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# VARIETIES COVERED BY AFFINE SPACES AND THEIR CONES

I. ARZHANTSEV, S. KALIMAN AND M. ZAIDENBERG

ABSTRACT. It was shown in [KZ23b] that the affine cones over flag manifolds and rational smooth projective surfaces are elliptic in the sense of Gromov. The latter remains true after successive blowups of points on these varieties. In the present note we extend this to smooth projective spherical varieties (in particular, toric varieties) successively blown up along linear subvarieties. The same also holds, more generally, for projective varieties covered by affine spaces.

## 1. INTRODUCTION

We work over algebraically closed field  $\mathbb{K}$  of characteristic zero;  $\mathbb{P}^n$  and  $\mathbb{A}^n$  stand for the projective resp. affine  $n$ -space over  $\mathbb{K}$ ; all the varieties below are algebraic varieties defined over  $\mathbb{K}$ .

The notion of Gromov ellipticity appeared first in analytic geometry where it is used in order to establish the Oka-Grauert Principle in the most general form, see [Gro89] and [For17]. Gromov considered as well an analogous notion of ellipticity in the setup of algebraic varieties. It is known that an elliptic smooth algebraic variety  $X$  of dimension  $n$  admits a surjective morphism from  $\mathbb{A}^{n+1}$  which also is surjective and smooth on an open subset of  $\mathbb{A}^{n+1}$ , see [Kus22a]. This implies that the endomorphism monoid  $\text{End}(X)$  is highly transitive on  $X$ , see [KZ23b, Appendix A]. Furthermore, for  $\mathbb{K} = \mathbb{C}$  the fundamental group  $\pi_1(X)$  is finite, see [Kus22b].

Recall that a smooth algebraic variety  $X$  is called *elliptic* if it admits a dominating Gromov spray  $(E, p, S)$  where  $p: E \rightarrow X$  is a vector bundle with zero section  $Z$  and  $s: E \rightarrow X$  is a morphism such that  $s|_Z = p|_Z$  and  $s$  is dominating at any point  $x \in X$ , that is, the restriction  $s|_{E_x}$  to the fiber  $E_x = p^{-1}(x)$  is dominant at the origin  $0_x \in E_x$ . The image  $O_x = s(E_x) \subset X$  is called the *s-orbit* of  $x$ .

According to [Gro89, 3.5.B] if the ellipticity holds locally on an open covering of  $X$ , then it holds globally. Moreover, the ellipticity of  $X$  holds if  $X$  is subelliptic, that is, there is a dominating collection of sprays on  $X$  instead of a single spray, see [For17] and [KZ23a]. In other words, one can always replace a dominating collection of sprays on  $X$  by a single dominating spray.

Let  $X \subset \mathbb{P}^N$  be a smooth projective variety of dimension  $n$ . The affine cone  $\text{cone}(X)$  in  $\mathbb{A}^{N+1}$  blown up at the origin gives rise to a line bundle  $F = \mathcal{O}_X(-1)$  on  $X$  whose zero section  $Z_F$  is the exceptional divisor of the blowup. The associated principal  $\mathbb{G}_m$ -fiber bundle  $Y \rightarrow X$  with fiber  $\mathbb{A}_*^1 = \mathbb{A}^1 \setminus \{0\}$  is isomorphic to  $F \setminus Z_F$  and so, to the

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punctured affine cone over  $X$  that is, the affine cone with its vertex removed:

$$Y = F \setminus Z_F \simeq_X \text{cone}(X) \setminus \{0\}.$$

Our aim is to establish the ellipticity of  $Y$  provided  $X$  is elliptic, under certain additional assumptions on  $X$ . In [KZ23b] the second and the third authors suggested criteria of ellipticity of  $Y$  based on the so called *curve-orbit property* for some families of smooth rational curves and sprays on  $X$ . In particular, it was shown in [KZ23b] that the punctured affine cones over a flag variety  $G/P$  blown up in several points and infinitesimally near points are elliptic and the same holds for any rational smooth projective surface, see [KZ23b, Theorem 0.1]. In the present note we develop further the technique of [KZ23b]. This allows us to establish similar facts for varieties of class  $\mathcal{A}_0$ , in particular, for smooth projective toric and more generally, spherical varieties. Recall that a spherical variety is a normal  $G$ -variety which contains an open  $B$ -orbit, where  $G$  is a reductive algebraic group and  $B$  is a Borel subgroup in  $G$ . A flag variety  $G/P$  and a normal toric variety are spherical varieties. For  $G/P$  the latter follows from the Bruhat decomposition  $G = BWB$ , and for a toric  $T$ -variety it suffices to choose  $G = B = T$ .

One says that a variety  $X$  belongs to class  $\mathcal{A}_0$  if there is an open cover  $\{A_i\}$  on  $X$  by affine cells  $A_i \simeq \mathbb{A}^n$  where  $n = \dim(X)$ . It is well known that a variety of class  $\mathcal{A}_0$  is elliptic; see Lemma 3.1 below. The blowup of a variety of class  $\mathcal{A}_0$  in a point is again a variety of class  $\mathcal{A}_0$ . More generally, suppose  $X$  is a variety of class  $\mathcal{A}_0$  and  $Z \subset X$  is a closed subvariety such that the pair  $(A_i, Z \cap A_i)$  is isomorphic for any  $i$  to a pair  $(\mathbb{A}^n, \mathbb{A}^k)$  with  $n - k \geq 2$ ; in this case  $Z$  is called a *linear subvariety of  $X$* . The blowup of a linear subvariety  $Z$  in  $X$  results again in a variety of class  $\mathcal{A}_0$ , see [APS14, Section 4, Statement 9].

We prove the following theorem.

**Theorem 1.1** (Theorem 3.4). *Let  $X$  be a projective variety of class  $\mathcal{A}_0$  with  $\dim(X) > 0$ , let  $D$  be an ample divisor on  $X$  and  $Y = F \setminus Z_F$  be the principal  $\mathbb{G}_m$ -fiber bundle associated with a line bundle  $F$  where either  $F = \mathcal{O}_X(-D)$  or  $F = \mathcal{O}_X(D)$ . Then  $Y$  is elliptic.*

Notice that up to isomorphism over  $X$  that inverses the  $\mathbb{G}_m$ -action,  $Y = F \setminus Z_F$  stays the same under replacing  $D$  by  $-D$ . Observe also that for a trivial line bundle  $F$  the variety  $Y \simeq_X X \times \mathbb{A}_*^1$  is not elliptic and  $\pi_1(Y)$  is infinite for  $\mathbb{K} = \mathbb{C}$ .

Smooth complete spherical varieties and smooth complete rational  $T$ -varieties of complexity one belong to class  $\mathcal{A}_0$ , see [APS14]. Hence, we have the following corollary.

**Corollary 1.2** (cf. Corollary 4.3). *Let a smooth projective variety  $X$  be either toric, or spherical, or rational  $T$ -variety of complexity one. Then the conclusion of Theorem 1.1 holds for  $X$  blown up successively along linear subvarieties.*

By Theorem 1.1 the punctured affine cones over projective varieties of class  $\mathcal{A}_0$  are elliptic. Let us mention further examples.

Recall that an effective  $\mathbb{G}_m$ -action  $\lambda: \mathbb{G}_m \times \bar{Y} \rightarrow \bar{Y}$  on a normal affine variety  $\bar{Y}$  is called *good* if there exists a point  $y_0 \in \bar{Y}$  which belongs to the closure of any

$\lambda$ -orbit. The structure algebra  $A = \mathcal{O}_{\bar{Y}}(\bar{Y})$  of such a  $\mathbb{G}_m$ -variety  $\bar{Y}$  is positively graded:  $A = \bigoplus_{k \geq 0} A_k$  where  $A_0 = \mathbb{K}$  and  $A_k$  for  $k > 0$  consists of  $\lambda$ -homogeneous elements of weight  $k$ . Let  $Y = \bar{Y} \setminus \{y_0\}$  and  $X = \text{Proj}(A) = Y/\lambda$ . Then  $X$  is a normal projective variety, see [Dem88, Proposition 3.3]. According to [Dem88, Theorem 3.5], see also [Dol07, Theorem 3.3.4], there exists an ample  $\mathbb{Q}$ -Cartier divisor  $D = \sum_i p_i/q_i D_i$  on  $X$ , where the  $D_i$  are prime divisors and the integers  $p_i, q_i$  are coprime, such that

$$A_k = H^0(X, \mathcal{O}_X(\lfloor kD \rfloor)) \quad \text{for every } k \geq 0.$$

Furthermore, the  $\mathbb{G}_m$ -action  $\lambda$  on  $Y$  is free if and only if  $D$  is a Cartier divisor that is,  $q_i = 1 \forall i$ , see [Dem88, Corollaire 2.8.1].

Conversely, given a smooth projective variety  $X$  and an ample Cartier divisor  $D$  on  $X$  one can consider the generalized affine cone

$$\bar{Y} = \text{Spec} \left( \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD)) \right).$$

This is a normal affine variety equipped with a good  $\mathbb{G}_m$ -action, see [Dem88, Sec. 3] or [Dol07, Proposition 3.3.5]. Letting  $Y = \bar{Y} \setminus \{y_0\}$  where  $y_0 \in \bar{Y}$  is the unique  $\mathbb{G}_m$ -fixed point, one gets a morphism  $\pi: Y \rightarrow X$ . Every fiber of  $\pi$  is reduced, irreducible and isomorphic to  $\mathbb{A}_*^1$ , see [Dem88, Proposition 2.8] and [Dol07, Proposition 3.4.5]. Furthermore, the  $\mathbb{G}_m$ -action on  $Y$  is free and  $\pi$  is locally trivial, see [Dem88, the proof of Proposition 2.8].

Consider also the line bundle  $F = \mathcal{O}_X(-D) \rightarrow X$  equipped with the associated  $\mathbb{G}_m$ -action. We have a birational morphism  $F \rightarrow \bar{Y}$  contracting the zero section  $Z_F$  to a normal point  $y_0 \in \bar{Y}$ , cf. [Dem88, 3.4]. It restricts to an equivariant isomorphism of smooth quasiaffine varieties

$$F \setminus Z_F \simeq \bar{Y} \setminus \{y_0\}$$

equipped with free  $\mathbb{G}_m$ -actions, see [Dem88, Corollaire 2.9], [Dol07, Sec. 3.4, p. 49] and [KPZ13, Sec. 1.15]; cf. [Pin77, p. 183].

Notice that while  $\bar{Y}$  above is normal, for  $X \subset \mathbb{P}^N$  the affine cone  $\text{cone}(X)$  is normal if and only if the embedding  $X \hookrightarrow \mathbb{P}^N$  is projectively normal, that is, for every  $d \geq 1$  the linear system cut out on  $X$  by the hypersurfaces of degree  $d$  is complete, see [Har04, Chap. II, Example 7.8.4]. Thus, if  $D$  is a hyperplane sections of  $X \subset \mathbb{P}^N$  then  $\bar{Y}$  as above is the normalization of the affine cone  $\text{cone}(X)$ . In particular, the punctured cone  $\text{cone}(X) \setminus \{0\}$  is  $\mathbb{G}_m$ -equivariantly isomorphic to  $\bar{Y} \setminus \{y_0\}$ .

## 2. THE CURVE-ORBIT PROPERTY

We need the following more general analog of the curve-orbit property (\*) for  $\mathbb{G}_a$ -sprays defined in [KZ23b, Definition 2.7].

**Definition 2.1.** Given a smooth variety  $B$  of dimension  $n - 1$  consider the  $\mathbb{G}_a$ -action on the cylinder  $V = B \times \mathbb{A}^1$  by shifts on the second factor:

$$s_V: \mathbb{G}_a \times V \rightarrow V, \quad (t, (b, v)) \mapsto (b, v + t)$$

along with the associated  $\mathbb{G}_a$ -spray  $(E_V, p_V, s_V)$  on  $V$  where  $E_V = V \times \mathbb{A}^1$  and  $p_V: E_V \rightarrow V$  is the first projection.

Let  $X$  be a smooth variety of dimension  $n$ . Assume that  $X$  admits a birational morphism  $\psi: V \rightarrow X$  biregular on an open dense subset  $V_0 \subset V$  with image  $X_0 \subset X$ . Consider the spray  $(E_0, p_0, s_0)$  on  $X_0$  with values in  $X$  conjugate to  $(E_V, p_V, s_V)|_{V_0}$  via  $\psi$ . That is,  $E_0 = X_0 \times \mathbb{A}^1$ ,  $p_0: E_0 \rightarrow X_0$  is the first projection and

$$s_0: E_0 \rightarrow X, \quad (x, t) \mapsto \psi(s_V(t, \psi^{-1}(x))).$$

Extend  $(E_0, p_0, s_0)$  to a rank 1 spray  $(E, p, s)$  on  $X$ ; the latter spray exists due to Gromov's Extension Lemma, see [Gro89, 3.5B], [For17, Propositions 6.4.1-6.4.2] or [KZ23b, Proposition 6.1]. We call  $(E, p, s)$  a  $\mathbb{G}_a$ -like spray on  $X$ . This spray is associated with the birational  $\mathbb{G}_a$ -action on  $X$  conjugate via  $\psi$  to the standard  $\mathbb{G}_a$ -action on the cylinder  $V$ , see [Dem70, Chap. 1].

**Remark 2.2.** The closure  $C_x = \overline{(O_x)}$  in  $X$  of the  $s_0$ -orbit  $O_x$  of a point  $x \in X_0$  is a rational curve. By construction, the intersection  $C_x \cap X_0$  is smooth and  $(E, p, s)$  restricts to a dominating spray on  $O_x \cap X_0$ , cf. Lemma 2.4 below. Moreover, the morphism  $s: E_x \simeq \mathbb{A}^1 \rightarrow C_x$  admits a lift  $\mathbb{A}^1 \rightarrow \mathbb{P}^1$ , and the latter morphism to normalization is an embedding.

However, the curve  $C_x$  can have singularities off  $X_0$ . Thus, the setup of Definition 2.1 does not guarantee that  $X$  verifies on  $X_0$  either the curve-orbit property (\*) of [KZ23b, Definition 2.7], or the enhanced curve-orbit property (\*\*) of [KZ23b, Proposition 3.1]. Indeed, the latter properties postulate the smoothness of  $C_x$  for  $x \in X_0$ , which occurs to be a rather restrictive condition for our purposes. The question arises whether any smooth complete variety of class  $\mathcal{A}_0$  verifies the curve-orbit property (\*) of [KZ23b] with smooth rational curves; cf. Proposition 3.3 below. Notice that this property holds for smooth complete rational surfaces and for flag varieties  $G/P$ , see [KZ23b].

In order to use a criterion of ellipticity of cones over projective varieties from [KZ23b, Corollary 2.9] we introduce the following objects.

**Definition 2.3.** Let  $X$  and  $B$  be as in Definition 2.1. Consider a  $\mathbb{P}^1$ -cylinder  $W = B \times \mathbb{P}^1$ . Assume that  $X$  admits a birational morphism  $\varphi: W \rightarrow X$  biregular on an open dense subset  $W_0 \subset W$  with image  $X_0 \subset X$ . Given a point  $u \in \mathbb{P}^1$  consider the cylinder  $V_u := B \times (\mathbb{P}^1 \setminus \{u\}) \simeq B \times \mathbb{A}^1$ , the birational morphism  $\psi_u = \varphi|_{V_u}: V_u \rightarrow X$  and the open dense subsets  $V_u \cap W_0 \subset V_u$  and  $X_u = \psi_u(V_u \cap W_0) \subset X_0$ . Thus, the data  $(W, \varphi, X_0)$  yields a one-parameter family of  $\mathbb{G}_a$ -like sprays  $(E_u, p_u, s_u)$  on  $X$  where  $u \in \mathbb{P}^1$ , see Definition 2.1.

**Lemma 2.4.** *Under the setup of Definition 2.3 let for  $x \in X_0$ ,*

$$(1) \quad w = \varphi^{-1}(x) = (b, u_x) \in W_0 \quad \text{and} \quad C_x = \varphi(\{b\} \times \mathbb{P}^1) \subset X.$$

*Then  $C_x$  is a complete rational curve in  $X$  through  $x$  such that  $C_x \cap X_0$  is smooth. If  $x \in X_u$ , that is  $\varphi(b, u) \neq x$ , then the  $s_u$ -orbit  $O_{u,x}$  of  $x$  is one-dimensional and  $(E_u, p_u, s_u)$  restricts to a spray on  $C_x$  dominating at  $x$  and such that  $s_u|_{E_{u,x}}: E_{u,x} \rightarrow O_{u,x}$  is a birational morphism étale over  $x$ .*

*Proof.* The  $\mathbb{G}_a$ -like spray  $(E_u, p_u, s_u)$  inherits a kind of the composition property of a  $\mathbb{G}_a$ -action. Namely, for any  $x' \in O_{u,x'} \cap X_0$  the  $s_u$ -orbits  $O_{u,x'}$  and  $O_{u,x}$  coincide. This implies that  $(E_u, p_u, s_u)$  restricts to a spray on  $C_x$ . The rest of the proof is easy and is left to the reader.  $\square$

**Definition 2.5.** Modifying [KZ23b, Definition 2.7] we say that a complete rational curve  $C$  on a smooth variety  $X$  verifies the *strengthen two-orbit property* at a smooth point  $x \in C$  if

- (\*) there exists a pair of rank 1 sprays  $(E_i, p_i, s_i)$  ( $i = 1, 2$ ) on  $X$  such that  $C$  is covered by the one-dimensional  $s_i$ -orbits  $O_{i,x}$ ,  $s_i: E_{i,x} \rightarrow O_{i,x}$  is a birational morphism étale over  $x$  and  $(E_i, p_i, s_i)$  restricts to a spray on  $O_{i,x}$  dominating at  $x$ .

If for any  $x \in X$  there exists a curve  $C = C_x$  as above, then we say that  $X$  verifies the *strengthen curve-orbit property*.

**Remark 2.6.** In fact, to show ellipticity of cones over an elliptic variety the following weaker condition could be used:

- for each point  $x \in X$  there exists a rational curve  $C$  in  $X$  and a pair of sprays  $\{(E_i, p_i, s_i)\}_{i=1,2}$  on  $X$  such that  $x \in C$  is a smooth point,  $C = O_{1,x} \cup O_{2,x}$  and  $ds_i(0_{i,x}) \neq 0$  where  $0_{i,x} \in E_{i,x}$  is the origin.

However, in the setup of the present paper the stronger property (\*) holds for each  $x \in X$ .

We have the following corollary.

**Corollary 2.7.** *Let  $X$  be a smooth projective variety, and let  $(W, \varphi, X_0)$  be a data as in Definition 2.3. For  $x \in X_0$  let  $C_x$  be a curve as in (1). Then  $C_x$  verifies the strengthen two-orbit property (\*) at  $x$  with a pair of  $\mathbb{G}_a$ -like sprays. If for each  $x \in X$  there exists a data  $(W, \varphi, X_0)$  such that  $x \in X_0$  then the strengthen curve-orbit property holds on  $X$  with pairs of  $\mathbb{G}_a$ -like sprays.*

*Proof.* Let  $\varphi^{-1}(x) = (b, u_x) \in W_0$ . Pick two distinct points  $u_1, u_2 \in \mathbb{P}^1$  different from  $u_x$  and consider the corresponding  $\mathbb{G}_a$ -like sprays  $(E_i, p_i, s_i) = (E_{u_i}, p_{u_i}, s_{u_i})$ ,  $i = 1, 2$ , see Definition 2.3. Due to Lemma 2.4 these sprays fit in Definition 2.5 of the strengthen two-orbit property. This yields the first assertion. Now the second is immediate.  $\square$

### 3. ELLIPTICITY OF CONES OVER $\mathcal{A}_0$ -VARIETIES

The following lemma is well known, cf. e.g. [Gro89, 3.5B']. For the convenience of the reader we recall the argument.

**Lemma 3.1.** *A variety of class  $\mathcal{A}_0$  is elliptic.*

*Proof.* Let  $\{A_i\}$  be an open covering of  $X$  such that  $A_i \simeq \mathbb{A}^n$ . Fix an isomorphism  $A_i \simeq \mathbb{A}^n$  and consider the  $(\mathbb{G}_a)^n$ -action on  $A_i$  by translations,

$$s_i: (\mathbb{G}_a)^n \times A_i \rightarrow A_i, \quad (v, x) \mapsto x + v.$$

Treating  $p_i = \text{pr}_2: (\mathbb{G}_a)^n \times A_i \rightarrow A_i$  as a trivial vector bundle on  $A_i$  of rank  $n$  this action yields a dominating spray  $(E_i, p_i, s_i)$  on  $A_i$  which extends to  $X$  by Gromov's Localization Lemma. This gives a dominating collection of sprays  $\{(E_i, p_i, s_i)\}$  on  $X$ . Thus,  $X$  is subelliptic and so, elliptic, see [KZ23a].  $\square$

Using Definitions 2.1 and 2.3 we can generalize the ellipticity criterion for cones in [KZ23b, Corollary 2.9] as follows. The proof repeats verbatim the proof of Corollary 2.9 in [KZ23b] with minor changes.

**Proposition 3.2.** *Let  $X$  be a smooth projective variety and  $\varrho: F \rightarrow X$  be an ample line bundle. Suppose that  $X$  is elliptic and for any point  $x \in X$  there exists a data  $(W, \varphi, X_0)$  as in Definition 2.3 such that  $x \in X_0$ . Then  $Y = F \setminus Z_F$  is elliptic.*

*Proof.* A dominating spray  $(E, p, s)$  on  $X$  lifts to a spray  $(\hat{E}, \hat{p}, \hat{s})$  of rank  $n = \dim(X)$  on  $Y$  where  $\hat{E}$  fits in the commutative diagrams

$$(2) \quad \begin{array}{ccc} \hat{E} & \xrightarrow{\hat{p}} & Y \\ \downarrow \hat{\rho} & & \downarrow \varrho|_Y \\ E & \xrightarrow{p} & X \end{array} \quad \text{and} \quad \begin{array}{ccc} \hat{E} & \xrightarrow{\hat{s}} & Y \\ \downarrow \hat{\rho} & & \downarrow \varrho|_Y \\ E & \xrightarrow{s} & X \end{array}$$

see [KZ23b, Lemma 2.3].

Given  $y \in Y$  we let  $x = \varrho(y) \in X$  and let  $C_x$  be a rational curve on  $X$  passing through  $x$  which is smooth at  $x$  and such that  $(C_x, x)$  satisfies the strengthened two-orbit property with a pair of  $\mathbb{G}_a$ -like sprays  $(E_i, p_i, s_i)$  on  $X$ ,  $i = 1, 2$ , see Definition 2.5 and Corollary 2.7. Due to [KZ23b, Lemma 2.3] these sprays admit a lift to a pair of rank 1 sprays  $(\hat{E}_i, \hat{p}_i, \hat{s}_i)$  on  $Y$ .

We will show, following the lines of the proof of Proposition 2.6 in [KZ23b], that the triplet of sprays  $(\hat{E}, \hat{p}, \hat{s})$  and  $(\hat{E}_i, \hat{p}_i, \hat{s}_i)$ ,  $i = 1, 2$  is dominating at  $y$ . This implies the assertion.

Notice that the tangent space at  $y$  to the  $\hat{s}$ -orbit is a hyperplane  $H \subset T_y Y$  such that  $d\varrho(H) = T_x X$ . We claim that the pair of tangent vectors at  $y$  to the  $\hat{s}_i$ -orbits  $\hat{O}_{i,y}$  on  $Y$  span a plane  $P$  in  $T_y Y$  such that  $d\varrho(P) = T_x C_x$ . Accepting this claim, there exists a nonzero vector  $v \in P$  such that  $d\varrho(v) = 0$ . Hence  $v \notin H$  and so,  $\text{span}(H, P) = T_y Y$ , which gives the desired domination.

To show the claim, consider a morphism of normalization  $\varphi_x: \mathbb{P}^1 \rightarrow C_x$  and the pullback line bundle  $\tilde{F} = \tilde{F}(x) = \varphi_x^*(F|_{C_x}) \rightarrow \mathbb{P}^1$ . Since  $F \rightarrow X$  is ample one has  $[F] \cdot [C_x] \neq 0$ . Hence,  $\tilde{F} \rightarrow \mathbb{P}^1$  is nontrivial and so is the associated principal  $\mathbb{G}_m$ -fiber bundle  $\tilde{Y} = \tilde{Y}(x) = \varphi_x^*(Y|_{C_x}) \rightarrow \mathbb{P}^1$ . For  $i = 1, 2$  consider the pullback  $\tilde{p}_i: \tilde{E}_i \rightarrow \tilde{Y}$  of the line bundle  $\hat{p}_i: \hat{E}_i \rightarrow Y$  via the induced morphism  $\tilde{Y} \rightarrow Y|_{C_x}$ .

According to the second diagram in (2) we have

$$\varrho(\hat{O}_{i,y}) = \varrho \circ \hat{s}_i(\hat{E}_{i,y}) = s_i(E_{i,x}) = O_{i,x} \subset C_x.$$

The birational morphism  $s_i|_{E_{i,x}}: E_{i,x} = \mathbb{A}^1 \rightarrow C_x$  admits a lift to normalization  $\tilde{s}_{i,x}: \mathbb{A}^1 \rightarrow \mathbb{P}^1$ . In fact,  $\tilde{s}_{i,x}$  is a birational morphism smooth at 0. Hence,  $\tilde{s}_{i,x}$  sends  $\mathbb{A}^1$  isomorphically onto  $U_i := \mathbb{P}^1 \setminus \{u_i\}$ , the notation being as in the proof of Corollary 2.7.



By Lemma 2.4  $(E_i, p_i, s_i)$  restricts to a  $\mathbb{G}_a$ -like spray on  $C_x$ . It is easily seen that there is a pullback  $\varphi_x^*((E_i, p_i, s_i)|_{C_x})$  to a  $\mathbb{G}_a$ -like spray  $(E'_i, p'_i, s'_i)$  on  $\mathbb{P}^1$  dominating at  $x$  and such that the  $s'_i$ -orbit of  $x$  coincides with  $U_i$ . Now, the pullback of  $(E'_i, p'_i, s'_i)$  to  $\tilde{Y}$  via  $\tilde{Y} \rightarrow \mathbb{P}^1$  gives a  $\mathbb{G}_a$ -like spray  $(\tilde{E}_i, \tilde{p}_i, \tilde{s}_i)$  on  $\tilde{Y}$  with  $\tilde{p}_i: \tilde{E}_i \rightarrow \tilde{Y}$  as above.

There are trivializations  $\tilde{Y}|_{U_i} \cong_{U_i} U_i \times \mathbb{A}_*^1$ ,  $i = 1, 2$ . The standard  $\mathbb{G}_a$ -action on  $U_i \simeq \mathbb{A}^1$  lifts to a  $\mathbb{G}_a$ -action on  $\tilde{Y}|_{U_i}$  whose orbits are the constant sections. Since any morphism  $\mathbb{A}^1 \rightarrow \mathbb{A}_*^1$  is constant, the one-dimensional  $\tilde{s}_i$ -orbits in  $\tilde{Y}$  are also the constant sections. Since the  $\mathbb{G}_m$ -fiber bundle  $\tilde{Y} \rightarrow \mathbb{P}^1$  is nontrivial, it admits no global section. In an appropriate affine coordinate  $z$  on  $U_1 \cap U_2$  the transition function equals  $z^k$  with  $k \neq 0$ . It follows that the constant sections over  $U_1$  meet transversally the ones over  $U_2$ .

The normalization morphism  $\varphi: \mathbb{P}^1 \rightarrow C_x$  is étale over  $x$ . Hence, also the morphism  $\tilde{Y} \rightarrow Y|_{C_x}$  is étale over  $y$ . Finally, the  $\hat{s}_i$ -orbits  $\hat{O}_{1,y}$  and  $\hat{O}_{2,y}$  meet transversally at  $y$ . This proves our claim.  $\square$

**Proposition 3.3.** *Suppose that a smooth complete variety  $X$  contains an open dense subset  $U$  isomorphic to  $\mathbb{A}^n$ . Then for any point  $x \in U$  there exists a data  $(W, \varphi, X_0)$  as in Definition 2.3 with  $X_0 = U$ .*

*Proof.* Let  $H$  be a hyperplane in  $\mathbb{P}^n$ ; we simply write  $\mathbb{A}^n$  for the complement  $\mathbb{P}^n \setminus H$ . The isomorphism  $\mathbb{A}^n \xrightarrow{\sim} U$  extends to a birational map  $\alpha: \mathbb{P}^n \dashrightarrow X$ . Consider the normalization  $\hat{X}$  of the closure of the graph  $\Gamma(\alpha) \subset \mathbb{P}^n \times X$ . The canonical projections of the product  $\mathbb{P}^n \times X$  to the factors yield proper birational morphisms  $f$  and  $g$  fitting in the commutative diagram

$$\begin{array}{ccc} & \hat{X} & \\ f \swarrow & & \searrow g \\ \mathbb{P}^n & \xrightarrow{\alpha} & X \end{array}$$

Let  $\hat{U} = f^{-1}(\mathbb{A}^n) \subset \hat{X}$ . Then the restrictions

$$f|_{\hat{U}}: \hat{U} \rightarrow \mathbb{A}^n \quad \text{and} \quad g|_{\hat{U}}: \hat{U} \rightarrow U$$

are isomorphisms. Since  $X$  is normal the codimension of the singular locus of  $\hat{X}$  is at least 2 and the same holds for the codimension in  $\mathbb{P}^n$  of the indeterminacy locus of  $\alpha: \mathbb{P}^n \dashrightarrow X$ . It follows from the Zariski Main Theorem, see [Har04, Corollary III.11.4] that for a general point  $p \in H$  there is a point  $\hat{p} \in \hat{X}$  such that  $f(\hat{p}) = p$  and  $g^{-1}(g(\hat{p})) = \hat{p}$ . Therefore,  $\alpha$  extends regularly to  $p$  and  $\alpha(p) = q := g(\hat{p})$ .

Consider now the free  $\mathbb{G}_a$ -action  $\lambda$  on  $\mathbb{A}^n$  by translations along the family of parallel lines in  $\mathbb{A}^n$  directed toward  $p \in H$ . Letting  $\lambda_U$  be the free  $\mathbb{G}_a$ -action on  $U$  conjugate with  $\lambda$  via  $\alpha$ , the closure in  $X$  of each  $\lambda_U$ -orbit passes through the common point  $q \in \bar{U} \setminus U$ . The action of  $\lambda_U$  on  $U$  yields an isomorphism  $B \times \mathbb{A}^1 \xrightarrow{\sim} U \simeq \mathbb{A}^n$  where  $B = \mathbb{A}^{n-1}$ . By our construction, it extends to a birational morphism  $\varphi: W = B \times \mathbb{P}^1 \rightarrow X$  which



sends  $W_0 = B \times \mathbb{A}^1$  biregularly onto  $U$  and contracts the divisor  $B \times \{\infty\}$  to the point  $q$ . This provides the desired data  $(W, \varphi, X_0)$  with  $X_0 = U$ .  $\square$

Due to Lemma 3.1, Propositions 3.2 and 3.3 lead to our main result.

**Theorem 3.4.** *Let  $X$  be a smooth projective variety of class  $\mathcal{A}_0$  and let  $F \rightarrow X$  be an ample or anti-ample line bundle on  $X$  with zero section  $Z_F$ . Then the variety  $Y = F \setminus Z_F$  is elliptic. In particular, any punctured affine cone  $Y$  over  $X$  is elliptic.*

#### 4. EXAMPLES

We start with the following well known example. For the reader's convenience we provide a proof.

**Lemma 4.1.** *A smooth complete toric variety  $X$  belongs to class  $\mathcal{A}_0$ .*

*Proof.* Let  $N$  be a lattice of rank  $n = \dim(X)$  and  $\Sigma$  be the associate complete fan of  $X$  in  $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ . Since  $X$  is smooth every  $n$ -cone  $\sigma \in \Sigma$  is simplicial generated by vectors  $v_1, \dots, v_n \in N$  which form a base of  $N$ . The corresponding affine variety  $X_{\sigma}$  is isomorphic to  $\mathbb{A}^n$  and the affine charts  $\{X_{\sigma}\}$  cover  $X$ , see e.g. [Ful93, Sec. 1.4].  $\square$

There are two classes of smooth complete varieties which include the toric ones where the assertion of Lemma 4.1 also holds. The first class consists of smooth complete rational varieties  $X$  with a torus action of complexity 1. The latter means that the general orbits of the torus have codimension 1 in  $X$ . The proof of this fact is somewhat involved, so we send the reader to the source, see [APS14, Theorem 5]. The second class consists of smooth complete spherical varieties. For the reader's convenience we sketch a proof.

**Lemma 4.2** ([BLV86, Sec. 1.5, Corollaire]). *A smooth complete spherical variety  $X$  belongs to class  $\mathcal{A}_0$ .*

*Proof.* Recall first the Local Structure Theorem, see [BLV86, Théorème 1.4]. Consider a normal  $G$ -variety  $Z$  and a point  $z \in Z$  such that the orbit  $Gz$  is a projective variety. Then the stabilizer  $G_z$  is a parabolic subgroup. Let  $P \subset G$  be the opposite parabolic subgroup with the Levi decomposition  $P = LP^u$  where  $P^u$  is the unipotent radical of  $P$  and  $L = P \cap G_z$  is the Levi subgroup in  $P$ . The Local Structure Theorem asserts that there is a locally closed affine subset  $V \subset Z$  such that  $z \in V$ ,  $LV = V$ ,  $P^uV$  is open in  $Z$  and the action of  $P^u$  on  $Z$  defines an isomorphism  $P^uV \cong P^u \times V$ .

Returning to Lemma 4.2 we let  $Z = X$  and we apply the notation above. It suffices to show that any point  $x \in X$  whose orbit  $Gx$  is closed in  $X$  admits a neighborhood in  $X$  isomorphic to  $\mathbb{A}^n$ . Indeed, the closure of any orbit  $Gy$  contains such a point  $x$ .

In the setup of the Local Structure Theorem the reductive Levi subgroup  $L$  of  $P = G_x^-$  acts on the corresponding smooth affine variety  $V$  with an open orbit and with an  $L$ -fixed point  $x$ . By Luna's Slice Étale theorem, see e.g. [PV94, Corollary of Theorem 6.7], one can conclude that  $V$  is equivariantly isomorphic to an  $L$ -module. It follows that  $x$  has a neighborhood in  $X$  isomorphic to  $P^u \times V$ , which is an affine space.  $\square$

The following corollary of Theorem 3.4 is immediate.

**Corollary 4.3.** *The punctured affine and generalized affine cones over a smooth projective toric or spherical variety are elliptic. This remains true after successively blowing up such a variety in linear subvarieties. The same holds for a smooth projective rational variety  $X$  with a torus action of complexity 1.*

**Remark 4.4.** Recall that the normalization of the affine cone  $\text{cone}(X)$  over a smooth toric variety  $X$  in  $\mathbb{P}^n$  is a normal affine toric variety with no torus factor. It is known that such a variety is flexible, see [AKZ12, Theorem 0.2(2)]. It follows that  $Y = \text{cone}(X) \setminus \{0\}$  is elliptic. However, it is not clear whether the flexibility of  $\text{cone}(X)$  survives under blowing-up a point in  $X$ .

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#### REFERENCES

- [AKZ12] I. V. Arzhantsev, K. G. Kuyumzhiyan, and M. G. Zaidenberg, *Flag varieties, toric varieties, and suspensions: Three instances of infinite transitivity*, Sbornik: Mathematics 203:7 (2012), 923–949.
- [APS14] I. Arzhantsev, A. Perepechko, and H. Süß, *Infinite transitivity on universal torsors*, J. London Math. Soc. 89:3 (2014), 762–778.
- [BLV86] M. Brion, D. Luna, and Th. Vust, *Espaces homogènes sphériques*, Invent. Math. 84 (1986), 617–632.
- [Dem70] M. Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*. Ann. Sci. Ecole Norm. Sup. (4) 3 (1970), 507–588.
- [Dem88] M. Demazure, *Anneaux gradués normaux. Introduction à la théorie des singularités*, II, 35–68, Travaux en Cours 37, Hermann, Paris, 1988.
- [Dol07] I. Dolgachev, *McKay correspondence*. Lecture cours, Winter 2006/07, available at: <http://www.math.lsa.umich.edu/~idolga/lecturenotes.html>.
- [For17] F. Forstnerič, *Stein Manifolds and Holomorphic Mapping. The Homotopy Principle in Complex Analysis*, Springer-Verlag, Berlin-Heidelberg, 2011.
- [Ful93] W. Fulton, *Introduction to toric varieties*. Annals of Math. Studies 131, Princeton University Press, Princeton, NJ, 1993.
- [Gro89] M. Gromov, *Oka’s principle for holomorphic sections of elliptic bundles*, J. Amer. Math. Soc. 2 (1989), 851–897. MR1001851. DOI10.2307/1990897.
- [Har04] R. Hartshorne, *Algebraic Geometry*. Graduate Texts in Mathematics 52, Springer 2004.
- [KZ23a] S. Kaliman and M. Zaidenberg, *Gromov ellipticity and subellipticity*. arXiv:2301.03058 (2023).
- [KZ23b] S. Kaliman and M. Zaidenberg, *Gromov ellipticity of cones over projective manifolds*. arXiv:2303.02036 (2023).
- [KPZ13] T. Kishimoto, Yu. Prokhorov, and M. Zaidenberg,  $\mathbb{G}_a$ -actions on affine cones. Transform. Groups 18:4 (2013), 1137–1153.
- [Kus22a] Y. Kusakabe. *Surjective morphisms onto subelliptic varieties*. arXiv:2212.06412 (2022).
- [Kus22b] Y. Kusakabe. *On the fundamental groups of subelliptic varieties*. arXiv:2212.07085 (2022).
- [Pin77] H. Pinkham, *Normal surface singularities with  $\mathbb{C}^*$ -action*. Math. Ann. 227 (1977), 183–193.
- [PV94] V. L. Popov and E. B. Vinberg, *Invariant Theory*. In *Algebraic Geometry, IV*, Encyclopedia of Mathematical Sciences, Vol. 55, pp. 123–278. Springer-Verlag, Berlin, 1994. doi:10.1007/978-3-662-03073-8\_2. ISBN 978-3-642-08119-4.

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