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Didier Dubois, Henri Prade

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Didier Dubois and Henri Prade

Institut de Recherche en Informatique de Toulouse, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex, France

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Abstract: This short note improves a result of Roubens and Vincke on possibility graphs, i.e. valued ordering relations defined on a finite set of fuzzy numbers, using possibility theory. Especially, the Ferrers and min–max transitivity property for possibility graphs are obtained as a straightforward consequence of an inference rule in possibilistic logic.

Keywords: Possibility graphs; Ferrers property; min–max transitivity.

Introduction

Roubens and Vincke [7] have recently studied the properties of 'possibility graphs' obtained from pairwise comparisons of fuzzy intervals [2]; these comparisons are performed by means of an index originally proposed by Baas and Kwakernaak [1]. This index is in agreement with possibility theory (Zadeh [9], Dubois and Prade [3]). It is proved in [7] that possibility graphs are a many-valued extension of interval graphs [6]. This note extends these results to the case when the fuzzy quantities to be compared are not convex, as a simple application of possibilistic logic [4]. The first section presents background on possibility graphs and possibilistic logic. Section 2 presents the main result, i.e. a new proof of the Ferrers property for possibility graphs defined on general fuzzy quantities. Section 3 briefly discusses the case when the usual ordering on the real line becomes fuzzy.

1. Background

Some basic notions and results are recalled here, especially possibilistic logic [4] which is derived from studies in automated reasoning, and possibility graphs [7] which are useful to investigate the properties of ordering relations between fuzzy intervals.

1.1. *Possibilistic logic*

A possibility measure Π on a finite Boolean algebra of proposition $\mathcal{B}$ is a mapping from $\mathcal{B}$ to $[0, 1]$ such that

$$\Pi(\mathbf{0}) = 0, \quad \Pi(\mathbf{1}) = 1$$
$$\Pi(p \vee q) = \max(\Pi(p), \Pi(q)), \quad \forall p, q \in \mathcal{B}, \tag{1}$$

where $\mathbf{0}$ is the contradiction, $\mathbf{1}$ the tautology, and $\vee$ the disjunction. The set-function N defined by $N(p) = 1 - \Pi(\neg p)$, where $\neg$ denotes negation, is called a necessity measure. Informally, $\Pi(p)$ expresses to what extent p is possibly true and $N(p)$ to what extent p is certainly true. The following conventions are adopted:

$N(p) = 1 \Leftrightarrow \Pi(\neg p) = 0$ means that p is true,

$\Pi(p) = 0 \Leftrightarrow N(\neg p) = 1$ means that p is false,

$\Pi(p) = \Pi(\neg p) = 1$ means total ignorance about the truth of p.

The well-known modus ponens rule extends to possibilistic logic under the form [4]

$$N(q) \geq \min(N(p \to q), N(p)), \tag{2}$$

where $p \to q$ means $\neg p \vee q$, p and q being any Boolean propositions. It is a consequence of (1), noticing that $N(p \wedge q) = \min(N(p), N(q))$, $\forall p, q$. Stronger inference rules such as the resolution principle can be extended as well [4]. Note that (2) is equivalent to

$$\Pi(\neg q) \leq \max(\Pi(p \wedge \neg q), \Pi(\neg p)). \tag{3}$$

Viewing the Boolean algebra as the set of subsets of a set X, each proposition $p \in \mathcal{B}$ corresponds to a subset $M(p) \subseteq X$ such that for all $x \in X$, p is true if and only if $x \in M(p)$.

Let π be a possibility distribution on X, i.e. a function $X \to [0, 1]$ with $\pi(x) = 1$ for some x. A possibility measure Π on $\mathcal{B}$ is generated by π through the equation

$$\forall p \in \mathcal{B}, \quad \Pi(p) = \max\{\pi(x) \mid x \in M(p)\}. \tag{4}$$

When $\mathcal{B}$ is not finite axiom (1) is extended to the union of any indexed family of propositions, changing max into sup.

Note that generally $\Pi(p \wedge q) \leq \min(\Pi(p), \Pi(q))$ and $N(p \vee q) \geq \max(N(p), N(q))$ and the equalities may not hold. The inequalities turn into equalities when X is a Cartesian product $X_1 \times X_2$, π is of the form $\min(\pi_1, \pi_2)$ where π_i is a possibility distribution on X_i, $i = 1, 2$, and for propositions p_1 and p_2 such that $M(p_1) = \{A_1\} \times X_2$, $M(p_2) = X_1 \times \{A_2\}$, where A_i is a subset of X_i, $i = 1, 2$. Then

$$\Pi(p_1 \wedge p_2) = \min(\Pi(p_1), \Pi(p_2)),$$
$$N(p_1 \vee p_2) = \max(N(p_1), N(p_2)).$$

In this case π_1 and π_2 are said to be non-interactive and π is said to be decomposable [8, 3].

1.2. *Possibility graphs*

A fuzzy interval A is a fuzzy subset of the real line defined by a membership function μ_A that satisfies

$$\text{Normality:} \quad \exists x \in \mathbb{R}, \quad \mu_A(x) = 1, \tag{5}$$
$$\text{Convexity:} \quad x \leq y \leq z \Rightarrow \mu_A(y) \geq \min(\mu_A(x), \mu_A(z)). \tag{6}$$

In other words, any level-cut $A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}$ is an interval, and A_α is not empty for $\alpha = 1$. μ_A can be viewed as the fuzzy set of possible values of a variable a, i.e. a possibility distribution [3, 9].

Let $\Omega = \{A, B, \ldots\}$ be a finite set of fuzzy intervals describing the ranges of non-interactive (i.e. unrelated) variables $a, b, \ldots$. A possibility graph $G_{\geq}$ on Ω is a fuzzy relation on Ω defined by the possibility measure $\Pi(\cdot \geq \cdot)$:

$$\Pi(a \geq b) = \sup_{x \geq y} \min(\mu_A(x), \mu_B(y)). \tag{7}$$

Index $\Pi(a \geq b)$, introduced by Baas and Kwakernaak [1] for comparing fuzzy intervals, measures to what extent it is possible that a variable a (fuzzily restricted by A) is at least as great as a variable b (fuzzily restricted by B), in the sense of possibility theory [3]. Indeed it evaluates the degree $\Pi(a \geq b)$ that the proposition $a \geq b$ holds true, where the possibility distribution of the pair (a, b) is $\min(\mu_A, \mu_B)$. It assumes that a and b are unrelated variables [8]. Clearly the fuzzy relation defined by (7) results from applying the extension principle [8, 3] to the usual ordering $\geq$.

The properties of the possibility graph $G_{\geq}$ are

(i) $G_{\geq}$ is reflexive, i.e. if $A = B$, then $\Pi(a \geq b) = 1$;

(ii) $G_{\geq}$ is complete, i.e. $\max(\Pi(a \geq b), \Pi(b \geq a)) = 1$;

(iii) $G_{\geq}$ is a Ferrers graph, i.e. $\forall a, b, c, d$, $\min(\Pi(a \geq b), \Pi(c \geq d)) \leq \max(\Pi(a \geq d), \Pi(c \geq b))$;

(iv) $G_{\geq}$ is min–max transitive, i.e. $\forall a, b, c$, $\Pi(a \geq b) \leq \max(\Pi(a \geq c), \Pi(c \geq b))$.

In Roubens and Vincke [7], $\Pi(a \geq b)$ is denoted by $\mu(A, B)$, i.e. involving the names of the fuzzy intervals instead of their underlying variables. This notation is slightly ambiguous when $A = B$ since it does not imply $a = b$. For instance expressing the reflexivity property as $\mu(A, A) = 1$ is not satisfactory because it may mean only the trivial equality $\Pi(a \geq a) = 1$ instead of (i). Fortunately, $\Pi(a \geq b) = \Pi(b \geq a)$ whenever $A = B$. It is interesting to consider the companion index

$$N(a \geq b) = \inf_{x < y} [1 - \min(\mu_A(x), \mu_B(y))] = 1 - \Pi(a < b)$$

for further comments. Namely, if $A = B$, then $N(a \geq b) < 1$ as soon as A is not a real number. On the contrary, $N(a \geq a) = 1$ trivially holds because $a \geq a$ is always true. This distinction is more difficult to perceive when the possibilistic index is denoted by $\mu(A, B)$.

Properties (iii) and (iv) are established by Roubens and Vincke [7]. The

transitivity property is actually a special case of the Ferrers property, letting $d = c$ in the expression of the latter.

Lastly, letting $h(A, B)$ be the height of the intersection of A and B obtained by changing the inequality $\geq$ into an equality in (7), it is well known that if A and B are convex in the sense of (6), then

$$h(A, B) = \min(\Pi(a \geq b), \Pi(b \geq a)). \tag{8}$$

2. The main result

The proof of the Ferrers graph property in [7] explicitly uses the convexity assumption (6). This note indicates that this assumption can be disposed of, and proposes a new, simpler proof of this property, when the fuzzy sets $A, B, \ldots$ are defined on a general ordered space.

Let X be a set equipped with a total ordering $\geq$ ($\geq$ is reflexive, transitive, antisymmetric and complete). Let Ω be a set of fuzzy sets on X, where the only assumption is that $\forall A_i \in \Omega$, A_i possesses the normality property (5). Let $G_{\geq}$ be a possibility graph defined on Ω via (7). It is easy to check that $G_{\geq}$ is reflexive and complete. Moreover, the following result holds.

Theorem. *A possibility graph induced from normal fuzzy sets defined on a finite set X with an antisymmetric linear ordering has the Ferrers property.*

The proof uses the following obvious lemma.

Lemma. *Let $<$ be the strict ordering relation based on $\geq$, i.e. $x < y$ if and only if $y \geq x$ but not $x \geq y$. Then*

$$(x < t \text{ and } z < y) \text{ implies } (z < t \text{ or } x < y).$$

Proof. It is obvious. For instance if $x < t$, $z < y$ and $t \leq z$, then $x < t \leq z < y$, i.e. $x < y$. □

Proof of theorem. Let a, b, c, d be the variables fuzzily restricted by A, B, C, D, respectively. Let p, q, r, s be propositions modeling the statements $a < d$, $c < b$, $a < b$ and $c < d$, respectively. From the lemma, the proposition $(p \wedge q) \rightarrow (r \vee s)$ is always true; hence for any possibility distribution π on X^4, the degree of necessity

$$N((p \wedge q) \rightarrow (r \vee s)) = 1.$$

Hence, applying the inference rule (2) in possibilistic logic, we obtain

$$N(p \wedge q) \leq N(r \vee s), \quad \forall \pi \text{ on } X^4. \tag{9}$$

Now $N(p \wedge q) = \min(N(p), N(q))$ by definition of necessity measures, and due to (4),

$$\begin{aligned} N(r \vee s) &= 1 - \Pi(\neg r \wedge \neg s) \\ &= \inf_{x \geq y \text{ and } z \geq t} [1 - \min(\mu_A(x), \mu_B(y), \mu_C(z), \mu_D(t))] \\ &= \max(N(r), N(s)) \end{aligned}$$

because $\pi = \min(\mu_A, \mu_B, \mu_C, \mu_D)$ is decomposable (see the end of Section 1.1). Substituting these values in (9), noticing that $N(p) = 1 - \Pi(\neg p) = 1 - \Pi(a \geq d)$, etc., leads exactly to the Ferrers inequality. □

Corollary. *$G_{\geq}$ is a min–max transitive fuzzy relation.*

3. When the usual ordering becomes fuzzy

We might consider the same problem as above, turning the ordering on X into a fuzzy relation R. This type of generalization makes sense for instance in temporal reasoning, when linguistic temporal relations must be modeled (such as 'much before', 'a little after', ...) (see Dubois and Prade [5]). More generally, R may model concepts such as a 'much larger than', 'almost equal to', 'very different from', etc.

The quantity $\Pi(a \geq b)$ is extended into $\Pi(a\, R\, b)$ such that

$$\Pi(a\, R\, b) = \sup_{x,y} \min(\mu_R(x, y), \mu_A(x), \mu_B(y)). \tag{10}$$

The corresponding degree of necessity $N(a\, R\, b)$ is defined by

$$N(a\, R\, b) = 1 - \Pi(a \text{ not } R\, b) = \inf_{x,y} \max(\mu_R(x, y), 1 - \mu_A(x), 1 - \mu_B(y)), \tag{11}$$

where relation not R has membership function $1 - \mu_R$. Equations (10) and (11) make sense for any fuzzy relation R on X.

Given two relations R and S on X, the following result has been proved [5]:

$$N(a\, [R \circ S]\, b) \geq \min(N(a\, R\, c), N(c\, S\, b)). \tag{12}$$

where $R \circ S$ is the sup–min composition of fuzzy relations

$$\mu_{R \circ S}(x, z) = \sup_y \min(\mu_R(x, y), \mu_S(y, z)). \tag{13}$$

The proof, not recalled here for the sake of brevity (see [5]), is direct and cannot take advantage of possibilistic logic because the property deals with fuzzy propositions such as $a\, R\, b$, etc., while possibilistic logic as described in Section 1 only deals with Boolean propositions. The inequality (12) holds for any pair (R, S) and is equivalent to

$$\Pi(a\, R \circ S\, b) \leq \max(\Pi(a\, R\, c), \Pi(c\, S\, b)), \tag{14}$$

for all R's and S's, where $\bar{\circ}$ is the inf–max composition of fuzzy relations obtained by changing max into min and sup into inf in (13). The min–max transitivity property would be attained in the case when $\mu_{R\circ S} \leq \mu_{R\bar{\circ}S}$. In fact, generally the inequality

$$\sup_z \min(\mu_R(x, z), \mu_S(z, y)) \leq \inf_{z'} \max(\mu_R(x, z'), \mu_S(z', y)) \tag{15}$$

does not hold. However when $R = S$ and means $\geq$ on X, where $\geq$ is transitive and complete, (15) is true and means

$$\text{if } \exists z,\ x \geq z \geq y \quad \text{then} \quad \forall z',\ x \geq z' \text{ or } z' \geq y.$$

In order to establish the transitivity property of the possibility graph induced by a fuzzy relation, we need the following lemma.

Lemma. *Let R be a fuzzy relation on X which is complete (i.e. $\forall x, \forall y, \mu_R(x, y) = 1$ or $\mu_R(y, x) = 1$) and max–min transitive, then R is also min–max transitive, i.e. $\max(\mu_R(x, z'), \mu_R(z', y)) \geq \mu_R(x, y)$, $\forall x, \forall y, \forall z'$.*

Proof. The max–min transitivity of R writes

$$\mu_R(x, y) \geq \min(\mu_R(x, z), \mu_R(z, y)).$$

Let us assume that $\exists z'$, $\mu_R(x, z') < \mu_R(x, y)$ and $\mu_R(z', y) < \mu_R(x, y)$. Then $\max(\mu_R(x, z'), \mu_R(z', y)) < 1$ and due to completeness, $\mu_R(z', x) = \mu_R(y, z') = 1$; but $\mu_R(z', y) \geq \min(\mu_R(z', x), \mu_R(x, y)) = \mu_R(x, y)$. This is a contradiction. Hence $\max(\mu_R(x, z'), \mu_R(z', y)) \geq \mu_R(x, y)$. □

As a consequence, inequality (15) holds for $R = S$, and it is true that $\mu_{R\circ R} \leq \mu_{R\bar{\circ}R}$. Moreover if R is reflexive then $R \circ R = R$. We have nearly proved the following.

Proposition 2. *Let R be a reflexive, max–min transitive and complete fuzzy relation on X, then the fuzzy possibility graph induced from (X, R) to a set Ω of normal fuzzy sets on X is min–max transitive. Moreover, the fuzzy possibility graph is reflexive and complete.*

Proof. Use (14) and the inequality $\mu_R \leq \mu_{R\circ R}$ justified by the lemma and the equality $R \circ R = R$. Then $\Pi(a\,R\,b) \leq \Pi(a\,R\,\bar{\circ}\,R\,b)$. This proves the min–max transitivity property. The other properties are obvious. □

Conclusion

This short note has two purposes: (i) to show the usefulness of inference rules in possibilistic logic for proving mathematical results in possibility theory; especially the proof of the Ferrers property has been simplified and its applicability enhanced; (ii) to indicate that possibility graph properties might

extend to the case when it is induced by a complete fuzzy ordering relation, instead of a crisp one.

In the general case, the extension of the Ferrers property might fail to hold, however. For instance if two crisp relations R and S are defined on X, the extended Ferrers property could read

$$\text{if } (a\,R\,b) \text{ and } (c\,S\,d) \text{ then } a\,[R \circ S]\,d \text{ or } c\,[S \circ R]\,b. \tag{16}$$

This property does not hold generally; for instance let $X = \mathbb{R}$ and

$$a\,R\,b \Leftrightarrow a < b - 2,$$
$$b\,S\,c \Leftrightarrow b < c - 3,$$
$$a\,R \circ S\,c \Leftrightarrow a < c - 5 \Leftrightarrow a\,S \circ R\,c.$$

Equation (16) means: if $(a < b - 2)$ and $(c < d - 3)$ then $(a < d - 5)$ or $(c < b - 5)$. This is wrong: for instance $b = a + 3$, $c = a + 1$, $d = a + 5$ which implies $b = c + 2$, so that neither $a < d - 5$ nor $c < b - 5$ hold.

For further research is the extension of the Ferrers property to fuzzy possibility graphs when $R = S$ is a complete fuzzy ordering.

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