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► **To cite this version:**

Bachir Bekka. THE BOHR COMPACTIFICATION OF AN ARITHMETIC GROUP. Transactions of the American Mathematical Society, 2025, pp.3755-3777. <10.1090/tran/9422>. <hal-04073352>

HAL Id: hal-04073352

<https://hal.science/hal-04073352v1>

Submitted on 18 Apr 2023

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THE BOHR COMPACTIFICATION OF AN ARITHMETIC GROUP

BACHIR BEKKA

ABSTRACT. Given a group Γ , its Bohr compactification $\text{Bohr}(\Gamma)$ and its profinite completion $\text{Prof}(\Gamma)$ are compact groups naturally associated to Γ ; moreover, $\text{Prof}(\Gamma)$ can be identified with the quotient of $\text{Bohr}(\Gamma)$ by its connected component $\text{Bohr}(\Gamma)_0$. We study the structure of $\text{Bohr}(\Gamma)$ for an arithmetic subgroup Γ of an algebraic group \mathbf{G} over \mathbf{Q} . When \mathbf{G} is unipotent, we show that $\text{Bohr}(\Gamma)$ can be identified with the direct product $\text{Bohr}(\Gamma^{\text{Ab}})_0 \times \text{Prof}(\Gamma)$, where $\Gamma^{\text{Ab}} = \Gamma/[\Gamma, \Gamma]$ is the abelianization of Γ . In the general case, using a Levi decomposition $\mathbf{G} = \mathbf{U} \rtimes \mathbf{H}$ (where \mathbf{U} is unipotent and \mathbf{H} is reductive), we show that $\text{Bohr}(\Gamma)$ can be described as the semi-direct product of a certain quotient of $\text{Bohr}(\Gamma \cap \mathbf{U})$ with $\text{Bohr}(\Gamma \cap \mathbf{H})$. When \mathbf{G} is simple and has higher \mathbf{R} -rank, $\text{Bohr}(\Gamma)$ is isomorphic, up to a finite group, to the product $K \times \text{Prof}(\Gamma)$, where K is the maximal compact factor of $\mathbf{G}(\mathbf{R})$.

1. INTRODUCTION

Given a topological group G , the **Bohr compactification** of G is a pair $(\text{Bohr}(G), \beta)$ consisting of a compact (Hausdorff) group $\text{Bohr}(G)$ and a continuous homomorphism $\beta : G \rightarrow \text{Bohr}(G)$ with dense image, satisfying the following universal property: for every compact group K and every continuous homomorphism $\alpha : G \rightarrow K$, there exists a continuous homomorphism $\alpha' : \text{Bohr}(G) \rightarrow K$ such that the diagram

$$\begin{array}{ccc}
 & & \text{Bohr}(G) \\
 & \nearrow \beta & \downarrow \alpha' \\
 G & \xrightarrow{\alpha} & K
 \end{array}$$

commutes. The pair $(\text{Bohr}(G), \beta)$ is unique in the following sense: if (K', β') is a pair consisting of a compact group K' and a continuous homomorphism $\beta' : G \rightarrow K'$ with dense image satisfying the same universal property (such a pair will be called a Bohr compactification of

2000 *Mathematics Subject Classification.* 22D10; 22C05; 20E18.

The author acknowledges the support by the ANR (French Agence Nationale de la Recherche) through the project Labex Lebesgue (ANR-11-LABX-0020-01).

G), then there exists an isomorphism $\alpha : \text{Bohr}(G) \rightarrow K'$ of topological groups such that $\beta' = \alpha \circ \beta$.

The compact group $\text{Bohr}(G)$ was first introduced by A. Weil ([Wei40, Chap.VII]) as a tool for the study of almost periodic functions on G , a subject initiated by H. Bohr ([Boh25a], [Boh25b]) in the case $G = \mathbf{R}$ and generalized to other groups by J. von Neumann ([vN34]) among others. For more on this subject, see [Dix77, §16] or [BH, 4.C]).

The group $\text{Bohr}(\Gamma)$ has been determined for only very few non abelian *discrete* groups Γ (for some general results, see [HK01] and [Hol64]; for the well-known case of abelian groups, see [AK43] and Section 11).

In contrast, there is a second much more studied completion of Γ , namely the **profinite completion** of Γ , which is a pair $(\text{Prof}(\Gamma), \alpha)$ consisting of a profinite group (that is, a projective limit of finite groups) $\text{Prof}(\Gamma)$ satisfying a similar universal property with respect to such groups, together with a homomorphism with $\alpha : \Gamma \rightarrow \text{Prof}(\Gamma)$ with dense image. The group $\text{Prof}(\Gamma)$ can be realized as the projective limit $\varprojlim \Gamma/H$, where H runs over the family of the normal subgroups of finite index of Γ . For all this, see [RZ00].

The universal property of $\text{Bohr}(\Gamma)$ gives rise to a continuous epimorphism $\alpha' : \text{Bohr}(\Gamma) \rightarrow \text{Prof}(\Gamma)$. It is easy to see (see Proposition 7 below) that the kernel of α' is $\text{Bohr}(\Gamma)_0$, the connected component of $\text{Bohr}(\Gamma)$; so, we have a short exact sequence

$$1 \longrightarrow \text{Bohr}(\Gamma)_0 \longrightarrow \text{Bohr}(\Gamma) \longrightarrow \text{Prof}(\Gamma) \longrightarrow 1.$$

In this paper, we will deal with the case where Γ is an arithmetic subgroup in a linear algebraic group. The setting is as follows. Let \mathbf{G} be a connected linear algebraic group over \mathbf{Q} with a fixed faithful representation $\rho : \mathbf{G} \rightarrow GL_m$. We consider the subgroup $\mathbf{G}(\mathbf{Z})$ of the group $\mathbf{G}(\mathbf{Q})$ of \mathbf{Q} -points of \mathbf{G} , that is,

$$\mathbf{G}(\mathbf{Z}) = \rho^{-1}(\rho(\mathbf{G}) \cap GL_m(\mathbf{Z})).$$

A subgroup Γ of $\mathbf{G}(\mathbf{Q})$ is called an **arithmetic subgroup** if Γ is commensurable to $\mathbf{G}(\mathbf{Z})$, that is, $\Gamma \cap \mathbf{G}(\mathbf{Z})$ has finite index in both Γ and $\mathbf{G}(\mathbf{Z})$. Observe that Γ is a discrete subgroup of the real Lie group $\mathbf{G}(\mathbf{R})$.

We first deal with the case where \mathbf{G} is unipotent. More generally, we describe the Bohr compactification of any finitely generated nilpotent group. Observe that an arithmetic subgroup in a unipotent algebraic \mathbf{Q} -group is finitely generated (see Corollary 2 of Theorem 2.10 in [Rag72]).

For two topological groups H and L , we write $H \cong L$ if H and L are topologically isomorphic. We observe that, when Δ is a finitely

generated abelian group, $\text{Bohr}(\Delta)$ splits as a direct sum $\text{Bohr}(\Delta) = \text{Bohr}(\Delta)_0 \oplus \text{Prof}(\Delta)$; see Proposition 11.

Theorem 1. *Let Γ be a finitely generated nilpotent group. We have a direct product decomposition*

$$\text{Bohr}(\Gamma) \cong \text{Bohr}(\Gamma^{\text{Ab}})_0 \times \text{Prof}(\Gamma),$$

where $\Gamma^{\text{Ab}} = \Gamma/[\Gamma, \Gamma]$ is the abelianization of Γ . This isomorphism is induced by the natural maps $\Gamma \rightarrow \text{Bohr}(\Gamma^{\text{Ab}})$ and $\Gamma \rightarrow \text{Prof}(\Gamma)$, together with the projection $\text{Bohr}(\Gamma^{\text{Ab}}) \rightarrow \text{Bohr}(\Gamma^{\text{Ab}})_0$.

A crucial tool in the proof of Theorem 1 is the fact that elements in the commutator subgroup $[\Gamma, \Gamma]$ of a nilpotent group Γ are distorted (see Proposition 15).

We now turn to the case of a general algebraic group \mathbf{G} over \mathbf{Q} . Let \mathbf{U} be the unipotent radical of \mathbf{G} . Then \mathbf{U} is defined over \mathbf{Q} and there exists a connected reductive \mathbf{Q} -subgroup \mathbf{H} such that we have a **Levi decomposition** as semi-direct product $\mathbf{G} = \mathbf{U} \rtimes \mathbf{H}$ (see [Mos56]).

The group $\Lambda = \mathbf{H}(\mathbf{Z})$ acts by automorphisms on $\Delta = \mathbf{U}(\mathbf{Z})$ and hence on $\text{Bohr}(\Delta)$, by the universal property of $\text{Bohr}(\Delta)$. In general, this action does not extend to an action of $\text{Bohr}(\Lambda)$ on $\text{Bohr}(\Delta)$. However, as we will see below (proof of Theorem 2), $\text{Bohr}(\Lambda)$ acts naturally by automorphisms on an appropriate quotient of $\text{Bohr}(\Delta)$.

Observe that (see [BHC62, Corollary 4.6]) every arithmetic subgroup of $\mathbf{G}(\mathbf{Q})$ is commensurable to $\Delta(\mathbf{Z}) \rtimes \mathbf{H}(\mathbf{Z})$. Recall that two topological groups G_1 and G_2 are (abstractly) commensurable if there exist finite index subgroups H_1 and H_2 of G_1 and G_2 such that H_1 is topologically isomorphic to H_2 . If this is the case, then $\text{Bohr}(G_1)$ and $\text{Bohr}(G_2)$ are commensurable; in fact, each one of the groups $\text{Bohr}(G_1)$ or $\text{Bohr}(G_2)$ can be described in terms of the other (see Propositions 8 and 9). For this reason, we will often deal with only one chosen representative of the commensurability class of an arithmetic group.

Theorem 2. *Let \mathbf{G} be a connected linear algebraic group over \mathbf{Q} , with Levi decomposition $\mathbf{G} = \mathbf{U} \rtimes \mathbf{H}$. Set $\Lambda := \mathbf{H}(\mathbf{Z})$, $\Delta := \mathbf{U}(\mathbf{Z})$, and $\Gamma := \Delta \rtimes \Lambda$. Let $\widehat{\Delta^{\text{Ab}}}_{\Lambda\text{-fin}}$ be the subgroup of the dual group $\widehat{\Delta^{\text{Ab}}}$ of Δ^{Ab} consisting of the characters with finite Λ -orbit. We have a semi-direct decomposition*

$$\text{Bohr}(\Gamma) \cong (Q \times \text{Prof}(\Delta)) \rtimes \text{Bohr}(\Lambda),$$

where Q is the connected component of $\text{Bohr}(\Delta^{\text{Ab}})/N$ and N is the annihilator of $\widehat{\Delta^{\text{Ab}}}_{\Lambda\text{-fin}}$ in $\text{Bohr}(\Delta^{\text{Ab}})$. This isomorphism is induced by the natural homomorphisms $\Delta \rightarrow \text{Bohr}(\Delta^{\text{Ab}})/N$ and $\Lambda \rightarrow \text{Bohr}(\Lambda)$.

Theorems 1 and 2 reduce the determination of $\text{Bohr}(\Gamma)$ for an arithmetic group Γ in \mathbf{G} to the case where \mathbf{G} is reductive. We have a further reduction to the case where \mathbf{G} is simply connected and almost simple. Indeed, recall that a group L is the **almost direct product** of subgroups L_1, \dots, L_n if the product map $L_1 \times \dots \times L_n \rightarrow L$ is a surjective homomorphism with finite kernel.

Let \mathbf{G} be a connected reductive algebraic group over \mathbf{Q} . The commutator subgroup $\mathbf{L} := [\mathbf{G}, \mathbf{G}]$ of \mathbf{G} is a connected semi-simple \mathbf{Q} -group and \mathbf{G} is an almost direct product $\mathbf{G} = \mathbf{T}\mathbf{L}$ for a central \mathbf{Q} -torus \mathbf{T} (see (14.2) and (18.2) in [Bor91]) Moreover, \mathbf{L} is an almost direct product $\mathbf{L} = \mathbf{L}_1 \cdots \mathbf{L}_n$ of connected almost \mathbf{Q} -simple \mathbf{Q} -subgroups \mathbf{L}_i , called the almost \mathbf{Q} -simple factors of \mathbf{L} (see [Bor91, (22.10)]). For every $i \in \{1, \dots, n\}$, let $\widetilde{\mathbf{L}}_i$ be the simply connected covering group \mathbf{L}_i . Set $\widetilde{\mathbf{G}} = \mathbf{T} \times \widetilde{\mathbf{L}}_1 \times \dots \times \widetilde{\mathbf{L}}_n$. Let $\widetilde{\Gamma}$ be the arithmetic subgroup $\mathbf{T}(\mathbf{Z}) \times \widetilde{\mathbf{L}}_1(\mathbf{Z}) \times \dots \times \widetilde{\mathbf{L}}_n(\mathbf{Z})$ in $\widetilde{\mathbf{G}}(\mathbf{Q})$. The image Γ of $\widetilde{\Gamma}$ under the isogeny $p : \widetilde{\mathbf{G}} \rightarrow \mathbf{G}$ is an arithmetic subgroup of $\mathbf{G}(\mathbf{Q})$ (see Corollaries 6.4 and 6.11 in [BHC62]). The map $p : \widetilde{\Gamma} \rightarrow \Gamma$ induces an isomorphism $\text{Bohr}(\Gamma) \cong \text{Bohr}(\widetilde{\Gamma})/F$, where F is the finite normal subgroup $F = \widetilde{\beta}(\ker p)$ and $\widetilde{\beta} : \widetilde{\Gamma} \rightarrow \text{Bohr}(\widetilde{\Gamma})$ is the natural map (see Proposition 10).

As an easy consequence of Margulis' superrigidity results, we give a description of the Bohr compactification of an arithmetic lattice in a simple algebraic \mathbf{Q} -group \mathbf{G} under a higher rank assumption. Such a description does not seem possible for arbitrary \mathbf{G} . For instance, the free non abelian group F_2 on two generators is an arithmetic lattice in $SL_2(\mathbf{Q})$, but we know of no simple description of $\text{Bohr}(F_2)$.

Theorem 3. *Let \mathbf{G} be a connected, simply connected, and almost simple \mathbf{Q} -group. Assume that the real semisimple Lie group $\mathbf{G}(\mathbf{R})$ is not locally isomorphic to any group of the form $SO(m, 1) \times K$ or $SU(m, 1) \times K$ for a compact Lie group K . Let \mathbf{G}_{nc} be the product of the almost \mathbf{R} -simple factors \mathbf{G}_i of \mathbf{G} for which $\mathbf{G}_i(\mathbf{R})$ is non compact. Let $\Gamma \subset \mathbf{G}(\mathbf{Q})$ be an arithmetic subgroup. We have a direct product decomposition*

$$\text{Bohr}(\Gamma) \cong \text{Bohr}(\Gamma)_0 \times \text{Prof}(\Gamma)$$

and an isomorphism

$$\text{Bohr}(\Gamma)_0 \cong \mathbf{G}(\mathbf{R})/\mathbf{G}_{\text{nc}}(\mathbf{R}),$$

induced by the natural maps $\Gamma \rightarrow \mathbf{G}(\mathbf{R})/\mathbf{G}_{\text{nc}}(\mathbf{R})$ and $\Gamma \rightarrow \text{Prof}(\Gamma)$.

A group Γ as in Theorem 3 is an irreducible lattice in the Lie group $G = \mathbf{G}(\mathbf{R})$, that is, the homogeneous space G/Γ carries a G -invariant

probability measure; moreover, Γ is cocompact in G if and only if \mathbf{G} is anisotropic over \mathbf{Q} (for all this, see [BHC62, (7.8), (11.6)]). The following corollary is a direct consequence of Theorem 3 and of the fact that a non cocompact arithmetic lattice in a semisimple Lie group has nontrivial unipotent elements (see [Mor15, (5.5.14)]).

Corollary 4. *With the notation as in Theorem 3, assume that \mathbf{G} is isotropic over \mathbf{Q} . For every arithmetic subgroup Γ of $\mathbf{G}(\mathbf{Q})$, the natural map $\text{Bohr}(\Gamma) \rightarrow \text{Prof}(\Gamma)$ is an isomorphism.*

As shown in Section 6, it may happen that $\text{Bohr}(\mathbf{G}(\mathbf{Z})) \cong \text{Prof}(\mathbf{G}(\mathbf{Z}))$, even when $\mathbf{G}(\mathbf{Z})$ is cocompact in $\mathbf{G}(\mathbf{R})$.

A general arithmetic lattice Γ has a third completion: the **congruence completion** $\text{Cong}(\Gamma)$ of Γ is the projective limit $\varprojlim \Gamma/H$, where H runs over the family of the congruence subgroups of Γ ; recall that a normal subgroup of Γ is a congruence subgroup if it contains the kernel of the map $\mathbf{G}(\mathbf{Z}) \rightarrow \mathbf{G}(\mathbf{Z}/N\mathbf{Z})$ of the reduction modulo N , for some integer $N \geq 1$. There is a natural surjective homomorphism $\pi : \text{Prof}(\Gamma) \rightarrow \text{Cong}(\Gamma)$. The so-called **congruence subgroup problem** asks whether π is injective and hence an isomorphism of topological groups; more generally, one can ask for a description of the kernel of π . This problem has been extensively studied for arithmetic subgroups (and, more generally, for S -arithmetic subgroups) in various algebraic groups; for instance, it is known that π is an isomorphism when $\Gamma = SL_n(\mathbf{Z})$ for $n \geq 3$ or $\Gamma = Sp_{2n}(\mathbf{Z})$ for $n \geq 2$ (see [BMS67]); moreover, the same conclusion is true when $\Gamma = \mathbf{T}(\mathbf{Z})$ for a torus \mathbf{T} (see [Che51]) and when $\Gamma = \mathbf{U}(\mathbf{Z})$ for a unipotent group \mathbf{U} (see Proposition 16 below). For more on the congruence subgroup problem, see for instance [Rag76] or [PR94, §9.5].

This paper is organized as follows. In Section 2, we establish some general facts about the Bohr compactifications of commensurable groups and the relationship between Bohr compactifications and unitary representations; we also give an explicit description of the Bohr compactification for a finitely generated abelian group. In Section 3, we give the proof of Theorem 1. Section 4 contains the proof of Theorem 2 and Section 5 the proof of Theorem 3. Section 6 is devoted to the explicit computation of the Bohr compactification for various examples of arithmetic groups.

2. SOME PRELIMINARIES

2.1. Bohr compactifications and unitary representations. Given a topological group G , we will consider finite dimensional unitary representations of G , that is, continuous homomorphisms $G \rightarrow U(n)$. Two

such representations are equivalent if they are conjugate by a unitary matrix. A representation π is irreducible if \mathbf{C}^n and $\{0\}$ there are only $\pi(G)$ -invariant subspaces of \mathbf{C}^n . We denote by $\text{Rep}_{\text{fd}}(G)$ the set of equivalence classes of finite dimensional unitary representations of G and by \widehat{G}_{fd} the subset of irreducible ones. Every $\pi \in \text{Rep}_{\text{fd}}(G)$ is a direct sum of representations from \widehat{G}_{fd}

When K is a compact group, every irreducible unitary representation of K is finite dimensional and $\widehat{K}_{\text{fd}} = \widehat{K}$ is the unitary dual space of K . By the Peter-Weyl theorem, \widehat{K} separates the points of K .

Let $\beta : G \rightarrow H$ be a continuous homomorphism of topological groups G and H with dense image; then β induces *injective* maps

$$\widehat{\beta} : \text{Rep}_{\text{fd}}(H) \rightarrow \text{Rep}_{\text{fd}}(G) \quad \text{and} \quad \widehat{\beta} : \widehat{H}_{\text{fd}} \rightarrow \widehat{G}_{\text{fd}},$$

given by $\widehat{\beta}(\pi) = \pi \circ \beta$ for $\pi \in \text{Rep}_{\text{fd}}(H)$. The following proposition, which may be considered as well-known, is a useful tool for identifying the Bohr compactification of a group.

Proposition 5. *Let G be a topological group, K a compact group, and $\beta : G \rightarrow K$ a continuous homomorphism with dense image. The following properties are equivalent:*

- (i) (K, β) is a Bohr compactification of G ;
- (ii) the induced map $\widehat{\beta} : \widehat{K} \rightarrow \widehat{G}_{\text{fd}}$ is surjective;
- (iii) the induced map $\widehat{\beta} : \text{Rep}_{\text{fd}}(K) \rightarrow \text{Rep}_{\text{fd}}(G)$ is surjective.

Proof. Assume that (i) holds and let $\pi : G \rightarrow U(n)$ be an irreducible representation of G ; by the universal property of the Bohr compactification, there exists a continuous homomorphism $\pi' : K \rightarrow U(n)$ such that $\pi = \widehat{\beta}(\pi')$ and (ii) follows.

Conversely, assume that (ii) holds. Let L be a compact group and $\alpha : G \rightarrow L$ a continuous homomorphism with dense image. Choose a family $\pi_i : L \rightarrow U(n_i)$ of representatives of \widehat{L} . By the Peter-Weyl theorem, we may identify L with its image in $\prod_i U(n_i)$ under the map $x \mapsto \bigoplus_i \pi_i(x)$. For every i , we have $\pi_i \circ \alpha \in \widehat{G}_{\text{fd}}$ and hence $\pi_i \circ \alpha = \widehat{\beta}(\pi'_i) = \pi'_i \circ \beta$ for some representation $\pi'_i : K \rightarrow U(n_i)$ of K . Define a continuous homomorphism

$$\alpha' : K \rightarrow \prod_i U(n_i) \quad x \mapsto \bigoplus_i \pi'_i(x).$$

We have $\alpha' \circ \beta = \alpha$ and hence

$$\alpha'(K) = \alpha'(\overline{\beta(G)}) \subset \overline{\alpha(G)} = L.$$

So, (i) and (ii) are equivalent. It is obvious that (ii) is equivalent to (iii). □

The profinite completion $(\text{Prof}(G), \alpha)$ of G may be similarly characterized in terms of certain unitary representations of G . Recall first that $(\text{Prof}(G), \alpha)$ is a pair consisting of a profinite group $\text{Prof}(G)$ and a continuous homomorphism $\alpha : G \rightarrow \text{Prof}(G)$ with dense image, satisfying the following universal property: for every profinite group K and every continuous homomorphism $f : G \rightarrow K$, there exists a continuous homomorphism $f' : \text{Prof}(G) \rightarrow K$ such that the diagram

$$\begin{array}{ccc}
 & & \text{Prof}(G) \\
 & \nearrow \alpha & \downarrow f' \\
 G & \xrightarrow{f} & K
 \end{array}$$

commutes. Recall that the class of profinite groups coincides with the class of totally disconnected compact groups (see [BH, Proposition 4.C.10]).

Denote by $\text{Rep}_{\text{finite}}(G)$ the set of equivalence classes of finite dimensional unitary representations π of G for which $\pi(G)$ is finite; let $\widehat{G}_{\text{finite}}$ be the subset of irreducible representations from $\text{Rep}_{\text{finite}}(G)$.

If $\alpha : G \rightarrow H$ is a continuous homomorphism of topological groups G and H with dense image, then α induces *injective* maps

$$\widehat{\alpha} : \text{Rep}_{\text{finite}}(H) \rightarrow \text{Rep}_{\text{finite}}(G) \quad \text{and} \quad \widehat{\alpha} : \widehat{H}_{\text{finite}} \rightarrow \widehat{G}_{\text{finite}}.$$

Observe that $\widehat{K} = \widehat{K}_{\text{finite}}$ if K is a profinite group. (Conversely, it follows from Peter-Weyl theorem that, if K is a compact group with $\widehat{K} = \widehat{K}_{\text{finite}}$, then K is profinite.) The proof of the following proposition is similar to the proof of Proposition 5 and will be omitted.

Proposition 6. *Let K be a totally disconnected compact group and $\alpha : G \rightarrow K$ a continuous homomorphism with dense image. The following properties are equivalent:*

- (i) (K, α) is a profinite completion of G ;
- (ii) the induced map $\widehat{\alpha} : \widehat{K} \rightarrow \widehat{G}_{\text{finite}}$ is surjective;
- (ii) the induced map $\widehat{\beta} : \text{Rep}_{\text{finite}}(K) \rightarrow \text{Rep}_{\text{finite}}(G)$ is surjective.

The universal property of $\text{Bohr}(G)$ implies that there is a continuous epimorphism $\alpha' : \text{Bohr}(G) \rightarrow \text{Prof}(G)$ such that the diagram

$$\begin{array}{ccc} & & \text{Bohr}(G) \\ & \nearrow \beta & \downarrow \alpha' \\ G & \xrightarrow{\alpha} & \text{Prof}(G) \end{array}$$

commutes. We record the following elementary but basic fact mentioned in the introduction.

Proposition 7. *The kernel of $\alpha' : \text{Bohr}(G) \rightarrow \text{Prof}(G)$ coincides with the connected component $\text{Bohr}(G)_0$ of $\text{Bohr}(G)$.*

Proof. Since $\text{Bohr}(G)_0$ is connected and $\text{Prof}(G)$ is totally disconnected, $\text{Bohr}(G)_0$ is contained in $\text{Ker}\alpha'$. So, α' factorizes to a continuous epimorphism $\alpha'' : K \rightarrow \text{Prof}(G)$, where $K := \text{Bohr}(G)/\text{Bohr}(G)_0$ and we have a commutative diagram

$$\begin{array}{ccc} & & K \\ & \nearrow p \circ \beta & \downarrow \alpha'' \\ G & \xrightarrow{\alpha} & \text{Prof}(G). \end{array}$$

where $p : \text{Bohr}(G) \rightarrow K$ is the canonical epimorphism. Since K is a totally disconnected compact group, there exists a continuous epimorphism $f : \text{Prof}(G) \rightarrow K$ and we have a commutative diagram

$$\begin{array}{ccc} & & K \\ & \nearrow p \circ \beta & \uparrow f \\ G & \xrightarrow{\alpha} & \text{Prof}(G). \end{array}$$

For every $g \in G$, we have

$$f(\alpha''(p \circ \beta(g))) = f(\alpha(g)) = p \circ \beta(g);$$

since $p \circ \beta(G)$ is dense in K , it follows that $f \circ \alpha''$ is the identity on K . This implies that α'' is injective and hence an isomorphism. \square

2.2. Bohr compactifications of commensurable groups. Let G be a topological group and H be a closed subgroup of finite index in G . We first determine $\text{Bohr}(H)$ in terms of $\text{Bohr}(G)$.

Proposition 8. *Let $(\text{Bohr}(G), \beta)$ be the Bohr compactification of G . Set $K := \overline{\beta(H)}$.*

- (i) K is a subgroup of finite index of $\text{Bohr}(G)$.
- (ii) $(K, \beta|_H)$ is a Bohr compactification of H .

- (iii) K and $\text{Bohr}(G)$ have the same connected component of the identity.

Proof. Item (i) is obvious and Item (iii) follows from Item (i). To show Item (ii), let π be a unitary representation of H on \mathbf{C}^n . Since H has finite index in G , the induced representation $\rho := \text{Ind}_H^G \pi$, which is a unitary representation of G , is finite dimensional. Hence, there exists $\rho' \in \text{Rep}_{\text{fd}}(\text{Bohr}(G))$ such that $\rho = \rho' \circ \beta$. Now, π is equivalent to a subrepresentation of the restriction of ρ to H (see [BH, 1.F]); so, we may identify π with the representation of H defined by a $\rho(H)$ -invariant subspace W of the space of ρ . Then W is $\rho'(K)$ -invariant and defines therefore a representation π' of K . We have $\pi = \pi' \circ (\beta|_H)$ and Proposition 5 shows that Item (ii) holds. \square

Next, we want to determine $\text{Bohr}(G)$ in terms of $\text{Bohr}(H)$.

Given a compact group K and a finite set X , we define another compact group, we call the **induced group** of (K, X) , as

$$\text{Ind}(K, X) := K^X \rtimes \text{Sym}(X),$$

where the group $\text{Sym}(X)$ of bijections of X acts by permutations of indices on K^X :

$$\sigma((g_x)_{x \in X}) = (g_{\sigma^{-1}(x)})_{x \in X} \quad \text{for all } \sigma \in \text{Sym}(X), (g_x)_{x \in X} \in K^X$$

Observe that, if $\pi : K \rightarrow U(n)$ is a representation of K on $V = \mathbf{C}^n$, then a unitary representation $\text{Ind}(\pi)$ of $\text{Ind}(K, X)$ on V^X is defined by

$$\text{Ind}(\pi)((g_x)_{x \in X}, \sigma)(v_x)_{x \in X} = (\pi(g_x)v_{\sigma^{-1}(x)})_{x \in X},$$

for $((g_x)_{x \in X}, \sigma) \in \text{Ind}(K, X)$ and $(v_x)_{x \in X} \in V^X$.

Coming back to our setting, where H is a closed subgroup of finite index in G , we fix a transversal X for the right cosets of H ; so, we have a disjoint union $G = \bigsqcup_{x \in X} Hx$. For every $g \in G$ and $x \in X$, let $x \cdot g$ and $c(x, g)$ be the unique elements in X and H such that $xg = c(x, g)(x \cdot g)$. Observe that

$$X \times G \rightarrow X, \quad (x, g) \mapsto x \cdot g$$

is an action of G on X (on the right), which is equivalent to the natural action of G on $H \backslash G$ given by right multiplication. In particular, for every $g \in G$, the map $\sigma(g) : x \mapsto x \cdot g^{-1}$ belongs to $\text{Sym}(X)$ and we have a homomorphism

$$G \mapsto \text{Sym}(X), \quad g \mapsto \sigma(g).$$

Proposition 9. *Let $(\text{Bohr}(H), \beta)$ be the Bohr compactification of H . Let $\text{Ind}(\text{Bohr}(H), X)$ be the compact group defined as above. Consider the map $\tilde{\beta} : G \rightarrow \text{Ind}(\text{Bohr}(H), X)$ defined by*

$$\tilde{\beta}(g) = (\beta(c(x, g)))_{x \in X}, \sigma(g) \quad \text{for all } g \in G.$$

The closure of $\tilde{\beta}(G)$ in $\text{Ind}(\text{Bohr}(H), X)$, together with the map $\tilde{\beta}$, is a Bohr compactification of G .

Proof. It is readily checked that $\tilde{\beta} : G \rightarrow \text{Ind}(\text{Bohr}(H), X)$ is a continuous homomorphism. Let $\rho : G \rightarrow U(n)$ be a finite dimensional unitary representation of G . Set $\pi := \rho|_H \in \text{Rep}_{\text{fd}}(H)$. There exists $\pi' \in \text{Rep}_{\text{fd}}(\text{Bohr}(H))$ such that $\pi = \pi' \circ \beta$. Let $\tilde{\pi} := \text{Ind}_H^G \pi$. As is well-known (see [BH, 1.F]), $\tilde{\pi}$ can be realized on V^X for $V := \mathbf{C}^n$ by the formula

$$\tilde{\pi}(g)(v_x)_{x \in X} = (\pi(c(x, g))v_{x \cdot g})_{x \in X} = (\pi(c(x, g))v_{\sigma(g^{-1})x})_{x \in X},$$

for all $g \in G$ and $(v_x)_{x \in X} \in V^X$. With the unitary representation $\text{Ind}(\pi')$ of $\text{Ind}(\text{Bohr}(H), X)$ defined as above, we have therefore

$$(*) \quad \tilde{\pi}(g) = \text{Ind}(\pi')(\tilde{\beta}(g)) \quad \text{for all } g \in G,$$

that is, $\tilde{\pi} = \text{Ind}(\pi') \circ \tilde{\beta}$. Now,

$$\tilde{\pi} = \text{Ind}_H^G \pi = \text{Ind}_H^G(\rho|_H)$$

is equivalent to the tensor product representation $\rho \otimes \lambda_{G/H}$, where $\lambda_{G/H}$ is the regular representation of G/H (see [BHV08, E.2.5]). Since $\lambda_{G/H}$ contains the trivial representation of G , it follows that ρ is equivalent to a subrepresentation of $\tilde{\pi}$; so, we can identify ρ with the representation of G defined by a $\tilde{\pi}(G)$ -invariant subspace W of V^X . Denoting by L the closure of $\tilde{\beta}(G)$, it follows from $(*)$ that W is invariant under $\text{Ind}(\pi')(L)$ and so defines a representation ρ' of L . Then $\rho = \rho' \circ \tilde{\beta}$ and the claim follows from Proposition 5. \square

We will also need the following well-known (see [HK01, Lemma 2.2]) description of the Bohr compactification of a quotient of G in terms of the Bohr compactification of G .

Proposition 10. *Let $(\text{Bohr}(G), \beta)$ be the Bohr compactification of the topological group G and let N be a closed normal subgroup of G . Let K_N be the closure of $\beta(N)$ in $\text{Bohr}(G)$*

- (i) K_N is a normal subgroup of $\text{Bohr}(G)$ and β induces a continuous homomorphism $\bar{\alpha} : G/N \rightarrow \text{Bohr}(G)/K_N$
- (ii) $(\text{Bohr}(G)/K_N, \bar{\alpha})$ is a Bohr compactification of G/N .

Proof. Let $(\text{Bohr}(G/N), \bar{\beta})$ be the Bohr compactification of G/N . The canonical homomorphism $\alpha : G \rightarrow G/N$ induces a continuous homomorphism $\alpha' : \text{Bohr}(G) \rightarrow \text{Bohr}(G/N)$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\beta} & \text{Bohr}(G) \\ \downarrow \alpha & & \downarrow \alpha' \\ G/N & \xrightarrow{\bar{\beta}} & \text{Bohr}(G/N) \end{array}$$

commutes. It follows that $\beta(N)$ and hence K_N is contained in $\text{Ker } \alpha'$. So, we have induced homomorphisms $\bar{\alpha} : G/N \rightarrow \text{Bohr}(G)/K_N$ and $\bar{\alpha}' : \text{Bohr}(G)/K_N \rightarrow \text{Bohr}(G/N)$, giving rise to a commutative diagram

$$\begin{array}{ccc} & & \text{Bohr}(G)/K_N \\ & \nearrow \bar{\alpha} & \downarrow \bar{\alpha}' \\ G/N & \xrightarrow{\bar{\beta}} & \text{Bohr}(G/N). \end{array}$$

It follows that $(\text{Bohr}(G)/K_N, \bar{\alpha})$ has the same universal property for G/N as $(\text{Bohr}(G/N), \bar{\beta})$. Since $\bar{\alpha}$ has dense image, $(\text{Bohr}(G)/K_N, \bar{\alpha})$ is therefore a Bohr compactification of G/N . \square

2.3. Bohr compactification of finitely generated abelian groups.

Let G be a locally compact abelian group. Its dual group \widehat{G} consists of the continuous homomorphism from G to the circle group \mathbf{S}^1 ; equipped with the topology of uniform convergence on compact subsets, \widehat{G} is again a locally compact abelian group. Let $\widehat{G}_{\text{disc}}$ be the group \widehat{G} equipped with the discrete topology. It is well-known (see e.g. [BH, Proposition 4.C.4]) that the Bohr compactification of G coincides with the dual group K of $\widehat{G}_{\text{disc}}$, together with the embedding $i : G \rightarrow K$ given by $i(g)(\chi) = \chi(g)$ for all $g \in G$ and $\chi \in \widehat{G}$. Notice that this implies that, by Pontrjagin duality, the dual group of $\text{Bohr}(G)$ coincides with $\widehat{G}_{\text{disc}}$.

A more precise information on the structure of the Bohr compactification is available in the case of a (discrete) finitely generated abelian group. As is well-known, such a group Γ splits a direct sum $\Gamma = F \oplus A$ of a finite group F (which is its torsion subgroup) and a free abelian group A of finite rank $k \geq 0$, called the rank of Γ . Recall that \mathbf{Z}_p denotes the ring of p -adic integers for a prime p and \mathbf{A} the ring of adèles over \mathbf{Q} .

Proposition 11. *Let Γ be a finitely generated abelian group of rank k .*

(i) *We have a direct sum decomposition*

$$\text{Bohr}(\Gamma) \cong \text{Bohr}(\Gamma)_0 \oplus \text{Prof}(\Gamma).$$

(ii) *We have*

$$\text{Prof}(\Gamma) \cong F \oplus \prod_{p \text{ prime}} \mathbf{Z}_p^k,$$

where F is a finite group.

(iii) *We have*

$$\text{Bohr}(\Gamma)_0 \cong \prod_{\omega \in \mathfrak{c}} \mathbf{A}^k / \mathbf{Q}^k,$$

a product of uncountably many copies of the adelic solenoid $\mathbf{A}^k / \mathbf{Q}^k$.

Proof. We have $\Gamma \cong F \oplus \mathbf{Z}^k$ for a finite group F and $\text{Bohr}(\mathbf{Z}^k) = \text{Bohr}(\mathbf{Z})^k$. So, it suffices to determine $\text{Bohr}(\mathbf{Z})$. As mentioned above, $\text{Bohr}(\mathbf{Z})$ can be identified with the dual group of the circle \mathbf{S}^1 viewed as discrete group. Choose a linear basis $\{1\} \cup \{x_\omega \mid \omega \in \mathfrak{c}\}$ of \mathbf{R} over \mathbf{Q} . Then $\mathbf{S}^1 \cong \mathbf{R}/\mathbf{Z}$ is isomorphic to the abelian group

$$(\mathbf{Q}/\mathbf{Z}) \oplus \bigoplus_{\omega \in \mathfrak{c}} \mathbf{Q}.$$

Hence,

$$\text{Bohr}(\mathbf{Z}) \cong \widehat{\mathbf{Q}/\mathbf{Z}} \oplus \prod_{\omega \in \mathfrak{c}} \widehat{\mathbf{Q}}.$$

Now,

$$\mathbf{Q}/\mathbf{Z} = \bigoplus_{p \text{ prime}} Z(p^\infty),$$

with $Z(p^\infty) = \varinjlim_k \mathbf{Z}/p^k \mathbf{Z}$ the p -primary component of \mathbf{Q}/\mathbf{Z} . Hence,

$$\widehat{Z(p^\infty)} \cong \varprojlim_k \mathbf{Z}/p^k \mathbf{Z} = \mathbf{Z}_p.$$

On the other hand, $\widehat{\mathbf{Q}}$ can be identified with the solenoid \mathbf{A}/\mathbf{Q} (see e.g. [HR79, (25.4)]). It follows that

$$\text{Bohr}(\Gamma) \cong \prod_{p \text{ prime}} \mathbf{Z}_p \oplus \prod_{\omega \in \mathfrak{c}} \mathbf{A}/\mathbf{Q}.$$

□

2.4. Restrictions of representations to normal subgroups. Let Γ be a group and N a normal subgroup of Γ . Recall that Γ acts on \widehat{N}_{fd} : for $\sigma \in \widehat{N}_{\text{fd}}$ and $\gamma \in \Gamma$, the conjugate representation $\sigma^\gamma \in \widehat{N}_{\text{fd}}$ is defined by

$$\sigma^\gamma(n) = \sigma(\gamma^{-1}n\gamma), \quad \text{for all } n \in N.$$

The stabilizer Γ_σ of σ is the subgroup consisting of all $\gamma \in \Gamma$ for which σ^γ is equivalent to σ ; observe that Γ_σ contains N .

Given a unitary representation ρ of N on a finite dimensional vector space V and $\sigma \in \widehat{N}_{\text{fd}}$, we denote by V^σ the σ -isotypical component of ρ , that is, the sum of all ρ -invariant subspaces W for which the restriction of ρ to W is equivalent to σ . Observe that V decomposes as direct sum $V = \bigoplus_{\sigma \in \Sigma_\rho} V^\sigma$, where Σ_ρ is the finite set of $\sigma \in \widehat{N}_{\text{fd}}$ with $V^\sigma \neq \{0\}$.

Proposition 12. *Let π be an irreducible unitary representation of Γ on a finite dimensional vector space V . Let $V = \bigoplus_{\sigma \in \Sigma_{\pi|_N}} V^\sigma$ be the decomposition of the restriction $\pi|_N$ of π to N into isotypical components. Then $\Sigma_{\pi|_N}$ coincides with a Γ -orbit: there exists $\sigma \in \widehat{N}_{\text{fd}}$ such that $\Sigma_{\pi|_N} = \{\sigma^\gamma : \gamma \in \Gamma\}$; in particular, Γ_σ has finite index in Γ .*

Proof. Let $\sigma \in \Sigma_{\pi|_N}$ and fix a transversal T for the left cosets of Γ_σ with $e \in T$. Then $V^{\sigma^t} = \pi(t)V^\sigma$ for all $t \in T$. Since π is irreducible and $\sum_{t \in T} \pi(t)V^\sigma$ is $\pi(\Gamma)$ -invariant, it follows that $\Sigma_{\pi|_N}$ is a Γ -orbit. \square

3. PROOF OF THEOREM 1

3.1. Distortion and Bohr compactification. Let Γ be a finitely generated group with a finite set S of generators. For $\gamma \in \Gamma$, denote by $\ell_S(\gamma)$ the word length of γ with respect to $S \cup S^{-1}$ and set

$$t(\gamma) = \liminf_{n \rightarrow \infty} \frac{\ell_S(\gamma^n)}{n}.$$

The number $t(\gamma)$ is called the *translation number* of γ in [GS91]

Definition 13. An element $\gamma \in \Gamma$ is said to be **distorted** if $t(\gamma) = 0$.

In fact, since the sequence $n \mapsto \ell_S(\gamma^n)$ is subadditive, we have, by Fekete's lemma,

$$t(\gamma) = \lim_{n \rightarrow \infty} \frac{\ell_S(\gamma^n)}{n} = \inf \left\{ \frac{\ell_S(\gamma^n)}{n} : n \in \mathbf{N}^* \right\}$$

The property of being distorted is independent of the choice of the set of generators. Distorted elements are called *algebraically parabolic* in [BGS85, (7.5), p.90], but we prefer to use the terminology from [FH06]. The relevance of distortion to the Bohr compactification lies

in the following proposition; for a related result with a similar proof, see [LMR00, (2.4)].

Proposition 14. *Let Γ be a finitely generated group and $\gamma \in \Gamma$ a distorted element. Then, for every finite dimensional unitary representation $\pi : \Gamma \rightarrow U(N)$ of Γ , the matrix $\pi(\gamma) \in U(N)$ has finite order.*

Proof. It suffices to show that all eigenvalues of the unitary matrix $\pi(\gamma)$ are roots of unity. Assume, by contradiction, that π has an eigenvalue $\lambda \in \mathbf{S}^1$ of infinite order.

Let S be a finite set of generators of Γ with $S = S^{-1}$. The group $\pi(\Gamma)$ is generated by the set $\{\pi(s) \mid s \in S\}$. Hence, $\pi(G)$ is contained in $GL_N(L)$, where L is the subfield of \mathbf{C} generated by the matrix coefficients of the $\pi(s)$'s. It follows that λ is contained in a finitely generated extension ℓ of L . By a lemma of Tits ([Tit72, Lemma 4.1]), there exists a locally compact field k endowed with an absolute value $|\cdot|$ and a field embedding $\sigma : \ell \rightarrow k$ such that $|\sigma(\lambda)| \neq 1$. Upon replacing γ by γ^{-1} , we may assume that $|\sigma(\lambda)| > 1$.

Define a function (“norm”) $\xi \mapsto \|\xi\|$ on k^N by

$$\|\xi\| = \max\{|\xi_1|, \dots, |\xi_N|\} \quad \text{for all } \xi = (\xi_1, \dots, \xi_N) \in k^N.$$

For a matrix $A \in GL_N(k)$, set $\|A\| = \sup_{\xi \neq 0} \|A\xi\|/\|\xi\|$. It is obvious that $\|A\xi\| \leq \|A\|\|\xi\|$ for all $\xi \in k^N$ and hence

$$(*) \quad \|AB\| \leq \|A\|\|B\| \quad \text{for all } A, B \in GL_N(k).$$

In particular, we have $\|A^n\| \leq \|A\|^n$ for all $A \in GL_N(k)$ and $n \in \mathbf{N}$.

For a matrix $w \in GL_n(\ell)$, denote by $\sigma(w)$ the matrix in $GL_n(k)$ obtained by applying σ to the entries of w . Set $A_s = \sigma(\pi(s))$ for $s \in S$ and $A := \sigma(\pi(\gamma))$. With

$$C := \max\{\|A_s\| : s \in S\},$$

it is clear that Inequality (*) implies that

$$(**) \quad \|A^n\| = \|\sigma(\pi(\gamma^n))\| \leq C^{\ell_S(\gamma^n)} \quad \text{for all } n \in \mathbf{N}.$$

On the other hand, $\sigma(\lambda)$ is an eigenvalue of A ; so, there exists $\xi \in k^N \setminus \{0\}$ such that $A\xi = \sigma(\lambda)\xi$ and hence $A^n\xi = \sigma(\lambda)^n\xi$ for all $n \in \mathbf{N}$. So, for every $n \in \mathbf{N}$, we have

$$\|A^n\xi\| = |\sigma(\lambda)|^n \|\xi\|$$

and this implies that

$$\|A^n\| \geq |\sigma(\lambda)|^n.$$

In view of (**), we obtain therefore

$$\frac{\ell_S(\gamma^n) \log C}{n} \geq \log |\sigma(\lambda)| \quad \text{for all } n \in \mathbf{N}.$$

Since $|\sigma(\lambda)| > 1$, this contradicts the fact that $\liminf_{n \rightarrow \infty} \frac{\ell_S(\gamma^n)}{n} = 0$. \square

3.2. Distorted elements in nilpotent groups. Let Γ be a finitely generated nilpotent subgroup. For subsets A, B in Γ , we let $[A, B]$ denote the subgroup of Γ generated by all commutators $[a, b] = aba^{-1}b^{-1}$, for $a \in A$ and $b \in B$. Let

$$\Gamma^{(0)} \supset \Gamma^{(1)} \supset \dots \supset \Gamma^{(d-1)} \supset \Gamma^{(d)} = \{e\}$$

be the lower central series of Γ , defined inductively by $\Gamma^{(0)} = \Gamma$ and $\Gamma^{(k+1)} = [\Gamma^{(k)}, \Gamma]$. The step of nilpotency of Γ is the smallest $d \geq 1$ such that $\Gamma^{(d-1)} \neq \{e\}$ and $\Gamma^{(d)} = \{e\}$.

Proposition 15. *Let Γ be a finitely generated nilpotent subgroup. Every $\gamma \in \Gamma^{(1)} = [\Gamma, \Gamma]$ is distorted.*

Proof. Let S be a finite set of generators of Γ with $S = S^{-1}$. Let $d \geq 1$ be the step of nilpotency of Γ . The case $d = 1$ being trivial, we will assume that $d \geq 2$. We will show by induction on $i \in \{1, \dots, d-1\}$ that every $\gamma \in \Gamma^{(d-i)}$ is distorted.

- *First step.* Assume that $i = 1$. It is well-known that every element γ in $\Gamma^{(d-1)}$ is distorted (see for instance [BGS85, (7.6), p. 91]); in fact, more precise estimates are available: for every $\gamma \in \Gamma^{(d-1)}$, we have $\ell_S(\gamma^n) = O(n^{1/d})$ as $n \rightarrow \infty$ (see [Tit81, 2.3 Lemme] or [DK18, Lemma 14.15]).

- *Second step.* Assume that, for every finitely generated nilpotent subgroup Λ of step $d' \geq 2$, every element $\delta \in \Lambda^{(d'-i)}$ is distorted for $i \in \{1, \dots, d'-2\}$. Let $\gamma \in \Gamma^{(d-i-1)}$ and fix $\varepsilon > 0$.

The quotient group $\bar{\Gamma} = \Gamma/\Gamma^{(d-1)}$ is nilpotent of step $d' = d-1$ and $p(\gamma) \in \bar{\Gamma}^{(d'-i)}$, where $p : \Gamma \rightarrow \bar{\Gamma}$ is the quotient map. By induction hypothesis, $p(\gamma)$ is distorted in $\bar{\Gamma}$ with respect to the generating set $\bar{S} := p(S)$. So, we have $\lim_{n \rightarrow \infty} \frac{\ell_{\bar{S}}(p(\gamma)^n)}{n} = 0$; hence, we can find an integer $N \geq 1$ such that

$$(*) \quad \forall n \geq N, \exists \delta_n \in \Gamma^{(d-1)} : \frac{\ell_S(\gamma^n \delta_n)}{n} \leq \varepsilon.$$

By the first step, we have $\lim_{k \rightarrow \infty} \frac{\ell_S(\delta_N^k)}{k} = 0$, since $\delta_N \in \Gamma^{(d-1)}$; so, there exists $K \geq 1$ such that

$$(**) \quad \forall k \geq K : \frac{\ell_S(\delta_N^k)}{k} \leq \varepsilon.$$

Let $k \geq K$. We have

$$(***) \quad \frac{\ell_S(\gamma^{Nk})}{Nk} = \frac{\ell_S((\gamma^{Nk}\delta_N^k)(\delta_N^{-1})^k)}{Nk} \leq \frac{\ell_S(\gamma^{Nk}\delta_N^k)}{Nk} + \frac{\ell_S(\delta_N^k)}{Nk}.$$

Now, since $\Gamma^{(d-1)}$ is contained in the center of Γ , the elements δ_N and γ_N commute and hence, by (*), we have

$$\frac{\ell_S(\gamma^{Nk}\delta_N^k)}{Nk} = \frac{\ell_S((\gamma^N\delta_N)^k)}{Nk} \leq k \frac{\ell_S(\gamma^N\delta_N)}{Nk} = \frac{\ell_S(\gamma^N\delta_N)}{N} \leq \varepsilon.$$

So, together with (***) and (**), we obtain

$$\forall k \geq K : \frac{\ell_S(\gamma^{Nk})}{Nk} \leq 2\varepsilon.$$

This shows that $t(\gamma) = 0$. □

3.3. Congruence subgroups in unipotent groups. The following result, which shows that the congruence subgroup problem has a positive solution for unipotent groups, is well-known (see the sketch in [Rag76, p.108]); for the convenience of the reader, we reproduce its short proof.

Proposition 16. *Let \mathbf{U} be a unipotent algebraic group over \mathbf{Q} . Let Γ be an arithmetic subgroup of $\mathbf{U}(\mathbf{Q})$. Then every finite index subgroup of Γ is a congruence subgroup.*

Proof. We can find a sequence

$$\mathbf{U} = \mathbf{U}_0 \supset \mathbf{U}_1 \supset \cdots \supset \mathbf{U}_{d-1} \supset \mathbf{U}_d = \{e\}$$

of normal \mathbf{Q} -subgroups of \mathbf{U} such that the factor groups $\mathbf{U}_i/\mathbf{U}_{i+1}$ are \mathbf{Q} -isomorphic to \mathbf{G}_a , the additive group of dimension 1 (see [Bor63, (15.5)]).

We proceed by induction on $d \geq 1$. If $d = 1$, then Γ is commensurable to \mathbf{Z} and the claim is obvious true. Assume that $d \geq 2$. Then \mathbf{U} can be written as semi-direct product $\mathbf{U} = \mathbf{U}_1 \rtimes \mathbf{G}_a$. By [BHC62, Corollary 4.6], Γ is commensurable to $\mathbf{U}_1(\mathbf{Z}) \rtimes \mathbf{Z}$. Let H a subgroup of finite index in Γ . Then $H \cap \mathbf{U}_1(\mathbf{Z})$ has finite index in $\mathbf{U}_1(\mathbf{Z})$ and hence, by induction hypothesis, contains the kernel of the reduction $\mathbf{U}_1(\mathbf{Z}) \rightarrow \mathbf{U}_1(\mathbf{Z}/N_1\mathbf{Z})$ modulo some $N_1 \geq 1$. Moreover, $H \cap \mathbf{Z} = N_2\mathbf{Z}$ for some $N_2 \geq 1$. Hence, H contains the kernel of the reduction $\mathbf{U}(\mathbf{Z}) \rightarrow \mathbf{U}(\mathbf{Z}/N_1N_2\mathbf{Z})$ modulo N_1N_2 . □

3.4. Proof of Theorem 1. Let Γ be a finitely generated nilpotent group and $\alpha : \Gamma \rightarrow \text{Prof}(\Gamma)$ the canonical homomorphism. Recall (see Proposition 11) that the Bohr compactification of $\Gamma^{\text{Ab}} = \Gamma/[\Gamma, \Gamma]$ splits as a direct sum

$$\text{Bohr}(\Gamma^{\text{Ab}}) = \text{Bohr}(\Gamma^{\text{Ab}})_0 \oplus B_1,$$

for a closed subgroup $B_1 \cong \text{Prof}(\Gamma^{\text{Ab}})$. Let $p : \text{Bohr}(\Gamma^{\text{Ab}}) \rightarrow \text{Bohr}(\Gamma^{\text{Ab}})_0$ be the corresponding projection. Denote by $\beta_0 : \Gamma \rightarrow \text{Bohr}(\Gamma^{\text{Ab}})$ the map induced by the quotient homomorphism $\Gamma \rightarrow \Gamma^{\text{Ab}}$. Set

$$K := \text{Bohr}(\Gamma^{\text{Ab}})_0 \times \text{Prof}(\Gamma),$$

and let $\beta : \Gamma \rightarrow K$ be the homomorphism $\gamma \mapsto (p \circ \beta_0(\gamma), \alpha(\gamma))$. We claim that (K, β) is a Bohr compactification for Γ .

• *First step.* We claim that $\beta(\Gamma)$ is dense in K . Indeed, let L be the closure of $\beta(\Gamma)$ in K and L_0 its connected component. Since $\text{Prof}(\Gamma)$ is totally disconnected, the projection of L_0 on $\text{Prof}(\Gamma)$ is trivial; hence $L_0 = K_0 \times \{1\}$ for a connected closed subgroup K_0 of $\text{Bohr}(\Gamma^{\text{Ab}})_0$. The projection of L on $\text{Bohr}(\Gamma^{\text{Ab}})_0$ induces then a continuous homomorphism

$$f : L/L_0 \rightarrow \text{Bohr}(\Gamma^{\text{Ab}})_0/K_0.$$

Observe that f has dense image, since $p \circ \beta_0(\Gamma)$ is dense in $\text{Bohr}(\Gamma^{\text{Ab}})_0$; so, f is surjective by compactness of L/L_0 . It follows, by compactness again, that $\text{Bohr}(\Gamma^{\text{Ab}})_0/K_0$ is topologically isomorphic to a quotient of L/L_0 . As L/L_0 is totally disconnected, this implies (see [Bou71, Chap. 3, §4, Corollaire 3]) that $\text{Bohr}(\Gamma^{\text{Ab}})_0/K_0$ is also totally disconnected and hence that $K_0 = \text{Bohr}(\Gamma^{\text{Ab}})_0$. So, $\text{Bohr}(\Gamma^{\text{Ab}})_0 \times \{1\}$ is contained in L . It follows that L is the product of $\text{Bohr}(\Gamma^{\text{Ab}})_0$ with a subgroup of $\text{Prof}(\Gamma)$. Since $\alpha(\Gamma)$ is dense in $\text{Prof}(\Gamma)$, this subgroup coincides with $\text{Prof}(\Gamma)$, that is, $L = K$ and the claim is proved.

• *Third step.* We claim that every irreducible unitary representation $\pi : \Gamma \rightarrow U(N)$ of Γ is of the form $\chi \otimes \rho$ for some $\chi \in \widehat{\Gamma^{\text{Ab}}}$ and $\rho \in \widehat{\Gamma}_{\text{finite}}$.

Indeed, Propositions 14 and 15, imply that $\pi([\Gamma, \Gamma])$ is a periodic subgroup of $U(N)$. Since Γ is finitely generated, $[\Gamma, \Gamma]$ is finitely generated (in fact, every subgroup of Γ is finitely generated; see [Rag72, 2.7 Theorem]). Hence, by Schur's theorem (see [Weh73, 4.9 Corollary]), $\pi([\Gamma, \Gamma])$ is finite. It follows that there exists a finite index normal subgroup H of $[\Gamma, \Gamma]$ so that $\pi|_H$ is the trivial representation of H .

Next, we claim that there exists a normal subgroup Δ of finite index in Γ such that $\Delta \cap [\Gamma, \Gamma] = H$. Indeed, since $\Gamma/[\Gamma, \Gamma]$ is abelian and finitely generated, we have $\Gamma/[\Gamma, \Gamma] \cong \mathbf{Z}^k \oplus F$ for some finite subgroup F and some integer $k \geq 0$. Let Γ_1 be the inverse image in Γ of the copy

of \mathbf{Z}^k in $\Gamma/[\Gamma, \Gamma]$. Then Γ_1 is a normal subgroup of finite index of Γ . Moreover, Γ_1 can be written as iterated semi-direct product

$$\Gamma_1 = (\dots (([\Gamma, \Gamma] \rtimes \mathbf{Z}) \rtimes \mathbf{Z}) \rtimes \mathbf{Z}).$$

Set

$$\Delta := (\dots ((H \rtimes \mathbf{Z}) \rtimes \mathbf{Z}) \rtimes \mathbf{Z}).$$

Then Δ is a normal subgroup of finite index of Γ with $\Delta \cap [\Gamma, \Gamma] = H$.

Since $\pi|_H$ is trivial on H and since $[\Delta, \Delta] \subset H$, the restriction $\pi|_\Delta$ of π to Δ factorizes through Δ^{Ab} . So, by Proposition 12, there exists a finite Γ -orbit \mathcal{O} in $\widehat{\Delta^{\text{Ab}}}$ such that we have a direct sum decomposition $V = \bigoplus_{\chi \in \mathcal{O}} V^\chi$, where V^χ is the χ -isotypical component of $\pi|_\Delta$.

Fix $\chi \in \mathcal{O}$. Since χ is trivial on H and since $\Delta \cap [\Gamma, \Gamma] = H$, we can view χ as a unitary character of the subgroup $\Delta/(\Delta \cap [\Gamma, \Gamma])$ of Γ^{Ab} . Hence, χ extends to a character $\tilde{\chi} \in \widehat{\Gamma^{\text{Ab}}}$ (see, e.g. [HR79, (24.12)]). This implies that $\Gamma_\chi = \Gamma$; indeed,

$$\chi^\gamma(\delta) = \tilde{\chi}(\gamma^{-1}\delta\gamma) = \tilde{\chi}(\delta) = \chi(\delta)$$

for every $\gamma \in \Gamma$ and $\delta \in \Delta$. This shows that \mathcal{O} is a singleton and so $V = V^\chi$. We write

$$\pi = \tilde{\chi} \otimes (\overline{\tilde{\chi}} \otimes \pi).$$

Then $\rho := \overline{\tilde{\chi}} \otimes \pi$ is an irreducible unitary representation of Γ which is trivial on Δ ; so, ρ has finite image and $\pi = \tilde{\chi} \otimes \rho$.

• *Third step.* Let $\pi \in \widehat{\Gamma}_{\text{fd}}$. We claim that there exists a representation $\pi' \in \widehat{K}$ such that $\pi = \pi' \circ \beta$. Once proved, Proposition 5 will imply that (K, β) is a Bohr compactification for Γ .

By the second step, we can write $\pi = \chi \otimes \rho$ for some $\chi \in \widehat{\Gamma^{\text{Ab}}}$ and $\rho \in \widehat{\Gamma}_{\text{finite}}$. On the one hand, we can write $\rho = \rho' \circ \alpha$ for some $\rho' \in \widehat{\text{Prof}(\Gamma)}$, by the universal property of $\widehat{\text{Prof}(\Gamma)}$. On the other hand, we can decompose χ as $\chi = \chi_0 \chi_1$ with $\chi_0 \in \widehat{\Gamma^{\text{Ab}}}$ of infinite order and $\chi_1 \in \widehat{\Gamma^{\text{Ab}}}$ of finite order. We have $\chi_0 = \chi'_0 \circ (p \circ \beta_0)$ and $\chi_1 = \chi'_1 \circ \alpha$ for unitary characters χ'_0 of $\text{Bohr}(\Gamma^{\text{Ab}})_0$ and χ'_1 of $\text{Prof}(\Gamma^{\text{Ab}})$. For $\pi' = \chi_0 \otimes (\chi'_1 \otimes \rho')$, we have $\pi' \in \widehat{K}$ and $\pi = \pi' \circ \beta$.

4. PROOF OF THEOREMS 2

Let $\mathbf{G} = \mathbf{U} \rtimes \mathbf{H}$ be a Levi decomposition of \mathbf{G} and set

$$\Lambda = \mathbf{H}(\mathbf{Z}), \quad \Delta = \mathbf{U}(\mathbf{Z}), \quad \text{and} \quad \Gamma = \Delta \rtimes \Lambda.$$

Denote by $\beta_\Delta : \Delta \rightarrow \text{Bohr}(\Delta)$ and $\beta_\Lambda : \Lambda \rightarrow \text{Bohr}(\Lambda)$ the natural homomorphisms. Observe that, by the universal property of $\text{Bohr}(\Delta)$, every element $\lambda \in \Lambda$ defines a continuous automorphism $\theta_b(\lambda)$ of $\text{Bohr}(\Delta)$

such that

$$\theta_b(\lambda)(\delta) = \beta_\Delta(\lambda\delta\lambda^{-1}) \quad \text{for all } \delta \in \Delta.$$

The corresponding homomorphism $\theta_b : \Lambda \rightarrow \text{Aut}(\text{Bohr}(\Delta))$ defines an action of Λ on $\text{Bohr}(\Delta)$. By Theorem 1, we have

$$\text{Bohr}(\Delta) = \text{Bohr}(\Delta^{\text{Ab}})_0 \times \text{Prof}(\Delta).$$

The group Λ acts naturally on Δ^{Ab} and, by duality, on $\widehat{\Delta^{\text{Ab}}}$. Let

$$H := \widehat{\Delta^{\text{Ab}}}_{\Lambda\text{-fin}} \subset \widehat{\Delta^{\text{Ab}}}$$

be the subgroup of characters of Δ^{Ab} with finite Λ -orbits. Observe that H contains the torsion subgroup of $\widehat{\Delta^{\text{Ab}}}$.

Let

$$\alpha : \Lambda \rightarrow \text{Aut}(H)$$

be the homomorphism given by the action of Λ on H .

For a locally compact group G , the group $\text{Aut}(G)$ of continuous automorphisms of G will be endowed with the compact-open topology for which it is also a (not necessarily locally compact) topological group (see [HR79, (26.3)]).

• *First step.* We claim that the closure of $\alpha(\Lambda)$ in $\text{Aut}(H)$ is compact. Indeed, let us identify $\text{Aut}(H)$ with a subset of the product space H^H . The topology of $\text{Aut}(H)$ coincides with the topology induced by the product topology on H^H . Viewed this way, $\alpha(\Lambda)$ is a subspace of the product $\prod_{\chi \in H} \chi^\Lambda$ of the finite Λ -orbits χ^Λ . Since $\prod_{\chi \in H} \chi^\Lambda$ is compact and hence closed, the claim is proved.

Next, let N be the annihilator of H in $\text{Bohr}(\Delta^{\text{Ab}})$. Then N is Λ -invariant and the induced action of Λ on $\text{Bohr}(\Delta^{\text{Ab}})/N$ is a quotient of the action given by θ_b .

Let C be the connected component of $\text{Bohr}(\Delta^{\text{Ab}})/N$. Then C coincides with the image of $\text{Bohr}(\Delta^{\text{Ab}})_0$ in $\text{Bohr}(\Delta^{\text{Ab}})/N$ (see [Bou71, Chap. 3, §4, Corollaire 3]) and so

$$C \cong \text{Bohr}(\Delta^{\text{Ab}})_0 / (N \cap \text{Bohr}(\Delta^{\text{Ab}})_0).$$

Since C is invariant under Λ , we obtain an action of Λ on C ; let

$$\widehat{\alpha} : \Lambda \rightarrow \text{Aut}(C)$$

be the corresponding homomorphism.

• *Second step.* We claim that the action $\widehat{\alpha}$ of Λ on C extends to an action of $\text{Bohr}(\Lambda)$; more precisely, there exists a continuous homomorphism

$$\widehat{\alpha}' : \text{Bohr}(\Lambda) \rightarrow \text{Aut}(C)$$

such that the diagram

$$\begin{array}{ccc} & & \text{Bohr}(\Lambda) \\ & \nearrow \beta_\Lambda & \downarrow \widehat{\alpha}' \\ \Lambda & \xrightarrow{\widehat{\alpha}} & \text{Aut}(C) \end{array}$$

commutes. Indeed, by the first step, the closure K of $\alpha(\Lambda)$ in $\text{Aut}(H)$ is a compact group. Hence, by the universal property of $\text{Bohr}(\Lambda)$, there exists a continuous homomorphism

$$\alpha' : \text{Bohr}(\Lambda) \rightarrow K \subset \text{Aut}(H)$$

such that the diagram

$$\begin{array}{ccc} & & \text{Bohr}(\Lambda) \\ & \nearrow \beta_\Lambda & \downarrow \alpha' \\ \Lambda & \xrightarrow{\alpha} & \text{Aut}(H) \end{array}$$

commutes. Since $\widehat{H} = \text{Bohr}(\Delta^{\text{Ab}})/N$, we obtain by duality a continuous homomorphism $\widehat{\alpha}' : \text{Bohr}(\Lambda) \rightarrow \text{Aut}(\text{Bohr}(\Delta^{\text{Ab}})/N)$. The connected component C of $\text{Bohr}(\Delta^{\text{Ab}})/N$ is invariant under $\text{Bohr}(\Lambda)$. This proves the existence of the map $\widehat{\alpha}' : \text{Bohr}(\Lambda) \rightarrow \text{Aut}(C)$ with the claimed property.

Next, observe that, by the universal property of $\text{Prof}(\Delta)$, every element $\lambda \in \Lambda$ defines a continuous automorphism $\theta_p(\lambda)$ of $\text{Prof}(\Delta)$ such that

$$\theta_p(\lambda)(\delta) = \beta_\Delta(\lambda\delta\lambda^{-1}) \quad \text{for all } \delta \in \Delta.$$

The corresponding homomorphism $\theta_p : \Lambda \rightarrow \text{Aut}(\text{Prof}(\Delta))$ defines an action of Λ on $\text{Prof}(\Delta)$.

• *Third step.* We claim that the action θ_p of Λ on $\text{Prof}(\Delta)$ extends to an action of $\text{Bohr}(\Lambda)$; more precisely, there exists a homomorphism $\theta' : \text{Bohr}(\Lambda) \rightarrow \text{Aut}(\text{Prof}(\Delta))$ such that the diagram

$$\begin{array}{ccc} & & \text{Prof}(\Delta) \\ & \nearrow \beta_\Lambda & \downarrow \theta' \\ \Lambda & \xrightarrow{\theta_p} & \text{Aut}(\text{Prof}(\Delta)) \end{array}$$

commutes. Indeed, since Δ is finitely generated and since its image in $\text{Bohr}(\Delta)$ is dense, the profinite group $\text{Bohr}(\Delta)$ is finitely generated (that is, there exists a finite subset of $\text{Bohr}(\Delta)$ which generates a dense subgroup). This implies that $\text{Aut}(\text{Bohr}(\Delta))$ is a profinite

group (see [RZ00, Corollary 4.4.4]) and so there exists a homomorphism $\theta'_p : \text{Prof}(\Lambda) \rightarrow \text{Aut}(\text{Prof}(\Delta))$ such that $\theta'_p \circ \alpha_\Lambda = \theta_p$. We then lift θ'_p to a homomorphism $\theta' : \text{Bohr}(\Lambda) \rightarrow \text{Aut}(\text{Prof}(\Delta))$.

We set

$$Q := \text{Bohr}(\Delta)/(N \cap \text{Bohr}(\Delta^{\text{Ab}})_0) = C \times \text{Prof}(\Delta);$$

we have an action of Λ on Q given by the homomorphism

$$\widehat{\alpha} \oplus \theta_p : \Lambda \rightarrow \text{Aut}(C) \times \text{Aut}(\text{Prof}(\Delta)) \subset \text{Aut}(Q)$$

and, by the second and third step, an action of $\text{Bohr}(\Lambda)$ on Q given by

$$\widehat{\alpha}' \oplus \theta' : \text{Bohr}(\Lambda) \rightarrow \text{Aut}(C) \times \text{Aut}(\text{Prof}(\Delta))$$

such that the diagram

$$\begin{array}{ccc} & & \text{Bohr}(\Lambda) \\ & \nearrow \beta_\Lambda & \downarrow \widehat{\alpha}' \oplus \theta' \\ \Lambda & \xrightarrow{\widehat{\alpha} \oplus \theta_p} & \text{Aut}(C) \times \text{Aut}(\text{Prof}(\Delta)) \end{array}$$

commutes.

Let

$$B := (C \times \text{Prof}(\Delta)) \rtimes \text{Bohr}(\Lambda)$$

be the semi-direct product defined by $\widehat{\alpha}' \oplus \theta'$. Let

$$p : \text{Bohr}(\Delta) \rightarrow C = \text{Bohr}(\Delta^{\text{Ab}})_0/(N \cap \text{Bohr}(\Delta^{\text{Ab}})_0)$$

be the quotient epimorphism.

• *Fourth step.* We claim that B , together with the map $\beta : \Gamma \rightarrow B$, given by

$$\beta(\delta, \lambda) = (p(\beta_\Delta(\delta)), \beta_\Lambda(\lambda)) \quad \text{for all } (\delta, \lambda) \in \Gamma,$$

is a Bohr compactification for $\Gamma = \Delta \rtimes \Lambda$.

First, we have to check that β is a homomorphism with dense image. Since $p \circ \beta_\Delta$ and β_Λ are homomorphisms with dense image, it suffices to show that

$$\beta(\lambda\delta\lambda^{-1}, e) = ((\widehat{\alpha}' \oplus \theta')(\beta_\Lambda(\lambda))(p(\beta_\Delta(\delta)), e) \quad \text{for all } (\delta, \lambda) \in \Gamma.$$

This is indeed the case: since p is equivariant for the Λ -actions, we have

$$p(\beta_\Delta(\lambda\delta\lambda^{-1})) = p(\theta_b(\lambda)\beta_\Delta(\delta)) = (\widehat{\alpha}' \oplus \theta')(\beta_\Lambda(\lambda))p(\beta_\Delta(\delta)).$$

Next, let π be a unitary representation of Γ on a finite dimensional vector space V . By Proposition 5, we have to show that there exists a unitary representation $\tilde{\pi}$ of B on V such that $\pi = \tilde{\pi} \circ \beta$.

Consider a decomposition of $V = V_1 \oplus \cdots \oplus V_s$ into irreducible $\pi(\Delta)$ -invariant subspaces V_i ; denote by $\sigma_1, \dots, \sigma_s$ the corresponding irreducible representations of Δ . By Theorem 1, every σ_i is of the form $\sigma_i = \chi_i \otimes \rho_i$ for some $\chi_i \in \widehat{\Delta^{\text{Ab}}}$ and $\rho_i \in \widehat{\Delta}_{\text{finite}}$.

We decompose every χ_i as a product $\chi_i = \chi'_i \chi''_i$ with $\chi'_i \in \widehat{\Delta^{\text{Ab}}}$ of finite order and $\chi''_i \in \widehat{\Delta^{\text{Ab}}}$ of infinite order. Since χ'_i has finite image, upon replacing ρ_i by $\chi'_i \otimes \rho_i$, we may and will assume that every non trivial χ_i has infinite order.

Fix $i \in \{1, \dots, s\}$. We can extend χ_i and ρ_i to unitary representations of $\text{Bohr}(\Delta)$, that is, we can find representations $\tilde{\chi}_i$ and $\tilde{\rho}_i$ of $\text{Bohr}(\Delta)$ on V_i such that $\chi_i = \tilde{\chi}_i \circ \beta_\Delta$ and $\rho_i = \tilde{\rho}_i \circ \beta_\Delta$. By Proposition 12, the stabilizer Γ_{σ_i} of σ_i has finite index in Γ . It follows that the Λ -orbit of σ_i is finite, and this implies that $\chi_i \in H$; hence, $\tilde{\chi}_i$ factorizes through

$$C = \text{Bohr}(\Delta^{\text{Ab}})_0 / (N \cap \text{Bohr}(\Delta^{\text{Ab}})_0)$$

and we have $\chi_i = \tilde{\chi}_i \circ (p \circ \beta_\Delta)$. Since ρ_i has finite image, $\tilde{\rho}_i$ factorizes through $\text{Prof}(\Delta)$. So, $\tilde{\sigma}_i := \tilde{\chi}_i \otimes \tilde{\rho}_i$ is a unitary representation of $C \times \text{Prof}(\Delta)$ on V_i . Set

$$\tilde{\pi}_\Delta := \bigoplus_{i=1}^s \tilde{\sigma}_i.$$

Then $\tilde{\pi}_\Delta$ is a unitary representation of $C \times \text{Prof}(\Delta)$ on V such that $\pi|_\Delta = \tilde{\pi}_\Delta \circ (\beta|_\Delta)$.

On the other hand, since $\pi|_\Lambda$ is a finite dimensional representation of Λ , we can find a representation $\tilde{\pi}_\Lambda$ of $\text{Bohr}(\Lambda)$ on V such that $\pi|_\Lambda = \tilde{\pi}_\Lambda \circ (\beta|_\Lambda)$. For $\lambda \in \Lambda$ and $\delta \in \Delta$, we have

$$\begin{aligned} \tilde{\pi}_\Delta(\beta(\lambda)\beta(\delta)\beta(\lambda)^{-1}) &= \tilde{\pi}_\Delta(\beta(\lambda\delta\lambda)^{-1}) \\ &= \pi(\lambda\delta\lambda)^{-1} \\ &= \pi(\lambda)\pi(\delta)\pi(\lambda)^{-1} \\ &= \tilde{\pi}_\Lambda(\beta(\lambda))\tilde{\pi}_\Delta(\beta(\delta))\tilde{\pi}_\Lambda(\beta(\lambda))^{-1}. \end{aligned}$$

Since β has dense image in B , it follows that

$$\tilde{\pi}_\Delta(bab^{-1}) = \tilde{\pi}_\Lambda(b)\tilde{\pi}_\Delta(a)\tilde{\pi}_\Lambda(b)^{-1} \quad \text{for all } (a, b) \in B$$

and therefore the formula

$$\tilde{\pi}(a, b) = \tilde{\pi}_\Delta(a)\tilde{\pi}_\Lambda(b) \quad \text{for all } (a, b) \in B$$

defines a unitary representation of B on V such that $\pi = \tilde{\pi} \circ \beta$.

5. PROOF OF THEOREM 3

Recall that we are assuming that \mathbf{G} is a connected, simply-connected and almost \mathbf{Q} -simple algebraic group. The group \mathbf{G} can be obtained from an absolutely simple algebraic group \mathbf{H} by the so-called restriction of scalars; more precisely (see [BT65, 6.21, (ii)]), there exists a number field K and an absolutely simple algebraic group \mathbf{H} over K which is absolutely simple with the following property: \mathbf{G} can be written as (more precisely, is \mathbf{Q} -isomorphic to) the \mathbf{Q} -group $\mathbf{H}^{\sigma_1} \times \cdots \times \mathbf{H}^{\sigma_s}$, where the σ_i 's are the different (non conjugate) embeddings of K in \mathbf{C} . Assuming that $\sigma_1, \dots, \sigma_{r_1}$ are the embeddings such that $\sigma_i(K) \subset \mathbf{R}$, we can identify $\mathbf{G}(\mathbf{R})$ with

$$\mathbf{H}^{\sigma_1}(\mathbf{R}) \times \cdots \times \mathbf{H}^{\sigma_{r_1}}(\mathbf{R}) \times \mathbf{H}^{\sigma_{r_1+1}}(\mathbf{C}) \times \cdots \times \mathbf{H}^{\sigma_{r_s}}(\mathbf{C}).$$

Let \mathbf{G}_c be the product of the \mathbf{H}^{σ_i} 's for which $\mathbf{H}^{\sigma_i}(\mathbf{R})$ is compact.

We assume now that the real semisimple Lie group $\mathbf{G}(\mathbf{R})$ is not locally isomorphic to a group of the form $SO(m, 1) \times L$ or $SU(m, 1) \times L$ for a compact Lie group L . Let $\Gamma \subset \mathbf{G}(\mathbf{Q})$ be an arithmetic subgroup.

Set $K := \mathbf{G}_c(\mathbf{R}) \times \text{Prof}(\Gamma)$ and let $\beta : \Gamma \rightarrow K$ be defined by $\beta(\gamma) = (p(\gamma), \alpha(\gamma))$, where $p : \mathbf{G}(\mathbf{R}) \rightarrow \mathbf{G}_c(\mathbf{R})$ is the canonical projection and $\alpha : \Gamma \rightarrow \text{Prof}(\Gamma)$ the map associated to $\text{Prof}(\Gamma)$. We claim that (K, β) is a Bohr compactification of Γ ,

First, we show that $\beta(\Gamma)$ has dense image. Observe that $\mathbf{G}_c(\mathbf{R})$ is connected (see [Bor91, (24.6.c)]). By the Strong Approximation Theorem (see [PR94, Theorem 7.12]), $p(\mathbf{G}(\mathbf{Z}))$ is dense in $\mathbf{G}_c(\mathbf{R})$. Since $\mathbf{G}_c(\mathbf{R})$ is connected and since Γ is commensurable to $\mathbf{G}(\mathbf{Z})$, it follows that $p(\Gamma)$ is dense in $\mathbf{G}_c(\mathbf{R})$. Now, $\alpha(\Gamma)$ is dense in $\text{Prof}(\Gamma)$ and $\text{Prof}(\Gamma)$ is totally disconnected. As in the first step of the proof of Theorem 1, we conclude that $\beta(\Gamma)$ is dense in K .

Let $\pi : \Gamma \rightarrow U(n)$ be a finite dimensional unitary representation of Γ . Then, by Margulis' superrigidity theorem (see [Mar91, Chap. VIII, Theorem B]), [Mor15, Corollary 16.4.1]), there exists a continuous homomorphism $\rho_1 : \mathbf{G}(\mathbf{R}) \rightarrow U(n)$ and a homomorphism $\rho_2 : \Gamma \rightarrow U(n)$ such that

- (i) $\rho_2(\Gamma)$ is finite;
- (ii) $\rho_1(g)\rho_2(\gamma) = \rho_2(\gamma)\rho_1(g)$ for all $g \in \mathbf{G}(\mathbf{R})$ and $\gamma \in \Gamma$;
- (iii) $\pi(\gamma) = \rho_1(\gamma)\rho_2(\gamma)$ for all $\gamma \in \Gamma$.

By a classical result of Segal and von Neumann [SvN50], ρ_1 factorizes through $\mathbf{G}_c(\mathbf{R})$, that is, $\rho_1 = \rho'_1 \circ p$ for a unitary representation ρ'_1 of $\mathbf{G}_c(\mathbf{R})$. It follows from (i) that $\rho_2 = \rho'_2 \circ \alpha$ for a unitary representation ρ'_2 of $\text{Prof}(\Gamma)$. Moreover, (ii) and (iii) show that $\pi = (\rho_1|_\Gamma) \otimes \rho_2$. Hence,

$\pi = (\rho'_1 \otimes \rho'_2) \circ \otimes \beta$. We conclude by Proposition 5 that (K, β) is a Bohr compactification of Γ .

6. A FEW EXAMPLES

We compute the Bohr compactification for various examples of arithmetic groups.

- (1) For an integer $n \geq 1$, the $(2n+1)$ -dimensional Heisenberg group is the unipotent \mathbf{Q} -group \mathbf{H}_{2n+1} of matrices of the form

$$m(x_1, \dots, x_n, y_1, \dots, y_n, z) := \begin{pmatrix} 1 & x_1 & \dots & x_n & z \\ 0 & 1 & \dots & 0 & y_1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & y_n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

The arithmetic group $\Gamma = \mathbf{H}_{2n+1}(\mathbf{Z})$ is nilpotent of step 2; its commutator subgroup $[\Gamma, \Gamma]$ coincides with its center $\{m(0, 0, z) : z \in \mathbf{Z}\}$. So, $\Gamma^{\text{Ab}} \cong \mathbf{Z}^{2n}$. We have, by Theorem 1,

$$\text{Bohr}(\Gamma) \cong \text{Bohr}(\mathbf{Z}^{2n})_0 \times \text{Prof}(\Gamma)$$

and hence, by Proposition 11 and Proposition 16

$$\text{Bohr}(\Gamma) \cong \left(\prod_{\omega \in \mathfrak{c}} \mathbf{A}/\mathbf{Q} \right) \times \prod_{p \text{ prime}} \mathbf{H}_{2n+1}(\mathbf{Z}_p).$$

- (2) Let $\mathbf{G} = SL_n$ for $n \geq 3$ or $\mathbf{G} = Sp_{2n}$ for $n \geq 2$. Then $SL_n(\mathbf{Z})$ and $Sp_{2n}(\mathbf{Z})$ are non cocompact arithmetic lattices in $SL_n(\mathbf{R})$ and $Sp_{2n}(\mathbf{R})$, respectively. Hence, we have, by Corollary 4, $\text{Bohr}(SL_n(\mathbf{Z})) = \text{Prof}(SL_n(\mathbf{Z}))$ and $\text{Bohr}(Sp_{2n}(\mathbf{Z})) = \text{Prof}(Sp_{2n}(\mathbf{Z}))$. Since $SL_n(\mathbf{Z})$ and $Sp_{2n}(\mathbf{Z})$ have the congruence subgroup property, it follows that

$$\text{Bohr}(SL_n(\mathbf{Z})) \cong \prod_{p \text{ prime}} SL_n(\mathbf{Z}_p) \cong SL_n(\text{Prof}(\mathbf{Z}))$$

and similarly

$$\text{Bohr}(Sp_{2n}(\mathbf{Z})) \cong \prod_{p \text{ prime}} Sp_{2n}(\mathbf{Z}_p) \cong Sp_{2n}(\text{Prof}(\mathbf{Z})).$$

- (3) The group $\Gamma = SL_2(\mathbf{Z}[\sqrt{2}])$ embeds as a non cocompact arithmetic lattice of $SL_2(\mathbf{R}) \times SL_2(\mathbf{R})$. So, by Corollary 4, we have

$$\text{Bohr}(SL_2(\mathbf{Z}[\sqrt{2}])) \cong \text{Prof}(SL_2(\mathbf{Z}[\sqrt{2}])).$$

Moreover, since Γ has the congruence subgroup property (see [Ser70, Corollaire 3]), it follows that

$$\text{Bohr}(SL_2(\mathbf{Z}[\sqrt{2}])) \cong \text{Cong}(SL_2(\mathbf{Z}[\sqrt{2}])).$$

- (4) For $n \geq 4$, consider the quadratic form

$$q(x_1, \dots, x_n) = x_1^2 + \dots + x_{n-1}^2 - \sqrt{2}x_n^2 - \sqrt{2}x_{n+1}^2$$

The group $\mathbf{G} = SO(q)$ of unimodular $(n+1) \times (n+1)$ -matrices which preserve q is an almost simple algebraic group over the number field $\mathbf{Q}[\sqrt{2}]$. The subgroup $\Gamma = SO(q, \mathbf{Z}[\sqrt{2}])$ of $\mathbf{Z}[\sqrt{2}]$ -rational points in \mathbf{G} embeds as a cocompact lattice of the semisimple real Lie group $SO(n+1) \times SO(n-1, 2)$ via the map

$$SO(q, \mathbf{Q}[\sqrt{2}]) \rightarrow SO(n+1) \times SO(n-1, 2), \gamma \mapsto (\gamma^\sigma, \gamma),$$

where σ is the field automorphism of $\mathbf{Q}[\sqrt{2}]$ given by $\sigma(\sqrt{2}) = -\sqrt{2}$; so, $SO(n+1) \times SO(n-1, 2)$ is the group of real points of the \mathbf{Q} -group $R_{\mathbf{Q}[\sqrt{2}]/\mathbf{Q}}(\mathbf{G})$ obtained by restriction of scalars from the $\mathbf{Q}[\sqrt{2}]$ -group \mathbf{G} . Observe that $R_{\mathbf{Q}[\sqrt{2}]/\mathbf{Q}}(\mathbf{G})$ is almost $\mathbf{Q}[\sqrt{2}]$ -simple since \mathbf{G} is almost \mathbf{Q} -simple. By Theorem 3, we have

$$\text{Bohr}(SO(q, \mathbf{Z}[\sqrt{2}])) \cong SO(n+1) \times \text{Prof}(SO(q, \mathbf{Z}[\sqrt{2}])).$$

- (5) For $d \geq 2$, let D be a central division algebra over \mathbf{Q} such that $D \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic to the algebra $M_d(\mathbf{R})$ of real $d \times d$ -matrices. There exists a subring \mathcal{O} of D which is a \mathbf{Z} -lattice in D (a so-called order in D). There is an embedding $\varphi : D \rightarrow M_d(\mathbf{R})$ such that $\varphi(SL_1(D)) \subset SL_d(\mathbf{Q})$ and such that $\Gamma := \varphi(SL_1(\mathcal{O}))$ is an arithmetic cocompact lattice in $SL_d(\mathbf{R})$, where $SL_1(D)$ is the group of norm one elements in D ; for all this, see [Mor15, §6.8.i]. For $d \geq 2$, we have

$$\text{Bohr}(\Gamma) \cong \text{Prof}(\Gamma).$$

So, this is an example of a *cocompact* lattice Γ in a simple real Lie group for which there exists no homomorphism $\Gamma \rightarrow U(n)$ with infinite image; the existence of such examples was mentioned in [Mor15, (16.4.3)]

- (6) For $n \geq 3$, let Γ be the semi-direct product $\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})$, induced by the usual linear action of $SL_n(\mathbf{Z})$ on \mathbf{R}^n . The dual action of $SL_n(\mathbf{Z})$ on $\widehat{\mathbf{Z}^n} \cong \mathbf{R}^n/\mathbf{Z}^n$ is given by

$$SL_n(\mathbf{Z}) \times \mathbf{R}^n/\mathbf{Z}^n \rightarrow \mathbf{R}^n/\mathbf{Z}^n, (g, x + \mathbf{Z}^n) \mapsto {}^t gx + \mathbf{Z}^n.$$

It is well-known and easy to show that the subgroup of $SL_n(\mathbf{Z})$ -periodic orbits in $\widehat{\mathbf{Z}}^n$ corresponds to $\mathbf{Q}^n/\mathbf{Z}^n$, that is, to the characters of finite image. It follows from Theorem 2 that

$$\text{Bohr}(\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})) \cong \text{Bohr}(SL_n(\mathbf{Z}))_0 \times \text{Prof}(\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})).$$

For $n \geq 3$, we have therefore

$$\text{Bohr}(\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})) \cong \text{Prof}(\mathbf{Z}^n \rtimes SL_n(\mathbf{Z})) \cong \prod_{p \text{ prime}} \mathbf{Z}_p \rtimes SL_n(\mathbf{Z}_p).$$

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