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# Perturbed Unicycle Mobile Robots: A Second–Order Sliding–Mode Trajectory Tracking Control

Héctor Ríos<sup>†,\*</sup>, Manuel Mera<sup>‡</sup> and Andrey Polyakov<sup>§</sup>

**Abstract**—This paper contributes to the design of a second–order sliding–mode controller for the trajectory tracking problem in perturbed unicycle mobile robots. The proposed strategy takes into account the design of two particular sliding variables, which ensure the convergence of the tracking error to the origin in a finite time despite the effect of some external perturbations. The straightforward structure of the controller is simple to tune and implement. The global, uniform and finite–time stability of the closed–loop tracking error dynamics is demonstrated by means of Lyapunov functions. Furthermore, the performance of the proposed approach is validated through some experiments using a QBot2 unicycle mobile robot.

**Index Terms**—Sliding–Mode Control, Trajectory Tracking, Unicycle Mobile Robots.

## I. INTRODUCTION

THE unicycle mobile robots (UMRs) have been studied extensively in the last decades due to their capability of moving freely from one point to another one and to the wide diversity of possible applications in controlled and non–controlled environments (see, *e.g.*, [1] and [2]).

However, it is well–known that the kinematic model of this class of systems does not fulfill the Brockett’s necessary condition for smooth state–feedback stabilization [3]. Therefore, the design of non–smooth or time–varying feedback controllers is a requirement for this class of mobile robots. Additionally, as it is shown in [4] and [5], even if external forces cannot be considered in the kinematic model, there exist some other signals or non–modeled phenomena, *e.g.*, the skidding and slipping of the wheels and corrupt control signals, that must be taken into account for the controller design. In this sense, the trajectory tracking control design, considering the non–holonomic constraints and external perturbations, is still a challenging problem, and the sliding–mode control theory is a good option to deal with such problems [6].

It is worth mentioning that most of the works based on the kinematic model do not consider the effect of the external perturbations. For instance, in [7] an asymptotic sliding–mode control approach is proposed to solve the trajectory tracking problem for two–wheeled mobile robots. However, the authors provide only local asymptotic stability of the tracking error. In the adaptive control framework, in [8], under a persistently exciting condition, a smooth nonlinear time–varying controller is proposed for non–holonomic UMRs. The proposed controller guarantees uniform global asymptotic stability of the tracking error in a leader–follower context. In the same context, in [9], the authors proposed a fixed–time leader–follower formation control for a swarm of UMRs with visibility and performance constraints. It is shown that the formation tracking error converge to a neighborhood of the origin in a fixed time. In [10], a first–order sliding–mode controller is presented for a differential–drive robot following a path. The proposed approach is able to ensure local asymptotic stability of the path tracking in terms of the Frénet–Serret frame. In [11], a dynamic feedback linearization controller, together with attitude and velocity observers, is proposed for the trajectory tracking problem in UMRs. However, as we previously mentioned, these works do not consider the effect of the external perturbations.

On the other hand, there do exist some works that consider such perturbations. For instance, in [12] two dynamic control laws are proposed to deal with perturbations in velocities and to ensure global asymptotic stability of the tracking error. Nevertheless, only simulation results are provided. In [13], the problem of finite–time posture stabilization is addressed for UMRs by means of sliding–mode control and multi–rate output–feedback techniques. The proposed approach considers side–slipping effects characterized by additive perturbations but no perturbations are taken into account in the angular velocity. In [14], a first–order sliding–mode control approach is proposed to deal with the trajectory tracking problem in perturbed UMRs. This approach considers some skidding and slipping effects on the wheels and it guarantees the asymptotic convergence to zero of the tracking error. In [15], a predefined–time stabilization controller is proposed for a class of uncertain non–holonomic systems. Such an approach is just applied to the stabilization of a UMR and the

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considered additive perturbations do not have any physical meaning for the UMR. Based on an MPC approach, in [16], the authors propose a trajectory tracking controller for UMRs, which is able to deal with some additive perturbations. Nevertheless, the proposed approach is computationally complex and the considered additive perturbations lack physical meaning. In [17], two robust control techniques, *i.e.*, sliding-mode control and the attractive ellipsoid method, are used to deal with the trajectory tracking problem in perturbed UMRs. The proposed approach only ensures asymptotic convergence to a region around the origin for the tracking error. Recently, a robust controller, based on the Super-Twisting algorithm, was presented in [18], which guarantees asymptotic convergence of the tracking error to zero, despite the presence of some skidding and slipping effects. The proposed controller is continuous but local. In [19], the authors introduce a robust finite-time stabilizing controller for arbitrary-order non-holonomic systems. The proposed controller can be applied to a UMR but the considered external perturbations have no physical meaning for such a particular system. Another alternative to deal with the skidding and slipping effects is to apply perturbation estimators. For instance, in [20], the authors proposed different side-slipping estimators in order to get robust linear controllers. However, the complexity of the control approaches increases.

Motivated by the above mentioned issues (external perturbations with no physical meaning, complexity, local stability and asymptotic convergence rates) and contrary to most of the proposed controllers, in this paper, a straightforward second-order sliding-mode controller is proposed to solve the trajectory tracking problem globally, uniformly and in a finite time for perturbed unicycle mobile robots. The proposed control approach possesses the following features: a) The designed control law guarantees global, uniform and finite-time convergence of the tracking error to zero for any desired trajectory satisfying the non-holonomic constraint of the unicycle mobile robot; b) The considered external perturbations are completely compensated; c) The controller parameters selection is simple and this facilitates its experimental implementability. Some experimental results highlight the feasibility and performance of the proposed second-order sliding-mode controller.

The rest of the paper is organized as follows. The problem statement is given in Section III. The proposed controller is presented in Section IV. Some experimental results, using a QBot2, are presented in Section V. Concluding remarks are provided in Section VI. Finally, the proof of the main result is postponed to the Appendix.

**Notation:** Denote the trigonometric functions  $\sin(\theta)$ ,  $\cos(\theta)$ ,  $\arcsin(\theta)$  and  $\arctan(\theta)$  as  $s(\theta) = \sin(\theta)$ ,  $c(\theta) = \cos(\theta)$ ,  $\arcs(\theta) = \arcsin(\theta)$  and  $\arct(\theta) = \arctan(\theta)$ , respectively. Define the function  $[s]^\gamma = |s|^\gamma \text{sign}(s)$ , for  $\gamma \in \mathbb{R}_{\geq 0}$  and any  $s \in \mathbb{R}$ . The set of real numbers is defined by  $\mathbb{R}$ , and then,  $\mathbb{R}_{\geq 0} = \{s \in \mathbb{R} : s \geq 0\}$ .

## II. PRELIMINARIES

Consider the system

$$\dot{x} = f(t, x), \quad t \in \mathbb{R}_{\geq 0}, \quad x(0) = x_0, \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state vector. The function  $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is assumed to be locally bounded uniformly in  $t$ . For  $f$  locally measurable but discontinuous with respect to  $x$ , the solutions are understood in the sense of Filippov [21]. That is,  $x(t, x_0)$  is a solution to (1) if it is absolutely continuous, and if it satisfies almost everywhere the differential inclusion

$$\dot{x} \in K[f](t, x) = \overline{\text{co}} \bigcap_{\varepsilon > 0} \bigcap_{\mu N = 0} f(t, B(x, \varepsilon) \setminus N),$$

where  $K[f](t, x)$  is an upper semi-continuous, nonempty, compact and convex valued map,  $\overline{\text{co}}$  represents the convex closure of a set,  $B(x, \varepsilon) = \{v \in \mathbb{R}^n : \|x - v\| < \varepsilon\}$  and  $\mu$  is the Lebesgue measure. Note that the intersections are taken over all the sets  $N$  of Lebesgue measure zero, over all  $\varepsilon > 0$ .

**Definition 1.** [22], [23]. *Let the origin be an equilibrium of (1). Then, the system (1) is Globally Uniformly Finite-Time Stable (FTS) if: a) there exists a function  $\alpha \in \kappa$  such that  $\|x(t, x_0)\| \leq \alpha(\|x_0\|)$ , for all  $t \geq 0$ , for any  $x_0 \in \mathbb{R}^n$ ; and b) there exists a locally bounded function  $T : \mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}_{\geq 0}$  such that  $x(t, x_0) = 0$ , for all  $t \geq T(x_0)$  and any  $x_0 \in \mathbb{R}^n$ .*

The function  $T$  estimates a settling time of the system.

## III. PROBLEM STATEMENT

Consider the perturbed kinematic model of an UMR (see, Fig. 1):

$$\dot{\theta} = (1 + d_1(t))\omega, \quad (2a)$$

$$\dot{x} = (1 + d_2(t))c(\theta)v, \quad (2b)$$

$$\dot{y} = (1 + d_2(t))s(\theta)v, \quad (2c)$$

where  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$  denote the midpoint between the wheels and  $\theta \in \mathbb{R}$  represents the orientation angle of the UMR. The terms  $v$  and  $\omega$  contain the linear and angular velocities of the UMR, and represent the control inputs. The terms  $d_1$  and  $d_2$  represent some time-varying perturbations, which are multiplicative to the inputs and that may come from the settling time of the internal controller that translates the velocity commands in current/voltage inputs and sends them to the motors [24] or non-modeled kinematics phenomena proportional to the control inputs, such as slipping of the wheels [5]. It is assumed that such time-varying perturbations  $d_i(t)$  are unknown but uniformly bounded, *i.e.*,  $-1 < d_i(t) \leq d_{\max} < 1$ , for  $i = 1, 2$ , with a known positive constant  $d_{\max}$ . Note that the constraint  $d_i(t) > -1$  ensures that the perturbations do not cause a change of sign in the control inputs.

The aim of this work is to design a trajectory tracking control for the UMR able to compensate some multiplicative perturbations and reach the desired trajectory in a finite time.

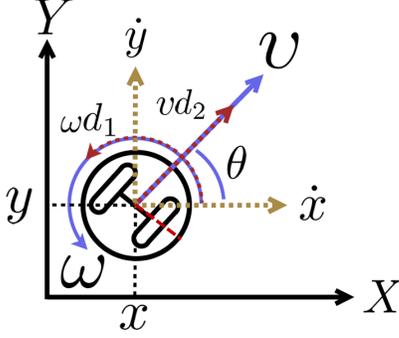


Figure 1. Schematic Diagram of the Perturbed UMR

#### IV. FINITE-TIME CONTROLLER DESIGN

Let us define the tracking errors as follows

$$e_1 = \theta_d - \theta, \quad (3a)$$

$$e_2 = c(\theta)(x_d - x) + s(\theta)(y_d - y), \quad (3b)$$

$$e_3 = c(\theta)(y_d - y) - s(\theta)(x_d - x), \quad (3c)$$

where  $x_d$ ,  $y_d$  and  $\theta_d$  come from a reference kinematic model for the UMR, *i.e.*,

$$\dot{\theta}_d = \omega_d, \quad (4a)$$

$$\dot{x}_d = c(\theta_d)v_d, \quad (4b)$$

$$\dot{y}_d = s(\theta_d)v_d, \quad (4c)$$

where  $v_d$  and  $\omega_d$  are the linear and angular reference velocities, respectively. These are assumed continuous and bounded by some positive constants  $\underline{v}_d$ ,  $\bar{v}_d$  and  $\bar{\omega}_d$ , *i.e.*,  $0 < \underline{v}_d < v_d(t) \leq \bar{v}_d$ , and  $|\omega_d(t)| \leq \bar{\omega}_d$ .

Therefore, the tracking error dynamics can be calculated as

$$\dot{e}_1 = -\omega d_1(t) + \tau_1, \quad (5a)$$

$$\dot{e}_2 = (1 + d_1(t))\omega e_3 - v d_2(t) + \tau_2, \quad (5b)$$

$$\dot{e}_3 = -(1 + d_1(t))\omega e_2 + v d s(e_1), \quad (5c)$$

with the virtual control inputs  $\tau_1$  and  $\tau_2$  satisfying

$$\tau_1 = \omega_d - \omega, \quad (6a)$$

$$\tau_2 = v_d c(e_1) - v. \quad (6b)$$

Let us propose the following new sliding variables

$$s_1 = e_1 + \left[ [e_1]^2 + k_3 \arcs \left( \frac{\sqrt{2}}{\pi} \arct(e_3) \right) \right]^{\frac{1}{2}}, \quad (7a)$$

$$s_2 = e_2, \quad (7b)$$

with a positive design controller parameter  $k_3 > 0$ . Then, the virtual inputs  $\tau_1$  and  $\tau_2$  are designed as

$$\tau_1 = -(v_d(t)|e_3| + k_1(t)) [s_1]^0, \quad (8a)$$

$$\tau_2 = -(\alpha|\omega e_3| + k_2(t)) [s_2]^0, \quad (8b)$$

where  $\alpha > 0$ ,  $k_1 > 0$  and  $k_2 > 0$  represent the rest of design controller parameters, which will be designed further on.

*Remark 1.* The purpose of the structure given in (7) and (8) is to enforce a sliding-mode on the surface  $s_1 = 0$ , and then, a special relation between  $e_1$  and  $e_3$  is ensured. The restricted dynamics, on the sliding surface  $s_1 = 0$ , of  $e_1$  with respect to  $e_3$ , has a virtual control input through the term  $s(e_1)$ , which helps to steer  $e_3$  to zero in a finite time.

Therefore, taking into account the design of the virtual inputs (8) and the expressions given in (6), the real control inputs  $\omega$  and  $v$  are designed as follows

$$\omega = \omega_d(t) + (v_d(t)|e_3| + k_1(t)) [s_1]^0, \quad (9a)$$

$$v = v_d(t)c(e_1) + (\alpha|\omega e_3| + k_2(t)) [s_2]^0. \quad (9b)$$

The following theorem provides the main result of this work.

**Theorem 1.** Let the control law (9), with the sliding variables (7), be applied to the perturbed kinematic model of the UMR (2), with  $d_i(t) \leq d_{\max} < 1$ , for  $i = 1, 2$ . If the controller parameters are designed as

$$\alpha = \frac{1 + d_{\max}}{1 - d_{\max}}, \quad (10)$$

$$k_1(t) = \frac{k_3 v_d(t) + \bar{\omega}_d \pi d_{\max}}{\pi(1 - d_{\max})}, \quad (11)$$

$$k_2(t) = \frac{v_d(t)d_{\max} + \gamma}{1 - d_{\max}}, \quad \gamma > 0, \quad (12)$$

$$k_3 \in \left( 0, \frac{\pi}{4} \right), \quad (13)$$

then, for a given desired trajectory satisfying (4), the tracking error system (5) is Globally Uniformly FTS.

The proof is postponed to the Appendix. Note that the FT stability of system (5) implies, by the bijectivity of the transformation (3), that  $x(t) = x_d(t)$ ,  $y(t) = y_d(t)$  and  $\theta(t) = \theta_d(t)$ , for all  $t \geq T > 0$ . Moreover, the controller parameters  $\gamma$  and  $k_3$  are proportional to the size of  $k_1$  and  $k_2$ , and then, to the convergence rate of the tracking error  $e_2$ , and the sliding surface  $s_1$  and the tracking error  $e_3$ , respectively (please, see the Appendix). However, it is also clear that there is a trade-off between the convergence speed and the control effort that must be taken into account in the selection of  $\gamma$  and  $k_3$ .

Note that it is possible to implement this controller considering saturation on the actuators, limiting the maximal admissible perturbation with respect to the maximum linear and angular velocities of the UMR, *i.e.*, with respect to saturation constraints. Let us consider that  $|\omega(t)| \leq \omega_{\max}$  and  $|v(t)| \leq v_{\max}$ , for some  $\omega_{\max}, v_{\max} > 0$ . Then, some additional constraints, over the controller gains and the linear and angular reference velocities, can be found to provide an admissible perturbation level  $d_{\max}$ , taking into account saturation constraints.

It should also be noticed that the proposed controller (9) is discontinuous and this could imply an implementation issue for slow actuator dynamics. However, it is possible to implement it by some continuous approximations of the sign function. One well-known function that can be used for this objective is  $\lceil s \rceil^0 \approx s/(|s| + \delta)$ , where the tuning parameter  $\delta > 0$  is a small constant.

A block diagram describing the proposed control design is given in Fig. 2.

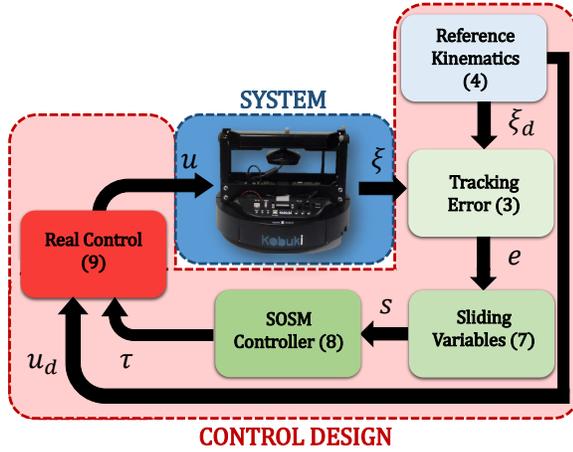


Figure 2. Diagram of the Control Design. Notation:  $\xi = (\theta, x, y)^\top$ ,  $u = (\omega, v)^\top$ ,  $\xi_d = (\theta_d, x_d, y_d)^\top$ ,  $u_d = (\omega_d, v_d)^\top$ ,  $e = (e_1, e_2, e_3)^\top$ ,  $s = (s_1, s_2)^\top$  and  $\tau = (\tau_1, \tau_2)^\top$ .

## V. EXPERIMENTAL RESULTS

The experimental results are obtained using the QBot2 platform by Quanser (see Fig. 3). The QBot2 possesses a processing embedded system, which communicates through a real-time control software called QUARC with a sampling time equal to 1[ms]. Such software allows us to build a direct interface with Matlab-Simulink and build different algorithms and controllers. The QBot2 posture and orientation are obtained through odometry, *i.e.*, the wheel spinning is measured by the robot encoders, and through the kinematic model of the UMR, we can compute the total displacement and orientation angle. Note that the linear and angular velocities are the control inputs.



Figure 3. QBot2 by Quanser

The initial conditions for the kinematics are  $x_0 = 0.5$ [m],  $y_0 = 0$ [m] and  $\theta_0 = 0$ [rad]. The desired trajectory is given by  $\omega_d(t) = (\dot{x}_d \dot{y}_d - \dot{y}_d \dot{x}_d) / (\dot{x}_d^2 + \dot{y}_d^2)$ ,  $v_d(t) = \sqrt{\dot{x}_d^2 + \dot{y}_d^2}$ ,  $x_d(t) = c(0.13t)$ ,  $y_d(t) = s(0.26t)$ ,  $\theta_d(t) = \int_0^t \omega_d(\tau) d\tau$ , and thus,  $\bar{\omega}_d = 0.7580$ . For robustness purposes, additional to the intrinsic disturbances that the experimental platform possesses (*e.g.*, the effect of the difference between the angular velocity of the wheel and the linear input velocity  $v$ , caused by the wheel slipping on the surface; or the rate deviation of change of the orientation angle  $\theta$  due to the wheel slippage), some external signals, added by software, are considered. The external perturbations are taken as  $d_1(t) = 0.01s(t) + 0.01$  and  $d_2(t) = 0.03c(t) + 0.03$ , and hence,  $d_{\max} = 0.06$ . Then, according to Theorem 1, the parameters of the proposed second-order sliding-mode (SOSM) controller are selected as  $\alpha = 1.1277$ ,  $\gamma = 0.001$ ,  $k_3 = 0.7069$  and

$$k_1(t) = \frac{0.7069v_d(t) + 0.1429}{2.9531},$$

$$k_2(t) = \frac{0.06v_d(t) + 0.001}{0.94}.$$

In order to illustrate the performance of the proposed controller, we compare the proposed controller with the robust first-order sliding-mode (FOSM) controller given in [14], *i.e.*,

$$s_1 = e_1 + \arcs(\min\{\delta_1|e_3|^{-1}, \delta_2\}e_3), \quad (14a)$$

$$s_2 = e_2, \quad (14b)$$

with positive design controller parameters  $\delta_1 \in (0, 1)$  and  $\delta_2 > 0$ , and the virtual inputs  $\tau_1$  and  $\tau_2$  as

$$\tau_1 = -k_1 \lceil s_1 \rceil^0, \quad (15a)$$

$$\tau_2 = -k_2 \lceil s_2 \rceil^0, \quad (15b)$$

where  $k_1 > 0$  and  $k_2 > 0$  represent the rest of design controller parameters, which are designed according to [14]. Then, the parameters of the FOSM controller are selected as  $\delta_1 = 0.3$ ,  $\delta_2 = 6$ ,  $k_1 = 0.04$  and  $k_2 = 0.45$ , which guarantee the asymptotic convergence to zero of the tracking error. It is worth mentioning that both controllers are designed taken into account the same time-varying perturbations  $d_1$  and  $d_2$ . Moreover, both controllers are able to deal with the same class of perturbations.

The results are shown in Figs. 4 and 5. We can see that the FOSM controller provides a slightly faster convergence rate than the SOSM controller. This is mainly due to the fact that the time-varying gains of the SOSM are somehow adapted to the reference linear velocity  $v_d(t)$ , which does not necessarily improve the rate of convergence. However, both controllers provide a similar trajectory tracking performance using similar control efforts. Both controllers are able to properly track the desired trajectory despite the intrinsic disturbances and the considered external perturbations. Additionally, it is worth highlighting that the proposed SOSM controller only requires for 2 parameters tuning, *i.e.*,  $\gamma$  and  $k_3$ , and thus, its

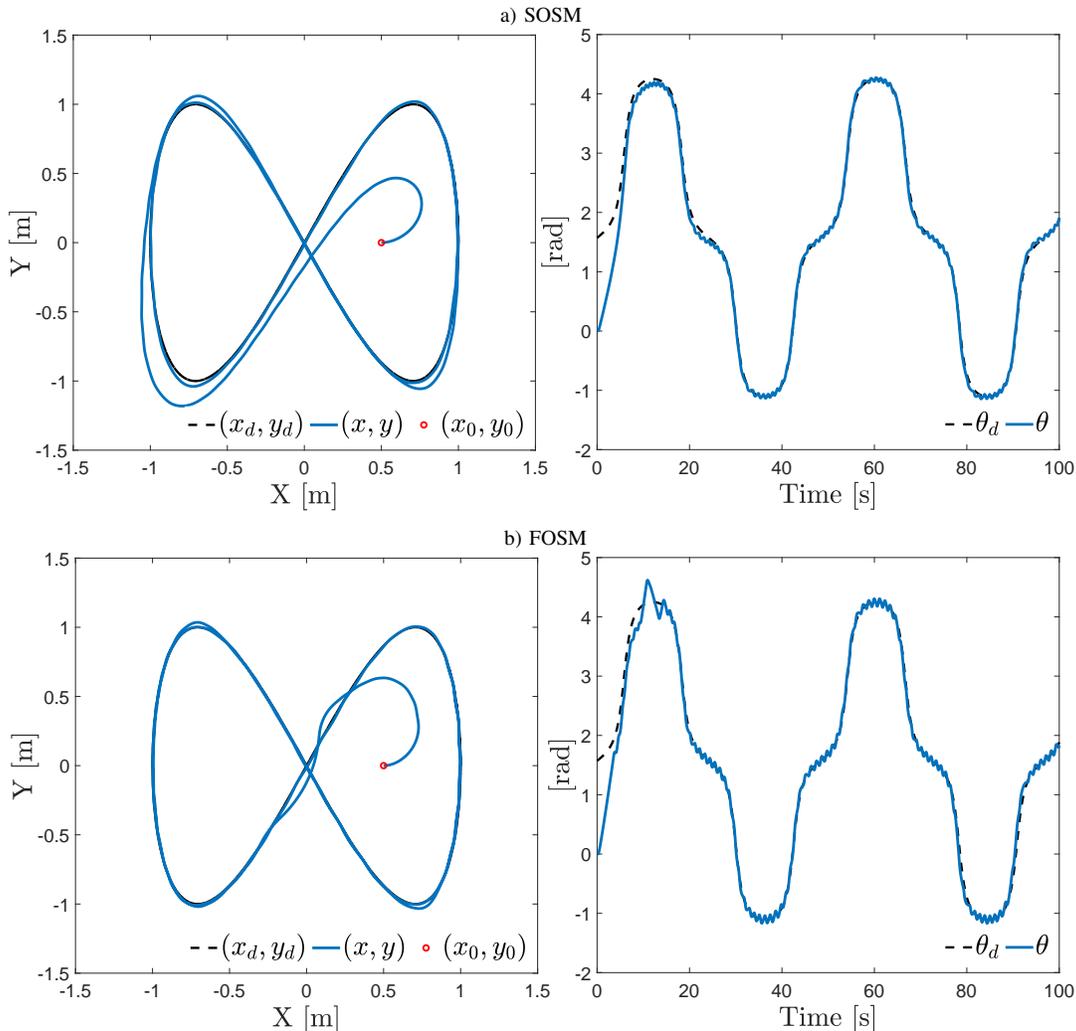


Figure 4. Experimental Results for System Trajectories

synthesis is simpler than the above FOSM controller, which needs for 4 parameters tuning, *i.e.*,  $\delta_1$ ,  $\delta_2$ ,  $k_1$  and  $k_2$ .

In order to better analyze the performance of the controllers, in terms of the tracking error and the control effort, we provide the following performance indexes  $e_{rms}(t) = \sqrt{\frac{1}{T} \int_{t-T}^t \|\bar{e}(\tau)\|^2 d\tau}$  and  $u_{rms}(t) = \sqrt{\frac{1}{T} \int_{t-T}^t \|u(\tau)\|^2 d\tau}$ , with  $\bar{e} = (x - x_d, y - y_d, \theta - \theta_d)^\top$ ,  $u = (\omega, v)^\top$  and  $T = 0.1$ , which provide measures of the deviation of the signals  $\bar{e}(t)$  and  $u(t)$  from zero and its power. The behavior of the performance indexes is illustrated in Fig. 6 and some properties of such indexes are illustrated in Table I. The first and second columns provide the minimum and maximum values of the corresponding performance indexes while the third column provides its mean value, respectively. Based on these results, we can confirm that the trajectory tracking performance is practically the same but the control effort required by the proposed SOSM controller is slightly less than the FOSM controller.

Table I  
PERFORMANCE INDEXES  $e_{rms}$  AND  $u_{rms}$

$e_{rms}$	min	max	mean
FOSM	0.1679	1.5961	0.2374
SOSM	0.1658	1.6145	0.2564
$u_{rms}$	min	max	mean
FOSM	0.1992	1.0905	0.5649
<b>SOSM</b>	<b>0.1324</b>	<b>0.9229</b>	<b>0.3928</b>

## VI. CONCLUSIONS

This paper contributes to the design of a second-order sliding-mode controller for the trajectory tracking problem in perturbed unicycle mobile robots. The proposed strategy takes into account the design of two particular sliding variables, which ensure the convergence of the tracking error to the origin in a finite time despite the effect of some external perturbations. The straightforward structure of the controller is simple to tune and to implement. The global, uniform and finite-time stability of the closed-loop tracking error

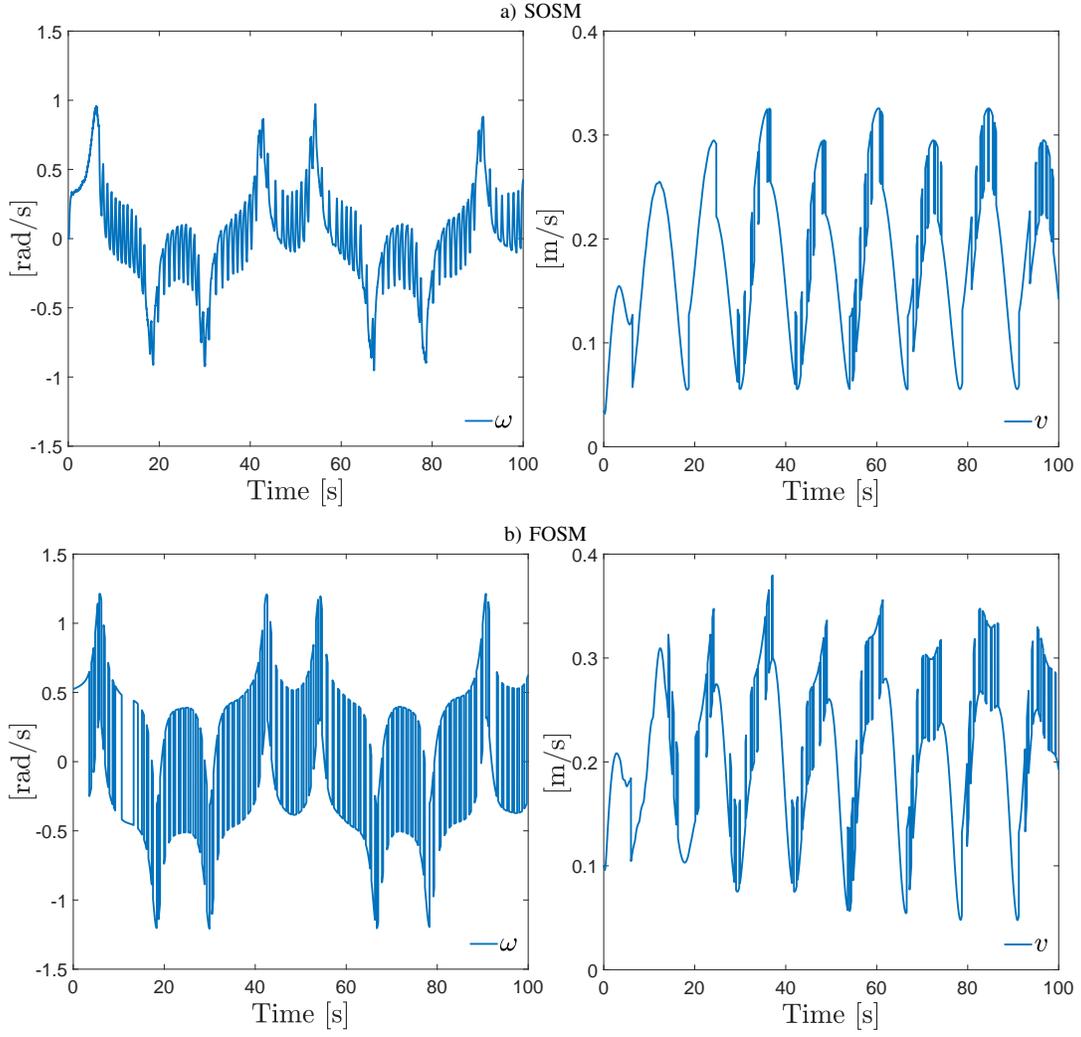


Figure 5. Experimental Results for Control Signals

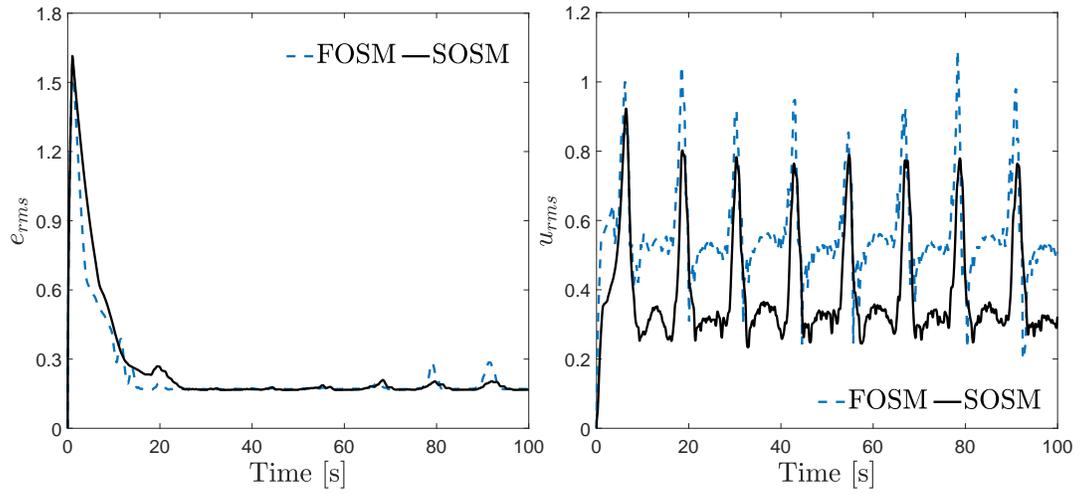


Figure 6. Performance Indexes

dynamics is demonstrated by means of Lyapunov functions. Furthermore, the performance of the proposed approach is validated through some experiments using a QBot2 unicycle mobile robot.

#### ACKNOWLEDGMENT

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#### APPENDIX

##### A. Proof of Theorem 1

The proof is split into three stages. The first one is to prove that  $s_2 = 0$  is FTS. Then, we will prove that  $s_1 = 0$  is also FTS, and finally, it will be proven that the trajectories of the tracking error, within the sliding surface  $s_1 = 0$ , are attracted to the origin in a finite time.

**Convergence Analysis for  $s_2 = 0$ :** According to (5b), (7b) and (8b), the closed-loop dynamics of  $s_2$  is given by

$$\dot{s}_2 = (1 + d_1(t))\omega e_3 - v d_2(t) - (\alpha|\omega e_3| + k_2(t))|s_2|^0. \quad (16)$$

Let us consider the candidate Lyapunov function  $V_2 = s_2^2/2$ . Thus, the time derivative of  $V_2$ , along the trajectories of the system (16), satisfies

$$\dot{V}_2 \leq [(1 + d_{\max})|\omega e_3| + v d_{\max} - (\alpha|\omega e_3| + k_2(t))]|s_2|,$$

and, since  $|v| \leq v_d(t) + \alpha|\omega e_3| + k_2(t)$ , it follows that

$$\dot{V}_2 \leq [(1 + d_{\max})|\omega e_3| + (v_d(t) + \alpha|\omega e_3| + k_2(t))d_{\max} - (\alpha|\omega e_3| + k_2(t))]|s_2|.$$

Then, selecting  $\alpha$  and  $k_2$  as in (10) and (12), respectively, i.e.,  $\alpha = (1 + d_{\max})/(1 - d_{\max})$ ,  $k_2(t) = (v_d(t)d_{\max} + \gamma)/(1 - d_{\max})$ ,  $\gamma > 0$ , and since  $d_{\max} < 1$ , the time derivative of  $V_2$  satisfies

$$\dot{V}_2 \leq -\gamma|s_2| \leq -\sqrt{2}\gamma V_2^{\frac{1}{2}},$$

which implies that  $s_2(t) = e_2(t) = 0$ , for all  $t \geq T_1$ , with the time  $T_1$  upper bounded as  $T_1 \leq \sqrt{2}\gamma^{-1}V_2^{\frac{1}{2}}(e_2(0))$ .

**Convergence Analysis for  $s_1 = 0$ :** Recall that  $s_1$  is given as

$$s_1 = e_1 + \left[ |e_1|^2 + k_3 \arcs\left(\frac{\sqrt{2}}{\pi} \arct(e_3)\right) \right]^{\frac{1}{2}}.$$

Thus, in order to prove that  $s_1 = 0$  is FTS, let us consider the non-smooth candidate Lyapunov function  $V_1 = s_1^2/2$

[25]. Hence, the time derivative of  $V_1$ , along the trajectories of system (5a) and (5c), is given by

$$\begin{aligned} \dot{V}_1 = & s_1 [-\omega d_1(t) + \tau_1 \\ & + \frac{1}{2} \left| |e_1|^2 + k_3 \arcs\left(\frac{\sqrt{2}}{\pi} \arct(e_3)\right) \right|^{-\frac{1}{2}} \\ & \frac{d}{dt} \left[ |e_1|^2 + k_3 \arcs\left(\frac{\sqrt{2}}{\pi} \arct(e_3)\right) \right] \Big]. \end{aligned}$$

Note that

$$\begin{aligned} \frac{d}{dt} \left[ |e_1|^2 + k_3 \arcs\left(\frac{\sqrt{2}}{\pi} \arct(e_3)\right) \right] = & 2|e_1|\dot{e}_1 \\ & + k_3 \frac{\sqrt{2}}{\pi} \frac{\dot{e}_3}{(1 + e_3^2)\sqrt{1 - \frac{2}{\pi^2} \arct^2(e_3)}}. \end{aligned}$$

According to (5a), (5c), (7a) and (8a), the time derivative of  $V_1$ , for  $\phi_2 \neq 0$ , satisfies

$$\begin{aligned} \dot{V}_1 = & s_1 \left[ -\omega d_1(t) - (v_d(t)|e_3| + k_1(t))|s_1|^0 \right. \\ & \left. - \frac{1}{2} \left( \frac{\phi_3(t, e_1, e_3)}{\phi_2(e_1, e_3)} - \frac{\phi_4(t, e_1, e_2)}{\phi_1(e_3)\phi_2(e_1, e_3)} \right) |s_1|^0 \right], \end{aligned}$$

where

$$\phi_1(e_3) = (1 + e_3^2)\sqrt{1 - \frac{2}{\pi^2} \arct^2(e_3)}^{\frac{1}{2}},$$

$$\phi_2(e_1, e_3) = \left| |e_1|^2 + k_3 \arcs\left(\frac{\sqrt{2}}{\pi} \arct(e_3)\right) \right|^{\frac{1}{2}},$$

$$\phi_3(t, e_1, e_3) = 2|e_1| \left[ \omega d_1(t)|s_1|^0 + v_d(t)|e_3| + k_1(t) \right],$$

$$\phi_4(t, e_1, e_2) = k_3 \frac{\sqrt{2}}{\pi} \left[ -(1 + d_1(t))\omega e_2 + v_d s(e_1) \right] |s_1|^0.$$

Before proceeding with the convergence analysis, we will show that

$$\frac{\phi_3(t, e_1, e_3)\phi_1(e_3) - \phi_4(t, e_1, e_2)}{\phi_1(e_3)\phi_2(e_1, e_3)} \geq 0,$$

for all  $t \geq T_1$  and all  $(e_1, e_3) : \phi_2(e_1, e_3) \neq 0$ . Note that

$$\begin{aligned} \frac{1}{\sqrt{2}} \leq \sqrt{1 - \frac{2}{\pi^2} \arct^2(e_3)} \leq 1, \quad \forall e_3 \in \mathbb{R}, \\ 1 \leq (1 + e_3^2), \quad \forall e_3 \in \mathbb{R}. \end{aligned}$$

Therefore,  $\phi_1(e_3) \geq 1/\sqrt{2}$ , for all  $e_3 \in \mathbb{R}$ , and by definition, also  $\phi_2(e_1, e_3) \geq 0$ . Then, due to  $|\omega| \leq \bar{\omega}_d + v_d(t)|e_3| + k_1(t)$ , it follows that  $\phi_3$  satisfies the following lower bound

$$\begin{aligned} \phi_3(t, e_1, e_3) \geq & 2|e_1| \left[ -(\bar{\omega}_d + v_d(t)|e_3| + k_1(t))d_{\max} \right. \\ & \left. + v_d(t)|e_3| + k_1(t) \right]. \quad (17) \end{aligned}$$

Moreover, since  $s_2(t) = e_2(t) = 0$ , for all  $t \geq T_1$ ,  $\phi_4$  satisfies the following upper bound

$$\phi_4(t, e_1, 0) \leq k_3 \frac{\sqrt{2}}{\pi} v_d(t) |e_1|, \quad (18)$$

for all  $t \geq T_1$ . Additionally, due to the fact that  $e_2$  and  $\omega$  are bounded for all  $t < T_1$ ,  $\phi_4(t, e_1, e_2)$  is bounded for all  $t \geq 0$ . Therefore, selecting  $k_1$  as in (11), i.e.,  $k_1(t) = (k_3 v_d(t) + \bar{\omega}_d \pi d_{\max}) / [\pi(1 - d_{\max})]$ , recalling that  $\phi_1(e_3) \geq 1/\sqrt{2}$  and  $d_{\max} < 1$ , and taking into account (17) and (18), one obtains that

$$\frac{\phi_3(t, e_1, e_3) - \sqrt{2}\phi_4(t, e_1, 0)}{\phi_2(e_1, e_3)} \geq 0,$$

for all  $t \geq T_1$  and all  $(e_1, e_3) : \phi_2(e_1, e_3) \neq 0$ . Hence, the time derivative of  $V_1$  satisfies

$$\dot{V}_1 \leq [|\omega|d_{\max} - v_d(t)|e_3| - k_1(t)] |s_1|,$$

for all  $t \geq T_1$  and all  $(e_1, e_3) : \phi_2(e_1, e_3) \neq 0$ . Then, due to the fact that  $|\omega| \leq \bar{\omega}_d + v_d(t)|e_3| + k_1(t)$ , it follows that

$$\dot{V}_1 \leq [(\bar{\omega}_d + v_d(t)|e_3| + k_1(t))d_{\max} - v_d(t)|e_3| - k_1(t)] |s_1|.$$

Note that

$$k_1(t)(1 - d_{\max}) = \frac{k_3 v_d(t) + \bar{\omega}_d \pi d_{\max}}{\pi},$$

then, since  $d_{\max} < 1$  and  $0 < \underline{v}_d < v_d(t)$ , the time derivative of  $V_1$  is upper bounded as

$$\dot{V}_1 \leq -\frac{k_3 \underline{v}_d}{\pi} |s_1| \leq -\frac{\sqrt{2} k_3 \underline{v}_d}{\pi} V_1^{\frac{1}{2}}.$$

Since  $\dot{V}_1 \rightarrow -\infty$  as  $\phi_2(e_1, e_3) \rightarrow 0$ , by [25, Corollary 1], the set  $\{t > 0 : \dot{V}_1(e_1(t), e_3(t)) = \{-\infty\}\}$  has measure zero. Using [25, Theorem 12], we derive  $s_1(t) = 0$ , for all  $t \geq T_2 > T_1$ , with the time  $T_2$  upper bounded as  $T_2 \leq \sqrt{2\pi}(k_3 \underline{v}_d)^{-1} V_1^{\frac{1}{2}}(s_1(T_1)) + T_1$ .

**Terminal Attractor within  $s_1 = 0$ :** As soon as a sliding mode appears on the surface  $s_1 = 0$ , according to (7a), it holds that

$$s \left( \frac{2}{k_3} [e_1]^2 \right) = -\frac{\sqrt{2}}{\pi} \arctan(e_3),$$

and considering that  $s(2\theta) = 2s(\theta)c(\theta)$ , one obtains

$$2s \left( \frac{[e_1]^2}{k_3} \right) c \left( \frac{[e_1]^2}{k_3} \right) = -\frac{\sqrt{2}}{\pi} \arctan(e_3).$$

On the other hand, based on (7a), it is given that  $[s_1]^0 = [e_1]^0$ , for  $|e_1|^2/k_3 > \pi/4$ , and hence; the closed-loop dynamics of  $e_1$  satisfies

$$\dot{e}_1 = -\omega d_1(t) - (v_d(t)|e_3| + k_1(t)) [e_1]^0,$$

which implies, due to  $k_1$  is selected as in (11), that the trajectory of  $e_1$  holds

$$|e_1(t)| \leq \frac{\sqrt{\pi k_3}}{2}, \quad \forall t \geq T_3 > T_2,$$

and thus

$$\frac{1}{\sqrt{2}} \leq c \left( \frac{[e_1]^2}{k_3} \right) \leq 1.$$

Let us define

$$\phi(e_1) = \frac{s \left( \frac{[e_1]^2}{k_3} \right) |e_1|^2}{\left( \frac{[e_1]^2}{k_3} \right) s^2(e_1)}.$$

Therefore, if  $k_3 \in (0, \pi/4]$ , then  $|e_1| \leq \pi/4$ , and thus

$$s \left( \frac{[e_1]^2}{k_3} \right) = \frac{\phi(e_1)}{k_3} [s(e_1)]^2 = -\frac{\arctan(e_3)}{\sqrt{2\pi} c \left( \frac{[e_1]^2}{k_3} \right)},$$

is satisfied for all  $|e_1| \leq \pi/4$ . Moreover, note that if both  $|e_1|^2/k_3 \leq \pi/4$  and  $|e_1| \leq \pi/4$  hold; then  $2\sqrt{2}/\pi \leq \phi(e_1) \leq \pi^2/8$ , and hence, on the sliding surface  $s_1 = 0$ , it holds that

$$s(e_1) = -\sqrt{k_3} \left[ \frac{\arctan(e_3)}{\sqrt{2\pi} c \left( \frac{[e_1]^2}{k_3} \right) \phi(e_1)} \right]^{\frac{1}{2}}.$$

Therefore, the closed-loop dynamics of  $e_3$ , on the sliding surface  $s_1 = 0$ , satisfies

$$\dot{e}_3 = -\sqrt{k_3} v_d(t) \left[ \frac{\arctan(e_3)}{\sqrt{2\pi} c \left( \frac{[e_1]^2}{k_3} \right) \phi(e_1)} \right]^{\frac{1}{2}},$$

for all  $t \geq T_3$ . Moreover, it is possible to rewrite the dynamics of  $e_3$  as follows

$$\dot{e}_3 = -\sqrt{k_3} v_d(t) \bar{\phi}(e_1, e_3) [e_3]^{\frac{1}{2}}, \quad (19)$$

where

$$\bar{\phi}(e_1, e_3) = \left[ \frac{\arctan(e_3)}{\sqrt{2\pi} c \left( \frac{[e_1]^2}{k_3} \right) \phi(e_1) e_3} \right]^{\frac{1}{2}} \geq \phi^+ > 0.$$

Thus, let us consider the candidate Lyapunov function  $V_3 = e_3^2/2$ . Hence, the time derivative of  $V_3$ , along the trajectories of the system (19), satisfies

$$\dot{V}_3 \leq -2^{\frac{3}{4}} \sqrt{k_3} \underline{v}_d \bar{\phi}(e_1, e_3) V_3^{\frac{3}{4}},$$

which implies that  $e_3(t) = 0$ , for all  $t \geq T_4 > T_3$ , and thus,  $e_1 = 0$  is also Globally Uniformly FTS. Therefore, based on all the previous analysis, it is proven that  $(e_1, e_2, e_3)^T = 0$  is Globally Uniformly FTS. This concludes the proof. ■

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