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Relaxed formulation for Controlled Branching Diffusions, Existence of an Optimal Control and HJB Equation

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Abstract

The focus of this article is studying an optimal control problem for branching diffusion processes. Initially, we introduce the problem in its strong formulation and expand it to include linearly growing drifts. Then, we present a relaxed formulation that provides a suitable characterization based on martingale measures. Considering weak controls, we prove they are equivalent to strong controls in the relaxed setting, and establish the equivalence between the strong and relaxed problem, under a Filippov-type convexity condition. Furthermore, by defining control rules, we can restate the problem as the minimization of a lower semi-continuous function over a compact set, leading to the existence of optimal controls both for the relaxed problem and the strong one. Finally, with a useful embedding technique, we show that the value function solves a system of HJB equations, establishing a verification theorem. We then apply it to a linear-quadratic example and a kinetic one.

MSC Classification- 93E20, 60J60, 60J80, 35K10, 60J70, 60J85

Keywords— Stochastic control, relaxed control, branching diffusion processes, martingale representation.

1 Introduction

The focus of this paper is on populations that are optimally controlled. Specifically, we aim to show the presence of a strong control for controlled branching diffusions and to describe the optimal dynamics.

The class of branching diffusion processes describes the evolution of particles, whose spatial movement is modelled by a SDE. Introduced in [40], [27], [25], [26], their study has been developed extensively, especially for their use in the probabilistic representation of semilinear PDEs (see, *e.g.*, [24]) and in the regularized unbalanced optimal transport (see, *e.g.*, [3]).

Several examples of optimal control for branching processes are discussed in the literature (see, *e.g.*, [44],[36], [7], [31]). They have been introduced in [44], wherein their modelling employs a topological sum of Euclidean space. The control, living within a compact space, solely affects the drift of spatial movement. The author permits each particle to potentially be influenced by any other living particle, without imposing any additional assumptions on the structure of these interactions. Moreover, the running cost yields a high degree of generality as well, leading to a correspondingly complex differential characterization. By selecting the cost function as the product of functions associated with the living particles at the terminal time, [36] employs controlled branching processes as a probabilistic tool to examine a specific group of parabolic Bellman equations. In this study, the control, still confined to a compact set, influences both drift and volatility. A Hamilton-Jacobi-Bellmann (HJB) equation is identified, establishing that the value function represents its unique (viscosity) solution.

In [7], the author goes further in the analysis of this setting. Initially, the controlled processes are described as measure-valued processes. Using Ulam-Harris-Neveu labelling (see, *e.g.*, [2]) to describe the genealogy of the particles, the author introduces a label set that assists in defining the branching events. A set of Brownian motions and Poisson random measures, indexed by these labels, are used to provide a strong formulation for the controlled branching processes. This facilitates the well-posedness for dynamics where drift, volatility, branching rate, and branching mechanisms are not only controlled but also dependent on the position of each particle. While these coefficients are still assumed to be bounded, the control space is no longer necessarily compact. Since the dynamics are coupled only through the control, the product structure of the cost yields a branching property that converts the problem into a finite-dimensional one. A PDE characterization of the value function is then obtained, leveraging the differential properties of the Euclidean space where each single particle is defined. In [31], a similar approach is also employed. Here, the symmetry of the reward function is again used to establish a different branching property that allows for finite-dimensional rewriting.

This article expands on previous work on optimal control of branching diffusions. Firstly, we introduce a coupling between the particle dynamics via the empirical measure of the population, similar to the interactions in mean field control literature. Secondly, we consider unbounded control space, and we allow the drift to have linear growth in both space and control while keeping the other coefficients bounded. We derive an HJB equation to characterize the value function, taking advantage of the homeomorphism between the topological sum of Euclidean spaces, as in [44], and the subset of finite measures, as in [7] and [31]. This results in a verification theorem that we later rewrite as a (sub)martingale condition, similar to [37], to verify optimality. This brings us closer to the description of these processes as measure-valued and facilitates intuition for solving optimization problems, applying these results to a linear quadratic example and a kinetic one.

The first part of this paper addresses the issue of the existence of optimal controls. We follow the approach of [13] and [23], which involves a relaxed formulation of the problem. This formalism introduces different descriptions of the control problem, namely control rules and natural controls, allowing for greater flexibility and easier manipulation of the controlled dynamics. Proving that a control rule (resp. natural control) with a lower cost can be constructed from any relaxed control (resp. control rule), we establish the equivalence between strong and relaxed problems. Furthermore, we show that the cost function is lower semicontinuous for the control rule case, and, under some coercivity assumptions, we confine the search for minima to a compact set under a suitable topology. This rewrites the original optimization as the minimization of a lower semicontinuous function over a compact space, establishing the existence of optimal values and controls.

Similar methodology has been used in mean-field control theory (see, *e.g.*, [33], [1]) or branching populations dynamics (see, *e.g.*, [8]). Our approach differs from [8] as they make large use of the

indexation with respect to the label set. Nonetheless, we use the topology introduced in this article, to apply it to measures with finite first-order moments.

The study of measure-valued processes in \mathbb{R}^d has been ongoing since the late nineties. In seminal works such as [35], [38], and [39], these processes were introduced as solutions to martingale problems. This strategy, detailed for the case of diffusions in [19], allows for a more abstract yet clearer manipulation of these objects. In [11], this point of view is applied to describe various dynamics, such as Fleming-Viot processes and superprocesses. This point of view provides useful convergence criteria and methods for characterizing their uniqueness in law, which will be extensively used in the remainder of the paper. In particular, the relaxed formulation of a control problem relies on the martingale problem formulation, as described in [17]. By exploiting the symmetry of the cost function with respect to the labelling, we can confine controls to an admissible class that preserves this symmetry. This restriction does not affect the problem's value function under mild assumptions, but it is crucial for defining relaxed controls, which, to the best of our knowledge, is the first of its kind. The control is seen as a probability measure of the action space that depends not only on time but also on space. We begin by presenting the connection between strong and relaxed controls through Dirac measures, identifying the class of weak controls. We prove their law uniqueness and use Doob's functional representation theorem to refer to the strong formulation. Under a Filippov-type convexity condition is satisfied (see, *e.g.*, [20]), any relaxed control can be associated with a weak one with lower cost. This gives the equivalence between the strong and relaxed characterisations and provides optimal strong control via the identification of weak and strong controls.

Finally, when attempting to optimize trajectories, we consider the concept of the kinetic energy of the system. This is the case of the Schrödinger bridge problem, as in [22], where one seeks to identify the random evolution (*i.e.*, a probability measure on path-space) that is closest to a prior Markov diffusion evolution in the relative entropy sense, while also satisfying certain initial and final marginals. It has been noted that this problem can be framed as a stochastic control problem, see, *e.g.*, [10], [9], [5], [6], where the kinetic energy plays a fundamental role in the cost function. Continuing along this line of reasoning, we present an example involving a comparable cost function and proceed to solve it with the help of the verification theorem.

The rest of the paper is structured as follows. In Section 2, we provide an introduction to the setting and the strong formulation for controlled branching processes. The control problem is defined and its well-posedness is proven. In Section 3, we introduce the relaxed formulation, presenting equivalent representations and characterizing them using martingale measures. Section 4 establishes the equivalence between the relaxed and strong formulations under a Filippov-type convexity condition. We introduce natural controls in this setting and show that we can restrict the problem to this class by conditioning on measures. Then, we compare the embedding of strong controls with weak ones and show their equivalence via uniqueness in law for these objects and Doob's functional representation theorem. Section 5 introduces the set of control rules and uses it to prove the lower semicontinuity of the cost functions in this set. Here, we show there exists a minimal solution of the strong control problem, after restricting to a compact set found using the coercivity assumption of the cost. Finally, in Section 6, we present the system of HJB equations and use it to solve a linear quadratic example and a kinetic one.

2 The control problem

2.1 The set of measures

For a Polish space (E, d) with $\mathcal{B}(E)$ its Borelian σ -field, we write $C_b(E)$ (resp. $C_0(E)$) for the subset of the continuous functions that are bounded (resp. that vanish at infinity), and $M(E)$ (resp. $\mathcal{P}(E)$) for the set of Borel positive finite measures (resp. probability measures) on E . We equip $M(E)$ with weak* topology, *i.e.*, the weakest topology that makes continuous the maps $M(E) \ni \lambda \mapsto \int_E \varphi(x) \lambda(dx)$ for any $\varphi \in C_b(\mathbb{R}^d)$. We denote $\langle \varphi, \lambda \rangle = \int_E \varphi(x) \lambda(dx)$ for $\lambda \in M(E)$ and $\varphi \in C_b(E)$.

Denote also by $M^p(E)$ the subspace of measures with finite p -th moment for $p \geq 1$, *i.e.*, the collection of all $\lambda \in M(E)$ such that $\int_E d(x, x_0)^p \lambda(dx) < \infty$ for some $x_0 \in E$. The weak* topology can be metrized in $M^p(E)$ by the Wasserstein type metric $\mathbf{d}_{p,E}$, as introduced in [8, Appendix B]. This means that, if ∂ is a cemetery point, we consider first \bar{E} the enlarged space $\bar{E} := E \cup \{\partial\}$. Defining $d(x, \partial) := d(x, x_0) + 1$, we have that (\bar{E}, d) is Polish. On the space

$$M_m^p(\bar{E}) := \{\lambda \in M^p(\bar{E}) : \lambda(\bar{E}) = m\},$$

consider the Wasserstein distance as follows

$$\mathbf{d}_{p,E,m}(\lambda, \lambda') = \left(\inf_{\pi \in \Pi(\lambda, \lambda')} \int_{\bar{E} \times \bar{E}} d(x, y)^p \pi(dx, dy) \right)^{1/p}, \quad \text{for } \lambda, \lambda' \in M_m^p(\bar{E}),$$

where $\Pi(\lambda, \lambda')$ denotes the collection of all non-negative measures on $\bar{E} \times \bar{E}$ with marginals λ and λ' . The distance $\mathbf{d}_{p,E}$ on $M^p(E)$ is defined as

$$\mathbf{d}_{p,E}(\lambda, \lambda') = \mathbf{d}_{p,E,m}(\bar{\lambda}_m, \bar{\lambda}'_m), \quad \text{for } \lambda, \lambda' \in M_m^p(E),$$

where $m \geq \lambda(E) \vee \lambda'(E)$ and

$$\bar{\lambda}_m(\cdot) := \lambda(\cdot \cap E) + (m - \lambda(E))\delta_{\partial}(\cdot), \quad \bar{\lambda}'_m(\cdot) := \lambda'(\cdot \cap E) + (m - \lambda'(E))\delta_{\partial}(\cdot).$$

As proven in [8, Lemma B.1], this definition does not depend on the choice of m . Moreover, for some $x_0 \in E$, we have the natural bound

$$\mathbf{d}_{p,E}^p(\lambda, \delta_{x_0}) \leq \int_E d(x, x_0)^p \lambda(dx) + \langle 1, \lambda \rangle^p, \quad \text{for } \lambda \in M^p(E). \quad (2.1)$$

We can remark that all the results in [8, Appendix B], about the convergence under $\mathbf{d}_{1,E}$, can be directly generalised for $\mathbf{d}_{p,E}$.

Finally, we write $\mathcal{N}[E]$ for the space of atomic measures in E , *i.e.*,

$$\mathcal{N}[E] := \left\{ \sum_{i=1}^m \delta_{x_i} : m \in \mathbb{N}, x_i \in E \text{ for } i \leq m \right\},$$

a weakly* closed subset of $M(E)$. In particular, we remark that $\mathcal{N}[\mathbb{R}^d]$ is also a closed set of $M^p(\mathbb{R}^d)$ with respect to the distance $\mathbf{d}_{p,E}$. This is due to the fact that $\mathcal{N}[\mathbb{R}^d]$ is weakly*-closed and, from [8, Lemma B.2], convergence in $M^1(\mathbb{R}^d)$ entails weak*-convergence to some $\lambda \in \mathcal{N}[\mathbb{R}^d] \subseteq M^1(\mathbb{R}^d)$.

Remark 2.1. *Each vector $\vec{x}^m = (x_1, \dots, x_m) \in \mathbb{R}^{dm}$ can be embedded in $\mathcal{N}[\mathbb{R}^d]$ as $\iota(\vec{x}^m) := \sum_{i=1}^m \delta_{x_i}$. Fix $\vec{x}^m, \vec{y}^m \in \mathbb{R}^{dm}$. We use the characterisation of the distance $\mathbf{d}_{1,E}$ of [8, Lemma B.1] and obtain*

$$\mathbf{d}_{1,E}(\iota(\vec{x}^m), \iota(\vec{y}^m)) = \sup_{\varphi \in Lip_1^0(\mathbb{R}^d)} \sum_{i=1}^m |\varphi(x_i) - \varphi(y_i)| \leq \sum_{i=1}^m |x_i - y_i| = |\vec{x}^m - \vec{y}^m|.$$

where $Lip_1^0(\mathbb{R}^d)$ denote the collection of all functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ with Lipschitz constant smaller or equal to 1 and such that $\varphi(0) = 0$.

2.2 Strong formulation

Fix a finite time horizon $T > 0$. Let $\mathbf{D}^d = \mathbb{D}([0, T]; M^1(\mathbb{R}^d))$ be the set of càdlàg functions (right continuous with left limits) from $[0, T]$ to $M^1(\mathbb{R}^d)$. We endow this space with Skorohod metric $d_{\mathbf{D}^d}$ associated with the metric $\mathbf{d}_{\mathbb{R}^d}$, which makes it complete (see, e.g., [4]). For $\mathbb{P} \in \mathcal{P}(\mathbf{D}^d)$, $\mathbb{P}_t \in \mathcal{P}(M^1(\mathbb{R}^d))$ denotes the time- t marginal of \mathbb{P} , i.e., the image of \mathbb{P} under the map $\mathbf{D}^d \ni \mu \mapsto \mu_t \in M^1(\mathbb{R}^d)$.

Assumptions We are given dimensions $d, d' \in \mathbb{N}$, a closed subset A of \mathbb{R}^m representing the set of actions, and the following continuous functions

$$(b, \sigma, \gamma, p_k) : \mathbb{R}^d \times M^1(\mathbb{R}^d) \times A \rightarrow \mathbb{R}^d \times \mathbb{R}^{d \times d'} \times \mathbb{R}_+ \times [0, 1]$$

for $k \geq 0$, such that $\sum_{k \geq 0} p_k(x, \lambda, a) = 1$ for any $(x, \lambda, a) \in \mathbb{R}^d \times M^1(\mathbb{R}^d) \times A$. Assume that b and σ are Lipschitz continuous in (x, λ) , i.e., there exists $L > 0$ such that

$$|b(x, \lambda, a) - b(x', \lambda', a)| + |\sigma(x, \lambda, a) - \sigma(x', \lambda', a)| \leq L(|x - x'| + \mathbf{d}_{\mathbb{R}^d}(\lambda, \lambda')), \quad (2.2)$$

for any $x, x' \in \mathbb{R}^d$, $\lambda, \lambda' \in M^1(\mathbb{R}^d)$, and $a \in A$. Suppose also that σ and γ are uniformly bounded, and b has linear growth in (x, a) while bounded in λ , i.e., there exists $C_\sigma, C_\gamma, C_b > 0$ such that

$$|b(x, \lambda, a)| \leq C_b(1 + |x| + |a|), \quad |\sigma(x, \lambda, a)| \leq C_\sigma, \quad \gamma(x, \lambda, a) \leq C_\gamma, \quad (2.3)$$

for $(x, \lambda, a) \in \mathbb{R}^d \times M^1(\mathbb{R}^d) \times A$. Let Φ be the generating function of $(p_k)_k$, i.e.,

$$\Phi(s, x, \lambda, a) = \sum_{k=0}^{\infty} p_k(x, \lambda, a) s^k, \quad \text{for } (s, x, \lambda, a) \in [0, 1] \times \mathbb{R}^d \times M^1(E) \times A.$$

Assume that the first and second order moments related to $(p_k)_k$ are uniformly bounded, i.e., there exist two constants $C_\Phi^1, C_\Phi^2 > 0$ such that

$$\partial_s \Phi(1, x, \lambda, a) = \sum_{k \geq 1} k p_k(x, \lambda, a) \leq C_\Phi^1, \quad \partial_{ss}^2 \Phi(1, x, \lambda, a) = \sum_{k \geq 1} k(k-1) p_k(x, \lambda, a) \leq C_\Phi^2, \quad (2.4)$$

for any $(x, \lambda, a) \in \mathbb{R}^d \times M^1(\mathbb{R}^d) \times A$. The generalization to time-dependent coefficients is straightforward. We do not address it explicitly not to make the notation heavier. We will make use of this setting in Section 6.3.

Strong controls We consider the set of labels $\mathcal{I} = \{\emptyset\} \cup \bigcup_{n=1}^{+\infty} \mathbb{N}^n$ and use Ulam-Harris labelling to consider the genealogy of the particles. Denote by \emptyset the mother particle, and $i = i_1 \cdots i_n$ the multi-integer $i = (i_1, \dots, i_n) \in \mathbb{N}^n$, $n \geq 1$. For $i = i_1 \cdots i_n \in \mathbb{N}^n$ and $j = j_1 \cdots j_m \in \mathbb{N}^m$, we define their concatenation is $ij \in \mathbb{N}^{n+m}$ by $ij = i_1 \cdots i_n j_1 \cdots j_m$, and extend it to the entire \mathcal{I} by $\emptyset i = i \emptyset = i$ for all $i \in \mathcal{I}$. When a particle $i = i_1 \cdots i_n \in \mathbb{N}^n$ gives birth to k particles, the off-springs are labelled $i0, \dots, i(k-1)$.

Let $(\Omega^s, \mathbb{F}^s = \{\mathcal{F}_t^s\}_{t \geq 0}, \mathcal{F}^s, \mathbb{P}^s)$ be a filtered probability space satisfying the usual conditions. Suppose that this space supports two independent families $\{W^i\}_{i \in \mathcal{I}}$ and $\{Q^i\}_{i \in \mathcal{I}}$ of mutually independent processes. Let W^i be a d' -dimensional Wiener processes, and $Q^i(ds dz)$ a Poisson random measure on $[0, T] \times \mathbb{R}_+$ with intensity measure $ds dz$.

Definition 2.1 (Standard strong control). *We say that $\beta = (\beta^i)_{i \in \mathcal{I}}$ is a standard strong control if β is an \mathbb{F}^s -predictable $A^{\mathcal{I}}$ -valued process, such that*

$$\mathbb{E}^{\mathbb{P}^s} \left[\sup_{i \in \mathcal{I}} \int_t^T |\beta_s^i|^2 ds \right] < \infty. \quad (2.5)$$

Fix a standard control $\beta = (\beta^i)_{i \in \mathcal{I}}$. We describe the *controlled branching diffusion* ξ^β as the measure-valued process

$$\xi_t^\beta = \sum_{i \in V_t} \delta_{Y_t^{i,\beta}},$$

where $Y_t^{i,\beta}$ is the position of the member with label $i \in \mathcal{I}$, and V_t the set of alive particles at time t . This process takes values in $\mathcal{N}[\mathbb{R}^d]$ and the behaviour of each alive particle i is characterized by the following three properties:

- *Spatial motion*: during its lifetime, it moves in \mathbb{R}^d according to the following stochastic differential equation

$$dY_s^{i,\beta} = b \left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i \right) ds + \sigma \left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i \right) dW_s;$$

- *Branching rate γ* : given a position $Y_s^{i,\beta}$ at time s , the probability it dies in the time interval $[s, s + \delta s)$ is $\gamma \left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i \right) \delta s + o(\delta s)$.
- *Branching mechanism*: when it dies at a time s , it leaves behind (at the location where it died) a random number of offspring with probability $\left(p_k \left(Y_s^{i,\beta}, \xi_s^\beta, \beta_s^i \right) \right)_{k \in \mathbb{N}}$.

If the control is constant, *i.e.*, we are in the uncontrolled setting, conditionally on time and place of birth, the offspring evolve independently of each other in the same way as their parent.

Let L be the generator (associated with the spatial motion of each particle) defined on $\varphi \in C_b^2(\mathbb{R}^d)$ as

$$L\varphi(x, \lambda, a) = b(x, \lambda, a)^\top D\varphi(x) + \frac{1}{2} \text{Tr} \left(\sigma \sigma^\top(x, \lambda, a) D^2\varphi(x) \right),$$

where D and D^2 denote gradient and Hessian. The representation of previous properties is given by the following SDE

$$\begin{aligned} \langle \varphi, \xi_s^\beta \rangle &= \langle \varphi, \xi_t^\beta \rangle + \int_t^s \sum_{i \in V_u} D\varphi(Y_u^{i,\beta})^\top \sigma \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) dB_u^i + \int_t^s \sum_{i \in V_u} L\varphi \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) du \\ &+ \int_{(t,s] \times \mathbb{R}_+} \sum_{i \in V_{u-}} \sum_{k \geq 0} (k-1) \varphi(Y_u^{i,\beta}) \mathbf{1}_{I_k \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right)}(z) Q^i(dudz), \end{aligned} \quad (2.6)$$

with

$$I_k(x, \lambda, a) = \left[\gamma(x, \lambda, a) \sum_{\ell=0}^{k-1} p_\ell(x, \lambda, a), \gamma(x, \lambda, a) \sum_{\ell=0}^k p_\ell(x, \lambda, a) \right),$$

for all $(x, \lambda, a) \in \mathbb{R}^d \times M^1(\mathbb{R}^d) \times A$, $k \geq 0$, with the value of an empty sum being zero by convention. Notice that $(I_k(x, \lambda, a))_{k \in \mathbb{N}}$ forms a partition of the interval $[0, \gamma(x, \lambda, a))$.

2.2.1 Existence of branching processes and moment estimates

We aim at showing the existence of controlled branching diffusions for any standard strong control and giving bounds on their moments. These two aspects are proved in the following two propositions, adapting [7, Proposition 2.1] to our context.

Proposition 2.1. *Let $t \in [0, T]$, $\lambda := \sum_{i \in V} \delta_{x^i} \in \mathcal{N}[\mathbb{R}^d]$ with $V \subseteq \mathcal{I}$ finite, and β be a standard strong control. There exists a unique (up to indistinguishability) càdlàg and adapted process $(\xi_s^\beta)_{s \geq t}$ satisfying (2.6) such that $\xi_t^\beta = \lambda$. In addition, there exists a constant $C > 0$ depending only on T and on the coefficients b, σ, γ and $(p_k)_k$ such that*

$$\mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, t+h]} |V_u| \right] \leq \langle 1, \lambda \rangle e^{C_\gamma C_\Phi^1 h}, \quad (2.7)$$

$$\mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, t+h]} |V_u|^2 \right] \leq \langle 1, \lambda \rangle e^{C_\gamma (C_\Phi^1 + C_\Phi^2) h}, \quad (2.8)$$

$$\mathbb{E}^{\mathbb{P}^s} \left[\int_t^{t+h} \sum_{i \in V_u} |\beta_u^i| du \right] \leq C, \quad (2.9)$$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, t+h]} \sum_{i \in V_u} |Y_u^{i, \beta}| \right] &\leq C \left(\sum_{i \in V} |x^i| + \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{t+h} |V_u| du \right] \right. \\ &\quad \left. + \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{t+h} \sum_{i \in V_u} |\beta_u^i| du \right] \right), \end{aligned} \quad (2.10)$$

for any $h > 0$, where $|V|$ denotes the cardinality of $V \subseteq \mathcal{I}$.

Proof. Fix $(t, \lambda = \sum_{i \in V} \delta_{x^i}) \in \mathbb{R}_+ \times \mathcal{N}[\mathbb{R}^d]$, and β be a standard strong control. Using induction, we build the branching events of the population. We later show that such a process satisfies (2.6) and is well-defined. From (2.1), we have that (2.7) and (2.10) entail well-posedness for the process ξ^β .

Define by induction an increasing sequence of stopping time $(\tau_k)_{k \in \mathbb{N}}$, a sequence of random variables $(V_k)_{k \in \mathbb{N}}$ valued in the set of finite subsets of \mathcal{I} and a sequence of processes $\left((Y_s^{i, \beta})_{s \in [\tau_{k-1}, \tau_k]}, i \in V_k \right)_{k \in \mathbb{N}}$ such that

$$\xi_s^\beta = \sum_{k \geq 1} \mathbb{1}_{\tau_{k-1} \leq s < \tau_k} \sum_{i \in V_k} \delta_{Y_s^{i, \beta}}.$$

We set $\tau_0 = t$, $V_0 = V$, and $Y_t^{i, \beta} := x^i$ for all $i \in V$. Then, given τ_{k-1} and V_{k-1} , define τ_k as

$$\tau_k = \inf \{ s \in (\tau_{k-1}, T] : \exists i \in V_{k-1}, Q^i((\tau_{k-1}, s] \times [0, C_\gamma]) = 1 \}.$$

Define \mathcal{Y}^k , $\mathfrak{b}^k(\mathcal{Y}^k, \beta_s)$, $\Sigma^k(\mathcal{Y}^k, \beta_s)$, and \mathcal{W}^k , as

$$\begin{aligned} \mathcal{Y}_s^k &:= \begin{pmatrix} Y_s^{i_1, \beta} \\ \vdots \\ Y_s^{i_{|V_{k-1}|}, \beta} \end{pmatrix}, \quad \mathfrak{b}^k(\mathcal{Y}_s^k, \beta_s) := \begin{pmatrix} b\left(Y_s^{i_1, \beta}, \sum_{i \in V_{k-1}} \delta_{Y_s^{i, \beta}}, \beta_s^{i_1}\right) \\ \vdots \\ b\left(Y_s^{i_{|V_{k-1}|}, \beta}, \sum_{i \in V_{k-1}} \delta_{Y_s^{i, \beta}}, \beta_s^{i_{|V_{k-1}|}}\right) \end{pmatrix}, \\ \Sigma^k(\mathcal{Y}_s^k, \beta_s) &:= \begin{pmatrix} \sigma\left(Y_s^{i_1, \beta}, \sum_{i \in V_{k-1}} \delta_{Y_s^{i, \beta}}, \beta_s^{i_1}\right) \\ \vdots \\ \sigma\left(Y_s^{i_{|V_{k-1}|}, \beta}, \sum_{i \in V_{k-1}} \delta_{Y_s^{i, \beta}}, \beta_s^{i_{|V_{k-1}|}}\right) \end{pmatrix}, \quad \mathcal{W}_s^k = \begin{pmatrix} W_s^{i_1} \\ \vdots \\ W_s^{i_{|V_{k-1}|}} \end{pmatrix}, \end{aligned}$$

taking values in $\mathbb{R}^{|V_{k-1}|}$, $\mathbb{R}^{|V_{k-1}|}$, $\mathbb{R}^{|V_{k-1}| \times d'}$, and $\mathbb{R}^{|V_{k-1}|}$ respectively. As recalled in Remark 2.1, \mathfrak{b}^k and Σ^k are Lipschitz continuous in $\mathbb{R}^{|V_{k-1}|}$. Therefore, \mathcal{Y}^k is uniquely (up to indistinguishability) defined as the continuous and adapted process satisfying

$$\mathcal{Y}_s^k = \mathcal{Y}_{\tau_{k-1}}^k + \int_{\tau_{k-1}}^s \mathfrak{b}^k(\mathcal{Y}_u^k, \beta_u) du + \int_{\tau_{k-1}}^s \Sigma^k(\mathcal{Y}_u^k, \beta_u) d\mathcal{W}_u^k, \quad \mathbb{P} - \text{a.s.}$$

Describing what happens at branching events τ_k , we can conclude the construction of the branching process. Given the definition of τ_k , there is an (almost surely) unique label, that we denote $\hat{i}_k \in V_{k-1}$, such that

$$Q^{\hat{i}_k}((\tau_{k-1}, \tau_k] \times [0, C_\gamma]) = 1.$$

Let χ_k the $[0, C_\gamma]$ -valued random variable such that (τ_k, χ_k) belongs to the support of $Q^{\hat{i}_k}$. We set V_k as

$$V_k := \begin{cases} V_{k-1}, & \text{if } \chi_k \in [\gamma(Y_{\tau_k}^{\hat{i}_k, \beta}, \sum_{i \in V_{k-1}} \delta_{Y_{\tau_k}^{i, \beta}}, \beta_{\tau_k}^{\hat{i}_k}), C_\gamma], \\ V_{k-1} \setminus \{\hat{i}_k\}, & \text{if } \chi_k \in I_0(Y_{\tau_k}^{\hat{i}_k, \beta}, \sum_{i \in V_{k-1}} \delta_{Y_{\tau_k}^{i, \beta}}, \beta_{\tau_k}^{\hat{i}_k}), \\ V_{k-1} \setminus \{\hat{i}_k\} \cup \{\hat{i}_k 0, \dots, \hat{i}_k(\ell-1)\}, & \text{if } \chi_k \in I_\ell(Y_{\tau_k}^{\hat{i}_k, \beta}, \sum_{i \in V_{k-1}} \delta_{Y_{\tau_k}^{i, \beta}}, \beta_{\tau_k}^{\hat{i}_k}) \text{ for } \ell \geq 1, \end{cases}$$

where we impose the continuity of the flow for the off-spring, *i.e.*, $Y_{\tau_k}^{i, \beta} := Y_{\tau_k}^{\hat{i}_k, \beta}$ for $i \in V_k \setminus V_{k-1}$.

We prove that this process satisfies the SDE (2.6) by induction. Suppose it holds true up to τ_{k-1} , we have

$$\langle \varphi, \xi_{s \wedge \tau_k}^\beta \rangle = \mathbf{1}_{s \leq \tau_{k-1}} \langle \varphi, \xi_s^\beta \rangle + \mathbf{1}_{\tau_{k-1} < s < \tau_k} \sum_{i \in V_{k-1}} \varphi(Y_s^{i, \beta}) + \mathbf{1}_{s \geq \tau_k} \sum_{i \in V_k} \varphi(Y_{\tau_k}^{i, \beta}). \quad (2.11)$$

The first term on the r.h.s. satisfies (2.6) by induction hypothesis. We apply Itô's formula for each branch to deal with the second one. Finally, the third term is equal to

$$\begin{aligned} \sum_{i \in V_k} \varphi(Y_{\tau_k}^{i, \beta}) &= \sum_{i \in V_{k-1}} \varphi(Y_{\tau_k}^{i, \beta}) - \mathbf{1}_{\chi_k \in [0, \gamma(Y_{\tau_k}^{\hat{i}_k, \beta}, \sum_{i \in V_{k-1}} \delta_{Y_{\tau_k}^{i, \beta}}, \beta_{\tau_k}^{\hat{i}_k}))} \varphi(Y_{\tau_k}^{\hat{i}_k, \beta}) \\ &\quad + \sum_{\ell \geq 1} \mathbf{1}_{\chi_k \in I_\ell(Y_{\tau_k}^{\hat{i}_k, \beta}, \sum_{i \in V_{k-1}} \delta_{Y_{\tau_k}^{i, \beta}}, \beta_{\tau_k}^{\hat{i}_k}))} \sum_{l=1}^{\ell-1} \varphi(Y_{\tau_k}^{\hat{i}_k l, \beta}), \end{aligned}$$

which coincides with the integral w.r.t. the Poisson random measures over $(\tau_{k-1}, \tau_k]$. Therefore, (2.6) is satisfied up to τ_k and we can conclude by induction.

We focus now on estimates (2.7)-(2.10). Let $\{\theta_n\}_{n \in \mathbb{N}}$ be defined as follows

$$\theta_n := \inf \{s \geq t : |V_s| \geq n\} \wedge \inf \left\{ s \geq t : \sum_{i \in V_u} |Y_u^{i,\beta}| \geq n \right\}. \quad (2.12)$$

The first part of the proof ensures that $\xi_{\cdot \wedge \theta_n}^\beta$ well-defined and satisfies (2.6). Apply (2.6) to the function $x \mapsto 1$, obtaining

$$|V_{s \wedge \theta_n}| = |V_t| + \int_{(t, s \wedge \theta_n] \times \mathbb{R}_+} \sum_{i \in V_u} \sum_{k \geq 0} (k-1) \mathbf{1}_{I_k(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i)}(z) Q^i(dudz).$$

Applying Itô's formula, we also obtain

$$\begin{aligned} |V_{s \wedge \theta_n}|^2 &= |V_t|^2 + \int_{(t, s \wedge \theta_n] \times \mathbb{R}_+} \sum_{i \in V_u} \sum_{k \geq 0} \left((|V_u| + k - 1)^2 - |V_u|^2 \right) \mathbf{1}_{I_k(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i)}(z) Q^i(dudz) \\ &= |V_t|^2 + \int_{(t, s \wedge \theta_n] \times \mathbb{R}_+} \sum_{i \in V_u} \sum_{k \geq 0} (2(k-1)|V_u| + (k-1)^2) \mathbf{1}_{I_k(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i)}(z) Q^i(dudz). \end{aligned}$$

Therefore, we get

$$\begin{aligned} \sup_{u \in [t, s]} |V_{u \wedge \theta_n}| &\leq |V_t| + \int_{(t, s \wedge \theta_n] \times \mathbb{R}_+} \sum_{i \in V_u} \sum_{k \geq 1} (k-1) \mathbf{1}_{I_k(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i)}(z) Q^i(dudz), \\ \sup_{u \in [t, s]} |V_{u \wedge \theta_n}|^2 &\leq |V_t|^2 + \int_{(t, s \wedge \theta_n] \times \mathbb{R}_+} \sum_{i \in V_u} \sum_{k \geq 1} (2(k-1)|V_u| + (k-1)^2) \mathbf{1}_{I_k(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i)}(z) Q^i(dudz), \end{aligned}$$

and, taking the expectation,

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, s]} |V_{u \wedge \theta_n}| \right] &\leq |V_t| + \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{s \wedge \theta_n} \sum_{i \in V_u} \gamma(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i) \sum_{k \geq 1} (k-1) p_k(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i) du \right] \\ &\leq |V_t| + C_\gamma C_\Phi^1 \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{s \wedge \theta_n} \sup_{z \in [t, u]} |V_{z \wedge \theta_n}| \right], \\ \mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, s]} |V_{u \wedge \theta_n}|^2 \right] &\leq |V_t|^2 + C_\gamma (C_\Phi^1 + C_\Phi^2) \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{s \wedge \theta_n} \sup_{z \in [t, u]} |V_{z \wedge \theta_n}|^2 \right]. \end{aligned}$$

Applying Grönwall's lemma, we obtain

$$\mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, s]} |V_{u \wedge \theta_n}| \right] \leq |V_t| e^{C_\gamma C_\Phi^1 (s-t)}, \quad \mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, s]} |V_{u \wedge \theta_n}|^2 \right] \leq |V_t|^2 e^{C_\gamma (C_\Phi^1 + C_\Phi^2) (s-t)}.$$

Since the bound is uniform in n , θ_n converges almost surely to infinity, and by Fatou's lemma, we retrieve (2.7) and (2.8). This implies also (2.9), since

$$\mathbb{E}^{\mathbb{P}^s} \left[\int_t^s \sum_{i \in V_u} |\beta_u^i| du \right] \leq \mathbb{E}^{\mathbb{P}^s} \left[\int_t^s |V_u| \sup_{i \in \mathcal{I}} |\beta_u^i| du \right] \leq \mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, s]} |V_u| \int_t^s \sup_{i \in \mathcal{I}} |\beta_u^i| du \right] \leq C,$$

where in the last inequality we used Cauchy-Schwartz inequality, (2.5) and (2.8).

Proving (2.10) would be more tricky since the SDE (2.6) cannot be applied directly. We see that (2.11) is still valid for $\varphi(x) = |x|$. Itô's formula yields, for $s \in (\tau_{k-1}, \tau_k)$,

$$\begin{aligned}
\sum_{i \in V_{k-1}} \left| Y_s^{i,\beta} \right| &= \sum_{i \in V_{k-1}} \left| Y_{\tau_k}^{i,\beta} + \int_{\tau_{k-1}}^s b \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) du + \int_{\tau_{k-1}}^s \sigma \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) dW_u^i \right| \\
&\leq \sum_{i \in V_{k-1}} \left| Y_{\tau_k}^{i,\beta} \right| + \sum_{i \in V_{k-1}} \int_{\tau_{k-1}}^s \left| b \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) \right| du + \sum_{i \in V_{k-1}} \left| \int_{\tau_{k-1}}^s \sigma \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) dW_u^i \right| \\
&\leq \sum_{i \in V_{k-1}} \left| Y_{\tau_k}^{i,\beta} \right| + C_b \int_{\tau_{k-1}}^s |V_u| du + C_b \sum_{i \in V_{k-1}} \int_{\tau_{k-1}}^s \left(\left| Y_u^{i,\beta} \right| + \left| \beta_u^i \right| \right) du + \\
&\quad \sum_{i \in V_{k-1}} \left| \int_{\tau_{k-1}}^s \sigma \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) dW_u^i \right|,
\end{aligned}$$

where we have used the bound (2.3) over the coefficient b in the last inequality. Since the family of Brownian motions $\{W^i\}_{i \in \mathcal{I}}$ are independent from the one of Poisson measures $\{Q^i\}_{i \in \mathcal{I}}$, we have that taking the conditional expectation with respect to $\mathcal{F}_{\tau_{k-1}}$, we can apply the Burkholder-Davis-Gundy's inequalities (see, e.g., [12, Theorem 92]). This means that there exists a constant $C > 0$ (which may change from line to line) such that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [\tau_{k-1} \wedge \theta_n, s \wedge \tau_k \wedge \theta_n]} \sum_{i \in V_{k-1}} \left| \int_{\tau_{k-1} \wedge \theta_n}^u \sigma \left(Y_r^{i,\beta}, \xi_r^\beta, \beta_r^i \right) dW_r^i \right| \middle| \mathcal{F}_{\tau_{k-1}} \right] \\
&\leq C \mathbb{E}^{\mathbb{P}^s} \left[\sum_{i \in V_{k-1}} \left(\int_{\tau_{k-1} \wedge \theta_n}^{s \wedge \tau_k \wedge \theta_n} \text{Tr} \left(\sigma \sigma^\top \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) \right) du \right)^{1/2} \middle| \mathcal{F}_{\tau_{k-1}} \right] \\
&\leq C \mathbb{E}^{\mathbb{P}^s} \left[(s \wedge \tau_k \wedge \theta_n - \tau_{k-1} \wedge \theta_n) |V_{k-1}| \middle| \mathcal{F}_{\tau_{k-1}} \right] = C \mathbb{E}^{\mathbb{P}^s} \left[\int_{\tau_{k-1} \wedge \theta_n}^{s \wedge \tau_k \wedge \theta_n} |V_u| du \middle| \mathcal{F}_{\tau_{k-1}} \right],
\end{aligned}$$

using (2.3) in the last line. Therefore, by induction, there exists a constant $C > 0$ (which may change from line to line) such that

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, s]} \sum_{i \in V_u \wedge \theta_n} \left| Y_u^{i,\beta} \right| \right] &\leq \sum_{i \in V} |x^i| + C \left(\mathbb{E}^{\mathbb{P}^s} \left[\int_t^{s \wedge \theta_n} |V_u| du \right] + \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{s \wedge \theta_n} \sum_{i \in V_u} \left| Y_u^{i,\beta} \right| du \right] \right. \\
&\quad \left. + \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{s \wedge \theta_n} \sum_{i \in V_u} \left| \beta_u^i \right| du \right] \right),
\end{aligned}$$

using (2.7) and (2.9) to bound the mass of the population. Applying Grönwall's lemma, we obtain

$$\mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, s]} \sum_{i \in V_u \wedge \theta_n} \left| Y_u^{i,\beta} \right| \right] \leq C \left(\sum_{i \in V} |x^i| + \mathbb{E}^{\mathbb{P}^s} \left[\int_t^s |V_u| du \right] + \mathbb{E}^{\mathbb{P}^s} \left[\int_t^s \sum_{i \in V_u} \left| \beta_u^i \right| du \right] \right).$$

Since the estimate is uniform in n and θ_n converges almost surely to infinity, applying Fatou's lemma, we retrieve (2.10). \square

2.2.2 Control problem

We are given the continuous functions $\psi : \mathbb{R}^d \times M^1(\mathbb{R}^d) \times A \rightarrow \mathbb{R}$, $\Psi : M^1(\mathbb{R}^d) \rightarrow \mathbb{R}$. We suppose that there exists $C_\Psi, c_\psi > 0$ such that

$$\Psi(\lambda) \leq C_\Psi \left(1 + \int_{\mathbb{R}^d} |x|^2 \lambda(dx) + \langle 1, \lambda \rangle^2 \right) \quad (2.13)$$

$$\Psi(\mu) \geq -C_\Psi \left(1 + \int_{\mathbb{R}^d} |x| \lambda(dx) + \langle 1, \lambda \rangle \right) \quad (2.14)$$

$$\psi(x, \lambda, a) \leq C_\Psi \left(1 + |x|^2 + \int_{\mathbb{R}^d} |x| \lambda(dx) + |a|^2 \right) \quad (2.15)$$

$$\psi(x, \lambda, a) \geq -C_\Psi (1 + |x|) + c_\psi |a|^2 \quad (2.16)$$

for $\lambda \in M^1(\mathbb{R}^d)$.

Fix a standard strong control β and $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$ a starting condition. We define the cost function as

$$J(t, \lambda; \beta) := \mathbb{E}^{\mathbb{P}^\beta} \left[\int_t^T \sum_{i \in V_s} \psi \left(Y_s^{i, \beta}, \xi_s^\beta, \beta_s^i \right) ds + \Psi \left(\xi_T^\beta \right) \middle| \xi_t^\beta = \lambda \right].$$

As the dependence of the cost J on the label is solely through the spatial components and the control, we limit the set of controls. This restriction is implemented to maintain symmetry between positions in \mathbb{R}^d and the chosen control in A , enabling a natural embedding of strong controls into relaxed ones.

Definition 2.2 (Admissible strong control). *Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$. We say that $\beta = (\beta^i)_{i \in \mathcal{I}}$ is an admissible strong control, and we denote $\beta \in \mathcal{R}_{(t, \lambda)}^s$, if β is a standard strong control and*

$$\mathbb{E}^{\mathbb{P}^\beta} \left[\int_t^T \sum_{i, j \in V_s, i \neq j} \mathbf{1}_{Y_s^{i, \beta} = Y_s^{j, \beta}, \beta_s^i \neq \beta_s^j} ds \right] = 0. \quad (2.17)$$

We can now state the *strong control problem* as

$$v^s(t, \lambda) = \inf \left\{ J(t, \lambda; \beta) : \beta \in \mathcal{R}_{(t, \lambda)}^s \right\}, \quad (2.18)$$

for $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$.

Remark 2.2. *Under additional assumptions, restricting from standard to admissible controls does not impact the value function. For example, whenever σ is uniformly elliptic, i.e., there exist $\varepsilon > 0$ such that $\sigma \sigma^\top(x, \lambda, a) \geq \varepsilon \mathbb{I}_d$, with \mathbb{I}_d being the identity matrix of dimension $d \times d$, all alive particles take different positions $dt \otimes d\mathbb{P}$ -a.s. Therefore, all standard controls are admissible.*

2.3 Well-posedness of the control problem

To finally give a well-posedness of the control problem, we must prove the finite second order of the Branching Processes, at least close to an optimal value. We apply the techniques used to prove Proposition 2.1 to get the next lemma.

Lemma 2.1. *Let $(t, \lambda) \in \mathbb{R}_+ \times \mathcal{N}[\mathbb{R}^d]$, and β be a standard strong control. There exists a constant $C > 0$ depending only on T and on the coefficients b, σ, γ and $(p_k)_k$ such that*

$$\mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, t+h]} \sum_{i \in V_u} |Y_u^{i, \beta}|^2 \right] \leq C \left(\sum_{i \in V} |x^i|^2 + \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{t+h} |V_u| du \right] \right. \\ \left. + \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{t+h} \sum_{i \in V_u} |\beta_u^i|^2 du \right] \right), \quad (2.19)$$

for any $h > 0$.

Proof. Fix $(t, \lambda = \sum_{i \in V} \delta_{x^i}) \in \mathbb{R}_+ \times \mathcal{N}[\mathbb{R}^d]$, and β be a standard strong control. Let $\{\theta_n\}_{n \in \mathbb{N}}$ be as in (2.12). We have that $\xi_{s \wedge \theta_n}^\beta$ satisfies (2.6). Applying (2.6) to the function $x \mapsto |x|^2$, we get

$$\sum_{i \in V_{s \wedge \theta_n}} |Y_{s \wedge \theta_n}^{i, \beta}|^2 = \sum_{i \in V} |x^i|^2 + \int_t^{s \wedge \theta_n} \sum_{i \in V_u} 2 \left(Y_u^{i, \beta} \right)^\top \sigma \left(Y_u^{i, \beta}, \xi_u^\beta, \beta_u^i \right) dB_u^i \\ + \int_t^{s \wedge \theta_n} \sum_{i \in V_u} 2 \left(Y_u^{i, \beta} \right)^\top b \left(Y_u^{i, \beta}, \xi_u^\beta, \beta_u^i \right) du + \\ + \int_t^{s \wedge \theta_n} \sum_{i \in V_u} \text{Tr} \left(\sigma \sigma^\top \left(Y_u^{i, \beta}, \xi_u^\beta, \beta_u^i \right) \right) du \\ + \int_{(t, s \wedge \theta_n] \times \mathbb{R}_+} \sum_{i \in V_u} \sum_{k \geq 0} (k-1) |Y_u^{i, \beta}|^2 \mathbf{1}_{I_k \left(Y_u^{i, \beta}, \xi_u^\beta, \beta_u^i \right)}(z) Q^i(du dz),$$

Taking the supremum in the interval $[t, s]$ and taking the expectation, we bound each term in the r.h.s. Applying Burkholder-Davis-Gundy's inequalities to the second term, there exists a constant $C > 0$ (which may change from line to line) such that

$$\mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, s]} \int_t^{u \wedge \theta_n} \sum_{i \in V_r} 2 \left(Y_r^{i, \beta} \right)^\top \sigma \left(Y_r^{i, \beta}, \xi_r^\beta, \beta_r^i \right) dB_r^i \right] \\ \leq C \mathbb{E}^{\mathbb{P}^s} \left[\left(\int_t^{s \wedge \theta_n} \sum_{i \in V_u} |Y_u^{i, \beta}|^2 \text{Tr} \left(\sigma \sigma^\top \left(Y_u^{i, \beta}, \xi_u^\beta, \beta_u^i \right) \right) du \right)^{1/2} \right] \leq C \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{s \wedge \theta_n} \sum_{i \in V_u} |Y_u^{i, \beta}|^2 du \right].$$

From (2.3) on the growth of b and σ , the third and the fourth terms can be bounded as follows

$$\mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, s]} \int_t^{u \wedge \theta_n} \sum_{i \in V_r} \left(2 \left(Y_r^{i, \beta} \right)^\top b \left(Y_r^{i, \beta}, \xi_r^\beta, \beta_r^i \right) + \text{Tr} \left(\sigma \sigma^\top \left(Y_r^{i, \beta}, \xi_r^\beta, \beta_r^i \right) \right) \right) dr \right] \\ \leq C \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{s \wedge \theta_n} |V_u| + \sum_{i \in V_u} |Y_u^{i, \beta}|^2 + |\beta_u^i|^2 du \right],$$

using that $a^\top b \leq \frac{1}{2}(|a|^2 + |b|^2)$ for $a, b \in \mathbb{R}^d$. Finally, the last term gives

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, s]} \int_{(t, u \wedge \theta_n] \times \mathbb{R}^+} \sum_{i \in V_{r-}} \sum_{k \geq 0} (k-1) \left| Y_r^{i, \beta} \right|^2 \mathbf{1}_{I_k(Y_r^{i, \beta}, \xi_r^\beta, \beta_r^i)}(z) Q^i(drdz) \right] \\ & \leq \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{s \wedge \theta_n} \sum_{i \in V_{u-}} \gamma(Y_u^{i, \beta}, \xi_u^\beta, \beta_u^i) \sum_{k \geq 1} (k-1) \left| Y_u^{i, \beta} \right|^2 p_k(Y_u^{i, \beta}, \xi_u^\beta, \beta_u^i) du \right] \\ & \leq C \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{s \wedge \theta_n} \sum_{i \in V_u} \left| Y_u^{i, \beta} \right|^2 du \right]. \end{aligned}$$

Combining all the terms and using Gronwall's inequality first and Fatou's lemma then, we obtain (2.19). \square

This lemma tells us that whenever $\mathbb{E}^{\mathbb{P}^s} \left[\int_t^T \sum_{i \in V_u} |\beta_u^i|^2 du \right] < \infty$, we have $|J(t, \lambda; \beta)| < \infty$ from the coercivity bounds. Therefore, ε -optimal controls must satisfy this condition, as shown in the following proposition.

Proposition 2.2. Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$. Let $\varepsilon > 0$, and let $\mathcal{R}_{(t, \lambda)}^{s, \varepsilon}$ be the set of $\beta \in \mathcal{R}_{(t, \lambda)}^s$ satisfying

$$J(t, \lambda; \beta) \leq v^s(t, \lambda) + \varepsilon.$$

Then

$$\sup_{\beta \in \mathcal{R}_{(t, \lambda)}^{s, \varepsilon}} \mathbb{E}^{\mathbb{P}^s} \left[\int_t^T \sum_{i \in V_u} |\beta_u^i|^2 du \right] < \infty. \quad (2.20)$$

Moreover, $v^s(t, \lambda) > -\infty$.

Proof. We use (2.14) and (2.16) along with Lemma 2.1 to find a constant $C > 0$ (which may change from line to line) such that, for all $\beta \in \mathcal{R}_{(t, \lambda)}^s$,

$$\begin{aligned} J(t, \lambda; \beta) & \geq -C \mathbb{E}^{\mathbb{P}^s} \left[1 + \sup_{u \in [t, T]} |V_u|^2 + \sup_{u \in [t, T]} \sum_{i \in V_u} |Y_u^{i, \beta}| \right] + c_\psi \mathbb{E}^{\mathbb{P}^s} \left[\int_t^T \sum_{i \in V_u} |\beta_u^i|^2 du \right] \\ & \geq -C \mathbb{E}^{\mathbb{P}^s} \left[1 + \int_t^T \sum_{i \in V_u} |\beta_u^i| du \right] + c_\psi \mathbb{E}^{\mathbb{P}^s} \left[\int_t^T \sum_{i \in V_u} |\beta_u^i|^2 du \right] \end{aligned} \quad (2.21)$$

This already proves $v^s(t, \lambda) > -\infty$, as the function $a \mapsto c_\psi |a|^2 - C|a|$ is bounded from above. To prove the first claim, fix arbitrarily a constant control $\beta_s^{a_0, i} := a_0 \in A$. Lemma 2.1 and Proposition 2.1 imply

$$\mathbb{E}^{\mathbb{P}^s} \left[\sup_{u \in [t, t+h]} \sum_{i \in V_u} \left| Y_u^{i, \beta^{a_0}} \right|^2 \right] \leq C \left(1 + \mathbb{E}^{\mathbb{P}^s} \left[\int_t^{t+h} \sum_{i \in V_u} |\beta_u^{a_0, i}|^2 du \right] \right) \leq C (1 + |a_0|^2).$$

Then, from (2.13) and (2.15), we have show $J(t, \lambda; \beta^{a_0}) < \infty$. Therefore, for $\beta \in \mathcal{R}_{(t, \lambda)}^{s, \varepsilon}$, we have $J(t, \lambda; \beta) \leq J(t, \lambda; \beta^{a_0}) + \varepsilon$. This and (2.21) yield

$$\sup_{\beta \in \mathcal{R}_{(t, \lambda)}^{s, \varepsilon}} \mathbb{E}^{\mathbb{P}^s} \left[\int_t^T \sum_{i \in V_u} \left(|\beta_u^i|^2 - C |\beta_u^i| \right) du \right] < \infty.$$

This gives (2.20), by Proposition 2.1. \square

3 Relaxed formulation

We give the relaxed formulation for the branching diffusion control problem by working with relaxed controls and weak solutions of the previous SDE.

We equip the product space $[0, T] \times \mathbb{R}^d \times A$ with the σ -algebra $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(A)$. Let $\mathcal{A}^{\text{Leb}} \subseteq M^1([0, T] \times \mathbb{R}^d \times A)$ be the set of measures, whose projection on $[0, T]$ is the Lebesgue measure. Each $\alpha \in \mathcal{A}^{\text{Leb}}$ can be identified with its disintegration (see, e.g., [30, Corollary 1.26, Chapter 1]). In particular, we have $\alpha(ds, dx, da) = ds \mathbf{y}_s(dx) \bar{\alpha}_s(x, da)$, for a process $(\mathbf{y}_s(dx))_s$ (resp. $(\bar{\alpha}_s(x, da))_s$) taking values in the set of functions from $[0, T]$ (resp. $[0, T] \times \mathbb{R}^d$) to $M^1(\mathbb{R}^d)$ (resp. $M^1(A)$). Let $\mathcal{A}^{\text{Leb}, \cdot, 1} \subseteq \mathcal{A}^{\text{Leb}, \cdot, \cdot}$ be the set of elements α such that $\bar{\alpha}_s(x, da) \in \mathcal{P}^1(A)$ for any $(s, x) \in [0, T] \times \mathbb{R}^d$. For $\mathbf{x} = (\mathbf{x}_s)_s \in \mathbf{D}^d$ fixed, we denote the *space of relaxed controls* $\mathcal{A}^{\text{Leb}, \mathbf{x}, 1}$ as

$$\mathcal{A}^{\text{Leb}, \mathbf{x}, 1} := \left\{ \alpha \in \mathcal{A}^{\text{Leb}, \cdot, 1} : \alpha(ds, dx, da) = ds \mathbf{x}_s(dx) \bar{\alpha}_s(x, da) \text{ a.e. } s \in [0, T] \right\},$$

which is *weakly* closed*.

3.1 Martingale model

Let \mathcal{L} be the generator defined on the cylindrical functions $F_\varphi = F(\langle \varphi, \cdot \rangle)$, for $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$, as

$$\begin{aligned} \mathcal{L}F_\varphi(x, \lambda, a) &= F'(\langle \varphi, \lambda \rangle) L\varphi(x, \lambda, a) + \frac{1}{2} F''(\langle \varphi, \lambda \rangle) |D\varphi(x)\sigma(x, \lambda, a)|^2 \\ &\quad + \gamma(x, \lambda, a) \left(\sum_{k \geq 0} F(\langle \varphi, \lambda \rangle + (k-1)\varphi(x)) p_k(x, \lambda, a) - F_\varphi(\lambda) \right). \end{aligned}$$

For simplicity, we write $F'_\varphi(\lambda)$ for $F'(\langle \varphi, \lambda \rangle)$ and $F''_\varphi(\lambda)$ for $F''(\langle \varphi, \lambda \rangle)$. Moreover, for $\mathbb{F} = \{\mathcal{F}_s\}_{s \geq 0}$ a filtration, we denote $\hat{\mathbb{F}} = \{\hat{\mathcal{F}}_s\}_{s \geq 0}$ the filtration such that $\hat{\mathcal{F}}_s := \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_s$ for any $s \geq 0$.

Definition 3.3 (Relaxed control). *Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$. We say that \mathcal{C} is a relaxed control, and we denote $\mathcal{C} \in \mathcal{R}_{(t, \lambda)}^r$, if*

$$\mathcal{C} = \left(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s\}_{s \geq 0}, (X_s)_{s \geq 0}, (\bar{\alpha}_s)_{s \geq 0} \right)$$

where

- (i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with filtration \mathbb{F} ;
- (ii) $(X_s)_{s \geq 0}$ is an \mathbb{F} -progressively measurable process living in \mathbf{D}^d such that $\mathbb{P}(X_t = \lambda) = 1$;
- (iii) $\bar{\alpha} : [0, T] \times \mathbb{R}^d \times \Omega \rightarrow \mathcal{P}^1(A)$ is a $\hat{\mathbb{F}}$ -predictable process associated with $\alpha \in \mathcal{A}^{\text{Leb}, \cdot, 1}$ such that $\mathbb{P}(\alpha \in \mathcal{A}^{\text{Leb}, X, 1}) = 1$, i.e.,

$$\begin{aligned} \mathbb{P}(\alpha(ds, dx, da) = ds X_s(dx) \bar{\alpha}_s(x, da) \text{ a.e. } s \in [0, T]) &= 1, \\ \mathbb{E}^{\mathbb{P}} \left[\int_t^T \int_{\mathbb{R}^d \times A} |a| \bar{\alpha}_s(x, da) X_s(dx) ds \right] &< \infty; \end{aligned}$$

(iv) for any $F_\varphi = F(\langle \varphi, \cdot \rangle)$, with $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$, the process

$$M_s^{F_\varphi} = F_\varphi(X_s) - \int_t^s \int_{\mathbb{R}^d \times A} \mathcal{L}F_\varphi(x, X_u, a) \bar{\alpha}_u(x, da) X_u(dx) du \quad (3.22)$$

is a (\mathbb{P}, \mathbb{F}) -martingale for $s \geq t$.

Remark 3.3. We highlight two main aspects of this definition.

1. For $\mathcal{C} \in \mathcal{R}_{(t, \lambda)}^v$, we are only interested in the time interval $[t, T]$. Therefore, X_s and α_s can be redefined for $s \in [0, t)$ as $X_s = \lambda$ and $\alpha_s = \delta_{a_0}$ for some $a_0 \in A$.
2. For $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$, admissible strong controls are embedded in $\mathcal{R}_{(t, \lambda)}^v$. Indeed, it suffices to consider $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ as in Section 2 and define $(\bar{\alpha}_s)_s$ as $\bar{\alpha}_s(x, da) = \delta_{\mathbf{a}(s, x)}$ for

$$\mathbf{a}(s, x) := \frac{\sum_{i \in V_{s-}} \beta_{s-}^i \mathbf{1}_{Y_{s-}^{i, \beta} = x}}{\sum_{i \in V_{s-}} \mathbf{1}_{Y_{s-}^{i, \beta} = x}} \mathbf{1}_{\{|V_{s-}| > 0\}} + a_0 \mathbf{1}_{\{|V_{s-}| = 0\} \cup \{s \leq t\}}, \quad (3.23)$$

for some $a_0 \in A$ and with the convention $0/0 := a_0$. The SDE (2.6), combined with Itô's formula for semimartingales, implies (3.22). Hence, it is a relaxed control, and, with abuse of notation we denote $\beta \in \mathcal{R}_{(t, \lambda)}^v$.

We can find equivalent representations of (3.22), an important tool in the manipulation of these objects. It is given using the quadratic variation of a martingale (see, e.g., [28, Chapter I-4e]).

Lemma 3.2. Given $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$, let $\mathcal{C} = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s\}_s, (X_s)_s, (\alpha_s)_s)$ be such that conditions (i), (ii), and (iii) in the definition 3.3 are satisfied. The following are equivalent.

(i) We have $\mathcal{C} \in \mathcal{R}_{(t, \lambda)}^v$.

(ii) For any $\varphi \in C_b^2(\mathbb{R}^d)$ such that $\varphi > \varepsilon$ for some $\varepsilon > 0$ and $\sup_{\mathbb{R}^d} \varphi \leq 1$,

$$M_s^{\exp \log \varphi} = e^{\langle \log \varphi, X_s \rangle} - \int_t^s \int_{\mathbb{R}^d \times A} \left(\frac{L\varphi(x, X_u, a) + \gamma(x, X_u, a)(\Phi(\varphi(x), x, X_u, a) - \varphi(x))}{\varphi(x)} \right) \bar{\alpha}_u(x, da) X_u(dx) e^{\langle \log \varphi, X_u \rangle} du \quad (3.24)$$

is a (\mathbb{P}, \mathbb{F}) -martingale for $s \geq t$.

(iii) For any $\varphi \in C_b^2(\mathbb{R}^d)$ the process

$$\begin{aligned} \bar{M}_s^\varphi = \langle \varphi, X_t \rangle & - \int_t^s \int_{\mathbb{R}^d \times A} L\varphi(x, X_u, a) \bar{\alpha}_u(x, da) X_u(dx) du \\ & - \int_t^s \int_{\mathbb{R}^d \times A} \gamma(x, X_u, a) (\partial_s \Phi(1, x, X_u, a) - 1) \varphi(x) \\ & \bar{\alpha}_u(x, da) X_u(dx) du, \quad s \in [t, T]. \end{aligned} \quad (3.25)$$

is a (\mathbb{P}, \mathbb{F}) -martingale with quadratic variation process

$$\begin{aligned} [\bar{M}^\varphi]_s & = \int_t^s \int_{\mathbb{R}^d \times A} \left(\text{Tr} \left(\sigma \sigma^\top(x, X_u, a) D\varphi D\varphi^\top(x) \right) \right. \\ & \left. + \gamma(x, X_u, a) (\partial_{ss}^2 \Phi(1, x, X_u, a) - \partial_s \Phi(1, x, X_u, a) + 1) \varphi^2(x) \right) \\ & \bar{\alpha}_u(x, da) X_u(dx) du, \quad s \in [t, T]. \end{aligned} \quad (3.26)$$

Proof. (i) \implies (ii): We need to prove that (3.22) is a martingale for the function $F_{\log \varphi}$ with $F(x) = \exp(x)$ and $\varphi \in C_b^2(\mathbb{R}^d)$ such that $\varphi > \varepsilon$ for some $\varepsilon > 0$ and $\sup_{\mathbb{R}^d} \varphi \leq 1$. The process $M^{\exp \log \varphi}$, as in (3.24), is a local martingale. To prove that it is a martingale, we show its quadratic variation has a finite expectation. Since the compensator of $(M^{\exp \log \varphi})^2$ is the same of $M^{\exp 2 \log \varphi} = M^{\exp \log \varphi^2}$, we get the quadratic variation of $M^{\exp \log \varphi}$ applying (3.22) to $F \in C_b^2(\mathbb{R})$ and φ^2 . Therefore, it is equal to

$$[M^{\exp \log \varphi}]_s = \int_t^s \int_{\mathbb{R}^d \times A} \left(\frac{L\varphi^2(x, X_u, a) + \gamma(x, X_u, a)(\Phi(\varphi^2(x), x, X_u, a) - \varphi^2(x))}{\varphi^2(x)} \right) \bar{\alpha}_u(x, da) X_u(dx) e^{\langle \log \varphi^2, X_u \rangle} du.$$

Since $[M^{\exp \log \varphi}]$ is uniformly bounded, using Itô's isometry, $M^{\exp \log \varphi}$ is a martingale.

(ii) \implies (iii): Fix $f \in C_b^2(\mathbb{R}^d)$. For $\theta > 0$, and $M_f := \sup_{\mathbb{R}^d} |f|$, we define $\varphi_1 := e^{\theta(f-M_f)}$ and $\varphi_2 := e^{-\theta M_f}$. Since f is bounded, there exists $\varepsilon > 0$ such that $\varphi_1 > \varepsilon$ and $\sup_{\mathbb{R}^d} \varphi_1 \leq 1$. Applying (3.24) to φ_1 and φ_2 , we get

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left[e^{\langle \theta(f-M_f), X_{s+h} \rangle} - e^{\langle \theta(f-M_f), X_s \rangle} \right. \\ & \quad - \int_s^{s+h} \int_{\mathbb{R}^d \times A} \left(\theta Lf(x, X_u, a) + \theta^2 \text{Tr} \left(\sigma \sigma^\top(x, X_u, a) Df Df^\top(x) \right) \right. \\ & \quad \left. \left. + \gamma(x, X_u, a) \frac{\Phi \left((e^{\theta(f(x)-M_f)}), x, X_u, a \right) - e^{\theta(f(x)-M_f)}}{e^{\theta(f(x)-M_f)}} \right) \right. \\ & \quad \left. \left. \bar{\alpha}_u(x, da) X_u(dx) e^{\langle \theta(f-M_f), X_u \rangle} du \right) \Big| \mathcal{F}_s \right] = 0, \end{aligned} \quad (3.27)$$

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left[e^{\langle -\theta M_f, X_{s+h} \rangle} - e^{\langle -\theta M_f, X_s \rangle} - \int_s^{s+h} \int_{\mathbb{R}^d \times A} \gamma(x, X_u, a) \right. \\ & \quad \left. \frac{\Phi(e^{-\theta M_f}, x, X_u, a) - (e^{-\theta M_f})}{e^{-\theta M_f}} \bar{\alpha}_u(x, da) X_u(dx) e^{\langle -\theta M_f, X_u \rangle} du \right] \Big| \mathcal{F}_s = 0. \end{aligned} \quad (3.28)$$

Since all the functions are bounded, we are allowed to differentiate with respect to θ . Dividing by θ , subtracting (3.27) and (3.28), and setting $\theta = 0$, we get (3.25). Differentiating twice with respect to θ , dividing by θ^2 subtracting (3.27) and (3.28) and setting $\theta = 0$, we get (3.26).

(iii) \implies (i): We prove the last implication using Itô's formula for semimartingales. Fix $F \in C^2(\mathbb{R}^n)$ and $f \in C_b^2(\mathbb{R}^n)$. We have that $\langle f, X_s \rangle_{s \geq t}$ is a \mathbb{P} -semimartingale, and so, by Itô's formula, we have (3.22). \square

3.2 Representation and relaxed control problem

In this section, we show that relaxed controls can be expressed as solutions to stochastic differential equations. This representation proves valuable in establishing the non-explosion property and, subsequently, the well-posedness of the control problem. This characterization relies on martingale measures within extensions of the designated space. Succinct definitions and pertinent results concerning these entities are summarized in [15] (for a comprehensive study on the subject, refer to [42]). Here, we provide a brief recap of their definition.

Definition 3.4. Let (G, \mathcal{G}) be a Lusin space with its σ -algebra, and $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s\}_s)$ a filtered space satisfying the usual condition, where we define \mathcal{P} the predictable σ -field. A process \mathcal{M} on $\Omega \times [0, T] \times \mathcal{G}$ is called martingale measure on G if

- (i) $\mathcal{M}_0(E) = 0$ a.s. for any $E \in \mathcal{G}$;
- (ii) \mathcal{M}_t is a σ -finite, $L^2(\Omega)$ -valued measure for all $t \in [0, T]$;
- (iii) $(\mathcal{M}_t(E))_{t \in [0, T]}$ is an \mathbb{F} -martingale for any $E \in \mathcal{G}$.

We say that \mathcal{M} is orthogonal if the product $\mathcal{M}_t(E)\mathcal{M}_t(E')$ is a martingale for any two disjoint sets $E, E' \in \mathcal{G}$. We also say, on one hand, that is continuous if $(\mathcal{M}_t(E))_{t \geq 0}$ is continuous, purely discontinuous, on the other hand, if $(\mathcal{M}_t(E))_{t \geq 0}$ is a purely discontinuous martingale for any $E \in \mathcal{G}$.

For a strong representation of relaxed controls, we rely on the notion of *predictable projection* and *intensity* that we briefly recall. For an \mathbb{R} -valued \mathbb{F} -adapted process Y , there exists (see, e.g., [28, Theorem 2.28, Chapter I]) a $(-\infty, \infty]$ -valued process, called the *predictable projection* of Y and denoted by ${}^P Y$. It is determined uniquely up to a negligible set by the following two conditions:

- (i) it is predictable;
- (ii) ${}^P Y_T = \mathbb{E}^{\mathbb{P}} [Y_T | \mathcal{F}_{T-}]$ on $\{T < \infty\}$ for all predictable stopping times T .

For a continuous orthogonal martingale measure \mathcal{M} on G , there exists a random, predictable real-valued measure I on $\mathcal{B}([0, T]) \otimes \mathcal{G}$, called *intensity* of \mathcal{M} , defined by: $[\mathcal{M}(E)]_s = \int_0^t \int_E I(dx, ds)$ \mathbb{P} -a.s., for all $t > 0$. We can construct a stochastic integral with respect to \mathcal{M} for all functions φ defined on $\Omega \times [0, T] \times G$, $\mathcal{P} \otimes \mathcal{G}$ measurable, such that

$$\mathbb{E}^{\mathbb{P}} \left[\int_0^t \int_E \varphi^2(\omega, s, x) I(\omega, dx, ds) \right] < \infty,$$

denoted by $\int_0^t \int_E \varphi(s, x) \mathcal{M}(dx, ds)$. We refer to [42, Chapter 2] for the proofs.

The representation of these processes is grounded in the representation theorems for continuous and purely discontinuous martingale measures, as done in [35]. We apply her construction in our context and get the following proposition.

Proposition 3.3. *Let $\mathcal{C} = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s\}_s, (X_s)_s, (\alpha_s)_s) \in \mathcal{R}_{(t, \lambda)}^c$. There exists an extension $(\hat{\Omega} = \Omega \times \tilde{\Omega}, \hat{\mathcal{F}} = \mathcal{F} \otimes \tilde{\mathcal{F}}, \hat{\mathbb{P}} = \mathbb{P} \otimes \tilde{\mathbb{P}}, \{\hat{\mathcal{F}}_s = \mathcal{F}_s \otimes \tilde{\mathcal{F}}_s\}_s)$ of $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, where we naturally extend X and α , that satisfies the following properties.*

1. $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\mathbb{F}})$ is a filtered probability space supporting a continuous $\hat{\mathbb{F}}$ -martingale measures \mathcal{M}^c on $\hat{\Omega} \times [0, T] \times \mathbb{R}^d \times A$, with intensity measure $ds X_s(dx) \bar{\alpha}_s(x, da)$, and a purely discontinuous $\hat{\mathbb{F}}$ -martingale measure \mathcal{M}^d on $\hat{\Omega} \times [0, T] \times \mathbb{R}^d \times \mathbb{R}_+ \times A$, with dual predictable projection measure $ds X_s(dx) dz \bar{\alpha}_s(x, da)$.
2. $\hat{\mathbb{P}} \circ X_t^{-1} = \lambda$.
3. $\hat{\mathbb{P}}(\alpha \in \mathcal{A}^{Leb, X, 1}) = 1$.
4. X satisfies the following dynamics

$$\begin{aligned} \langle f, X_s \rangle &= \langle f, \lambda \rangle + \int_t^s \int_{\mathbb{R}^d \times A} (Lf(x, X_r, a) + \\ &\quad \gamma(x, X_r, a) (\partial_s \Phi(1, x, X_r, a) - 1) f(x)) \bar{\alpha}_r(x, da) X_r(dx) dr \\ &+ \int_t^s \int_{\mathbb{R}^d \times A} Df(x) \sigma(x, X_s, a) \mathcal{M}^c(dr, dx, da) \\ &+ \int_t^s \int_{\mathbb{R}^d \times \mathbb{R}_+ \times A} \sum_{k \geq 0} \langle f, (k-1)\delta_x \rangle \mathbb{1}_{I_k(x, X_r, a)}(z) \mathcal{M}^d(dr, dx, dz, da) . \end{aligned} \quad (3.29)$$

for all $f \in C_b^\infty(\mathbb{R}^d)$ and all $[t, s] \subseteq [0, T]$.

Proof. We follow the ideas in [35, Theorem 2.7] and [35, Theorem 2.9] to characterize the martingale \bar{M}_s^f in (3.25). From [28, Theorem 4.18], every square integrable martingale starting at 0 can be uniquely decomposed in the sum of a continuous martingale $\bar{M}^{f,c}$ and a purely discontinuous martingale $\bar{M}^{f,d}$, which is the compensated sum of its jumps. We show the connection of these two processes with X and α .

First, we focus on $\bar{M}^{f,d}$. Since a purely discontinuous martingale $\bar{M}^{f,d}$ is the compensated sum of its jumps, we look at $\Delta X_s = X_s - X_{s-}$. Let \tilde{N} be the Lévy system of X , *i.e.*, a measure on $M^1(\mathbb{R}^d) \times \mathbb{R}_+$ given by $N_s(X_s, dv)ds$ where $N_s(\bar{X}, dv)$ is the image measure of the measure $\nu_s(x, \bar{X}, du)\bar{X}(dx)$ by the mapping $(u, x) \mapsto u\delta_x$ from $\mathbb{R}_+ \times \mathbb{R}^d$ to $M^1(\mathbb{R}^d)$, and a certain kernel ν . Comparing the last term in expressions (3.22) and [16, Théorème 7 (4)], we identify ν as

$$\nu_s(x, \lambda, dz) = \int_A \sum_{k \geq 0} (k-1) \mathbb{1}_{I_k(x, \lambda, a)}(z) \bar{\alpha}_s(x, da) dz.$$

This means that, for F bounded positive measurable function on $\mathbb{R}_+ \times M^1(\mathbb{R}^d)$, we have that

$$\begin{aligned} & \sum_{t < r \leq s} F(r, \Delta X_r) \mathbb{1}_{\{\Delta X_r \neq 0\}} \\ & \quad - \int_t^s \int_{\mathbb{R}^d} \int_{(0, \infty)} \int_A \sum_{k \geq 0} F(r, (k-1)\delta_x) \mathbb{1}_{I_k(x, X_r, a)}(z) \bar{\alpha}_r(x, da) dz X_r(dx) dr \\ & = \sum_{t < r \leq s} F(r, \Delta X_r) \mathbb{1}_{\{\Delta X_r \neq 0\}} \\ & \quad - \int_t^s \int_{\mathbb{R}^d \times A} \sum_{k \geq 0} F(r, (k-1)\delta_x) \gamma(x, X_r, a) p_k(x, X_r, a) \bar{\alpha}_r(x, da) X_r(dx) dr \end{aligned} \tag{3.30}$$

is a \mathbb{F} -martingale. With this description of ν and $N_s(X_s, dv)ds$, we use [35, Proposition 2.8] to prove that we satisfy the hypothesis of [14, Theorem 12]. Therefore, there exists an extension $(\bar{\Omega}^1 = \Omega \times \Omega^1, \bar{\mathcal{F}}^1 = \mathcal{F} \otimes \mathcal{F}^1, \bar{\mathbb{P}}^1 = \mathbb{P} \otimes \mathbb{P}^1, \{\bar{\mathcal{F}}_s^1 = \mathcal{F}_s \otimes \mathcal{F}_s^1\}_s)$, and martingale measures \mathcal{M}^d on $[0, T] \times \mathbb{R}^d \times \mathbb{R}_+ \times A$ in it, such that its dual predictable projection measure is $dr X_r(dx) dz \bar{\alpha}_r(x, da)$, and

$$\bar{M}_s^{f,d} = \int_t^s \int_{\mathbb{R}^d \times \mathbb{R}_+ \times A} \sum_{k \geq 0} \langle f, (k-1)\delta_x \rangle \mathbb{1}_{I_k(x, X_r, a)}(z) \mathcal{M}^d(dr, dx, dz, da).$$

Focus now on $\bar{M}^{f,c}$. The first term in (3.26) comes from the continuous martingale, *i.e.*,

$$\left[\bar{M}^{f,c} \right]_s = \int_t^s \int_{\mathbb{R}^d \times A} \text{Tr} \left(\sigma \sigma^\top(x, X_r, a) D\varphi D\varphi^\top(x) \right) \bar{\alpha}_r(x, da) X_r(dx) dr.$$

Since $\sigma \in L^2(X_s(dx) \alpha_s(da) ds)$, from [15, Theorem III-7], there exist an extension $(\bar{\Omega}^2 = \bar{\Omega}^1 \times \Omega^2, \bar{\mathcal{F}}^2 = \bar{\mathcal{F}}^1 \otimes \mathcal{F}^2, \bar{\mathbb{P}}^2 = \bar{\mathbb{P}}^1 \otimes \mathbb{P}^2, \{\bar{\mathcal{F}}_s^2 = \bar{\mathcal{F}}_s^1 \otimes \mathcal{F}_s^2\}_s)$, and a continuous martingale measure \mathcal{M}^c on $[0, T] \times \mathbb{R}^d \times A$ on this space, such that its intensity is $ds X_s(dx) \bar{\alpha}_s(x, da)$, and we have

$$\bar{M}_s^{f,c} = \int_t^s \int_{\mathbb{R}^d \times A} Df(x) \sigma(x, X_r, a) \mathcal{M}^c(dr, dx, da).$$

The imposed dependence on X and α over \mathcal{M}^d and \mathcal{M}^c implies that (3.29) is satisfied.

Conversely, if a $M^1(\mathbb{R}^d)$ -valued process satisfies (3.29), applying Itô's formula, we have (3.24). \square

We can now define the relaxed control problem. For $\mathcal{C} \in \mathcal{R}_{(t,\lambda)}^r$, we define the cost function as

$$J(t, \lambda; \mathcal{C}) = \mathbb{E}^{\mathbb{P}} \left[\int_t^T \int_{\mathbb{R}^d \times A} \psi(s, X_s, a) \bar{\alpha}_s(x, da) X_s(dx) ds + \Psi(X_T) \right], \quad (3.31)$$

and the *relaxed control problem* as

$$v^r(t, \lambda) = \inf \left\{ J(t, \lambda; \mathcal{C}) : \mathcal{C} \in \mathcal{R}_{(t,\lambda)}^r \right\}, \quad (3.32)$$

for any $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$.

To achieve the well-posedness of this problem, akin to the case of strong controls, it is necessary to get non-explosion bounds, as presented in Proposition 2.1 and Proposition 2.2. However, we choose an alternative approach instead of replicating similar results within this new framework. Firstly, we establish an equivalence between the strong and relaxed formulations. Subsequently, we employ this equivalence to retrieve estimates for the relaxed formulation, thereby ensuring the well-posedness of the relaxed control problem.

4 Equivalence between strong and relaxed formulation

We state the following straightforward adaptation of [23, Lemma 3.7]. This enables the process X to be reduced to its canonical filtration. It is important to emphasize that the following lemma is presented in relation to the filtration generated by the processes, rather than its right-continuous extension or its completion with respect to a specific probability measure. This construction aligns with the approach described in [23], where the only requirement is the existence of a countably dense set of test functions that define the martingale problem.

Lemma 4.3. *Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$ and $\mathcal{C} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_s\}_{s \geq 0}, (X_s)_{s \geq 0}, (\bar{\alpha}_s)_{s \geq 0}) \in \mathcal{R}_{(t,\lambda)}^r$. If $\{\mathcal{F}_s^X\}_s$ is the filtration generated by X and $\{\mathcal{G}_s\}_{s \geq 0}$ another filtration such that $\mathcal{F}_s^X \subseteq \mathcal{G}_s \subseteq \mathcal{F}_s$ for any $s \geq 0$. Then, there exists $(\bar{\alpha}_s^{\mathcal{G}})_{s \geq 0}$ such that*

$$\bar{\mathcal{C}} = \left(\Omega, \mathcal{G}_T, \mathbb{P}, \{\mathcal{G}_s\}_{s \geq 0}, (X_s)_{s \geq 0}, (\bar{\alpha}_s^{\mathcal{G}})_{s \geq 0} \right)$$

is in $\mathcal{R}_{(t,\lambda)}^r$ and $J(t, \lambda; \mathcal{C}) = J(t, \lambda; \bar{\mathcal{C}})$.

Denoting the canonical process on \mathbf{D}^d as μ , we define $\mathbb{F}^\mu = \{\mathcal{F}_s^\mu\}_{s \geq 0}$ as the filtration generated by this process. The previous lemma hints at considering a subset of relaxed controls as follows.

Definition 4.5. *Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$. $\mathcal{C} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_s\}_s, (X_s)_s, (\bar{\alpha}_s)_s)$ in $\mathcal{R}_{(t,\lambda)}^r$ is a natural control, and we say that \mathcal{C} is in $\mathcal{R}_{(t,\lambda)}^n$, if $\Omega = \mathbf{D}^d$, $\mathcal{F} = \mathcal{F}_T^\mu$, $\mathcal{F}_s = \mathcal{F}_s^\mu$ for $s \in [t, T]$, $X = \mu$, and*

$$\mathbb{P}(\mu_s = \lambda, s \in [0, t]) = 1.$$

We observe that the pair $(\mathbb{P}, \bar{\alpha})$ determine natural controls, consisting in a probability measure on \mathbf{D}^d , i.e., the distribution of μ , and the control process $(\bar{\alpha}_s)_s$. With abuse of notation, we use $(\mathbb{P}, \bar{\alpha})$ to refer to $\mathcal{C}^{\mathbb{P}, \bar{\alpha}} := (\mathbf{D}^d, \mathcal{F}_T^\mu, \mathbb{P}, \{\mathcal{F}_s^\mu\}_s, (\mu_s)_s, (\bar{\alpha}_s)_s)$ in $\mathcal{R}_{(t,\lambda)}^n$.

4.1 Weak controls

Considering the implications highlighted in Remark 3.3, we can focus on a subset of controls known as weak controls. Notably, the elements within this class exhibit uniqueness in terms of their probability distributions. This particular property serves as the crucial connection for identifying the class of strong controls within the realm of relaxed controls.

For a fixed $\mathbf{x} \in \mathbf{D}^d$, the set of measurable functions $\mathbf{a} : [0, T] \times \mathbb{R}^d \rightarrow A$ is canonically embedded in $\mathcal{A}^{\text{Leb}, \mathbf{x}, 1}$ by $\alpha^{\mathbf{a}}(ds, dx, da) := ds \mathbf{x}_s(dx) \delta_{\mathbf{a}(s, x)}(da)$.

Definition 4.6. Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$. We say that (\mathbb{P}, \mathbf{a}) is a weak control, and we write $(\mathbb{P}, \mathbf{a}) \in \mathcal{R}_{(t, \lambda)}^0$, if $\mathbf{a} : [0, T] \times \mathbb{R}^d \times \Omega \rightarrow A$ is $\hat{\mathbb{F}}^\mu$ -predictable, and $(\mathbb{P}, \alpha^{\mathbf{a}}) \in \mathcal{R}_{(t, \lambda)}^n$.

Therefore, for $\mathbb{P} \in \mathcal{R}_{(t, \lambda)}^0$, we have that

$$F_\varphi(\mu_s) - \int_t^s \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x, \mathbf{a}(u, x), \mu_u) \mu_u(dx) du$$

is a $(\mathbb{P}, \mathbb{F}^\mu)$ -martingale for $s \geq t$, $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$.

We now prove how to restrict the class of controls from $\mathcal{R}_{(t, \lambda)}^n$ to $\mathcal{R}_{(t, \lambda)}^0$ without impacting the value function. This is done by showing that we can always associate natural and weak control with the same cost under the following assumption.

Assumption A1. The following set

$$K(x, \lambda) := \left\{ \left(b(x, \lambda, a), \sigma \sigma^\top(x, \lambda, a), ((\gamma p_k)(x, \lambda, a))_{k \geq 0}, z \right) : a \in A, z \geq \psi(x, \lambda, a) \right\} \\ \subseteq \mathbb{R}^d \times \mathbb{R}^{d \times d} \times \mathbb{R}_+^\infty \times \mathbb{R}$$

is convex for all $(x, \lambda) \in \mathbb{R}^d \times M^1(\mathbb{R}^d)$.

This convexity assumption is the so-called *Filippov condition*, common in the control literature. It holds, for example, when A is a convex subset of a vector space, and the parameters are affine in a , which is the case of the Linear-Quadratic example presented in Section 6.3.

Proposition 4.4. Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$. Suppose that Assumption A1 holds. For $(\mathbb{P}, (\alpha_s)_s) \in \mathcal{R}_{(t, \lambda)}^n$, there exists \mathbf{a} such that $(\mathbb{P}, \mathbf{a}) \in \mathcal{R}_{(t, \lambda)}^0$ and $J(t, \lambda; \mathcal{C}^{\mathbb{P}, \delta_{\mathbf{a}}}) \geq J(t, \lambda; \mathcal{C}^{\mathbb{P}, \bar{\alpha}})$.

Proof. Given $(\mathbb{P}, (\alpha_s)_s)$ in $\mathcal{R}_{(t, \lambda)}^n$, we define c by

$$c^1(s, x, \lambda, \omega) = \int_A \left(b, \sigma \sigma^\top, (\gamma p_k)_{k \geq 0} \right) (x, \lambda, a) \bar{\alpha}_s(x, da), \\ c^2(s, x, \lambda, \omega) = \int_A \psi(x, \lambda, a) \bar{\alpha}_s(x, da).$$

All the functions defining K are continuous, therefore, for almost all $(x, \lambda) \in \mathbb{R}^d \times M^1(\mathbb{R}^d)$, $K(x, \lambda)$ is closed. Since $K(x, \lambda)$ is closed and convex, $(c^1, c^2)(s, x, \lambda, \omega)$ is in $K(x, \lambda)$ for any (x, λ) and almost all (s, ω) . Moreover, from [23, Lemma A.1], we can take (c^1, c^2) to be $\hat{\mathbb{F}}^\mu$ -predictable. We apply [23, Theorem A.9] and obtain that there is a $\hat{\mathbb{F}}^\mu$ -predictable A -valued process \mathbf{a} such that

$$c^1(s, x, \lambda, \omega) = \left(b, \sigma \sigma^\top, (\gamma p_k)_{k \geq 0} \right) (x, \lambda, \mathbf{a}(s, x, \lambda, \omega)), \quad (4.33)$$

$$c^2(s, x, \lambda, \omega) \geq \psi(x, \lambda, \mathbf{a}(s, x, \lambda, \omega)) \quad (4.34)$$

for any (x, λ) and for almost all (s, ω) . For $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$, we must have

$$\int_{\mathbb{R}^d \times A} \mathcal{L}F_\varphi(x, \mu_u, a_u) \bar{\alpha}_u(x, da) \mu_u(dx) = \int_{\mathbb{R}^d} \mathcal{L}F_\varphi(x, \mathbf{a}(s, x, \mu_u), \mu_u) \mu_u(dx)$$

for almost all (s, ω) . Hence, $F_\varphi(\mu_s) - \int_t^s \int_{\mathbb{R}^d \times A} \mathcal{L}F_\varphi(x, \mathbf{a}(s, x, \mu_u), \mu_u) \mu_u(dx) du$ is a martingale, for all $s \geq t$. Therefore, $(\mathbb{P}, \mathbf{a}) \in \mathcal{R}_{(t, \lambda)}^0$, and, from (4.33), we get $J(t, \lambda; \mathcal{C}^{\mathbb{P}, \delta_a}) \leq J(t, \lambda; \mathcal{C}^{\mathbb{P}, \bar{\alpha}})$. \square

4.2 Uniqueness in law for weak controls

We introduce the domain \mathcal{D} as the set of function $h : \mathbb{R}_+ \times \mathbf{D}^d \rightarrow \mathbb{R}$ of the form

$$h(s, \mathbf{x}) = F(\langle f_1(s \wedge t_1, \cdot), \mathbf{x}_{s \wedge t_1} \rangle, \dots, \langle f_p(s \wedge t_p, \cdot), \mathbf{x}_{s \wedge t_p} \rangle), \quad (s, \mathbf{x}) \in \mathbb{R}_+ \times \mathbf{D}^d,$$

for some $p \geq 1$, $0 \leq t_1 < \dots < t_p \leq T$, $F \in C_b^2(\mathbb{R}^p)$, and $f_1, \dots, f_p \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$. For $f \in C_b^{1,2}([0, T] \times \mathbb{R}^d)$, we use the notation $Lf(s, x, \mu, a) = Lf(s, \cdot)(x, \mu, a)$. For a measurable function $\mathbf{a} : \mathbb{R}^d \rightarrow A$, define the operator $\mathbb{L}^{\mathbf{a}}$ on \mathcal{D} by

$$\begin{aligned} \mathbb{L}^{\mathbf{a}} h(s, \mathbf{x}) &= DF(\langle f_1(s \wedge t_1, \cdot), \mathbf{x}_{s \wedge t_1} \rangle, \dots, \langle f_p(s \wedge t_p, \cdot), \mathbf{x}_{s \wedge t_p} \rangle)^\top \mathcal{L}^{\mathbf{a}} \mathbf{f}(s, \mathbf{x}) \\ &\quad + \frac{1}{2} \text{Tr} \left(\left\langle \mathfrak{S}^{\mathbf{a}} \mathbf{f}(\mathfrak{S}^{\mathbf{a}} \mathbf{f})^\top(s, \cdot), \mathbf{x}_s \right\rangle D^2 F(\langle f_1(s \wedge t_1, \cdot), \mathbf{x}_{s \wedge t_1} \rangle, \dots, \langle f_p(s \wedge t_p, \cdot), \mathbf{x}_{s \wedge t_p} \rangle) \right) \\ &\quad + \sum_{j=1}^p \mathbb{1}_{t_{j-1} < s \leq t_j} \int_{\mathbb{R}^d} \sum_{k \geq 0} \gamma(x, \mathbf{a}(s, x), \mathbf{x}_s) p_k(x, \mathbf{a}(s, x), \mathbf{x}_s) \\ &\quad \left(F(\langle f_1(s \wedge t_1, \cdot), \mathbf{x}_{s \wedge t_1} \rangle, \dots, \langle f_{j-1}(s \wedge t_{j-1}, \cdot), \mathbf{x}_{s \wedge t_{j-1}} \rangle, \mathfrak{G}_k^1 f_j(s, x, \mathbf{x}_s), \dots, \mathfrak{G}_k^1 f_p(s, x, \mathbf{x}_s)) \right. \\ &\quad \left. - F(\langle f_1(s \wedge t_1, \cdot), \mathbf{x}_{s \wedge t_1} \rangle, \dots, \langle f_p(s \wedge t_p, \cdot), \mathbf{x}_{s \wedge t_p} \rangle) \right) \mathbf{x}_s(dx) \end{aligned}$$

with $t_0 = 0$, where

$$\begin{aligned} \mathcal{L}^{\mathbf{a}} \mathbf{f}(s, \mathbf{x}) &:= \begin{pmatrix} \mathbb{1}_{s \leq t_1} \int_{\mathbb{R}^d} \partial_t f_1(s, x) + Lf_1(s, x, \mathbf{x}_s, \mathbf{a}(s, x)) \mathbf{x}_s(dx) \\ \vdots \\ \mathbb{1}_{s \leq t_p} \int_{\mathbb{R}^d} \partial_t f_p(s, x) + Lf_p(s, x, \mathbf{x}_s, \mathbf{a}(s, x)) \mathbf{x}_s(dx) \end{pmatrix}, \\ \mathfrak{S}^{\mathbf{a}} \mathbf{f}(s, x, \mathbf{x}) &:= \begin{pmatrix} \mathbb{1}_{s \leq t_1} |Df_1(s, x) \sigma(x, \mathbf{x}_s, \mathbf{a}(s, x))| \\ \vdots \\ \mathbb{1}_{s \leq t_p} |Df_p(s, x) \sigma(x, \mathbf{x}_s, \mathbf{a}(s, x))| \end{pmatrix}, \\ \mathfrak{G}_k^n f_j(s, x, \mathbf{x}) &:= \langle f_j(s, \cdot), \mathbf{x}_s \rangle + \frac{k-1}{n} f_j(s, x), \end{aligned}$$

for $(s, x, \mathbf{x}) \in [0, T] \times \mathbb{R}^d \times \mathbf{D}^d$, and $k, j, n \geq 0$.

Considering the canonical process $\mu \in \mathbf{D}^d$, we take the extended process \mathfrak{r} defined by

$$\mathfrak{r}_s = (s, (\mu_{u \wedge s})), \quad s \in [t, T],$$

valued in $\mathbb{R} \times \mathbf{D}^d$, which is separable. Note that for $(\mathbb{P}, \mathbf{a}) \in \mathcal{R}_{(t, \lambda)}^0$ the process

$$h(\mathfrak{r}_s) - \int_t^s \mathbb{L}^{\mathbf{a}} h(\mathfrak{r}_u) du, \quad t \leq u \leq T, \quad (4.35)$$

is a \mathbb{F}^μ -martingale under \mathbb{P} . Therefore, we have that this condition gives information about the marginals.

Proposition 4.5. Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$ and $(\mathbb{P}, \mathbf{a}) \in \mathcal{R}_{(t, \lambda)}^0$. For any $(\mathbb{P}', \mathbf{a}) \in \mathcal{R}_{(t, \lambda)}^0$, \mathbb{P} and \mathbb{P}' have the same one dimensional marginals:

$$\mathbb{P}(\mathbf{x}_s \in B) = \mathbb{P}'(\mathbf{x}_s \in B) \quad (4.36)$$

for $s \in [t, T]$ and $B \in \mathcal{B}([0, T] \times \mathbf{D}^d)$.

Proof. We first endow the measurable space $(\mathbf{D}^d \times \mathbf{D}^d, \mathcal{F}_T^\mu \otimes \mathcal{F}_T^\mu)$ with the probability measure $\mathbb{Q} = \mathbb{P} \otimes \mathbb{P}'$. For $h \in \mathcal{D}$, we have

$$\mathbb{E}^{\mathbb{Q}}[h \otimes h(\mathbf{x}_s, \mathbf{x}_t)] = \mathbb{E}^{\mathbb{Q}}[h \otimes h(\mathbf{x}_t, \mathbf{x}_s)]$$

Indeed, the processes

$$h \otimes h(\mathbf{x}_s, \mathbf{x}_t) - \int_t^s \mathbb{L}^\alpha h(\mathbf{x}_u) h(\mathbf{x}_t) du, \quad t \leq s \leq T$$

and

$$h \otimes h(\mathbf{x}_t, \mathbf{x}_s) - \int_t^s h(\mathbf{x}_t) \mathbb{L}^\alpha h(\mathbf{x}_u) du, \quad t \leq s \leq T$$

are both martingales under \mathbb{Q} . Since all the considered functions are bounded, we can take the expectation and get

$$\mathbb{E}^{\mathbb{Q}}[h \otimes h(\mathbf{x}_t, \mathbf{x}_s)] = \mathbb{E}^{\mathbb{Q}}[h \otimes h(\mathbf{x}_s, \mathbf{x}_t)]$$

and

$$\mathbb{E}^{\mathbb{P}}[h(\mathbf{x}_s)] = \mathbb{E}^{\mathbb{P}'}[h(\mathbf{x}_s)].$$

Since any bounded $\mathcal{B}(\mathfrak{X})$ -measurable function can be approximated almost everywhere for \mathbb{P} and \mathbb{P}' by a sequence of \mathcal{D} we get (4.36). \square

Theorem 4.1. Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$ and \mathbf{a} a $\hat{\mathbb{F}}^\mu$ -predictable process from $[0, T] \times \mathbb{R}^d$ to A . There exists at most one $\mathbb{P} \in \mathcal{P}^1(\mathbf{D}^d)$ such that $(\mathbb{P}, \mathbf{a}) \in \mathcal{R}_{(t, \lambda)}^0$, and we denote it \mathbb{P}^α .

Proof. The proof is a direct consequence of [19, Theorem 4.2, Chapter 4] and Proposition 4.5. \square

4.3 Equivalence between relaxed and strong formulations

Proposition 4.6. Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$. For \mathbf{a} a $\hat{\mathbb{F}}^\mu$ -predictable process from $[0, T] \times \mathbb{R}^d$ to A , there exist $\beta \in \mathcal{R}_{(t, \lambda)}^s$ and $\mathbb{P}^\alpha \in \mathcal{P}^1(\mathbf{D}^d)$ such that $(\mathbb{P}^\alpha, \mathbf{a}) \in \mathcal{R}_{(t, \lambda)}^0$, and the law of ξ^β under \mathbb{P}^α is the same of the one on μ under \mathbb{P}^α .

Proof. Since \mathbf{a} is $\hat{\mathbb{F}}^\mu$ -predictable, from Doob's functional representation theorem (see, e.g., Lemma 1.13 in [29]), there exists a $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbf{D}^d)$ -measurable function $\kappa^\alpha : [0, T] \times \mathbb{R}^d \times \mathbf{D}^d \rightarrow A$ such that $\mathbf{a}(s, x, \omega) = \kappa^\alpha(s, x, \mu(\omega_{\wedge s})) = \kappa^\alpha(s, x, \mu(\omega))$ for any $s \in [0, T]$, $x \in \mathbb{R}^d$, and $\omega \in \Omega$.

Fix some $a_0 \in A$. We consider the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ as in Section 2 and define the standard strong control β^α as

$$\beta_s^{\alpha, i} = \kappa^\alpha \left(s, Y_s^{i, \beta}, \left(\xi_{u \wedge s}^\beta \right)_{u \in [0, T]} \right) \mathbf{1}_{i \in V_s} + a_0 \mathbf{1}_{i \notin V_s},$$

where ξ^β (resp. $Y^{i,\beta}$ for $i \in V_s$) is the strongly controlled population (resp. particle) associated with β^α . From Proposition 2.1, there exists a unique càdlàg process that satisfies (2.6) associated with this control β^α . Moreover, condition (2.17) is satisfied, hence $\beta^\alpha \in \mathcal{R}_{(t,\lambda)}^s$.

With the embedding given in Remark 3.3, we can associate to β^α the relaxed control

$$\mathcal{C}^\alpha = (\Omega^\alpha, \mathcal{F}^\alpha, \mathbb{P}^\alpha, \{\mathcal{F}_s^\alpha\}_s, (X_s^\alpha)_s, (\bar{\alpha}_s^\alpha)_s).$$

From Lemma 4.3, we get a natural control $(\mathbb{P}^{n,\alpha}, \bar{\alpha}^{n,\alpha})$. Following [23, Lemma 3.7], since $\bar{\alpha}^\alpha$ is a Dirac measure \mathbb{P}^α -a.s., we have that $\bar{\alpha}^{n,\alpha}$ is a Dirac measures $\mathbb{P}^{n,\alpha}$ -a.s. Moreover, we can see that $\bar{\alpha}_s^{n,\alpha}(x, da) = \delta_{\kappa^\alpha(s,x,\mu(\omega_{\cdot \wedge s}))} = \delta_{\mathbf{a}(s,x)}$ $\mathbb{P}^{n,\alpha}$ -a.s., hence $(\mathbb{P}^{n,\alpha}, \mathbf{a}) \in \mathcal{R}_{(t,\lambda)}^0$. \square

Combining Theorem 4.1 and Proposition 4.6, we have that a weak control is specified by the $\hat{\mathbb{F}}^\mu$ -predictable control \mathbf{a} . With abuse of notation, we write $\mathbf{a} \in \mathcal{R}_{(t,\lambda)}^0$ (resp. $J(t, \lambda; \mathbf{a})$) to denote $\mathcal{C}^\alpha := (\mathbf{D}^d, \mathcal{F}_T^\mu, \mathbb{P}^\alpha, \{\mathcal{F}_s^\mu\}_s, (\mu_s)_s, (\delta_{\mathbf{a}(s,\cdot)})_s) \in \mathcal{R}_{(t,\lambda)}^r$ (resp. $J(t, \lambda; \mathcal{C}^\mathbb{P})$).

Proposition 4.7. *Suppose Assumption A1 holds. For $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$, we have*

$$\begin{aligned} v(t, \lambda) &:= \inf \left\{ J(t, \lambda; \mathcal{C}) : \mathcal{C} \in \mathcal{R}_{(t,\lambda)}^r \right\} = \inf \left\{ J(t, \lambda; \mathbf{a}) : \mathbf{a} \in \mathcal{R}_{(t,\lambda)}^0 \right\} \\ &= \inf \left\{ J(t, \lambda; \beta) : \beta \in \mathcal{R}_{(t,\lambda)}^s \right\}. \end{aligned}$$

Proof. We denote $v^r(t, \lambda) = \inf \left\{ J(t, \lambda; \mathcal{C}) : \mathcal{C} \in \mathcal{R}_{(t,\lambda)}^r \right\}$, $v^0(t, \lambda) = \inf \left\{ J(t, \lambda; \mathbf{a}) : \mathbf{a} \in \mathcal{R}_{(t,\lambda)}^0 \right\}$ and $v^s(t, \lambda) = \inf \left\{ J(t, \lambda; \beta) : \beta \in \mathcal{R}_{(t,\lambda)}^s \right\}$. From the embedding of Remark 3.3, we have that $v^r(t, \lambda) \leq v^s(t, \lambda)$. Using Lemma 4.3 and Proposition 4.4, for each relaxed control, there exists a weak control that does not increase the value functions. This means that $v^r(t, \lambda) = v^0(t, \lambda)$. Finally, from Proposition 4.6, any weak control finds a representation in the strong controls set. This means that $v^s(t, \lambda) \leq v^0(t, \lambda)$. \square

We can now give the bounds on the moments of the controlled processes in the relaxed framework.

Proposition 4.8. *Let $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$, and*

$$\mathcal{C} = \left(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s\}_{s \geq 0}, (X_s)_{s \geq 0}, (\bar{\alpha}_s)_{s \geq 0} \right) \in \mathcal{R}_{(t,\lambda)}^r.$$

There exists a constant $C > 0$ depending only on T and on the coefficients b, σ, γ and $(p_k)_k$ such that

$$\mathbb{E}^\mathbb{P} \left[\sup_{u \in [t, t+h]} \langle 1, X_u \rangle \right] \leq \langle 1, \lambda \rangle e^{C_\gamma C_\Phi^1 h}, \quad (4.37)$$

$$\mathbb{E}^\mathbb{P} \left[\sup_{u \in [t, t+h]} \langle 1, X_u \rangle^2 \right] \leq \langle 1, \lambda \rangle e^{C_\gamma (C_\Phi^1 + C_\Phi^2) h}, \quad (4.38)$$

$$\begin{aligned} \mathbb{E}^\mathbb{P} \left[\sup_{u \in [t, t+h]} \langle |\cdot|, X_u \rangle \right] &\leq C \left(\langle |\cdot|, \lambda \rangle + \mathbb{E}^\mathbb{P} \left[\int_t^{t+h} \langle 1, X_u \rangle du \right] \right. \\ &\quad \left. + \mathbb{E}^\mathbb{P} \left[\int_t^{t+h} \int_{\mathbb{R}^d \times A} |a| \bar{\alpha}_u(x, da) X_u(dx) du \right] \right), \end{aligned} \quad (4.39)$$

$$\begin{aligned} \mathbb{E}^\mathbb{P} \left[\sup_{u \in [t, t+h]} \langle |\cdot|^2, X_u \rangle \right] &\leq C \left(\langle |\cdot|^2, \lambda \rangle + \mathbb{E}^\mathbb{P} \left[\int_t^{t+h} \langle 1, X_u \rangle du \right] \right. \\ &\quad \left. + \mathbb{E}^\mathbb{P} \left[\int_t^{t+h} \int_{\mathbb{R}^d \times A} |a|^2 \bar{\alpha}_u(x, da) X_u(dx) du \right] \right), \end{aligned} \quad (4.40)$$

for any $h > 0$, where $|\cdot|$ (resp. $|\cdot|^2$) denote the function $x \mapsto |x|$ (resp. $x \mapsto |x|^2$). Moreover, for $\varepsilon > 0$, if $\mathcal{R}_{(t,\lambda)}^{\varepsilon}$ denotes the set of $\mathcal{C} \in \mathcal{R}_{(t,\lambda)}^r$ satisfying $J(t, \lambda; \mathcal{C}) \leq v(t, \lambda) + \varepsilon$. Then

$$\sup_{\beta \in \mathcal{R}_{(t,\lambda)}^{\varepsilon}} \mathbb{E}^{\mathbb{P}} \left[\int_t^{t+h} \int_{\mathbb{R}^d \times A} |a|^2 \bar{\alpha}_u(x, da) X_u(dx) du \right] < \infty. \quad (4.41)$$

Proof. From Lemma 4.3, any bound established on relaxed control transposes exactly on natural controls. Fix $(\mathbb{P}, (\alpha_s)_s) \in \mathcal{R}_{(t,\lambda)}^n$. From the proof of Proposition 4.4, we see that the weak control $(\mathbb{P}, \mathbf{a}) \in \mathcal{R}_{(t,\lambda)}^0$ associated with this natural control does not modify the probability measure \mathbb{P} , nor the law of μ , using Assumption A1. In particular, this procedure can be applied for any kind of cost functions (ψ, Ψ) as soon as they satisfy the bounds (2.13)-(2.16).

Define now $\psi^1(x, \lambda, a) := |a|$ (resp. $\psi^2(x, \lambda, a) := |a|^2$). Since ψ^1 (resp. ψ^2) satisfies (2.13)-(2.16), we consider \mathbf{a}^1 (resp. \mathbf{a}^2) the weak control associated with the couple $(\psi^1, 0)$ (resp. $(\psi^2, 0)$). In the notation of the paper, the cost functions associated with these couples are respectively

$$J_p(t, \lambda; \mathcal{C}) = \mathbb{E}^{\mathbb{P}} \left[\int_t^T \int_{\mathbb{R}^d \times A} |a|^p \bar{\alpha}_s(x, da) \mu_s(dx) ds \right], \quad \text{for } p = 1, 2.$$

Using the identification between weak, controls and strong controls, we have that (2.7), (2.8), (2.10), and (2.19) extend directly to the framework of weak controls. Therefore, since the first two depend only on the parameters of the model and the initial condition (t, λ) , we get (4.37) and (4.38).

Since the association from α to \mathbf{a}^1 (resp. \mathbf{a}^2) given by Proposition 4.4 is non-increasing in the cost function, we have that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\int_t^T \int_{\mathbb{R}^d} |\mathbf{a}^1(s, x)| \mu_s(dx) ds \right] &\leq \mathbb{E}^{\mathbb{P}} \left[\int_t^T \int_{\mathbb{R}^d \times A} |a| \bar{\alpha}_s(x, da) \mu_s(dx) ds \right], \\ \mathbb{E}^{\mathbb{P}} \left[\int_t^T \int_{\mathbb{R}^d} |\mathbf{a}^2(s, x)|^2 \mu_s(dx) ds \right] &\leq \mathbb{E}^{\mathbb{P}} \left[\int_t^T \int_{\mathbb{R}^d \times A} |a|^2 \bar{\alpha}_s(x, da) \mu_s(dx) ds \right]. \end{aligned}$$

Therefore, combining these inequalities with (2.10) and (2.19), we get exactly (4.39) and (4.40).

Finally, to retrieve (4.41), we argue exactly as in Proposition 2.2 directly in the relaxed control setting. This is again a consequence that the function $a \mapsto |a|^2 - C|a|$ is bounded below and (2.13)-(2.16). □

5 Existence of Optimal Controls

We look for canonic relaxed controls to show the existence of optimal controls. From Lemma 4.3, we can define the control problem 3.31-3.32 with respect to any class \mathcal{R}' such that $\mathcal{R}^n \subseteq \mathcal{R}' \subseteq \mathcal{R}^r$ without increasing the value function. Since we focus on the pair (X, α) in the definition of relaxed controls, canonic relaxed controls are defined in $\Omega = \mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1}$. Let (μ, \mathbf{a}) be the projection maps (or canonical processes) on $\mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1}$, and $\mathbb{F}^{\mu, \mathbf{a}} = \{\mathcal{F}_s^{\mu, \mathbf{a}}\}_s$ the filtration generated by them, *i.e.*,

$$\sigma \left(\mu_s(B_1), \mathbf{a}([0, s'] \times B_2 \times B_3), \text{ for } s, s' \in [0, T], B_1, B_2 \in \mathcal{B}(\mathbb{R}^d), B_3 \in \mathcal{B}(A) \right).$$

Definition 5.7 (Control rule). Fix $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$. $\mathcal{C} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_s\}_s, (X_s)_s, (\bar{\alpha}_s)_s) \in \mathcal{R}_{(t, \lambda)}^r$ is a control rule, and we write $\mathcal{C} \in \mathcal{R}_{(t, \lambda)}$, if $\Omega = \mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1}$, $\mathcal{F} = \mathcal{F}_T^\mu, \mathbb{P} = \mathbb{P}$, $\{\mathcal{F}_s^\mu\}_s, (\mu_s)_s, (\bar{\mathbf{a}}_s)_s \in \mathcal{R}_{(t, \lambda)}$ (resp. $J(t, \lambda; \mathcal{C}^\mathbb{P})$).

$$\mathbb{P}(\mu_s = \lambda, s \in [0, t]) = 1.$$

A control rule is specified by $\mathbb{P} \in \mathcal{P}^1(\mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1})$, i.e., the distribution of (μ, \mathbf{a}) . With abuse of notation, we write $\mathbb{P} \in \mathcal{R}_{(t, \lambda)}$ (resp. $J(t, \lambda; \mathbb{P})$) to denote $\mathcal{C}^\mathbb{P} := (\mathbf{D}^d, \mathcal{F}_T^\mu, \mathbb{P}, \{\mathcal{F}_s^\mu\}_s, (\mu_s)_s, (\bar{\mathbf{a}}_s)_s) \in \mathcal{R}_{(t, \lambda)}$ (resp. $J(t, \lambda; \mathcal{C}^\mathbb{P})$).

From Lemma 4.3, any relaxed control is associated with a control rule with the same cost function J . Therefore,

$$v(t, \lambda) = \inf \left\{ J(t, \lambda; \mathcal{C}) : \mathcal{C} \in \mathcal{R}_{(t, \lambda)}^r \right\} = \inf \left\{ J(t, \lambda; \mathbb{P}) : \mathbb{P} \in \mathcal{R}_{(t, \lambda)} \right\}.$$

We aim at applying the same procedure, as in [23] and [33], to exhibit the existence of a relaxed control. This means proving the optimization problem consists of minimizing a lower semicontinuous function on a compact set. Therefore, we aim at showing that J is lower semicontinuous and $\mathcal{R}_{(t, \lambda)}^\varepsilon := \mathcal{R}_{(t, \lambda)}^{r, \varepsilon} \cap \mathcal{R}_{(t, \lambda)}$ is compact in $\mathcal{P}^1(\mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1})$ for $\varepsilon > 0$.

Lemma 5.4. For $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$, $J(t, \lambda; \cdot)$ is lower semicontinuous on $\mathcal{P}^1(\mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1})$.

Proof. Consider $f : \mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1} \rightarrow \mathbb{R}$, defined as

$$f(\mathbf{x}, \alpha) := \int_t^T \int_{\mathbb{R}^d \times A} \psi(x, \mathbf{x}_s, a) \bar{\alpha}_s(x, da) \mathbf{x}_s(dx) ds + \Psi(\mathbf{x}_T).$$

This function is lower semicontinuous as a consequence of the continuity of ψ and Ψ and their growth conditions (2.16) and (2.14). This means that $J(t, \lambda; \mathbb{P}) = \int f d\mathbb{P}$ is lower semicontinuous. \square

For a Polish space (E, d) and $\mathbb{P} \in \mathcal{P}(M(E))$, we define the mean measure $m\mathbb{P} \in \mathcal{P}(E)$ by

$$m\mathbb{P}(C) := \int_{M(E)} \lambda(C) \mathbb{P}(d\lambda).$$

Since $\mathbf{d}_{p, E}$ is a Wasserstein type distance, from (2.1), the results from [32, Appendix B] can be naturally extended to this setting. As the primary focus is on convergence in weak* topology in the first part, we will examine an alternative metrization, simpler than $\mathbf{d}_{p, E}$.

A family $\mathcal{F} \subseteq C_b(E)$ is said to be *separating* if, whenever $\langle \varphi, \lambda \rangle = \langle \varphi, \lambda' \rangle$ for all $\varphi \in \mathcal{F}$, and some $\lambda, \lambda' \in M(E)$, we necessarily have $\lambda = \lambda'$. Since E is Polish, from the Portmanteau theorem (see, e.g., [41, Theorem 1.1.1]), the set of uniformly continuous functions, for any metric equivalent to d , is separating. Using Tychonoff's embedding theorem (see, e.g., [43, Theorem 17.8]), $C_b(E)$ is also separable. Therefore, there exists a countable and separating family $\mathcal{F}_E = \{\varphi_k, k \in \mathbb{N}\}$ subset of $C_b(E)$ such that the function $E \ni x \mapsto 1$ belongs to \mathcal{F}_E and $\|\varphi_k\|_\infty := \sup_E |\varphi_k| \leq 1$ for all $k \in \mathbb{N}$ since multiplying by a positive constant do not impact the property of being separating. With the use of this family,

$$\mathbf{d}_{\text{weak}^*, E}(\lambda, \lambda') = \sum_{\varphi_k \in \mathcal{F}_E} \frac{1}{2^k} |\langle \varphi_k, \lambda \rangle - \langle \varphi_k, \lambda' \rangle|,$$

for $\lambda, \lambda' \in M(E)$. As in [41, Theorem 1.1.2], this distance $\mathbf{d}_{\text{weak}^*, E}$ induces on $M(E)$ the weak* topology. Whenever $E = \mathbb{R}^d$, we can adjust this metric to take into account useful differential

properties. Let $\mathcal{F}_{\mathbb{R}^d}$ be taken as a subset of $C_b^2(\mathbb{R}^d)$, the set of real functions with bounded, continuous derivatives over \mathbb{R}^d up to order two. Without loss of generality, since C^2 is dense in C^0 , this set is separating under local uniform convergence (application of [21, Theorem 8.14]). Moreover, since $\mathbf{x} \mapsto 1$ belongs to $\mathcal{F}_{\mathbb{R}^d}$, adding a constant or multiplying by a non-negative constant to each function does not change the property of being a separating set, we assume $\varphi_k \geq 0$. We define the distance

$$\mathbf{d}_{\text{weak}^*, \mathbb{R}^d}(\lambda, \lambda') = \sum_{\varphi_k \in \mathcal{F}_{\mathbb{R}^d}} \frac{1}{2^k q_k} |\langle \varphi_k, \lambda \rangle - \langle \varphi_k, \lambda' \rangle|, \quad (5.42)$$

with $q_k = \max\{1, \|D\varphi_k\|_\infty, \|D^2\varphi_k\|_\infty\}$.

Proposition 5.9. *Given $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$ and $\varepsilon > 0$, $\mathcal{R}_\lambda^\varepsilon$ is compact in $\mathcal{P}^1(\mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1})$.*

Proof. The proof of this lemma breaks into four steps.

Step 1. First, we aim at proving that $\{m_{\mathbb{D}^d}^{\mathbb{P}} : \mathbb{P} \in \mathcal{R}_{(t, \lambda)}^\varepsilon\} \subseteq \mathcal{P}(\mathbf{D}^b)$ is tight. To do that, we verify Aldous' criterion (see, e.g., [29, Theorem 14.11]), i.e., proving

$$\lim_{\delta \downarrow 0} \sup_{\mathbb{P} \in \mathcal{R}_{(t, \lambda)}^\varepsilon} \sup_{\tau} \mathbb{E}^{\mathbb{P}} [\mathbf{d}_{\text{weak}^*, \mathbb{R}^d}(\mu_{(\tau+\delta) \wedge T}, \mu_\tau)] = 0, \quad (5.43)$$

where the innermost supremum is over stopping times τ valued in $[t, T]$.

From Proposition 3.3, we know there exists an extension $\hat{\Omega}$ of $\mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1}$ where μ can be represented as the solution of (3.29). This SDE is driven by \mathcal{M}^c orthogonal continuous martingale measure on $\hat{\Omega} \times [0, T] \times \mathbb{R}^d \times A$, with intensity measure $ds\mu_s(dx)\bar{\mathbf{a}}_s(x, da)$, and a purely discontinuous martingale measure \mathcal{M}^d on $\hat{\Omega} \times [0, T] \times \mathbb{R}^d \times \mathbb{R}_+ \times A$, with dual predictable projection measure $ds\mu_s(dx)dz\bar{\mathbf{a}}_s(x, da)$. Applying (3.29) to $\varphi_k \in \mathcal{F}_{\mathbb{R}^d}$, we get

$$\begin{aligned} \langle \varphi_k, \mu_{(s+\delta) \wedge T} \rangle &= \langle \varphi_k, \mu_s \rangle + \int_s^{(s+\delta) \wedge T} \int_{\mathbb{R}^d \times A} (L\varphi_k(x, \mu, a_r) + \\ &\quad \gamma(x, \mu, a_r) (\partial_s \Phi(1, x, \mu, a_r) - 1) \varphi_k(x)) \bar{\mathbf{a}}_r(x, da) \mu_r(dx) dr + \\ &\quad + \int_s^{(s+\delta) \wedge T} \int_{\mathbb{R}^d \times A} D\varphi_k(x) \sigma(x, X_r, a) \mathcal{M}^c(dr, dx, da) \\ &\quad + \int_s^{(s+\delta) \wedge T} \int_{\mathbb{R}^d \times \mathbb{R}_+ \times A} \sum_{k \geq 0} \langle \varphi_k, (k-1)\delta_x \rangle \mathbf{1}_{I_k(x, \mu_r, a)}(z) \mathcal{M}^d(dr, dx, dz, da). \end{aligned}$$

for $s \in [0, T]$, $k \in \mathbb{N}$. Therefore, to bound the quantity $\mathbb{E}^{\mathbb{P}} [|\langle \varphi_k, \mu_{(s+\delta) \wedge T} \rangle - \langle \varphi_k, \mu_s \rangle|]$, it suffices to bound the last three terms in the r.h.s. There is a constant $C > 0$ that depends only on b, σ, γ and Φ (which may change from line to line) such that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}} \left[\left| \int_s^{(s+\delta) \wedge T} \int_{\mathbb{R}^d \times A} (L\varphi_k(x, \mu, a_r) + \gamma(x, \mu, a_r) (\partial_s \Phi(1, x, \mu, a_r) - 1) \varphi_k(x)) \bar{\mathbf{a}}_r(x, da) \mu_r(dx) dr \right| \right] &\leq \\ &\leq Cq_k \mathbb{E}^{\mathbb{P}} \left[\int_s^{(s+\delta) \wedge T} (\langle 1, \mu_u \rangle + \langle |\cdot|, \mu_u \rangle) du + \int_s^{(s+\delta) \wedge T} \int_{\mathbb{R}^d \times A} |a| \bar{\mathbf{a}}_u(x, da) \mu_u(dx) du \right]. \end{aligned}$$

Applying Burkholder-Davis-Gundy inequality, we obtain

$$\mathbb{E}^{\mathbb{P}} \left[\left| \int_s^{(s+\delta) \wedge T} \int_{\mathbb{R}^d \times A} D\varphi_k(x) \sigma(x, X_r, a) \mathcal{M}^c(dr, dx, da) \right| \right] \leq Cq_k \mathbb{E}^{\mathbb{P}} \left[\int_s^{(s+\delta) \wedge T} \langle 1, \mu_u \rangle du \right].$$

Finally, since $\varphi_k \geq 0$, we have

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} \left[\left| \int_s^{(s+\delta)\wedge T} \int_{\mathbb{R}^d \times \mathbb{R}_+ \times A} \sum_{k \geq 0} \langle \varphi_k, (k-1)\delta_x \rangle \mathbf{1}_{I_k(x, \mu_r, a)}(z) \mathcal{M}^d(dr, dx, dz, da) \right| \right] \\ & \leq \mathbb{E}^{\mathbb{P}} \left[\left| \int_s^{(s+\delta)\wedge T} \int_{\mathbb{R}^d \times A} \varphi_k(x) \sum_{k \geq 1} (k-1)\gamma(x, \mu_r, a) p_k(x, \mu_r, a) \bar{\mathbf{a}}_r(x, da) \mu_r(dx) dr \right| \right] \\ & \leq Cq_k \mathbb{E}^{\mathbb{P}} \left[\int_s^{(s+\delta)\wedge T} \langle 1, \mu_u \rangle du \right]. \end{aligned}$$

Combining these inequalities, we get

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} [|\langle \varphi_k, \mu_{(s+\delta)\wedge T} \rangle - \langle \varphi_k, \mu_s \rangle|] \\ & \leq Cq_k \mathbb{E}^{\mathbb{P}} \left[\int_s^{(s+\delta)\wedge T} (\langle 1, \mu_u \rangle + \langle |\cdot|, \mu_u \rangle) du + \int_s^{(s+\delta)\wedge T} \int_{\mathbb{R}^d \times A} |a| \bar{\mathbf{a}}_u(x, da) \mu_u(dx) du \right] \\ & \leq \delta Cq_k \left(\mathbb{E}^{\mathbb{P}} \left[\sup_{u \in [0, T]} (\langle 1, \mu_u \rangle + \langle |\cdot|, \mu_u \rangle) \right] + \mathbb{E}^{\mathbb{P}} \left[\int_s^{(s+\delta)\wedge T} \int_{\mathbb{R}^d \times A} |a| \bar{\mathbf{a}}_u(x, da) \mu_u(dx) du \right] \right). \end{aligned}$$

Combining (4.37) and (4.40), together with the uniform bound (4.41), we obtain $\mathbb{E}^{\mathbb{P}} [|\langle \varphi_k, \mu_{(s+\delta)\wedge T} \rangle - \langle \varphi_k, \mu_s \rangle|] \leq Cq_k \delta$. Multiplying for $\frac{1}{2^k q_k}$, summing over $k \in \mathbb{N}$ and applying the monotone convergence theorem, we get $\mathbb{E}^{\mathbb{P}} [\mathbf{d}_{\mathbb{R}^d}(\mu_{(s+\delta)\wedge T}, \mu_s)] \leq \delta C$, which gives us (5.43).

Step 2. Secondly, we prove that $\{\mathbb{P}_{\mathbf{D}^d} : \mathbb{P} \in \mathcal{R}_{(t, \lambda)}^\varepsilon\} \subseteq \mathcal{P}^1(\mathbf{D}^b)$ is relatively compact. Combining the bound (4.40) with (4.41) and (4.37), we get

$$\sup_{\mathbb{P} \in \mathcal{R}_{(t, \lambda)}^\varepsilon} \mathbb{E}^{\mathbb{P}} \left[\sup_{u \in [t, T]} \int_{\mathbb{R}^d} |x|^2 \mu_u(dx) \right] < \infty.$$

This bound, together with (4.38) and (2.1), gives that

$$\sup_{\mathbb{P} \in \mathcal{R}_{(t, \lambda)}^\varepsilon} \mathbb{E}^{\mathbb{P}} \left[\sup_{u \in [t, T]} \mathbf{d}_{2, \mathbb{R}^d}^2(\mu_u, \delta_0) \right] < \infty. \quad (5.44)$$

Putting together *Step 1* and this bound, we have from [32, Corollary B.2] that $\{\mathbb{P}_{\mathbf{D}^d} : \mathbb{P} \in \mathcal{R}_{(t, \lambda)}^\varepsilon\} \subseteq \mathcal{P}^1(\mathbf{D}^b)$ is relatively compact.

Step 3. From the first step, we have that $\{\mathbb{P} \circ \mu^{-1} : \mathbb{P} \in \mathcal{R}_{(t, \lambda)}^\varepsilon\}$ is tight in $\mathcal{P}^1(\mathbf{D}^b)$. Adding this to (4.41) and (5.44), we have that $\{\mathbb{P}_{\mathcal{A}^{\text{Leb}, \cdot, 1}} : \mathbb{P} \in \mathcal{R}_{(t, \lambda)}^\varepsilon\}$ is compact in $\mathcal{P}^1(\mathcal{A}^{\text{Leb}, \cdot, 1})$. This entails that $\mathcal{R}_{(t, \lambda)}^\varepsilon$ is relatively compact in $\mathcal{P}^1(\mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1})$ since $\{\mathbb{P}_{\mathbf{D}^d} : \mathbb{P} \in \mathcal{R}_{(t, \lambda)}^\varepsilon\}$ and $\{\mathbb{P}_{\mathcal{A}^{\text{Leb}, \cdot, 1}} : \mathbb{P} \in \mathcal{R}_{(t, \lambda)}^\varepsilon\}$ are relatively compact in $\mathcal{P}^1(\mathbf{D}^d)$ and $\mathcal{P}^1(\mathcal{A}^{\text{Leb}, \cdot, 1})$ respectively.

Step 4. Finally, we prove $\mathcal{R}_{(t, \lambda)}^\varepsilon$ is closed. To do that, we show that \mathbb{P}^∞ belongs to $\mathcal{R}_{(t, \lambda)}^\varepsilon$ for $\mathbb{P}^n \rightarrow \mathbb{P}^\infty$ in $\mathcal{P}^1(\mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1})$, with $\mathbb{P}^n \in \mathcal{R}_{(t, \lambda)}^\varepsilon$. Since μ_t has law λ under \mathbb{P}^n , the same is true under \mathbb{P}^∞ . Analogously, since $\mathbb{P}^n(\alpha \in \mathcal{A}^{\text{Leb}, \mu, 1}) = 1$, the same is true under \mathbb{P}^∞ . For any $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$ and $\mathbb{P} \in \mathcal{P}^1(\mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1})$, define $M_s^{\mathbb{P}, F, \varphi} : \mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1} \rightarrow \mathbb{R}$ by

$$M_s^{\mathbb{P}, F, \varphi}(\mathbf{x}, \alpha) = F_\varphi(\mathbf{x}_s) - \int_t^s \int_{\mathbb{R}^d \times A} \mathcal{L}F_\varphi(y, \mathbf{y}_u, a) \bar{\alpha}_u(y, da) \mathbf{y}_u(dy) \delta_{\mathbf{y}_u = \mathbf{x}_u} du.$$

Recalling the definition of \mathcal{L} , we see that there exists a constant $C > 0$ depending only on the bounds of F , φ and the constants C_b, C_σ, C_γ such that

$$|\mathcal{L}F_\varphi(y, \lambda, a)| \leq C(1 + |y| + |a|).$$

This implies

$$\left| M_s^{\mathbb{P}, F_\varphi}(\mathbf{x}, \alpha) \right| \leq C \left(1 + \sup_{u \in [t, T]} \mathbf{d}_{1, \mathbb{R}^d}(\mathbf{x}_u, \delta_0) + \int_t^T \int_{\mathbb{R}^d \times A} |a| \bar{\alpha}_u(x, da) \mathbf{y}_u(dx) du \right).$$

Combining this with the continuity of b, σ, γ and p_k for $k \in \mathbb{N}$, we have that $(\mathbb{P}, \mathbf{x}, \alpha) \mapsto M_s^{\mathbb{P}, F_\varphi}(\mathbf{x}, \alpha)$ is a continuous function for each $s \in [t, T]$, $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$ using [32, Corollary A.5]. Since $\mathbb{P}^n \rightarrow \mathbb{P}^\infty$ in $\mathcal{P}^1(\mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1})$, it follows that

$$\mathbb{E}^{\mathbb{P}^\infty} \left[\left(M_{s+u}^{\mathbb{P}^\infty, F_\varphi} - M_s^{\mathbb{P}^\infty, F_\varphi} \right) \Lambda \right] = \lim_{n \rightarrow \infty} \mathbb{E}^{\mathbb{P}^n} \left[\left(M_{s+u}^{\mathbb{P}^n, F_\varphi} - M_s^{\mathbb{P}^n, F_\varphi} \right) \Lambda \right],$$

for every $s \in [t, T]$, $u \geq 0$ such that $s + u \leq T$, any $F \in C_b^2(\mathbb{R})$ and $\varphi \in C_b^2(\mathbb{R}^d)$, and any bounded continuous function Λ on $\mathbf{D}^d \times \mathcal{A}^{\text{Leb}, \cdot, 1}$, measurable with respect to $\sigma(\mu_u, \bar{\mathbf{a}}_u : u \in [t, s])$. Since $\mathbb{P}^n \in \mathcal{R}_{(t, \lambda)}^\varepsilon$, the process $\left(M_s^{\mathbb{P}^n, F_\varphi}(\mu, \mathbf{a}) \right)_{s \in [0, T]}$ is a martingale under \mathbb{P}^n , and the above quantity is zero. This shows that $\left(M_s^{\mathbb{P}^\infty, \varphi}(\mu, \mathbf{a}) \right)_{s \in [0, T]}$ is a martingale under \mathbb{P}^∞ , and so $\mathbb{P}^\infty \in \mathcal{R}_{(t, \lambda)}$.

Moreover, by Lemma 5.4 we get since J is lower semicontinuous. Therefore,

$$J(t, \lambda; \mathbb{P}^\infty) \leq \liminf_{n \rightarrow \infty} J(t, \lambda; \mathbb{P}^n) \leq v(t, \lambda) + \varepsilon,$$

which means that $\mathbb{P}^\infty \in \mathcal{R}_{(t, \lambda)}^\varepsilon$. □

Theorem 5.2. *For $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$, there exists an optimal control $\beta^* \in \mathcal{R}_{(t, \lambda)}^5$ such that*

$$v(t, \lambda) = J(t, \lambda; \beta^*). \tag{5.45}$$

Proof. Fix $\varepsilon > 0$. We have that $\inf_{\mathbb{P} \in \mathcal{R}_{(t, \lambda)}} J(t, \lambda; \mathbb{P}) = \inf_{\mathbb{P} \in \mathcal{R}_{(t, \lambda)}^\varepsilon} J(t, \lambda; \mathbb{P})$. By Proposition 5.9, $\mathcal{R}_{(t, \lambda)}^\varepsilon$ is compact and, by Lemma 5.4, J is lower-semicontinuous. Therefore, since $v(t, \lambda)$ is the infimum of a continuous function over a nonempty compact set, it exists $\mathbb{P}^* \in \mathcal{R}_{(t, \lambda)}$ such that $v(t, \lambda) = J(t, \lambda; \mathbb{P}^*)$. From Lemma 4.3 and Proposition 4.4, under Assumption A1, we have the existence of optimal weak control \mathbf{a}^* such that $J(t, \lambda; \mathbf{a}^*) \leq J(t, \lambda; \mathbb{P}^*)$. Immersing this weak control in the class of strong controls, we find β^* that satisfies (5.45). □

6 HJB equation

6.1 Homeomorphisms with $\cup_{m \geq 0} \mathbb{R}^{dm}$

We have established the existence of an optimal control for the problem under consideration, which holds true under general assumptions. However, the formalism we have discussed thus far does not provide guidance on how to determine these optimal controls. A step towards addressing this is the differential characterization of the value function, commonly referred to as the Hamilton-Jacobi-Bellman (HJB) equation.

Though the problem has been stated in terms of finite measures, this depiction cannot be employed directly to tackle the task at hand. Indeed, the subset $\mathcal{N}[\mathbb{R}^d]$ where our processes live

is not open in $(M^1(\mathbb{R}^d), \mathbf{d}_{1, \mathbb{R}^d})$. As recalled in Remark 2.1, we can embed \mathbb{R}^{dm} to $\mathcal{N}[\mathbb{R}^d]$ for any $m \in \mathbb{N}$ via ι . Denoting $(\mathbb{R}^d)^0 := \{\emptyset\}$, and $\iota(\emptyset) := \mathbf{0}$, which is the measure equal to 0, we see that $\iota\left(\bigcup_{m \geq 0} (\mathbb{R}^d)^m\right) = \mathcal{N}[\mathbb{R}^d]$. Therefore, we can define a HJB system exploiting the differential structure of each \mathbb{R}^{dm} .

For each $m \in \mathbb{N}$, let $v^m : [0, T] \times (\mathbb{R}^d)^m \rightarrow \mathbb{R}$ be

$$v^m(t, x_1, \dots, x_m) := v\left(t, \sum_{i=1}^m \delta_{x_i}\right) = v(t, \iota(\vec{x}^m)), \quad (6.46)$$

with $\vec{x}^m = (x_1, \dots, x_m)^\top$. Analogously, we define $(\mathfrak{b}^m, \Sigma^m) : (\mathbb{R}^d)^m \times A^m \rightarrow \mathbb{R}^{dm} \times \mathbb{R}^{dm \times d'}$ as

$$\mathfrak{b}^m(\vec{x}^m, \vec{a}^m) := \begin{pmatrix} b(x_1, \iota(\vec{x}^m), a_1) \\ \vdots \\ b(x_m, \iota(\vec{x}^m), a_m) \end{pmatrix}, \quad \Sigma^m(\vec{x}^m, \vec{a}^m) := \begin{pmatrix} \sigma(x_1, \iota(\vec{x}^m), a_1) \\ \vdots \\ \sigma(x_m, \iota(\vec{x}^m), a_m) \end{pmatrix}.$$

For any $m \in \mathbb{N}$, let \mathbf{L}^m be the generator as follows

$$\begin{aligned} \mathbf{L}^m v^m(\vec{x}^m, \vec{a}^m) &:= \mathfrak{b}^m(\vec{x}^m, \vec{a}^m)^\top Dv^m(\vec{x}^m) + \frac{1}{2} \text{Tr}\left(\Sigma^m(\Sigma^m)^\top(\vec{x}^m, \vec{a}^m) D^2 v^m(\vec{x}^m)\right) \\ &+ \sum_{i=1}^m \gamma(x_i, \iota(\vec{x}^m), a_i) \left(\sum_{k \geq 0} v^{m+(k-1)}\left(x_1, \dots, x_{i-1}, \underbrace{x_i, \dots, x_i}_{(k-1)\text{-times}}, x_{i+1}, \dots, x_m\right) \right. \\ &\quad \left. p_k(x_i, \iota(\vec{x}^m), a_i) - v^m(\vec{x}^m) \right). \end{aligned}$$

Remark 6.4. *These notations look like the one used in Proposition 2.1. As seen in their construction, branching processes behave as diffusion processes between two different branching events, that are defined via a Poisson random measure independent of each Brownian motion. This is why the first two terms of \mathbf{L}^m are Itô's-like terms while the last one takes into account the results of the branching events.*

Since our aim is giving a Verification Theorem, we associate an admissible control from a set of functions $\hat{a}^m : [0, T] \times (\mathbb{R}^d)^m \rightarrow A^m$ in the following way. As done in [7] and [31], we consider the partial ordering relation \preceq (resp. \prec) by

$$j \preceq i \Leftrightarrow \exists \ell \in \mathcal{I} : i = j\ell \quad (\text{resp. } j \prec i \Leftrightarrow \exists \ell \in \mathcal{I} \setminus \{\emptyset\} : i = j\ell)$$

for all $i, j \in \mathcal{I}$. With respect to this partial ordering, for $i = i_0 \dots i_p, j = j_0 \dots j_q \in \mathcal{I}$, we define $i \wedge j$ as \emptyset in the case $i_0 \neq j_0$, and as $i_0 \dots i_{\ell-1}$ with $\ell \leq \min\{p, q\}$ if $j_k = i_k$ for $k = 0, \dots, \ell-1$ and $j_k \neq i_k$. If \mathcal{I}^\preceq is defined as

$$\mathcal{I}^\preceq = \{V \subseteq \mathcal{I} : |V| < \infty, i \not\prec j \text{ for } i, j \in V\},$$

the set of labels that could describe a population in $\mathcal{N}[\mathbb{R}^d]$ must belong to \mathcal{I}^\preceq . For any $V \subset \mathcal{I}^\preceq$, we can give a total order. If $i = i_0 \dots i_p, j = j_0 \dots j_q \in V$ and $i \wedge j = i_0 \dots i_{\ell-1}$, we denote $i < j$ if $i_\ell < j_\ell$. This means that for any $V \subset \mathcal{I}^\preceq$, there exists a bijection $\phi^V : V \rightarrow \{1, \dots, |V|\}$ associated with this total order in V .

Let $\hat{a}^m : [0, T] \times (\mathbb{R}^d)^m \rightarrow A^m$ be a function that is symmetric in the last m variables, for any $m \geq 1$. Let $\hat{\beta}$ be the control defined as follows

$$\hat{\beta}_s^i := \sum_{k \geq 1} \mathbf{1}_{\tau_{k-1} \leq s < \tau_k} \left(a_0 \mathbf{1}_{i \neq V_k} + \hat{a}_{\phi^{V_k}(i)}^{|V_k|} \left(s, Y_s^{(\phi^{V_k})^{-1}(1), \beta}, \dots, Y_s^{(\phi^{V_k})^{-1}(|V_k|), \beta} \right) \right). \quad (6.47)$$

Remark 6.5. *The connection between a control and a sequence of functions \hat{a}^m provides insight into approaching the problem of optimal control through the examination of the corresponding HJB equation. The equation itself is dependent on v^m , where each branching event is associated with the switching of regime m .*

6.2 Verification Theorem

Theorem 6.3. *Let w be a function in $C^0([0, T] \times \mathcal{N}[\mathbb{R}^d])$ such that*

$$-C_w(1 + \langle 1, \lambda \rangle + \langle |\cdot|, \lambda \rangle) \leq w_t(\lambda) \leq C_w(1 + \langle 1, \lambda \rangle^2 + \langle |\cdot|^2, \lambda \rangle). \quad (6.48)$$

for some constant $C_w > 0$. Assume that w^m , defined as in (6.46), is in $C^{1,2}([0, T] \times \mathbb{R}^{dm})$ for any $m \in \mathbb{N}$.

(i) *Suppose that*

$$-\partial_t w^m(t, \vec{x}^m) - \inf_{\vec{a}^m \in A^m} \left\{ L^m w^m(\vec{x}^m, \vec{a}^m) + \sum_{i=1}^m \psi(x_i, \iota(\vec{x}^m), a_i) \right\} \leq 0, \quad (6.49)$$

$$w^m(T, \vec{x}^m) \leq \Psi(\iota(\vec{x}^m))$$

for any $m \in \mathbb{N}$, $t \in [0, T]$, and $\vec{x}^m \in \mathbb{R}^{dm}$. Then $w \leq v$ on $[0, T] \times \mathcal{N}[\mathbb{R}^d]$.

(ii) *Suppose further $w^m(T, \vec{x}^m) = \Psi(\iota(\vec{x}^m))$, for any $m \in \mathbb{N}$, and $\vec{x}^m \in \mathbb{R}^{dm}$, and there exist measurable functions $\vec{a}^m(t, \vec{x}^m)$, for $m \in \mathbb{N}$, and $(t, \vec{x}^m) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$, valued in A^m such that*

$$-\partial_t w^m(t, \vec{x}^m) - \inf_{\vec{a}^m \in A^m} \left\{ L^m w^m(\vec{x}^m, \vec{a}^m) - \sum_{i=1}^m \psi(x_i, \iota(\vec{x}^m), a_i) \right\} \quad (6.50)$$

$$= -\partial_t w^m(t, \vec{x}^m) - \left\{ L^m w^m(\vec{x}^m, \vec{a}^m(t, \vec{x}^m)) - \sum_{i=1}^m \psi(x_i, \iota(\vec{x}^m), \mathbf{a}_i^m(t, \vec{x}^m)) \right\}$$

$$= 0.$$

Defining $\hat{\beta}$ as in (6.47) associated with the functions \vec{a}^m for $m \geq 1$, we assume that the following SDE admits a unique solution

$$\langle \varphi, \xi_s^{\hat{\beta}} \rangle = \langle \varphi, \lambda \rangle + \int_t^s \sum_{i \in V_u} D\varphi(Y_u^{i, \hat{\beta}})^\top \sigma(Y_u^{i, \hat{\beta}}, \xi_u^{\hat{\beta}}, \hat{\beta}_u^i) dB_u^i + \int_t^s \sum_{i \in V_u} L\varphi(Y_u^{i, \hat{\beta}}, \xi_u^{\hat{\beta}}, \hat{\beta}_u^i) du$$

$$+ \int_{(t, s] \times \mathbb{R}_+} \sum_{i \in V_{u-}} \sum_{k \geq 0} (k-1) \varphi(Y_u^{i, \hat{\beta}}) \mathbf{1}_{I_k(Y_u^{i, \hat{\beta}}, \xi_u^{\hat{\beta}}, \hat{\beta}_u^i)}(z) Q^i(dudz).$$

Suppose, moreover, that $\hat{\beta} \in \mathcal{R}_{(t, \lambda)}^s$ for any $(t, \lambda) \in \mathcal{N}[\mathbb{R}^d]$. Then, $w = v$ on $[0, T] \times \mathcal{N}[\mathbb{R}^d]$, and $\hat{\beta}$ is an optimal Markov control.

Proof. (i) We consider the notation adopted in Proposition 2.1. Fix a starting condition $(t, \vec{x}^m) \in [0, T] \times \mathbb{R}^{dm}$ and an admissible control $\beta \in \mathcal{R}_{(t, \iota(\vec{x}^m))}^s$. Define the two sequences of stopping times $(\tau_k)_{k \in \mathbb{N}}$ and $(\theta_n)_{n \in \mathbb{N}}$

$$\tau_k = \inf \{ s \in (\tau_{k-1}, T] : \exists i \in V_{k-1}, Q^i((\tau_{k-1}, s] \times [0, C_\gamma]) = 1 \},$$

$$\theta_n := \inf \{ s \in [t, T] : |V_s| \geq n \} \wedge \inf \left\{ s \in [t, T] : \sum_{i \in V_u} |Y_u^{i, \beta}| \geq n \right\}.$$

With these stopping times, we can describe ξ^β as

$$\xi_s^\beta = \sum_{k \geq 1} \mathbb{1}_{\tau_{k-1} \leq s < \tau_k} \sum_{i \in V_k} \delta_{Y_s^{i,\beta}} = \sum_{k \geq 1} \mathbb{1}_{\tau_{k-1} \leq s < \tau_k} \iota \left(\vec{Y}_s^{\beta, |V_k|} \right).$$

As noted in Remark 6.4, between the branching events τ_{k-1} and τ_k , the population behave like a controlled diffusion living in $\mathbb{R}^{|V_{k-1}|}$. Therefore, Itô's formula describes here the evolution of a function valued in ξ^β in each interval $[\tau_{k-1} \wedge \theta_n, \tau_k \wedge \theta_n]$.

Using the embedding ι , we have that (2.6) translates into

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^s} \left[w^{\mathbf{m}_k^n} \left(s \wedge \tau_k \wedge \theta_n, \vec{Y}_{s \wedge \tau_k \wedge \theta_n}^{\beta, \mathbf{m}_k^n} \right) - w^{\mathbf{m}_{k-1}^n} \left(s \wedge \tau_{k-1} \wedge \theta_n, \vec{Y}_{s \wedge \tau_{k-1} \wedge \theta_n}^{\beta, \mathbf{m}_{k-1}^n} \right) \right] \\ &= \mathbb{E}^{\mathbb{P}^s} \left[\int_{s \wedge \tau_{k-1} \wedge \theta_n}^{s \wedge \tau_1 \wedge \theta_n} \left\{ \partial_t w^{\mathbf{m}_{k-1}^n} \left(t, \vec{Y}_u^{\beta, \mathbf{m}_{k-1}^n} \right) + \mathbf{L}^{\mathbf{m}_{k-1}^n} w^{\mathbf{m}_{k-1}^n} \left(\vec{Y}_u^{\beta, \mathbf{m}_{k-1}^n}, \vec{\beta}_u^{\mathbf{m}_{k-1}^n} \right) \right\} du \right], \end{aligned}$$

where $\mathbf{m}_k^n := |V_{\tau_k \wedge \theta_n}|$ and $\vec{\beta}_u^{\mathbf{m}_{k-1}^n} := (\beta_u^i)_{i \in V_{\tau_{k-1} \wedge \theta_n}}$. Therefore, we have that

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^s} \left[w^{|V_{s \wedge \theta_n}|} \left(s \wedge \theta_n, \vec{Y}_{s \wedge \theta_n}^{\beta, |V_{s \wedge \theta_n}|} \right) \right] - w^m(t, \vec{x}^m) \tag{6.51} \\ &= \mathbb{E}^{\mathbb{P}^s} \left[\sum_{k \geq 1} \left(w^{\mathbf{m}_k^n} \left(s \wedge \tau_k \wedge \theta_n, \vec{Y}_{s \wedge \tau_k \wedge \theta_n}^{\beta, \mathbf{m}_k^n} \right) - w^{\mathbf{m}_{k-1}^n} \left(s \wedge \tau_{k-1} \wedge \theta_n, \vec{Y}_{s \wedge \tau_{k-1} \wedge \theta_n}^{\beta, \mathbf{m}_{k-1}^n} \right) \right) \right] \\ &= \mathbb{E}^{\mathbb{P}^s} \left[\sum_{k \geq 1} \int_{s \wedge \tau_{k-1} \wedge \theta_n}^{s \wedge \tau_1 \wedge \theta_n} \left\{ \partial_t w^{\mathbf{m}_{k-1}^n} \left(t, \vec{Y}_u^{\beta, \mathbf{m}_{k-1}^n} \right) + \mathbf{L}^{\mathbf{m}_{k-1}^n} w^{\mathbf{m}_{k-1}^n} \left(\vec{Y}_u^{\beta, \mathbf{m}_{k-1}^n}, \vec{\beta}_u^{\mathbf{m}_{k-1}^n} \right) \right\} du \right]. \end{aligned}$$

Since w satisfies (6.50), we have

$$\partial_t w^{\mathbf{m}_k^n} \left(t, \vec{Y}_u^{\beta, \mathbf{m}_k^n} \right) + \mathbf{L}^{\mathbf{m}_k^n} w^{\mathbf{m}_k^n} \left(\vec{Y}_u^{\beta, \mathbf{m}_k^n}, \vec{\beta}_u^{\mathbf{m}_k^n} \right) + \sum_{i \in V_{\tau_k \wedge \theta_n}} \psi \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) \geq 0,$$

for any $\beta \in \mathcal{R}_{(t, \iota(\vec{x}^m))}^s$, $k \geq 0$ and $u \in [\tau_k \wedge \theta_n, \tau_{k+1} \wedge \theta_n]$. Thus,

$$\mathbb{E}^{\mathbb{P}^s} \left[w^{|V_{s \wedge \theta_n}|} \left(s \wedge \theta_n, \vec{Y}_{s \wedge \theta_n}^{\beta, |V_{s \wedge \theta_n}|} \right) \right] - w^m(t, \vec{x}^m) \geq -\mathbb{E}^{\mathbb{P}^s} \left[\int_t^{s \wedge \theta_n} \sum_{i \in V_u} \psi \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) du \right]. \tag{6.52}$$

From (2.15)-(2.16), we get

$$\left| \int_t^{s \wedge \theta_n} \sum_{i \in V_u} \psi \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) du \right| \leq C_\Psi \left(1 + \int_t^T \left(|V_u|^2 + \sum_{i \in V_u} |Y_u^{i,\beta}|^2 + \sum_{i \in V_u} |\beta_u^i|^2 \right) du \right),$$

therefore the r.h.s. in (6.52) is integrable for $\beta \in \mathcal{R}_{(t, \iota(\vec{x}^m))}^{s, \varepsilon}$ using (2.8), (2.19) and (2.20). Analogously, from (6.48), we also have that l.h.s. in (6.52) explodes to infinity or is integrable for $\beta \in \mathcal{R}_{(t, \iota(\vec{x}^m))}^{s, \varepsilon}$. We can then apply the dominated convergence theorem, and send n to infinity into (6.52), obtaining

$$\mathbb{E}^{\mathbb{P}^s} \left[w^{|V_s|} \left(s, \vec{Y}_s^{\beta, |V_s|} \right) \right] - w^m(t, \vec{x}^m) \geq -\mathbb{E}^{\mathbb{P}^s} \left[\int_t^s \sum_{i \in V_u} \psi \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) du \right], \text{ for } \beta \in \mathcal{R}_{(t, \iota(\vec{x}^m))}^{s, \varepsilon}.$$

Since w is continuous on $[0, T] \times \mathcal{N}[\mathbb{R}^d]$, by sending s to T , we obtain by the dominated convergence theorem and by (6.49) that

$$\mathbb{E}^{\mathbb{P}^s} \left[\Psi \left(\xi_T^\beta \right) \right] - w^m(t, \bar{x}^m) \geq -\mathbb{E}^{\mathbb{P}^s} \left[\int_t^T \sum_{i \in V_u} \psi \left(Y_u^{i, \beta}, \xi_u^\beta, \beta_u^i \right) du \right], \text{ for } \beta \in \mathcal{R}_{(t, \iota(\bar{x}^m))}^{s, \varepsilon}.$$

From the arbitrariness of $\beta \in \mathcal{R}_{(t, \iota(\bar{x}^m))}^{s, \varepsilon}$, we deduce that $w^m(t, \bar{x}^m) \leq v^m(t, \bar{x}^m)$, for any $m \geq 1$, and $(t, \bar{x}^m) \in [0, T] \times \mathbb{R}^{dm}$, *i.e.*, $w(t, \lambda) \leq v(t, \lambda)$ for any $(t, \lambda) \in [0, T] \times \mathcal{N}[\mathbb{R}^d]$.

(ii) From the definition of the control $\hat{\beta}$, we have that

$$-\partial_t w^m(t, \bar{x}^m) - \left\{ \mathbf{L}^m v^m(\bar{x}^m, \bar{\mathbf{a}}^m(t, \bar{x}^m)) - \sum_{i=1}^m \psi(x_i, \iota(\bar{x}^m), \mathbf{a}_i^m(t, \bar{x}^m)) \right\} = 0.$$

Applying this to (6.51), we get

$$w^m(t, \bar{x}^m) = \mathbb{E}^{\mathbb{P}^s} \left[w^{|V_{s \wedge \theta_n}|} \left(s \wedge \theta_n, \bar{Y}_{s \wedge \theta_n}^{\hat{\beta}, |V_{s \wedge \theta_n}|} \right) + \int_t^{s \wedge \theta_n} \sum_{i \in V_u} \psi \left(Y_u^{i, \hat{\beta}}, \xi_u^{\hat{\beta}}, \hat{\beta}_u^i \right) du \right],$$

for any $n \geq 1$. From Fatou's lemma, we obtain

$$w^m(t, \bar{x}^m) \geq \mathbb{E}^{\mathbb{P}^s} \left[w^{|V_s|} \left(s, \bar{Y}_s^{\hat{\beta}, |V_s|} \right) + \int_t^s \sum_{i \in V_u} \psi \left(Y_u^{i, \hat{\beta}}, \xi_u^{\hat{\beta}}, \hat{\beta}_u^i \right) du \right].$$

Sending s to T and using again Fatou's lemma, together with the fact $w^p(T, \bar{y}^p) = \Psi(\iota(\bar{y}^p))$, for any $p \in \mathbb{N}$, and $\bar{y}^p \in \mathbb{R}^{dp}$, we see that

$$w^m(t, \bar{x}^m) \geq \mathbb{E}^{\mathbb{P}^s} \left[\Psi \left(\xi_T^{\hat{\beta}} \right) + \int_t^s \sum_{i \in V_u} \psi \left(Y_u^{i, \hat{\beta}}, \xi_u^{\hat{\beta}}, \hat{\beta}_u^i \right) du = J \left(t, (\bar{x}^m); \hat{\beta} \right) \right].$$

This shows that $w^m(t, \bar{x}^m) \geq J \left(t, (\bar{x}^m); \hat{\beta} \right) \geq v^m(t, \bar{x}^m)$, and finally that $w = v$ with $\hat{\beta}$ as an optimal Markovian control. \square

The verification theorem presented here offers the advantage of not only establishing the optimality of a solution but also showing that a certain function is smaller than the value function. This characterization serves as a generalization of [44, Theorem II.3.1] for value functions in a broader context. However, this description differs significantly from the one used to introduce the controlled processes. Hence, to ensure the proof of optimality, we provide an equivalent verification theorem. The subsequent proposition establishes a characterization of optimality without relying on the embedding to $\cup_{m \geq 0} \mathbb{R}^{dm}$. Instead, it employs a (sub)martingale criterion similar to [37, Lemma 2.1].

Proposition 6.10. *Let w be a function in $C^0([0, T] \times \mathcal{N}[\mathbb{R}^d])$ such that*

$$-C_w (1 + \langle 1, \lambda \rangle + \langle |\cdot|, \lambda \rangle) \leq w_t(\lambda) \leq C_w (1 + \langle 1, \lambda \rangle^2 + \langle |\cdot|^2, \lambda \rangle). \quad (6.53)$$

for some constant $C_w > 0$. Fix $(t, \bar{\lambda}) \in \mathcal{N}[\mathbb{R}^d]$, and assume the following

(i) $w_T(\lambda) = g(\lambda)$, for $\lambda \in \mathcal{N}[\mathbb{R}^d]$;

(ii) $\left\{ w_s \left(\xi_s^\beta \right) + \int_t^s \sum_{i \in V_u} \psi \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) du : s \in [t, T] \right\}$ is a \mathbb{P}^s -local submartingale, for any $\beta \in \mathcal{R}_{(t,\bar{\lambda})}^s$;

(iii) there exists $\hat{\beta} \in \mathcal{R}_{(t,\bar{\lambda})}^s$ such that $\left\{ w_s \left(\xi_s^{\hat{\beta}} \right) + \int_t^s \sum_{i \in V_u} \psi \left(Y_u^{i,\hat{\beta}}, \xi_u^{\hat{\beta}}, \hat{\beta}_u^i \right) du : s \in [t, T] \right\}$ is a \mathbb{P}^s -local martingale.

Then, $\bar{\beta}$ is an optimal control for $v(t, \bar{\lambda})$, i.e., $v(t, \bar{\lambda}) = J(t, \bar{\lambda}; \bar{\beta})$, and $v(t, \bar{\lambda}) = w_t(\bar{\lambda})$.

Proof. By the local submartingale property in condition (ii), there exists a nondecreasing sequence of stopping times $(\tau_n)_n$ such that $\tau_n \uparrow T$ a.s. and

$$\mathbb{E} \left[w_{s \wedge \tau_n} \left(\xi_{s \wedge \tau_n}^\beta \right) + \int_t^{s \wedge \tau_n} \sum_{i \in V_u} \psi \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) du \right] \geq w_t(\bar{\lambda}), \quad \text{for } \beta \in \mathcal{R}_{(t,\bar{\lambda})}^s. \quad (6.54)$$

We fix $\varepsilon > 0$ and restrict to consider $\beta \in \mathcal{R}_{(t,\bar{\lambda})}^{s,\varepsilon}$. From (6.53) and (2.15)-(2.16), we see that for all n and $\beta \in \mathcal{R}_{(t,\bar{\lambda})}^{s,\varepsilon}$, the r.h.s. is integrable and bounded by an integrable quantity. Applying dominated convergence theorem, by sending n to infinity into (6.54), we get

$$\begin{aligned} w_t(\bar{\lambda}) &\leq \mathbb{E} \left[w_T \left(\xi_T^\beta \right) + \int_t^T \sum_{i \in V_u} \psi \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) du \right] \\ &\leq \mathbb{E} \left[g \left(\xi_T^\beta \right) + \int_t^T \sum_{i \in V_u} \psi \left(Y_u^{i,\beta}, \xi_u^\beta, \beta_u^i \right) du \right] = J(t, \bar{\lambda}; \beta), \end{aligned}$$

using the terminal condition (i), and (3.31). Since β is arbitrary in $\mathcal{R}_{(t,\bar{\lambda})}^{s,\varepsilon}$, this shows that $v(t, \bar{\lambda}) \geq w_t(\bar{\lambda})$. To obtain the reverse inequality when the local martingale property for $\bar{\beta}$ in condition (iii) holds, we need to proceed as in the point (iii) of Theorem 6.3. This means that (6.54) is an equality and we conclude by applying Fatou's lemma. \square

6.3 Examples

We include two examples within the linear-quadratic framework. By establishing the equivalence between weak controls and strong controls, we choose to utilize the former formalism due to its simpler notation, avoiding unnecessary complexities.

6.3.1 Standard Linear-Quadratic case

We follow the path outlined in [37] and [34]. Let $A := \mathbb{R}^q$, $d' = d$ and let the coefficients be as follows

$$\begin{aligned} b_t(x, \lambda, a) &= B_t x + \bar{B}_t a, & \sigma_t(x, \lambda, a) &= \sigma_t \mathbb{I}, \\ \gamma_t(x, \lambda, a) &= \gamma_t, & p_k(x, \lambda, a) &= p_k, \end{aligned}$$

with \mathbb{I} being the identity matrix, and $B, \bar{B}, \bar{\sigma}, \bar{\gamma}$ are bounded valued in $\mathbb{R}^{d \times d}$, $\mathbb{R}^{d \times p}$, $\mathbb{R}^{d \times d}$ and \mathbb{R}_+ respectively. Since the control does not impact the coefficients that describe the branching, the search for a minimal control in (6.50) just focuses on each function w^m , without involving w^{m+k-1} for $k \geq 0$.

Let ψ and Ψ be as

$$\begin{aligned}\psi_t(x, \lambda, a) &= x^\top C_t x + c_t \langle 1, \lambda \rangle + a^\top \bar{C}_t a \\ \Psi(\lambda) &= \int_{\mathbb{R}^d} x^\top H x + h \langle 1, \lambda \rangle^2,\end{aligned}$$

where $t \mapsto C_t$ (resp. $t \mapsto \bar{C}_t$) is a bounded function in \mathbb{S}^d (resp. \mathbb{S}^m), the set of symmetric matrices in $\mathbb{R}^{d \times d}$ (resp. $\mathbb{R}^{m \times m}$), $t \mapsto c_t \in \mathbb{R}_+$ is bounded, $H \in \mathbb{S}^d$ and $h \geq 0$.

We shall make the following assumptions

- (i) C and H are non-negative a.s.;
- (ii) \bar{C} is uniformly positive definite, *i.e.*, $\bar{C}_t \geq \epsilon \mathbb{I}_m$ for some $\epsilon > 0$.

We are now ready to use Proposition (6.10) by seeking a field $\{w_t(\lambda) : \lambda \in \mathcal{N}[\mathbb{R}^d], t \in [0, T]\}$ that satisfies the local (sub)martingality conditions. Let w be as follows

$$\begin{aligned}w_t(\lambda) &= w_t^1(\lambda) + w_t^2(\lambda) + w_t^3(\lambda), \quad \text{with } w_t^1(\lambda) = \int_{\mathbb{R}^d} x^\top Q_t x \lambda(dx), \\ &w_t^2(\lambda) = p_t \langle 1, \lambda \rangle^2, \quad w_t^3(\lambda) = \bar{p}_t \langle 1, \lambda \rangle,\end{aligned}$$

for some functions (Q, p, \bar{p}) with values in $\mathbf{S}^d \times \mathbb{R} \times \mathbb{R}$ such that

$$\begin{cases} dQ_t = \dot{Q}_t dt, & \text{for } t \in [0, T], \quad Q_T = H, \\ dp_t = \dot{p}_t dt, & \text{for } t \in [0, T], \quad p_T = h, \\ d\bar{p}_t = \dot{\bar{p}}_t dt, & \text{for } t \in [0, T], \quad \bar{p}_T = 0. \end{cases}$$

The terminal conditions ensure that $w_t(\lambda) = \Psi(\lambda)$. Now, we need to determine the generators \dot{Q} , \dot{p} and $\dot{\bar{p}}$ to satisfy (6.50). Generalizing (3.22) to time-dependent functions, we have

$$\begin{aligned}w(t, \mu_t) &+ \int_0^t \int_{\mathbb{R}^d} \psi(x, \mu_u, \mathbf{a}_u(x)) \mu_u(dx) du \\ &= w(0, \mu_0) + \int_0^t \int_{\mathbb{R}^d} \mathcal{D}_u(x, \mu_u, \mathbf{a}_u(x), Q_u, p_u, \bar{p}_u) \mu_u(dx) du + \mathbb{M}_t,\end{aligned} \quad (6.55)$$

with

$$\begin{aligned}\mathcal{D}_u(x, \lambda, a, Q, p, \bar{p}) &:= x^\top \dot{Q} x + \dot{p} \langle 1, \lambda \rangle + \dot{\bar{p}} + (B_u x + \bar{B}_u a)^\top Q x \\ &+ x^\top Q (B_u x + \bar{B}_u a) + \sigma_u^2 \text{Tr}(Q) + (\bar{\gamma}_u M_1) x^\top Q x \\ &+ p \gamma_u (M_2 + M_1 \langle 1, \lambda \rangle) + \bar{p} \gamma_u M_1 + x^\top C_u x + c_u \langle 1, \lambda \rangle + a^\top \bar{C}_u a\end{aligned}$$

and \mathbb{M} is a martingale (after an eventual localization), and $M_1 := \sum_{k \geq 0} (k-1) p_k$, $M_2 := \sum_{k \geq 0} (k-1)^2 p_k$. Completing the square in \mathcal{D} , we obtain

$$\begin{aligned}\mathcal{D}_u(x, \lambda, a, Q, p, \bar{p}) &:= (\dot{p} + p \gamma_u M_1 + c_u) \langle 1, \lambda \rangle + (\dot{\bar{p}} + \sigma_u^2 \text{Tr}(Q) + \bar{p} \gamma_u M_1 + p \gamma_u M_2) \\ &+ x^\top \left(\dot{Q} + B_u^\top Q + Q B_u + (\bar{\gamma}_u M_1) Q + C_u + (\bar{B}_u Q + \bar{B}_u^\top Q)^\top \bar{C}_u^{-1} (\bar{B}_u Q + \bar{B}_u^\top Q) \right) x \\ &+ (a - \hat{a}_u(x, Q))^\top \bar{C}_u (a - \hat{a}_u(x, Q)),\end{aligned}$$

where

$$\hat{a}_u(x, Q) := -\bar{C}_u^{-1} (\bar{B}_u Q + \bar{B}_u^\top Q) x.$$

Therefore, whenever

$$\dot{Q} + B_u^\top Q + QB_u + (\bar{\gamma}_u M_1)Q + C_u + 2Q \left(\bar{B}_u \bar{C}_u^{-1} \bar{B}_u + \bar{B}_u^\top \bar{C}_u^{-1} \bar{B}_u \right) Q = 0, \quad (6.56)$$

$$\dot{p} + p\gamma_u M_1 + c_u = 0, \quad (6.57)$$

$$\dot{\bar{p}} + \sigma_u^2 \text{Tr}(Q) + \bar{p}\gamma_u M_1 + p\gamma_u M_2 = 0, \quad (6.58)$$

holds for $t \in [0, T]$, we have

$$\mathcal{D}_u(x, \lambda, a, Q, p, \bar{p}) = (a - \hat{a}_u(x, Q))^\top \bar{C}_u (a - \hat{a}_u(x, Q)).$$

Therefore, $\mathcal{D} \geq 0$ for any $a \in A$ and it is zero for $a = \hat{a}_u(x, Q)$. Additionally, it is worth noting that equations (6.56)-(6.58) have a solution since the first equation is a conventional Riccati equation, while the remaining two are linear ODEs.

This means that if the system of equations (6.58)-(6.58) is satisfied, from (6.55) and the fact that $\mathcal{D} \geq 0$, we get the local submartingale property (ii) of Proposition 6.10. Moreover, it is clear that it is zero for $\mathbf{a}_u(x) := \hat{a}_u(x, Q)$, with Q solution to (6.56), satisfying the local martingale property (iii) of Proposition 6.10. Therefore, such a control is an optimal one.

6.3.2 A Kinetic Example

In the case of a standard diffusion, we talk about *kinetic energy* when considering the following optimization setting. Consider controls β such that the diffusion satisfies the following SDE

$$dX_t = (b(t, X_t) + \beta_s)dt + \sigma dB_t,$$

with b Lipschitz in x uniformly in t and σ a positive constant. We look for a minimization of the cost function $\mathbb{E} \left[\frac{1}{2} \int_0^T |\beta_s|^2 \right]$, usually called the kinetic energy of the controlled diffusion.

We adapt this framework to the case of branching processes. Let $A := \mathbb{R}^q$, $d' = d$ and consider the following

$$\begin{aligned} b_t(x, \lambda, a) &= b(t, x) + a, & \sigma_t(x, \lambda, a) &= \mathbb{I}, \\ \gamma_t(x, \lambda, a) &= \gamma_t(x), & p_k(x, \lambda, a) &= p_k(x), \end{aligned}$$

with b , γ and p_k satisfying (2.2), (2.3) and (2.4). Taking the running cost as $\psi(x, \lambda, a) := \frac{1}{2}|a|^2$, we seek for a field $\{w_t(\lambda) : \lambda \in \mathcal{N}[\mathbb{R}^d], t \in [0, T]\}$ of the following form

$$w_t(\lambda) = \int_{\mathbb{R}^d} h(t, x) \lambda(dx),$$

for a certain function h . From (3.22), we have

$$\begin{aligned} w(t, \mu_t) &+ \int_0^t \int_{\mathbb{R}^d} \psi(x, \mu_u, \mathbf{a}_u(x)) \mu_u(dx) du \\ &= w(0, \mu_0) + \int_0^t \int_{\mathbb{R}^d} \mathcal{D}_u(x, \mu_u, \mathbf{a}_u(x), h) \mu_u(dx) du + \mathbb{M}_t, \end{aligned} \quad (6.59)$$

where

$$\mathcal{D}_t(x, \lambda, a, h) := \partial_t h + b(t, x)^\top Dh + a^\top Dh + \frac{1}{2} \Delta h + \frac{1}{2} |a|^2 + \phi(t, x)h,$$

with $\phi(x) := \gamma_t(x) \left(\sum_{k \geq 0} k p_k(x) - 1 \right)$, \mathbb{M} a martingale (after an eventual localization), and Δ the Laplacian. Operating as in the previous example, we see that whenever h satisfies the following PDE

$$\begin{cases} \partial_t h + b(t, x)^\top Dh - \frac{1}{2} |Dh|^2 + \frac{1}{2} \Delta h + \phi(t, x) h = 0 \\ h(T, x) = 0 \end{cases}, \quad (6.60)$$

we have

$$\mathcal{D}_u(x, \lambda, a, h) = \frac{1}{2} |a + Dh|^2.$$

This means that under (6.60), $\mathcal{D} \geq 0$ for any $a \in A$ and is zero for $a = -Dh$. Therefore, under (6.60), we get property (ii) of Proposition 6.10, and property (iii), for $\mathbf{a}_s(x) := -Dh(s, x)$, showing that this control is an optimal one. Under sufficient regularity of the function ϕ , solution of (6.60) can be established and found as an application of the Hopf-Cole transformation.

7 Conclusion

Our study focused on proving the existence of an optimal solution for controlled branching diffusions with final and running costs. We presented the strong formalism, expanding it to cover controlled populations with linearly growing drifts. Furthermore, we established bounds that ensure proper problem definition, which strengthens and broadens the existing literature on the subject.

Given appropriate conditions, we introduced the concept of relaxed controls in this new setting. This differs from [8] on how we deal with the label of each particle and is more focused on the law of the process living in $M(\mathbb{R}^d)$. By defining natural and weak controls, we were able to narrow down the scope of the problem. Uniqueness was proved for the class of weak controls, with strong controls being associated with them. Through a Filippov-type convexity condition, we showed equivalence among all formulations. Shifting our focus to control rules, we deal with this class for its topological properties. We showed that the optimization problem can be confined to a compact set and that the cost function is lower semicontinuous. This guarantees the existence of an optimal value for the relaxed problem, and subsequently, for the strong problem as well.

An homeomorphism is established between $\mathcal{N}[\mathbb{R}^d]$ and $\cup_{m \geq 0} \mathbb{R}^{dm}$. Leveraging the differential properties of the latter space, we derive a system of HJB equations for the problem and establish a verification theorem by extracting a control from the minimization of the HJB equations. Finally, two linear-quadratic examples are presented with the use of these results.

We believe that describing strong controls as weak controls provides a useful framework for introducing the rescaled version of the problem. In future research, we will concentrate on rescaling such processes, as done in [11] and [18], to develop controlled superprocesses. Since these processes will no longer have the advantage of the homeomorphism with $\cup_{m \geq 0} \mathbb{R}^{dm}$, it will be necessary to find a differential characterisation in $M(\mathbb{R}^d)$ directly.

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