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# A RANDOMNESS DEVICE TO CREATE THE CONDITIONS OF UNCERTAINTY

Loïc Berger\*

## Abstract

The experimental study of decision-making under uncertainty typically builds on Ellsberg's (1961) setting. Yet, as the total number of balls is known in standard Ellsberg's urns, an implicit constraint is put on the specification of the probability models to consider. In practice, this restricts the ability of Ellsberg's urns to characterize situations going beyond those of *model ambiguity*. In this note, I present a simple and easy-to-implement device that creates the initial conditions of *uncertainty*, which constitute a critical prerequisite for the study of model misspecification.

**Keywords:** Ambiguity aversion, experiment, Ellsberg paradox, model uncertainty

**JEL Classification:** D8

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# 1 Introduction

According to a well-known distinction attributed to Knight (1921), there exists a difference between the notions of *risk* and *uncertainty*. Specifically, while the former identifies situations where the uncertainty is “measurable” (i.e., it may be represented by numerical *objective* probabilities), the latter refers to situations where the uncertainty is “unmeasurable” (i.e., numerical *objective* probabilities are inapplicable). Following the seminal work of Ellsberg (1961), the term “ambiguity” has emerged to characterize situations of uncertainty in which the probabilities are *unknown*. Nowadays, the terms “ambiguity”, “uncertainty”, and “Knightian uncertainty” are used interchangeably in opposition to “risk” (Gilboa and Marinacci, 2013).

In his seminal work, Ellsberg (1961) proposed two distinct thought experiments aiming at challenging the expected utility theory of Savage (1954). These experiments, giving rise to the *Ellsberg Paradox*, have been the primary motivating ingredients of a remarkably rich literature on ambiguity and ambiguity aversion. In particular, Ellsberg’s examples have been extensively used in (laboratory) experiments, while ambiguity aversion has been one of the most intensively experimentally studied phenomena in decision theory in the last decades (Trautmann and Van De Kuilen, 2015).

In a nutshell, one of the two examples used by Ellsberg (1961)—and known as the Two-Color Problem—involves two colors and two urns. The first urn contains exactly 50 black and 50 red balls. The second urn contains 100 red and black balls but in unknown proportions. There are four bets possible  $\{r_1, b_1, r_2, b_2\}$ , where  $r_1$ , for example, consists of drawing a red ball from the first urn. A correct bet gives \$100, and nothing happens if the bet is incorrect. The four bets are represented in Table 1. The decision maker (DM) is asked to rank these bets. Typically, individuals are indifferent between  $r_1$  and  $b_1$ , and between  $r_2$  and  $b_2$ , but prefer  $r_1$  over  $r_2$ , and  $b_1$  over  $b_2$ . The reason is that the first urn offers known probabilities of winning, whereas the second urn does not. Yet, such preferences are incompatible with the existence of subjective probabilities and the subjective expected utility theory of Savage (1954).<sup>1</sup>

While the beauty of Ellsberg’s examples lies in their ability to show how ambiguity may arise naturally even in extremely simplified environments, one has to

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<sup>1</sup>Indeed, the ranking  $r_1 \succ r_2$  would imply  $\text{Prob}(\text{red in Urn 2}) < 1/2$ , but the ranking  $b_1 \succ b_2$  would imply  $\text{Prob}(\text{black in Urn 2}) < 1/2$ , which is inconsistent.

	Urn 1		Urn 2	
	Red	Black	Red	Black
$r_1$	\$100	\$0		
$b_1$	\$0	\$100		
$r_2$			\$100	\$0
$b_2$			\$0	\$100

Table 1: Ellsberg two-color problem

note that the state of information in Ellsberg’s examples is also highly structured. Ellsberg (1961, p 657) himself wrote<sup>2</sup>

“But the state of information in our urn example can be characterized neither as “ignorance” nor “risk” in these senses. Each subject does know enough about the problem to *rule out* a number of possible distributions”.

In Ellsberg’s setting, indeed, the potential compositions of the unknown urn are clearly identified: the probability of drawing a red ball necessarily belongs to the well-specified set  $\{\frac{0}{100}, \frac{1}{100}, \frac{2}{100}, \dots, \frac{100}{100}\}$ . The other potential compositions are ruled out by construction. In consequence, standard Ellsberg’s urns cannot be used to study decision-making under uncertainty going beyond *model ambiguity*.

In this note, I show how a slight modification of the standard Ellsberg’s setting enables to create the initial conditions of uncertainty needed to study *model misspecification*. The randomness device that I propose is both simple and easy-to-implement in laboratory environments.

## 2 Uncertainty decomposed into layers

Uncertainty may be decomposed into different layers of analysis. Following Hansen (2014); Marinacci (2015); Hansen and Marinacci (2016); Aydogan et al. (2022), a distinction can be made between the layers of (i) *risk*, (ii) *model uncertainty*, and (iii) *model misspecification*. *Risk* refers to the uncertainty regarding the outcomes *within* a particular probability model; *model ambiguity* refers to the

<sup>2</sup>Note that if this sentence initially concerned Ellsberg’s second thought experiment (which features a single urn with 30 yellow balls and 60 green and blue balls but in unknown proportion), it equally applies to Urn 2 in the Two-Color Problem.

uncertainty *across* probability models (among a set of possible models) to be used to assign these probabilities; and *model misspecification* refers to the uncertainty *about* models and the fact that the set of models under consideration might not include the correct model. These layers of uncertainty are inherent to any decision problem under uncertainty.

An important distinction between the layers of uncertainty is the nature of the uncertainty they feature. A risk is characterized by an *objective* probability measure, representing the randomness of a phenomenon. It refers to the physical quantification of uncertainty by means of a probability model. For example, a risk may be characterized by a specific composition of an urn containing balls of different colors (e.g., Urn 1). Alternatively, under ambiguity (e.g., in Ellsberg’s Urn 2), the uncertainty about the correct probability model to consider has an *epistemic* nature: no objective probability measure may be associated with the potential compositions of the urn. So, if the DM assigns probabilities to the possible compositions, they are necessarily subjectively determined, reflecting thereof the DM’s degree of belief.

In Ellsberg’s standard setup, the DM knows that the “correct” probability model belongs to a well-specified collection  $M$ .<sup>3</sup> For example, when there are 100 balls in the urn, the correct urn composition necessarily belongs to one of the 101 compositions physically possible. In other words, the (objective) information concerning the total number of balls in standard Ellsberg urns enables the DM to posit a restricted set  $M$  of potential models, thus making it impossible to study concerns for model misspecification.

### 3 A randomness device to create uncertainty

Studying model misspecification requires to be able to consider, *à priori*, any probability distribution. A natural starting point should therefore be a situation of *uncertainty* distinct from *model ambiguity*. Such a situation would, in principle, correspond to the case where the urn contains an infinite number of balls, so that any probability between 0 and 1 would be *à priori* possible.

Using a slight modification of Ellsberg’s canonical example, I propose a randomness device that mimics the situation of uncertainty by compelling DM to consider all the possible probabilities in  $[0, 1]$ . It consists in using an urn where the total

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<sup>3</sup>Formally, by letting  $(S, \Sigma)$  be a measurable space, where  $\Sigma$  is an algebra of events of the state space  $S$ , and let  $\Delta(S)$  be the collection of all probability measures  $m : \Sigma \rightarrow [0, 1]$ , we say that states are generated by a probability model  $m \in \Delta(S)$  belonging to a finite subset  $M$  of  $\Delta(S)$ .

number of balls is itself unknown.

To illustrate, consider the case in which the number of balls is told to be comprised between 1 and  $N$ . In such a situation, the total number of potential objective probability models to consider corresponds to the cardinality of the Farey sequence of order  $N$ .<sup>4</sup> Table 2 presents the sets of potential models when the *maximum* number of balls  $N$  varies between 1 and 8. As can be observed, when  $N = 1$ , the set of

Table 2: SETS OF MODELS AND THEIR CORRESPONDING CARDINALITY WHEN THE MAXIMUM NUMBER OF BALLS IN THE URN IS  $N$

$N$	Set of possible models: $M_N = \{P(r)\}$																	$ M_N $												
1	$\frac{0}{1}$																$\frac{1}{1}$	2												
2	$\frac{0}{1}$																$\frac{1}{2}$	$\frac{1}{1}$	3											
3	$\frac{0}{1}$																$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{1}$	5									
4	$\frac{0}{1}$																$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{1}$	7							
5	$\frac{0}{1}$																$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{1}{1}$	11			
6	$\frac{0}{1}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$													$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{1}{1}$	13				
7	$\frac{0}{1}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$											$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{1}{1}$	19
8	$\frac{0}{1}$	$\frac{1}{8}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	$\frac{1}{1}$	23						

probability models  $M$  consists of two elements:  $M_1 = \{m \in \Delta(S) : P(r) \in \{0, 1\}\}$ , where  $P(R)$  denotes the probability of drawing a red ball. When  $N = 2$ , the cardinality of  $M$  increases to  $|M| = 3$ , and  $M_2 = \{m \in \Delta(S) : P(r) \in \{0, \frac{1}{2}, 1\}\}$ .

By choosing a  $N$  that is large enough, this device may be viewed as sufficiently complicated to prevent subjects from doing any calculation of probability distribution over the possible compositions of the urn and their corresponding weights. Indeed, note that, even in the case in which all the possible total numbers of balls and all their possible compositions are assumed to be equally probable, the models are not weighted uniformly. To illustrate this point, consider the case where the maximum number of balls is 3, so that five different models exist:  $M_3 = \{m \in \Delta(S) : P(r) \in \{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}\}$ . Assuming that the total number of balls in the urn is uniformly distributed between 1 and 3, and that for each case the different potential models are weighted equally, one ends up with weights attached to the possible probability models that are  $q = \{\frac{13}{36}, \frac{1}{12}, \frac{1}{9}, \frac{1}{12}, \frac{13}{36}\}$ .<sup>5</sup>

<sup>4</sup>A Farey sequence of order  $N$ , denoted  $\mathfrak{F}_N$ , is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed  $N$  (Hardy et al., 1979).

<sup>5</sup>To see where these numbers come from, consider for example the case of the probability

When the maximum number of balls in the urn  $N$  is large enough, and absent of any additional information, a DM confronted with this randomness device will most likely be unable to compute the set of possible objective probability models, and will thus be induced to consider all the possible probability models, making the situation virtually identical to one of complete *uncertainty*:

$$M_N = \{m \in \Delta(S) : P(r) \in \{\mathfrak{F}_N\}\} \sim \{m \in \Delta(S) : P(r) \in [0, 1]\}. \quad (1)$$

Of course, what “large enough” refers to may be debated. In order to get a sense of what could be this  $N$ , Table 3 provides the number of possible compositions of the ambiguous urn for different values of  $N$ . As can be observed, the number of

Table 3: NUMBER OF MODELS FOR DIFFERENT VALUES OF  $N$

Maximum number of balls in the urn ( $N$ )	Number of possible compositions ( $ M_N $ )
1	2
5	11
10	33
20	129
30	279
40	491
50	775
100	3045
1000	304 193

probability models quickly explodes as the maximum total number of balls increases. In particular, 3045 different probability models need, in principle, to be considered when the maximum number of balls in the urn is  $N = 100$ .

One can therefore reasonably assume that subjects would be unable (or unwilling) to consider all the particular possible compositions in such a situation, and would rather view the situation close to one of uncertainty.

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model  $m \in \Delta(S) : P(r) = 1$ . If there is only one ball in the urn, the symmetry condition implies that this model arises with probability  $1/2$ , if there are two balls in the urn, it arises with probability  $1/3$ , and if there are three balls in the urn, it arises with probability  $1/4$ . Now, each of these three possibilities also arises with probability  $1/3$  (assuming once again the distribution is uniform). This means that altogether, this probability model is associated with a probability  $\frac{13}{36} (= \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{4})$ .

## 4 Discussion

Standard Ellsberg's urns are simple and elegant devices able to generate uncertainty using the layers of *risk* and *model ambiguity* only. They are particularly useful for their ability to make clear in what precise way ambiguity-sensitive preferences violate key axioms of expected utility theory. However, standard Ellsberg's urns are, by construction, unable to create the initial conditions of complete uncertainty, which constitute a critical prerequisite for the study of model misspecification.

In this note, I propose a slight modification of Ellsberg's example that allows for mimicking an unconstrained set of potential probability models. This device has the advantage of being extremely easy to implement. It permits to replicate the ex-ante conditions of uncertainty that would otherwise be difficult—or impossible—to implement in practice in a laboratory. Starting from such a situation of uncertainty, it is then possible to provide information about the set of models to be considered for the purpose of a particular study (e.g., endowing subject participants with potential distributions, such as expert's opinions, model output results, etc.) and to allow for this set to be misspecified.



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