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# Detailed derivation of the generalized Snell-Descartes laws from the Fermat's principle

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**Abstract:** Starting from the Fermat principle, we provide a detailed derivation of the generalized laws of refraction and reflection for a geometry realizing a metasurface. We first solve the Euler-Lagrange equations for a light ray propagating across the metasurface. The ray-path equation is found analytically and the results are supported by numerical calculations. We get generalized laws of refraction and reflection whose main features are: (i) they are relevant in gradient-index optics and in geometrical optics, (ii) a collection of rays emerge from the metasurface as a result of multiple reflections inside the metasurface, (iii) while deriving from Fermat's principle, they differ from the previously published results.

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## 1. Introduction

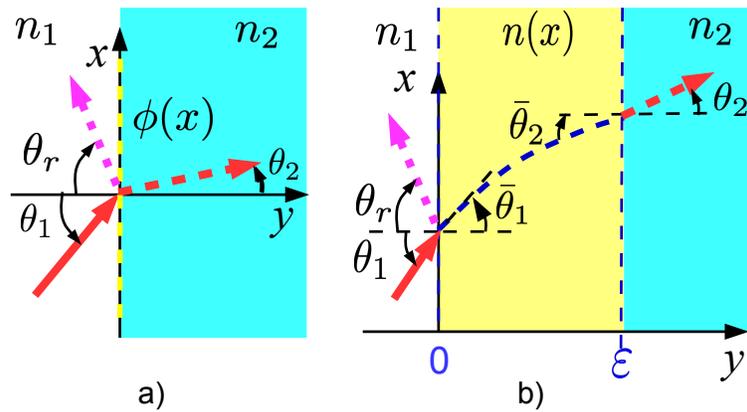


Fig. 1. a) Schematics of a metasurface (dashed yellow line) separating two media with refractive indices  $n_1$  and  $n_2$ . The incident light ray (plain red line) makes an angle  $\theta_1$  with the normal to the interface. The refracted ray (dashed red line) makes an angle  $\theta_2$  with the normal to the interface. The reflected ray (dashed magenta line) makes an angle  $\theta_r$  with the normal to the interface. b) Geometry of the metasurface considered in this article. The metasurface is the central layer (yellow) with refractive index  $n(x)$ . Its width is  $\varepsilon$ . The dashed blue line is the ray trajectory inside the metasurface. The angle between the ray trajectory and the normal to the interface at position  $y = 0$  is  $\theta(0) = \bar{\theta}_1$  and at position  $y = \varepsilon$  is  $\theta(\varepsilon) = \bar{\theta}_2$ . Other quantities are defined in a).

Metasurfaces were introduced in optics in reference [1] as optical devices enabling a control of light in terms of propagation [1, 2], phase [3] or polarisation [4]. The originality of metasurfaces over conventional materials or conventional optical devices is their thickness that is small as compared to the wavelength of light in vacuum. As a consequence, metasurfaces are thought of as just an interface acting on the incident light. Fig.(1-a) represents a metasurface (dashed yellow line) with an incident ray of light making the angle  $\theta_1$  with the normal to the interface between

two materials of refractive index  $n_1$  and  $n_2$ . Light can be reflected with the angle  $\theta_r$  or refracted with the angle  $\theta_2$ . The action of the metasurface on the light trajectory is described by a function  $\phi(x)$  called the phase shift [1]. A precise definition of this quantity is made in ref [5] in the specific situation where the metasurface is realized by diffractive objects. A noteworthy property of metasurfaces is that they lead to a generalization of the Snell-Descartes laws of refraction and reflection [1, 6]. The generalized Snell-Descartes laws for metasurfaces proposed in [1, 6] read as:

$$n_1 \sin \theta_r - n_1 \sin \theta_1 = \frac{1}{k_0} \frac{d}{dx} \phi(x) \quad (1)$$

$$n_2 \sin \theta_2 - n_1 \sin \theta_1 = \frac{1}{k_0} \frac{d}{dx} \phi(x) \quad (2)$$

Where  $\theta_1$  is the angle of incidence,  $\theta_2$  is the angle of refraction and  $\theta_r$  is the angle of reflection.  $k_0 = 2\pi/\lambda$  is the wavenumber in vacuum,  $\lambda$  being the wavelength of light in vacuum. We restrict the discussion to metasurfaces invariant in one direction. Then the discussion can be limited to the description of light trajectories in the plane of incidence. The 3D version of the generalized Snell-Descartes laws can be found in [6]. It is worth noting that the action of a metasurface on light propagation is described by a macroscopic quantity  $\phi(x)$  similarly to a refractive index that describes the propagation of light in a bulk medium. Metasurfaces can be considered as two-dimensional counter-parts of metamaterials [7].

Ref. [1] claims to derive the generalized Snell-Descartes laws from Fermat's principle. The authors also claimed an experimental proof of these laws in the mid-infrared while ref [2] claimed an experimental verification in the near-infrared. In both cases, the studied samples were made by diffractive nanostructures. As a consequence, the physical principle at the heart of their experimental studies does not rely on geometrical optics but on diffraction, a phenomenon that is not captured by Fermat's principle. In a previous article [5], we found that a phenomenological model describing their samples does not lead to eq.(1,2). Furthermore, we found that if the phase shift  $\phi(x)$  is designed to be periodic, as this is the case in [1, 2], then the light distribution is given by the grating equation. This last conclusion was strengthened in ref. [8] where a similar conclusion was reached based on the principles of Fourier optics. This reference even recommends avoiding the terminology "generalized Snell-Descartes Laws" when metasurfaces of diffractive nature are considered. Consequently, our goal is not to model metasurfaces made by diffractive nanostructures. We rather want to decide whether or not the set of equations (1,2) can be derived from Fermat's principle as it was stated in [1]. We think that such a proof is necessary since the belief that eq.(1,2) derive from Fermat's principle is widespread in the literature [4, 9–14].

The Fermat's principle states that the path followed by a light-ray is the one for which the optical path length, also called action  $\mathcal{S}$ , is extremal [15]. Assuming cartesian coordinates, the optical path-length  $\mathcal{S}$  is given by [16]:

$$\mathcal{S}[x(y)] = \int_{y_i}^{y_f} n_t(x, y) \sqrt{1 + \left[ \frac{d}{dy} x(y) \right]^2} dy \quad (3)$$

Where  $(y_i, x(y_i))$  [resp.  $(y_f, x(y_f))$ ] is the initial [resp. final] position of the light ray.  $n_t(x, y)$  is a position-dependent index of refraction. The Fermat principle applies to any medium  $n_t(x, y)$  but describes light propagation in the ray-optics approximation. As a consequence, it does not describe diffraction effects. Metasurfaces made by diffractive elements were considered in ref. [5], where it was shown that a model based on the Fraunhofer approximation does not lead to eq.(1-2). So it is worth asking if indeed eq.(1-2) can emerge from a model based on the Fermat principle only. In the present article, we consider such a model consistent with the assumptions of geometrical optics:

- The metasurface is made by a material with a refractive index  $n(x)$ , varying smoothly along the axis  $x$  on a scale larger than the wavelength of the incident light.
- It is surrounded by two homogeneous and isotropic media with refractive index  $n_1$  and  $n_2$ .
- The metasurface width is  $\varepsilon$ . At the end of the calculations, we make the thickness tends towards zero  $\varepsilon \rightarrow 0$ .

The corresponding metasurface is depicted in Fig.(1-b). The physics of this model is captured by the action  $S$  [eq.(3)] and the Fermat principle. We show in the following that it does not lead to the generalized Snell-Descartes laws [eq.(1-2)] found by [1, 6].

## 2. Ray trajectories in a metasurface

In this section we solve the equation describing the propagation of light rays in a medium with a refractive index  $n_t(x, y)$ . The rays are parametrized by the quantity  $y$ . The dynamical variables are the ray-coordinate  $x(y)$  and its derivative  $x'(y) = dx(y)/dy$ . A ray trajectory in the plane of incidence is the set of points  $(y, x(y))$  satisfying the equation of motion eq.(4) derived from the Fermat principle. Indeed the Fermat principle  $\delta S[x(y)] = 0$  leads to the following equation of motion [16, p.14] (Euler-Lagrange equation):

$$\frac{1}{\sqrt{1+x'(y)^2}} \frac{d}{dy} \left[ n_t(x, y) \frac{x'(y)}{\sqrt{1+x'(y)^2}} \right] = \frac{\partial}{\partial x} n_t(x, y) \quad (4)$$

$$x(0) = \bar{x}_1 \quad (5)$$

$$x'(0) = \tan \bar{\theta}_1 \quad (6)$$

The eq.(5-6) are the initial conditions at  $y = 0$ , the eq.(6) comes from the definition of the angle  $\theta(y)$  such as  $\tan \theta(y) = dx(y)/dy$ . This is the angle between the ray trajectory at position  $(y, x(y))$  and the horizontal axis.

First we consider a simple diopter separating two media with isotropic and homogeneous refractive indexes  $n_1$  and  $n_2$ , respectively. The refractive index  $n_t(x, y)$  describing this geometry reads:

$$n_t(x, y) = \begin{cases} n_1 & \forall y < 0 \\ n_2 & \forall y > 0 \end{cases} \quad (7)$$

The resulting equations of motions are the (usual) laws of refraction and reflection [17]:

$$\theta_r = \theta_1 \quad (8)$$

$$n_2 \sin \theta_2 = n_1 \sin \theta_1. \quad (9)$$

We now consider the metasurface geometry depicted in Fig.(1-b) with a refractive index  $n_t(x, y)$  that reads:

$$n_t(x, y) = \begin{cases} n_1 & \forall y < 0 \\ n(x) & y \in [0, \varepsilon] \\ n_2 & \forall y > \varepsilon \end{cases} \quad (10)$$

The metasurface starts at position  $y = 0$  and ends at position  $y = \varepsilon$ . It is bracketed by two uniform and homogeneous media with refractive indexes  $n_1$  and  $n_2$  respectively.

Assuming that  $\partial_y n_t(x, y) = 0$  inside the metasurface, we get the equation of motion for  $y \in [0, \varepsilon]$ :

$$x''(y) = \frac{1}{n(x)} \partial_x n(x) [1 + x'(y)^2] \quad (11)$$

The notations  $\partial_x = \partial/\partial x$  and  $\partial_y = \partial/\partial y$  are used as shorthand. The equation of motion eq.(11) is exact. No approximation was performed in obtaining it. In the limit  $\varepsilon \rightarrow 0$ , for  $y \in [0, \varepsilon]$ , the following equations

$$x'(y) \simeq \partial_x \{\ln n[x(y)]\} \Big|_{y=0} y + \tan \bar{\theta}_1 \quad (12)$$

$$x(y) \simeq \frac{1}{2} \partial_x \{\ln n[x(y)]\} \Big|_{y=0} y^2 + y \tan \bar{\theta}_1 + \bar{x}_1 \quad (13)$$

are approximate solutions of the equation (11) with the appropriate boundary conditions [eq.(5-6)]. See Supplement 1 for calculations details. Note that, in order to derive these solutions, the paraxial approximation is not assumed. Consequently, equations eq.(12-13) are valid for any initial angles  $\bar{\theta}_1$ . Furthermore, it is not assumed that  $n(x) - 1$  is a small quantity, only the first and second derivatives of  $n(x)$  should remain finite,  $\|\partial_x n(x)\|_\infty < \infty$  and  $\|\partial_x^2 n(x)\|_\infty < \infty$ .

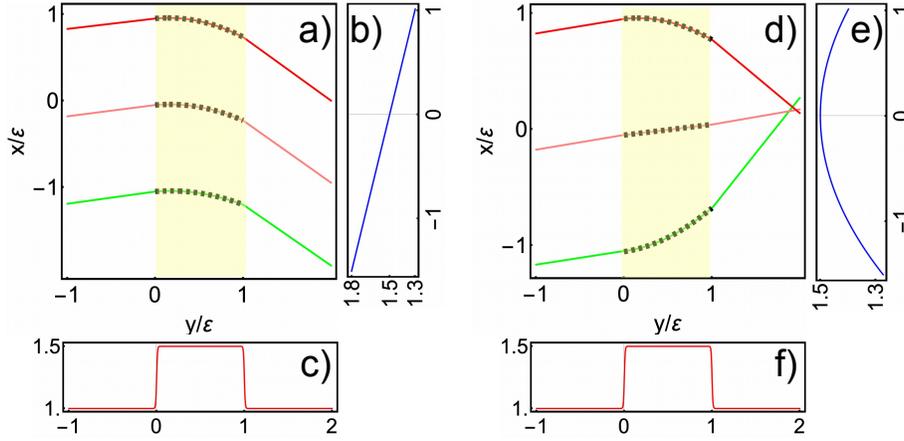


Fig. 2. Propagation of light rays inside a metasurface. a) Three light-rays impinging the metasurface with the same angle of incidence. The refractive index varies linearly, *i.e.*  $(\bar{k}\varepsilon)n(x) = 1 + \varphi_1 x$ . The dashed black lines are the analytical results given by the eq.(13). b) Profile of the refractive index  $n_t(x, 0)$  along the line  $y = 0$  in case a). c) Profile of the refractive index  $n_t(0, y)$  along the line  $x = 0$  in case a). d) Three light-rays impinging on the metasurface with the same angle of incidence. The refractive index varies quadratically, *i.e.*  $(\bar{k}\varepsilon)n(x) = 1 + \varphi_2 x^2$ . The dashed black lines are the analytical results given by the eq.(13). e) Profile of the refractive index  $n_t(x, 0)$  along the line  $y = 0$  in case d). f) Profile of the refractive index  $n_t(0, y)$  along the line  $x = 0$  in case d). The numerical calculations are made by assuming  $n_1 = n_2 = 1$ .

These solutions are compared with a numerical integration of eq.(4) for two different refractive index profiles. In Figures (2-a) and (2-d), the ray trajectories are shown for three different positions of impact on the metasurface (three different values of  $\bar{x}_1$ ). The angle of incidence  $\theta_1$  is

the same for the three rays. Fig.(2-a) considers a metasurface with a refractive index that varies linearly with the coordinate  $x$ , *i.e.*  $(\bar{k}\varepsilon)n(x) = 1 + \varphi_1 x$  with  $\varphi_1 = -0.4$  in the numerical example [the quantity  $\bar{k}$  is defined by the eq.(18)]. On the other hand, the Fig.(2-d) considers a metasurface with a refractive index that varies quadratically with the coordinate  $x$ , *i.e.*  $(\bar{k}\varepsilon)n(x) = 1 + \varphi_2 x^2$  with  $\varphi_2 = -0.2$  in the numerical example. In order to find the ray trajectories, we have solved numerically the equation of motion eq.(4). The equation (4) is form-invariant by the scale transformation  $\{x \rightarrow \varepsilon\bar{x}, y \rightarrow \varepsilon\bar{y}\}$ . So we can freely chose  $\varepsilon = 1$  in the numerical calculations. We also chose  $\bar{k} = 1$ . Changing the value of  $\bar{k}$  is equivalent to changing the coefficients  $\varphi_1$  or  $\varphi_2$ . A numerical solution for eq.(11) requires the smoothing of the refractive index profile  $n_t(x, y)$  [eq.(10)] near the metasurface boundaries. The profile of the refractive indexes along the  $y$ -axis are shown in Fig.(2-c) and Fig.(2-f). Despite the smoothing, the profile of the refractive index remains sharp near the metasurface boundaries. The profile of the refractive index along the coordinate  $x$  are shown in Fig.(2-b) for the linear case and in Fig.(2-e) for the quadratic case. For the numerical calculations, we consider that the metasurface is surrounded by a vacuum  $n_1 = n_2 = 1$ . The analytical predictions eq.(13) for the ray trajectories inside the metasurface are plotted as the black dashed lines. They superimpose accurately with the numerical calculations. The refractive index with a linear shape acts as a prism. It leads to a negative refraction. The refractive index with quadratic profile acts as a convergent lens. It leads to focusing since  $\varphi_2 < 0$ . These numerical calculations confirm that eq.(12) and eq.(13) correctly describes the propagation of a light ray inside the metasurface.

Since by definition,  $\tan \theta(y) = x'(y)$ , the equation (12) allows to find the angle between the ray trajectory and the horizontal axis at the position  $y = \varepsilon$ . It is denoted  $\bar{\theta}_2 = \theta(\varepsilon)$  and reads in first order in  $\varepsilon$ :

$$\bar{\theta}_2 = \bar{\theta}_1 + \cos^2 \bar{\theta}_1 \partial_x \{\ln n[x(y)]\} \Big|_{y=0} \varepsilon + o(\varepsilon) \quad (14)$$

### 3. Derivation of the generalized laws of refraction and reflection - First chapter

#### 3.1. The generalized law of refraction

In order to derive the generalization of the law of refraction for metasurfaces we are considering in this article, we need to connect the angles on both side of the metasurface, that is to say, the angle of incidence on the metasurface  $\theta_1$  and the angle of the ray refracted by the metasurface  $\theta_2$ . Then the derivation requires three steps:

1. Refraction at the interface  $y = 0$  between a medium of refractive  $n_1$  and a medium of refractive index  $n[x(0)]$ .
2. Propagation of the ray of light in a medium with refractive index  $n(x)$ . The equations of motion have been computed in the preceding section.
3. Refraction at the interface  $y = \varepsilon$  between a medium with refractive index  $n[x(\varepsilon)]$  and a medium with refractive index  $n_2$ .

At the interface  $y = 0$ , the angle between the ray of light and the normal to the boundary is the angle of incidence  $\theta(0^-) = \theta_1$ . The ray is refracted inside the metasurface with an angle  $\theta(0^+) = \bar{\theta}_1$ . These two angles are related by the (usual) law of refraction [eq.(9)]:

$$n_1 \sin \theta_1 = n[x(0)] \sin \bar{\theta}_1 \quad (15)$$

Similarly, we can connect the angles on both side of the interface  $y = \varepsilon$ . The relationship reads:

$$n[x(\varepsilon)] \sin \bar{\theta}_2 = n_2 \sin \theta_2, \quad (16)$$

with  $\bar{\theta}_2 = \theta(\varepsilon^-)$  and  $\theta_2 = \theta(\varepsilon^+)$ . Using the set of equations (14,15,16), the angle of incidence  $\theta_1$  can be connected to the angle  $\theta_2$  of emergence from the metasurface, through the relation:

$$n_2 \sin \theta_2 = n_1 \sin \theta_1 + \varepsilon \left\{ \cos^3 \bar{\theta}_1 \partial_x n[x(y)] \Big|_{y=0} + \tan \bar{\theta}_1 \sin \bar{\theta}_1 \partial_x n[x(y)] \Big|_{y=0} \right\} \quad (17)$$

This expression is valid up to first order in the metasurface's width  $\varepsilon$ . It shows that, in the limit  $\varepsilon \rightarrow 0$ , the metasurface acts only perturbatively on the propagation of light, except if the following condition is assumed:

$$\lim_{\varepsilon \rightarrow 0} (\bar{k}\varepsilon)n(x) = \phi(x). \quad (18)$$

Here  $\phi(x)$  is a function of order 0 in the metasurface width  $\varepsilon$ , *i.e.*  $\phi(x) = o(\varepsilon^0)$ , and  $\bar{k}$  is a quantity whose units are the inverse of a length.  $\phi(x)$  has no unit. The equation (18) is the fundamental relation for metasurfaces in geometrical optics. It implies that the metasurface acts on the light propagation. Only materials satisfying this equation lead to a modification of the law of refraction. It means that the optical path inside the metasurface has to remain finite as the metasurface width vanishes, *i.e.*  $\varepsilon n(x) = o(\varepsilon^0)$  when  $\varepsilon \rightarrow 0$ . This condition does not result from a physical principle. Indeed, the refractive index profile  $n(x)$  can be engineered to realize the desired function  $\phi(x)$  for a fixed (but small) metasurface width  $\varepsilon$ .

The assumption eq.(18) implies the following result based on eq.(15):

$$\bar{\theta}_1 = \frac{\bar{k}\varepsilon}{\phi[x(0)]} n_1 \sin \theta_1 + o(\varepsilon) \quad (19)$$

Since the refractive index of the metasurface  $n(x)$  becomes very large in the limit  $\varepsilon \rightarrow 0$ , all rays of light propagate inside the metasurface very close to the normal of the interface, *i.e.*  $\bar{\theta}_1 \rightarrow 0$ .

As a consequence, up to first order in  $\varepsilon$ , the angle at the output of the metasurface is given by the equation:

$$n_2 \sin \theta_2 = n_1 \sin \theta_1 + \frac{1}{\bar{k}} \partial_x \phi[x(y)] \Big|_{y=0} + o(\varepsilon). \quad (20)$$

Eq.(20) is the generalization of the law of refraction we are seeking for up to first order in the metasurface width. Since  $\phi(x) = o(\varepsilon^0)$ , the metasurface has an action on light propagation and the eq.(20) deviates significantly from the usual Snell-Descartes law. The quantity  $\bar{k}$  does not derive from the Fermat principle. It can be freely chosen and would result from the engineering of the quantity  $n(x)$ . The authors of the references [1,6] have chosen the wavenumber  $\bar{k} \equiv k_0$  but the laws of geometrical optics captured by the Fermat principle do not impose this choice.

Let us also stress that more general conditions than eq.(18) could be considered, also leading to a deviation from the usual Snell-Descartes laws. In such cases, the generalization of the Snell-Descartes laws could depend on the metasurface's width. For example the condition  $\lim_{\varepsilon \rightarrow 0} (\bar{k}\varepsilon)^s n(x) = \phi(x)$  with  $s \geq 1$  leads to a generalization of the law of refraction eq.(17) that varies as  $\varepsilon^{1-s}$  with the metasurfaces's width. We did not explore other choices than  $s = 1$ .

Instead, although fermat's principle does not impose this choice, we focus the discussion on the generalizations of the Snell-Descartes laws that are independent of the width of the metasurface's width Ref. [1, 6] and only considered the special case  $s = 1$ . As a consequence, depending on the engineering of the metasurface, the generalization of the law of refraction eq.(17) can strongly depart from the result published by [1, 6].

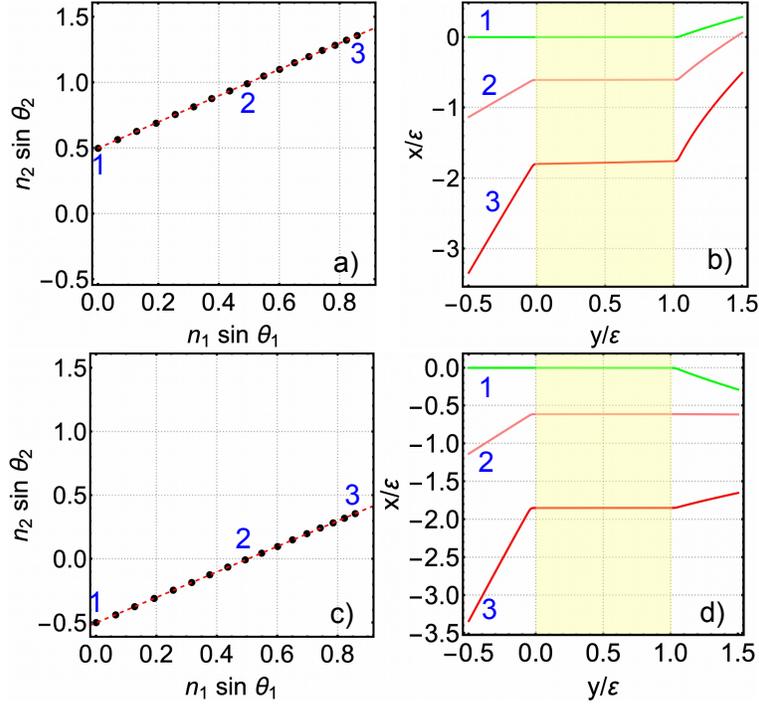


Fig. 3. Generalized law of refraction for a linear function  $\phi(x)$ . a)  $n_2 \sin \theta_2$  as a function of  $n_1 \sin \theta_1$  with  $\phi(x) = 0.5x$ . Dots are numerical results and the dashed red line is a linear fit. b) Ray trajectories corresponding to a). The number indicates the corresponding angles conditions. c)  $n_2 \sin \theta_2$  as a function of  $n_1 \sin \theta_1$  with  $\phi(x) = -0.5x$ . Dots are numerical results and the dashed red line is a linear fit. b) Ray trajectories corresponding to c). The number indicates the corresponding angles conditions.

The generalization of the law of refraction eq.(20) has been numerically checked by solving the eq.(4) with an index profile  $n_t(x, y)$  modeling a metasurface [eq.(10)]. The refracted angle  $\theta_2$  is computed with the help of  $\tan \theta(y) = x'(y)$  outside the metasurface that's to say for  $y > \epsilon$ . We have plotted in the Fig.(3) the quantity  $n_2 \sin \theta_2$  as a function of the quantity  $n_1 \sin \theta_1$ . In the computation, we have chosen  $n_1 = 1$  and  $n_2 = 2$ . The simulation assumes  $\epsilon = 0.01$  and  $\bar{k} = 1$ . The generalized law of refraction has been plotted for two functions  $\phi(x)$ , namely  $\phi(x) = 1 + 0.5x$  in Fig.(3-a) and  $\phi(x) = 1 - 0.5x$  in Fig.(3-c). In both case the generalized law of refraction eq.(20) is satisfied. The numerical results are the black dots in Fig.(3-a) and Fig.(3-c). They are fitted by a linear function with two parameters  $a$  and  $b$  such as  $n_2 \sin \theta_2 = a(n_1 \sin \theta_1) + b$ . In both case, the slope is  $a = 1.000$  and the vertical intercept is either  $b = 0.500$  in the case of Fig.(3-a) or  $b = -0.500$  in the case of Fig.(3-c). A sample of three trajectories are plotted in Fig.(3-b) and Fig.(3-d). They illustrate that the ray trajectories are very close to the horizontal axis inside the metasurface because of the eq.(19). The blue numbers 1, 2 or 3 in Fig(3-b) [resp. in Fig(3-d)] refer to condition  $(n_1 \sin \theta_1, n_2 \sin \theta_2)$  in Fig.(3-a) [resp. Fig.(3-c)]. Comparing Fig.(3-b) and

Fig.(3-d) shows the consequence of the refractive index profile inside the metasurface on the ray trajectories .

### 3.2. The generalized law of reflection

We now consider the ray of light that is reflected by the metasurface. This ray does not propagate inside the metasurface. This is illustrated by Fig.(1-b). The angle of incidence  $\theta_1$  and the angle of reflection  $\theta_r$  are related through the eq.(8) deriving from Fermat principle [17]:

$$\theta_r = \theta_1 \quad (21)$$

We conclude that the angle of reflection is still given by the usual Snell-Descartes law despite the presence of a metasurface. This result occurs because the ray of light does not propagate inside the metasurface. This is in strong contrast with the publication [6].

## 4. Derivation of the generalized laws of refraction and reflection - Second chapter: The whole story

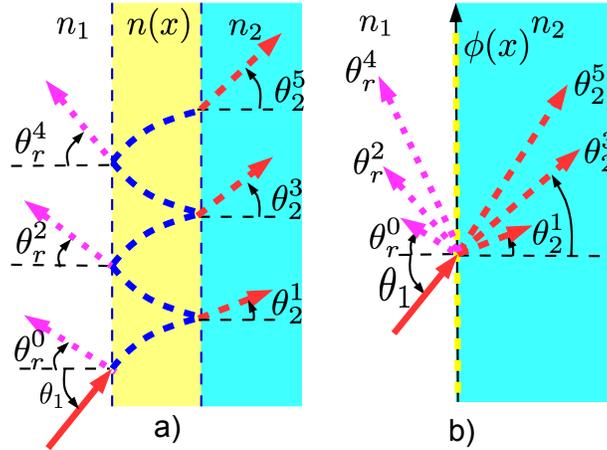


Fig. 4. a) Schematic of the propagation of light rays in a metasurface illustrating a few possible paths that a ray of light can follow before escaping from the metasurface. An incident light ray falls upon the metasurface with an angle of incidence  $\theta_1$ . After propagation inside the metasurface, it is reflected with the angle  $\theta_r^{2p}$  or refracted with the angle  $\theta_2^{2p+1}$ , with  $p \in \mathbb{N}$ . The superscript indicates the number of trips the ray makes before leaving the metasurface. b) Same as in a) in the limit  $\varepsilon \rightarrow 0$ .

The equation (20) is similar to previously published results apart from the quantity  $\bar{k}$  that differs from them. On the other hand, the eq.(21) differs completely. Nevertheless none of these two formulas are the complete story describing the propagation of a light beam across a metasurface. Indeed, when a light beam propagates in transparent materials, at the boundary between two transparent materials, a fraction of light is reflected and a fraction of light is refracted. When considering the propagation of a ray of light across the metasurface depicted in fig.1-b), one need to consider that the ray of light can be either reflected or transmitted at each of the interfaces between the metasurface and the materials with refractive index  $n_1$  or  $n_2$ . This leads to several possible trajectories for an incident light ray as shown by the Fig.4-a). Consequently the picture that emerges is that of a ray undergoing multiple internal reflections before leaving the metasurface either in the materials with refractive index  $n_1$  (the ray is said to be reflected by the

metasurface) or in the materials with refractive index  $n_2$  (the ray is said to be refracted by the metasurface). The angles of reflection or refraction depend on the whole trajectory inside the metasurface, particularly on the total distance the ray has travelled inside the metasurface. From geometrical optics principle, the probability for a light ray to follow a prescribed path cannot be computed. On the other hand, one can compute all the possible angles of refraction  $\theta_2^p$  and all the possible angles of reflection  $\theta_r^p$  with  $p \in \mathbb{N}^*$ ,  $p$  being the number of trips done by the ray of light inside the metasurface.

The equation (20) assumes that the ray of light is first refracted at the boundary  $y = 0$ , propagates to the second boundary located at coordinate  $y = \varepsilon$  and is finally refracted to the medium with refractive index  $n_2$ . It does one trip inside the metasurface. Now we consider a similar process with refraction at boundary  $y = 0$ ,  $(2p + 1)$  trips inside the metasurface (that is to say  $2p$  internal reflections with  $p \in \mathbb{N}$ ) and finally a refraction at boundary  $y = \varepsilon$ . The ray of light leaves the metasurface into the medium with refractive index  $n_2$ . Thanks to the symmetry properties implied by the law of reflection, (*i.e.* the angle of reflection equals the angle of incidence [eq.(8)]), the angle before the ray leaves the metasurface and the coordinate of the impact at the metasurface boundaries can be computed with the help of the set of equations (12) and (13), respectively. They reads:

$$\begin{aligned}\bar{\theta}_2^{2p+1} &\simeq \partial_x \{\ln n[x(y)]\} \Big|_{y=-\varepsilon} (2p+1)\varepsilon + \bar{\theta}_1 \\ x^{2p+1} &\simeq \frac{1}{2} \partial_x \{\ln n[x(y)]\} \Big|_{y=-\varepsilon} (2p+1)^2 \varepsilon^2 + \bar{\theta}_1 (2p+1)\varepsilon + \bar{x}_1\end{aligned}$$

These equations assume  $\bar{\theta}_1 = o(\varepsilon)$  as proved previously. Inserting these equations into eq.(17) with the condition given by eq.(18) leads to the following set of angles of refraction:

$$n_2 \sin \theta_2^{2p+1} = n_1 \sin \theta_1 + (2p+1) \frac{1}{k} \partial_x \phi[x(y)] \Big|_{y=0} \text{ with } p \in \mathbb{N}. \quad (22)$$

The limit  $\varepsilon \rightarrow 0$  has been taken. In the first order of  $\varepsilon$ , all the coordinates of impact  $x^{2p+1}$  collapse to the same value  $x^{2p+1} = \bar{x}_1 + O(\varepsilon)$ . The metasurface can be seen as an optical device inducing an optical path  $\phi(x)/\bar{k}$  from which several rays emerge. This is illustrated by the Fig.(4-b). The case  $p = 0$  leads to the eq.(20).

Similarly, a set of rays are reflected by the metasurface [see Fig.(4) for an illustration]. They perform  $2p$  trips,  $p \in \mathbb{N}$ , before leaving the metasurface into the medium with refractive index  $n_1$ . The angles of reflection are given by:

$$n_1 \sin \theta_r^{2p} = n_1 \sin \theta_1 + 2p \frac{1}{k} \partial_x \phi[x(y)] \Big|_{y=0} \text{ with } p \in \mathbb{N} \quad (23)$$

The case  $p = 0$  obviously leads to the eq.(21). We recall that the only assumption is  $\varepsilon \rightarrow 0$ . As a consequence the eq.(22-23) are valid for any value of the incident angle  $\theta_1$  and for any refractive index profile satisfying the relation  $(\bar{k}\varepsilon)n(x) = \phi(x)$ .

## 5. Conclusion

We have studied in details the propagation of optical rays through a metasurface. The derivation is performed in the framework of geometrical optics and is based on Fermat's principle. We have found that the relation (18) is required otherwise the metasurface only slightly affects the ray light propagation. This relation states that the optical path length should remain finite despite the vanishing of the metasurface width. The condition (18) leads to a generalization of the laws

of refraction and reflection that differ from the usual Snell-Descartes laws. The main result is that a collection of rays should emerge from the metasurface. The set of angles of refraction and reflection are given by the eq.(22) and eq.(23) respectively. These equations differ from the results published in [1, 6] proving that the Fermat principle cannot be invoked for their derivation.

In the article [5], we have studied the case where the metasurface is made by a collection of scatterers and consider a microscopic model based on diffraction and interferences. Such a model, close to the experimental realizations, does not lead to the generalized Snell-Descartes laws either. Our conclusion was strengthened by the recent study [8] where a similar conclusion was reached for diffractive metasurfaces with a sawtooth phase-profile. As a consequence, we conclude that eq.(1,2) given in [1, 6] are not compatible with the theoretical framework given either by Fermat's principle or by Fourier optics.

## 6. Backmatter

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**Disclosures.** The authors declare no conflicts of interest.

**Data Availability Statement.** Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the authors upon reasonable request.

**Supplemental document.** See Supplement 1 for supporting content.

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# A detailed derivation of the generalized Snell-Descartes laws from the Fermat principle: Supplemental document

**Abstract:** This file contains the calculations to derive the trajectory of a ray of light  $x(y)$  in a metasurface with refractive index  $n(x)$  as well as its derivative  $x'(y) = dx(y)/dy$ . They are equations (12) and (13) in the primary article.

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## A. Derivation of the equation (12)

This appendix aims at finding the eq.(12) in the primary article, that's to say at solving the equation

$$x''(y) = \frac{1}{n(x)} \partial_x n(x) [1 + x'(y)^2] \quad (24)$$

As an ansatz, we propose the following solution valid for all  $y \in [0, \varepsilon]$ , in the limit where  $\varepsilon \rightarrow 0$ :

$$x'(y) = \tan \left[ \frac{1}{n[x(y)]} \partial_x n[x(y)] y + c_1 \right] \quad (25)$$

Where  $c_1$  is a constant to be determined.

Injecting this solution in the eq.(24), we get:

$$x''(y) - \frac{1}{n(x)} \partial_x n(x) [1 + x'(y)^2] = y \partial_x^2 \ln\{n[x(y)]\} x'(y)$$

We now perform the following assumption:

- The quantities  $\partial_x n(x)$  and  $\partial_x^2 n(x)$  do not diverge in the range  $y \in [0, \varepsilon]$ . This assumption assumes smooth variations of the refractive index profile inside the metasurface.

Consequently from this assumption, we deduce the following results:

- The quantity  $\partial_x \ln[n(x)] = \frac{1}{n(x)} \partial_x n(x)$  is bounded in the range  $y \in [0, \varepsilon]$ . As a consequence, for  $y \in [0, \varepsilon]$ ,  $\|x'(y)\|_\infty = \left\| \tan \left[ \frac{1}{n[x(y)]} \partial_x n[x(y)] y + c_1 \right] \right\|_\infty < \infty$
- The quantity  $\partial_x^2 \ln\{n[x(y)]\}$  is bounded and we denote by  $M$  its supremum value  $M = \sup_{y \in [0, \varepsilon]} \{\partial_x^2 \ln\{n[x(y)]\}\}$

As a consequence, we get the following inequality:

$$\|y \partial_x^2 \ln\{n[x(y)]\} x'(y)\|_\infty \leq \varepsilon M \|x'(y)\|_\infty \quad (26)$$

We have chosen the supremum norm  $\|\bullet\|_\infty$  as the norm for functions. In the limit  $\varepsilon \rightarrow 0$ , we find that

$$\|y \partial_x^2 \ln\{n[x(y)]\} x'(y)\|_\infty \xrightarrow{\varepsilon \rightarrow 0} 0$$

Eq.(25) is a solution of the equation (24) for metasurfaces, *i.e.* for geometries assuming  $\varepsilon \rightarrow 0$ . Note that this equation does not assume that  $\partial_x \ln\{n[x(y)]\}$  is a small quantity. It only assumes that it is finite. Also it does not assume the paraxial approximation  $|x'(y)| \ll 1$ . In the range  $y \in [0, \varepsilon]$ , we can expand  $\partial_x \ln\{n[x(y)]\}$  in series as well as the eq.(25) up to first order in  $y$  around  $y = 0$ . Restricting to first order, we get

$$x'(y) = \partial_x \{\ln n[x(y)]\} \Big|_{y=0} y + \tan \bar{\theta}_1 + o(\varepsilon)$$

This is the eq.(12) in the article with the appropriate boundary conditions.

### **B. Derivation of the equation (13)**

The light-ray position  $x(y)$  as a function of the parameter  $y \in [0, \varepsilon]$  can be obtained by integrating the eq.(12) since  $y$  remains small. The solution reads:

$$x(y) = \frac{1}{2} \partial_x \{\ln n[x(y)]\} \Big|_{y=0} y^2 + \tan \bar{\theta}_1 y + \bar{x}_1 + o(\varepsilon^2)$$

The solution is valid for metasurfaces assuming that  $y \in [0, \varepsilon]$  with  $\varepsilon \rightarrow 0$ . We do not make any assumptions on the value of the refractive index  $n(x)$  or on the value of the initial angle  $\bar{\theta}_1$ .