \rho'_2) \circ \otimes \beta$ . We conclude by Proposition 5 that  $(K, \beta)$  is a Bohr compactification of  $\Gamma$ .

## 6. A FEW EXAMPLES

We compute the Bohr compactification for various examples of arithmetic groups.

- (1) For an integer  $n \geq 1$ , the  $(2n+1)$ -dimensional Heisenberg group is the unipotent  $\mathbf{Q}$ -group  $\mathbf{H}_{2n+1}$  of matrices of the form

$$m(x_1, \dots, x_n, y_1, \dots, y_n, z) := \begin{pmatrix} 1 & x_1 & \dots & x_n & z \\ 0 & 1 & \dots & 0 & y_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & y_n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

The arithmetic group  $\Gamma = \mathbf{H}_{2n+1}(\mathbf{Z})$  is nilpotent of step 2; its commutator subgroup  $[\Gamma, \Gamma]$  coincides with its center  $\{m(0, 0, z) : z \in \mathbf{Z}\}$ . So,  $\Gamma^{\text{Ab}} \cong \mathbf{Z}^{2n}$ . We have, by Theorem 1,

$$\text{Bohr}(\Gamma) \cong \text{Bohr}(\mathbf{Z}^{2n})_0 \times \text{Prof}(\Gamma)$$

and hence, by Proposition 11 and Proposition 16

$$\text{Bohr}(\Gamma) \cong \left( \prod_{\omega \in \mathfrak{c}} \mathbf{A}/\mathbf{Q} \right) \times \prod_{p \text{ prime}} \mathbf{H}_{2n+1}(\mathbf{Z}_p).$$

- (2) Let  $\mathbf{G} = SL_n$  for  $n \geq 3$  or  $\mathbf{G} = Sp_{2n}$  for  $n \geq 2$ . Then  $SL_n(\mathbf{Z})$  and  $Sp_{2n}(\mathbf{Z})$  are non cocompact arithmetic lattices in  $SL_n(\mathbf{R})$  and  $Sp_{2n}(\mathbf{R})$ , respectively. Hence, we have, by Corollary 4,  $\text{Bohr}(SL_n(\mathbf{Z})) = \text{Prof}(SL_n(\mathbf{Z}))$  and  $\text{Bohr}(Sp_{2n}(\mathbf{Z})) = \text{Prof}(Sp_{2n}(\mathbf{Z}))$ . Since  $SL_n(\mathbf{Z})$  and  $Sp_{2n}(\mathbf{Z})$  have the congruence subgroup property, it follows that

$$\text{Bohr}(SL_n(\mathbf{Z})) \cong \prod_{p \text{ prime}} SL_n(\mathbf{Z}_p) \cong SL_n(\text{Prof}(\mathbf{Z}))$$

and similarly

$$\text{Bohr}(Sp_{2n}(\mathbf{Z})) \cong \prod_{p \text{ prime}} Sp_{2n}(\mathbf{Z}_p) \cong Sp_{2n}(\text{Prof}(\mathbf{Z})).$$

- (3) The group  $\Gamma = SL_2(\mathbf{Z}[\sqrt{2}])$  embeds as a non cocompact arithmetic lattice of  $SL_2(\mathbf{R}) \times SL_2(\mathbf{R})$ . So, by Corollary 4, we have

$$\text{Bohr}(SL_2(\mathbf{Z}[\sqrt{2}])) \cong \text{Prof}(SL_2(\mathbf{Z}[\sqrt{2}])).$$

Moreover, since  $\Gamma$  has the congruence subgroup property (see [Ser70, Corollaire 3]), it follows that

$$\text{Bohr}(SL_2(\mathbf{Z}[\sqrt{2}])) \cong \text{Cong}(SL_2(\mathbf{Z}[\sqrt{2}])).$$

- (4) For  $n \geq 4$ , consider the quadratic form

$$q(x_1, \dots, x_n) = x_1^2 + \dots + x_{n-1}^2 - \sqrt{2}x_n^2 - \sqrt{2}x_{n+1}^2$$

The group  $\mathbf{G} = SO(q)$  of unimodular  $(n+1) \times (n+1)$ -matrices which preserve  $q$  is an almost simple algebraic group over the number field  $\mathbf{Q}[\sqrt{2}]$ . The subgroup  $\Gamma = SO(q, \mathbf{Z}[\sqrt{2}])$  of  $\mathbf{Z}[\sqrt{2}]$ -rational points in  $\mathbf{G}$  embeds as a cocompact lattice of the semisimple real Lie group  $SO(n+1) \times SO(n-1, 2)$  via the map

$$SO(q, \mathbf{Q}[\sqrt{2}]) \rightarrow SO(n+1) \times SO(n-1, 2), \gamma \mapsto (\gamma^\sigma, \gamma),$$

where  $\sigma$  is the field automorphism of  $\mathbf{Q}[\sqrt{2}]$  given by  $\sigma(\sqrt{2}) = -\sqrt{2}$ ; so,  $SO(n+1) \times SO(n-1, 2)$  is the group of real points of the  $\mathbf{Q}$ -group  $R_{\mathbf{Q}[\sqrt{2}]/\mathbf{Q}}(\mathbf{G})$  obtained by restriction of scalars from the  $\mathbf{Q}[\sqrt{2}]$ -group  $\mathbf{G}$ . Observe that  $R_{\mathbf{Q}[\sqrt{2}]/\mathbf{Q}}(\mathbf{G})$  is almost  $\mathbf{Q}[\sqrt{2}]$ -simple since  $\mathbf{G}$  is almost  $\mathbf{Q}$ -simple. By Theorem 3, we have

$$\text{Bohr}(SO(q, \mathbf{Z}[\sqrt{2}])) \cong SO(n+1) \times \text{Prof}(SO(q, \mathbf{Z}[\sqrt{2}])).$$

- (5) For  $d \geq 2$ , let  $D$  be a central division algebra over  $\mathbf{Q}$  such that  $D \otimes_{\mathbf{Q}} \mathbf{R}$  is isomorphic to the algebra  $M_d(\mathbf{R})$  of real  $d \times d$ -matrices. There exists a subring  $\mathcal{O}$  of  $D$  which is a  $\mathbf{Z}$ -lattice in  $D$  (a so-called order in  $D$ ). There is an embedding  $\varphi : D \rightarrow M_d(\mathbf{R})$  such that  $\varphi(SL_1(D)) \subset SL_d(\mathbf{Q})$  and such that  $\Gamma := \varphi(SL_1(\mathcal{O}))$  is an arithmetic cocompact lattice in  $SL_d(\mathbf{R})$ , where  $SL_1(D)$  is the group of norm one elements in  $D$ ; for all this, see [Mor15, §6.8.i]. For  $d \geq 2$ , we have

$$\text{Bohr}(\Gamma) \cong \text{Prof}(\Gamma).$$

So, this is an example of a *cocompact* lattice  $\Gamma$  in a simple real Lie group for which there exists no homomorphism  $\Gamma \rightarrow U(n)$  with infinite image; the existence of such examples was mentioned in [Mor15, (16.4.3)]

- (6) For  $n \geq 3$ , let  $\Gamma$  be the semi-direct product  $\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})$ , induced by the usual linear action of  $SL_n(\mathbf{Z})$  on  $\mathbf{R}^n$ . The dual action of  $SL_n(\mathbf{Z})$  on  $\widehat{\mathbf{Z}^n} \cong \mathbf{R}^n/\mathbf{Z}^n$  is given by

$$SL_n(\mathbf{Z}) \times \mathbf{R}^n/\mathbf{Z}^n \rightarrow \mathbf{R}^n/\mathbf{Z}^n, (g, x + \mathbf{Z}^n) \mapsto {}^t gx + \mathbf{Z}^n.$$

It is well-known and easy to show that the subgroup of  $SL_n(\mathbf{Z})$ -periodic orbits in  $\widehat{\mathbf{Z}}^n$  corresponds to  $\mathbf{Q}^n/\mathbf{Z}^n$ , that is, to the characters of finite image. It follows from Theorem 2 that

$$\text{Bohr}(\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})) \cong \text{Bohr}(SL_n(\mathbf{Z}))_0 \times \text{Prof}(\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})).$$

For  $n \geq 3$ , we have therefore

$$\text{Bohr}(\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})) \cong \text{Prof}(\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})) \cong \prod_{p \text{ prime}} \mathbf{Z}_p \rtimes SL_n(\mathbf{Z}_p).$$

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