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► **To cite this version:**

Andrey Polyakov, Denis Efimov, Xubin Ping. Consistent discretization of homogeneous finite/fixed-time controllers for LTI systems. *Automatica*, 2023, 155, pp.111118. hal-04071575

**HAL Id: hal-04071575**

**<https://hal.science/hal-04071575>**

Submitted on 17 Apr 2023

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# Consistent discretization of homogeneous finite/fixed-time controllers for LTI systems

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## Abstract

The paper proposes a discretization (sampled-time implementation) algorithm for a class of homogeneous controllers for linear time-invariant (LTI) systems preserving the finite-time and nearly fixed-time stability properties. The sampling period is assumed to be constant. Both single-input and multiple-input cases are considered. The robustness (Input-to-State Stability) of the obtained sampled-time control system is studied as well. Theoretical results are supported by numerical simulations.

*Key words:* nonlinear control; sampled-time control; homogeneous systems.

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## 1 Introduction

By definition, the homogeneity is a dilation symmetry introduced by Leonhard Euler in 18th century as follows:  $f(\lambda x) = \lambda^\nu f(x), \forall \lambda > 0$ , where the coordinate transformation  $x \mapsto \lambda x$  is known today as a standard (or Euler) dilation. A weighted (generalized) dilation is studied since 1950s. An introduction to stability theory of weighted homogeneous Ordinary Differential Equations (ODEs) can be found in [60]. Extensions of the homogeneity theory to various finite-dimensional and infinite-dimensional dynamical models are proposed in [25], [24], [12], [45]. Homogeneous differential equations/inclusions form an important class of control system models [51], [43], [3], [48]. They appear as local approximations [16] or set-valued extensions [29] of nonlinear systems and include models of process control [57], mechanical systems with frictions [42], fluid dynamics [45], etc. Stability and stabilization problems were studied for both standard [59], [2] and weighted homogeneous [9], [17], [50], [53], [13], [38] systems which are the most popular today [42], [29], [43], [3], [48], [34]. A homogeneous model predictive control is introduced in [8].

An asymptotically stable homogeneous system is finite-time stable in the case of negative homogeneity degree and nearly fixed-time stable in the case of the positive

homogeneity degree (see, e.g. [37], [4], [3]). However, the finite/fixed-time stability is a fragile property, since an arbitrary small measurement delay or an improper discretization of a finite-time or a fixed-time stable ODE may result in a chattering [1], [30] or even in a finite-time blow up [31]. Existing approaches to sampled-time implementation of nonlinear controllers (see, e.g. [41], [40], [18]) do not study these issues. The explicit discretization (sampled-time implementation) of a finite-time control yields a chattering even if the original control law is a continuous function of state [11], [23]. That is why the discretization issues are very important for practical implementation of finite/fixed-time control/estimation algorithms [1], [26], [32], [27], [21], [5], [35], [15].

The concept of consistent discretization introduced in [46] postulates that stability properties of a continuous-time system must be preserved in its discrete-time counterpart (approximation). Consistent discretizations for stable generalized homogeneous ODEs were developed in [46], [52] based on Lyapunov function theory. Some schemes with state dependent discretization step were given in [10]. Being efficient for numerical simulations, the mentioned schemes do not allow a consistent discretization (sampled-time implementation) of *finite-time controllers* in the general case. To the best of authors' knowledge, such implementations are developed only for some particular first order (see, [1], [20]) and some particular second order systems (see, [21], [6]). This paper presents a consistent discretization

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\* This work was supported by the National Natural Science Foundation of China under Grant 62050410352.

for a homogeneous controller designed in [47], [45], [58] for multidimensional linear plants. It is shown that the sampled-time implementation of the controller according to the developed scheme preserves the finite-time and nearly fixed-time stability property of the original closed-loop continuous-time system in the disturbance-free case. We also prove an Input-to-State Stability (ISS) of the obtained sampled-time control system with respect to bounded additive perturbations and measurement noises. Algorithms are developed for both single-input and multi-input models. Numerical simulations show an efficiency of this scheme and a complete rejection of the so-called *numerical chattering* [1] caused by a sampled-time implementation of a continuous-time control algorithm. The preliminary version of this paper [49] considers only the case of a finite-time stabilizer for a planar (two-dimensional) system.

*Notation:*  $\mathbb{N}$  is the set of natural numbers including 0;  $\mathbb{R}$  is the field of real numbers;  $\mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha \geq 0\}$ ;  $\mathbf{0}$  is the zero of a vector space;  $I_n \in \mathbb{R}^{n \times n}$  is the identity matrix;  $e_i = (0, \dots, 0, 1, 0, \dots, 0)^\top \in \mathbb{R}^n$  is the  $i$ -th element of the Euclidean basis;  $W \succ 0$  means positive definiteness of a matrix  $W = W^\top \in \mathbb{R}^{n \times n}$  and  $\lambda_{\max}(W)$  is a maximal eigenvalue of  $W$ ;  $\|x\| = \sqrt{x^\top P x}$  is the weighted Euclidean norm in  $\mathbb{R}^n$  with  $P \succ 0$  specified below in each case when  $P$  is not arbitrary;  $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$  for  $A \in \mathbb{R}^{n \times n}$ ;  $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$ ;  $\mathcal{K}$  is the class of strictly increasing positive definite continuous functions  $\mathbb{R}_+ \mapsto \mathbb{R}_+$ ;  $\gamma \in \mathcal{K}$  is of the class  $\mathcal{K}_\infty$  if  $\gamma(s) \rightarrow +\infty$  as  $s \rightarrow +\infty$ ;  $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$  is of the class  $\mathcal{KL}$  if the function  $s \mapsto \sigma(s, \tau)$  is of the class  $\mathcal{K}$  for any fixed  $\tau \in \mathbb{R}_+$  and the function  $\tau \mapsto \sigma(s, \tau)$  is monotonically decreasing to zero for any fixed  $s \in \mathbb{R}_+$ ;  $L^\infty(\mathbb{R}, \mathbb{R}^n)$  is the space of the essentially bounded measurable functions  $q : \mathbb{R} \mapsto \mathbb{R}^n$  and  $\|q\|_{L^\infty((a,b), \mathbb{R}^n)} = \text{ess sup}_{t \in (a,b)} \|q(t)\|$ ;  $\ell^\infty$  is a space uniformly bounded sequences;  $\text{diag}\{a_1, \dots, a_n\} \in \mathbb{R}^{n \times n}$  is a diagonal matrix.

## 2 Problem Statement

Let us consider a linear control system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad t > 0, \quad x(0) = x_0 \in \mathbb{R}^n, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the system state,  $u(t) \in \mathbb{R}^m$  is the control input,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  are known matrices.

**Definition 1** *Let the system (1) with a feedback control  $u(t) = \tilde{u}(x(t))$ ,  $\tilde{u} \in C(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R}^m)$  be globally uniformly finite-time<sup>1</sup> (resp. nearly fixed-time) stable. A family of*

<sup>1</sup> A system  $\dot{x} = f(t, x)$ ,  $x(0) = x_0$  is globally uniformly  
• *Lyapunov stable* if  $\exists \sigma \in \mathcal{K}_\infty : \|x(t, x_0)\| \leq \sigma(\|x_0\|), \forall t \geq 0, \forall x_0 \in \mathbb{R}^n$  and for any solution  $x(t, x_0)$  of the system;  
• *finite-time stable* if it is globally uniformly Lyapunov stable and there exists a locally bounded function  $T : \mathbb{R}^n \mapsto \mathbb{R}_+$  such that any trajectory of the system vanishes to zero

functions  $\tilde{u}_h : \mathbb{R}^n \mapsto \mathbb{R}^m$  parameterized by a scalar  $h > 0$  is said to be a consistent discretization of  $\tilde{u}$  if

- **Consistency of Stability:** the system (1) with

$$u(t) = \tilde{u}_h(x(t_i)), \quad t \in [t_i, t_{i+1}), \quad t_i = ih, i \in \mathbb{N} \quad (2)$$

is globally uniformly finite-time (resp., nearly fixed-time) stable for any  $h > 0$ ;

- **Control Approximation:**  $\forall r_1 > 0, \forall r_2 > r_1, \exists \omega_r \in \mathcal{K} :$

$$\sup_{r_1 \leq \|x\| \leq r_2} \|\tilde{u}_h(x) - \tilde{u}(x)\| \leq \omega_r(h), \quad \forall h > 0. \quad (3)$$

The first condition of Definition 1 asks the sampled-time control system to preserve the stability property of the original system for any fixed sampling period  $h > 0$ . The second condition guarantees that the control  $\tilde{u}_h$  is, indeed, an approximation of  $\tilde{u}$ , i.e.,  $\tilde{u}_h(x) \rightarrow \tilde{u}(x)$  as  $h \rightarrow 0^+$  uniformly on compacts from  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ . The origin is excluded since a finite-time stabilizing feedback is always non-smooth or even discontinuous at zero.

The aim of the paper is to propose a possible method of consistent discretization of finite/fixed-time controllers for LTI systems and to design a consistent discretization for one particular class of generalized homogeneous controllers studied in [47], [48], [45], [58]. First, we design a universal discretization being a mixture of feed-forward/feedback algorithms, which guarantees an exact tracking of the states of the original continuous-time closed-loop system at time instances  $t_{nk}, k \in \mathbb{N}$ . Next, we present the consistent (in the sense of the above definition) discretization scheme and study its robustness under the condition:

**Assumption 1** *The pair  $\{A, B\}$  is controllable, the matrix  $A$  is nilpotent and  $m = 1$ .*

Recall [58] that a linear system is generalized homogeneous of *non-zero degree* if and only if  $A$  is nilpotent. Finally, we generalize both schemes to multi-input systems, which can be decomposed into a cascade of single-input subsystems satisfying Assumption 1.

**Assumption 2** *Let us assume that*

$$A = \begin{bmatrix} A_1 & * & \dots & * & * \\ \mathbf{0} & A_2 & \dots & * & * \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & A_{m-1} & * \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & A_m \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_2 & \dots & \mathbf{0} & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & B_{m-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & B_m \end{bmatrix}, \quad (4)$$

where  $A_i \in \mathbb{R}^{n_i \times n_i}$ ,  $B_i \in \mathbb{R}^{n_i}$ ,  $n_1 + n_2 + \dots + n_m = n$ ,  $*$  is a (possibly) nonzero block. The pair  $\{A_i, B_i\}$  is controllable and the matrices  $A_i$  are nilpotent,  $i = 1, \dots, m$ .

in a finite time:  $\|x(t, x_0)\| = 0, \forall t \geq T(x_0), \forall x_0 \in \mathbb{R}^n$ ;  
• *nearly fixed-time stable* if it is globally uniformly Lyapunov stable and  $\forall r > 0, \exists T_r > 0 : \|x(t, x_0)\| < r, \forall t \geq T_r, \forall x_0 \in \mathbb{R}^n$ .

If the pair  $\{A, B\}$  is controllable and  $\text{rank}(B) = m$  then there exists a coordinate transformation [33] to a canonical form similar to (4). Assumption 2 asks that  $\{A, B\}$  is controllable,  $A$  is nilpotent,  $\text{rank}(B) = m$  and the system admits a transformation to the block form (4).

### 3 Preliminaries: Homogeneous systems

#### 3.1 Linear dilation and homogeneous norm

The so-called linear (geometric) dilation [45, Chapter 6] in  $\mathbb{R}^n$  is given by

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}} = \sum_{i=0}^{\infty} \frac{(sG_{\mathbf{d}})^i}{i!}, \quad s \in \mathbb{R}, \quad (5)$$

where  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is an anti-Hurwitz matrix<sup>2</sup> known as the *generator of linear dilation*. The latter guarantees that  $\mathbf{d}$  satisfies the limit property,  $\|\mathbf{d}(s)x\| \rightarrow 0$  as  $s \rightarrow -\infty$  and  $\|\mathbf{d}(s)x\| \rightarrow +\infty$  as  $s \rightarrow +\infty$ , required for a group  $\mathbf{d}$  to be a dilation in  $\mathbb{R}^n$  (see, e.g., [24]). The linear dilation introduces an alternative norm topology in  $\mathbb{R}^n$  by means the so-called canonical homogeneous norm.

**Definition 2** [45] *The function  $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto \mathbb{R}_+$  given by  $\|x\|_{\mathbf{d}} = 0$  for  $x = \mathbf{0}$  and*

$$\|x\|_{\mathbf{d}} = e^{s_x}, \quad \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0} \quad (6)$$

*is called the canonical homogeneous norm in  $\mathbb{R}^n$ , where  $\mathbf{d}$  is a linear monotone dilation<sup>3</sup>.*

Notice that  $\|x\| = 1$  (resp.  $\|x\| \leq 1$ ) is equivalent to  $\|x\|_{\mathbf{d}} = 1$  (resp.  $\|x\|_{\mathbf{d}} \leq 1$ ). For the uniform dilation  $\mathbf{d}(s) = e^s I_n$ ,  $s \in \mathbb{R}$  we have  $\|\cdot\| = \|\cdot\|_{\mathbf{d}}$ .

**Theorem 1** [44] *If  $\mathbf{d}$  is a monotone dilation and  $\|x\| = \sqrt{x^\top P x}$  with a symmetric matrix  $P \in \mathbb{R}^{n \times n}$  satisfying  $P G_{\mathbf{d}} + G_{\mathbf{d}}^\top P \succ 0$ ,  $P \succ 0$  then the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  is continuous on  $\mathbb{R}^n$  and smooth on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ :*

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \frac{\|x\|_{\mathbf{d}} x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d}})}{x^\top \mathbf{d}^\top (-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x}, \quad \forall x \neq \mathbf{0}; \quad (7)$$

Moreover,  $\underline{\sigma}(\|x\|) \leq \|x\|_{\mathbf{d}} \leq \bar{\sigma}(\|x\|)$ ,  $\forall x \in \mathbb{R}^n$ , with

$$\underline{\sigma}(r) = \begin{cases} r^{1/\alpha} & \text{if } r \geq 1, \\ r^{1/\beta} & \text{if } r < 1, \end{cases} \quad \bar{\sigma}(r) = \begin{cases} r^{1/\beta} & \text{if } r \geq 1, \\ r^{1/\alpha} & \text{if } r < 1, \end{cases}$$

where  $\alpha = 0.5 \lambda_{\max}(P^{1/2} G_{\mathbf{d}} P^{-1/2} + P^{-1/2} G_{\mathbf{d}}^\top P^{1/2}) > 0$  and  $\beta = 0.5 \lambda_{\min}(P^{1/2} G_{\mathbf{d}} P^{-1/2} + P^{-1/2} G_{\mathbf{d}}^\top P^{1/2}) > 0$ .

Below the canonical homogeneous norm is utilized as a Lyapunov function for analysis and control design.

<sup>2</sup> A matrix  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is anti-Hurwitz if  $-G_{\mathbf{d}}$  is Hurwitz.

<sup>3</sup> A dilation in  $\mathbb{R}^n$  is monotone if for any  $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$  the function  $s \mapsto \|\mathbf{d}(s)x\|$ ,  $s \in \mathbb{R}$  is strictly increasing.

**Remark 1 (On computation of  $\|\cdot\|_{\mathbf{d}}$ )** *Since the canonical homogeneous norm is defined implicitly, a computational algorithm is required for its practical implementation. Issues of numerical estimation of  $\|\cdot\|_{\mathbf{d}}$  are studied in [47], [48] based on a bisection method. In [45, Chapter 9.2.4] a scheme for an approximation of  $\|\cdot\|_{\mathbf{d}}$  by an explicit homogeneous function is presented.*

#### 3.2 Homogeneous continuous-time systems

**Definition 3** [24] *A vector field  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  (resp. a function  $h : \mathbb{R}^n \mapsto \mathbb{R}$ ) is said to be  $\mathbf{d}$ -homogeneous of degree  $\mu \in \mathbb{R}$  if  $f(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s) f(x)$  (resp.  $h(\mathbf{d}(s)x) = e^{\mu s} h(x)$ ), for all  $x \in \mathbb{R}^n$ ,  $s \in \mathbb{R}$ .*

If  $f$  is  $\mathbf{d}$ -homogeneous of degree  $\mu$  then solutions of  $\dot{x} = f(x)$  are symmetric [24]:  $x(e^{-\mu s} t, \mathbf{d}(s)x_0) = \mathbf{d}(s)x(t, x_0)$ , where  $x(t, z)$  denotes a solution with  $x(0) = z$ . For instance [58], the linear vector field  $x \mapsto Ax$ ,  $A \in \mathbb{R}^{n \times n}$  is  $\mathbf{d}$ -homogeneous of the degree  $\mu \neq 0 \Leftrightarrow A$  is nilpotent  $\Leftrightarrow A G_{\mathbf{d}} = (\mu I_n + G_{\mathbf{d}})A$ .

The homogeneity degree specifies the convergence rate.

**Theorem 2** [4], [36] *Let  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  be  $\mathbf{d}$ -homogeneous of a degree  $\mu \in \mathbb{R}$ . If the system  $\dot{x} = f(x)$  is asymptotically stable then it is globally uniformly finite-time (nearly fixed-time) stable for  $\mu < 0$  ( $\mu > 0$ ).*

The homogeneous control systems are robust (ISS) with respect to a rather large class of perturbations [19], [3].

#### 3.3 Homogeneous stabilization of linear plant

The following theorem merges results of [47], [58], [39].

**Theorem 3** *Let a pair  $\{A, B\}$  be controllable. Then*

1) *any solution  $Y_0 \in \mathbb{R}^{m \times n}$ ,  $G_0 \in \mathbb{R}^{n \times n}$  of the linear algebraic equation*

$$A G_0 - G_0 A + B Y_0 = A, \quad G_0 B = \mathbf{0} \quad (8)$$

*is such that the matrix  $G_0 - I_n$  is invertible, the matrix  $G_{\mathbf{d}} = I_n + \mu G_0$  is anti-Hurwitz for any  $\mu \in [-1, 1/\tilde{n}]$ , where  $\tilde{n}$  is a minimal natural number such that  $\text{rank}[B, AB, \dots, A^{\tilde{n}-1}B] = n$ , the matrix  $A_0 = A + B Y_0 (G_0 - I_n)^{-1}$  satisfies the identity*

$$A_0 G_{\mathbf{d}} = (G_{\mathbf{d}} + \mu I_n) A_0, \quad G_{\mathbf{d}} B = B; \quad (9)$$

2) *the linear algebraic system*

$$A_0 X + X A_0^\top + B Y + Y^\top B^\top + \rho (G_{\mathbf{d}} X + X G_{\mathbf{d}}^\top) = \mathbf{0}, \quad (10)$$

$$G_{\mathbf{d}} X + X G_{\mathbf{d}}^\top \succ 0, \quad X = X^\top \succ 0$$

*has a solution  $X \in \mathbb{R}^{n \times n}$ ,  $Y \in \mathbb{R}^{m \times n}$  for any  $\rho > 0$ ;*

3) the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  induced by the weighted Euclidean norm  $\|x\| = \sqrt{x^\top P x}$  with  $P = X^{-1}$  is a Lyapunov function of the system (1) with

$$u = K_0 x + \|x\|_{\mathbf{d}}^{1+\mu} K \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x, \quad (11)$$

$$K_0 = Y_0(G_0 - I_n)^{-1}, \quad K = Y X^{-1}, \quad (12)$$

where  $\mathbf{d}$  is a dilation generated by  $G_{\mathbf{d}}$ ; moreover,

$$\frac{d}{dt} \|x\|_{\mathbf{d}} = -\rho \|x\|_{\mathbf{d}}^{1+\mu}, \quad x \neq \mathbf{0}; \quad (13)$$

4) the feedback law  $u$  given by (11) is continuously differentiable on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ ,  $u$  is continuous at zero if  $\mu > -1$  and  $u$  is discontinuous at zero if  $\mu = -1$ ;

5) the system (1), (11) is  $\mathbf{d}$ -homogeneous of degree  $\mu$ .

Obviously, due to (13) the closed-loop system (1), (11) is uniformly finite-time stable if  $\mu < 0$  and it is nearly fixed-time stable if  $\mu > 0$ . For  $\mu = 0$  the control (11) becomes  $u = K_0 x + K x$ . Such a control law (under some variations and/or simplifications) has been presented in the literature as a solution to a finite-time stabilization problem for linear plants [28], [50], [47].

**Remark 2** Under Assumption 1, the equation (8) has a unique solution such that  $Y_0 = \mathbf{0}$  (i.e.,  $A_0 = A$ ) and  $\exists J \in \mathbb{R}^{n \times n} : J^{-1} G_0 J = -\text{diag}\{n-1, \dots, 1, 0\}$ . This follows from the fact then the system (1), in this case, is equivalent to a controlled integrator chain.

A topological equivalence of any stable  $\mathbf{d}$ -homogeneous system to a standard homogeneous one [44] allows an explicit representation of solution for (1), (11) to be derived.

**Corollary 1 (Explicit representation of solutions)** Under conditions of Theorem 3 with  $\mu \neq 0$ , a solution of the closed-loop system (1), (11) is unique and

$$x(t + \tau) = Q_\tau(\|x(t)\|_{\mathbf{d}}) x(t), \quad (14)$$

where  $\tau, t \geq 0$ ,  $Q_\tau(0) = \mathbf{0}$  and for  $r > 0$  one has

$$Q_\tau(r) = \begin{cases} e^{G_{\mathbf{d}} \ln r} \hat{Q} \left( \frac{\ln(1+\mu\rho\tau r^\mu)}{\rho\mu} \right) e^{-G_{\mathbf{d}} \ln r} & \text{if } \frac{1}{\tau\mu} > -\mu\rho\tau, \\ \mathbf{0} & \text{if } \frac{1}{\tau\mu} \leq -\mu\rho\tau, \end{cases} \quad (15)$$

$$\hat{Q}(\hat{s}) = e^{-\rho G_{\mathbf{d}} \hat{s}} e^{(A+B(K_0+K)+\rho G_{\mathbf{d}})\hat{s}}, \quad \hat{s} \geq 0. \quad (16)$$

The proof of this corollary, as well as the proofs of other main and auxiliary results, are given in the Appendix. The matrix-valued function  $Q_{nh}(\cdot)$  can be easily computed, since elements of a matrix exponential  $e^{sM}$  can always be represented as polynomial functions of  $s$ ,  $e^{\rho_i}$ ,  $\cos(\omega_i s)$  and  $\sin(\omega_i s)$ , where  $\rho_i + i\omega_i$  are eigenvalues of the matrix  $M \in \mathbb{R}^{n \times n}$ . Moreover, if  $A_0 = A + BK_0$ ,  $B, K, G_{\mathbf{d}}$  satisfy (10) then the matrix  $X^{1/2}(A_0 + BK + \rho G_{\mathbf{d}})X^{-1/2}$  is skew-symmetric and

$$e^{(A_0+BK+\rho G_{\mathbf{d}})\phi} = X^{-1/2} R(\phi) X^{1/2}, \quad (17)$$

where  $R(\phi)$  is a rotation matrix for any  $\phi \in \mathbb{R}$ , i.e.,  $R(\phi)R^\top(\phi) = R^\top(\phi)R(\phi) = I_n$ .

## Corollary 2 (On cascade homogeneous control)

Let Assumption 2 be fulfilled. Let  $G_{\mathbf{d}_i} \in \mathbb{R}^{n_i \times n_i}$ ,  $K_i \in \mathbb{R}^{1 \times n_i}$ ,  $P_i \in \mathbb{R}^{n_i \times n_i}$  and the control  $u_i(x_i)$  with  $x_i \in \mathbb{R}^{n_i}$  be defined by Theorem 3 for the pairs  $\{A_i, B_i\}$  and some  $\mu_i \in [-1, 1/\bar{n}_i]$ ,  $\rho_i > 0$ , respectively. Then the system (1) with the control  $u = (u_1, \dots, u_m)$  is globally uniformly finite-time stable if  $\mu_i < 0$  (resp., nearly fixed-time stable if  $\mu_i > 0$ ) for all  $i = 1, 2, \dots, m$ .

## 4 Discretization of Homogeneous Control

### 4.1 Single-input case

Let us represent the system (1) with the sampled-time control  $u(t) = u(t_k)$  for  $t \in [t_k, t_{k+1})$  in the form:

$$x_{k+1} = A_h x_k + B_h u(t_k), \quad k \in \mathbb{N}, \quad (18)$$

where  $x_k = x(t_k)$ ,  $t_k = kh$ ,  $A_h = e^{hA}$  and  $B_h = \int_0^h e^{sA} B ds$ . The system (18) can be rewritten as follows:

$$x_{k+n} = B_h u(t_{k+n-1}) + \dots + A_h^{n-1} B_h u(t_k) + A_h^n x_k. \quad (19)$$

The controllability of the pair  $\{A, B\}$  implies the controllability of the pair  $\{A_h, B_h\}$  and the invertability of

$$W_h = [B_h, A_h B_h, \dots, A_h^{n-1} B_h] \quad (20)$$

(see the formulas (45), (44) and Lemma 5 in Appendix).

Let parameters of a stabilizing homogeneous controller (11) be designed as in Theorem 3 for  $\mu \neq 0$ . By Corollary 1, to track a trajectory of the continuous-time closed-loop system (1), (11), the sampled-time control just has to fulfill the following identity

$$Q_{nh}(\|x_k\|_{\mathbf{d}}) x_k = B_h u(t_{k+n-1}) + \dots + A_h^{n-1} B_h u(t_k) + A_h^n x_k.$$

Indeed, if a sampled-time control is implemented as

$$\begin{bmatrix} u(t_{k+n-1}) \\ \vdots \\ u(t_k) \end{bmatrix} = W_h^{-1} (Q_{nh}(\|x_k\|_{\mathbf{d}}) - A_h^n) x_k, \quad (21)$$

then the discrete-time system (18), (21) tracks any trajectory of the continuous-time system (1), (11) at  $t = t_{kn}$ ,  $k \in \mathbb{N}$ . In particular, if  $\mu < 0$  then for any  $x_0 \in \mathbb{R}^n$  there exists  $k^* \in \mathbb{N}$  such that  $x_k = \mathbf{0}$ ,  $\forall k \geq k^*$ .

**Theorem 4** The system (1) with the sampled-time control (21) is globally uniformly finite-time stable if  $\mu < 0$  (nearly fixed-time stable if  $\mu > 0$ ).

Since  $u(t_{k+i-1})$  depends on  $x_k = x(t_k)$  but not on  $x(t_{k+i-1})$ , then the discretization (21) of the control (11) could be useful, for example, if the control sampling is  $n$  times faster than a measurement sampling. In other cases, the control (21) is a mixture of feedforward and feedback algorithms, where the state measurements  $x(t_{k+i-1})$  for  $i = 2, \dots, n-1$  are simply omitted during

the control implementation. This could badly impact to a robustness and to a precision of the sampled-time controller. To avoid this drawback, let us consider the static feedback law

$$\tilde{u}_h(x_k) = \tilde{K}_h(\|x_k\|_{\mathbf{d}})x_k, \quad (22)$$

$$\tilde{K}_h(\|x_k\|_{\mathbf{d}}) = e_n^\top W_h^{-1} (Q_{nh}(\|x_k\|_{\mathbf{d}}) - A_h^n), \quad (23)$$

which is obtained from (21) selecting only  $u(t_k)$ .

**Proposition 1 (Approximation property)** *Let  $u$  be a homogeneous control (11) designed by Theorem 3 under Assumption 1. Then  $\tilde{u}_h(x) \rightarrow u(x)$  as  $h \rightarrow 0^+$  uniformly on compacts from  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ .*

This proposition, in particular, implies that for a sufficiently small  $h > 0$  the system (18), (22) behaves similarly to the continuous-time system (1), (11) at least on small intervals of time. Let us denote

$$L_h = B_h e_n^\top W_h^{-1}, \quad F_h = A_h - L_h A_h^n, \quad h > 0, \quad (24)$$

$$M_h(\|x\|_{\mathbf{d}})x = (F_h + L_h Q_{nh}(\|x\|_{\mathbf{d}}))x, \quad x \in \mathbb{R}^n, \quad (25)$$

and rewrite the discrete-time system (18), (22) as follows

$$x_{k+1} = z_h(x_k) := M_h(\|x_k\|_{\mathbf{d}})x_k. \quad (26)$$

**Lemma 1 (Homogeneity of discretization)** *The system (18), (22) is  $\mathbf{d}$ -homogeneous as follows :*

$$z_h(\mathbf{d}(s)x) = \mathbf{d}(s)z_{e^{\mu s}h}(x), \quad (27)$$

$$\tilde{u}_h(\mathbf{d}(s)x) = e^{s(1+\mu)}\tilde{u}_{e^{\mu s}h}(x), \quad (28)$$

for all  $s \in \mathbb{R}$ , for all  $h > 0$  and for all  $x \in \mathbb{R}^n$

The dilation symmetry established by Lemma 1 guarantees that a global asymptotic stability of the discrete-time system (18), (22) for some  $h = \hat{h} > 0$  is equivalent to the global asymptotic stability of this system for any  $h > 0$ . For simplicity, we select

$$\hat{h} := (|\mu|\rho n)^{-1}. \quad (29)$$

As shown below, the key feature of the proposed control discretization is the nilpotence of the matrix  $F_h$ , i.e.,  $F_h^n = \mathbf{0}$ . Together with properties of  $Q_{nh}(\|x_k\|_{\mathbf{d}})x_k$ , this allows the controller (22) to preserve stability properties of the original system in its sampled-time counterpart.

**Lemma 2** *Let  $u$  be a homogeneous control (11) designed by Theorem 3 under Assumption 1. Then the closed-loop discrete-time system (18), (22) is*

1) *locally uniformly finite-time stable for  $\mu < 0$  and*

$$\|x_0\|_{\mathbf{d}} \leq \underline{r}^-(\hat{h}/h)^{1/\mu} \Rightarrow x_k = \mathbf{0}, \quad \forall k \geq n,$$

where  $\|x\|_{\mathbf{d}}$  is the canonical homogeneous norm induced by the weighted Euclidean norm  $\|x\| = \sqrt{x^\top P x}$  with  $P = X^{-1}$  and  $\underline{r}^- > 0$  is any number such that

$$\max_{i \in \{1, \dots, n\}} \|F_h^{i-1} \mathbf{d}(\ln \underline{r}^-)\| < 1;$$

2) *globally practically<sup>4</sup> finite-time stable for  $\mu < 0$  and the set  $\Omega^- = \{x \in \mathbb{R}^n : \|x\|_{\mathbf{d}} \leq \bar{r}^-(\hat{h}/h)^{1/\mu}\}$  is invariant and finite-time stable for some  $\bar{r}^- \geq \underline{r}^-$ ;*

3) *globally practically fixed-time stable for  $\mu > 0$ :*

$$\|x_k\|_{\mathbf{d}} \leq \bar{r}^+(\hat{h}/h)^{1/\mu}, \quad \forall k \geq n$$

for all  $x_0 \in \mathbb{R}^n$ , where  $\bar{r}^+ > \max_{\|v_i\|_{\mathbf{d}} \leq 1} \left\| \sum_{i=1}^n F_h^{i-1} L_{\hat{h}} v_i \right\|_{\mathbf{d}}$ ;

4) *locally asymptotically stable for  $\mu > 0$  and the set  $\Omega^+ = \{x \in \mathbb{R}^n : \|x\|_{\mathbf{d}} \leq \underline{r}^+(\hat{h}/h)^{1/\mu}\}$  is an invariant attraction domain for some  $\underline{r}^+ \in (0, \bar{r}^+]$ .*

The latter lemma proves that the discretization (22) of the controller (11), indeed, preserves a stability property of the original system at least locally (close to zero and close to infinity). The discrete-time system with  $h = \hat{h}$  behaves similarly to the continuous-time system for  $\|x_k\|_{\mathbf{d}} < \underline{r}^\pm$  and  $\|x_k\|_{\mathbf{d}} > \bar{r}^\pm$ . Hence, if the set  $\tilde{\Omega}^\pm = \{x : \underline{r}^\pm < \|x\|_{\mathbf{d}} < \bar{r}^\pm\}$  does not contain an invariant set of the discrete-time system then the discretization is globally consistent. Lemma 2 does not provide explicit estimates for  $\bar{r}^-$  and  $\underline{r}^+$ . So, a criterion of the global consistency of the discretization (22) has been obtained in a different manner.

Let us consider a family of mappings  $\Theta_k : (0, +\infty) \times S \mapsto \mathbb{R}^{n \times n}$  defined recursively as follows:  $\Theta_0(\delta, v) = I_n$  and

$$\Theta_{k+1}(\delta, v) = M_{\delta \hat{h}}(\|\Theta_k(\delta, v)v\|_{\mathbf{d}})\Theta_k(\delta, v), \quad k \in \mathbb{N}, \quad (30)$$

where  $\delta \in \mathbb{R}$ ,  $v \in S$ , the parameter  $\hat{h} > 0$  is defined by (29) and  $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$  is the unit sphere.

**Lemma 3** *Any solution of the discrete-time system (18), (22) with  $h = \hat{h}$  and  $x_0 \neq \mathbf{0}$  admits the representation*

$$x_k = \mathbf{d}(\ln \|x_0\|_{\mathbf{d}})\Theta_k(\|x_0\|_{\mathbf{d}}^\mu, v_0)v_0, \quad (31)$$

where  $v_0 = \mathbf{d}(-\ln \|x_0\|_{\mathbf{d}})x_0 \in S$ .

The following theorem presents a necessary and sufficient condition of the consistency of the discretization (22) for the controller (11).

**Theorem 5 (Consistent discretization)** *Let  $u$  be a homogeneous control (11) designed by Theorem 3 under Assumption 1 for  $\mu \neq 0$ . Then  $\tilde{u}_h$  given by (22) is a consistent discretization of the control  $u$  **if and only if** there exists  $k^* \geq 1$  such that*

$$\|\Theta_{k^*}(\delta, v)v\|_{\mathbf{d}} < 1, \quad \forall \delta \in (0, r^*], \quad \forall v \in S, \quad (32)$$

where  $r^* = (\underline{r}^-)^\mu$  if  $\mu < 0$  and  $r^* = (\bar{r}^+)^\mu$  if  $\mu > 0$ .

<sup>4</sup> *Practical finite-time and fixed-time stability is introduced using the same definitions by replacing a norm (distance to  $\mathbf{0}$ ) with a distance to a set being a neighborhood of zero.*

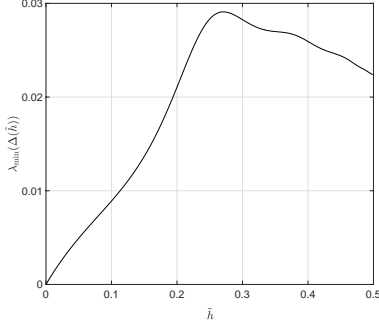


Fig. 1. The minimal eigenvalue of the matrix  $\Delta(\delta)$  for  $\mu = -1/2, \rho = 2$

Therefore, the discrete-time system (18), (22) is uniformly finite-time (or nearly fixed-time) stable if and only if the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  is a kind of a homogeneous Lyapunov function. Indeed, (31) and (32) simply means that  $\|x_{k^*}\|_{\mathbf{d}} < \|x_0\|_{\mathbf{d}}, \forall x_0: \|x_0\|_{\mathbf{d}}^\mu \in (0, r^*)$ .

**Remark 3 (On feasibility of condition (32))** For  $k^* = 1$  the condition (32) is equivalent to the nonlinearly parameterized matrix inequality

$$M_{\delta\hat{h}}(1)^\top X^{-1} M_{\delta\hat{h}}(1)^\top < X^{-1}, \quad \forall \delta \in (0, r^*), \quad (33)$$

which can be checked numerically on a sufficiently dense grid in  $(0, r^*]$  because of a continuous dependence of  $M_{\delta\hat{h}}(1)$  on the parameter  $\delta$ . Denoting  $\Delta(\delta) = X^{-1} - M_{\delta\hat{h}}(1)^\top X^{-1} M_{\delta\hat{h}}(1)$  we conclude that the condition (33) is fulfilled if  $\lambda_{\min}(\Delta(\delta)) > 0$  for all  $\delta \in (0, r^*]$ . For example, for  $n = 2, \mu = -1, \rho = 2, A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  Theorem 3 gives  $G_{\mathbf{d}} = \text{diag}\{2, 1\}, K = YX^{-1}$  with  $X = x_{11}\tilde{X}, x_{11} > 0, \tilde{X} = \begin{bmatrix} 1 & -\rho(1-\mu) \\ -\rho(1-\mu) & 7(2-\mu)^2\rho^2(1-\mu)/8 \end{bmatrix}, Y = \rho^2(2-\mu)(1-\mu)x_{11} \begin{bmatrix} \frac{8-7(2-\mu)}{8} & -\frac{7\rho(2-\mu)}{8} \end{bmatrix}$ . In this case, we have  $r^* = 1$  and Figure 1 depicts the evolution of the function  $\delta \mapsto \lambda_{\min}(\Delta(\delta))$ , which confirms that (33) is fulfilled. For  $k^* > 1$  a similar (but a bit more complicated) numerical procedure can be developed.

## 4.2 Robustness analysis

It is well known [19], [3] that homogeneous systems are Input-to-State Stable (ISS) with respect to sufficiently large class of perturbations. Recall [54] that a system

$$\dot{x} = f(t, x, q), \quad t > t_0 \quad (34)$$

is practically ISS with respect to  $q \in L^\infty(\mathbb{R}, \mathbb{R}^l)$  if there exist,  $c > 0, \xi \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that

$$\|x(t)\| \leq c + \xi(\|x_0\|, t - t_0) + \gamma(\|q\|_{L^\infty((t_0, t), \mathbb{R}^l)}). \quad (35)$$

If  $c = 0$  then the system is ISS. Local ISS restricts additionally the set of initial conditions  $\|x_0\| \leq r_x$  and/or the maximal magnitude of the input  $\|q\| \leq r_q$ .

Let us consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + q(t), & t > 0, \\ u(t) &= u(t_k), & t \in [t_k, t_{k+1}), \end{aligned} \quad (36)$$

where  $u(t_k)$  is given by (21) and  $q \in L^\infty(\mathbb{R}, \mathbb{R}^n)$  is an exogenous input.

**Theorem 6** The system (36) is 1) locally ISS; 2) practically fixed-time stable if  $\mu > 0$ ; 3) ISS if  $\mu > -\beta$ , where  $\beta > 0$  is defined in Theorem 1.

The ISS can be established for consistent discretization of homogeneous controller. Let us consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + q_p(t), & t > 0, \\ u(t) &= \tilde{u}_h(x(t_k) + q_m(t_k)), & t \in [t_k, t_{k+1}) \end{aligned} \quad (37)$$

where  $\tilde{u}_h$  given by (22) is a discretization of a control (11) designed by Theorem 3 under Assumption 1 and the exogenous input  $q = (q_p^\top, q_m^\top)^\top \in L^\infty(\mathbb{R}, \mathbb{R}^{2n})$  is such that  $\{q_m(t_k)\} \in \ell^\infty$ . Here  $q_p$  models an external perturbation and  $q_m$  is a measurement noise.

**Theorem 7** The system (37) is 1) locally ISS; 2) practically fixed-time stable if  $\mu > 0$ ; 3) practically ISS if  $\mu > -\beta$ ; 4) ISS if  $\mu > -\beta$  and the unperturbed system ( $q = \mathbf{0}$ ) is globally asymptotically stable.

Notice that  $q$  may contain an output of another system. In this case, a stability analysis of a cascade system can be based on ISS [54], [7]. We follow this idea for the design of the consistent discretization in the multi-input case.

## 4.3 Multi-input case

Let the control  $u$  be designed using Corollary 2 under Assumption 2. Since  $A_i$  is nilpotent then, as before,  $K_0 = \mathbf{0}$  in Theorem 3 and in Corollary 1. Let  $Q_{n_i h}(\cdot)$  be defined by the formula (15) for  $A_i, B_i, K_i, G_{\mathbf{d}_i}, P_i, \mu_i, i = 1, \dots, m$ . Let us denote

$$W_{h,i} = [B_{h,i}, A_{h,i}B_{h,i}, \dots, A_{h,i}^{n_i-1}B_{h,i}], \quad (38)$$

$A_{h,i} = e^{hA_i}, B_{h,i} = \int_0^h e^{\tau A_i} d\tau B$ , and introduce the following discretization of the controllers  $u_i$ :

$$\begin{bmatrix} u_i(t_{k+n-1}) \\ \vdots \\ u_i(t_k) \end{bmatrix} = W_{h,i}^{-1} (Q_{n_i h}(\|x_i(t_k)\|_{\mathbf{d}_i}) - A_{h,i}^{n_i}) x_i(t_k). \quad (39)$$

**Corollary 3** Under Assumption 2, the system (18) with the control (39) is globally uniformly finite-time stable if  $\mu_i < 0$  (nearly fixed-time stable if  $\mu_i > 0$ ),  $\forall i = 1, \dots, m$ .

Similarly to the single input case, a consistent discretization of the multi-input control system is designed as

$$u_i(x_{k,i}) = \tilde{K}_i(\|x_{k,i}\|_{\mathbf{d}_i}) x_{k,i}, \quad (40)$$

$$\tilde{K}_i(\|x_{k,i}\|_{\mathbf{d}_i}) = e_{n_i}^\top W_{h,i}^{-1} (Q_{n_i h}(\|x_{k,i}\|_{\mathbf{d}_i}) - A_{h,i}^{n_i}).$$

**Corollary 4** Let a control law  $u$  be designed by Corollary 2 under Assumption 2. Then (40) is a consistent discretization of  $u$  provided that conditions of Theorem 5 are fulfilled for  $\mu = \mu_i, \forall i=1, \dots, m$  and  $\max \mu_i < 0$  or  $\min \mu_i > 0$ .

## 5 Numerical Examples

### 5.1 Single-input system

Let  $n = 3, A = \begin{bmatrix} 0 & I_2 \\ 0 & 0 \end{bmatrix}, \mu = -0.25, \rho = 2$ . By Theorem 3 we derive a finite-time homogeneous control (11) with parameters  $K_0 = \mathbf{0}, G_d = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  and  $X = \begin{bmatrix} 1.0000 & -3.0000 & 6.4250 \\ -3.0000 & 10.0750 & -25.1875 \\ 6.4250 & -25.1875 & 75.2507 \end{bmatrix}, K = [-34.2751 \ -25.175 \ -7.5]$ . The numerical simulation has been done in MATLAB<sup>5</sup>. Simulation results of the system (1) with the consistently discretized control (2), (40) for  $x_0 = [1 \ -1 \ 0]^T, h = 0.1$  are presented on Fig. 2 (right). The system states converge to zero in a finite time in the sense that  $\|x_k\| \leq \epsilon, \forall k \geq k^* = 24$ , where  $\epsilon = 1.11 \cdot 10^{-16}$  is the machine zero. Since  $\|x_{k^*-1}\| \approx 10^{-6}$  then  $t_{k^*} = 2.4$  is the settling time estimate obtained by the numerical simulation, which coincides with the theoretical settling time  $T(x_0) = \|x_0\|_d^{-\mu} / (-\mu\rho) \approx 2.4526$  up to an error of the order  $h$ . This property is observed in all simulations, for other initial conditions and other sampling periods, confirming very high precision of the approximation of the continuous time system by its sampled-time counterpart. The results for the *explicit discretization*

$$u(t) = \tilde{u}(x(t_k)), \quad t \in [t_k, t_{k+1}) \quad (41)$$

of the control (11) are depicted on Fig. 2 (left) for comparison reasons. The system (1) with the explicitly discretized homogeneous controller is not asymptotically stable and suffers of a chattering. For  $\mu = 0.25$ ,

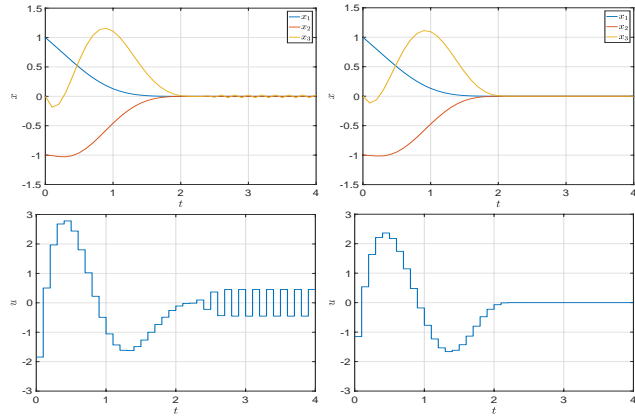


Fig. 2. The simulation results for the system (1) with explicitly (left) and consistently (right) discretized finite-time control (11) for  $n = 3, m = 1, \mu = -0.25, \rho = 2, h = 0.1$ .

<sup>5</sup> <https://gitlab.inria.fr/polyakov/hcs-toolbox-for-matlab>

$\rho = 2$  we similarly derive a nearly fixed-time homogeneous control (11) with parameters  $K_0 = \mathbf{0}, G_d = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.75 & 1 \\ 0 & 0 & 1 \end{bmatrix}$  and  $X = \begin{bmatrix} 1.0000 & -1.0000 & 0.4583 \\ -1.0000 & 2.0417 & -3.0625 \\ 0.4583 & -3.0625 & 7.9863 \end{bmatrix}, K = [-3.2569 \ -7.0069 \ -4.5000]$ . The simulation results for the explicitly discretized controller with  $x_0 = 4.96873515 \cdot [1 \ -1 \ 0]^T$  are depicted in Fig. 3 (left). With the explicit discretization, the system simply blows up for larger initial conditions. Simulations of the consistently discretized control were made for initial conditions with various magnitudes up to  $\|x_0\| \approx 10^{10}$ . They show the nearly fixed-time stability of the closed-loop system as in Fig. 3 (right). ISS of both controllers (for  $\mu = -0.25$  and  $\mu = 0.25$ ) with respect to additive perturbations and measurement noises was also confirmed by simulations.

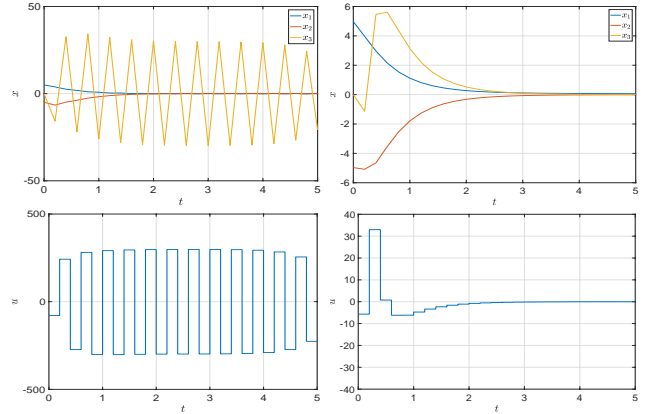


Fig. 3. The simulation results for the system (1) with explicitly (left) and consistently (right) discretized fixed-time control (11) for  $n = 3, m = 1, \mu = 0.25, \rho = 2, h = 0.2$ .

### 5.2 Multi-input system

For the multi-input case we consider the above single-input system ( $n_1 = 3$ ) with the finite-time control ( $\mu_1 = -0.25, \rho_1 = 1$ ) in the cascade with the second order system ( $n_2 = 2$ ) with the finite-time control ( $\mu_2 = -1, \rho_2 = 1$ ) considered in Remark 3, i.e.,  $A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & I_2 \\ 0 & 0 \end{bmatrix}, A_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . The simulation results for the cascade system are shown on Fig. 4 for  $x_1(0) = [1 \ -1 \ 0]^T, x_2(0) = [1 \ 0]^T$ . They confirm finite-time stability of the closed-loop system with the consistently discretized control.

## 6 Conclusions

In the paper, two types of discretization (sampled-time implementation) schemes for a homogeneous control law are developed. Both preserves the finite/fixed-time stability properties of the original continuous-time closed-loop system. The first scheme gives a mixture of feed-



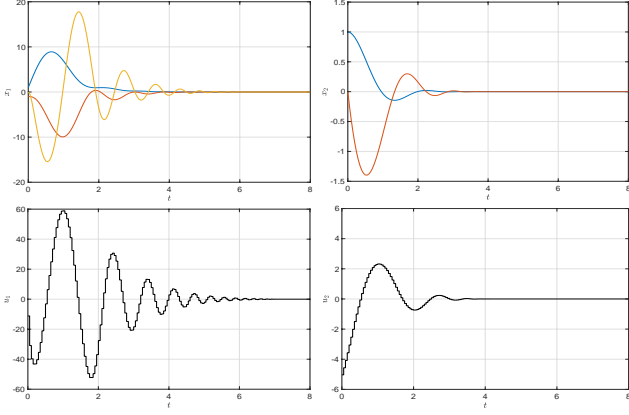


Fig. 4. The simulation results for the cascade system (1) with consistently discretized finite-time controllers with  $n_1=3, n_2=2, \mu_1=-0.25, \rho_1=1, \mu_2=-1, \rho_2=-1, h=0.05$ .

forward and feedback algorithms. It can be utilized if the control sampling is much faster than the sampling of measurements, or in the model-predictive framework. The second scheme (called consistent) provides a static feedback law, which always preserves the stability properties, at least, locally (close to zero and close to infinity). A necessary and sufficient condition of global consistency is presented. A particular sufficient condition is formalized in terms of a parametric LMI, which, by numerical simulations, is shown to be feasible in some cases. A development of numerical algorithms for control parameters tuning based on the obtained conditions of the consistency is an interesting problem for the future research.

## 7 Appendix

### 7.1 Auxiliary results

**Lemma 4** [14, page 136] If  $Z_1 Z_2 - Z_2 Z_1 = q Z_2$  with  $q \in \mathbb{R}$  and  $Z_1, Z_2 \in \mathbb{R}^{n \times n}$  then  $e^{Z_1} e^{Z_2} = e^{Z_1 + \frac{q}{1-e^{-q}} Z_2}$ .

**Lemma 5** If Assumption 1 is fulfilled and  $\mathbf{d}$  is a linear dilation defined in Theorem 3 then

1) for all  $s \in \mathbb{R}$  the following identities hold:

$$\mathbf{A} \mathbf{d}(s) = e^{\mu s} \mathbf{d}(s) \mathbf{A}, \quad \mathbf{d}(s) \mathbf{B} = e^s \mathbf{B}, \quad \mathbf{d}(s) e^{\tau A} = e^{-\mu s \tau} \mathbf{d}(s) e^{\tau A}, \quad (42)$$

$$\mathbf{d}(s) \int_0^h e^{\tau A} d\tau = e^{\mu s} \int_0^{e^{-\mu s} h} e^{\tau A} d\tau \mathbf{d}(s); \quad (43)$$

2) the matrix  $W_h$  given by (20) satisfies

$$W_h = \mathbf{d}_*(\ln h) \int_0^1 e^{\tau A} d\tau [B \ e^A B \ \dots \ e^{(n-1)A} B], \quad (44)$$

$$W_h^{-1} \mathbf{d}_*(\ln h) \int_0^n e^{\tau A} d\tau B = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \quad (45)$$

where  $\mathbf{d}_*$  corresponds to  $\mathbf{d}$  with  $\mu = -1$ ;

- the parameter  $\beta$  defined in Theorem 1 satisfies  $\beta = 1$  for any  $\mu < 0$ .

**Proof. 1)** Under Assumption 1 of Theorem 3 the dilation  $\mathbf{d}(s) = e^{s G_{\mathbf{d}}}$  is uniquely defined such that  $G_{\mathbf{d}}$  satisfies (9) with  $A_0 = A$ . We derive  $\mathbf{A} \mathbf{d}(s) = A \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!} = \sum_{i=0}^{+\infty} \frac{s^i (\mu I_n + G_{\mathbf{d}})^i A}{i!} = e^{\mu s} \mathbf{d}(s) A$ ,  $\mathbf{d}(s) B = \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!} B = \sum_{i=0}^{+\infty} \frac{s^i}{i!} B$ ,  $\forall s \in \mathbb{R}$ . Hence, for all  $s \in \mathbb{R}$  we have  $\mathbf{d}(s) e^{\tau A} = \mathbf{d}(s) \sum_{i=0}^{n-1} \frac{h^i A^i}{i!} = \sum_{i=0}^{n-1} \frac{e^{-\mu i s} h^i A^i \mathbf{d}(s)}{i!} = e^{-\mu s \tau} \mathbf{d}(s) e^{\tau A}$  and  $\mathbf{d}(s) \int_0^h e^{\tau A} d\tau = \int_0^h e^{-\mu s \tau} \mathbf{d}(s) e^{\tau A} d\tau$ .

and changing of the integration variable we derive (43). **2)** On the one hand, since  $e^{sA}$  and  $\int_0^h e^{\tau A} d\tau$  commute for any  $s \in \mathbb{R}$  then taking into account  $A_h^i = e^{ihA}$  we derive  $W_h = [B_h \ A_h B \ A_h^2 B_h \ \dots \ A_h^{n-1} B_h] = \left( \int_0^h e^{sA} ds \right) \cdot [B; e^{hA} B \ e^{2hA} B \ \dots \ e^{(n-1)hA} B]$ . On the other hand, the homogeneity identities (42) imply  $hB = \mathbf{d}_*(\ln h) B$  and  $e^{ihA} \mathbf{d}_*(\ln h) = \mathbf{d}_*(\ln h) e^{iA}$ ,  $\forall i \in \mathbb{N}, \forall h > 0$ . Hence, we have

$$W_h = \left( \frac{1}{h} \int_0^h e^{sA} ds \right) \mathbf{d}_*(\ln h) [B, e^A B, e^{2A} B, \dots, e^{(n-1)A} B].$$

Using the identity (43) we derive

$$\mathbf{d}_*(-\ln h) \left( \frac{1}{h} \int_0^h e^{sA} ds \right) = \int_0^1 e^{sA} ds \mathbf{d}_*(-\ln h),$$

so the formula (44) holds. Finally, since  $\int_0^n e^{\tau A} d\tau = \sum_{i=0}^{n-1} \int_i^{i+1} e^{\tau A} d\tau = \int_0^1 e^{\tau A} d\tau \sum_{i=0}^{n-1} e^{iA}$  then  $W_h^{-1} \mathbf{d}_*(\ln h) \cdot \int_0^n e^{\tau A} d\tau B = [B e^A B \ \dots \ e^{(n-1)A} B]^{-1} \sum_{i=0}^{n-1} e^{iA} B = \sum_{i=0}^{n-1} e_i$ .

**3)** For  $m = 1$ , under conditions of Theorem 3, we have  $G_{\mathbf{d}} = U^{-1} \text{diag}\{1 - \mu(n-1), \dots, 1\} U$  for some nonsingular matrix  $U \in \mathbb{R}^{n \times n}$ . The latter means that  $\beta = 1$

**Lemma 6** The function  $V : \mathbb{R}^n \mapsto \mathbb{R}_+$  defined as  $V(x) = \bar{\sigma}^{-1}(\|x\|_{\mathbf{d}})$ ,  $x \in \mathbb{R}^n$  is positive definite radially unbounded and globally Lipschitz continuous with the Lipschitz constant 1, where  $\bar{\sigma} \in \mathcal{K}_{\infty}$  is as in Theorem 1.

**Proof.** The Lipschitz continuity for  $\|x\| \geq 1$  follows from [44, Proposition 1]. For  $\|x\| \leq 1$  this fact can be proven similarly (just replacing  $\beta$  with  $\alpha$ ). All other properties of  $V$  are obvious.

**Lemma 7** For any  $C_* > 0$  and any  $\varepsilon \in (0, 1)$  there exist  $C > 0$  and  $r > 0$  such that

$$|u(x) - u_{\tilde{h}}(\tilde{x})| \leq C \min\{\|x\|_{\mathbf{d}}^{1+2\mu}, \|\tilde{x}\|^{1+2\mu}\}, \quad (46)$$

for all  $x, \tilde{x}$  satisfying  $(1 - \varepsilon)\|\tilde{x}\|_{\mathbf{d}} \leq \|x\|_{\mathbf{d}} \leq (1 + \varepsilon)\|\tilde{x}\|_{\mathbf{d}}$ ,  $\|x\|_{\mathbf{d}}^{\mu} \leq r$ ,  $\|\mathbf{d}(-\ln \|x\|_{\mathbf{d}})(x - \tilde{x})\| \leq C_* \|x\|^{\mu}$ .

**Proof.** Using approximation of  $\tilde{u}_{\tilde{h}}$  (see, the proof of Proposition 1) we conclude  $\exists h_0 \in (0, 1), \exists C_0 > 0$  such that  $\sup_{y \in S} |u(y) - \tilde{u}_{\tilde{h}}(y)| \leq C_0 \tilde{h}$ ,  $\forall \tilde{h} < h_0$ . Since  $u$  is locally Lipschitz continuous on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$  then for any  $\delta \in (0, 1)$  there exists  $C_1$  such that  $|u(y) - u(z)| \leq C_1 \|y - z\|$ ,  $\forall y \in S$  and  $\forall z : \|y - z\| \leq \delta$ . Using the Lipschitz condition and the dilation symmetry of  $u$  and  $\tilde{u}_{\tilde{h}}$  (see (28)) we derive

$$\begin{aligned}
& |u(x) - \tilde{u}_{\hat{h}}(\tilde{x})| \leq |u(x) - u(\tilde{x})| + |u(\tilde{x}) - \tilde{u}_{\hat{h}}(\tilde{x})| = \\
& \|x\|_{\mathbf{d}}^{1+\mu} |u(\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x) - u(\mathbf{d}(-\ln \|\tilde{x}\|_{\mathbf{d}})\tilde{x})| + \\
& \|\tilde{x}\|_{\mathbf{d}}^{1+\mu} |u(\mathbf{d}(-\ln \|\tilde{x}\|_{\mathbf{d}})\tilde{x}) - u_{\|\tilde{x}\|_{\mathbf{d}}^{\hat{h}}}(\mathbf{d}(-\ln \|\tilde{x}\|_{\mathbf{d}})\tilde{x})| \leq \\
& C_1 \|x\|_{\mathbf{d}}^{1+\mu} \|\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x - \mathbf{d}(-\ln \|\tilde{x}\|_{\mathbf{d}})\tilde{x}\| + \|\tilde{x}\|_{\mathbf{d}}^{1+\mu} C_0 \|\tilde{x}\|_{\mathbf{d}}^{\hat{h}} \hat{h} = \\
& C_1 \|x\|_{\mathbf{d}}^{1+\mu} \|\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x - \mathbf{d}(-\ln \|\tilde{x}\|_{\mathbf{d}})\tilde{x}\| + C_2 \|\tilde{x}\|_{\mathbf{d}}^{1+2\mu}
\end{aligned}$$

provided that  $\|\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x - \mathbf{d}(-\ln \|\tilde{x}\|_{\mathbf{d}})\tilde{x}\| \leq \delta$  and  $\|\tilde{x}\|_{\mathbf{d}}^{\hat{h}} \leq h_0/\hat{h}$ , where  $C_2 = C_0\hat{h}$ . Taking  $r \leq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{|\mu|} \min\left\{\frac{\delta}{C_*}, \frac{h_0}{\hat{h}}\right\}$  and  $C = (C_1C_* + C_2) \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1+2|\mu|}$  we complete the proof.

### 7.2 The proof of Corollary 1

Denoting  $y = \|x\|_{\mathbf{d}}\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$ , we derive  $\|y\| = \|x\|_{\mathbf{d}}\|\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x\| = \|x\|_{\mathbf{d}}$  and conclude that the closed-loop system (1), (11) is topologically equivalent (homeomorphic on  $\mathbb{R}^n$  and diffeomorphic on  $\mathbb{R}^n \setminus \{\mathbf{0}\}$ , see [44]) to the standard homogeneous system:

$$\dot{y} = \|y\|^{\mu}(A_0 + BK + \rho(G_{\mathbf{d}} - I_n))y,$$

where the identities  $\mathbf{d}(s)A_0 = e^{-\mu s}A_0\mathbf{d}(s)$  and  $\mathbf{d}(s)B = e^sB$ ,  $\forall s \in \mathbb{R}$  are utilized on the last step. In this case, using (10) we conclude

$$\frac{d}{dt}\|y(t)\| = -\rho\|y(t)\|^{\mu+1}$$

and  $\|y(t+\tau)\|^{-\mu} = \|y(t)\|^{-\mu} + \mu\rho\tau$ , for  $\|y(t)\|^{-\mu} + \mu\rho\tau \geq 0$ . Obviously,  $\|y(t+\tau)\| = 0$  if  $\|y(t)\|^{-\mu} + \mu\rho\tau \leq 0$ . The latter corresponds to the negative homogeneity degree  $\mu < 0$  and the finite-time stability of the closed-loop system. Hence, denoting  $\tilde{A} = A_0 + BK + \rho G_{\mathbf{d}}$  we obtain

$$\begin{aligned}
y(t+\tau) &= e^{(\tilde{A}-\rho I_n)\tau} \int_0^{\tau} \|y(t+\sigma)\|^{\mu} d\sigma y(t) \\
&= e^{(\tilde{A}-\rho I_n)\frac{1}{\mu\rho} \ln(1+\mu\rho\tau)} \|y(t)\|^{\mu} y(t).
\end{aligned}$$

Since  $\|y(t)\| = \|x(t)\|_{\mathbf{d}}$  then returning to the original coordinates we derive

$$x(t+\tau) = \mathbf{d}(\ln \|y(t+\tau)\|) \frac{y(t+\tau)}{\|y(t+\tau)\|} = Q_{\tau}(\|x(t)\|_{\mathbf{d}})x(t)$$

for all  $t \geq 0$  and all  $\tau \geq 0$ .

### 7.3 The proof of Corollary 2

Under Assumption 2, the model (1) is a system of interconnected subsystems with state vectors  $x_i \in \mathbb{R}^{n_i}$  and control inputs  $u_i \in \mathbb{R}$ . By the formula (7), we derive

$$\begin{aligned}
\frac{d}{dt}\|x_i\|_{\mathbf{d}_i} &= -\rho_i\|x_i\|_{\mathbf{d}_i}^{1+\mu_i} + \\
\|x_i\|_{\mathbf{d}_i} &\frac{x_i^{\top} \mathbf{d}_i^{\top} (-\ln \|x_i\|_{\mathbf{d}_i}) P_i \mathbf{d}_i (-\ln \|x_i\|_{\mathbf{d}_i}) \sum_{j=i+1}^m A_{ij} x_j}{x_i^{\top} \mathbf{d}_i^{\top} (-\ln \|x_i\|_{\mathbf{d}_i}) P_i G_{\mathbf{d}_i} \mathbf{d}_i (-\ln \|x_i\|_{\mathbf{d}_i}) x_i},
\end{aligned}$$

where  $A_{ij}$  are over diagonal blocks of the matrix  $A$ . By Theorem 3 the norm in  $\mathbb{R}^{n_i}$  is defined as  $\|z\| = \sqrt{z^{\top} P_i z}$ ,  $z \in \mathbb{R}^{n_i}$  and  $\|\mathbf{d}_i(-\ln \|x_i\|_{\mathbf{d}_i})x_i\| = 1$ . Taking into account  $P_i G_{\mathbf{d}_i} + G_{\mathbf{d}_i}^{\top} P_i \succ 0$  we derive

$$x_i^{\top} \mathbf{d}_i^{\top} (-\ln \|x_i\|_{\mathbf{d}_i}) P_i G_{\mathbf{d}_i} \mathbf{d}_i (-\ln \|x_i\|_{\mathbf{d}_i}) x_i \geq \beta_i > 0,$$

where  $\beta_i = 0.5\lambda_{\min}(P_i^{1/2}G_{\mathbf{d}_i}P_i^{-1/2} + P_i^{-1/2}G_{\mathbf{d}_i}^{\top}P_i^{1/2})$ . Applying the Cauchy-Schwartz inequality we obtain

$$\frac{x_i^{\top} \mathbf{d}_i^{\top} (-\ln \|x_i\|_{\mathbf{d}_i}) P_i \mathbf{d}_i (-\ln \|x_i\|_{\mathbf{d}_i}) \sum_{j=i+1}^m A_{ij} x_j}{x_i^{\top} \mathbf{d}_i^{\top} (-\ln \|x_i\|_{\mathbf{d}_i}) P_i G_{\mathbf{d}_i} \mathbf{d}_i (-\ln \|x_i\|_{\mathbf{d}_i}) x_i} \leq$$

$$\frac{1}{\beta_i} \|\mathbf{d}_i(-\ln \|x_i\|_{\mathbf{d}_i}) \sum_{j=i+1}^m A_{ij} x_j\| \leq \frac{\|\sum_{j=i+1}^m A_{ij} x_j\|}{\beta_i \sigma_i (\|x_i\|_{\mathbf{d}_i})},$$

where  $\sigma_i = \bar{\sigma}^{-1}$  and  $\bar{\sigma}$  is given in Theorem 1 with  $\alpha = \alpha_i = 0.5\lambda_{\max}(P_i^{1/2}G_{\mathbf{d}_i}P_i^{-1/2} + P_i^{-1/2}G_{\mathbf{d}_i}^{\top}P_i^{1/2})$  and  $\beta = \beta_i$ . The estimate obtained for  $\frac{d}{dt}\|x_i\|_{\mathbf{d}_i}$ , the stability of the  $m$ -subsystem and the cascade structure imply the forward completeness of the whole system.

If  $V_i(x_i) = \sigma_i(\|x_i\|_{\mathbf{d}_i})$  then for  $\|x_i\|_{\mathbf{d}_i} > 1$  we have

$$\frac{dV_i}{dt} = \beta_i \frac{V_i \cdot \frac{d}{dt}\|x_i\|_{\mathbf{d}_i}}{\|x_i\|_{\mathbf{d}_i}} \leq -\beta_i \rho_i V_i^{1+\frac{\mu_i}{\beta_i}} + \left\| \sum_{j=i+1}^m A_{ij} x_j \right\|.$$

For  $0 < \|x_i\|_{\mathbf{d}_i} < 1$  we derive

$$\frac{dV_i}{dt} = \alpha_i \frac{V_i \cdot \frac{d}{dt}\|x_i\|_{\mathbf{d}_i}}{\|x_i\|_{\mathbf{d}_i}} \leq -\alpha_i \rho_i V_i^{1+\frac{\mu_i}{\alpha_i}} + \frac{\alpha_i}{\beta_i} \left\| \sum_{j=i+1}^m A_{ij} x_j \right\|$$

Since  $V_i$  locally Lipschitz continuous on  $\mathbb{R}^{n_i} \setminus \{\mathbf{0}\}$  then using the Clarke's gradient for  $\|x_i\|_{\mathbf{d}_i} = 1$  we have

$$\begin{aligned}
\frac{dV_i}{dt} &\leq -\lambda_i(x_i) \alpha_i \rho_i V_i^{1+\frac{\mu_i}{\alpha_i}} - (1 - \lambda_i(x_i)) \beta_i \rho_i V_i^{1+\frac{\mu_i}{\beta_i}} \\
&\quad + \left( \lambda_i(x_i) \frac{\alpha_i}{\beta_i} + 1 - \lambda_i(x_i) \right) \left\| \sum_{j=i+1}^m A_{ij} x_j \right\|
\end{aligned}$$

with some  $\lambda(x_i) \in [0, 1]$ . The latter means that  $x_i \mapsto V_i(x_i)$  is an ISS Lyapunov function [56] of the  $i$ -th subsystem with respect to the input  $\sum_{j=i+1}^m A_{ij} x_j$  provided that  $\mu_i > -\beta_i$ . Since the  $m$ -th subsystem is globally uniformly asymptotically stable then using the cascade structure and the ISS property of each subsystem we conclude that the whole system is globally uniformly asymptotically stable as well. Moreover, the obtained estimate of  $\frac{d}{dt}\|x_i\|_{\mathbf{d}_i}$  implies that the finite-time, exponential and nearly fixed-time convergence rates are preserved as well dependently of the sign of  $\mu_i$  for all  $i = 1, \dots, m$ . If  $\mu_i \leq -\beta_i$ , then the each subsystem with the zero input is finite-time stable. Taking into account forward completeness and the cascade structure we conclude that the whole system is finite-time stable too.

### 7.4 The proof of Theorem 4

On the one hand, since  $x_{k+n} = Q_{nh}(\|x_k\|_{\mathbf{d}})x_k$  then, in the view of Corollary 1, the states of the discrete-time system (18) with the control (21) coincides with the states of the original continuous-time system (1), (11) at time instances  $t_{kn}, k = 0, 1, \dots$ . In this case, if the discrete-time system is globally Lyapunov stable then the finite-time or nearly fixed-time stability property of the original continuous-time system is preserved.

On the one hand, in the view of Theorem 3 and Corollary 1, we have  $\|Q_{\tau}(\|x\|_{\mathbf{d}})x\|_{\mathbf{d}}^{-\mu} = \|x\|_{\mathbf{d}}^{-\mu} + \mu\rho\tau$  for all  $x \in \mathbb{R}^n : \|x\|_{\mathbf{d}}^{-\mu} + \mu\rho\tau \geq 0$  and  $\forall \tau > 0$ , and  $Q_{\tau}(\|x\|_{\mathbf{d}})x = \mathbf{0}$  otherwise. Hence, we conclude

$\|Q_\tau(\|x\|_{\mathbf{d}})x\|_{\mathbf{d}} = (\|x\|_{\mathbf{d}}^{-\mu} + \mu\rho\tau)^{-1/\mu} \leq \|x\|_{\mathbf{d}} \leq \bar{\sigma}(\|x\|)$ , where  $\bar{\sigma} \in \mathcal{K}_\infty$  is defined in Theorem 1. In this case, there exists  $\xi \in \mathcal{K}_\infty$  such that  $|u(t_{k+j})| \leq \xi(\|x_{k+j}\|)$ ,  $j = 0, 1, \dots, n-1$  and there exists  $\sigma_1 \in \mathcal{K}_\infty$  such that  $\|x_{k+j+1}\|_{\mathbf{d}} \leq \sigma_1(\|x_{k+j}\|_{\mathbf{d}})$ ,  $j = 0, 1, \dots, n-1$ . Consequently, there exists  $\sigma_n \in \mathcal{K}_\infty$  such that  $\|x_{k+n}\|_{\mathbf{d}} \leq \sigma_n(\|x_k\|_{\mathbf{d}}), \forall k \in \mathbb{N}$ . On the other hand, by construction, we have  $x_{k+n} = Q_{nh}(\|x_k\|_{\mathbf{d}})x_k$ , so, taking into account  $\|Q_{nh}(\|x_k\|_{\mathbf{d}})x_k\|_{\mathbf{d}} \leq \|x_k\|_{\mathbf{d}}$ , we derive  $\|x_{k+n}\|_{\mathbf{d}} \leq \min\{\|x_k\|_{\mathbf{d}}, \sigma_n(\|x_k\|_{\mathbf{d}})\}$  and  $\|x_k\|_{\mathbf{d}} \leq \min\{\|x_0\|_{\mathbf{d}}, \sigma_n(\|x_0\|_{\mathbf{d}})\}, \forall k \geq 0$ . Using Theorem 1 we substantiated the global Lyapunov stability of the system (18) (as well as the system (1)) with the sampled-time control (21).

### 7.5 The proof of Proposition 1

First of all, notice that under Assumption 1 we have  $K_0 = \mathbf{0}$  and  $A_0 = A$  (see Theorem 3). Let us denote  $s_h = \ln(1 + \mu\rho nh r^\mu)^{1/(\rho\mu)}$  with  $r = \|x\|_{\mathbf{d}}$  and show  $\tilde{K}_h(r) \rightarrow r^{1+\mu}K\mathbf{d}(-\ln r)$  as  $h \rightarrow 0$ .

On the one hand, if  $\mathbf{d}_*(s)$  is defined by Lemma 5 then

$$\begin{aligned} \mathbf{d}_*(-\ln h)e^{(A+BK+\rho G_{\mathbf{d}})s_h} &= \sum_{i=0}^{\infty} \frac{s_h^i \mathbf{d}_*(-\ln h)(A+BK+\rho G_{\mathbf{d}})^i}{i!} = \\ &= \sum_{i=0}^{\infty} \frac{s_h^i (A+BK\mathbf{d}_*(\ln h) + \rho h G_{\mathbf{d}})^i \mathbf{d}_*(-\ln h)}{h^i i!} = \\ &= e^{\frac{s_h(A+\rho h G_{\mathbf{d}})}{h}} \mathbf{d}_*(-\ln h) + \sum_{i=1}^{n-1} \frac{s_h^i}{h^i i!} A^{i-1} BK + O(h). \end{aligned}$$

Indeed, for  $i = 2$  we have  $(A + BK\mathbf{d}_*(\ln h) + \rho h G_{\mathbf{d}})^2 \mathbf{d}_*(-\ln h) = (A + \rho h G_{\mathbf{d}})^2 \mathbf{d}_*(-\ln h) + ABK + O(h)$ , and, by induction, we conclude  $(A + BK\mathbf{d}_*(\ln h) + \rho h G_{\mathbf{d}})^i \mathbf{d}_*(-\ln h) = (A + \rho h G_{\mathbf{d}})^i \mathbf{d}_*(-\ln h) + A^{i-1} BK + O(h)$ . Since  $A$  is nilpotent then for  $i \geq n+1$  we have  $A^{i-1} = \mathbf{0}$  and  $(A + BK\mathbf{d}_*(\ln h) + \rho h G_{\mathbf{d}})^i \mathbf{d}_*(-\ln h) = (A + \rho h G_{\mathbf{d}})^i \mathbf{d}_*(-\ln h) + O(h)$ .

On the other hand, since  $\mathbf{d}(-s)A\mathbf{d}(s) = e^{\mu s}A$  then  $\mathbf{d}(-s)e^{\tau A}\mathbf{d}(s) = e^{e^{\mu\tau}A}$  for all  $s, \tau \in \mathbb{R}$  and  $Q_{nh}(r) - A_h^n = \mathbf{d}(\ln r)\hat{Q}(s_h)\mathbf{d}(-\ln r) - e^{nhA} = \mathbf{d}(\ln r)(\hat{Q}(s_h) - e^{nh r^\mu A})\mathbf{d}(-\ln r)$  where  $\hat{Q}(s_h)$  is given by (16). Since  $\mathbf{d}^*(s)$  commutes with  $\mathbf{d}(\tau)$ ,  $\forall s, \tau \in \mathbb{R}$  then using the identities (42) and the estimate of  $\mathbf{d}_*(-\ln h)e^{(A+BK+\rho G_{\mathbf{d}})s_h}$  we derive

$$\begin{aligned} \mathbf{d}_*(-\ln h)(e^{(A+BK+\rho G_{\mathbf{d}})s_h} - e^{\rho s_h G_{\mathbf{d}}} e^{nh r^\mu A}) &= \\ \mathbf{d}_*(-\ln h)(e^{(A+BK+\rho G_{\mathbf{d}})s_h} - e^{\rho s_h G_{\mathbf{d}}} e^{nr^\mu A}) \mathbf{d}_*(-\ln h) &= \\ \sum_{i=0}^{n-1} \frac{s_h^i A^{i-1} BK}{h^i i!} + \left( e^{\frac{s_h(A+\rho h G_{\mathbf{d}})}{h}} - e^{\rho s_h G_{\mathbf{d}}} e^{nr^\mu A} \right) \mathbf{d}_*(-\ln h) &+ O(h). \end{aligned}$$

If  $Z_1 = \rho s_h G_{\mathbf{d}}$ ,  $Z_2 = nr^\mu A$ ,  $q = -\mu\rho s_h$  then the condition  $Z_1 Z_2 - Z_2 Z_1 = q Z_2$  of Lemma 4 is fulfilled, so  $e^{Z_1} e^{Z_2} = e^{Z_1 + \frac{q}{1-e^{-q}} Z_2}$  or, equivalently,  $e^{\rho s_h G_{\mathbf{d}}} e^{nr^\mu A} = e^{\rho s_h G_{\mathbf{d}} + \frac{s_h}{h} A}$ . Therefore, since  $s_h = \frac{\ln(1+\mu\rho nh r^\mu)}{\rho\mu} = \frac{\mu\rho nh r^\mu - \frac{(\mu\rho 2hr^\mu)^2}{2} + O(h^3)}{\rho\mu} = hnr^\mu + O(h^2)$

then  $\mathbf{d}_*(-\ln h)(e^{(A+BK+\rho G_{\mathbf{d}})s_h} - e^{\rho s_h G_{\mathbf{d}}} e^{nh r^\mu A}) = \sum_{i=0}^{n-1} \frac{s_h^i A^{i-1} BK}{h^i i!} + O(h) = \sum_{i=0}^{n-1} \frac{(nr^\mu)^i A^{i-1} BK}{i!} + O(h) = \int_0^{nr^\mu} e^{\tau A} d\tau BK + O(h)$ . Since  $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$  then

$$\begin{aligned} \mathbf{d}_*(-\ln h)(Q_{nh}(r) - A_h^n) &= \\ \mathbf{d}(\ln r)e^{-\rho s_h G_{\mathbf{d}}} \int_0^{nr^\mu} e^{\tau A} d\tau BK \mathbf{d}(-\ln r) + O(h) &= \\ \int_0^{nr^\mu} \mathbf{d}(\ln r - \rho s_h) e^{\tau A} d\tau BK \mathbf{d}(-\ln r) + O(h) &= \\ \int_0^{nr^\mu} e^{-\mu(\ln r - \rho s_h)\tau A} d\tau \mathbf{d}(\ln r - \rho s_h) BK \mathbf{d}(-\ln r) + O(h) &= \\ \int_0^{nr^\mu} e^{\frac{1+\mu\rho nh r^\mu}{r^\mu}\tau A} d\tau \frac{r}{(1+\mu\rho nh r^\mu)^{1/\mu}} BK \mathbf{d}(-\ln r) + O(h) &= \\ \int_0^{nr^\mu} e^{\frac{1}{r^\mu}\tau A} d\tau BrK \mathbf{d}(-\ln r) + O(h) &= \\ \int_0^n e^{\tau A} d\tau Br^{1+\mu} K \mathbf{d}(-\ln r) + O(h). \end{aligned}$$

Therefore, using Lemma 5 and Remark 2 we conclude

$$\begin{aligned} \tilde{K}_h(r) &= e_n^\top W_n^{-1}(Q_{nh}(s_h) - A_h^n) = \\ e_n^\top W_n^{-1} \mathbf{d}_*(\ln h) \left( \int_0^n e^{\tau A} d\tau Br^{1+\mu} K \mathbf{d}(-\ln r) + O(h) \right) &= \\ r^{1+\mu} K \mathbf{d}(-\ln r) + O(h^2), \end{aligned}$$

and  $\tilde{K}_h(r) \rightarrow r^{1+\mu} K \mathbf{d}(-\ln r)$  as  $h \rightarrow 0$  uniformly on  $r$  from compacts belonging to  $(0, +\infty)$ .

### 7.6 The proof of Lemma 1

Let us show that (27) holds. Indeed, on the one hand, since  $\|\mathbf{d}(s)x\|_{\mathbf{d}} = e^s \|x\|_{\mathbf{d}}$  then

$$\hat{Q}\left(\frac{\ln(1+\mu\rho nh e^{-\mu s} \|\mathbf{d}(s)x\|_{\mathbf{d}}^\mu)}{\rho\mu}\right) = \hat{Q}\left(\frac{\ln(1+\mu\rho nh \|x\|_{\mathbf{d}}^\mu)}{\rho\mu}\right)$$

and  $\forall s \in \mathbb{R}, \forall x \in \mathbb{R}^n$  we have

$$Q_{nh e^{-\mu s}}(\|\mathbf{d}(s)x\|_{\mathbf{d}}) = \mathbf{d}(s)Q_{nh}(\|x\|_{\mathbf{d}})\mathbf{d}(-s).$$

On the other hand, using Lemma 5 we derive

$$W_{e^{-\mu s} h} = \mathbf{d}_*(-\mu s)W_h, \quad B_{e^{-\mu s} h} = e^{-\mu s - s} \mathbf{d}(s)B_h,$$

$L_{e^{-\mu s} h} \mathbf{d}(s) = \mathbf{d}(s)L_h$ ,  $\mathbf{d}(s)F_h = F_{e^{-\mu s} h} \mathbf{d}(s)$ , for all  $s \in \mathbb{R}$  and for all  $h > 0$ , where the identities  $\mathbf{d}_*(\tau) = e^{\tau I_n + \tau G_0}$  and  $\mathbf{d}(s) = e^{s I_n - s \mu G_0}$  are utilized for the analysis  $L_h$  in order to conclude that  $e^{-s - \mu s} \mathbf{d}_*(\mu s) \mathbf{d}(s) = I_n$ , where  $G_0$  is defined in Theorem 3. Hence, we derive (27). The identity (28) can be obtained in the same way.

### 7.7 The proof of Lemma 2

**1) Local Finite-Time Stability for  $\mu < 0$ .** Let us show that the matrix  $F_h$  is nilpotent. Notice that  $F_h$  can be rewritten as follows

$$\begin{aligned} F_h &= (I_n - [\mathbf{0} \ \mathbf{0} \ \dots \ \mathbf{0} \ B_h] W_h^{-1} A_h^{n-1}) A_h = \\ (A_h^{1-n} W_h W_h^{-1} A_h^{n-1} - [\mathbf{0} \ \mathbf{0} \ \dots \ \mathbf{0} \ B_h] W_h^{-1} A_h^{n-1}) A_h &= \\ A_h^{1-n} (W_h - [\mathbf{0} \ \mathbf{0} \ \dots \ \mathbf{0} \ A_h^{n-1} B_h]) W_h^{-1} A_h^n &= \\ A_h^{1-n} [B_h \ A_h B_h \ \dots \ A_h^{n-2} B_h \ \mathbf{0}] W_h^{-1} A_h^n \end{aligned}$$

Since  $W_h^{-1}[A_h B_h \dots A_h^{n-1} B_h \mathbf{0}] = [e_2 \dots e_n \mathbf{0}]$  then

$$F_h^2 = A_h^{1-n} [B_h A_h B_h \dots A_h^{n-2} B_h \mathbf{0}] [e_2 e_3 \dots e_n \mathbf{0}] W_h^{-1} A_h^n = A_h^{1-n} [A_h B_h \dots A_h^{n-2} B_h \mathbf{0} \mathbf{0}] W_h^{-1} A_h^n.$$

Continuing the same considerations we derive  $F_h^n = \mathbf{0}$ . On the one hand, since for  $\mu < 0$  we have  $Q(\|x\|_{\mathbf{d}}) = \mathbf{0}$  if  $\|x\|_{\mathbf{d}} \leq (-\mu\rho h)^{-1/\mu}$  then the closed-loop system becomes linear  $x_{k+1} = F_h x_k$  for  $\|x_k\|_{\mathbf{d}} \leq (-\mu\rho h)^{-1/\mu} = (\hat{h}/h)^{1/\mu}$ . On the other hand, the inequality  $\|F_h^i x_0\|_{\mathbf{d}} \leq (\hat{h}/h)^{1/\mu}$  is equivalent to  $\|\mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) F_h^i x_0\| = \|F_h^i \mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) x_0\| \leq 1$ , and the inequality  $\|x_0\|_{\mathbf{d}} \leq \underline{r}^- (\hat{h}/h)^{1/\mu}$  is equivalent to  $\|\mathbf{d}(-\ln \underline{r}^-) \mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) x_0\| \leq 1$ . Therefore, the inequality  $\|F_h^i \mathbf{d}(\ln \underline{r}^-)\| \leq 1$  yields  $\|F_h^i \mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) x_0\| \leq \|\mathbf{d}(-\ln \underline{r}^-) \mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) x_0\|$ . Hence, we derive  $\|x_i\|_{\mathbf{d}} = \|F_h^i x_0\|_{\mathbf{d}} \leq (\hat{h}/h)^{1/\mu}$  for  $i = 1, \dots, n-1$  provided that  $\|x_0\|_{\mathbf{d}} \leq \underline{r}^- (\hat{h}/h)^{1/\mu}$ . Taking into account the nilpotence of  $F_h$ , the latter implies local Lyapunov stability of the closed-loop system and the finite-time convergence of solutions to zero.

**2) Practical Finite-time Stability for  $\mu < 0$ .** The proof repeats the proof of Theorem 7, the case 1) for  $q = \mathbf{0}$  and gives  $\frac{d\|x\|_{\mathbf{d}}}{dt} \leq -0.5\rho\|x\|_{\mathbf{d}}^{1+\mu}$  for all  $x : \|x\|_{\mathbf{d}}^{-\mu} \geq \tilde{r}$ . Using Lemma 1 we derive  $\Omega^-$  for  $h \neq \hat{h}$  with  $\tilde{r}^- = \tilde{r}^{-1/\mu}$ , where  $\tilde{r}$  is defined in the proof of Theorem 7.

**3) Practical Fixed-time Stability for  $\mu > 0$**  Let us prove, now, the practical fixed-time stability. On the one hand, since, by Theorem 3 the canonical homogeneous norm is a Lyapunov function of the system satisfying

$$\frac{d}{dt} \|x(t)\|_{\mathbf{d}} = -\rho \|x(t)\|_{\mathbf{d}}^{1+\mu},$$

then  $\|x(t + nh)\|_{\mathbf{d}}^{-\mu} = \|x(t)\|_{\mathbf{d}}^{-\mu} + \mu\rho nh$ , and for  $\mu > 0$  we have  $\|x(t + nh)\|_{\mathbf{d}} < (\mu\rho nh)^{-1/\mu} = (\frac{\hat{h}}{h})^{1/\mu}$  independently of  $x(t)$ . On the other hand, by Corollary 1, we have  $x(t + nh) = Q_{nh}(\|x(t)\|_{\mathbf{d}})x(t)$ , so  $\|Q_{nh}(\|\tilde{x}\|_{\mathbf{d}})\tilde{x}\|_{\mathbf{d}} \leq (\frac{\hat{h}}{h})^{1/\mu}$ ,  $\forall \tilde{x} \in \mathbb{R}^n$ . Since the right-hand side of the system can be represented as follows

$$z_h(x) = F_h x + L_h Q_{nh}(\|x\|_{\mathbf{d}})x, \quad L_h := B_h e_n^\top W_h^{-1},$$

then, for any  $x_0 \in \mathbb{R}^n$  the solution  $x_k, k = 0, 1, 2, \dots$  of the discrete-time system (18), (22) satisfies

$$x_1 = F_h x_0 + L_h y_1,$$

$$x_2 = F_h^2 x_0 + F_h L_h y_1 + L_h y_2,$$

...

$$x_n = F_h^n x_0 + F_h^{n-1} L_h y_1 + F_h^{n-2} L_h y_2 + \dots + L_h y_n,$$

where  $y_i = Q_{nh}(\|x_{i-1}\|_{\mathbf{d}})x_{i-1}$ ,  $i = 1, 2, \dots$ . Since the matrix  $F_h$  is nilpotent then  $F_h^n = \mathbf{0}$  and

$$x_k = F_h^{n-1} L_h y_{k-n+1} + F_h^{n-2} L_h y_{k-n+2} + \dots + L_h y_k, \quad \forall k \geq n.$$

Since  $\|y_i\|_{\mathbf{d}} \leq (\frac{\hat{h}}{h})^{1/\mu} \Leftrightarrow \|\mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) y_i\|_{\mathbf{d}} \leq 1 \Leftrightarrow \|\mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) y_i\| \leq 1$  then

$$\|\mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) x_k\|_{\mathbf{d}} = \|\sum_{i=1}^n \mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) F_h^{i-1} L_h \mathbf{d}(-\frac{1}{\mu} \ln \frac{\hat{h}}{h}) v_i\|_{\mathbf{d}},$$

where  $v_i = \mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) y_{k+n-i}$ . Taking into account  $F_{\hat{h}} = \mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) F_h \mathbf{d}(-\frac{1}{\mu} \ln \frac{\hat{h}}{h})$ ,  $L_{\hat{h}} = \mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) L_h \mathbf{d}(-\frac{1}{\mu} \ln \frac{\hat{h}}{h})$  and  $\|v_i\| \leq 1$  we derive  $\|\mathbf{d}(\frac{1}{\mu} \ln \frac{\hat{h}}{h}) x_k\|_{\mathbf{d}} < \tilde{r}^+, \forall k \geq n$ .

**4) Local Asymptotic Stability for  $\mu > 0$ .** Since  $\frac{d\|x\|_{\mathbf{d}}}{dt} \leq -0.5\rho\|x\|_{\mathbf{d}}^{1+\mu}$  for all  $x : \|x\|_{\mathbf{d}}^{-\mu} \leq \tilde{r}^{-1}$  then for  $\mu > 0$  the closed-loop system is locally asymptotically stable. Using Lemma 1 we derive  $\Omega^+$  with  $\tilde{r}^+ = \tilde{r}^{-1/\mu}$ .

### 7.8 The proof of Lemma 3

The symmetry proven by Lemma 1 yields  $M_h(\|x\|_{\mathbf{d}})x = \mathbf{d}(s)M_{e^{\mu s}h}(\|\mathbf{d}(-s)x\|_{\mathbf{d}})\mathbf{d}(-s)x$  for all  $x \in \mathbb{R}^n$ , for all  $s \in \mathbb{R}$  and  $\forall h > 0$ . Hence, for any  $x_0 \in \mathbb{R}^n \setminus \{\mathbf{0}\}$  we have  $x_1 = M_h(\|x_0\|_{\mathbf{d}})x_0 = \mathbf{d}(\ln \|x_0\|_{\mathbf{d}})\Theta_1(\|x_0\|_{\mathbf{d}}^\mu \hat{h}, v_0)v_0$ . Since  $\|x_1\|_{\mathbf{d}} = \|x_0\|_{\mathbf{d}} \|\Theta_1(\|x_0\|_{\mathbf{d}}^\mu, v_0)v_0\|_{\mathbf{d}}$  then  $x_2 = M_h(\|x_1\|_{\mathbf{d}})x_1 = M_h(\|x_1\|_{\mathbf{d}})\mathbf{d}(\ln \|x_1\|_{\mathbf{d}})\Theta_1(\|x_1\|_{\mathbf{d}}^\mu \hat{h}, v_0)v_0 = \mathbf{d}(\ln \|x_0\|_{\mathbf{d}})M_{\|x_0\|_{\mathbf{d}}^\mu \hat{h}}(\|x_1\|_{\mathbf{d}}/\|x_0\|_{\mathbf{d}})\Theta_1(\|x_0\|_{\mathbf{d}}^\mu, v_0)v_0 = \mathbf{d}(\ln \|x_0\|_{\mathbf{d}})\Theta_2(\|x_0\|_{\mathbf{d}}^\mu, v_0)v_0$ . Repeating the above considerations we derive (31).

### 7.9 The proof of Theorem 5

The approximation property is proven by Proposition 1. Let us prove the consistency of stability properties. If the discrete-time system (26) is globally uniformly finite-time stable for some  $h > 0$  then due to dilation symmetry (see Lemma 1) it is globally uniformly finite-time stable for any  $h > 0$ , in particular, for  $h = \hat{h}$ .

*Necessity.* Let us consider **the case  $\mu < 0$** . The uniformity of the finite-time stability and Lemma 2 guarantee that there exists  $k^* \geq 1$  such that for any  $x_0 : \|x_0\|_{\mathbf{d}} \geq \underline{r}^-$  we have  $\|x_{k^*}\|_{\mathbf{d}} < \|x_0\|_{\mathbf{d}}$ . The latter means that  $\|x_0\|_{\mathbf{d}} \|\Theta_{k^*}(\|x_0\|_{\mathbf{d}}^\mu, v_0)v_0\|_{\mathbf{d}} < \|x_0\|_{\mathbf{d}}$  and

$$\|\Theta_{k^*}(\underline{r}^\mu, v_0)v_0\|_{\mathbf{d}} < 1, \forall \underline{r} \geq \underline{r}^-, \forall v_0 \in S.$$

Denoting  $\delta = \underline{r}^\mu$  for  $\mu < 0$  we derive the inequality (32). **The case  $\mu > 0$**  can be treated similarly.

*Sufficiency.* Let us denote  $r_* = (\bar{r}^-)^\mu$  for  $\mu < 0$  and  $r_* = (\bar{r}^+)^\mu$  for  $\mu > 0$ . Let us consider the candidate Lyapunov function  $V : \mathbb{R}^n \mapsto \mathbb{R}_+$  defined as follows

$$V(x) = \begin{cases} \|\mathbf{d}(-\mu^{-1} \ln r_*)x\|_{\mathbf{d}}^p & \text{if } \|x\|_{\mathbf{d}}^\mu \leq r_*, \\ \|\mathbf{d}(-\mu^{-1} \ln r_*)x\| & \text{if } \|x\|_{\mathbf{d}}^\mu \geq r_*, \end{cases} \quad (47)$$

where  $p = \beta$  if  $\mu < 0$  and  $p = \alpha$  if  $\mu > 0$ , where  $\alpha, \beta$  are given by Theorem 1. By construction,  $V$  is positive definite, radially unbounded and globally Lipschitz continuous with the Lipschitz constant  $L = \|\mathbf{d}(-\mu^{-1} \ln r_*)\|$  due to Lemma 6.

a) Let us show that  $V$  is a Lyapunov function for  $\|x_0\|_{\mathbf{d}}^\mu \in [r_*, r^*]$ . Since  $(\delta, v) \mapsto \|\Theta_{k^*}(\delta, v)v\|_{\mathbf{d}}$  is a continuous function on the compact  $[r_*, r^*] \times S$  then using (32) we derive  $\gamma = \max_{\delta \in [r_*, r^*]} \|\Theta_{k^*}(\delta, v)v\|_{\mathbf{d}} < 1$  and for  $\|x_0\|_{\mathbf{d}}^\mu \in [r_*, r^*]$  we have  $\|x_{k^*}\|_{\mathbf{d}} \leq \gamma \|x_0\|_{\mathbf{d}}$ . Since  $\|x_{k^*}\|_{\mathbf{d}} \leq \gamma \|x_0\|_{\mathbf{d}} \Leftrightarrow \|\mathbf{d}(s)x_{k^*}\|_{\mathbf{d}} \leq \gamma \|\mathbf{d}(s)x_0\|_{\mathbf{d}}, \forall s \in \mathbb{R}$  then  $V(x_{k^*}) \leq \gamma^p V(x_0)$  for  $\|x_0\|_{\mathbf{d}}^\mu \in [r_*, r^*]$ .

b) Let us show that  $V$  is a Lyapunov function for  $\|x_0\|_{\mathbf{d}}^\mu \leq r_*$ . Since  $\frac{d\|x\|_{\mathbf{d}}^\mu}{dt} \leq -0.5\rho\|x\|_{\mathbf{d}}^{1+\mu}$  for all  $x : \|x\|_{\mathbf{d}}^\mu \leq r_*$  (see the proof of Lemma 2, case 2), then  $\|x_1\|_{\mathbf{d}} \leq \gamma_*(\|x_0\|_{\mathbf{d}}^\mu)\|x_0\|_{\mathbf{d}}, \forall x_0 : \|x_0\|_{\mathbf{d}}^\mu \leq r_*$ , or equivalently,  $V(x_1) \leq \gamma_*^p(\|x_0\|_{\mathbf{d}}^\mu)V(x_0), \forall x_0 : \|x_0\|_{\mathbf{d}}^\mu \leq r_*$ , where  $\gamma_*(s) = (1 + 0.5\rho\mu s)^{-1/\mu}$  with  $s > 0$ .

c) Let us show that  $V$  is a Lyapunov function for  $\|x_0\|_{\mathbf{d}}^\mu \geq r^*$ . If  $\|x_0\|_{\mathbf{d}}^\mu > r^*$  then for all  $k \geq n$  we have  $\|x_k\| = 0$  if  $\mu < 0$  and  $\|\mathbf{d}(-\mu^{-1} \ln r^*)x_k\| < 1$  if  $\mu > 0$  (see the proof of Lemma 2). This means that there exist  $\gamma^* \in (0, 1)$  such that  $V(x_k) \leq \gamma^* V(x_0), \|x_0\|_{\mathbf{d}}^\mu > r^*$ . Without loss of generality we may assume that  $k^* \geq n$  (otherwise we just take  $nk^*$  instead of  $k^*$  in all above considerations).

d) Therefore, for any  $\bar{r}_* \in (0, r_*)$  and for any finite  $\bar{r}^* > r_*$  there exists  $\bar{\gamma} \in (0, 1)$  such that

$$V(x_{k^*}) \leq \bar{\gamma} V(x_0), \quad \forall x_0 : \bar{r}_* \leq \|x_0\|_{\mathbf{d}}^\mu \leq \bar{r}^*.$$

Taking into account local finite-time (resp., asymptotic) stability and practical finite-time (resp., fixed-time) stability proven by Lemma 2 for  $\mu < 0$  (resp.,  $\mu > 0$ ) we complete the proof.

### 7.10 The proof of Theorem 6

In a discrete time, the system (36) can be rewritten as

$$x_{k+n}^q = Q_{nh}(\|x_k^q\|_{\mathbf{d}})x_k^q + \sum_{i=0}^{n-1} A_h^{n-i} \tilde{q}_{k+i}$$

where  $\tilde{q}_k = \int_0^h e^{A(h-\tau)} q_p(t_k + \tau) d\tau$  is the sampled-time realization of the external perturbation, so  $\{\tilde{q}_k\} \in \ell^\infty$  for any  $h > 0$ . Let  $V$  be defined as in Lemma 6. Since  $\|Q_{nh}(\|x\|_{\mathbf{d}})x\|_{\mathbf{d}}^{-\mu} = \|x\|_{\mathbf{d}}^{-\mu} + \mu\rho nh$  for all  $x \in \mathbb{R}^n$ :

$\|x\|_{\mathbf{d}}^{-\mu} + \mu\rho nh \geq 0$  then  $\|x_{k+n}^q - \bar{q}_k\|_{\mathbf{d}}^{-\mu} = \|x_k^q\|_{\mathbf{d}}^{-\mu} + \mu\rho nh$ , where  $\bar{q}_k = \sum_{i=0}^{n-1} A_h^{n-i} \tilde{q}_{k+i}$ . Moreover, since  $\bar{q}_k$  is uniformly bounded, then for  $\mu > 0$  it guarantees a practical fixed-time stability. In this case, we derive

$$\begin{aligned} V(x_{k+n}^q) - V(x_k^q) &= V(x_{k+n}^q) - V(x_{k+n}^q - \bar{q}_k) + V(x_{k+n}^q - \bar{q}_k) - V(x_k^q) \\ &\leq \|\bar{q}_k\| + V(x_{k+n}^q - \bar{q}_k) - V(x_k^q) = \|\bar{q}_k\| + \bar{\sigma}^{-1}(\|x_k^q - \bar{q}_k\|_{\mathbf{d}}) - V(x_k^q) \\ &= \|\bar{q}_k\| + W(V(x_k^q)), \end{aligned}$$

where  $W(V) = \bar{\sigma}^{-1}(\bar{\sigma}(V)^{-\mu} + \mu\rho nh)^{-1/\mu} - V$ . For  $\mu > -\beta$  we have  $W \in \mathcal{K}_\infty$  and  $V$  is an ISS Lyapunov function [22].

### 7.11 The proof of Theorem 7

In a discrete time, the system (37) can be rewritten as

$$x_{k+1}^q = A_h x_k^q + B_h \tilde{K}_h(\|x_k^q + \hat{q}_k\|)(x_k^q + \hat{q}_k) + \tilde{q}_k, \quad (48)$$

where  $\{\hat{q}_k\}, \{\tilde{q}_k\} \in \ell^\infty$  for any  $h > 0$ ,  $\hat{q}_k = q_m(t_k)$  is the sampled-time realization of the measurement noise and  $\tilde{q}_k = \int_0^h e^{A(h-\tau)} q_p(t_k + \tau) d\tau$  is the sampled-time realization of the external perturbation. Denote  $q_k = (\tilde{q}_k^\top, \hat{q}_k^\top)$ . Due to the dilation symmetry proven by Lemma 1 it is sufficient to analyze ISS of (48) for  $h = \hat{h}$ .

1) Let us prove local ISS and practical ISS of (37). If  $x = e^{A h t} x_k^q + B_{h t} \tilde{u}_{\hat{h}}(x_k^q + \hat{q}_k) + \tilde{q}_{h t}$ ,  $h t = t - t_k$  and  $\tilde{q}_{h t} = \int_0^{h t} e^{A(h t - \tau)} q_p(t_k + \tau) d\tau$  then  $x$  corresponds to a solution of the system (37) for  $t \in [t_k, t_k + \hat{h})$ . Let us denote  $q_k^1 = \mathbf{d}(-\ln \|x_k^q\|_{\mathbf{d}}) \hat{q}_k$ ,  $q_k^2 = \mathbf{d}(-\ln \|x_k^q\|_{\mathbf{d}}) \tilde{q}_{h t}$ .

a) Let us show that  $\|x_k^q\|_{\mathbf{d}}$  is close to  $\|x\|_{\mathbf{d}}$  for a sufficiently large  $\|x_k^q\|_{\mathbf{d}}^{-\mu}$  and sufficiently small  $q_k^i, i = 1, 2$ . Using dilation symmetry (see (28)) we derive  $x = \mathbf{d}(\ln \|x_k^q\|_{\mathbf{d}}) (e^{A h t} \|x_k^q\|_{\mathbf{d}}^\mu v_k + B_{h t} \|x_k^q\|_{\mathbf{d}}^\mu \tilde{u}_{\hat{h}} \|x_k^q\|_{\mathbf{d}}^\mu (v_k + q_k^1) + q_k^2)$ , with  $v_k = \mathbf{d}(-\ln \|x_k^q\|_{\mathbf{d}}) x_k^q \in S$ . Since  $e^{A h t s} \rightarrow I_n$  as  $s \rightarrow 0$  and  $B_{h t s} \rightarrow \mathbf{0}$  as  $s \rightarrow 0$  then for any  $\varepsilon \in (0, 1)$  there exist  $r_\varepsilon > 0$  and  $\delta_\varepsilon > 0$  such that

$$\left\| \left\| e^{A h t} \|x_k^q\|_{\mathbf{d}}^\mu v_k + B_{h t} \|x_k^q\|_{\mathbf{d}}^\mu \tilde{u}_{\hat{h}} (v_k + q_k^1) + q_k^2 \right\|_{\mathbf{d}} - 1 \right\| \leq \varepsilon$$

for  $\|x_k^q\|_{\mathbf{d}}^{-\mu} > r_\varepsilon$  and  $\|q_k^i\| \leq \delta_\varepsilon, i = 1, 2$ . Hence, we have  $(1 - \varepsilon) \|x_k^q\|_{\mathbf{d}} \leq \|x\|_{\mathbf{d}} \leq (1 + \varepsilon) \|x_k^q\|_{\mathbf{d}}$  and  $\exists \tilde{C}_0, \tilde{C}_1 > 0 : \|\mathbf{d}(-\ln \|x\|_{\mathbf{d}}) \hat{q}_k\| = \left\| \mathbf{d} \left( -\ln \frac{\|x\|_{\mathbf{d}}}{\|x_k^q\|_{\mathbf{d}}} \right) q_k^1 \right\| \leq \tilde{C}_0 \|q_k^1\|$  and  $\|\mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x_k^q\| < \tilde{C}_1$  for  $\|x_k^q\|_{\mathbf{d}}^{-\mu} > r_\varepsilon$  and  $\|q_k^i\| \leq \delta_\varepsilon$ .

b) Let us show that  $u_h(x_k^q + \hat{q}_k)$  is close to  $u(x)$  for a sufficiently large  $\|x_k^q\|_{\mathbf{d}}^{-\mu}$  and sufficiently small  $\|q_k^i\|$ . Using Lemma 5 and the identity  $x_k^q = e^{-A h t} x - e^{-A h t} B_{h t} \tilde{u}_{\hat{h}}(x_k^q + \hat{q}_k) - e^{-A h t} \tilde{q}_{h t}$  we derive

$$\begin{aligned}
& \mathbf{d}(-\ln \|x\|_{\mathbf{a}})x - \mathbf{d}(-\ln \|x\|_{\mathbf{a}})(x_k^q + \hat{q}_k) = \\
& \mathbf{d}(-\ln \|x\|_{\mathbf{a}})(I_n - e^{-A_{h_t}})x + \mathbf{d}(-\ln \|x\|_{\mathbf{a}})e^{-A_{h_t}}B_{h_t}\tilde{u}_{\hat{h}}(x_k^q + \hat{q}_k) \\
& + \mathbf{d}(-\ln \|x\|_{\mathbf{a}})e^{-A_{h_t}}\tilde{q}_{h_t} + \mathbf{d}(-\ln \|x\|_{\mathbf{a}})\hat{q}_k = \\
& -\mathbf{d}(-\ln \|x\|_{\mathbf{a}})\int_0^{h_t} A e^{A\tau} d\tau x + e^{-A_{h_t}\|x\|_{\mathbf{a}}^{\mu}} \mathbf{d}(-\ln \|x\|_{\mathbf{a}})B_{h_t}\tilde{u}_{\hat{h}}(x_k^q + \hat{q}_k) \\
& + e^{-A_{h_t}\|x\|_{\mathbf{a}}^{\mu}} \mathbf{d}(-\ln \|x\|_{\mathbf{a}})\tilde{q}_{h_t} + \mathbf{d}(-\ln \|x\|_{\mathbf{a}})\hat{q}_k = \\
& - \int_0^{h_t}\|x\|_{\mathbf{a}}^{\mu} A e^{A\tau} d\tau \mathbf{d}(-\ln \|x\|_{\mathbf{a}})x \\
& + e^{-A_{h_t}\|x\|_{\mathbf{a}}^{\mu}} \cdot B_{h_t}\|x\|_{\mathbf{a}}^{\mu} \cdot \tilde{u}_{\|x\|_{\mathbf{a}}^{\mu}\hat{h}}(\mathbf{d}(-\ln(\|x\|_{\mathbf{a}}/\|x_k^q\|_{\mathbf{a}}))(v_k + q_k^1)) \\
& + e^{-A_{h_t}\|x\|_{\mathbf{a}}^{\mu}} \mathbf{d}(-\ln(\|x\|_{\mathbf{a}}/\|x_k^q\|_{\mathbf{a}}))q_k^2 + \mathbf{d}(-\ln \|x\|_{\mathbf{a}})\hat{q}_k.
\end{aligned}$$

Since  $\exists \tilde{C}_2 > 0 : \left\| \int_0^{h_t}\|x\|_{\mathbf{a}}^{\mu} A e^{A\tau} d\tau \right\| \leq \tilde{C}_2\|x\|_{\mathbf{a}}^{\mu}$  and  $\exists \tilde{C}_3 > 0 : \left\| B_{h_t}\|x\|_{\mathbf{a}}^{\mu} \right\| = \left\| \int_0^{h_t}\|x\|_{\mathbf{a}}^{\mu} e^{A\tau} d\tau B \right\| \leq \tilde{C}_3\|x\|_{\mathbf{a}}^{\mu}$  then for any  $\tilde{C}_4 > 0$  there exist  $C_3 > 0$  such that  $\|\mathbf{d}(-\ln \|x\|_{\mathbf{a}})x - \mathbf{d}(-\ln \|x\|_{\mathbf{a}})(x_k^q + \hat{q}_k)\| \leq C_3\|x\|_{\mathbf{a}}^{\mu}$  for  $\|x\|_{\mathbf{a}}^{\mu} \geq \tilde{r}_{\varepsilon} = \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\mu} r_{\varepsilon}$ ,  $\|q_k^i\| \leq \delta_{\varepsilon}$  and  $\|\mathbf{d}(-\ln \|x\|_{\mathbf{a}})\hat{q}_k\| \leq \tilde{C}_4\|x\|_{\mathbf{a}}^{\mu}$ . In this case, by Lemma 7 there exists  $C > 0$  and  $r > 0$  such that  $\|\tilde{u}_{\hat{h}}(x_k^q + \hat{q}_k) - u(x)\| \leq C\|x\|^{1+2\mu}$ . for  $\|x\|_{\mathbf{a}}^{-\mu} \geq \max\{\tilde{r}_{\varepsilon}, r^{-1}\}$ , all  $t \in [t_i, t_{i+1})$  and  $\|q_k^i\| \leq \delta_{\varepsilon}$  and  $\|\mathbf{d}(-\ln \|x\|_{\mathbf{a}})\hat{q}_k\| \leq \tilde{C}_4\|x\|_{\mathbf{a}}^{\mu}$ .

c) Adding and subtracting  $u(x)$  we derive

$$\begin{aligned}
\frac{d\|x\|_{\mathbf{a}}}{dt} &= \frac{\|x\|_{\mathbf{a}}x^{\top} \mathbf{d}^{\top}(-\ln \|x\|_{\mathbf{a}})P \mathbf{d}(-\ln \|x\|_{\mathbf{a}})(Ax + B\tilde{u}_{\hat{h}}(x_k + \hat{q}_k) + q_p)}{x^{\top} \mathbf{d}^{\top}(-\ln \|x\|_{\mathbf{a}})P G \mathbf{d}(-\ln \|x\|_{\mathbf{a}})x} \\
&= \frac{x^{\top} \mathbf{d}^{\top}(-\ln \|x\|_{\mathbf{a}})P(B(\tilde{u}_{\hat{h}}(x_k + \hat{q}_k) - u(x)) + \|x\|_{\mathbf{a}} \mathbf{d}(-\ln \|x\|_{\mathbf{a}})q_p)}{x^{\top} \mathbf{d}^{\top}(-\ln \|x\|_{\mathbf{a}})P G \mathbf{d}(-\ln \|x\|_{\mathbf{a}})x} - \rho\|x\|_{\mathbf{a}}^{1+\mu} \\
&\leq \beta\|B(\tilde{u}_{\hat{h}}(x_k + \hat{q}_k) - u(x))\| + \beta\|x\|_{\mathbf{a}}\|\mathbf{d}(-\ln \|x\|_{\mathbf{a}})q_p\| - \rho\|x\|_{\mathbf{a}}^{1+\mu} \\
&\leq \|x\|_{\mathbf{a}}^{1+\mu}(\beta\|B\|C\|x\|_{\mathbf{a}}^{-\rho} + \beta\|x\|_{\mathbf{a}}\|\mathbf{d}(-\ln \|x\|_{\mathbf{a}})q_p\|)
\end{aligned}$$

that for  $\|x\|_{\mathbf{a}}^{-\mu} \geq r' = \max\{\tilde{r}_{\varepsilon}, r^{-1}\}$ ,  $t \in [t_i, t_{i+1})$ ,  $\|q_k^i\| \leq \delta_{\varepsilon}$  and  $\|\mathbf{d}(-\ln \|x\|_{\mathbf{a}})\hat{q}_k\| \leq \tilde{C}_4\|x\|_{\mathbf{a}}^{\mu}$ .

d) Let us show that  $\|\cdot\|_{\mathbf{a}}$  is an ISS Lyapunov function (close to zero for  $\mu > 0$  and close to infinity for  $-\beta < \mu < 0$ ). Let  $\sigma(s) = s^{\mu}\bar{\sigma}^{-1}(s)$ , where  $\bar{\sigma} \in \mathcal{K}_{\infty}$  is given in Theorem 1 and  $s \geq 0$ . For  $\|q_p\| \leq 0.25\beta^{-1}\rho\sigma(\|x\|_{\mathbf{a}})$  we derive  $\beta\|x\|_{\mathbf{a}}\|\mathbf{d}(-\ln \|x\|_{\mathbf{a}})q_p\| \leq 0.25\rho\|x\|_{\mathbf{a}}^{1+\mu}$ . Notice that  $\|q_m\| \leq \delta_{\varepsilon}\bar{\sigma}^{-1}(\|x\|_{\mathbf{a}})$  implies  $\|q_k^1\| \leq \delta_{\varepsilon}$ ,  $\|q_m\| \leq \delta_0\sigma(\|x\|_{\mathbf{a}})$  with  $\delta_0 = \tilde{C}_0^{-1}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{\mu}$  implies

$C_0\|q_k^1\| \leq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{\mu} \|x_k^q\|_{\mathbf{a}}^{\mu}$  and  $\|\mathbf{d}(-\ln \|x\|_{\mathbf{a}})\hat{q}_k\| \leq \|x\|_{\mathbf{a}}^{\mu}$  for  $\|x\|_{\mathbf{a}}^{-\mu} \geq r'$ . For a sufficiently small  $\delta' > 0$  the inequalities  $\|x_k^q\|_{\mathbf{a}}^{-\mu} \geq r_{\varepsilon}$  (or, equivalently,  $\|x\|_{\mathbf{a}}^{-\mu} \geq \tilde{r}_{\varepsilon}$ ) and  $\|q_p\| \leq \delta'\sigma(\|x\|_{\mathbf{a}})$  imply  $\|q_k^2\| \leq \delta_{\varepsilon}$ .

Selecting  $\tilde{r} = \max\{r', 0.5\rho(\beta\|B\|C)^{-1}\}$  we derive  $\frac{d\|x\|_{\mathbf{a}}}{dt} \leq -0.25\rho\|x\|_{\mathbf{a}}^{1+\mu}$  for  $\|x\|_{\mathbf{a}}^{-\mu} \geq \tilde{r}$ ,  $\|q_p\| \leq \tilde{\delta}\sigma(\|x\|_{\mathbf{a}})$ ,  $\|q_m\| \leq \delta_0\sigma(\|x\|_{\mathbf{a}})$  and  $\|q_m\| \leq \delta_{\varepsilon}\bar{\sigma}(\|x\|_{\mathbf{a}})$ , where  $\tilde{\delta} = \min\{0.25\beta^{-1}\rho, \delta'\}$ . The function  $\sigma$  belongs to the class  $\mathcal{K}_{\infty}$  if  $\mu > -\beta$ . Therefore, the system (37) is practically ISS if  $-\beta < \mu < 0$  and locally ISS if  $\mu > 0$  even when the conditions of Theorem 5 do not hold.

e) Let us show that the system (37) is practically fixed-time stable and practically ISS for  $\mu > 0$ . Since  $F_{\hat{h}}^n = \mathbf{0}$

(see the proof of Lemma 2) then

$$\begin{aligned}
x_1^q &= F_{\hat{h}}x_0^q + L_{\hat{h}}y_1 + \bar{q}_1, & \bar{q}_1 &= \bar{q}_0 - L_{\hat{h}}A_{\hat{h}}^n\bar{q}_0, \\
x_2^q &= F_{\hat{h}}^2x_0^q + F_{\hat{h}}L_{\hat{h}}y_1 + L_{\hat{h}}y_2 + \bar{q}_1, & \bar{q}_2 &= F_{\hat{h}}\bar{q}_1 + \bar{q}_1 - L_{\hat{h}}A_{\hat{h}}^n\bar{q}_1, \\
&\dots & & \dots \\
x_n^q &= F_{\hat{h}}^{n-1}L_{\hat{h}}y_1 + \dots + L_{\hat{h}}y_n + \bar{q}_n, & \bar{q}_n &= F_{\hat{h}}\bar{q}_{n-1} + \bar{q}_{n-1} - L_{\hat{h}}A_{\hat{h}}^n\bar{q}_{n-1},
\end{aligned}$$

where  $y_{i+1} = Q_{n\hat{h}}(\|x_i^q + \hat{q}_i\|_{\mathbf{a}})(x_i^q + \hat{q}_i)$ ,  $i = 0, \dots, n-1$ . Since for  $\mu > 0$  we have  $\|y_{k+i}\| \leq 1$  (see the proof of Lemma 2, case 3) and  $\|\bar{q}_i\| \leq C \max_{j=0, \dots, i-1} \|q_j\|$  for some  $C > 0$ , then the system (37) with  $\mu > 0$  is practically fixed-time stable and practically ISS.

f) Let us show that the system (37) is locally ISS for  $\mu \geq -1$ . For  $\|x_k^q + \hat{q}_k\|_{\mathbf{a}} \leq r^-$  we have  $x_{k+1}^q = F_{\hat{h}}x_k^q + \tilde{q}_k - L_{\hat{h}}A_{\hat{h}}^n\hat{q}_k$ , where  $F_{\hat{h}}$  is a Schur stable nilpotent matrix. Since asymptotically stable linear system is ISS w. r. t. additive perturbations then (37) is locally ISS.

2) Let us show that the system (37) is ISS for  $\mu > -\beta$  provided that the unperturbed system is globally asymptotically stable. Our goal is to show ISS of

$$x_{(p+1)k^*}^q = \Xi(x_{pk^*}^q, q_{pk^*}, \dots, q_{(p+1)k^*-1}), \quad p=0, 1, \dots \quad (49)$$

which describes evolution of (48) with the discrete step  $k^*$ . The latter would imply ISS of (37). Notice that the local and practical ISS of (37) guarantees the local and practical ISS of the system (49). Let  $k^* \geq 1$ ,  $V, \bar{r}_*, \bar{r}^*, \bar{\gamma} \in (0, 1)$ ,  $L = \|\mathbf{d}(-\mu^{-1} \ln r^*)\|$  be defined as in the proof of Theorem 5. Let  $x_k \in \mathbb{R}^n$  denote a solution of the non-perturbed system with  $x_0 = x_0^q$ .

a) Let us show there exists  $\omega_{k^*} \in \mathcal{K}_{\infty}$ :

$$\|x_{k^*}^q - x_{k^*}\| \leq \omega_{k^*}(\max\{\|q_0\|, \dots, \|q_{k^*-1}\|\})$$

for  $\bar{r}_* \leq \|x_0\|_{\mathbf{a}}^{\mu} \leq \bar{r}^*$ . Since the system (37) is practically ISS then for any  $\tilde{\sigma} \in \mathcal{K}_{\infty}$  there exists a compact set  $\tilde{\Omega} \subset \mathbb{R}^n$  such that  $x_i^q \in \tilde{\Omega}$  for all  $i \in 0, \dots, k^*$  provided that  $\bar{r}_* \leq \|x_0\|_{\mathbf{a}}^{\mu} \leq \bar{r}^*$  and  $\max_{j=0, \dots, k^*-1} \|q_j\| \leq \tilde{\sigma}(\|x_0\|_{\mathbf{a}})$ .

Denoting  $\bar{q}_0 = \mathbf{0}$  and  $\bar{q}_k = F_{\hat{h}}\bar{q}_{k-1} + \tilde{q}_k + L_{\hat{h}}(Q_{n\hat{h}}(\|x_{k-1} + \bar{q}_{k-1} + \hat{q}_k\|_{\mathbf{a}})(x_{k-1} + \bar{q}_{k-1} + \hat{q}_k) - Q_{n\hat{h}}(\|x_{k-1}\|_{\mathbf{a}})x_{k-1})$  for  $k \geq 1$  we derive  $x_k^q = x_k + \bar{q}_k$ . Since the function  $x \mapsto Q_{\tau}(\|x\|_{\mathbf{a}})x$  is continuous on  $\mathbb{R}^n$  then by Heine-Cantor Theorem it is uniformly continuous on  $\tilde{\Omega}$  and there exists  $\omega_0 \in \mathcal{K}_{\infty}$  such that  $\|Q_{\tau}(\|z_1\|_{\mathbf{a}})z_1 - Q_{\tau}(\|z_2\|_{\mathbf{a}})z_2\| \leq \omega_0(\|z_1 - z_2\|)$  for all  $z_1, z_2 \in \tilde{\Omega}$  and

$$\|\bar{q}_1\| \leq \|\bar{q}_0\| + \|L_{\hat{h}}\omega_0(\|\hat{q}_0\|).$$

Repeating the above consideration, on  $k^*$ -th step, we derive that  $\exists \omega_{k^*} \in \mathcal{K}_{\infty}$ :  $\|\bar{q}_{k^*}\| \leq \omega_{k^*}(\max_{j=0, \dots, k^*-1} \|q_j\|)$  for  $\bar{r}_* \leq \|x_0\|_{\mathbf{a}}^{\mu} \leq \bar{r}^*$  and  $\max_{j=0, \dots, k^*-1} \|q_j\| \leq \tilde{\sigma}(\|x_0\|_{\mathbf{a}})$ .

b) Since  $V(x_{k^*}^q) = V(x_{k^*}^q) - V(x_k) + V(x_k) \leq L\|\bar{q}_{k^*}\| + \bar{\gamma}V(x_0) \leq \frac{\bar{\gamma}+1}{2}V(x_0^q)$  for  $\bar{r}_* \leq \|x_0\|_{\mathbf{a}}^{\mu} \leq \bar{r}^*$  and  $\|\bar{q}_{k^*}\| \leq \frac{1-\bar{\gamma}}{2L}V(x_0)$  then  $V(x_{k^*}^q) \leq \frac{\bar{\gamma}+1}{2}V(x_0^q)$  for  $\bar{r}_* \leq \|x_0\|_{\mathbf{a}}^{\mu} \leq \bar{r}^*$

and  $\max_{j=0, \dots, k^*-1} \|q_j\| \leq \min \{ \tilde{\sigma}(\|x_0\|_d), \omega_{k^*}^{-1} \left( \frac{1-\tilde{\gamma}}{2L} V(x_0) \right) \}$ . Taking into account the local and practical ISS proven above, the latter guarantees global ISS of (49) by [22].

### 7.12 The proof of Corollaries 3 and 4

Denote  $\delta_i(t) = \sum_{j=i+1}^m A_{ij} x_j(t)$ , where  $i = 1, \dots, m$  is a number of subsystem in the system (1), (2), (39) and the matrices  $A_{ij}$  are defined in the proof of Corollary 2. By Theorem 4, each subsystem with  $\delta_i = 0$  is finite-time (for  $\mu_i < 0$ ) or nearly fixed-time (for  $\mu_i > 0$ ) stable. Moreover, it is forward complete<sup>6</sup> if  $\delta_i$  is uniformly bounded. **The case  $\mu_i < 0$ .** Since the  $m$ -th subsystem is finite-time stable then  $\exists T_m > 0$  such that  $\delta_{m-1}(t) = \mathbf{0}$  for all  $t \geq T_m$ . Considering subsequently the systems  $m-1, m-2, \dots, 1$  we conclude that the system (18), (39) is finite-time stable. **The case  $\mu_i > 0$ .** Since the  $m$ -th subsystem is fixed-time stable then  $d_{m-1}$  is uniformly bounded and  $(m-1)$ -th subsystem practically fixed-time stable (see Theorem 6), but the ISS property guarantees its global uniform asymptotic stability [55]. Using the cascade structure of the system we complete the proof of Corollary 3. The proof of Corollary 4 is literally the same but it uses Theorems 5 and 7 instead of Theorem 4 and 6, respectively.

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<sup>6</sup> A system is forward complete if all its solutions are defined globally in the forward time.

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