SPECTRAL ANALYSIS OF THE DISCRETE MAXWELL OPERATOR: THE LIMITING ABSORPTION PRINCIPLE

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Abstract. We make the beginning of the spectral analysis of the inhomogeneous discrete Maxwell operator $\hat{H}^D$ defined on the square lattice $\mathbf{Z}^3$: we prove that the limiting absorption principle holds. To this aim we construct a conjugate operator to the Fourier series of $\hat{H}^D$ at any not-zero real value. In particular, we analyse the case of thresholds of $\hat{H}^D$.

1. **Introduction.** We are interested in the limiting absorption principle (LAP) for the inhomogeneous discrete Maxwell operator $\hat{H}^D$ defined on the square lattice $\mathbf{Z}^3$, i.e. the limit in some usual or more abstract Besov spaces of the resolvents $\hat{R}(z) = (\hat{H}^D - z)^{-1}$ when $z \in \mathbb{C} \setminus \mathbb{R}$ tends to a spectral value λ of $\hat{H}^D$. It is equivalent and more convenient to deal with the Fourier series H^D of $\hat{H}^D$ instead of $\hat{H}^D$ since H^D is a multiplication operator on the three-dimensional real torus $\mathbb{T}^3 \approx (\mathbb{R}/(2\pi\mathbf{Z}))^3$. The operator H^D is so represented by the analytically fibered self-adjoint operator $\mathbb{T}^3 \ni x \mapsto H^D(x)$ on an Hilbert space $\mathcal{H}^D$ where $H^D(x)$ is a 6×6 real matrix. We prove the LAP using the conjugate operator technique, as developed in the greatest generality by Gerard and Nier in a first version [3]. Actually, denoting $\Sigma = \{(\lambda, x) \mid \lambda \in \sigma(H^D(x))\}$ the energy-momentum set, where $\sigma(H^D(x))$ denotes the spectrum of an $H^D(x)$, we have the stratification $\Sigma = \cup_{j=1}^6 \Sigma_j$ where Σ_i is the semi-analytical set of elements (λ, x) for which λ is an eigenvalue of multiplicity i of $H^D(x)$ so we can introduce the (finite) set $\mathcal{T}$ of thresholds. At this point we could simply apply [3, Theorem 3.1] and claim that for any interval $I \subset\subset \mathbb{R} \setminus \mathcal{T}$, there exists an operator A_I with domain $\mathcal{C}^\infty(\mathbb{T}^3; \mathbb{C}^6)$, essentially self-adjoint on $\mathcal{H}^D$, conjugated to H^D, and satisfying Points (i)-(iii) of [3, Theorem 3.1]. See Points (i)-(iv) of Theorem 3.1. However we go beyond this result. Moreover, the construction of the conjugate operator in [3] contains an error since, in fact, the analytically fibered self-adjoint operator for which the LAP is proved is not of class $\mathcal{C}^2(A_I)$. The authors of [3] have recently corrected it in the new version [4] at the cost of losing the (strict) globality of Mourre's inequality (3) which has to be replaced be a local one as (4). Nevertheless, following the ideas of the old version [3] and the new version [4] we construct an explicit conjugate operator A_ϕ to H^D with the same properties than the operator A_I of [3, Theorem 3.1]. Here we prefer to use as parameter for the conjugate a numerical smooth function ϕ instead of an interval I so A_ϕ is conjugated to H^D at each point of $\phi^{-1}(\mathbb{R}^*)$.

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We prove the LAP at any nonzero threshold. This result is optimal in the sense that there is no conjugate operator to H^D at 0 which is the obvious eigenvalue of H^D, and it shows also that H^D has no other eigenvalue. Denoting by $\mathcal{X}^*$ the finite set of points $x \in \mathbb{T}^3$ such that $\sigma(H^D(x)) \cap \mathcal{T}$ is not reduced to the eigenvalue 0, we construct A_ϕ as a first order symmetric differential operator with smooth coefficients outside $\mathcal{X}^*$ and with rational singularities at points $x \in \mathcal{X}^*$ such that $\sigma(H^D(x)) \cap \mathcal{T} \cap \phi^{-1}(\mathbb{R}^*)$ is not void. Then, the set of thresholds admits the partition $\mathcal{T} = \mathcal{T}_{sa} \cup \mathcal{T}_{sm}$: let $x^* \in \mathcal{X}^*$ and $\lambda \in \mathcal{T} \cap \sigma(H^D(x^*))$, if $\lambda \in \mathcal{T}_{sa}$ then the local part A_{x^*} of A_ϕ near x^* is essentially self-adjoint, and if $\lambda \in \mathcal{T}_{sm}$ then A_ϕ near x^* admits a maximal monotone extension. Thus, we obtain the LAP on $\mathbb{R}^* \setminus \mathcal{T}_{sm}$ in the same terms as [3, Theorem 3.3].
If $\lambda \in \phi^{-1}(\mathbb{R}^*) \cap \mathcal{T}_{sm}$ then A_ϕ may not have a maximal monotone extension since its singularities may come from several points of $\mathcal{X}^*$ and since the sum of two maximal monotone operators, even with disjoint supports, is not necessarily maximal monotone. Nevertheless we prove the LAP on $\mathcal{T}_{sm}$ by a slight extension of [2, Theorem 3.3].

In Part 2 we describe the operator H^D and the Hilbert space $\mathcal{H}^D$ on which it is self-adjoint. In Part 3 we state the main results of our work, notably, the existence of a conjugate operator A_ϕ for H^D at Theorem 3.1, then several LAP in terms of abstract or usual Besov spaces at Corollaries 1-5. In Part 4 we describe the spectrum of $H^D(x)$, the stratification of the energy-momentum set Σ; we define the thresholds and describe $\mathcal{T}$. It appears that $\mathcal{T}$ depends mainly on the parameter $\boldsymbol{\beta} = \boldsymbol{\varepsilon} \times \boldsymbol{\mu}$ where $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ denote respectively the permittivity and the permeability. In fact outside the first quite basic case $\boldsymbol{\beta} = 0$ we have to deal with many special cases for which the set $\mathcal{T}$ is no more obvious. In Part 5 we construct the conjugate operator A_ϕ according to the parameter $\boldsymbol{\beta}$. We prove also the main properties of A_ϕ. In Part 6 we prove the main results (see Part 3).

All along the text we use the following notations. Let $\mathcal{E} \in \{\mathbb{T}^3, \mathbb{R}^3\}$ where $\mathbb{T}^3 \approx (\mathbb{R}/(2\pi\mathbb{Z}))^3$ is the 3-dimensional real torus. If f is a numerical function (from $\mathbb{R}$ into $\mathbb{R}$) we then denote by f again the mapping $\mathcal{E} \ni x \mapsto (f(x_1), f(x_2), f(x_3)) \in \mathbb{R}^3$, so if $E \subset \mathcal{E}$ then $f(E) = \{f(x);\ x \in E\} \subset \mathbb{R}^3$ and if $E \subset \mathbb{R}^3$ then $f^{-1}(E) = \{x \in \mathcal{E};\ f(x) \in E\}$. In particular we set, for $x \in \mathcal{E}$,

$$\begin{aligned} y &= \sin x := (y_1 = \sin x_1, y_2 = \sin x_2, y_3 = \sin x_3) \in \mathbb{R}^3, \\ z &= y^2 := (z_1 = y_1^2, z_2 = y_2^2, z_3 = y_3^2) \in \mathbb{R}^3. \end{aligned}$$

(Although the letter z will be also used as a complex energy, there is no possible confusion with the notation above.) The other notations are standard. If $E \subset \mathcal{E} \in \{\mathbb{T}^3, \mathbb{R}^3\}$ and $\mathcal{F} \in \{\mathbb{R}^n, \mathbb{C}^n\}$, we denote by $\mathcal{C}_c^\infty(E, \mathcal{F})$ the real space of $\mathcal{C}^\infty$ functions with values in $\mathcal{F}$, defined on $\mathcal{E}$ and with compact support in E.
Let T a self-adjoint operator, then $\sigma(T)$ is its spectrum; if $E \subset \mathbb{R}$ then $1_E(T)$ operator $\chi_{[a,b)}(T)$ where χ_J is the characteristic function of a set $J \subset \mathbb{R}$. If X and Y are two metrics spaces $B(X, Y)$ is the space of bounded operators from X into Y and $B(X) := B(X, X)$.
For $n \geq 1$ the space $\mathbb{R}^n$ (respect., $\mathbb{C}^n$) is equipped with the scalar product $< \cdot, \cdot >_{\mathbb{R}^n}$ (respect., with the hermitian product

$$< f, g >_{\mathbb{C}^n} = \sum_{j=1}^n f_j \overline{g_j}, \quad f = (f_j)_{1 \leq j \leq n},\ g = (g_j)_{1 \leq j \leq n}).$$

Full notations are at the end of the text, Part 8.4.

2. The discrete Maxwell Operator.

2.1. The homogeneous discrete Maxwell Operator.

Let $\mathbb{Z}^3 = \{n = (n_1, n_2, n_3);\ n_j \in \mathbb{Z}\}$ the square lattice, $\mathbb{T}^3 \approx (\mathbb{R}/(2\pi\mathbb{Z}))^3$ the 3-dimensional real torus, U the unitary transform between $L^2(\mathbb{T}^3)$ and $l^2(\mathbb{Z}^3)$:

$$(Uf)(n) := \hat{f}(n) \equiv (2\pi)^{-\frac{3}{2}} \int_{\mathbb{T}^3} e^{inx} f(x) \mathrm{d}x, \quad n \in \mathbb{Z}^3,$$

so that any $f \in L^2(\mathbb{T}^3)$ can be written

$$f(x) = (U^* \hat{f})(x) \equiv (2\pi)^{-\frac{3}{2}} \sum_{n \in \mathbb{Z}^3} e^{-inx} \hat{f}(n), \quad x \in \mathbb{T}^3.$$

The homogeneous discrete Maxwell operator is the bounded operator H_0 on $\mathcal{H} = (L^2(\mathbb{T}^3))^6$ defined by

$$\hat{H}_0 = U H_0 U^*,$$

with $H_0(x)$ the real anti-symmetrical 6×6 matrix:

$$H_0(x) = \begin{pmatrix} 0_{3\times 3} & M(y) \\ -M(y) & 0_{3\times 3} \end{pmatrix} \in \mathbb{R}^6, \quad x \in \mathbb{T}^3,$$

where $y = \sin x$ and $M(y)$ is the real anti-symmetrical 3×3 matrix:

$$M(y) = \begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix}, \quad y \in \mathbb{R}^3. \tag{1}$$

The space $\mathcal{H} = L^2(\mathbb{T}^3, \mathrm{d}x; \mathbb{R}^6)$ can be written as the hilbertian sum

$$\mathcal{H} = \int_{\mathbb{T}^3}^{\oplus} \mathbb{R}^6 \mathrm{d}x,$$

with the scalar product

$$(u, v) := \int_{\mathbb{T}^3} < u(x), v(x) >_{\mathbb{R}^6} \mathrm{d}x.$$

We then have

$$H_0 = \int_{\mathbb{T}^3}^{\oplus} H_0(x) \mathrm{d}x$$

which shows that H_0 is a bounded self-adjoint operator on $\mathcal{H}$.

All along the article we consider that the vectorial functions u and v may be complex-valued so we denote by the same symbol $\mathcal{H}$ the hilbertian space $\int_{\mathbb{T}^3}^{\oplus} \mathbb{C}^6 \mathrm{d}x$ equipped with the hilbertian product

$$(u, v) := \int_{\mathbb{T}^3} < u(x), v(x) >_{\mathbb{C}^6} \mathrm{d}x = \int_{\mathbb{T}^3} \sum_{j=1}^{6} u_j(x) \overline{v_j(x)} \mathrm{d}x$$

and with norm $\|u\| := (u, u)^{\frac{1}{2}}$.

2.2. **The discrete Maxwell Operator.** Let $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ be 3×3 constant diagonal matrices with diagonal elements, respectively, $\varepsilon_1,\varepsilon_2,\varepsilon_3 \in (0,\infty)$ for $\boldsymbol{\varepsilon}$, and $\mu_1,\mu_2,\mu_3 \in (0,\infty)$ for $\boldsymbol{\mu}$. The inhomogeneous discrete-Maxwell operator is defined by

$$\hat{H}^D = \hat{D}\hat{H}_0,$$

where we set

$$\hat{D} = \begin{pmatrix} \boldsymbol{\varepsilon} & 0_{3\times 3} \\ 0_{3\times 3} & \boldsymbol{\mu} \end{pmatrix}.$$

The (unperturbed) discrete Maxwell operator is $H^D = U^*\hat{H}^D U$ so we have, since $\hat{D}$ is constant,

$$H^D = U^*(\hat{D}\hat{H}_0)U = \hat{D}U^*\hat{H}_0 U = \hat{D}H_0.$$

The relation

$$(\hat{D}^{-1}H^D u, v) \;=\; (H_0 u, v)$$

shows that the operator H^D is bounded and self-adjoint on the space $\mathcal{H}^D = \mathcal{H}$ equipped with the hilbertian product

$$(u,v)_{\mathcal{H}^D} := (\hat{D}^{-1}u, v) = \int_{\mathbb{T}^3} < \hat{D}^{-1}u(x), v(x) >_{\mathbb{C}^6} \mathrm{d}x$$

and with norm $\|u\|_{\mathcal{H}^D} := (u,u)^{\frac{1}{2}}_{\mathcal{H}^D}$. (Since the norms $\|\cdot\|$ and $\|\cdot\|_{\mathcal{H}^D}$ are equivalent, we can omit the index "$_{\mathcal{H}^D}$".) It appears that H^D is a multiplication operator and we write

$$H^D = \int_{\mathbb{T}^3}^{\oplus} H^D(x)\mathrm{d}x,$$

where $H^D(x)$ is self-adjoint on $\mathbb{C}^6$ equipped with the hermitian product:

$$< u(x), v(x) >_{\mathbb{C}^6,D} := < \hat{D}^{-1}u(x), v(x) >_{\mathbb{C}^6} .$$

Since $H^D(x)$ depends only on $y = \sin(x)$ we write it $H^D(x) = h^D(y)$. It is known (see [11]) that

$$\sigma(H^D) = \overline{\cup_{x\in\mathbb{T}^3}\sigma(H^D(x))}, \tag{2}$$

which is a compact set of $\mathbb{R}$.

3. Results.

3.1. **Main Theorem.** We introduce the following settings:

$$\begin{aligned} \mathcal{X}_0 &:= \{x\in\mathbb{T}^3;\, z=0\}, \quad \mathbb{T}^3_0 = \mathbb{T}^3\setminus\mathcal{X}_0, \\ \mathcal{X}^* &:= \{x\in\mathbb{T}^3;\, \sigma(H^D(x))\cap\mathcal{T}\neq\{0\}\}\subset\mathbb{T}^3_0. \end{aligned}$$

We denote $\mathcal{D}_0 := \mathcal{C}^\infty_c(\mathbb{T}^3\setminus\mathcal{X}^*)$ and $R(z) := (H^D - z)^{-1}$ the resolvent of H^D.

Theorem 3.1. Let $\mathcal{T}'\subset\mathcal{T}$ with $0\in\mathcal{T}'$, and $\phi\in\mathcal{C}^\infty_c(\mathbb{R}\setminus\mathcal{T}')$. Then there exists a symmetric operator A_ϕ defined on $\mathcal{C}^\infty(\mathbb{T}^3)$ if $\mathcal{T}'=\mathcal{T}$ and on $\mathcal{D}_0$ if $\mathcal{T}'\neq\mathcal{T}$, and satisfying the following properties:

(i) There exists a constant $\delta=\delta(\phi)>0$ so that we have

$$\phi(H^D)[H^D, iA_\phi]\phi(H^D) \geq \delta\phi^2(H^D). \tag{3}$$

(ii) The multi-commutators $ad^k_{A_\phi}(H^D)$ are bounded for all $k\in\mathbb{N}$.

(iii) The operator A_ϕ is a first order differential operator in x whose coefficients belong to $\mathcal{C}^\infty(\mathbb{T}^3;\mathcal{L}(\mathbb{C}^6))$ if $\mathcal{T}'=\mathcal{T}$ and to $\mathcal{C}^\infty(\mathbb{T}^3\setminus\mathcal{X}^*;\mathcal{L}(\mathbb{C}^6))$ if $\mathcal{T}'\neq\mathcal{T}$, and there exists $\tilde{\phi}\in\mathcal{C}^\infty_c(\mathbb{R}\setminus\mathcal{T}')$ so that $A_\phi = \tilde{\phi}(H^D)A_\phi = A_\phi\tilde{\phi}(H^D)$.

(iv) If $\mathcal{T}_{sm} \subset \mathcal{T}'$ then A_ϕ is essentially self-adjoint.
(v) If $\mathcal{T}' = \{0\}$ then A_ϕ has the form $\sum_{j=0}^{3} A_j$ where A_0 has smooth coefficients and is essentially self-adjoint, A_1 and A_2 have coefficients with rational singularities at some points of $\mathcal{X}^*$ and, defined on the domain $\mathcal{D}_0$, admit a maximal symmetric extension; moreover, $\operatorname{supp} A_1 \cap \operatorname{supp} A_2 = \emptyset$.

Remark 1.
- Since the coefficients of A_ϕ are smooth in $\mathbb{T}^3 \setminus \mathcal{X}^*$ and, since $\mathcal{D}_0$ is dense in $\mathcal{H}^D$, then, for $u \in \mathcal{H}^D$, the distribution $A_\phi u$ belongs to the topological dual space of $\mathcal{D}_0$ so, since A_ϕ is symmetric,

$$A_\phi u \in \mathcal{H}^D \Longleftrightarrow |(u, A_\phi v)_{\mathcal{H}^D}| \leq C\|v\| \quad \forall v \in \mathcal{D}_0.$$

Setting

$$\|u\|_{D(A_\phi)} := \|u\| + \|A_\phi u\| \quad \forall u \in \mathcal{D}_0,$$

the closure $\bar{A}_\phi$ of A_ϕ has domain

$$D(\bar{A}_\phi) := \overline{\mathcal{D}_0}^{\|\cdot\|_{D(A_\phi)}}.$$

The adjoint of $\bar{A}_\phi$ or of A_ϕ is the operator A_ϕ^* with domain $D(A_\phi^*) = \{v \in \mathcal{H}^D;\ |(A_\phi u, v)_{\mathcal{H}^D}| \leq C\|u\|,\ u \in D(A_\phi)\}$. If $\operatorname{supp}\phi \cap \mathcal{T} = \emptyset$ then A_ϕ is essentially self-adjoint and its self-adjoint extension has domain $\overline{\mathcal{C}^\infty(\mathbb{T}^3)}^{\|\cdot\|_{D(A_\phi)}} \supset H^1(\mathbb{T}^3)$ (with dense inclusion).
- Our set $\mathcal{T}$ of thresholds may defer to the set of thresholds in [3]. Anyway, $\mathcal{T}$ is a discrete (and finite) set. The sets $\mathcal{T}$, $\mathcal{T}_{sa}$ and $\mathcal{T}_{sm}$ are precisely described in Part 5.7.3.
- In the case $\mathcal{T}' = \mathcal{T}$, the result of [3] implies the existence of an essentially self-adjoint operator A_I with smooth coefficients such that Points (i), (iii) and (iv) with A_ϕ replaced by A_I hold. But the first commutator $[H^D, A_I]$ is not a multiplication operator so Point (ii) fails, and, in fact, $H^D \notin \mathcal{C}^{1,1}(A_I)$ (this set is defined at Point B of Corollary 1). The new version [4] of [3] provides an essentially self-adjoint operator A_{I,I_1} with smooth coefficients such that Points (ii)−(iv) and a local (weaker) version of Point (i) are maintained (with A_ϕ replaced by A_{I,I_1}).
- We give an explicite formula for A_ϕ which is easier to read than the general formula in [4] (which is only valid in the case $\mathcal{T}' = \mathcal{T}$).

3.2. **Main consequences and extensions.** The first obvious consequence of Theorem 3.1 is that the singular continuous spectrum of H^D is then empty. But it is actually a consequence of the general theorem in [3] revised in [4].

The second consequence is that we can state the LAP outside $\mathcal{T}_{sm} \cup \{0\}$ in the same terms as those of Gérard and Nier in the old version [3] of their work. See also [9, 10]. Let us consider a compact interval $I \subset \mathbb{R}^* \setminus \mathcal{T}_{sm}$, and fix $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^* \setminus \mathcal{T}_{sm})$ such that $\phi = 1$ on a neighborhood of I. We denote by $A_\phi^{sa} \subset A_\phi^*$ a self-adjoint extension of A_ϕ. We define the abstract Besov space $\mathcal{B}_A$ by

$$\mathcal{B}_A = \{f \in \mathcal{H};\ \|f\|_{\mathcal{B}_A} := \sum_{j=0}^{\infty} r_j^{1/2} \|1_{r_{j-1} \leq |A_\phi^{sa}| \leq r_j} f\| < \infty\}.$$

Its dual space $\mathcal{B}_A^*$ is the completion of $\mathcal{H}^D$ by the following norm

$$\|u\|_{\mathcal{B}_A^*} = \sup_{j \geq 0} r_j^{1/2} \|1_{r_{j-1} \leq |A_\phi^{sa}| < r_j} u\|.$$

For $s > 1/2$, the following inclusion relations hold :

$$D((1+|A_\phi^{sa}|)^s) \subset \mathcal{B}_A \subset D((1+|A_\phi^{sa}|)^{1/2}) \subset \mathcal{H}^D$$
$$\subset D((1+|A_\phi^{sa}|)^{-1/2}) \subset \mathcal{B}_A{}^* \subset D((1+|A_\phi^{sa}|)^{-s}).$$

We can claim

Corollary 1. (LAP on $\mathbb{R}^* \setminus \mathcal{T}_{sm}$ in abstract Besov spaces.) We have

$$\sup_{\lambda\in I,\,\mu>0} \|R(\lambda \pm i\mu)f\|_{\mathcal{B}_A{}^*} \leq C_I \|f\|_{\mathcal{B}_A} \quad \forall f \in \mathcal{B}_A.$$

Moreover letting $s > 1/2$ then the limits

$$\lim_{\varepsilon\to\pm 0} (1+|A_\phi^{sa}|)^{-s} R(\lambda + i\varepsilon)(1+|A_\phi^{sa}|)^{-s}$$

exist in $B(\mathcal{H}^D)$ and are bounded, with uniform convergence according to $\lambda \in I$. The mapping $\mathbb{R} \setminus \mathcal{T} \ni \lambda \mapsto R(\lambda \pm i0)$ is norm continuous in $B(D((1+|A_\phi^{sa}|)^s, D((1+|A_\phi^{sa}|)^{-s})$ and weakly continuous in $B(\mathcal{B}_A, \mathcal{B}_A{}^*)$.

For further developments we establish also the LAP in terms of the usual Besov spaces described by Isozaki and alii [12] with the restriction to spectral values outside the thresholds. Thus, we consider the case $\operatorname{supp}\phi \subset \mathbb{R} \setminus \mathcal{T}$ in Theorem 3.1. We set $N = (N_1, N_2, N_3)$, $N_j = i\partial/\partial x_j$ and the self-adjoint operators

$$|N| = \sqrt{N^2} = \sqrt{-\Delta}, \quad N^2 = \sum_{j=1}^3 N_j^2 = -\Delta \quad \text{on } \mathbb{T}^3,$$

where Δ denotes the Laplacian on $\mathbb{T}^3 = [-\pi, \pi]^3$ with periodic boundary condition. Let $D'(\mathbb{T}^3, \mathbb{C}) = (\mathcal{C}^\infty(\mathbb{T}^3, \mathbb{C}))'$ the space of distribution on $\mathbb{T}^3$ and consider $D'(\mathbb{T}^3, \mathbb{C}^6) \approx (D'(\mathbb{T}^3, \mathbb{C}))^6 \approx (D'(\mathbb{T}^3, \mathbb{R}))^{12}$. We introduce the normed spaces

$$\mathcal{H}^s = \{u \in D'(\mathbb{T}^3, \mathbb{C}^6),\ \|u\|_s < \infty\}, \quad \|u\|_s := \|(1+N^2)^{s/2}u\|, \quad s \in \mathbb{R},$$

so $\mathcal{H}^s$ is the completion of $D(|N|^s)$, the domain of $|N|^s$, with respect to the norm $\|u\|_s$ and we have $\mathcal{H}^D = \mathcal{H}^0 = L^2(\mathbb{T}^3, \mathbb{C}^6)$. For $s \geq 0$ and $u \in \mathcal{C}^\infty(\mathbb{T}^3, \mathbb{C}^6)$ we have $\|(1+|A_\phi^{sa}|)^s u\| \leq C\|u\|_s$ where C does not depend on u. Thus, the following inclusion relations hold :

$$\mathcal{H}^s \subset D((1+|A_\phi^{sa}|)^s) \subset \mathcal{H}^D \subset D((1+|A_\phi^{sa}|)^{-s}) \subset \mathcal{H}^{-s} \quad \forall s \geq 0.$$

Using the sequence $(r_j)_{j\geq -1}$ where $r_{-1} = 0$, $r_j = 2^j$ for $j \geq 0$ we define the Besov space $\mathcal{B}$ by

$$\mathcal{B} := \{f \in \mathcal{H}^D;\ \|f\|_{\mathcal{B}} := \sum_{j=0}^\infty r_j^{1/2} \|1_{r_{j-1}\leq |N| \leq r_j} f\| < \infty\}.$$

Its dual space $\mathcal{B}^*$ is the completion of $\mathcal{H}$ by the following norm

$$\|u\|_{\mathcal{B}^*} = \sup_{j\geq 0} r_j^{1/2} \|1_{r_{j-1}\leq |N| < r_j} u\|.$$

For $s > 1/2$, the following inclusion relations hold :

$$\mathcal{H}^s \subset \mathcal{B} \subset \mathcal{H}^{1/2} \subset \mathcal{H}^D \subset \mathcal{H}^{-1/2} \subset \mathcal{B}^* \subset \mathcal{H}^{-s}.$$

Moreover, Lemma 2.8 of [5] says that there is a constant $C > 0$ such that

$$\|f\|_{\mathcal{B}_A} \leq C\|f\|_{\mathcal{B}} \quad \forall f \in \mathcal{B},$$

that is, $\mathcal{B} \subset \mathcal{B}_A$, and so, $\mathcal{B}_A{}^* \subset \mathcal{B}^*$. Hence, Corollary 1 can be extended as

Corollary 2. (LAP on $\mathbb{R} \setminus \mathcal{T}$ in usual Besov spaces.) We have

$$\sup_{\lambda \in I,\ \mu > 0} \|R(\lambda \pm i\mu)\|_{B(\mathcal{B};\mathcal{B}^*)} < \infty,$$

for all compact set $I \subset \mathbb{R} \setminus \mathcal{T}$. Moreover letting $s > 1/2$ then the limits

$$R(\lambda \pm i0) := \lim_{\varepsilon \searrow 0} R(\lambda \pm i\varepsilon) \in B(\mathcal{H}^s; \mathcal{H}^{-s})$$

exist and are bounded, with uniform convergence according to $\lambda \in I$. The mapping $\mathbb{R} \setminus \mathcal{T} \ni \lambda \mapsto R(\lambda \pm i0)$ is norm continuous in $B(\mathcal{H}^s; \mathcal{H}^{-s})$ and weakly continuous in $B(\mathcal{B}; \mathcal{B}^*)$.

Another consequence of Theorem 3.1 is the following extension of the LAP to any nonzero spectral value, thanks to a slight adaptation of [2, Theorem 3.3].

Corollary 3. (LAP on $\mathbb{R}^*$.) Let $I \subset \mathbb{R}^*$ a compact interval. There exists a constant C_I such that

$$|(u, R(z)u)_{\mathcal{H}^D}| \leq C_I \|u\|^2_{D(\bar{A}_\phi)} \quad \forall u \in D(\bar{A}_\phi),$$

for all $z = \lambda + i\mu$ with $\lambda \in I$, $\mu \neq 0$ real. Moreover if $z_1 = \lambda_1 + i\mu_1$, $z_2 = \lambda_2 + i\mu_2$ are two such numbers, and if μ_1 and μ_2 have the same sign, then

$$|(u, (R(z_1) - R(z_2))u)_{\mathcal{H}^D}| \leq C_I |z_1 - z_2|^{1/2} \|u\|^2_{D(\bar{A}_\phi)} \quad \forall u \in D(\bar{A}_\phi).$$

In particular, if $u \in D(\bar{A}_\phi)$ then the limits

$$\lim_{\varepsilon \searrow 0^+} (u, R(\lambda \pm i\varepsilon)u)_{\mathcal{H}^D} =: (u, R(\lambda \pm i0)u)_{\mathcal{H}^D}$$

exist uniformly in $\lambda \in I$, and, for all $\lambda_1, \lambda_2 \in I$, we have

$$(u, (R(\lambda_1 \pm i0) - R(\lambda_2 \pm i0))u)_{\mathcal{H}^D} \leq C_I |\lambda_1 - \lambda_2|^{1/2} \|u\|^2_{D(\bar{A}_\phi)}.$$

An immediate consequence of Corollary 3 is

Corollary 4. The point spectrum $\sigma_p(H^D)$ of H^D is reduced to $\{0\}$.

Before giving the proof of Theorem 3.1, we state the results for some natural class of perturbed Hamiltonians $H^D_V = H^D + V$, as done in [3]. We will simply recall some well known results in the Mourre theory (see [9, 10]) and refer the reader to the book [1] for a complete exposition of the Mourre method. In particular a sharper version of Corollary 5 is given in [1, Prop. 7.5.6].

Corollary 5. Let a compact interval $I \subset \mathbb{R}^* \setminus \mathcal{T}_{sm}$, and fix $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^* \setminus \mathcal{T}_{sm})$ such that $\phi = 1$ on a neighborhood of I. Let V a symmetric operator on $\mathcal{H}^D$ so that

1. $VR(i)$ and $R(i)[V, iA_\phi]R(i)$ are compact.
2. $V \in C^{1,1}(A_\phi)$, i.e.,

$$\int \|R(i)(e^{itA_\phi}[V, iA_\phi]e^{-itA_\phi} - [V, iA_\phi])R(i)\| \frac{\mathrm{d}t}{t} < \infty.$$

Then, setting $H^D_V := H^D + V$, the following results hold:

1. There exists a constant $\delta > 0$ and a compact operator K so that,

$$\phi(H^D_V)[H^D_V, iA_\phi]\phi(H^D_V) \geq \delta\phi^2(H^D_V) + K.$$

   Consequently point spectrum $\sigma_p(H^D_V)$ is of finite multiplicity in $\mathbb{R}^* \setminus \mathcal{T}_{sm}$ and has no accumulation point in $\mathbb{R}^* \setminus \mathcal{T}_{sm}$.
2. For each $\lambda \in I \setminus \sigma_p(H^D_V)$ there exist $\varepsilon > 0$ and $c > 0$ so that,

$$1_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H^D_V)[H^D_V, iA_\phi]1_{[\lambda-\varepsilon,\lambda+\varepsilon]} \geq c1_{[\lambda-\varepsilon,\lambda+\varepsilon]}(H^D_V). \tag{4}$$

3. The LAP for H_V^D holds on $I \setminus \sigma_p(H_V^D)$: the limits
$$\lim_{\varepsilon \to \pm 0} (1+|A_\phi|)^{-s} R(\lambda \pm i\varepsilon)(1+|A_\phi|)^{-s}$$
exist and are bounded for all $s > 1/2$.
Consequently the singular continuous spectrum of H_V^D is empty.
4. If the operator $(1+|A_\phi|)^{-s} V (1+|A_\phi|)^{-s}$ is bounded for some $s > 1/2$, then the wave operators
$$s - \lim_{t \to \pm\infty} e^{itH_V^D} e^{itH^D} 1_I(H^D) =: \Omega_I^\pm$$
exist and are asymptotically complete:
$$1_I^c(H_V^D)\mathcal{H}^D = \Omega_I^\pm \mathcal{H}^D.$$

4. First Spectral Properties.

4.1. Spectrum of $h^D(y)$.

We introduce the new parameters:

$$\begin{aligned} \boldsymbol{\beta} &= \boldsymbol{\varepsilon} \times \boldsymbol{\mu} = (\beta_1, \beta_2, \beta_3) && (5)\\ \boldsymbol{\alpha} &= (\alpha_1, \alpha_2, \alpha_3), \quad \alpha_1 := (\varepsilon_2\mu_3 + \varepsilon_3\mu_2)/2 \quad \text{and, c.p.,} \\ \gamma_1 &= \varepsilon_2\varepsilon_3\mu_2\mu_3 \quad \text{and c.p. :} && (6)\\ \gamma_2 &= \varepsilon_1\varepsilon_3\mu_1\mu_3, \\ \gamma_3 &= \varepsilon_1\varepsilon_2\mu_1\mu_2. \end{aligned}$$

(The abreviation "c.p." means "circular permutation" so we have the other values by circular permutation, ex., $\beta_1 = \varepsilon_2\mu_3 - \varepsilon_3\mu_2$).

Let us describe the spectrum of $h^D(y)$.

Lemma 4.1. We have
$$\det(h^D(y) - k) = \det(\boldsymbol{\varepsilon} M(y) \boldsymbol{\mu} M(y) + k^2)$$
and the factorization
$$\det(h^D(y) - k) = k^2(\tau^+(z) - k^2)(\tau^-(z) - k^2),$$
with
$$\tau^\pm = \Psi_0 \pm \sqrt{K_0}, \tag{7}$$
where
$$\begin{aligned} K_0(z) &= \frac{1}{4}(\beta_1^2 z_1^2 - 2\beta_1\beta_2 z_1 z_2) + c.p., && (8)\\ \Psi_0(z) &= \boldsymbol{\alpha} \cdot z := \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3. && (9) \end{aligned}$$

Proof in Appendix A. Since the characteristic polynomial $\det(H^D(x) - \lambda)$ depends on x via the new variable $z = (z_1, z_2, z_3) = \sin^2 x \in [0,1]^3$ we set
$$p(z;\lambda) = \det(H^D(x) - \lambda) = \det(\boldsymbol{\varepsilon} M(y) \boldsymbol{\mu} M(y) + \lambda^2).$$

Remark 2. If $z \neq 0$ then
$$\tau^+(z) = \Psi_0(z) + \sqrt{K_0(z)} \geq \tau^-(z) = \Psi_0(z) - \sqrt{K_0(z)} > 0.$$
Moreover there exists $C > 0$ such that
$$\tau^-(z) \geq C|z|, \quad z \in [0,1]^3.$$

Since $\boldsymbol{\beta}\cdot\boldsymbol{\varepsilon}=0$ and $\varepsilon_i>0$ for all $i\in[[1,3]]$ then there exists $j\in[[1,3]]$ such that $\beta_j\beta_i\leq 0$ and $\beta_k\beta_i\geq 0$ for $i,k\neq j$. If two of the β_j's vanish then $\boldsymbol{\beta}$ vanishes. Moreover $\boldsymbol{\beta}$ is replaced by $-\boldsymbol{\beta}$ if $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are exchanged, which involves the same analysis. Hence if $\boldsymbol{\beta}\neq 0$ then we can assume without any restriction:

$$\textbf{(A0)}\text{: } \beta_1\geq\beta_2>0>\beta_3 \text{ or } \beta_1>\beta_2=0>\beta_3.$$

The functions Ψ_0 and K_0 are homogeneous polynomials. The relation $K_0\equiv 0$ is equivalent to $\boldsymbol{\beta}=0$ which is the special case where ε and μ are proportional. If one of the β_i's vanishes, so, under Asumption (A0), $\beta_2=0$, then $\sqrt{K_0(z)}$ is polynomial. Thus, the functions $\mathbb{R}^3\ni z\mapsto\tau^{\pm}(z)$ are homogeneous analytical complex functions with branch at $z=0$ and at points z for which $K_0(z)=0$ if $\beta_2\neq 0$ for all i, and with branch point at $z=0$ only if $\beta_2\neq 0$.

If $z=0$ then $h^D(y)=0_{6\times 6}$ and all the eigenvalues vanish. Let us consider the case $z\neq 0$.

Theorem 4.2. (Spectrum of $h^D(y)$.) Let $z\in[0,+\infty)^3\setminus\{0_{\mathbb{R}^3}\}$. Then 0 is a double eigenvalue with eigenvectors $(y_1,y_2,y_3,0,0,0)\equiv y\otimes 0_{\mathbb{C}^3}$ and $(0,0,0,y_1,y_2,y_3)\equiv 0_{\mathbb{C}^3}\otimes y$.

Assume $\boldsymbol{\beta}=0$. Then $K_0\equiv 0$ and all the eigenvalues have multiplicity two. Moreover, the nonzero eigenvalues of $h^D(y)$ are

$$\pm\sqrt{\tau^+(z)}=\pm\sqrt{\tau^-(z)}=\pm\sqrt{\varepsilon_2\mu_3z_1+\varepsilon_3\mu_1z_2+\varepsilon_1\mu_2z_3}.$$

Assume $\boldsymbol{\beta}\neq 0$ (so (A0) holds). Then the nonzero eigenvalues of $h^D(y)$ are

- $\pm\sqrt{\tau^+(z)}$, simple iff $K_0(z)\neq 0$,
- $\pm\sqrt{\tau^-(z)}$, simple iff $K_0(z)\neq 0$.
- $\pm\sqrt{\tau^+(z)}=\pm\sqrt{\tau^-(z)}$, double iff $K_0(z)=0$.

Assume $\boldsymbol{\beta}\neq 0$ (so (A0) holds) with $\beta_2=0$. Then, τ^+ and τ^- are linear according to z:

$$\tau^+(z) = \varepsilon_2\mu_3z_1+\varepsilon_3\mu_1z_2+\varepsilon_2\mu_1z_3, \tag{10}$$
$$\tau^-(z) = \varepsilon_3\mu_2z_1+\varepsilon_3\mu_1z_2+\varepsilon_1\mu_2z_3. \tag{11}$$

(Hence we observe that:
If $(z_1,z_3)\neq 0_{\mathbb{R}^2}$, then the nonzero eigenvalues of $h^D(y)$ are $\pm\sqrt{\tau^+(z)}$, simple. If $(z_1,z_3)=0_{\mathbb{R}^2}$ and $z_2\neq 0$, then the nonzero eigenvalues of $h^D(y)$ are

$$\pm\sqrt{\tau^+(z)}=\pm\sqrt{\tau^-(z)}=\pm\sqrt{\alpha_2}|y_2|=\pm\sqrt{\varepsilon_3\mu_1}|y_2|,$$

double.)

The Proof of Theorem 4.2 follows from (8) and (9).

Lemma 4.3. Let us assume $\boldsymbol{\beta}\neq 0$ (so (A0) holds). We have

$$K_0^{-1}(\{0\}):=\{z\in[0,1]^3\,;\,K_0(z)=0\}=\{t(\beta_2,\beta_1,0)\,;\,0\leq t\leq\frac{1}{\beta_1}\}.$$

The Proof is let to the reader.

We set

$$\lambda^*=\max\{\sqrt{\tau^+(z)}\,;\,z\in[0,1]^3\}\in(0,+\infty).$$

The following result is a consequence of Theorem 4.2 and of relation (2):

Proposition 1. 1) The operator H^D admits 0 as eigenvalue of infinite order.
2) The spectrum of H^D is

$$\sigma(H^D)=[-\lambda^*,\lambda^*].$$

Proof in Appendix A.

4.2. **Stratification and thresholds.** Following [3] the set of energy-momentum is

$$\Sigma = \{(\lambda, x) \,|\, \lambda \in \sigma(H^D(x))\} \subset \sigma(H^D) \times \mathbb{T}^3.$$

We have $(\lambda, x) \in \Sigma \Longleftrightarrow p(z; \lambda) = 0$. We consider the canonical projections:

$$\begin{aligned} P_M &: \mathbb{R} \times \mathbb{T}^3 \ni (\lambda, x) \mapsto x \in \mathbb{T}^3, \\ P_{\mathbb{R}} &: \mathbb{R} \times \mathbb{T}^3 \ni (\lambda, x) \mapsto \lambda \in \mathbb{R}. \end{aligned}$$

It is clear that $P_{\mathbb{R}}|_\Sigma$ is a proper map. The spectrum $\sigma(H^D(x))$ of $H^D(x)$ is discrete and depends continuously on x. The operators $H^D(x)$ are the fibers and the space $\mathbb{T}^3$ is the momentum space.
The set of energy-momentum Σ admits the partition

$$\Sigma = \cup_{i=1}^6 \Sigma_i,$$

where Σ_i is the semi-analytical set of elements (λ, x) for which λ is an eigenvalue of multiplicity i of $H^D(x)$. We set

$$\mathcal{X}_j = P_M(\Sigma_j), \quad j \geq 1.$$

We see that $\Sigma_j = \emptyset$ for $j = 3, 4, 5$, $\Sigma_6 = \{0\} \times \mathcal{X}_0$ so $\mathcal{X}_6 = \mathcal{X}_0$, $\mathcal{X}_j = \emptyset$ for $j = 3, 4, 5$. Moroever,

$$\begin{aligned} \Sigma_1 &= \Sigma_1^+ \cup \Sigma_1^-, \\ \Sigma_1^\pm &:= \{(\lambda, x);\, 0 \neq \lambda^2 = \tau^\pm(z) \neq \tau^\mp(z)\}, \\ \Sigma_2 &= \{(\lambda, x);\, 0 \neq \lambda^2 = \tau^+(z) = \tau^-(z)\}. \end{aligned}$$

If $\boldsymbol{\beta} = 0$ then $\Sigma_1 = \emptyset$ and $(\lambda, x) \in \Sigma_2$ iff $z \neq 0$ and $\lambda^2 = \Psi_0(z)$.
If $\boldsymbol{\beta} \neq 0$ holds then $(\lambda, x) \in \Sigma_1$ iff $K_0(z) \neq 0$ and $\lambda^2 \in \{\tau^+(z), \tau^-(z)\}$.

Let us define the set of thresholds, $\mathcal{T}$. For a more general definition of $\mathcal{T}$ (which may defer from ours), see [3]. We set

$$\begin{aligned} \Sigma_1^* &:= \Sigma_1^{*+} \cup \Sigma_1^{*-}, \\ \Sigma_1^{*\pm} &:= \{(\lambda, x) \in \Sigma_1;\, \lambda^2 = \tau^\pm(x), \nabla_x \tau^\pm(z) = 0\}, \\ \Sigma_2^* &:= \{(\lambda, x) \in \Sigma_2;\, \lambda^2 = \sqrt{\Psi_0(z)},\ \text{and } \nabla_x \Psi_0(z) \text{ is normal to } \mathcal{X}_2 \text{ at } x\}, \end{aligned}$$

then,

$$\begin{aligned} \mathcal{T}_j &:= P_{\mathbb{R}}(\Sigma_j^*), \\ \mathcal{X}_j^* &:= P_M(\Sigma_j^*), \\ \mathcal{Z}_j^* &:= \sin^2(\mathcal{X}_j^*), \quad j = 1, 2. \end{aligned}$$

Observing that $P_{\mathbb{R}}(\Sigma_6) = \{0\}$, we set the set of thresholds, $\mathcal{T}$, as

$$\mathcal{T} := \{0\} \cup \mathcal{T}_1 \cup \mathcal{T}_2.$$

Let us describe $\mathcal{T}$. Setting

$$\mathcal{X}^* = \mathcal{X}_1^* \cup \mathcal{X}_2^*,$$

we see that λ is a threshold iff there exists $z^* \in \sin^2(\mathcal{X}^*)$ such that $p(z^*; \lambda) = 0$. Setting

$$\begin{aligned} \mathcal{T}_1^\pm &:= P_{\mathbb{R}}(\Sigma_1^{*\pm}), \\ \mathcal{X}_1^{*\pm} &:= P_M(\Sigma_1^{*\pm}), \\ \mathcal{Z}_1^{*\pm} &:= \sin^2(\mathcal{X}_1^{*\pm}), \end{aligned}$$

we have

$$\begin{aligned} \mathcal{T}_1 &= \mathcal{T}_1^+ \cup \mathcal{T}_1^-, \\ \mathcal{X}_1^* &= \mathcal{X}_1^{*+} \cup \mathcal{X}_1^{*-}, \\ \sin^2(\mathcal{X}_1^*) &= \mathcal{Z}_1^{*+} \cup \mathcal{Z}_1^{*-}, \end{aligned}$$

so

$$\mathcal{T} = \mathcal{T}_1^+ \cup \mathcal{T}_1^- \cup \mathcal{T}_2 \cup \{0\}. \tag{12}$$

Obviously $\mathcal{T}$ is symmetric to 0 so we shall analyse H^D at the positive thresholds only.

Lemma 4.4. A) Assume $\boldsymbol{\beta} = 0$. Then we have $\partial_{z_i}\tau^{\pm}(z) = \partial_{z_i}\Psi_0(z) > 0$ for all i.

B) Assume $\boldsymbol{\beta} \neq 0$ (so (A0) holds.) We set

$$\nu := \frac{2\alpha_3\sqrt{\beta_1\beta_2} - \sqrt{\gamma_3}(\beta_1+\beta_2)}{|\beta_3|\sqrt{\gamma_3}}. \tag{13}$$

Let $z \in [0,1]^3$ such that $K_0(z) \neq 0$.

1) We have $\partial_{z_i}\tau^+(z) > 0$ for $i = 1,2,3$, and $\partial_{z_i}\tau^-(z) > 0$ for $i = 1,2$.

2) a) Assume $\beta_2 = 0$ (so $\nu < 0$). Then $\partial_{z_3}\tau^-(z) > 0$.

b) Assume $\beta_2 > 0$.

(i) If $z_1 = 0$ or $z_2 = 0$ then $\partial_{z_3}\tau^-(z) > 0$.

(ii) The derivative $\partial_{z_3}\tau^-(1,1,z_3)$ vanishes iff $z_3 = \nu \in [0,1]$, and if $z_3 \neq \nu$ then $\partial_{z_3}\tau^-(1,1,z_3)$ has the same sign than $z_3 - \nu$.

Proof in Appendix A.

Remark 3. Let $\tilde{\nu} \in \mathbb{R}$ there exist $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ such that $\nu = \tilde{\nu}$. Proof in Appendix A.

Lemma 4.4 implies that the thresholds of the analytically fibered family $(H^D(x), x \in \mathbb{T}^3)$ come from the values $x \in \mathbb{T}^3$ such that $\partial_{x_i} z_i(x) = 0$ at least for $i = 1,2$, so $z_1, z_2 \in \{0,1\}$, and, in addition, we have $z_3 \in \{0, \nu, 1\}$.

We can determine now the set $\mathcal{T}$ of thresholds. Setting

$$\begin{aligned} Z_{\{0,1\}} &= \{0,1\}^3, \quad Z^*_{\{0,1\}} = Z_{\{0,1\}} \setminus \{0_{\mathbb{R}^3}\}, \\ X_{\{0,1\}} &= \{x \in \mathbb{T}^3;\ z \in Z_{\{0,1\}}\}, \quad X^*_{\{0,1\}} = \{x \in \mathbb{T}^3;\ z \in Z^*_{\{0,1\}}\}, \end{aligned}$$

we obtain the following (remember also (12))

Lemma 4.5. 1) Case $\boldsymbol{\beta} = 0$. We have $K_0 \equiv 0$, $\mathcal{X}_1 = \emptyset$, $\Sigma_1 = \emptyset$, $\mathcal{X}_2 = \mathbb{T}^3_0$. Then $\sin^2(\mathcal{X}_2^*) = Z^*_{\{0,1\}}$, $\mathcal{T}_1 = \emptyset$, $\mathcal{T} = \{0\} \cup \mathcal{T}_2$ and

$$\mathcal{T}_2 \cap \mathbb{R}^+ = \{\sqrt{\Psi_0(z)}; z \in Z^*_{\{0,1\}}\}.$$

2) Case $\boldsymbol{\beta} \neq 0$ (so (A0) holds). We set $\mathcal{Z}^*_\nu = \{(1,1,\nu)\} \cap [0,1]^3$. Then the sets $\mathcal{X}_j$, $j = 1,2$, are not trivial (remember Lemma 4.3). We have $\mathcal{Z}_1^{*+} = Z^*_{\{0,1\}} \setminus \{(\frac{\beta_2}{\beta_1}, 1, 0)\}$, $\mathcal{Z}_1^{*-} = Z^*_{\{0,1\}} \cup \mathcal{Z}^*_\nu \setminus \{(\frac{\beta_2}{\beta_1}, 1, 0)\}$ and $\sin^2(\mathcal{X}_2^*) = \{(\frac{\beta_2}{\beta_1}, 1, 0)\}$,

so

$$\begin{aligned}\mathcal{T}_1^+ \cap \mathbb{R}^+ &= \{\sqrt{\tau^+(z)},\ z \in Z^*_{\{0,1\}},\ z \neq (\frac{\beta_2}{\beta_1},1,0)\},\\ \mathcal{T}_1^- \cap \mathbb{R}^+ &= \{\sqrt{\tau^-(z)},\ z \in Z^*_{\{0,1\}} \cup \mathcal{Z}^*_\nu,\ z \neq (\frac{\beta_2}{\beta_1},1,0)\},\\ \mathcal{T}_2 \cap \mathbb{R}^+ &= \{\sqrt{\Psi_0(\frac{\beta_2}{\beta_1},1,0)}\}.\end{aligned}$$

Proof in Appendix A.

Remark 4. We have $\mathcal{X}_1^{*+} \subset \mathcal{X}_1^{*-}$ and the inclusion is an equality if and only if $\nu \notin (0,1)$.

Remark 5. Let us consider the case $\beta_2 = 0$. By definition of $\mathcal{T}_2$ we have

$$\mathcal{T}_2 \cap \mathbb{R}^+ = \{\sqrt{\Psi_0(z)};\ z \neq 0,\ \nabla_x\sqrt{\Psi_0(z)} = 0\}.$$

Then, the eigenvalues $\sqrt{\tau^\pm}$ are analytic in $[0,+\infty)^3 \setminus \{0_{\mathbb{R}^3}\}$, since the functions $\tau^\pm$ are linear and positive in $[0,+\infty)^3 \setminus \{0_{\mathbb{R}^3}\}$. Hence it would be an alternative to replace $\mathcal{T}_2$ by $\mathcal{T}_2'$ where we set

$$\begin{aligned}\mathcal{T}_2' &:= \mathcal{T}_2'^+ \cup \mathcal{T}_2'^-, \quad \mathcal{T}_2'^- := -\mathcal{T}_2'^+,\\ \mathcal{T}_2'^+ &:= \{\sqrt{\tau^\pm(z)};\ z \neq 0,\ \nabla_x\sqrt{\tau^\pm(z)} = 0\} = \mathcal{T}_2' \cap \mathbb{R}^+.\end{aligned}$$

But, thanks to Points B) 1) and 2) a) of Lemma 4.4 we obtain

$$\mathcal{T}_2'^+ = \{\sqrt{\tau^\pm(z)};\ z \in \{0,1\}^3,\ z \neq 0_{\mathbb{R}^3}\} = \mathcal{T}_2 \cap \mathbb{R}^+,$$

so the sets $\mathcal{T}_2'$ and $\mathcal{T}_2$ coincide.

5. The conjugated operator.

In this section we consider a set $\mathcal{T}' \subset \mathcal{T}$ and a function $\phi \in \mathcal{C}_c^\infty((0,+\infty) \setminus \mathcal{T}';\mathbb{R})$. We construct an adequate conjugated operator A_ϕ to H on $\operatorname{supp}\phi$.

5.1. Eigenprojectors.

If $\boldsymbol{\beta} = 0$ then $\Sigma_1 = \emptyset$ and the function $\sqrt{\Psi_0(y^2)}$ is analytic in $\mathbb{R}^3 \setminus \{0_{\mathbb{R}^3}\}$. The associated orthogonal eigenprojection

$$\pi_2(y) := \frac{1}{2i\pi}\int_{\mathcal{C}}(h^D(y) - \zeta)^{-1}d\zeta \quad \forall y \neq 0, \tag{14}$$

where $\mathcal{C} \subset \mathbb{C}$ is a complex contour containing $\sqrt{\Psi_0(z)}$ but not 0, is then analytic in $\mathbb{R}^3 \setminus \{0_{\mathbb{R}^3}\}$ and has range two.

Let us assume $\boldsymbol{\beta} \neq 0$. Let us denote by $\pi_1^\pm(y)$ the orthogonal eigenprojection on $\ker(h^D(y) - \sqrt{\tau^\pm(z)})$, i.e.,

$$\pi_1^\pm(y) := \frac{1}{2i\pi}\int_{\mathcal{C}}(h^D(y) - \zeta)^{-1}d\zeta \quad \forall y \in \sin(\mathcal{X}_1),$$

where $\mathcal{C}$ is a contour containing $\sqrt{\tau^\pm(z)}$ but no other eigenvalue of $h^D(y)$. Let again $\pi_2(y)$ defined by (14) where now $\mathcal{C}$ is a contour containing both $\sqrt{\tau^+(z)}$ and $\sqrt{\tau^-(z)}$ but no other eigenvalue. Thus $\pi_2(y)$ is the orthogonal eigenprojection on $\ker(h^D(y) - \sqrt{\tau^+(z)}) + \ker(h^D(y) - \sqrt{\tau^-(z)})$, and

$$\pi_2(y) = \pi_1^+(y) \oplus \pi_1^-(y) \quad \forall y \in \sin(\mathcal{X}_1).$$

Each $\pi_1^{\pm}(y)$ has range one and $\pi_2(y)$ has range two. Each $\pi_1^{\pm}$ is analytic at $y \in \sin(\mathcal{X}_1)$ (as subset of $\mathbb{R}^3$), π_2 is analytic on $\sin(\mathbb{R}^3) \setminus \{0_{\mathbb{R}^3}\}$ ($\subset \mathbb{R}^3$), and

$$\begin{aligned} h^D(y)\pi_2(y) &= \sqrt{\tau^+(z)}\pi_1^+(y) + \sqrt{\tau^-(z)}\pi_1^-(y) \quad \forall y \in \sin(\mathcal{X}_1), \\ h^D(y)\pi_2(y) &= \sqrt{\Psi_0(z)}\pi_2(y) \quad \forall y \in \sin(\mathcal{X}_2). \end{aligned}$$

5.2. **Global tangent field to $\mathcal{X}_2$.** If $\beta_2 = 0$ then the $\tau^{\pm}$'s and $\pi_1^{\pm}$'s extend analytically into $\mathbb{R}^3 \setminus \{0_{\mathbb{R}^3}\}$ with the relation

$$\pi_2(y) = \pi_1^+(y) + \pi_1^-(y) \quad \forall y \neq 0_{\mathbb{R}^3}.$$

In addition, in Case (A0) (with $\beta_2 = 0$), the sum $\pi_1^+(y) + \pi_1^-(y)$ is direct.

Let us assume $\beta_2 \neq 0$ (with Assumption (A0)) and make a precise description of $\mathcal{X}_2$. A point $x \in \mathbb{T}^3$ belongs to $\mathcal{X}_2$ iff $z \neq 0$ and $z_3 = 0 = \beta_1 z_1 - \beta_2 z_2$. The last relation can be written

$$\beta_1 y_1^2 = \beta_2 y_2^2, \quad y_2 \in [-1, 1] \setminus \{0\}.$$

When an nonzero eigenvalue of $H^D(x)$ (respect., of $h^D(y)$) is not simple then the stratification method explained in [3] involves a tangential vector field to the set $\mathcal{X}_2$ (respect., to $\sin(\mathcal{X}_2)$): $w(x) := (\sin(x_1)\cos(x_2), \cos(x_1)\sin(x_2), 0)$, (respect., $\tilde{w}(y) := (y_1, y_2, 0)$). We observe that $|w(x)| \neq 0$ for all $x \in \mathcal{X}_2 \setminus \mathcal{X}^*$, and $|w(x)| \neq 0$ for all $x \in \mathcal{X}_2$ if $\beta_2 \in (0, \beta_1)$. If $\beta_2 = \beta_1$ then w vanishes at all $x^* \in \mathcal{X}_2^*$ since $z_1^* = z_2^* = 1$.

We introduce the following notations. Letting a function f: $\mathbb{T}^3$ or $\mathbb{R}^3 \to \mathbb{C}^n$ and a vector field $v(x) = (v_j(x))_{1 \leq j \leq n} \in \mathbb{C}^n$, then $v \cdot \nabla_x f$ is the vectorial function $x \mapsto \sum_{j=1}^n v_j(x)\partial_{x_j} f(x) \in \mathbb{C}^n$. We set also

$$\begin{aligned} f_w &:= w \cdot \nabla_x f, \\ \tilde{f}_{\tilde{w}} &:= \tilde{w} \cdot \nabla_y f. \end{aligned}$$

We thus have

$$f_w(x) = \cos(x_1)\cos(x_2)\tilde{f}_{\tilde{w}}(y). \tag{15}$$

5.3. **First cut–off functions.** We consider the following metric on $\mathbb{T}^3 \approx (\mathbb{R}/(2\pi\mathbf{Z}))^3$:

$$\mathrm{d}_0(x, x^*) = |e^{ix} - e^{ix^*}| \quad x^*,\, x \in \mathbb{T}^3.$$

We denote $\mathrm{d}_0(x, E) = \inf\{\mathrm{d}_0(x, x^*) \mid x^* \in E\}$ when $E \subset \mathbb{T}^3$. We consider a cut–off function $\varphi_1 \in \mathcal{C}^\infty(\mathbb{R}; [0, 1])$ such that $\operatorname{supp}\varphi_1 \subset \{s \in \mathbb{R};\ |s| < 1\}$ and $\varphi_1 = 1$ in $\{s;\ |s| < 1/2\}$. Let b, b_0 with $0 < b < b_0/2$ two small parameters which will be precised later. We separate the eigenvalue 0 from $\mathbb{R}^+$, and, equivalently, $\mathcal{X}_0$ from $\mathbb{T}^3$, with the cut–off function $\chi_0(x) := \varphi_1(|z|/b_0)$. Since $\operatorname{supp}\phi \subset (0, \infty)$, we can fix b_0 sufficiently small such that:

$$\{\sqrt{\tau^+(z)}, \sqrt{\tau^-(z)}\} \cap \operatorname{supp}\phi \neq \emptyset \Rightarrow \chi_0(x) = 1.$$

In addition we set

$$\begin{aligned} \chi_{x^*}(x) &:= \varphi_1(\mathrm{d}_0(x, x^*)/b) \quad x^*,\, x \in \mathbb{T}^3, \\ \chi^{*+}(x) &:= (1 - \chi_0(x))\Pi_{x^* \in \mathcal{X}_1^{*+}}(1 - \chi_{x^*}), \\ \chi^{*-}(x) &:= (1 - \chi_0(x))\Pi_{x^* \in \mathcal{X}_1^{*-}}(1 - \chi_{x^*}), \\ \chi^{*}(x) &:= (1 - \chi_0(x))\Pi_{x^* \in \mathcal{X}^{*}}(1 - \chi_{x^*}), \end{aligned}$$

so χ^* vanishes in $\{x \in \mathbb{T}^3; \mathrm{d}_0(x, \mathcal{X}^*) < b/2\}$ and in $\{x \in \mathbb{T}^3; |z| < b_0/2\}$; we have also $\chi^*(x) = 1$ if $\mathrm{d}_0(x, \mathcal{X}^*) > b$ and $|z| > b_0$. It means that χ^* is a smooth cut-off

function localizing in the complement of $\mathcal{X}^* \cup \mathcal{X}_0$, and, since $\mathcal{X}^* \cup \mathcal{X}_0$ is a discrete set (and finite), we then have, for $b_0 > 0$ sufficiently small,

$$1 - \chi^*(x) = \chi_0(x) + \sum_{x^* \in \mathcal{X}^*} \chi_{x^*}(x),$$
$$1 - \chi^{*\pm}(x) = \chi_0(x) + \sum_{x^* \in \mathcal{X}_1^{*\pm}} \chi_{x^*}(x).$$

5.4. **The conjugated operator outside thresholds.** Case $\boldsymbol{\beta} = 0$. Remember that we have $\pi_1^+ = \pi_1^- = \pi_2$ which is analytic in $\mathbb{R}^3 \setminus \{0_{\mathbb{R}^3}\}$. We set, for $u \in \mathcal{C}^\infty(\mathbb{T}^3)$, $x \in \mathbb{T}^3$,

$$\mathcal{A}_{out} u\,(x) := i\chi^*(x)\pi_2(y) \frac{\nabla_x \sqrt{\Psi_0(z)}}{|\nabla_x \sqrt{\Psi_0(z)}|^2} \nabla_x (\chi^*(x)\pi_2(y)u(x)).$$

Case $\boldsymbol{\beta} \neq 0$ and $\beta_2 = 0$ (with Assumption (A0)). Remember that the functions $\tau^\pm(\cdot)$ are analytic in $\mathbb{R}^3$ (see (10) and (11)) and are positive in $[0,+\infty)^3 \setminus \{0_{\mathbb{R}^3}\}$. Thus, the eigenvalues $\sqrt{\tau^\pm(\cdot)}$ are analytic in $[0,+\infty)^3 \setminus \{0_{\mathbb{R}^3}\}$. We set, for $u \in \mathcal{C}^\infty(\mathbb{T}^3)$, $x \in \mathbb{T}^3$,

$$\mathcal{A}_{out} u\,(x) := \sum_{\pm} i\chi^*(x)\pi_1^\pm(y) \frac{\nabla_x \sqrt{\tau^\pm(z)}}{|\nabla_x \sqrt{\tau^\pm(z)}|^2} \nabla_x (\chi^*(x)\pi_1^\pm(y)u(x)).$$

Case $\beta_2 \neq 0$ (with Assumption (A0)). Firstly, we have $\mathcal{X}_1^{*+} \subset \mathcal{X}_1^{*-}$, but not necessarily the converse inclusion. Actually, if $\nu \in (0,1)$ (remember Lemma 4.5) then the value $\sqrt{\tau^-(1,1,\nu)}$ is a threshold be not necessarily $\sqrt{\tau^+(1,1,\nu)}$, so we may have $x_\nu^* \in \mathcal{X}_1^{*-} \setminus \mathcal{X}_1^{*+}$.
Secondly, in aim to have $H^D \in \mathcal{C}^\infty(A_{out})$, we need to separate $\mathcal{X}_2 \subset \partial\mathcal{X}_1$ from $\mathcal{X}_1$, as explained in [4]. Since $\overline{\mathcal{X}_2} = \mathcal{X}_2 \cup \mathcal{X}_0 = \{x \in \mathbb{T}^3; K_0(z) = 0\}$ is compact then there exist two smooth cut-off functions, χ_1 and χ_2 in $\mathcal{C}^\infty(\mathbb{T}^3;[0,1])$, such that $\operatorname{supp}\chi_2 \subset \{x;\ \mathrm{d}_0(x,\mathcal{X}_2) \leq 2b\}$, $\chi_2(x) = 1$ if $\mathrm{d}_0(x,\mathcal{X}_2) \leq b$, $\chi_1(x) = 1$ if $\mathrm{d}_0(x,\mathcal{X}_2) \geq 3b$, and $\operatorname{supp}\chi_1 \subset \{x;\ \mathrm{d}_0(x,\mathcal{X}_2) \geq 2b\}$. Thus $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$ and $\chi_2 = 1$ on $\overline{\mathcal{X}_2}$. We then set $\chi_3 := 1 - \chi_1 - \chi_2$ so $\operatorname{supp}\chi_3 \subset \{x;\ b \leq \mathrm{d}_0(x,\mathcal{X}_2) \leq 3b\}$, and

$$\sum_{j=1}^3 \chi_j^2(x) > 0 \quad \forall x \in \mathbb{T}^3.$$

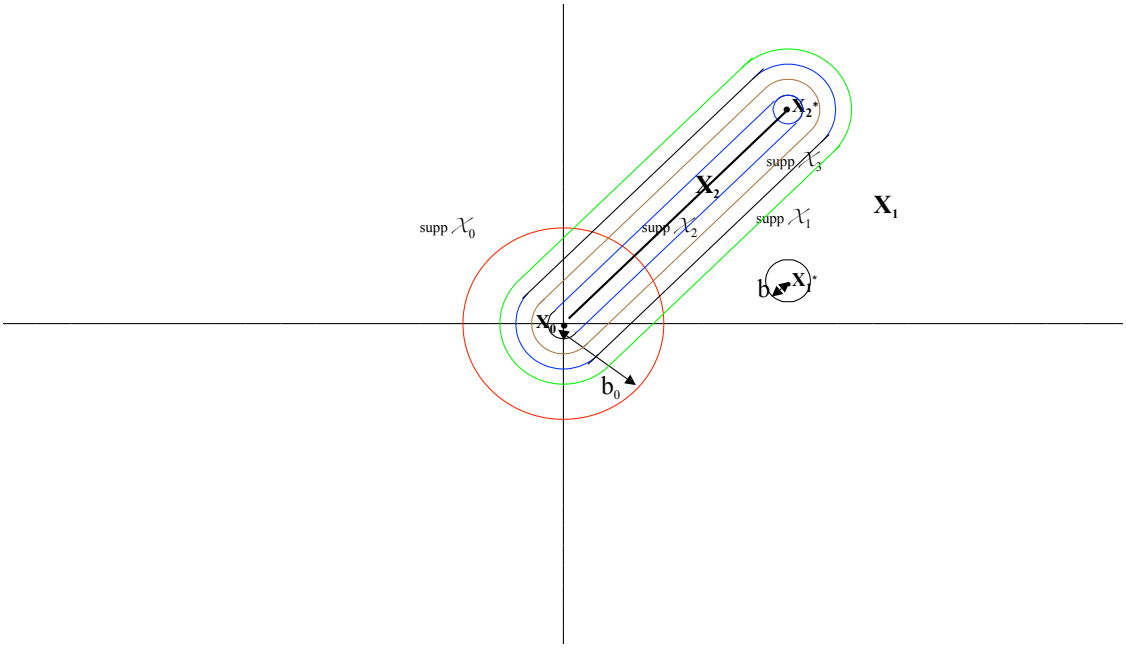
See Figure 1. We have in addition (b being sufficiently small)

$$\operatorname{supp}\chi_{x^*} \subset \operatorname{supp}\chi_1 \setminus \operatorname{supp}\chi_3 \quad \forall x^* \in \mathcal{X}_1^*, \tag{16}$$
$$\operatorname{supp}\chi_{x^*} \subset \operatorname{supp}\chi_2 \setminus \operatorname{supp}\chi_3 \quad \forall x^* \in \mathcal{X}_2^*. \tag{17}$$

$$\chi_j^*(x) := \chi^*(x)\chi_j(x) \quad j = 2,3$$
$$\chi_1^{*\pm}(x) := \chi^{*\pm}(x)\chi_1(x).$$

(In fact, Relations (17) and (17) imply $\chi_3^* = (1 - \chi_0)\chi_3$.) The function χ_2^* is a smooth cut-off localizing in $\mathcal{X}_2 \setminus \mathcal{X}^*$ while $\chi_1^{*\pm}$ is a smooth cut-off localizing in $\mathcal{X}_1 \setminus \mathcal{X}_1^{*\pm}$.

FIGURE 1. Cut-off



For $u \in \mathcal{C}^\infty(\mathbb{T}^3)$, $x \in \mathbb{T}^3$, we set,

$$\begin{aligned}\mathcal{A}_{out}u\,(x) \quad := \quad & \sum_{\pm} i\chi_1^{*\pm}(x)\pi_1^{\pm}(y)\frac{\nabla_x\sqrt{\tau^{\pm}(z)}}{|\nabla_x\sqrt{\tau^{\pm}(z)}|^2}\nabla_x(\chi_1^{*\pm}(x)\pi_1^{\pm}(y)u(x)) \\ & +i\chi_2^*(x)(\sqrt{\Psi_0(z)}_w)^{-1}\pi_2(y)\ (\chi_2^*(x)\pi_2(y)u(x))_w \\ & +\sum_{\pm} i\chi_3^*(x)(\sqrt{\Psi_0(z)}_w)^{-1}\pi_1^{\pm}(y)\ (\chi_3^*(x)\pi_1^{\pm}(y)u(x))_w.\end{aligned}$$

Remark 6. The function $\sqrt{\Psi_0(z)}_w$ may vanish at points of $\mathcal{X}^*$ but not of $\mathcal{X}_2 \setminus \mathcal{X}^*$ so $\chi_2^*(x)(\sqrt{\Psi_0(z)}_w)$ is well-defined (for b sufficiently small). For more details, see the proof of (19) below. Similarly, the function $x \mapsto |\nabla_x\sqrt{\tau^{\pm}(z)}|$ is positive in $\operatorname{supp}\chi_1^{*\pm}$.

In each case we symmetrize $\mathcal{A}_{out}$ by setting

$$A_{out} := \mathcal{A}_{out} + \mathcal{A}_{out}^*,$$

with domain $\mathcal{C}^\infty(\mathbb{T}^3)$. Here $\mathcal{A}_{out}^*$ is the hermitian conjugate of $\mathcal{A}_{out}$. By observing that the mappings $x \mapsto \chi_j(x)\pi_1^{\pm}(y)$ for $j = 1, 3$, and $x \mapsto (1-\chi_0(x))\pi_2(y)$ are smooth, then A_{out} is a symmetric first order differential operator in x whose coefficients belong to $\mathcal{C}^\infty(\mathbb{T}^3; \mathcal{L}(\mathbb{C}^6))$. It is then essentially self-adjoint on $\mathcal{H}^D$ (see [3, Lemma 3.10]). Since $D(H^D) = \mathcal{H}^D$, some possible problematic points of the Mourre Theory then become trivial (see [2]).

5.5. **"Punctual" Mourre's estimate outside thresholds.** We set

$$H_{1,out}(x) := [H^D, iA_{out}](x).$$

Similarly to proof of the Mourre's estimate in [3] we show that if the positive parameter b is sufficiently small then A_{out} is strictly conjugated to H on I.

Case $\beta_2 \neq 0$ under assumption (A0). Let $u \in \mathcal{C}^\infty(\mathbb{T}^3)$, we have

$$\begin{aligned}
&-i\mathcal{A}_{out} \circ H^D u\,(x) = \\
&\sum_{\pm} \chi_1^{*\pm}(x)\pi_1^{\pm}(y)\frac{\nabla_x\sqrt{\tau^{\pm}(z)}}{|\nabla_x\sqrt{\tau^{\pm}(z)}|^2}\nabla_x(\chi_1^{*\pm}(x)\pi_1^{\pm}(y)h^D(y)u(x)) \\
&+\chi_2^*(x)\pi_2(y)(\sqrt{\Psi_0(z)}_w)^{-1}\ (\chi_2^*(x)\pi_2(y)h^D(y)u(x))_w \\
&+\sum_{\pm} \chi_3^*(x)\pi_1^{\pm}(y)(\sqrt{\Psi_0(z)}_w)^{-1}\ (\chi_3^*(x)\pi_1^{\pm}(y)h^D(y))_w.
\end{aligned}$$

By using

$$(\pi_1^{\pm}(y))^2 = \pi_1^{\pm}(y), \quad \pi_1^{\pm}(y)h^D(y) = h^D(y)\pi_1^{\pm}(y), \quad \pi_2(y)h^D(y) = h^D(y)\pi_2(y),$$

we obtain the expression of $[H^D, i\mathcal{A}_{out}]$ as a multiplication operator:

$$\begin{aligned}
[H^D, i\mathcal{A}_{out}](x) &= (iH^D \circ \mathcal{A}_{out} - i\mathcal{A}_{out} \circ H^D)(x) \\
&= \sum_{\pm}(\chi_1^{*\pm}(x))^2\frac{\nabla_x\sqrt{\tau^{\pm}(z)}}{|\nabla_x\sqrt{\tau^{\pm}(z)}|^2}\pi_1^{\pm}(y)\nabla_x(\pi_1^{\pm}(y)h^D(y))\pi_1^{\pm}(y) \\
&\quad +(\chi_2^*(x))^2(\sqrt{\Psi_0(z)}_w)^{-1}\ \pi_2(y)(h^D(y)\pi_2(y))_w\pi_2(y) \\
&\quad +\sum_{\pm}(\chi_3^*(x))^2(\sqrt{\Psi_0(z)}_w)^{-1}\ \pi_1^{\pm}(y)(h^D(y)\pi_1^{\pm}(y))_w\pi_1^{\pm}(y).
\end{aligned}$$

In $\mathcal{X}_1$ we have

$$\pi_1^{\pm}(y)\nabla_x(h^D(y)\pi_1^{\pm}(y))\pi_1^{\pm}(y) = \nabla_x\sqrt{\tau^{\pm}(z)}\pi_1^{\pm}(y),$$

so,

$$\begin{aligned}
&|\nabla_x\sqrt{\tau^{\pm}}(z)|^{-2}\pi_1^{\pm}(y)\nabla_x\sqrt{\tau^{\pm}}(z)\cdot\nabla_x(h^D(y)\pi_1^{\pm}(y))\pi_1^{\pm}(y) = \pi_1^{\pm}(y), \\
&\sum_{\pm}\pi_1^{\pm}(y)(\sqrt{\tau^{\pm}}_w(x))^{-1}w\cdot\nabla_x(h^D(y)\pi_1^{\pm}(y)) = \sum_{\pm}\pi_1^{\pm}(y) = \pi_2(y).
\end{aligned}$$

Let us make the following computations near $\mathcal{X}_2$, precisely, in $\operatorname{supp}\chi_2^* \cup \operatorname{supp}\chi_3^*$. Setting $\xi(y) := h^D(y)\pi_2(y) - \sqrt{\Psi_0(z)}\pi_2(y)$, we have

$$\pi_2(y)\ w(x)\cdot\nabla_x(h^D(y)\pi_2(y))\pi_2(y) = \sqrt{\Psi_0(z)}(x)\pi_2(y) + \pi_2(y)\ \xi_w(x)\pi_2(y).$$

Thus,

$$\begin{aligned}
\frac{1}{2}H_{1,out}(x) &= \sum_{\pm}(\chi_1^{*\pm}(x))^2\pi_1^{\pm}(y) + ((\chi_2^*(x))^2 + (\chi_3^*(x))^2)\pi_2(y) \\
&\quad +(\chi_2^*(x))^2\pi_2(y)(\sqrt{\Psi_0(z)})^{-1}\ \xi_w(x)\pi_2(y). \qquad (18)
\end{aligned}$$

For $x \in \mathcal{X}_2$ we have $\xi(y) = 0$, so, since $\tilde{w}$ is a tangent field to $\sin(\mathcal{X}_2)$,

$$\tilde{\xi}_{\tilde{w}}(y) = 0, \quad \forall x \in \mathcal{X}_2.$$

For $x \in \mathcal{X}_2$ we have

$$\sqrt{\tilde{\Psi_0(z)}}_{\tilde{w}} = (\Psi_0(z))^{-1/2}(\alpha_1 z_1 + \alpha_2 z_2) > 0. \qquad (19)$$

In addition, since the relation

$$(\sqrt{\Psi_0(z)}_w)^{-1}\xi_w(x) = (\sqrt{\tilde{\Psi_0(z)}}_{\tilde{w}})^{-1}\tilde{\xi}_{\tilde{w}}(y)$$

holds true for $x \notin \mathcal{X}^*$ and $\sqrt{\tilde{\Psi_0(z)}}_{\tilde{w}} \neq 0$, then the function $(\sqrt{\Psi_0(z)}_w)^{-1}\xi_w$ is defined and is smooth in the compact set $\operatorname{supp}(1-\chi_0)\chi_2 \subset \operatorname{supp}\chi_2 \subset \{x \in \mathbb{T}^3;\ \mathrm{d}_0(x,\mathcal{X}_2) \leq 2b\}$, and vanishes on $\mathcal{X}_2$. Hence, for b sufficiently small, we have

$$\|(\sqrt{\Psi_0(z)}_w)^{-1}\xi_w(x)\|_\infty < \frac{1}{2} \quad \forall x \in \operatorname{supp}(1-\chi_0)\chi_2, \tag{20}$$

where $\|\cdot\|_\infty$ denotes here the usual infinite norm on matrices. From (18), (20), we then obtain

$$H_{1,out}(x) \geq \sum_{\pm} (\chi_1^{*\pm}(x))^2 \pi_1^{\pm}(y) + ((\chi_2^*(x))^2 + (\chi_3^*(x))^2)\pi_2(y). \tag{21}$$

Remark 7. In the two other cases where $\beta_2 = 0$ we obtain

$$\frac{1}{2}H_{1,out}(x) = (\chi^*(x))^2\pi_2(y), \tag{22}$$

so the punctual Mourre's estimate becomes simply

$$H_{1,out}(x) \geq 2(\chi^*(x))^2\pi_2(y).$$

5.6. **Smoothness.** Relations (22) and (18) show that the symmetric form $H_{1,out}$ defined on $\mathcal{C}^\infty(\mathbb{T}^3)$ is a multiplication operator on $\mathcal{H}^D$ by smooth coefficients, so is bounded and closeable. Thus, $[H_{1,out}, iA_{out}]$ is a differential operator of order one at most. But when computing it's first order term we have to check only that $H_{1,out}$ is commuting with each coefficient of the first order terms of $-i\mathcal{A}_{out}(x)$. In fact, the possible problematic bracket arising from the calculation of $[H_{1,out}, i\mathcal{A}_{out}]$ is, in the case $\beta_2 \neq 0$,

$$[(\chi_2^*(x))^2(\sqrt{\Psi_0(z)}_w)^{-1}\pi_2(y)\ \xi_w(x)\pi_2(y) \qquad , (\chi^{*\pm}(x))^2 \tfrac{\nabla_x\sqrt{\tau^\pm(z)}}{|\nabla_x\sqrt{\tau^\pm(z)}|^2} \cdot \\ \cdot\pi_1^\pm(y)\nabla_x(h^D(y)\pi_1^\pm(y))\pi_1^\pm(y)].$$

But since $\chi_1\chi_2 = 0$ then this bracket vanishes. Hence, $[H_{1,out}, iA_{out}]$ is a multiplication operator, is bounded in $\mathcal{H}$, and we have $H^D \in \mathcal{C}^2(A_{out})$. By induction we see that $H^D \in \mathcal{C}^\infty(A_{out})$. (See also [4].)

5.7. **The conjugated operator near thresholds.**

5.7.1. *Enumeration of the different cases.* Since our proof of the LAP at each threshold related to some $x^* \in \mathcal{X}^*$ requires a special treatment which depends on the values of $\boldsymbol{\beta}$ and of x^*, we enumerate the different cases as follows.

Case 1 $\boldsymbol{\beta} = 0$ and $x^* \in \mathcal{X}_2^*$.
Case 2 $\boldsymbol{\beta} \neq 0$ (so (A0) holds) and $x^* \in \mathcal{X}_1^*$.
  Subcase 2-1 $\beta_2 = 0$.
  Subcase 2-2 $\beta_2 > 0$.
    Subcase 2-2-a $x^* \in \mathcal{X}_1^{*-}$ and $z_1^* = z_2^* = 1$ and $z_3^* = \nu \in (0,1)$.
    Subcase 2-2-b $x^* \in \mathcal{X}_1^{*-}$ and $z_1^* = z_2^* = 1$ and $z_3^* \neq \nu$.
    Subcase 2-2-c $x^* \in \mathcal{X}_1^{*-}$ and $z_1^* = z_2^* = 1$ and $z_3^* = \nu \in \{0,1\}$.
    Subcase 2-2-d $x^* \in \mathcal{X}_1^{*-}$ and ($z_1^* = 0$ or $z_2^* = 0$).
    Subcase 2-2-e $x^* \in \mathcal{X}_1^{*+}$.
Case 3 $\boldsymbol{\beta} \neq 0$ and $x^* \in \mathcal{X}_2^*$.
  Subcase 3-1) $\beta_2 = 0$.
  SubCase 3-2 $\beta_2 = \beta_1$.
  SubCase 3-3 $\beta_2 \in (0, \beta_1)$.

Remark 8. In Case 2-2-c, if $\beta_1 = \beta_2$ then $(1,1,0) \in \sin^2 \mathcal{X}_2^*$ so $\nu = 1$.

5.7.2. *Behaviour of the eigenvalues of $H^D(x)$ at a threshold.* We set $s_j^* = 1 - 2z_j^*$ if $z_j^* \in \{0,1\}$, $j \in [[1,3]]$, so $s_j^* \in \{-1,1\}$. We set also $s_j = s_j^*$ for $j = 1,2$.
In Case 1 we set $V = \sqrt{} \circ \Psi_0 \circ \sin^2$ and $s_3 := s_3^*$.
In Case 2 with $x^* \in \mathcal{X}_1^{*\pm}$ and in Case 3-1 we set $V = \sqrt{} \circ \tau^{\pm} \circ \sin^2$ and
- in Cases 2-1 and 2-2-d and 2-2-e, and 3-1 we set $s_3 := s_3^*$;
- in Case 2-2-a we set $s_3 := 1$;
- in Case 2-2-b we set $s_3 := \mathrm{sgn}(z_3^* - \nu) s_3^*$.

Lemma 5.1. In Cases 1 and 2-1 and 2-2-a and 2-2-b and 2-2-d and 2-2-e and 3-1) we have

$$\mathrm{d}V(x) = (\sum_{j=1}^{3} C_j s_j (x_j - x_j^*) \mathrm{d}x_j)(1 + O(\mathrm{d}_0(x,x^*))), \tag{23}$$

as $x \to x^*$, where $C_j > 0$, $j = 1,2,3$.
In Case 2-2-c we have

$$\mathrm{d}V(x) \quad = \quad (-\sum_{j=1}^{2} C_j (x_j - x_j^*) \mathrm{d}x_j + C_3 (x_3 - x_3^*)^3 \mathrm{d}x_3)(1 + O(\mathrm{d}_0(x,x^*))), \tag{24}$$

as $x \to x^*$, where $C_j > 0$, $j = 1,2,3$.

Proof in Appendix B.

Lemma 5.2. Consider Cases 3-2 or 3-3 (i.e., Assumption (A0) with $\beta_2 \neq 0$ and $x^* \in \mathcal{X}_2^*$). We then have the following estimates.
In Case 3-2,

$$(\sqrt{\Psi_0(z)}_w) = C(x_1 - x_1^*)(x_2 - x_2^*)(1 + O(\mathrm{d}_0(x,x^*))), \tag{25}$$

and, in Case 3-3,

$$(\sqrt{\Psi_0(z)}_w) = C(x_2 - x_2^*)(1 + O(\mathrm{d}_0(x,x^*))), \tag{26}$$

for some $C \neq 0$ as $x \to x^*$.

Proof in Appendix B.

5.7.3. *Partition of the set of thresholds.* With the notations (the s_j's notably) of Part 5.7.2 we set

$$\begin{array}{rcl} \mathcal{T}_{sa} & := & \mathcal{T} \setminus (\mathcal{T}_{sm} \cup \{0\}), \\ \mathcal{T}_{sm} & := & \mathcal{T}_{sm}^+ \cup -\mathcal{T}_{sm}^+, \\ \mathcal{T}_{sm}^+ & := & \mathcal{T}_{ms,1}^+ \cup \mathcal{T}_{ms,2}^+, \end{array}$$

with, in Case $\boldsymbol{\beta} = 0$:

$$\begin{array}{rcl} \mathcal{T}_{ms,1}^+ & := & \emptyset, \\ \mathcal{T}_{ms,2}^+ & := & \{\sqrt{\Psi_0(z^*)};\ x^* \in \mathcal{X}_2^* (= X_{\{0,1\}}^*),\ s_1 = s_2 = s_3 = \pm 1\}, \end{array}$$

in Case $\boldsymbol{\beta} \neq 0$:

$$\begin{array}{rcl} \mathcal{T}_{ms,1}^+ & := & \{\sqrt{\tau^+(z^*)}, \sqrt{\tau^-(z^*)};\ x^* \in \mathcal{X}_1^*,\ s_1 = s_2 = s_3 = \pm 1\} \cup \mathcal{T}_\nu, \\ \mathcal{T}_{ms,2}^+ & := & \{\sqrt{\Psi_0(\frac{\beta_2}{\beta_1}, 1, 0)}\}, \end{array}$$

with

$$\begin{array}{rcl}\mathcal{T}_\nu & := & \{\sqrt{\tau^-(z^*)};\ x^*\in\mathcal{X}_1^*,\ s_1=s_2=1(=:s_3)\} \quad \nu\in(0,1),\\ \mathcal{T}_\nu & := & \emptyset \quad \nu\notin(0,1).\end{array}$$

5.7.4. *New coordinate near an element of $\mathcal{X}^*$.* We give an approximation of a vector proportional to $\nabla_x V(x)$ (where V is defined in Part 5.7.2) of the form $\nabla_x p_1$ near a point $x^*\in\mathcal{X}_j^*$. We then give an approximation of a vector proportional to $w(x)$ near a point $x^*\in\mathcal{X}_2^*$ in Cases 3-2 and 3-3.

With the notations of Lemma 5.1, in Cases 1, 2-1, 2-2-a, 2-2-b, 2-2-d, 2-2-e and 3-1, we set

$$p_1(x;x^*)=\frac{1}{2}\sum_{j=1}^{3}C_j s_j(x_j-x_j^*)^2;$$

in Case 2-2-c, we set

$$p_1(x;x^*)=\frac{1}{2}\sum_{j=1}^{2}C_j(x_j-x_j^*)^2-\frac{1}{4}C_3(x_3-x_3^*)^4.$$

Then, Relations (23) and (24) of Lemma 5.1 can be written

$$\mathrm{d}V(x)=(1+O(\mathrm{d}_0(x,x^*)))\mathrm{d}p_1(x;x^*),\quad x\to x^*.$$

5.7.5. *The conjugated operator near thresholds.* Let $x^*\in\mathcal{X}^*$. For simplicity we then write $p_1(x;x^*)=p_1(x)$. For $u\in\mathcal{D}_0$ and $x\in\mathbb{T}^3\setminus\{x^*\}$ we set,
- in Case 1 ($\boldsymbol{\beta}=0$, $x^*\in\mathcal{X}_2^*$):

$$\mathcal{A}_{x^*}u\,(x):=i\chi_{x^*}(x)\pi_2(y)\frac{\nabla_x p_1(x)}{\nabla_x p_1(x)\cdot\nabla_x\sqrt{\Psi_0(z)}}\nabla_x(\chi_{x^*}(x)\pi_2(y)u(x)),$$

- in Case 2 ($\boldsymbol{\beta}\neq0$, $x^*\in\mathcal{X}_1^{*\pm}$) and Case 3-1) ($\boldsymbol{\beta}\neq0$, $\beta_2=0$, $x^*\in\mathcal{X}_2^*$):

$$\mathcal{A}_{x^*}^{\pm}u\,(x):=i\chi_{x^*}(x)\pi_1^{\pm}(y)\frac{\nabla_x p_1(x)}{\nabla_x p_1(x)\cdot\nabla_x\sqrt{\tau^{\pm}(z)}}\nabla_x(\chi_{x^*}(x)\pi_1^{\pm}(y)u(x)),$$

and $\mathcal{A}_{x^*}:=\mathcal{A}_{x^*}^{+}+\mathcal{A}_{x^*}^{-}$,
- in Cases 3-2 and 3-3 ($\beta_2\neq0$, $x^*\in\mathcal{X}_2^*$):

$$\mathcal{A}_{x^*}u\,(x):=i\chi_{x^*}(x)\pi_2(y)(\sqrt{\Psi_0(z)}_w)^{-1}(\chi_{x^*}(x)\pi_2(y)u(x))_w.$$

In each case we symmetrize $\mathcal{A}_{x^*}$ and $\mathcal{A}_{x^*}^{\pm}$ by setting

$$A_{x^*}:=\mathcal{A}_{x^*}+\mathcal{A}_{x^*}{}^*,\quad A_{x^*}^{\pm}:=\mathcal{A}_{x^*}^{\pm}+(\mathcal{A}_{x^*}^{\pm})^*,$$

where $\mathcal{A}_{x^*}{}^*$ (respect., $(\mathcal{A}_{x^*}^{\pm})^*$) denotes the formal adjoint to $\mathcal{A}_{x^*}$ (respect., to $\mathcal{A}_{x^*}^{\pm}$). It is defined on $\mathcal{D}_0$ too.

We set

$$\begin{array}{rcl}\mathcal{T}_{in} & := & (\mathcal{T}\setminus\mathcal{T}')\cap(0,+\infty),\\ \mathcal{X}_{1,in}^{*\pm} & := & \{x\in\mathcal{X}_1^{*\pm};\sqrt{\tau^{\pm}(z)}\in\mathcal{T}_{in}\},\\ \mathcal{X}_{2,in}^{*} & := & \{x\in\mathcal{X}_2^{*};\sqrt{\Psi_0(z)}\in\mathcal{T}_{in}\}.\end{array}$$

(We have, in Case 1, $\mathcal{X}_{1,in}^{*\pm}=\emptyset$.) We set

$$A_{in}:=\sum_{x^*\in\mathcal{X}_{2,in}^*}A_{x^*}+\sum_{\pm}\sum_{x^*\in\mathcal{X}_{1,in}^{*\pm}}A_{x^*}^{\pm}.$$

Then the operator A_{in} with domain $\mathcal{D}_0$ is symmetric, closable and densely defined on $\mathcal{H}^D$.

We set, as quadratic forms defined on $\mathcal{D}_0$,

$$H_{1,x^*} := [H^D, iA_{x^*}], \quad H_{1,in} := [H^D, iA_{in}].$$

By a straight calculation as in Part 5.5 we obtain

Lemma 5.3. We have for $x \neq x^*$, in Cases 1 and 2:

$$\frac{1}{2} H_{1,x^*}(x) = (\chi_{x^*}(x))^2 \pi_2(y),$$

and, in Cases 3-2 and 3-3,

$$\frac{1}{2} H_{1,x^*}(x) = (\chi_{x^*}(x))^2 \pi_2(y) + (\chi_{x^*}(x))^2 \pi_2(y) (\sqrt{\Psi_0(z)}_w)^{-1} \, \xi_w(x) \pi_2(y).$$

Lemma 5.3 shows that the quadratic forms H_{1,x^*} and $H_{1,in}$ extend continuously as bounded quadratic forms on $\mathcal{H}^D$ which are associated with bounded self-adjoint operators, as multiplication operators by smooth real symmetric coefficients, denoted, respectively, H_{1,x^*} and $H_{1,in}$. In addition, these coefficients (as functions of x) are commuting with $\pi_2(y)$. Then, an obvious iteration shows that $H^D \in \mathcal{C}^\infty(A_{x^*})$ for all $x^* \in \mathcal{X}^*$. Since $x \neq x'$ implies $\operatorname{supp} \chi_x \cap \operatorname{supp} \chi_{x'} = \emptyset$ then $H^D \in \mathcal{C}^\infty(A_{in})$.

We set

$$A_\phi := A_{out} + A_{in}.$$

The argumentation to prove the property $H^D \in \mathcal{C}^\infty(A_{out})$ at Part 5.6 still holds with A_{out} replaced by A_ϕ, so we obtain

Lemma 5.4. The quadratic form $H_{1,\phi} := [H^D, iA_\phi]$ defined on $\mathcal{D}_0$ defines a bounded self-adjoint multiplication operator on $\mathcal{H}^D$. In addition, $H^D \in \mathcal{C}^\infty(A_\phi)$.

5.8. **"Punctual" Mourre's estimate.** Let us prove that

$$\phi(H^D)(x) H_{1,\phi}(x) \phi(H^D)(x) \geq C \phi^2(H^D)(x), \tag{27}$$

for all $x \in \mathbb{T}^3$, where $C > 0$ does not depend on x but on ϕ only.

We consider the case $\beta_2 \neq 0$ (under assumption (A0)) only. The other case $\beta_2 \neq 0$ is more simple and omitted. As in Part 5.5 (see (18)) the calculation of $H_{1,\phi}$ yields

$$\begin{aligned}
\frac{1}{2} H_{1,\phi}(x) \quad = \quad & \sum_{\pm} (\chi_1^{*\pm}(x))^2 \pi_1^\pm(y) + ((\chi_2^*(x))^2 + (\chi_3^*(x))^2) \pi_2(y) \\
& + (\chi_2^*(x))^2 \pi_2(y) T_w(x)^{-1} \, \xi_w(x) \pi_2(y) \\
& + \sum_{\pm} \sum_{x^* \in \mathcal{X}_{1,in}^{*\pm} \cup \mathcal{X}_{1,in}^{*-}} (\chi_{x^*}(x))^2 \pi_1^\pm(y) + \sum_{x^* \in \mathcal{X}_{2,in}^*} (\chi_{x^*}(x))^2 \pi_2(y) \\
& + \sum_{x^* \in \mathcal{X}_{2,in}^*} (\chi_{x^*}(x))^2 \pi_2(y) (\sqrt{\Psi_0(z)}_w)^{-1} \, \xi_w(x) \pi_2(y) \quad x \notin \mathcal{X}^* \cup \mathcal{X}_0.
\end{aligned}$$

Thus, as for Inequality (21), we get,

$$\begin{aligned}
H_{1,\phi}(x) \quad \geq \quad & \sum_{\pm} (\chi_1^{*\pm}(x))^2 \pi_1^\pm(y) + ((\chi_2^*(x))^2 + (\chi_3^*(x))^2) \pi_2(y) \\
& + \sum_{\pm} \sum_{x^* \in \mathcal{X}_{1,in}^{*\pm} \cup \mathcal{X}_{1,in}^{*-}} (\chi_{x^*}(x))^2 \pi_1^\pm(y) + \sum_{x^* \in \mathcal{X}_{2,in}^*} (\chi_{x^*}(x))^2 \pi_2(y).
\end{aligned} \tag{28}$$

Let us fix $x \in \operatorname{supp}\phi(H^D(x))$. Thus $\chi_0(x) = 1$. We consider the following cases.

1. Case $\mathrm{d}_0(x, \mathcal{X}^*) \geq b$. Then, $x \notin \operatorname{supp}\chi_{x^*}$ for any $x^* \in \mathcal{X}^*$, and $\chi^*(x) = \chi^{*\pm}(x) = 1$. Hence (28) becomes
$$\begin{aligned} H_{1,\phi}(x) &\geq \sum_{\pm} \chi_1^2(x)\pi_1^{\pm}(y) + (\chi_2^2(x) + \chi_3^2(x))\pi_2(y) = \sum_{j=1}^{3} \chi_j^2(x)\pi_2(y) \\ &\geq \delta_0\pi_2(y), \end{aligned}$$
where $\delta_0 := \min_{\mathbb{T}^3} \sum_{j=1} \chi_j^2 > 0$. Since $\phi(H^D(x))\pi_2(x) = \phi(H^D(x))$, then (27) holds.
2. Case $\mathrm{d}_0(x, \mathcal{X}^*) < b$. Then there exists exactly one $x^* \in \mathcal{X}^*$ such that $x \in \operatorname{supp}\chi_{x^*}$ and $x \notin \operatorname{supp}\chi_{x'}$ if $x' \in \mathcal{X}^* \setminus \{x^*\}$. We set
$$\delta(x^*) := \min_{\{\chi_0=1\}} (1 - \chi_{x^*})^2 + (\chi_{x^*})^2 > 0.$$
If $x^* \notin \mathcal{X}^*_{2,in} \cup \mathcal{X}^{*+}_{1,in} \cup \mathcal{X}^{*-}_{1,in}$ then $x \notin \operatorname{sup}\phi(H^D)(x)$ so (27) is trivial. We thus assume $x^* \in \mathcal{X}^*_{2,in} \cup \mathcal{X}^{*+}_{1,in} \cup \mathcal{X}^{*-}_{1,in}$.
   (a) Case $x^* \in \mathcal{X}^*_{2,in}$. Thus $x \notin \operatorname{supp}\chi_1 \cup \operatorname{supp}\chi_3$ and
   $$\chi_2^*(x) = (1 - \chi_{x^*}(x))\chi_2(x) = (1 - \chi_{x^*}(x)).$$
   Hence (28) becomes
   $$H_{1,\phi}(x) \geq (\chi_2^*(x))^2\pi_2(y) + (\chi_{x^*}(x))^2\pi_2(y) \geq \delta(x^*).$$
   Thus (27) holds.
   (b) Case $x^* \in \mathcal{X}^{*+}_{1,in}$ (which is included in $\mathcal{X}^{*-}_{1,in}$ and does not intersect $\mathcal{X}^*_{2,in}$). Thus $x \notin \operatorname{supp}\chi_2 \cup \operatorname{supp}\chi_3$ and
   $$\chi_1^{*\pm}(x) = (1 - \chi_{x^*}(x))\chi_1(x) = 1 - \chi_{x^*}(x).$$
   Hence (28) becomes
   $$\begin{aligned} H_{1,\phi}(x) &\geq \sum_{\pm} ((1 - \chi_{x^*}(x))^2 + (\chi_{x^*}(x))^2)\pi_1^{\pm}(y) \\ &= ((1 - \chi_{x^*}(x))^2 + (\chi_{x^*}(x))^2)\pi_2(y) \\ &\geq \delta(x^*)\pi_2(y). \end{aligned}$$
   Thus (27) holds.
   (c) Case $x^* \in \mathcal{X}^{*-}_{1,in} \setminus \mathcal{X}^{*+}_{1,in}$ (which does not intersect $\mathcal{X}^*_{2,in}$). Thus $x \notin \operatorname{supp}\chi_2 \cup \operatorname{supp}\chi_3$ and
   $$\chi_1^{*-}(x) = (1 - \chi_{x^*}(x))\chi_1(x) = 1 - \chi_{x^*}(x), \quad \chi_1^{*+}(x) = 0$$
   Hence (28) becomes
   $$\begin{aligned} H_{1,\phi}(x) &\geq \sum_{\pm} ((1 - \chi_{x^*}(x))^2 + (\chi_{x^*}(x))^2)\pi_1^{-}(y) \\ &\geq \delta(x^*)\pi_1^{-}(y). \end{aligned}$$
   But we have also
   $$\phi(H^D)(x) = \phi(\sqrt{\tau^-(x)})\pi_1^-(y).$$
   Thus (27) holds.

As conclusion, (27) is proved with $C = \min(\delta_0, \min_{\mathcal{X}^*} \delta(x^*))$. □

5.9. **Self-adjointness and maximal monotonicity of parts of the conjugated operator.** The conjugate operator A_ϕ with domain $\mathcal{D}_0$ is a symmetric first order differential operator in x whose coefficients belong to $\mathcal{C}^\infty(\mathbb{T}^3 \setminus \mathcal{X}^*; \mathcal{L}(\mathbb{C}^6))$. As we already saw, A_{out} with domain $\mathcal{C}^\infty(\mathbb{T}^3)$ is essentially self-adjoint and a self-adjoint extension is $\overline{A_{out}} = A_{out}$ with domain $D(\overline{A_{out}}) = \{u \in \mathcal{H}^D;\, A_{out}u \in \mathcal{H}^D\}$. (We may observe that $D(\overline{A_{out}})$ is also the closure of $\mathcal{C}^\infty(\mathbb{T}^3)$ under the norm $\|u\| + \|A_{out}u\|$.) Let us check that for all $x^* \in \mathcal{X}^*$ the operator A_{x^*} (with the same domain $\mathcal{D}_0$) is essentially self-adjoint or, at least, admits a maximal symmetric extension.

Lemma 5.5. Remembering the notations of Section 5.7.2 we then claim:

A) Cases 1 and 2-1 and 2-2-a and 2-2-b and 3-1. If $\{s_1, s_2, s_3\} = \{-1, 1\}$, then A_{x^*} is essentially self-adjoint on $\mathcal{H}^D$. Otherwise, i.e., if all the s_j's have the same sign, then A_{x^*}admits a maximal symmetric extension on $\mathcal{H}^D$.
B) Case 2-2-c. The operator A_{x^*} is essentially self-adjoint on $\mathcal{H}^D$.
C) Cases 3-2 and 3-3 (so we have (A0) with $\beta_2 \in (0, \beta_1]$, $x^* \in \mathcal{X}_2^*$, $|y_2^*| = 1$). The operator A_{x^*} admits a maximal symmetric extension on $\mathcal{H}^D$.

Proof in Appendix B.

Remark 9. When A_{x^*} is essentially self-adjoint then the set $\mathcal{D}_0$ is not dense in the domain of the self-adjoint extension $\overline{A_{x^*}}$ of A_{x^*}. (The simple reason is that $\mathcal{C}_c^\infty(\mathbb{R}^*)$ is not dense in $H^1(\mathbb{R})$.)

We set

$$\mathcal{X}_{sa}^* := \{x^* \in \mathcal{X}^*;\ A_{x^*} \text{ is essentially self-adjoint}\}, \quad \mathcal{X}_{sm}^* := \mathcal{X}^* \setminus \mathcal{X}_{sa}^*,$$

and

$$A_{sa} := \sum_{x^* \in \mathcal{X}_{sa}^*} A_{x^*},\ D(A_{sa}) := \mathcal{D}_0.$$

Corollary 6. 1) The operator A_{sa} is essentially self-adjoint on $\mathcal{H}^D$.
2) If $\mathcal{T}_{sm} \subset \mathcal{T}'$ then the operator A_ϕ defined on $\mathcal{D}_0$ is essentially self-adjoint on $\mathcal{H}^D$.

Proof. 1) The operator A_{sa} is the finite sum of essentially self-adjoint operators A_{x^*} defined on $\mathcal{D}_0$ and with disjoint supports so A_{sa} is essentially self-adjoint too.
2) For simplicity we assume that $\mathcal{T}' = \mathcal{T}_{sm}$ and we consider the case $\beta_2 \neq 0$ only. The operator $\bar{A} := A_\phi$ with domain $D(\bar{A}) := \{u \in \mathcal{H}^D;\, A_\phi u \in \mathcal{H}^D\}$ is a symmetric extension of A_ϕ. Let us prove that it is self-adjoint. Let $v \in D(\bar{A}^*)$ so

$$|(\bar{A}u, v)_{\mathcal{H}^D}| \leq C\|u\| \quad \forall u \in D(\bar{A}).$$

Let $u \in D(\bar{A})$. Let $\varphi_1 \in \mathcal{C}^\infty(\mathbb{T}^3 \setminus \mathcal{X}_{sm}^*; [0,1])$ such that $\operatorname{supp}\varphi_1$ is a small neighbourhood of $\mathcal{X}_{sa}^*$ and $\varphi_1 = 1$ near $\mathcal{X}_{sa}^*$. Setting $B := \varphi_1 A_\phi \varphi_1$, since $\nabla\varphi_1$ vanishes near $\mathcal{X}^*$ then $B - A_\phi\varphi_1^2$ is bounded on $\mathcal{H}^D$, $\varphi_1^2 u \in D(\bar{A})$ and we get

$$|(Bu, v)_{\mathcal{H}^D}| \leq |(\bar{A}(\varphi_1^2 u), v)_{\mathcal{H}^D}| + C'\|u\| \leq C''\|u\|.$$

In addition, we have $B = \varphi_1 A_{sa} \varphi_1$ since $\bar{A}$ coincides with A_{sa} in $\operatorname{supp}\varphi_1$, so B is essentially self-adjoint (the proof is similar to those of A_{sa}). Hence, $Bv \in \mathcal{H}^D$ and then $\varphi_1^2 v \in D(\bar{A})$. Let $\varphi_2 \in \mathcal{C}_c^\infty(\mathbb{T}^3 \setminus \mathcal{X}^*; [0,1])$. Then, $\varphi_2 A_\phi \varphi_2 - A_\phi \varphi_2^2$ is bounded on $\mathcal{H}^D$, $\varphi_2^2 u \in D(\bar{A})$ and

$$|(\varphi_2 \bar{A}(\varphi_2 u), v)_{\mathcal{H}^D}| \leq |(A(\varphi_2^2 u), v)_{\mathcal{H}^D}| + C'\|u\| \leq C''\|u\|.$$

Since $\varphi_2 A_\phi \varphi_2$ is a symmetric first order differential operator with smooth coefficients it is so essentially self-adjoint and we get $\varphi_2 A_\phi \varphi_2 v \in \mathcal{H}^D$, and $\varphi_2^2 v \in D(\bar{A})$.

Letting φ_1 such that its derivatives at any order vanish on $\varphi_1^{-1}(\{1\})$ we can choose $\varphi_2 := \sqrt{1-\varphi_1^2}$. Then $v = \sum_{j=1}^2 \varphi_j^2 v \in D(\bar{A})$. □

6. **Proofs of the main results.**

6.1. **Proof of Theorem 3.1.** Clearly, it is not restrictive to consider that $\operatorname{supp}\phi \subset (0,+\infty)$ so $\phi \in \mathcal{C}_c^\infty((0,+\infty)\setminus\mathcal{T}')$. We then construct the operators A_{out}, A_{in}, A_ϕ as above. Thanks to Lemma 5.4, the operator A_ϕ satisfies Point (ii). Point (i) is a straight consequence of (27).

Proof of Point (iii). We consider the cases $\operatorname{supp}\phi \subset (0,+\infty)$ and $\beta_2 \neq 0$ only. We have $\operatorname{supp} A_\phi \subset \mathcal{X}_A$ where we set

$$\mathcal{X}_A := \operatorname{supp}\chi_1^{*+} \cup \operatorname{supp}\chi_1^{*-} \cup \operatorname{supp}\chi_2^* \cup \operatorname{supp}\chi_3^* \cup_{x^*\in\mathcal{X}_{1,in}^{*+}\cup\mathcal{X}_{1,in}^{*-}\cup\mathcal{X}_{2,in}^*} \operatorname{supp}\chi_{x^*}.$$

The set $\mathcal{K} := \cup_\pm \sqrt{\tau^\pm(\sin^2(\mathcal{X}_A))}$ is then a compact subset of $(0,\infty)\setminus\mathcal{T}'$. Thus there exists $\tilde{\phi} \in \mathbb{C}_c^\infty((0,\infty)\setminus\mathcal{T}')$ with $\tilde{\phi} = 1$ in $\mathcal{K}$. Thus if $x \in \operatorname{supp} A_\phi \cap \mathcal{X}_1$ then

$$\tilde{\phi}(H^D)(x) = \sum_\pm \tilde{\phi}(\sqrt{\tau^\pm(z)})\pi_1^\pm(y) = \pi_2(y),$$

and, if $x \in \operatorname{supp} A_\phi \cap \mathcal{X}_2$, then

$$\tilde{\phi}(H^D)(x) = \tilde{\phi}(\sqrt{\Psi_0(z)})\pi_2(y) = \pi_2(y).$$

Hence, $\tilde{\phi}(H^D)(x) = \pi_2(y)$ on $\operatorname{supp} A_\phi$. In addition, $A_\phi(x)$ is obviously commuting with $\pi_2(y)$ for all x. It shows that Point (iii) holds.

Point (iv) is Point 2) of Corollary 6.

Point (v) is the consequence of Corollary 6.

6.2. **Adaptation of the theory of Georgescu and alii.** Notation: if Q is a bounded quadratic form on $\mathcal{H}$ we denote by Q° the bounded operator associated with Q. Let us consider the case $\mathcal{T}' = \{0\}$ so A_{in} may not be essentially self-adjoint. We set

$$\mathcal{X}_{sm1}^* := \{x^* \in \mathcal{X}_{sm}^*;\ A_{x^*} \text{ has default index } (N^+, N^- = 0)\},$$
$$\mathcal{X}_{sm2}^* := \{x^* \in \mathcal{X}_{sm}^*;\ A_{x^*} \text{ has default index } (N^+ = 0, N^-)\}.$$

We write

$$A_\phi = A_0 + A_1 + A_2$$

where all the A_j are differential operators of first order defined at least on $\mathcal{D}_0$ by: $A_0 = A_{out}$, $A_1 = \sum_{x^*\in\mathcal{X}_{sm1}^*\cup\mathcal{X}_{sa}^*} A_{x^*}$, $A_2 = \sum_{x^*\in\mathcal{X}_{sm2}^*} A_{x^*}$. The proof of Corollary 6 shows that the operator A_0 is essentially self-adjoint.

Remark 10. We could have set more naturally $A_0 = A_{sa} + A_{out}$, $A_1 = \sum_{x^*\in\mathcal{X}_{sm1}^*}$ and A_2 unchanged. In such a choice some coefficients of A_0 have a rational singularity on $\mathcal{X}_{sa}^*$.

Since the supports of the A_{x^*}, $x^* \in \mathcal{X}^*$, are two-by-two disjoint then the operators $\pm A_0$ and $(-1)^j A_j$, $j = 1, 2$, admit a maximal symmetric extension with deficiency index of the form $(N, 0)$. We denote by A_j^{sm} with domain $D(A_j^{sm})$ the maximal symmetric extension of A_j (with domain $\mathcal{D}_0$).

Let us show that we can modify the main hypotheses (M1)-(M5) of [2, Theorem 3.3] and extend the statement of [2, Theorem 3.3] to our situation. We consider variables $z \in \rho(H^D)$ and ε real with $0 < |\varepsilon| < \varepsilon_0$ and $\Im m(z)\varepsilon \geq 0$. We set $H_\varepsilon := H^D - i\varepsilon H'$. (Thus $H_\varepsilon^* = H_{-\varepsilon}$.) Then, the resolvent $R_\varepsilon(z) := (H_\varepsilon - z)^{-1}$ is

well-defined if ε_0 is sufficiently small, see [2, Proposition 3.11]. Actually we make the following observations:

- The domain $\mathcal{D}_0$ of A_ϕ is dense in $\mathcal{H}^D$.
- Assumption [2, (M3)] becomes:
  (M3*): [$\pm A_0^{sm}$ (respect., $(-1)^j A_j^{sm}$) is the generator of a C_0-group $(W_t^{(0)})_{t\in\mathbb{R}}$ (respect., semigroup $(W_t^{(j)})_{t\geq 0}$) in $\mathcal{H}^D$.]
  Clearly, Condition (M3*) is satisfied.
- Setting $< H >:= (1+H^2)^{1/2}$, Assumption [2, (M2)] becomes:
  (M2*): [a bounded open set $J \subset \mathbb{R}$ is given and there are numbers $a > 0$, $b \geq 0$, such that $H' \geq (a1_J(H^D) - b1_{\mathbb{R}\setminus J}) < H^D >$ as forms on $\mathcal{H}^D$.]
  Thus, for all bounded open set $J \subset\subset (0,+\infty)$, Condition (M2*) is satisfied (by choosing ϕ such that $\phi = 1$ on J, and with $b = 0$), thanks to Mourre's Inequality (27).
- Assumption [2, (M4)] becomes:
  (M4*): [There is $H'_j \in B(\mathcal{H}^D)$ such that the limits

$$\lim_{t\to 0^+} \quad (-1)^j t^{-1}\{(H^D u, W_t^{(j)} u)_{\mathcal{H}^D} - (u, W_t^{(j)} H^D u)_{\mathcal{H}^D}\} \quad (j \neq 0),$$
$$\lim_{t\to 0} \quad t^{-1}\{(H^D u, W_t^{(0)} u)_{\mathcal{H}^D} - (u, W_t^{(0)} H^D u)_{\mathcal{H}^D}\} \quad (j = 0)$$

  exist and are respectively equal to $(u, H'_j u)$ for all $u \in \mathcal{H}^D$.]
  Clearly, Condition (M4*) is satisfied. We have $[H^D, iA_j]^\circ = H'_j$ and $[H^D, iA_\phi]^\circ = H' := \sum_{j=0}^2 H'_j \in B(\mathcal{H}^D)$, and we can write $H^D \in C^1(A_j^{sm})$, $H^D \in C^1(\bar{A}_\phi)$.
- The proofs of [2, Lemmas 3.13 and 3.14] with Conditions (M3*) and (M4*) satisfied imply the following relations:

$$\begin{aligned} [R_\varepsilon(z), iA_j^{sm}]^\circ &= R_\varepsilon(z)(iH'_j + \varepsilon H''_j)R_\varepsilon(z) \quad j = 0,1,2, \\ [R_\varepsilon(z), iA_\phi]^\circ &= R_\varepsilon(z)(iH' + \varepsilon H'')R_\varepsilon(z), \\ \frac{\mathrm{d}R_\varepsilon(z)}{d\varepsilon} &= [R_\varepsilon(z), iA_\phi]^\circ - \varepsilon R_\varepsilon(z) H'' R_\varepsilon(z). \end{aligned}$$

  In particular the map $\varepsilon \mapsto R_\varepsilon(z) \in B(\mathcal{H}^D)$ is C^1 in norm on $]0,1]$.
- Since H^D and the H'_j's are symmetric bounded self-adjoint operators on $\mathcal{H}^D$ (so H'_j is regular; see also [2, Remark 2.15]), then Assumption [2, (M1)] becomes:
  (M1*): [for all j, $H^D \in C^1(H'_j)$.]
  We see that (M1*) is obviously satisfied and $H^D \in C^1(H')$.
- Assumption [2, (M5)] becomes:
  (M5*): [for all $j = 0,1,2$, there is $H''_j \in B(\mathcal{H}^D)$ such that the limits

$$\lim_{t\to 0^+} \quad (-1)^j t^{-1}\{(H'u, W_t^{(j)} u) - (u, W_t^{(j)} H' u)\} \quad j \neq 0,$$
$$\lim_{t\to 0} \quad t^{-1}\{(H'u, W_t^{(0)} u) - (u, W_t^{(0)} H' u)\} \quad (j = 0),$$

  exist and are respectively equal to $(u, H''_j u)$ for all $u \in \mathcal{H}^D$.]
  Thanks to [2, Remark 3.1], Condition (M5*) is satisfied since it follows from the following facts: $H^D \in C^1(A_j^{sm})$ with $[H^D, iA_j^{sm}]^\circ = H'_j$, $H' \in C^1(A_j^{sm})$ with $[H', iA_j^{sm}]^\circ = H''_j$, so we can write $H^D \in C^2(\bar{A}_\phi)$.

Then the proof of [2, Theorem 3.3] implies the result of Corollary 3. □

Conclusion. The results of this work, notably the LAP outside thresholds, is the first step to futur developments as:

- Extension of the result of Isozaki and Morioka [8] on Rellich type theorem for discrete Schrodinger operators to the case of discrete Maxwell operators.
- The LAP for perturbed discrete Maxwell operator of the form $\hat{H}^D = \hat{D}\hat{H}_0$ where $\hat{D}$ is not constant but depends on $n \in \mathbb{Z}^3$.
- Conditions of radiation for perturbed discrete Maxwell operators. Actually, let $\hat{f}$ in a suitable subspace of $L^2(\mathbb{Z}^3)$, particularly the space of sequences with compact support. We have to characterize $\hat{u}^{\pm}(n)$ for $|n|$ large where $\hat{u}^{\pm} := (\hat{H}^D - \lambda \pm i0)^{-1}\hat{f}$.
- Extension of the result of Isozaki and Jensen [6] on the continuum limit for lattice Schrödinger operators to the case of discrete Maxwell operators.
- Extension of the result of Isozaki and [7] on the inverse scattering for lattice Schrödinger operators to the case of discrete Maxwell operators.

7. Appendix A.

7.1. Proof of Lemma 4.1.

We have

$$\hat{D}\hat{H}_0(x) - \lambda = \begin{pmatrix} -\lambda & \varepsilon M \\ -\mu M & -\lambda \end{pmatrix}.$$

Thus,

$$\det(\hat{D}\hat{H}_0(x) - \lambda) = \det(\lambda^2 + \varepsilon M(y)\mu M(y)) =: p(z;\lambda).$$

We have

$$\begin{aligned} \varepsilon M\mu M &= \begin{pmatrix} 0 & -\varepsilon_1 y_3 & \varepsilon_1 y_2 \\ \varepsilon_2 y_3 & 0 & -\varepsilon_2 y_1 \\ -\varepsilon_3 y_2 & \varepsilon_3 y_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mu_1 y_3 & \mu_1 y_2 \\ \mu_2 y_3 & 0 & -\mu_2 y_1 \\ -\mu_3 y_2 & \mu_3 y_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\varepsilon_1\mu_3 y_2^2 - \varepsilon_1\mu_2 y_3^2 & \varepsilon_1\mu_3 y_1 y_2 & \varepsilon_1\mu_2 y_1 y_3 \\ \varepsilon_2\mu_3 y_1 y_2 & -\varepsilon_2\mu_3 y_1^2 - \varepsilon_2\mu_1 y_3^2 & \varepsilon_2\mu_1 y_2 y_3 \\ \varepsilon_3\mu_2 y_1 y_3 & \varepsilon_3\mu_1 y_2 y_3 & -\varepsilon_3\mu_2 y_1^2 - \varepsilon_3\mu_1 y_2^2 \end{pmatrix}. \end{aligned}$$

Then, for $t = -\lambda^2 \in \mathbb{C}$,

$$\begin{aligned} \det(\varepsilon M\mu M - t) &= -t^3 - t^2\{((\varepsilon_2\mu_3 + \varepsilon_3\mu_2)y_1^2 + (\varepsilon_1\mu_3 + \varepsilon_3\mu_1)y_2^2 + (\varepsilon_1\mu_2 + \varepsilon_2\mu_1)y_3^2\} \\ &\quad -t\{\varepsilon_2\varepsilon_3\mu_2\mu_3 y_1^4 + \varepsilon_1\varepsilon_3\mu_1\mu_3 y_2^4 + \varepsilon_1\varepsilon_2\mu_1\mu_2 y_3^4 + (\varepsilon_2\varepsilon_3\mu_1\mu_3 + \varepsilon_1\varepsilon_3\mu_2\mu_3)y_1^2y_2^2 \\ &\quad +(\varepsilon_2\varepsilon_3\mu_1\mu_2 + \varepsilon_1\varepsilon_2\mu_2\mu_3)y_1^2y_3^2 + (\varepsilon_1\varepsilon_3\mu_1\mu_2 + \varepsilon_1\varepsilon_2\mu_1\mu_3)y_2^2y_3^2\} \\ &\equiv -t^3 - 2t^2\Psi_0 - t\Phi_0, \end{aligned}$$

where $\Psi_0(z)$ is defined by (9)

$$\Phi_0 := \varepsilon_2\varepsilon_3\mu_2\mu_3 z_1^2 + (\varepsilon_2\varepsilon_3\mu_1\mu_3 + \varepsilon_1\varepsilon_3\mu_2\mu_3)z_1z_2 + c.p..$$

We easily observe that the following relations hold (also, $+c.p.$):

$$\alpha_1^2 - \gamma_1 = \frac{1}{4}\beta_1^2, \tag{29}$$

$$\varepsilon_1\mu_1\alpha_1 - \alpha_2\alpha_3 = \frac{1}{4}\beta_2\beta_3, \tag{30}$$

$$\alpha_3\beta_2 + \alpha_2\beta_3 = -\varepsilon_1\mu_1\beta_1, \tag{31}$$

where γ is defined by (6) and $\boldsymbol{\beta}$ by (5). Thanks to (29), (30), (31), we compute:

$$\begin{aligned}\Psi_0^2-\Phi_0 &= (\alpha_1z_1+\alpha_2z_2+\alpha_3z_3)^2/4-\{\gamma_1z_1^2+\gamma_2z_2^2\\ &\quad +\gamma_3z_3^2+2\varepsilon_3\mu_3\alpha_3z_1z_2+2\varepsilon_1\mu_1\alpha_1z_2z_3+2\varepsilon_2\mu_2\alpha_2z_1z_3\}/4\\ &= K_0(z),\end{aligned}$$

where K_0 is defined by (8). Hence the eigenvalues of $\varepsilon M(y)\mu M(y)$ are 0 and

$$t=k^2=\Psi_0(z)\pm\sqrt{K_0(z)}.$$

Then Relation (7) follows.

Remark 11. From (8) we obtain the relations

$$\begin{aligned}K_0(z) &= \frac{1}{4}(\beta_1z_1-\beta_2z_2-\beta_3z_3)^2-\beta_2\beta_3z_2z_3,\\ K_0(z) &= \frac{1}{4}(\beta_3z_3-\beta_1z_1-\beta_2z_2)^2-\beta_1\beta_2z_1z_2.\end{aligned}\tag{32}$$

7.2. **Proof of Proposition 1.** Similarly to (2) we have

$$\sigma(h^D)=\overline{\cup_{y\in[-1,1]^3}\sigma(h^D(y))}.$$

In addition, since τ^+ and τ^- are continuous with $\tau^\pm(0)=0$ and $\tau^+\geq\tau^-\geq 0$ in $[0,1]^3$, then

$$\cup_{y\in[-1,1]^3}\sigma(h^D(y))=\cup_\pm\{\pm\tau^+(z);\ z\in[0,1]^3\}=[-\lambda^*,\lambda^*].$$

The conclusion follows.

7.3. **Proof of Lemma 4.4.** We set $z_i'=\beta_iz_i$ (so $z_1',z_2'\geq 0$ and $z_3'\leq 0$). Remember that

$$(4\alpha_1^2-\beta_1^2)=(2\alpha_1-\beta_1)(2\alpha_1+\beta_1)=4\varepsilon_3\mu_2\varepsilon_2\mu_3=4\gamma_1>0.$$

A) (Case $\boldsymbol{\beta}=0$). This point is obvious.

B) (Case $\boldsymbol{\beta}\neq 0$.) Thanks to (32) we have

$$\begin{aligned}\frac{\partial}{\partial z_1}K_0 &= \frac{1}{2}\beta_1(z_1'-z_2'-z_3'),\\ \frac{\partial}{\partial z_3}K_0 &= \frac{1}{2}\beta_3(z_3'-z_1'-z_2'),\end{aligned}\tag{33}$$

and

$$\sqrt{K_0(z)}\geq\frac{1}{2}|z_1'-z_2'-z_3'|.\tag{34}$$

1) We have

$$\sqrt{K_0}\nabla_z\tau^\pm=\sqrt{K_0}\nabla_z\Psi_0-\frac{1}{2}\nabla_zK_0,$$

so, by using (33), (34),

$$\begin{aligned}2\sqrt{K_0(z)}\partial_{z_1}\tau^\pm(z) &= 2\sqrt{K_0(z)}(\alpha_1-\frac{1}{2}|\beta_1|)+|\beta_1|\sqrt{K_0(z)}\pm\frac{1}{2}\beta_1|z_1'-z_2'-z_3'|\\ &\geq 2\sqrt{K_0(z)}(\alpha_1-\frac{1}{2}|\beta_1|)>0.\end{aligned}$$

Hence $\partial_{z_1}\tau^\pm(z)>0$. Similarly, $\partial_{z_2}\tau^\pm(z)>0$.

We have

$$2\sqrt{K_0}\partial_{z_3}\tau^\pm(z)=2\sqrt{K_0}\alpha_3\pm\frac{1}{2}\beta_3(z_3'-z_1'-z_2').$$

Since $\beta_3 < 0$ and $z_3' - z_1' - z_2' \leq 0$ then $\partial_{z_3}\tau^+(z) \geq 0$. Moreover if $\partial_{z_3}\tau^+(z) = 0$ then $\sqrt{K_0(z)} = 0$ which is forbidden. Hence $\partial_{z_3}\tau^+(z) > 0$.

2) a) (Case $\beta_2 = 0$.) Thanks to (32), we have

$$2\sqrt{K_0}\partial_{z_3}\tau^-(z) = \alpha_3(z_1' - z_3') - \frac{1}{2}\beta_3(z_3' - z_1') = 2\sqrt{K_0}(\alpha_3 - \frac{1}{2}\beta_3) > 0.$$

b) (Case $\beta_2 > 0$.)

(i) Assume $z_1 = 0$ or $z_2 = 0$. Then $\sqrt{K_0(z)} = \frac{1}{2}(z_1' + z_2' - z_3')$ and

$$2\sqrt{K_0}\partial_{z_3}\tau^-(z) = (\alpha_3 - \frac{1}{2}|\beta_3|)(z_1' + z_2' - z_3') > 0,$$

so, $\partial_{z_3}\tau^-(z) > 0$.

(ii) Assume $z_1 = z_2 = 1$. The functions $\xi := 4K_0(z)\partial_{z_3}\tau^-(z)\partial_{z_3}\tau^+(z)$ and $\partial_{z_3}\tau^-(z)$ have the same sign outside $K_0^{-1}(\{0\})$. We have

$$\begin{aligned}\xi &= 4\alpha_3^2 K_0(z) - (\partial_{z_3}K_0(z))^2\\ &= 4\alpha_3^2(\frac{1}{4}(\beta_1+\beta_2-z_3')^2 - \beta_1\beta_2) - \frac{1}{4}\beta_3^2(\beta_1+\beta_2-z_3')^2\\ &= \gamma_3(\beta_1+\beta_2-z_3')^2 - 4\alpha_3^2\beta_1\beta_2.\end{aligned}$$

Thus, if $\gamma_3 = 0$ then $\xi < 0$ and if $\gamma_3 \neq 0$ then

$$\begin{aligned}\xi = 0 &\iff \sqrt{\gamma_3}(\beta_1+\beta_2-\beta_3 z_3) = 2\alpha_3\sqrt{\beta_1\beta_2}\\ &\iff z_3 = \nu.\end{aligned}$$

Moreover, we have

$$\partial_{z_3}\xi = 0 = -2\sqrt{\gamma_3}\beta_3(\beta_1+\beta_2-\beta_3 z_3) > 0.$$

The conclusion follows.

7.4. **Proof of Lemma 4.5.** Lemma 4.5 is a straightforward consequence of Lemma 4.4 and of the following observations.

1) Case $\boldsymbol{\beta} = 0$. We have $K_0 \equiv 0$ so $\mathcal{X}_1 = \emptyset$, $\mathcal{T}_1 = \emptyset$, $\mathcal{X}_2 = \mathbb{T}^3 \setminus \{0\}$. In addition we have $\partial_{x_i}\Psi_0 = 2\alpha_i \sin x_i \cos x_i$. Hence $\nabla_x\Psi_0(z)$ vanishes iff $z \in \{0,1\}^3$. Hence, noting that $\tau^\pm = \Psi_0$, we obtain $Z_2^* = Z^*_{\{0,1\}}$ and $\mathcal{T}_2 = \Psi_0(Z^*_{\{0,1\}})$.

2) Case $\boldsymbol{\beta} \neq 0$ (so (A0) holds). Thanks to Lemma 4.3 we have

$$\sin^2(\mathcal{X}_1^*) \subset [0,1]^3 \setminus \{(\frac{\beta_2}{\beta_1}t, t, 0), t \in [0,1]\}.$$

For $t \in [0,1]$ we have $(\beta_2 t/\beta_1, t, 0) \in \{0,1\}^3 \setminus \{0_{\mathbb{R}^3}\}$ iff $t = 1$ and $\beta_2 = \beta_1$, or $t = 1$ and $\beta_2 = 0$. The characterization of $\sin^2(\mathcal{X}_1^*)$ follows, then those of $\mathcal{X}_1^*$ and of $\mathcal{T}_1$. Let us determine $\mathcal{X}_2^*$. We look for a tangent vector field to $\mathcal{X}_2$. A point $x \in \mathbb{T}^3$ belongs to $\mathcal{X}_2$ iff $z \neq 0$ and $z_3 = 0 = \beta_1 z_1 - \beta_2 z_2 = 0$. The last relation can be written

$$\beta_1 y_1^2 = \beta_2 y_2^2,$$

and $y_2 \neq 0$. (If $y_2 = 0$ then $y_1 = 0$ so $z = 0$ which is forbidden.) Then a tangent field to $\mathcal{X}_2$ (respect., to $\sin(\mathcal{X}_2)$) is then given by the vector field

$$w_0(x) := (\sin(x_1)\cos(x_2), \cos(x_1)\sin(x_2), 0), \tag{35}$$

(respect.,

$$\tilde{w}_0(y) := (y_1, y_2, 0)$$

).

Remark 12. In Case 3-1) (i.e, $\beta_2 = 0$) it is equivalent but more simple to set $w_0(x) = (0, 1, 0)$. However our choice in (35) is general.

We then observe that $|w(x)| \neq 0$ for all $x \in \mathcal{X}_2 \setminus \mathcal{X}^*$, and $|w(x)| \neq 0$ for all $x \in \mathcal{X}_2$ if $\beta_2 \in [0, \beta_1)$. If $\beta_2 = \beta_1$ then $w(x)$ vanishes at all $x^* \in \mathcal{X}_2^*$ since $z_1^* = z_2^* = 1$. The determination of $\sin^2(\mathcal{X}_2^*)$ follows, then those of $\mathcal{X}_2^*$ and of $\mathcal{T}_2$.

7.5. **Proof of the statement of Remark 3.** Step 1. We prove the following assertion. Let a real vector $\boldsymbol{\beta} \in \mathbb{R}^3$ and three positive real values ε_1, μ_1, α_3 such that $\beta_1 \geq \beta_2 > 0 > \beta_3$ and $\alpha_3 > |\beta_3|/2$. Then there exists positive values ε_j, μ_j, $j = 2, 3$, such that $\boldsymbol{\beta} = \boldsymbol{\varepsilon} \times \boldsymbol{\mu}$ and $2\alpha_3 = \varepsilon_1\mu_2 + \varepsilon_1\mu_2$.

Proof. We set successively

$$\begin{aligned} \delta^{\pm} &:= \alpha_3 \pm \frac{1}{2}\beta_3 > 0, \\ \varepsilon_2 &:= \frac{\delta^-}{\mu_1} > 0, \\ \mu_2 &:= \frac{\delta^+}{\varepsilon_1} > 0, \\ \varepsilon_3 &:= \frac{\varepsilon_1\beta_1 + \varepsilon_2\beta_2}{-\beta_3} > 0, \\ \mu_3 &:= \frac{\mu_1\beta_1 + \mu_2\beta_2}{-\beta_3} > 0. \end{aligned}$$

A direct calculation provides $\boldsymbol{\varepsilon} \times \boldsymbol{\mu} = \boldsymbol{\beta}$ and $\varepsilon_1\mu_2 + \varepsilon_2\mu_1 = 2\alpha_3$. □

Step 2. Let $\boldsymbol{\beta} \in \mathbb{R}^3$ such that $\beta_1 \geq \beta_2 > 0 > \beta_3$. Let us consider the function $(|\beta_3|/2, +\infty) \ni \alpha_3 \mapsto \nu(\alpha_3) = \nu$ defined by (13) and set

$$F(r) := \frac{2r}{\sqrt{r^2 - \frac{1}{4}\beta_3^2}} - 2 \quad \text{for } r > |\beta_3|/2,$$

$$\nu^*(\boldsymbol{\beta}) := -\frac{(\sqrt{\beta_1} - \sqrt{\beta_2})^2}{|\beta_3|}.$$

Then we have

$$\nu(\alpha_3) = \tilde{\nu} \Longleftrightarrow F(\alpha_3) = \frac{|\beta_3|}{\sqrt{\beta_1\beta_2}}(\tilde{\nu} - \nu^*(\boldsymbol{\beta})).$$

Obviously, the function F realizes a decreasing bijection from $(|\beta_3|/2, +\infty)$ into $(0, +\infty)$. Thus if $\nu^*(\boldsymbol{\beta}) < \tilde{\nu}$ then there exists a unique value $\alpha_3 > |\beta_3|/2$ such that $\nu(\alpha_3) = \tilde{\nu}$. But the condition $\nu^*(\boldsymbol{\beta}) < \tilde{\nu}$ is easily satisfied since $\nu^*(\boldsymbol{\beta}) \to -\infty$ as $\beta_3 \to 0^-$ if $\beta_1 > \beta_2$. The conclusion follows.

8. Appendix B.

8.1. **Proof of Lemma 5.1.** Firstly we observe that if $z_j^* \in \{0, 1\}$ then

$$\begin{aligned} z_j - z_j^* &= \sin^2 x_j - \sin^2 x_j^* = \sin(2x_j^*)(x_j - x_j^*) + 2\cos(2x_j^*)(x_j - x_j^*)^2 \\ &\quad + O((x_j - x_j^*)^3) \\ &= 2s_j^*(x_j - x_j^*)^2 + O((x_j - x_j^*)^4), \end{aligned} \tag{36}$$

and if $z_j \notin \{0, 1\}$ then

$$z_j - z_j^* = \sin(2x_j^*)(x_j - x_j^*) + O((x_j - x_j^*)^2), \tag{37}$$

with $\sin(2x_j^*) \neq 0$.

Case 1. Assume $\boldsymbol{\beta} = 0$ so $V(x) = \sqrt{\Psi_0(z)}$. We have $\partial_{z_j} V(x^*) = \frac{\alpha_j}{2\sqrt{\Psi_0(z^*)}} > 0$ for $j = 1, 2, 3$ so, by using (36),

$$\begin{aligned}
\partial_{x_j} V(x) &= \partial_{z_j} V(x) \partial_{x_j} z_j = \partial_{z_j} V(x) \sin(2x_j) \\
&= (\partial_{z_j} V(x^*) + O(z - z^*))(\sin(2x_j^*) + 2\cos(2x_j^*)(x_j - x_j^*) \\
&\quad + O((x_j - x_j^*)^2)) \\
&= 2(\partial_{z_j} V(x^*) + O(z - z^*))(s_j^*(x_j - x_j^*) + O((x_j - x_j^*)^2)) \\
&= C_j s_j^*(x_j - x_j^*)(1 + O(\mathrm{d}_0(x, x^*))),
\end{aligned}$$

where $C_j = 2\partial_{z_j} V(x^*) = \frac{\alpha_j}{\sqrt{\Psi_0(z^*)}} > 0$. Thus (23) is proved.

Case 3-1 is similar since the six partial derivatives $\partial_{z_j} \tau^{\pm}$ are all constant and positive, and $\tau^{\pm}(z^*) > 0$. (See (10) and (11).)

Cases 2-1 and 2-2-a and 2-2-b are similar to Case 1, the sign of C_j being a consequence of Lemma 4.4.

Let us be more precise in Case 2-2-a. Since $z_3^* = \nu \in (0, 1)$ then $\sin(2x_3^*) \neq 0$ so, by using (37), (36), we have

$$\begin{aligned}
\partial_{x_3} V^2(x) &= \partial_{z_3} \tau^-(z) \sin(2x_3) \\
&= (O(z_1 - z_1^*) + O(z_2 - z_2^*) + \partial_{z_3}^2 \tau^-(z^*))(z_3 - z_3^*) \\
&\quad (1 + O((z_3 - z_3^*)))(\sin(2x_3^*) + O(x_3 - x_3^*)).
\end{aligned}$$

Since $z_j - z_j^* = O((x_j - x_j^*)^2) = O(\partial_{x_j} \tau^-(z) \mathrm{d}_0(x, x^*))$ for $j = 1, 2$, and $z_3 - z_3^* = \sin(2x_3^*)(x_3 - x_3^*)(1 + O(x_3 - x_3^*))$ then

$$\begin{aligned}
\partial_{x_3} V(x) &= \frac{\partial_{x_3} V^2(x)}{2V(x)} = C_3(x_3 - x_3^*) + O(d_0^2(x, x^*)) \\
&= C_3(x_3 - x_3^*) + |\nabla_x V(x)| O(\mathrm{d}_0(x, x^*)),
\end{aligned}$$

where $C_3 = (2\sqrt{\tau^-(z^*)})^{-1} \partial_{z_3}^2 \tau^-(z^*) \sin^2(2x_3^*) > 0$. Thus (23) holds in Case 2-2-a too.

Case 2-2-c. The computation of the derivatives $\partial_{x_j} V(x)$, $j = 1, 2$, is similar to the other cases (with $s_j = s_j^* = -1$). By using (36) we have

$$\begin{aligned}
\partial_{x_3} V^2(x) &= \partial_{z_3} \tau^-(z) \sin(2x_3) \\
&= \Big((O(z_1 - z_1^*) + O(z_2 - z_2^*) + \partial_{z_3}^2 \tau^-(z^*))(z_3 - z_3^*) + O(z_3 - z_3^*)^2\Big) \\
&\quad \cdot \Big(2s_3^*(x_3 - x_3^*) + O((x_3 - x_3^*)^3)\Big).
\end{aligned}$$

Hence

$$\partial_{x_3} V(x) = C_3(x_3 - x_3^*)^3 + O(|\nabla_x V(x)|\, \mathrm{d}_0(x, x^*)),$$

where $C_3 = 2(\sqrt{\tau^-(z^*)})^{-1} \partial_{z_3}^2 \tau^-(z^*) > 0$. Thus (24) holds too.

The lemma is proved.

8.2. **Proof of Lemma 5.2.** We remember that $y_1^* \neq 0$, $z_2^* \neq 0$ and $z_3^* = 0$, so,

$$\sqrt{\Psi_0(z)}_w = \cos(x_1)\cos(x_2)\sqrt{\tilde{\Psi}_0(z)}_{\tilde{w}}.$$

Thanks to (19) the function $x \mapsto \sqrt{\tilde{\Psi}_0(z)}_{\tilde{w}} = 2\alpha_1 z_1 + 2\alpha_2 z_2$ is smooth and positive in $\operatorname{supp} \chi_2^*$.

In Case 3-3, we have $0 < z_1^* < z_2^* = 1$ then $\cos y_1^* \neq 0$ so $\cos(x_1) = \cos(x_1^*) + O(\mathrm{d}_0(x, x^*))$ with $\cos(x_1^*) \neq 0$, and $\cos(x_2) = -y_2^*(x_2 - x_2^*) + O(x_2 - x_2^*)^3$. Hence (26) holds.

In Case 3-2, we have $z_1^* = z_2^*$, $\cos y_1^* = 0$ so $\cos(x_j) = -y_j^*(x_j - x_j^*) + O(x_j - x_j^*)^3$, $j = 1, 2$. Hence (25) holds.

8.3. **Proof of Lemma 5.5.** We fix a representation of $x^* \in \mathbb{T}^3$ in $\mathbb{R}^3$ which we denote again x^*. Then, the multiplication by χ_{x^*} is an isometry (non surjective) from $\mathcal{H}^D$ into the Hilbert space $L^2_D(\mathbb{R}^3, \mathbb{C}^6) := L^2(\mathbb{R}^3, \mathbb{C}^6)$ equipped with the scalar product

$$(u, v)_{L^2_D(\mathbb{R}^3;\mathbb{C}^6)} := \int_{\mathbb{R}^3} < u(x), v(x) >_{\mathbb{C}^6, D} \, \mathrm{d}x = \int_{\mathbb{R}^3} < \hat{D}^{-1} u(x), v(x) >_{\mathbb{C}^6} \, \mathrm{d}x.$$

So, we can identify A_{x^*} with an unbounded symmetric operator on $L^2(\mathbb{R}^3; \mathbb{C}^6)$, which we denote A_{x^*} again.

We set $x'_j = \sqrt{C_j/2}(x_j - x_j^*)$ where the C_j's are the positive constants of Section 5.7.4, and in (23) or in (24) of Lemma 5.1. We set also $\rho' = \sqrt{x_1'^2 + x_2'^2}$, $r' = \sqrt{\rho'^2 + x_3'^2}$.

A-1) Let us consider Case 1 with $s_1 = s_2 = 1$ and $s_3 = -1$. Since $\boldsymbol{\beta} = 0$ then we have $\boldsymbol{\mu} = \kappa \boldsymbol{\varepsilon}$, with the scalar $\kappa > 0$. We have

$$p_1(x) = \rho'^2 - x_3'^2,$$

and we set

$$\begin{array}{rcl} p_2(x) & := & 2\rho' x'_3, \\ p_3(x) & := & \frac{(x'_1, x'_2)}{\rho'} \in \mathbb{S}^1 \approx \mathbb{R}/(2\pi\mathbf{Z}). \end{array}$$

The mapping $\mathbb{R}^2 \setminus \{(0,0)\} \ni (x'_1, x'_2) \mapsto (\rho', p_3) \in \mathbb{R}^+ \times \mathbb{S}^1$ is the polar change of coordinates. Since $p_1 + ip_2 = (\rho' + ix'_3)^2$, then the mapping $(\rho', x'_3) \mapsto (p_1, p_2)$ is a $\mathcal{C}^\infty$-diffeomorphism from $(0, \infty) \times \mathbb{R}$ onto $\mathcal{O} := \mathbb{R}^2 \setminus (\mathbb{R}^- \times \{0\})$. Thus the mapping

$$\Phi : \ x' = (x'_1, x'_2, x'_3) \mapsto p = (p_1, p_2, p_3)$$

is a $\mathcal{C}^\infty$-diffeomorphism from $\mathbb{R}^{2*} \times \mathbb{R}$ onto $\mathcal{U} := \mathcal{O} \times \mathbb{S}^1$, with jacobian

$$J_\Phi(x') = -\frac{r'^2}{\rho'}.$$

We set $\tilde{H} = L^2(\mathbb{R}^{2*} \times \mathbb{S}^1, \mathbb{C}^6; \mathrm{d}p)$ equipped with the following scalar product:

$$(\tilde{u}, \tilde{v})_{\tilde{H}} := \int_{\mathbb{R}^2 \times \mathbb{S}^1} < \tilde{u}(p), \tilde{v}(p) >_{\mathbb{C}^6, D} \, \mathrm{d}p.$$

For $\tilde{u} \in \tilde{H}$, $p \in \mathcal{U}$, we set

$$u(x) = |J_\Phi(x')|^{1/2} \tilde{u}(p), \quad x' = \Phi^{-1}(p) \in \mathbb{R}^3, \tag{38}$$

so the transform

$$T : \ L^2(\mathbb{R}^3, \mathbb{C}^6) \ni u \mapsto \tilde{u} \in \tilde{H}$$

is a bijective isometry. Setting $\tilde{\pi}(p) = \pi_2(x)$ and $\tilde{\chi}(p) = \chi_{x^*}(x)$, the partial derivatives $\partial^j_{p_1} \tilde{\chi}$, $j \geq 0$, are bounded in $\mathbb{R}^{2*} \times \mathbb{S}^1$ since $\tilde{\chi} = 1$ near $p(x^*) = 0$ and the function $|\nabla_x p_1|$ on $\mathrm{supp}\, \nabla \chi_{x^*}$ is smooth and bounded by below by a positive constant. For example if $j = 1$, we have

$$\sup_{x \in B(\mathbb{R}^3)} |\partial^j_{p_1} \tilde{\chi}(p(x))| = \sup_{\frac{r}{2} \leq \mathrm{d}_0(x, x^*) \leq r} |\partial^j_{p_1} \tilde{\chi}(p(x))| \leq C.$$

The projector $\tilde{\pi}$ is continuous but admits a singular of first order at $p = 0$. Observing that

$$\frac{\nabla p_1(x) \nabla u(x)}{|\nabla p_1|^2} = \frac{\partial u}{\partial p_1},$$

and denoting by "+ sym." the terms of symmetrization of $\mathcal{A}_{x^*}$, we have, for $u, v \in \mathcal{C}_c^\infty((\mathbb{R}^2 \setminus \{(x_1^*, x_2^*)\} \times \mathbb{R})$,

$$\begin{aligned}(A_{x^*}u, v)_{\mathcal{H}^D} &= \int_{\mathbb{R}^3} i\chi_{x^*}(x) < \pi_2(y)\frac{\nabla p_1(x)\nabla(\chi_{x^*}(x)\pi_2(y)u(x))}{|\nabla p_1(x)|^2}, v(x) >_{\mathbb{C}^6,D} \mathrm{d}x + sym.\\ &= \int_{\mathbb{R}^2\times\mathbb{S}^1} i < \frac{\partial(\tilde\pi\tilde u)}{\partial p_1}, \tilde\chi^2(p)\tilde\pi(p)\tilde v(p) >_{\mathbb{C}^6,D} \mathrm{d}p + sym.\\ &= \int_{\mathbb{R}^2\times\mathbb{S}^1} i < \frac{\partial(\tilde\chi\tilde\pi\tilde u)}{\partial p_1}, \tilde\chi\tilde\pi\tilde v >_{\mathbb{C}^6,D} \mathrm{d}p \equiv (\tilde A_{x^*}\tilde u, \tilde v)_{\tilde H}.\end{aligned}$$

The projection $\tilde\pi$ has range two. Since $\boldsymbol{\beta} = 0$, we have, for $z \neq 0$, a basis of the eigenspace $\ker(H^D(x) - \sqrt{\Psi_0(z)})$ of the form $(\varphi_1(p), \varphi_2(p) = \overline{\varphi_1(p)})^T$, with $\varphi_1 = (q, i\sqrt{\kappa}q)$ and $q(p)^T \in \ker(i\sqrt{\kappa}\boldsymbol{\varepsilon}M(y) - \Psi_0(z)I_3)$, where $x = x(p)$, I_3 denotes the identity matrix of size 3, and $M(y)$ is the 3×3 matrix defined at (1). Moreover we can choose $q(p)$ such that $x \mapsto q(p(x))$ is analytic in the support of χ_{x^*} at least, and with $< \boldsymbol{\varepsilon}^{-1}q(p), q(p) >_{\mathbb{C}^3} = 1/2$, so $(\varphi_1(p), \varphi_2(p))$ is orthonormal in $\mathbb{C}^6$ equipped with $<,>_{\mathbb{C}^6,D}$. We thus have $< \varphi_i(p), \varphi_j(p) >_{\mathbb{C}^6,D} = \delta_{i,j}$, $i, j \in \{1, 2\}$, but also

$$\begin{aligned}< \partial_p\varphi_1, \varphi_2 >_{\mathbb{C}^6,D} &= < \boldsymbol{\varepsilon}^{-1}\partial_p q, q >_{\mathbb{C}^3} + < \boldsymbol{\mu}^{-1}\partial_p(i\sqrt{\kappa}q), -i\sqrt{\kappa}q >_{\mathbb{C}^3}\\ &= < \boldsymbol{\varepsilon}^{-1}\partial_p q, q >_{\mathbb{C}^3} + < i\boldsymbol{\varepsilon}^{-1}\partial_p q, -iq >_{\mathbb{C}^3}\\ &= 0.\end{aligned}$$

Similarly, $< \varphi_1, \partial_p\varphi_2 >_{\mathbb{C}^6,D} = 0$. Hence we have

$$\tilde\pi(p)\tilde u(p) = \tilde\xi_1(\tilde u)(p)\varphi_1(p) + \tilde\xi_2(\tilde u)(p)\varphi_2(p),$$

where we set

$$\tilde\xi_j(\tilde u)(p) := < \tilde u(p), \varphi_j(p) >_{\mathbb{C}^6,D}. \tag{39}$$

We then have

$$(\tilde A_{x^*}\tilde u, \tilde v)_{\tilde H} = i\sum_{j=1}^{2}\int_{\mathbb{R}^2\times\mathbb{S}^1}\frac{\partial}{\partial p_1}(\tilde\chi\tilde\xi_j(\tilde u))\tilde\chi\overline{\tilde\xi_j(\tilde v)}\mathrm{d}p.$$

Let us set

$$D(\tilde A_{x^*}) = \{\tilde u \in \tilde H;\ \tilde\chi^2\partial_{p_1}\tilde\xi_j(\tilde u) \in L^2(\mathbb{R}^2\times\mathbb{S}^1, \mathbb{C}; \mathrm{d}p),\ j = 1, 2\}.$$

Let us show that $D(\tilde A_{x^*}^*) = D(\tilde A_{x^*})$. Let $\tilde v \in D(\tilde A_{x^*}^*)$, so we have:

$$|(\tilde A_{x^*}\tilde u, \tilde v)_{\tilde H}| \leq C\|\tilde u\|_{\tilde H}, \quad \forall \tilde u \in D(\tilde A_{x^*}), \tag{40}$$

that is,

$$|\sum_{j=1}^{2}\int_{\mathbb{R}^2\times\mathbb{S}^1}\frac{\partial\tilde\xi_j(\tilde u)}{\partial p_1}\tilde\chi^2(p)\overline{\tilde\xi_j(\tilde v)}\mathrm{d}p| \leq C\|\tilde u\|_{\tilde H}, \quad \forall \tilde u \in D(\tilde A_{x^*}).$$

We fix $j \in \{1, 2\}$ and choose $\tilde u(p) = f(p_1)g(p_2, p_3)\varphi_j(p)$ in the above estimate with arbitrary $f \in H^1(\mathbb{R}; \mathbb{C}; \mathrm{d}p_1)$ and $g \in L^2(\mathbb{R}\times\mathbb{S}^1; \mathbb{C}; \mathrm{d}p_2\mathrm{d}p_3)$. Then $\|\tilde u\|_{\tilde H} \leq C\|f\|_{H^1(\mathbb{C})}\|g\|_{L^2(\mathbb{R}\times\mathbb{S}^1)}$ so we have

$$\begin{aligned}|\int_{\mathbb{R}^2\times\mathbb{S}^1}\frac{\partial f(p_1)}{\partial p_1}g(p_2, p_3)\tilde\chi^2(p)\overline{\tilde\xi_j(\tilde v)}\mathrm{d}p| &\leq C\|f\|_{H^1(\mathbb{R})}\|g\|_{L^2(\mathbb{R}\times\mathbb{S}^1)},\\ &\forall f \in H^1(\mathbb{R}; \mathrm{d}p_1),\ g \in L^2(\mathbb{R}\times\mathbb{S}^1, \mathrm{d}p_2\mathrm{d}p_3).\end{aligned}$$

It shows that

$$K(p_1) := \int_{\mathbb{R}\times\mathbb{S}^1}\tilde\chi^2(p)\tilde\xi_j(\tilde v)g(p_2, p_3)\mathrm{d}p_2\mathrm{d}p_3 \in H^1(\mathbb{R}; \mathbb{C}; \mathrm{d}p_1)$$

with

$$\|\frac{\partial}{\partial p_1}K(p_1)\|_{L^2(\mathbb{R})} \leq C\|g\|_{L^2(\mathbb{R}\times\mathbb{S}^1)}.$$

But we have

$$\begin{aligned}\frac{\partial}{\partial p_1}K(p_1) &= \int_{\mathbb{R}\times\mathbb{S}^1}\tilde{\chi}^2(p)\frac{\partial}{\partial p_1}\tilde{\xi}_j(\tilde{v})g(p_2,p_3)\mathrm{d}p_2\mathrm{d}p_3 + L(p_1),\\ L(p_1) &:= \int_{\mathbb{R}\times\mathbb{S}^1}\tilde{\xi}_j(\tilde{v})(\frac{\partial}{\partial p_1}\tilde{\chi}^2(p))g(p_2,p_3)\mathrm{d}p_2\mathrm{d}p_3,\end{aligned}$$

with $\|L_1\|_{L^2(\mathbb{R})} \leq C\|g\|_{L^2(\mathbb{R}\times\mathbb{S}^1)}$. Hence we have

$$\tilde{\chi}^2\frac{\partial}{\partial p_1}\tilde{\xi}_j(\tilde{v}) \in L^2(\mathbb{R}^2\times\mathbb{S}^1,\mathbb{C};\mathrm{d}p).$$

Thus, $\tilde{v} \in D(\tilde{A}_{x^*})$ and so $\tilde{A}_{x^*}$ is self-adjoint. Consequently, A_{x^*} with domain $T^{-1}(D(\tilde{A}_{x^*}))$ is a self-adjoint operator.

Case 1 with the general situation $s_1s_2s_3 = -1$ is similar.
Cases 2-1 and 2-2-a and 2-2-b and 2-2-d and 2-2-d and 2-2-e, with $s_1s_2s_3 = -1$, are similar, except that the projection $\tilde{\pi}$ ($= \pi_1^+(y)$ or $= \pi_1^-(y)$) has range one, which simplifies the proof.

Case 3-1 with $s_1s_2s_3 = -1$. We set $\pi_1^\pm A_{x^*}\pi_1^\pm =: A^\pm$ so $A_{x^*} = \sum_\pm A^\pm$ with $D(A^\pm) = \mathcal{D}_0$. We prove that $A^\pm$ is essentially self-adjoint on $\mathcal{H}^D$. We set

$$k^\pm(x) := \frac{|\nabla_x p_1(x)|}{|\nabla_x p_1(x)\cdot\nabla_x\sqrt{\tau^\pm(z)}|^{1/2}}.$$

Thanks to Lemma 5.1 we have

$$k^\pm(x) = 1 + O(\mathrm{d}_0(x,x^*)).$$

Thus $k^\pm(x)$ is defined for $x \simeq x^*$ and $x \neq x^*$, extends as a positive lipschitzian function near x^*. We then consider the same transforms than in Case 1 with $\tilde{H}^\pm$ replacing $\tilde{H}$ so we have

$$\begin{aligned}(A^\pm u^\pm, v^\pm)_{\mathcal{H}^D} &= i\int_{\mathbb{R}^2\times\mathbb{S}^1} < \frac{\partial}{\partial p_1}(\tilde{\chi}\tilde{u}^\pm), \tilde{\chi}\overline{\tilde{v}^\pm} >_{\mathbb{C}^6,D}\ \mathrm{d}p\\ &\equiv (\tilde{A}^\pm\tilde{u}^\pm,\tilde{v}^\pm)_{\tilde{H}},\end{aligned}$$

where we set $\tilde{A}^\pm := i\tilde{\chi}\circ\frac{\partial}{\partial p_1}\circ\tilde{\chi}$, and

$$\tilde{u}^\pm(p) := |\nabla_x p_1(x)||J_\Phi(x')|^{-1/2}\tilde{k}^\pm(p)u(x) \quad x' = \Phi^{-1}(p) \in \mathbb{R}^3,$$

and $\tilde{k}^\pm(p) := k^\pm(x)$, $\tilde{\pi}_1^\pm(p) := \pi_1^\pm(y)$, $\tilde{\chi}(p) := \chi_{x^*}(x)$. Thus, as in Case 2-1 with $\pi_j s_j = 1$, $A^\pm = \pi_1^\pm A^\pm\pi_1^\pm$ is essentially self-adjoint on $\mathcal{H}^D$. We denote by $D^\pm$ the domain of the self-adjoint extension of $A^\pm$, so $D^\pm = \{u \in \mathcal{H}^D;\ A^\pm u \in \mathcal{H}^D\}$. Then, A_{x^*} extends as a symmetric operator, $A'_{x^*} = A_{x^*}$ with domain $D(A'_{x^*}) := D^+\cap D^-$. Now, let $v \in D((A'_{x^*})^*)$ so

$$|(A_{x^*}u,v)_{\mathcal{H}^D}| \leq C\|u\| \quad \forall u \in D(A'_{x^*}).$$

Let $u \in D^\pm$. Then $A_{x^*}\pi_1^\pm u = A^\pm u \in \mathcal{H}^D$, so $\pi_1^\pm u \in D(A'_{x^*})$. Thus

$$|(A^\pm u,v)_{\mathcal{H}^D}| = |(A_{x^*}\pi_1^\pm u,v)_{\mathcal{H}^D}| \leq C\|\pi_1^\pm u\| \leq C\|u\| \quad \forall u \in D^\pm.$$

Hence $v \in D^\pm$. Thus, $v \in D(A'_{x^*})$, so A'_{x^*} is self-adjoint.

Case 2-2-c. We have

$$p_1(x) = \rho'^2 - \frac{1}{2}x_3'^4,$$

and we set

$$\begin{aligned} p_2(x) &:= x_2' e^{-\frac{1}{2x_3'^2}}, \\ p_3(x) &:= x_1' e^{-\frac{1}{2x_3'^2}}, \end{aligned}$$

with $p_2|_{x_3'=0} = p_3|_{x_3'} = 0$, so $p_2, p_3 \in \mathcal{C}^\infty(\mathbb{R}^3)$. Then,

$$\begin{aligned} \nabla p_1 &= 2(x_1', x_2', -x_3'^3), \\ \nabla p_2 &= e^{-\frac{1}{2x_3'^2}}(0, 1, x_2'/x_3'^3), \\ \nabla p_3 &= e^{-\frac{1}{2x_3'^2}}(1, 0, x_1'/x_3'^3), \end{aligned}$$

$\nabla p_1 \perp \nabla p_j$, $j = 2, 3$, and the Jacobian of the mapping Φ: $x' \mapsto p$ is

$$J_\Phi(x') = \frac{\rho'^2}{x_3'^3} e^{-\frac{1}{x_3'^2}}.$$

It does not vanish if $x_3' \neq 0$ or $\rho' \neq 0$. Let us invert Φ. The sign of x_3' is not determined by p so we consider

$$\Phi^\pm : \quad \mathbb{R}^{2*} \times \mathbb{R}^{\pm *} \ni x' \to p \in \mathbb{R} \times \mathbb{R}^{2*}.$$

Let $p \in \mathbb{R} \times \mathbb{R}^{2*}$. We have $x_1' = p_3 e^{\frac{1}{2x_3'^2}}$, $x_2' = p_2 e^{\frac{1}{2x_3'^2}}$ so x_3' satisfies the equation $F(x_3'^2) = p_1$ where we set

$$F(t) := (p_2^2 + p_3^2)e^{1/t} - \frac{1}{2}t^2, \quad t > 0.$$

Since $F' > 0$, $F(+\infty) = -\infty$ and $F(0^+) = +\infty$, then the equation is uniquely solvable by some $t_0 > 0$ so we obtain $x_3' = \pm\sqrt{t_0} \in \mathbb{R}^*$. Hence $\Phi^\pm$ is bijective. We let the lector to check that $\Phi^\pm$ is an homeomorphism from $\mathbb{R}^{2*} \times \mathbb{R}^{\pm *}$ into $\mathbb{R} \times \mathbb{R}^{2*}$. Hence, $\Phi^\pm$ is a $\mathcal{C}^\infty$-diffeomorphism from $\mathbb{R}^{2*} \times \mathbb{R}^{\pm *}$ into $\mathbb{R} \times \mathbb{R}^{2*}$
We set the Hilbert spaces $\tilde{H}^\pm = L^2(\mathbb{R}^{2*} \times \mathbb{R}^{\pm *}, \mathbb{C}^6; \mathrm{d}p)$ equipped with the scalar product

$$(\tilde{u}^\pm, \tilde{v}^\pm)_{\tilde{H}^\pm} := \int_{\mathbb{R}^{2*} \times \mathbb{R}^{\pm *}} < \tilde{u}^\pm(p), \tilde{v}^\pm(p) >_{\mathbb{C}^6, D} \, \mathrm{d}p,$$

then $\tilde{H} := \tilde{H}^+ \oplus \tilde{H}^-$. For $\tilde{u} = (\tilde{u}^+, \tilde{u}^-) \in \tilde{H}$, $x' \in \mathbb{R}^{2*} \times \mathbb{R}^{\pm *}$, we set

$$u(x) = |J_\Phi(x')|^{1/2} \tilde{u}^\pm(\Phi^\pm(x')),$$

so the transform

$$T : \ L^2(\mathbb{R}^3) \ni u \mapsto \tilde{u} \in \tilde{H}$$

is a bijective isometry (up to a nonzero constant multiplicative factor).

Setting again $\tilde{\chi}(p) = \chi_{x*}(x)$, $\tilde{\pi}(p) = \pi_2(x)$, we have, for $u, v \in \mathcal{C}_c^\infty(\mathbb{R}^2 \setminus \{(x_1^*, x_2^*)\} \times \mathbb{R} \setminus \{x_3^*\})$,

$$(A_{x^*} u, v)_{\mathcal{H}^D} = \int_{\mathbb{R}^3} i < \frac{\partial(\tilde{\chi}\tilde{\pi}\tilde{u})}{\partial p_1}, \tilde{\chi}\tilde{\pi}\tilde{v} >_{\mathbb{C}^6, D} \, \mathrm{d}p \equiv (\tilde{A}_{x^*} \tilde{u}, \tilde{v})_{\tilde{H}}.$$

The projection $\tilde{\pi}$ has range one so this case is similar to case 2-1, so A_{x^*} is essentially self-adjoint.

A-2) Let us treat Case 1 with $s_1 = s_2 = -1$ (and $s_3 = -1$). We observe that

$$p_1(x) = \sum_{j=1}^{3} (x'_j)^2 \geq 0.$$

(where we set $x'_j = \sqrt{C_j/2}(x_j - x_j^*)$). We use the spherical coordinates: $x' = \rho\omega$ with $\rho = \sqrt{x'^2_1 + x'^2_2 + x'^2_3} > 0$, $\omega = \rho^{-1}x' \in \mathbb{S}^2$, so we have $p_1 = \rho^2$ and choose two other coordinates, p_2, p_3, on the sphere $\mathbb{S}^2$.
We then follow the above method (Case 1 with $s_1 = s_2 = 1 = -s_3$) with similar notations, notably, with the same couple (φ_1, φ_2) and coordinates $\tilde{\xi}_j$ (defined by (39)) $j = 1, 2$. The mapping

$$\Phi : \ x' = (x'_1, x'_2, x'_3) \mapsto p = (p_1, p_2, p_3)$$

is a $\mathcal{C}^\infty$-diffeomorphism from $\mathbb{R}^{3*}$ onto $\mathbb{R}^{+*} \times \mathbb{S}^2$. The jacobian of Φ has the form

$$J_\Phi(x') = j(p_2, p_3)\sqrt{p_1},$$

where j is a positive smooth function on $\mathbb{S}^2$. We set $\tilde{H} = L^2(\mathbb{R}^{+*} \times \mathbb{S}^2, \mathbb{C}^6; \mathrm{d}p)$ equipped with the following scalar product:

$$(\tilde{u}, \tilde{v})_{\tilde{H}} := \int_{\mathbb{R}^{+*}\times\mathbb{S}^2} < \tilde{u}(p), \tilde{v}(p) >_{\mathbb{C}^6, D} \ \mathrm{d}p.$$

For $\tilde{u} \in \tilde{H}$, $p \in \mathcal{U}$, we consider the transformation defined by (38) between u and $\tilde{u}$ so it is a bijective isometry (up to a positive constant multiplicative factor) between $L^2(\mathbb{R}^3, \mathbb{C}^6)$ and $\tilde{H}$ which we denote T again. Setting $\tilde{\pi}(p) = \pi_2(y)$, we have $\tilde{\chi} \in \mathcal{C}_c^\infty(\mathbb{R}^{+*} \times \mathbb{S}^2)$ and $\tilde{\chi} = 1$ near $p(x^*) = 0$. We thus have, for $u, v \in \mathcal{C}_c^\infty(\mathbb{R}^{+*} \times \mathbb{S}^2, \mathbb{C}^6)$,

$$(A_{x^*}u, v)_{\mathcal{H}^D} = \sum_{j=1}^{2} \int_{\mathbb{R}^{+*}\times\mathbb{S}^2} i\tilde{\chi}\frac{\partial(\tilde{\chi}\tilde{\xi}_j(\tilde{u}))}{\partial p_1}\overline{\tilde{\xi}_j(\tilde{v})}\mathrm{d}p \equiv (\tilde{A}_{x^*}\tilde{u}, \tilde{v})_{\tilde{H}}.$$

The above formula defines the symmetric operator $\tilde{A}_{x^*}$ on $\tilde{H}$ with domain $\mathcal{C}_c^\infty(\mathbb{R}^{+*} \times \mathbb{S}^2)$. Thus, $\tilde{A}_{x^*}$ extends to the operator with the same formula defined on

$$\begin{aligned} D(\tilde{A}_{x^*}) \quad = \quad & \mathcal{H}_{1,0} := \{\tilde{u} \in \tilde{H}; \ \tilde{\chi}^2\partial_{p_1}\tilde{\xi}_j(\tilde{u}) \in L^2(\mathbb{R}^{+*} \times \mathbb{S}^2, \mathbb{C}; \mathrm{d}p), \\ & \tilde{\chi}^2\partial_{p_1}\tilde{\xi}_j(\tilde{u})|_{p_1=0} = 0, \ j = 1, 2\}. \end{aligned}$$

Let us prove that the default index N^+ of $\tilde{A}_{x^*}$ vanishes. Firstly, observe that

$$D((\tilde{A}_{x^*})^*) = \mathcal{H}_1 := \{\tilde{u} \in \tilde{H}; \ \tilde{\chi}^2\partial_{p_1}\tilde{\xi}_j(\tilde{u}) \in L^2(\mathbb{R}^{+*} \times \mathbb{S}^2, \mathbb{C}; \mathrm{d}p), \ j = 1, 2\}. \quad (41)$$

In fact an integration by parts shows that $D((\tilde{A}_{x^*})^*)$ contains $\mathcal{H}_1$. Then, let $\tilde{v} \in D((\tilde{A}_{x^*})^*)$ so (40) holds. As in Case 1 with $\Pi_{k=1}^3 s_k = -1$, let $j \in \{1, 2\}$ and choose $\tilde{u}(p) = f(p_1)g(p_2, p_3)\varphi_j(p)$ with arbitrary $f \in H^1(\mathbb{R}^{+*}; \mathbb{C}; \mathrm{d}p_1)$ and $g \in L^2(\mathbb{S}^2; \mathbb{C}; \mathrm{d}p_2\mathrm{d}p_3)$, so we have

$$\left|\int_{\mathbb{R}^+\times\mathbb{S}^2} \frac{\partial f(p_1)}{\partial p_1} g(p_2, p_3)\tilde{\chi}^2(p)\overline{\tilde{\xi}_j(\tilde{v})}\mathrm{d}p\right| \leq C\|f\|_{H^1(\mathbb{R}^+)}\|g\|_{L^2(\mathbb{S}^2)},$$

$$\forall f \in H^1(\mathbb{R}; \mathrm{d}p_1), \ g \in L^2(\mathbb{S}^2, \mathrm{d}p_2\mathrm{d}p_3).$$

It implies $\frac{\partial}{\partial p_1}(\tilde{\chi}^2\tilde{\xi}_j(\tilde{v})) \in L^2(\mathbb{R}^{+*} \times \mathbb{S}^2, \mathbb{C}; \mathrm{d}p)$, then $\tilde{\chi}^2\frac{\partial}{\partial p_1}\tilde{\xi}(\tilde{v}) \in L^2(\mathbb{R}^{+*} \times \mathbb{S}^2, \mathbb{C}; \mathrm{d}p)$, so $\tilde{v} \in \mathcal{H}_1$. Therefore, (41) is proved. Now, let $\tilde{v} \in D((\tilde{A}_{x^*})^*)$ such that $(\tilde{A}_{x^*})^*\tilde{v} = i\tilde{v}$. Thus we have $(-i(\tilde{A}_{x^*})^*\tilde{v}, \tilde{v})_{\tilde{H}} = (\tilde{v}, \tilde{v})_{\tilde{H}}$. An integration by parts (according to the variable p_1) shows that $v = 0$. Consequently, A_{x^*} with domain $T^{-1}(D(\tilde{A}_{x^*}))$ is

a maximal symmetric operator with the default index $N^+ = 0$. (See also[2, Lemma 1.3] for results of the same kind).

C) (Cases 3-2 and 3-3). We set

$$p_1 = \cos(x_1)\cos(x_2),$$

so p_1 vanishes at $x = x^* \in \mathcal{X}_2^*$ (such that $z^* = (\frac{\beta_2}{\beta_1}, 1, 0)$). We have

$$\nabla_x p_1 = -w = (\sin x_1 \cos x_2, \sin x_2 \cos x_1, 0).$$

We set

$$p_2 = \frac{\sin x_1}{\sin x_2} - \frac{\sin x_1^*}{\sin x_2^*}, \quad p_3 = x_3,$$

so $\nabla_x p_i \cdot \nabla_x p_j = 0$ if $i \neq j$ and the jacobian of the map Φ: $x \mapsto p = (p_1, p_2, p_3)$ is

$$J_\Phi(x) = \Pi_{j=1}^3 \nabla_x p_j.$$

We have $w \cdot \nabla_x u = u_w = -|\nabla p_1|^2 \partial_{p_1} u$ and, thanks to (15), $\sqrt{\Psi_0(z)}_w = p_1 \sqrt{\tilde{\Psi_0(z)}}_{\tilde{w}}$ where $\sqrt{\tilde{\Psi_0(z)}}_{\tilde{w}}$ is analytic and does not vanishes at x^*.

Case 3-3. We have $\nabla_x p_1(x^*) \neq 0$ and $J_\Phi(x^*) \neq 0$, so Φ is a local diffeomorphism from a neighborhood of x^* in $\mathbb{R}^3$ into a neighborhood of $0_{\mathbb{R}^3}$ in $\mathbb{R}^3$. Hence, we have

$$(A_{x^*} u, v)_{\mathcal{H}^D} = -i \int_{\mathbb{R}^3} \frac{k(x)}{p_1} < \frac{\partial(\chi_2 \pi_2 u)}{\partial p_1}, \chi_2 \pi_2 v >_{\mathbb{C}^6, D} \, \mathrm{d}p + \mathrm{sym},$$

where k is smooth with $k(x^*) > 0$. As in the above cases, we thus set $q = (q_1, q_2, q_3)$, $q_1 = p_1|p_1|$, $q_j = p_j$ for $j = 2, 3$, $\tilde{\chi}(q) = \chi_{x^*}(x)$, $\tilde{\pi}(q) = \pi_2(y)$, $\tilde{u}(q) = |k(x)|^{1/2} u(x)$. We then obtain

$$(A_{x^*} u, v)_{\mathcal{H}^D} = -i \int_{\mathbb{R}^3} \mathrm{sgn}(q_1) < \frac{\partial(\tilde{\chi}\tilde{\pi}\tilde{u})}{\partial q_1}, \tilde{\chi}_2 \tilde{\pi}_2 \tilde{v} >_{\mathbb{C}^6, D} \, \mathrm{d}q \equiv (\tilde{A}_{x^*} \tilde{u}, \tilde{v})_{\tilde{H}},$$

where $\tilde{H} = L^2(\mathbb{R}^3, (\mathbb{C}^6, <, >_{\mathbb{C}^6, D}); \mathrm{d}q)$ is a usual Hilbert space. The projection $\tilde{\pi}$ has range two so Case 3-3 is similar to A-2) with A_{x^*} replaced by $-A_{x^*}$. Hence, A_{x^*} has default index $N^- = 0$ and admits a maximal symmetric extension.

Case 3-2. We have $\nabla_x p_1(x^*) = 0$ so $J_\Phi(x^*) = 0$. Let us "invert" $x \mapsto p$. For simplicity we assume $y_1^* = y_2^* = 1$. Set $x_j' = x_j - x_j^*$ for $j = 1, 2$. Since $\sin x_j \simeq 1 - (x_j')^2/2$ and $\cos(x_j) \simeq -x_j'$ for $j = 1, 2$ then $p_1 \simeq x_1' x_2'$ and $-2p_2 \simeq (x_1')^2 - (x_2')^2$. Thus $(x_1' + i x_2')^2 \simeq 2i(p_1 + i p_2)$.
It means that we have the same transform than in Case A-1), i.e., there exists an Hilbert space $\tilde{H}$ and an isometry $L^2(\mathbb{R}^3) \ni u \to \tilde{u} \in \tilde{H}$ such that

$$(A_{x^*} u, v)_{\mathcal{H}^D} = -i \int_{\mathbb{R}^2 \times \mathbb{S}^1} \mathrm{sgn}(q_1) < \frac{\partial(\tilde{\chi}\tilde{\pi}\tilde{u})}{\partial q_1}, \tilde{\chi}_2 \tilde{\pi}_2 \tilde{v} >_{\mathbb{C}^6, D} \, \mathrm{d}q,$$

where $\tilde{\chi}(q) := \chi_{x^*}(x)$, $\tilde{\pi}(q) := \pi_2(y)$. Hence, as in Case 3-3, A_{x^*} has default index $N^- = 0$) and admits a maximal symmetric extension.

8.4. **Notations.**

$$
\begin{aligned}
y &= \sin x \quad (y_j = \sin x_j),\\
z &= y^2 \quad (z_j = y_j^2),\\
\boldsymbol{\beta} &= \boldsymbol{\varepsilon} \times \boldsymbol{\mu} = (\beta_1, \beta_2, \beta_3),\\
\boldsymbol{\alpha} &= (\alpha_1, \alpha_2, \alpha_3), \quad \alpha_1 := (\varepsilon_2\mu_3 + \varepsilon_3\mu_2)/2 \quad \text{and c.p.},\\
\gamma_1 &= \varepsilon_2\varepsilon_3\mu_2\mu_3 \quad \text{and c.p.},\\
\nu &= \frac{2\alpha_3\sqrt{\beta_1\beta_2} - \sqrt{\gamma_3}(\beta_1+\beta_2)}{|\beta_3|\sqrt{\gamma_3}},\\
\Phi_0 &= \varepsilon_2\varepsilon_3\mu_2\mu_3 z_1^2 + (\varepsilon_2\varepsilon_3\mu_1\mu_3 + \varepsilon_1\varepsilon_3\mu_2\mu_3) z_1 z_2 + \text{c.p.},\\
K_0 &= \frac{1}{4}\left(\beta_1^2 z_1^2 + \beta_2^2 z_2^2 + \beta_3^2 z_3^2 - 2\beta_1\beta_2 z_1 z_2 - 2\beta_2\beta_3 z_2 z_3 - 2\beta_1\beta_3 z_1 z_3\right),\\
\Psi_0 &= \boldsymbol{\alpha} \cdot z \;=\; \alpha_1 z_1 + \alpha_2 z_2 + \alpha_3 z_3,\\
\tau^{\pm} &= \Psi_0 \pm \sqrt{K_0},\\
\lambda^* &= \max\{\sqrt{\tau^+(z)}|\; z \in [0,1]^3\},\\
P_M &: \mathbb{R} \times \mathbb{T}^3 \ni (\lambda, x) \mapsto x \in \mathbb{T}^3,\\
P_{\mathbb{R}} &: \mathbb{R} \times \mathbb{T}^3 \ni (\lambda, x) \mapsto \lambda \in \mathbb{R},\\
\Sigma &= \{(\lambda, x) \,|\, \lambda \in \sigma(H^D(x))\} = \cup_{j=1}^6 \Sigma_j,\\
\mathcal{X}_j &= P_M(\Sigma_j), \quad j = 1, 2,\\
\Sigma_1 &= \{(\lambda, x) \in \Sigma;\; K_0(z) \neq 0\;\},\\
\Sigma_1^{*\pm} &= \{(\lambda, x) \in \Sigma_1;\; \lambda^2 = \tau^{\pm}(z), \nabla_x \tau^{\pm}(z) = 0\},\\
\Sigma_1^* &= \Sigma_1^{*+} \cup \Sigma_1^{*-},\\
\Sigma_2 &= \{(\lambda, x) \in \Sigma;\; z \neq 0,\; K_0(z) = 0,\; \lambda^2 = \Psi_0(z)\},\\
\Sigma_2^* &= \{(\lambda, x) \in \Sigma_2;\; \nabla_x \Psi_0(z) \text{ is normal to } \mathcal{X}_2 \text{ at } x\},\\
\mathcal{X}_0 &= \{x \in \mathbb{T}^3;\; z = 0\},\\
\mathbb{T}_0^3 &= \mathbb{T}^3 \setminus \mathcal{X}_0,\\
Z_{\{0,1\}} &= \{0,1\}^3,\\
Z_{\{0,1\}}^* &= Z_{\{0,1\}} \setminus \{(0,0,0)\},\\
X_{\{0,1\}} &= \{x \in \mathbb{T}^3;\; z \in Z_{\{0,1\}}\},\\
X_{\{0,1\}}^* &= \{x \in \mathbb{T}^3;\; z \in Z_{\{0,1\}}^*\},\\
\mathcal{T} &= \mathcal{T}_1 \cup \mathcal{T}_2 \cup \{0\},\\
\mathcal{T}_j &= P_{\mathbb{R}}(\Sigma_j^*),\\
\mathcal{T}_1^{\pm} &= P_{\mathbb{R}}(\Sigma_1^{*\pm}),\\
\mathcal{X}_j^* &= P_M(\Sigma_j^*),\\
\mathcal{X}_1^{*\pm} &= P_M(\Sigma_1^{*\pm}),\\
\mathcal{X}^* &= \mathcal{X}_1^* \cup \mathcal{X}_2^*,\\
\mathcal{Z}_1^{*\pm} &= \sin^2(\mathcal{X}_1^{\pm}),\\
\sin^2(\mathcal{X}^*) &= \sin^2(\mathcal{X}_1^*) \cup \sin^2(\mathcal{X}_2^*),\\
A_{out} &= \mathcal{A}_{out} + \mathcal{A}_{out}^*,\\
A_{\phi} &= A_{out} + A_{in} = A_0 + A_1 + A_2,\\
A_0 &= A_{out}.
\end{aligned}
$$

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