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**Upper and Lower Images of a Fuzzy Set Induced by a Fuzzy Relation:
Applications to Fuzzy Inference and Diagnosis***

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and

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118 route de Narbonne, 31062 Toulouse Cedex, France***ABSTRACT**

In fuzzy set theory the image of a fuzzy set induced by a fuzzy relation is usually obtained by a sup-t-norm composition. This corresponds to the upper image which gathers the elements in relation with at least one element of the fuzzy set. However the dual point of view, leading to the definition of the lower image as the fuzzy set of elements in relation with all the elements of the fuzzy set whose image is computed, is not often considered. Fuzzy sets and fuzzy relations, depending on the situations, can be interpreted either in a conjunctive manner or as subsets of mutually exclusive possible values for variables whose precise values are ill-known (disjunctive view). The applications of upper and lower images are investigated in both interpretations. The generalized modus ponens, used in fuzzy rule-based systems, corresponds to the disjunctive view. The interest of upper and lower images is also emphasized for diagnosis problems where the conjunctive interpretation is encountered.

1. INTRODUCTION

Various schemes of fuzzy reasoning have been proposed in order to manage the uncertainty pervading expert knowledge and data. Two different kinds of models were introduced at the end of the 1970s and have been developed since that time. One, designed for solving diagnosis problems, is based on the

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association of causes (disorders, diseases, ...) and effects (manifestations, symptoms, ...) by means of a fuzzy relation; then possible causes of a set of observed effects are identified by solving a fuzzy relation equation. It can be viewed as a form of abductive reasoning.

The other kind of model corresponds to the extension of classical patterns of deductive reasoning, such as the modus ponens, to uncertain and fuzzy premises. From a knowledge representation point of view, it must be pointed out that the fuzzy sets, which appear in the two kinds of approaches, have a different interpretation. In the deductive reasoning model, fuzzy sets have a disjunctive meaning; they restrict the more or less possible values of variables and then these possible values are regarded as mutually exclusive since each variable is attached to a single-valued attribute of a given object (the attribute value is unique but ill known). By contrast, in the abductive reasoning model, fuzzy sets have usually a conjunctive interpretation; indeed there is no idea of mutual-exclusiveness between the symptoms, which are more or less certainly produced by a given disease, for instance.

In this paper the different compositions of a fuzzy set with a fuzzy relation, on which fuzzy reasoning techniques are based, are considered in great detail. More precisely the properties of the upper and lower images of a fuzzy set by a fuzzy relation or mapping are studied; the disjunctive and conjunctive interpretations of the mathematical model are presented. Weighted upper and lower images are also introduced. Then the application of upper and lower images to the generalized modus ponens, on the one hand, and to diagnosis problems, on the other hand, are discussed. The concluding remarks stress the importance of the results for the design of fuzzy expert systems.

2. UPPER AND LOWER IMAGES OF A SET BY A RELATION

2.1. CASE OF A CRISP SET AND OF A CRISP RELATION

2.1.1. Mathematical model

Let U and V be two sets, A be a crisp (i.e., ordinary) subset of U and R be a crisp relation defined on $U \times V$. R induces a multiple-valued mapping Γ from U to V defined by $\forall u \in U, \Gamma(u) = \{v \in V, (u, v) \in R\}$. Then we can define the upper image B^* of A induced by R (or Γ) in the following way:

$$B^* = \{v \in V \mid \exists u \in A, (u, v) \in R\} \quad (1)$$

The lower image B_* is defined by

$$B_* = \{v \in V \mid \forall u \in A, (u, v) \in R\} \quad (2)$$

Note that the upper and lower images are then respectively equal to

$$B^* = \{v \in V \mid A \cap \Gamma^{-1}(v) \neq \emptyset\} \quad (3)$$

or

$$B^* = \bigcup_{u \in A} \Gamma(u), \quad (4)$$

and

$$B_* = \{v \in V \mid A \subseteq \Gamma^{-1}(v)\} \quad (5)$$

or

$$B_* = \bigcap_{u \in A} \Gamma(u) \quad (6)$$

where

$$\Gamma^{-1}(v) = \{u \in U, v \in \Gamma(u)\}.$$

Observe also that B^* is the smallest B such that $B \supseteq \text{proj}_V[(A \times V) \cap R]$ where proj_V denotes the projection over V and B_* is the largest set B such that $A \times B \subseteq R$.

The upper image B^* is the set of elements of V , which are in relation R with *at least one* element of A ; the lower image B_* is the set of elements of V , which are in relation R with *all* the elements of A . When Γ is an ordinary mapping (i.e., $\forall u, \Gamma(u)$ is a singleton), $B_* = \emptyset$, except in the special case where $\forall u \in A, \Gamma(u) = v_0$ (then $B_* = \{v_0\}$). Thus the notion of lower image is interesting only with multiple-valued mappings or in the more general case of fuzzy mappings considered in section 2.2. We shall write symbolically

$$B^* = A \textcircled{3} R$$

$$B_* = A \textcircled{4} R$$

These notations have been recently proposed by Izumi et al. [19], who investigated some of the properties of these compositions.

Using (4) and (6) makes obvious the monotonicity properties

$$A \subseteq A' \Rightarrow A \textcircled{3} R \subseteq A' \textcircled{3} R \quad (7)$$

$$A \subseteq A' \Rightarrow A \textcircled{4} R \supseteq A' \textcircled{4} R \quad (8)$$

and the inclusion

$$B_* \subseteq B^* \quad (9)$$

The equality between B_* and B^* holds if and only if $\exists C \subseteq V, \forall u \in A, \Gamma(u) = C$; it is the case in particular if A is a singleton of U . When A becomes larger, B_* becomes larger and may even become equal to V , whereas B^* becomes smaller and may become empty. The following properties are also easy to establish using (4) and (6):

$$(A_1 \cup A_2) \oplus R = (A_1 \oplus R) \cup (A_2 \oplus R) \quad (10a)$$

$$(A_1 \cap A_2) \oplus R \subseteq (A_1 \oplus R) \cap (A_2 \oplus R) \quad (10b)$$

$$(A_1 \cup A_2) \odot R = (A_1 \odot R) \cap (A_2 \odot R) \quad (11a)$$

$$(A_1 \cap A_2) \odot R \supseteq (A_1 \odot R) \cup (A_2 \odot R). \quad (11b)$$

Let \bar{R} (resp. $\bar{\Gamma}$) be the complement of R (resp. Γ); i.e., $\bar{R} = \{(u, v), (u, v) \notin R\}$ and $\bar{\Gamma}(u) = \{v \in V, (u, v) \in R\} = \bar{\Gamma}(u)$. Let us denote by $B_*(\bar{R}), B^*(\bar{R}), B_*(\bar{R}), B^*(\bar{R})$ the upper and lower images of A induced by R and \bar{R} , respectively. Then using (4) and (6) we obtain

$$B_*(\bar{R}) = \overline{B_*(R)}; \quad B^*(\bar{R}) = \overline{B^*(R)} \quad (12)$$

2.1.2. Two kinds of interpretations

In this paper we consider two different interpretations of the preceding equations (which are extended to the fuzzy case in the next subsection), namely, the *conjunctive* interpretation and the *disjunctive* interpretation. In the conjunctive interpretation, a mathematical set is viewed as the value of a *multiple-valued* variable. For instance, the proposition "the tongues spoken by John are English, French, and Italian" expresses that John can speak *each* of the languages in the set {English, French, Italian}; the multiple-valued variable under consideration is here "tongue(s)-spoken-by-John." By contrast, in the disjunctive interpretation, a mathematical set is viewed as restricting the possible values of a *single-valued* variable. In other words, the singletons whose union constitutes the set under consideration are *mutually exclusive*

values for the variable. For example, the proposition "John is between 30 and 40 years old" expresses an imprecise knowledge about John's age and clearly only one value in the interval [30, 40] corresponds to the genuine value of his age. The reader is referred to Zadeh [33], Yager [29], or Dubois and Prade [12] for discussions about the conjunctive and the disjunctive interpretations.

A typical conjunctive interpretation of A, Γ, B_* and B^* is the following. A is a set of simultaneously observed symptoms in a particular case; $\Gamma(u)$ is the set of diagnoses associated with the symptom u ; then the lower image B_* is the set that contains the diagnose(s) that are compatible with the whole set of symptoms A , whereas the upper image B^* is the set of diagnoses that are compatible with at least one of the symptoms in A . Note that the elements in B_* or B^* are not mutually exclusive since it is possible to have several different diseases in the same time.

A disjunctive interpretation of A, Γ, B_* and B^* occurs in the following situation. U is the domain of an observed variable $X, A \subseteq U$ corresponds to partial information relative to X , i.e., X is equal to one of the values in A . $\Gamma(u)$ is the set of mutually exclusive possible decisions 'Y = v' that can be contemplated when $X = u$. In other words, when $X = u$, any value $v \in \Gamma(u)$ can be assigned to the variable Y . Knowing only that the value of X is in A , B_* is the set of decisions that apply whatever the precise value taken by X is, provided this is in A . If $B_* = \emptyset$, there is no value that can be chosen with certainty for Y ; then any value in B^* may be appropriate for Y with respect to the precise value of X (which remains unknown) but may be unsuitable as well.

Note 2.1. In the two preceding interpretations both A and R are either understood in a conjunctive or in a disjunctive way. It would be possible to consider hybrid situations where the set A or the relation R has a conjunctive interpretation whereas the other is interpreted in a disjunctive way.

2.2. FUZZY CASE

2.2.1. Mathematical model

Let A and R be a fuzzy set and a fuzzy relation defined on U and on $U \times V$, respectively, and represented by their membership functions μ_A and μ_R . A and R are supposed to be normalized, i.e., $\exists u, \mu_A(u) = 1$ and $\exists (u, v), \mu_R(u, v) = 1$. R induces a fuzzy mapping Γ from U to V whose membership function at point u is defined by

$$\forall u \in U, \quad \mu_{\Gamma(u)}(v) = \mu_R(u, v) \quad (13)$$

$\Gamma(u)$ is a fuzzy set on V associated with u . Then the fuzzy reverse image $\Gamma^{-1}(v)$ is defined by $\mu_{\Gamma^{-1}(v)}(u) = \mu_{\Gamma(v)}(u)$.

The definition of the upper image is extended to the fuzzy case in the following way (Zadeh, [31]):

$$\forall v \in V, \quad \mu_{B^*}(v) = \sup_{u \in U} \mu_A(u) * \mu_R(u, v) \quad (14)$$

where $*$ is a binary operation that defines a generalized intersection. The operation $*$ must be such that (i) $\forall x \in [0, 1], x * 1 = 1 * x = x$; (ii) $0 * 0 = 0$; (iii) $\forall(x, y, z, t) \in [0, 1]^4$, if $x \leq y$ and $z \leq t$, then $x * z \leq y * t$ (monotonicity). If we add the symmetry and associativity requirements, which are natural for an intersection operation, the operation $*$ is bound to be chosen among the so-called triangular norms; see the appendix for details. The main triangular norms are the min operation, the product, and the operation $x * y = \max(0, x + y - 1)$. Note that (14) can be viewed as an extension of (1), (3), or (4) since the maximization in (14) expresses that we are looking for at least one u whose membership degree in A and degree of relationship R with v are as great as possible. Moreover, when A is an ordinary set, expression (14) reduces to $\mu_{B^*}(v) = \sup_{u \in A} \mu_{\Gamma(v)}(u)$, which looks like (4); when R is also an ordinary relation, we recover the equivalent definitions of B^* given in 2.1.1.

The lower image B_* is still defined in the fuzzy case as the largest set B such that $A \times B \subseteq R$, the fuzzy set inclusion being defined as usual, i.e.,

$$F \subseteq G \Leftrightarrow \forall t, \mu_F(t) \leq \mu_G(t) \quad (15)$$

and the Cartesian product being defined by means of an intersection operation $*$ fulfilling the same requirements as in (14)), i.e.,

$$\forall u \in U, \quad \forall v \in V, \quad \mu_{A \times B}(u, v) = \mu_A(u) * \mu_B(v). \quad (16)$$

Then the membership function of the lower image B_* is given by

$$\begin{aligned} \forall v \in V, \quad \mu_{B_*}(v) &= \inf_{u \in U} \sup_{\beta \in [0, 1]} \{ \beta \mid \mu_A(u) * \beta \leq \mu_R(u, v) \} \\ &= \inf_{u \in U} \mu_A(u) * \mu_R(u, v) \end{aligned} \quad (17) \quad (18)$$

with

$$\forall(x, y) \in [0, 1]^2, \quad x * \rightarrow y = \sup_{s \in [0, 1]} \{s \mid x * s \leq y\}. \quad (19)$$

The subexpression in (17) beginning with "sup" yields the greatest possible value $\beta(u)$ of $\mu_{B_*}(v)$ such that $\mu_A(u) * \mu_B(v) \leq \mu_R(u, v)$ for a given u . Since $*$ is non-decreasing, $\mu_{B_*}(v)$ is finally obtained by taking the greatest lower bound of $\beta(u)$ for u ranging in U . Note that the operation $* \rightarrow$ defined by (19) is an implication connective function in multiple-valued logic when $*$ defines a conjunction (see Goguen [17]), but also, [26], [6], [7], [4]). See the appendix for the different examples of operations $* \rightarrow$ associated with a particular operation $*$. Thus (18) can be viewed as expressing the degree of inclusion of A into $\Gamma^{-1}(v)$ [i.e., to what extent any element u of A belongs to $\Gamma^{-1}(v)$] (see [31] and appears clearly to be an extension of (2), (5), or (6) (since when A is an ordinary set, (18) reduces to $\mu_{B_*}(v) = \inf_{u \in A} \mu_R(u, v)$ owing to $1 * \rightarrow y = y$ and $0 * \rightarrow y = 1$). When A and R are ordinary set and relation, we recover the equivalent definitions of B_* given in 2.1.1. (Inf, $* \rightarrow$) compositions have been studied by Sanchez [25], [26] for $*$ = min and Miyakoshi and Shimbo [21] in the general case.

Using (12), another definition for B_* (resp. B^*) can be derived in the fuzzy case from the one of B^* (resp. B_*), given a fuzzy complementation operation, when A is a fuzzy set. However when A is an ordinary set and R is a fuzzy relation, (12) still holds with definitions (14) and (18). We shall use the usual complementation operation defined by

$$\forall t, \quad \mu_{\bar{F}}(t) = 1 - \mu_F(t). \quad (20)$$

We obtain from (12), (20), and (14)

$$\forall v \in V, \quad \mu_{B_*}(v) = \inf_{u \in U} (1 - \mu_A(u)) \perp \mu_R(u, v) \quad (21)$$

where \perp is an operation, expressing a disjunction, associated with the operation $*$ according to the relation

$$x \perp y = 1 - (1 - x) * (1 - y); \quad (22)$$

\perp is called a *triangular co-norm* when $*$ is a triangular norm; see the

appendix for details. From (12), (20), and (18), we get

$$\forall v \in V, \mu_{B^*}(v) = \sup_{u \in U} \mu_A(u) \hat{*} \mu_R(u, v) \quad (23)$$

where

$$x \hat{*} y = \inf_{s \in \{0, 1\}} \{s \mid (1-x) \perp s \geq y\}. \quad (24)$$

Using a result proved in Dubois and Prade [7], the expressions (21) and (23) turn out to be very similar to (18) and (14), respectively. Indeed the implication connective function defined by $(1-x) \perp y$, appearing in (21), derives from the operation $\hat{*}$ via (19) (where $*$ is replaced by $\hat{*}$); see [7] or the appendix for a proof when $*$ is a continuous triangular norm. It can be checked that $\hat{*}$ defined by (24) is such that (i) $0 \hat{*} 1 = 0$; (ii) $\forall x \in [0, 1], 1 \hat{*} x = x$; (iii) $0 \hat{*} 0 = 0$; (iv) $\hat{*}$ is monotonously nondecreasing with respect to its two arguments. Thus $\hat{*}$ satisfies the three basic properties required for $*$, except $x \hat{*} 1 = x$, which may hold only for $x = 0$ (however, we always have $x \hat{*} 1 \geq x$). The operation $\hat{*}$ is an intersection operation which is nonsymmetrical in general; however in some cases $*$ and $\hat{*}$ are identical. The possible nonsymmetry of $\hat{*}$ does not cause any serious problem since the roles played by μ_A and μ_R are not interchangeable. See the appendix for a table giving the operations $*$, \perp , and $\hat{*}$ corresponding to the main triangular norms $*$. So, (14) and (23), as well as (18) and (21), present no essential differences. $\hat{*}$ is called the *adjoint* of $*$.

Since $*$ and $\hat{*}$ are nondecreasing with respect to their left operands, whereas $*$ and $\hat{*}$ (with $x \hat{*} \rightarrow y = (1-x) \perp y = 1 - x*(1-y)$) are nonincreasing with respect to their left operands, the monotonicity properties (7) and (8) still hold in the fuzzy case [the fuzzy set inclusion being defined by (15)], for any definition of the upper image and of the lower image, respectively.

Owing to the identities $\forall x \in [0, 1], 1 * x = 1 \hat{*} x = 1 \rightarrow x = 1 \hat{*} \rightarrow x = x$, which always hold, as soon as A is normalized (i.e., $\exists u_0, \mu_A(u_0) = 1$), we have

$$\forall v \in V, \mu_{B^*}(v) \leq \mu_R(u_0, v) \leq \mu_{B^*}(v)$$

and thus the inclusion (9) [extended in the sense of (15)] is satisfied in any case (even if B^* and B^* are not defined from the same operation $*$) when A and R are fuzzy. Note that when A is a singleton $\{u_0\}$, we have $B^* = B^* = \Gamma(u_0)$ as in the nonfuzzy case.

Owing to the distributivity properties,

$$\max(x, x') * y = \max(x * y, x' * y);$$

$$\max(x, x') \hat{*} y = \max(x \hat{*} y, x' \hat{*} y)$$

$$\min(x, x') * y = \min(x * y, x' * y);$$

$$\min(x, x') \hat{*} y = \min(x \hat{*} y, x' \hat{*} y);$$

$$\max(x, x') * \rightarrow y = \min(x * \rightarrow y, x' * \rightarrow y);$$

$$\max(x, x') \hat{*} \rightarrow y = \min(x \hat{*} \rightarrow y, x' \hat{*} \rightarrow y)$$

$$\min(x, x') * \rightarrow y = \max(x * \rightarrow y, x' * \rightarrow y);$$

$$\min(x, x') \hat{*} \rightarrow y = \max(x \hat{*} \rightarrow y, x' \hat{*} \rightarrow y);$$

the equalities (10a) and (11a) and the inclusions (10b) and (11b) still hold in the fuzzy case (the fuzzy set inclusion being defined by (15), the fuzzy set union and intersection being pointwisely defined by means of max and min operations, respectively), for any definition of the upper image and of the lower image. These properties have been noticed in a slightly different setting by Izumi et al. [18], [19].

2.2.2. Interpretations

In the conjunctive interpretation, $\Gamma(u)$ is the fuzzy set of elements of V which are more or less associated with u ; $\mu_{\Gamma(u)}(v) = \mu_R(u, v)$ is the degree of association of v with u . When A remains nonfuzzy, $\mu_{B^*}(v) = \inf_{u \in A} \mu_R(u, v)$ estimates to what extent v is associated with *all* elements u in A , whereas $\mu_{B^*}(v) = \sup_{u \in A} \mu_R(u, v)$ estimates to what extent v is associated with *at least one* element u in A . In a diagnosis problem, $\mu_R(u, v)$ represents a degree of association between the symptom u and the disease v . Note that when A is nonfuzzy, all the definitions proposed for the lower (or the upper) image reduce to a unique expression (since $\forall *, 1 * x = 1 \hat{*} x = x = 1 * \rightarrow x = 1 \hat{*} \rightarrow x$ and $0 * x = 0 \hat{*} x = 0$ and $0 * \rightarrow x = 0 \hat{*} \rightarrow x = 1$). When A is a fuzzy, $\mu_A(u)$ expresses, for instance, to what extent the symptom u can be considered as present (i.e., $\mu_A(u)$ can be viewed as the degree of truth of the statement "the symptom u is present." Then B^* and B^* represent, respectively, the fuzzy

set of diseases that are associated with all the present symptoms and the fuzzy set of diseases that are associated with at least one of the present symptoms. The operations * and the related ones correspond to slightly different ways of aggregating the degrees of presence and of association when their values are intermediary between 0 and 1.

In the disjunctive interpretation, R restricts the possible values of a variable Y when the value of another variable X is known; then $\Gamma(u)$ is a set of (mutually exclusive) values that are more or less possible for the single-valued variable Y when the other variable X is precisely equal to u ; conversely knowing that $Y = v$, the only more or less possible corresponding values for X are restricted by $\mu_{\Gamma^{-1}(v)}$. The more or less possible values of the single-valued variable X are a priori restricted by A . More briefly we shall say that " X is (in) A ". In other words, $\mu_A(u)$ is the degree of possibility that $X = u$ (the normalization of A guarantees that there exists at least one value that is completely possible for X), whereas $\mu_R(u, v)$ is the degree of possibility that $Y = v$ when $X = u$. Then $\mu_{B^*}(v)$ estimates the possibility (see Zadeh [32]) that " v belongs to the image of the value of X via Γ " when it is known that the value of X is in A . Indeed, (14) expresses a degree of intersection of A with $\Gamma^{-1}(v)$. For $* = \min$, (14) can be viewed as the usual definition of the possibility measure of the fuzzy event " X is in $\Gamma^{-1}(v)$," based on the possibility distribution μ_A representing the known fact " X is in A ." The possibility measure will be denoted by $\Pi(\Gamma^{-1}(v); A)$. The quantity $\mu_{B^*}(v)$ estimates the certainty (or the necessity) that " v belongs to the image of the value of X via Γ ," knowing that " X is in A ," since $\mu_{B^*}(v)$ can be considered as a degree of inclusion of A in $\Gamma^{-1}(v)$. In other words, $\mu_{B^*}(v)$ is the degree of certainty that X restricted by A is in relation R with v . If we use (21) with $\perp = \max$ for defining $\mu_{B^*}(v)$, we obtain the usual definition of the necessity of the fuzzy event " X is in $\Gamma^{-1}(v)$," on the basis of the possibility distribution μ_A . This necessity measure will be denoted by $N(\Gamma^{-1}(v); A)$ and owing to (12) is such that

$$N(\Gamma^{-1}(v); A) = 1 - \Pi(\overline{\Gamma^{-1}(v)}; A) \tag{25}$$

(25) is in agreement with the idea of necessity of an event as the impossibility of the opposite event.

To sum up, in the conjunctive interpretation, B^* is the fuzzy set of the elements that are more or less in relation R with at least one of the elements of the fuzzy set A ; B_* is the fuzzy set of the elements that are more or less in relation R with all the elements of the fuzzy set A . In the disjunctive interpretation B^* restricts the more or less possible values of a single-valued variable Y whose value is more or less in relation R with the value of a

single-valued variable X , whose more or less possible values are themselves restricted by A ; then B_* restricts the possible values of Y for which we are more or less certain that they are more or less in relation R with the value of X , whatever it is in A .

2.2.3. Remark: upper and lower images and inverse images

The upper and lower images of a crisp or fuzzy set by a crisp or fuzzy relation, which are defined in section 2.1.1 and 2.2.1, must not be confused with the notions of upper and lower inverse images that have been considered in the fuzzy set literature.

Using the same notations as in the previous section, the upper inverse image A^{-1*} of a subset B of V via a relation R (or its equivalent generalized mapping Γ) is defined

$$A^{-1*} = \{u \in U, B \cap \Gamma(u) \neq \emptyset\} \tag{26}$$

— in the fuzzy case by

$$\forall u \in U, \mu_{A^{-1*}}(u) = \sup_{v \in V} \mu_R(u, v) * \mu_B(v), \tag{27}$$

whereas the lower inverse image A_*^{-1} is defined

$$A_*^{-1} = \{u \in U, \Gamma(u) \neq \emptyset, \Gamma(u) \subseteq B\} \tag{28}$$

— in the crisp case by

$$\forall u \in U, \mu_{A_*^{-1}}(u) = \inf_{v \in V} \mu_R(u, v) * \mu_B(v). \tag{29}$$

The upper inverse image A^{-1*} is the (crisp or fuzzy) set of elements of U , which are in relation R with elements in B ; the lower inverse image A_*^{-1} is the (crisp or fuzzy) set of elements of U , which are in relation R only with elements in B . Note the difference between (5) and (28) (or (18) and (29)); Γ^{-1} is changed into Γ , but the inclusion is reversed also. When we interpret R in a disjunctive way, $\Gamma(u)$ acts as a restriction on the possible values of the image of u (which is an element of V); then A^{-1*} (resp. A_*^{-1}) is the (crisp

or fuzzy) set of elements of U , which are *possibly* (resp. *necessarily* or certainly) associated with elements in B . Upper and lower inverse images are naturally encountered when dealing with incomplete or fuzzy information data bases; then, U is a set of objects, $\Gamma(u)$ represents the available (possibly incomplete or fuzzy) knowledge concerning the value of a single-valued attribute applicable to the object u ; A^{-1*} (resp. A^*) is the set of objects whose attribute value is possibly (resp. necessarily) in B ; see Prade, Testemale [23]. More generally, upper and lower inverse images appear in the study of twofold fuzzy sets [13] and fuzzy rough sets [16].

Many interesting results can be obtained by combining upper or lower "direct" and inverse images together. For instance, it is easy to check that $(R \textcircled{\ominus} B) \textcircled{\ominus} R \supseteq B$, i.e., the image of the converse image of B contains B .

2.3. WEIGHTED UPPER AND LOWER IMAGES

As shown by their definitions (1) and (2) in the crisp case, the notions of upper and lower images are closely related to the ideas of existential and universal quantifications, respectively. Moreover, all the elements in U have a priori the same "importance" in these definitions. In this subsection we propose softened versions of the notions of upper and lower images. For instance we define the fuzzy set of elements of V that are in relation R with all the *important* elements in A or the fuzzy set of elements of V that are in relation R with at least *some* elements in A . First we present a general framework where we deal with the introduction of degrees of importance in fuzzy set union and intersection (to accommodate the definitions (4) and (6)).

2.3.1. Importance weighting

In Dubois and Prade [9], [10] the following weighting of the operations max and min, used for defining fuzzy set union and intersection, is proposed. This weighting is here extended to sup and inf operations. Let $F(u)$ be a family of fuzzy sets defined on V and indexed by $u \in U$ and let $w(u)$, $\forall u \in U$ be the a priori grade of importance of the element u in U , where it is assumed that $w(u) \geq 0$, $\forall u \in U$ and that $\sup_{u \in U} w(u) = 1$. Then the weighted union and intersection of the $F(u)$'s, denoted by $\bigcup_{w(u)} F(u)$ and by $\bigcap_{w(u)} F(u)$, respectively, are defined by

$$\forall v \in V, \mu_{\bigcup_{w(u)} F(u)}^{(v)} = \sup_{u \in U} \min(w(u), \mu_{F(u)}(v)) \tag{30}$$

$$\forall v \in V, \mu_{\bigcap_{w(u)} F(u)}^{(v)} = \inf_{u \in U} \max(1 - w(u), \mu_{F(u)}(v)) \tag{31}$$

Note that when U is finite and $\forall u \in U, w(u) = 1$, the expressions (30) and (31) reduce to the usual union and intersection operations, i.e., to $\max_u \mu_{F(u)}(v)$ and to $\min_u \mu_{F(u)}(v)$, respectively. When $w(u) = 0$, the fuzzy set $F(u)$ is not at all taken into account in the weighted union and intersection defined by (30) and (31). The formula (30) expresses that the extent to which v belongs to the union of important $F(u)$'s [i.e., belongs to at least one important $F(u)$], and corresponds to a degree of intersection of the fuzzy set of $F(u)$'s to which v belongs with the fuzzy set of important $F(u)$'s [defined by the $w(u)$'s]. Similarly, formula (31) expresses that the extent to which v belongs to the intersection of important $F(u)$'s [i.e., belongs to all the important $F(u)$'s], and corresponds to a degree of inclusion of the fuzzy set of important $F(u)$'s in the fuzzy set of $F(u)$'s to which v belongs. See [10] for a complete justification and discussion of this weighting procedure, which is already used for modeling degrees of importance in compound patterns for matching techniques; see [15].

Then we can define the fuzzy set of elements of V that are in relation R with at least one important element in A , when A is an ordinary set by

$$\forall v \in V, \mu_{B_w^*}(v) = \sup_{u \in A} \min(w(u), \mu_R(u, v)) \tag{32}$$

and when A is fuzzy by

$$\forall v \in V, \mu_{B_w^*}(v) = \sup_{u \in U} \min(w(u), \mu_A(u), \mu_R(u, v)) \tag{33}$$

choosing $*$ = min in (14). The fuzzy set of elements of V that are in relation R with all the important elements in A is defined, when A is an ordinary set, by

$$\forall v \in V, \mu_{B_w^*}(v) = \inf_{u \in A} \max(1 - w(u), \mu_R(u, v)) \tag{34}$$

and when A is a fuzzy by

$$\forall v \in V, \mu_{B_w^*}(v) = \inf_{u \in U} \max(1 - w(u), 1 - \mu_A(u), \mu_R(u, v)) \tag{35}$$

choosing \perp = max in (21). Note that we have the fuzzy set inclusions

$$B_* \subseteq B_{w^*} \quad \text{and} \quad B_w^* \subseteq B^* \tag{36}$$

where B_* and B^* are, respectively, defined by (21) (with \perp = max) and (14)

(with $*$ = min). Moreover, we have

$$B_{w*} \subseteq B_w^* \quad (37)$$

provided that $\exists u \in U$, $\mu_A(u) = 1 = w(u)$, i.e., the fuzzy set defined by w is normalized on the subset of elements of U that fully belong to A . Note that letting W be the fuzzy set of important elements in U ($\mu_W(u) = w(u)$) $B_w^* = (A \cap W) \textcircled{\ominus} R$ and $B_{w*} = (A \cap W) \textcircled{\ominus} R$.

2.3.2. Numerical and fuzzy quantifiers

In the following we assume that U is finite; namely $U = \{u_1, \dots, u_m\}$. To define the fuzzy set $B(k)$ of elements of V , which are in relation R with at least k elements in A , we have to calculate the extent to which there exists a subset of U with k elements such that each of them are in A (with a degree of membership as high as possible) and are in relation R^{-1} with an element v of V (also with a degree of relationship $\mu_{R^{-1}}(v, u) = \mu_R(u, v)$ as high as possible). Let σ denote any permutation of the elements in U . Then we have

$\forall v \in V$,

$$\mu_{B(k)}(v) = \max_{\sigma} \min_{1 \leq i \leq m} \max \left(1 - \mu_{\{1, k\}}(i), \min(\mu_A(u_{\sigma(i)}), \mu_R(u_{\sigma(i)}, v)) \right) \quad (38)$$

using an expression similar to (21) with $1 = \max$ and U finite. Indeed, $\mu_{\{1, k\}}$, the characteristic function of the set of integers $\{1, \dots, k\}$, plays the role of the membership function of a subset of U with k elements; the maximization on the set of permutations expresses that we are looking for the best subset of U whose elements are both in A and in relation R^{-1} with v . As shown in [15] for a similar expression, the maximum in (38) is reached for the permutation $\bar{\sigma}$ such that $\min(\mu_A(u_{\bar{\sigma}(i)}), \mu_R(u_{\bar{\sigma}(i)}, v)) \geq \dots \geq \min(\mu_A(u_{\bar{\sigma}(m)}), \mu_R(u_{\bar{\sigma}(m)}, v))$. This is true more generally, changing $[1, k]$ into a fuzzy integer I with a nonincreasing membership function (i.e., $\mu_I(1) = 1$ and $\mu_I(i) \geq \mu_I(i+1)$). The fuzzy set I can be viewed as an importance weighting that can be allocated to the elements of U in any order. Then we have

$$\forall v \in V, \quad \mu_{B(I)}(v) = \min_{1 \leq i \leq m} \max \left(1 - \mu_I(i), \min(\mu_A(u_{\bar{\sigma}(i)}), \mu_R(u_{\bar{\sigma}(i)}, v)) \right) \quad (39)$$

Using another result shown in [10], [15], we have the following equivalent representation of (39):

$$\forall v \in V, \quad \mu_{B(I)}(v) = \max_{1 \leq i \leq m} \min(\mu_Q(i), \mu_A(u_{\bar{\sigma}(i)}), \mu_R(u_{\bar{\sigma}(i)}, v)) \quad (40)$$

with $\mu_Q(i) = 1 - \mu_I(i+1)$, $i = 1, m-1$; $\mu_Q(m) = 1$. Q is an (absolute) fuzzy quantifier [35] with a nondecreasing membership function that models a fuzzy count of elements. $B(I)$ is the fuzzy set of elements of V that are in relation R with a number, whose possible values are fuzzily restricted by Q , of elements in A . The expression (40) when $A = U$ has been independently suggested by Yager [30]. Thus for instance v can be considered as in relation R with at least k elements in A to the degree $\min(\mu_A(u_{\bar{\sigma}(k)}), \mu_R(u_{\bar{\sigma}(k)}, v))$, applying (40) with $I = [1, k]$ and then $Q = [k, m]$. Obviously, this degree is equal to 0 when there are less than k elements in A with a positive degree of membership. For $k = 1$, i.e., $I = \{1\}$, (40) reduces

$$\mu_{B(I)}(v) = \max_{u \in U} \min(\mu_A(u), \mu_R(u, v))$$

where we recognize (14) with $*$ = min and U finite, which is satisfying. When A is an ordinary set whose cardinality is k and $I = \{1, \dots, k\}$, (40) reduces to

$$\mu_{B(I)}(v) = \min_{u \in A} \mu_R(u, v),$$

which is nothing but (18) or (21) when A is non fuzzy; i.e., we recover the fuzzy set of elements in V that are in relation R with all the elements in A , which is natural.

3. APPLICATIONS TO APPROXIMATE REASONING

In the following two applications the notions of upper and lower images are considered. First a dual view of the generalized modus ponens ([34], [6]) is presented in the framework of a disjunctive interpretation. Then an application to diagnostic systems is proposed, using a conjunctive interpretation.

3.1. A NEW VIEW OF THE GENERALIZED MODUS PONENS

Let us consider the rule "if X is A then Y is B ", where X and Y are single-valued and take their value on the domains U and V , respectively; A

and B are supposed to be normalized fuzzy sets defined on U and V , respectively. This rule may be interpreted in two different ways according to whether B is viewed as the upper or lower image of A induced by a fuzzy relation R . Thinking of B as an upper image leads us to look for the largest relation R in the sense of (15) such that

$$A \textcircled{3} R \subseteq B,$$

i.e.,

$$\forall v \in V, \sup_{u \in U} \mu_A(u) \textcircled{\circ} \mu_R(u, v) \leq \mu_B(v) \quad (41)$$

where $\textcircled{\circ}$ stands for a continuous triangular * norm or an operation $\hat{*}$ derived from a continuous triangular co-norm in the sense of (24). Viewing B as a lower image leads us to look for the smallest R [in the sense of (15)] such that

$$A \textcircled{4} R \supseteq B,$$

i.e.,

$$\forall v \in V, \inf_{u \in U} \mu_A(u) \textcircled{\circ} \mu_R(u, v) \geq \mu_B(v) \quad (42)$$

where $\textcircled{\circ} \rightarrow$ is defined, using (19), from the operation $\textcircled{\circ}$ considered in (41). Inequalities, rather than equalities, appear in (41) and in (42) to guarantee the existence of solutions, and since the fuzzy set B specified in the rule may not be the smallest (resp. largest) set of possible values for Y that are possibly (resp. necessarily) in relation R with the value of X .

The greatest solution of (41) is given by

$$\forall u \in U, \forall v \in V, \mu_R(u, v) = \mu_A(u) \textcircled{\circ} \mu_B(v) \triangleq \mu_{A \textcircled{\circ} \rightarrow B}(u, v) \quad (43)$$

where $\textcircled{\circ} \rightarrow$ is defined by (19), whether $\textcircled{\circ}$ is a triangular norm * or its adjoint $\hat{*}$.

Proof. When $\textcircled{\circ}$ is a triangular norm, this result is well known from the literature in fuzzy relations (e.g., [5], [16]). When $\textcircled{\circ} = \hat{*}$ the result is proved by the same technique as when $\textcircled{\circ}$ is a triangular norm, namely (Sanchez [26]):

- if R is a solution of (41), $R \subseteq A \textcircled{\circ} \rightarrow B$ because generally $R \subseteq A \textcircled{\circ} \rightarrow (A \textcircled{3} R) \subseteq A \textcircled{\circ} \rightarrow B$ since R is a solution. The first inclusion is obtained using the inequality $a \hat{*} \rightarrow (a \hat{*} r) \geq r$ proved in the appendix, where $a = \mu_A(u)$, $r = \mu_R(u, v)$.

- $A \textcircled{\circ} \rightarrow B$ is a solution of (41) due to the inequality $a \hat{*} (a \hat{*} \rightarrow b) \leq b$ proved in the appendix. Q.E.D.

When A is an ordinary subset of U , (43) reduces to

$$\forall v \in V, \mu_R(u, v) = \begin{cases} \mu_B(v) & \text{if } u \in A \\ 1 & \text{if } u \notin A \end{cases} \quad (44)$$

or if we prefer using (13), $\Gamma(u) = B$ if $u \in A$ and $\Gamma(u) = V$ otherwise.

The smallest solution of (42) is given by

$$\forall u \in U, \forall v \in V, \mu_R(u, v) = \mu_A(u) \textcircled{\circ} \mu_B(v) \quad (45)$$

whether $\textcircled{\circ}$ is a triangular norm * or its adjoint $\hat{*}$.

Proof. Equation (42) can be written $A \textcircled{4} R \subseteq \bar{B}$. But $A \textcircled{4} R = A \textcircled{3} \bar{R}$, where $\textcircled{3}$ is a sup- $\textcircled{\circ}$ composition, when $\textcircled{4}$ is an inf- $\textcircled{\circ} \rightarrow$ composition, $\textcircled{\circ}$ being the adjoint of $\textcircled{\circ}$. From the previous proof we know that the greatest solution of equation $A \textcircled{3} \bar{R} \subseteq \bar{B}$ is $\mu_{\bar{R}}(u, v) = \mu_A(u) \textcircled{\circ} \rightarrow (1 - \mu_B(v))$. This corresponds to taking the opposite of the representation of the rule "if X is A then Y is not B " to model the rule "if X is A then Y is B ." Hence the least solution of (42) is $\mu_R(u, v) = 1 - (\mu_A(u) \textcircled{\circ} \rightarrow (1 - \mu_B(v))) = \mu_A(u) \textcircled{\circ} \mu_B(v)$ where $\textcircled{\circ}$ is the adjoint of $\textcircled{\circ}$. But because the adjoint property is involutive (see appendix), $\textcircled{\circ} = \textcircled{\circ}$. Q.E.D.

When A is an ordinary subset, (45) reduces to

$$\forall v \in V, \mu_R(u, v) = \begin{cases} \mu_B(v) & \text{if } u \in A \\ 0 & \text{if } u \notin A \end{cases} \quad (46)$$

or in terms of Γ , $\Gamma(u) = B$ if $u \in A$ and $\Gamma(u) = \emptyset$ otherwise. The difference between the two interpretations of the rule, in terms of implication or in terms of conjunction, is particularly clear when A is nonfuzzy, i.e., considering (44) vs. (46). A generalized deductive inference can be performed in both cases.

The so-called generalized modus ponens, introduced by Zadeh [34], corresponds to the following pattern of inference:

$$\frac{\text{if } X \text{ is } A, \text{ then } Y \text{ is } B}{X \text{ is } A'}{Y \text{ is } B'}$$

where B' is to be computed from A' , A , and B . Using the upper image interpretation we obtain (see Dubois and Prade [6] and Trillas and Valverde [11]).

$$\forall v \in V, \mu_{B'_1}(v) = \sup_{u \in U} \mu_{A'}(u) * (\mu_A(u) * \rightarrow \mu_B(v)) \tag{47}$$

using (14) and (43). Clearly, we can also use $\hat{*}$ and $\hat{*} \rightarrow$ instead of $*$ and $* \rightarrow$, respectively. For the sake of brevity we shall write (47) as

$$B'_1 = A' \textcircled{\ominus} * (A * \rightarrow B).$$

With the lower image interpretation, we obtain, using (18) and (45)

$$\forall v \in V, \mu_{B'_2} = \inf_{u \in U} \mu_{A'}(u) * \rightarrow (\mu_A(u) * \mu_B(v)) \tag{48}$$

which will be written symbolically as

$$B'_2 = A' \textcircled{\forall} * (A * B)$$

We can also use $\hat{*} \rightarrow$ and $\hat{*}$ instead of $* \rightarrow$ and $*$ in (48). It can be easily checked that B'_1 and B'_2 bracket B . Namely, we have

$$\forall A', \forall A, A' \textcircled{\forall} * (A * B) \subseteq B \subseteq A' \textcircled{\ominus} * (A * \rightarrow B) \tag{49}$$

Proof. Since $1 * b = 1 \hat{*} b = b, 1 * \rightarrow b = 1 \hat{*} \rightarrow b = b$, we have

$$\mu_{B'_2}(v) \leq \inf_{u \in U} \mu_{A'}(u) * \rightarrow (1 * \mu_B(v)) = \mu_B(v)$$

provided that A' is normalized. The other inclusion in (49) is also easy to prove when A' is normalized and is well known [6].

Inference model (48) was first proposed by DiNola et al. [4] when $*$ = minimum. Noticeable properties of (47) are the following [(6), [8)]:

$$A' \subseteq A \Rightarrow B'_1 = B \tag{50}$$

$$A' = \bar{A} \Rightarrow B'_1 = V \tag{51}$$

$$\forall v \in V, \mu_{B'_1}(v) \geq \sup_{u \in U} \mu_{A'}(u) * (\mu_A(u) * \rightarrow 0) \tag{52}$$

(50) is easy to see since $A' \subseteq A \Rightarrow A' \textcircled{\ominus} * (A * \rightarrow B) \subseteq A \textcircled{\ominus} * (A * \rightarrow B)$ and $a * (a * \rightarrow b) \leq b$ if $*$ is continuous as well as $a \hat{*} [(1 - a) \perp b] \leq b$. Then $A' \textcircled{\ominus} * (A * \rightarrow B) \subseteq B$, which yields the result, taking into account (49). (51) holds provided that $\exists u_0, \mu_A(u_0) = 0$; indeed

$$\begin{aligned} & \sup_{u \in U} (1 - \mu_A(u)) * (\mu_A(u) * \rightarrow \mu_B(v)) \\ & \geq (1 - \mu_A(u_0)) * (\mu_A(u_0) * \rightarrow \mu_B(v)) = 1 \end{aligned}$$

owing to $0 * \rightarrow b = 0 \hat{*} \rightarrow b = 1$. (52) is obvious. Note that $a * \rightarrow 0$ defines a negation operation in the multiple-valued logic based on $*$. Thus (52) indicates that any value in V becomes somewhat possible for Y as soon as there exists a possible value for X (in A) that does not belong at all to A or whose degree of membership to A is small. For instance for $*$ = min, (52) gives $\mu_{B'_1}(v) \geq \Pi(s(A); A)$ where $s(A)$ denotes the support of A defined by $s(A) = \{u \in U, \mu_A(u) > 0\}$ and where Π is a possibility measure (see 2.2.2 for the notation). See [6] and [8] for details.

Analogous properties can be proved for (48). Namely

$$A' \subseteq A \Rightarrow B'_2 = B \tag{53}$$

$$A' = \bar{A} \Rightarrow B'_2 = 0 \tag{54}$$

$$\forall v \in V, \mu_{B'_2}(v) \leq \inf_{u \in U} \mu_{A'}(u) * \rightarrow \mu_A(u) \tag{55}$$

To prove (53), note that $A' \subseteq A \Rightarrow A' \textcircled{\forall} * (A * B) \supseteq A \textcircled{\forall} * (A * B)$ and $a * \rightarrow (a * b) \geq b$ as well as $(1 - a) \perp (a \hat{*} b) \geq b$ if $*$ is continuous. Then $A' \textcircled{\forall} * (A * B) \supseteq B$, which yields the result owing to (49). (54) is easy to prove as soon as $\exists u_0, \mu_A(u_0) = 0$, since

$$\begin{aligned} & \inf_{u \in U} (1 - \mu_A(u) * \rightarrow (\mu_A(u) * \mu_B(v))) \\ & \leq (1 - \mu_A(u_0) * \rightarrow (\mu_A(u_0) * \mu_B(v))) = 0 \end{aligned}$$

owing to $1 * \rightarrow 0 = 1 \hat{*} \rightarrow 0 = 0$. (55) is a straightforward consequence of (48). The upper bound of $\mu_{B'_2}(v)$ is a degree of inclusion of A' into A .

What is remarkable is that the lower bound of $\mu_{B_1}(v)$ in (52) and the upper bound of $\mu_{B_2}(v)$ in (55) convey information of the same nature. Indeed, the complement to 1 of the lower bound of μ_{B_1} in (52) is also a degree of inclusion of A' into A , since $1 - [\alpha' * (a * \rightarrow 0)]$ can be viewed as a multiple-valued logic implication ($a * \rightarrow 0$ being a negation). According to the choice made $*$ or $\hat{*}$, these lower and upper bounds may be more or less drastic. For instance, for $* = \min$, the lower bound of B_1' is $\sup_{u, \mu_A(u)=0} \mu_{A'}(u) = 1 - N(s(A); A)$, whereas the upper bound of B_2' is given by

$$\begin{cases} 1 & \text{if } A' \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

(provided that $\inf_{u \in U} \mu_A(u) = 0$). Contrastedly, with $a * b = \max(0, a + b - 1) = a \hat{*} b$, then $a * \rightarrow b = \min(1, 1 - a + b) = a \hat{*} \rightarrow b$ is Lukasiewicz' implication and the lower bound of B_1' is $\sup_{u \in U} \max(0, \mu_{A'}(u) - \mu_A(u))$, whereas the upper bound of B_2' is equal to $1 - \sup_{u \in U} \max(0, \mu_{A'}(u) - \mu_A(u))$; in this case the information conveyed by the lower and the upper bound is identical.

In conclusion, $B_1' = A' \textcircled{\ominus} * (A * \rightarrow B)$, is the set of values in V that are in relation R with at least one value in A' , where R represents the rule 'if X is A , then Y is B' '. B_1' is larger or equal to B on the support of B ; $\mu_{B_1'}(v)$ is constant and equal to the lower bound in (52) outside this support. This corresponds to the possibility that Y remains indeterminate if there exists a possibility of finding a value in U that belongs more to A' than to A .

$B_2' = A' \textcircled{\cup} * (A * B)$ restricts the values in V that are certainly in relation (in the sense of the rule "if X is A then Y is B' ") with the value X , i.e., with all values in U more or less compatible with A' . When A' is not perfectly included in A , the possibility of finding a set of values in V certainly in relation with the value of X decreases and then B_2' is no longer normalized and may become empty. In case of $* = \min$, $B_2' = \emptyset$ as soon as $A' \not\subseteq A$. For other operations $*$ the decreasing of B_2' from B to \emptyset is gradual.

Obviously, the upper image approach to the generalized modus ponens, which leads to (47), is easy to understand and to use. The rule "if X is A then Y is B' " defines a set of possible values (which are not a priori forbidden) for Y given a possible value of X . Then, knowing that the possible values of X are in A' , the corresponding fuzzy set of possible values for Y can be calculated. In the lower image approach, which leads to (48), the interpretation of the rule is slightly different. B is then a set of values associated with all the values in A , and there is no value associated with any value outside A . Knowing that the value of X is restricted by A' , the fuzzy

set of possible values for Y that are associated with all the possible values for X is then obtained. The existence of two dual extensions leading to a generalized modus ponens is worth noticing, as well as the two interpretations of the if-then rule. Besides, the representation of a rule "if X is A then Y is B' " in terms of a Cartesian product $A * B$ defined by (45) has been extensively used, in place of the implication-based representation (43), in fuzzy logic controllers [20] to compute B' as $A' \textcircled{\ominus} \min (A * B)$. This approach remains heuristic, since $A * B$ which is included in $A * \rightarrow B$ (owing to $V(a, b) \in [0, 1]^2$, $a * b \leq a * \rightarrow b$) is not the largest solution of (41).

Remark. In case of several rules "if X is A^i then Y is B^i ", $i = 1, m, B_1'$ and B_2' would be computed respectively as

$$B_1' = A' \textcircled{\ominus} * \left(\bigcap_{i=1, m} A^i * \rightarrow B^i \right)$$

and

$$B_2' = A' \textcircled{\cup} * \left(\bigcup_{i=1, m} A^i * B^i \right)$$

where the intersection \cap and the union \cup are respectively defined by means of min and max operations, since $A^i * \rightarrow B^i$ and $A^i * B^i$ are the largest and the smallest solutions of (41) and (42) respectively.

2.2. ABDUCTIVE REASONING USING (WEIGHTED) UPPER AND LOWER IMAGES

A diagnosis problem can be formalized in the following way. See [26] for instance. Let $V = \{v_1, \dots, v_n\}$ be a finite set of disorders and $U = \{u_1, \dots, u_m\}$ be a finite set of manifestations. Let R be a relation defined on $U \times V$ that captures the intuitive notion of causal association between disorders and manifestations. Let Γ be the multiple-valued mapping induced by R . $\Gamma^{-1}(v) = \{u \in U, (u, v) \in R\}$ is the set of manifestations associated with the disorder v . More generally, R will be a fuzzy relation and $\Gamma^{-1}(v)$ a fuzzy set. Then $\Gamma^{-1}(v)$ will be the fuzzy set of manifestations that more or less *certainly* occur when disorder v is present, and $\Gamma(u)$, defined by $\mu_{\Gamma(u)}(v) = \mu_R(u, v)$, will be the fuzzy set of disorders that are more or less *possible* when manifestation u occurs. Let A be a subset of U , corresponding to a set of observed manifestations. A may also be a fuzzy set. Then $\mu_A(u)$ can be interpreted as the certainty of the presence of manifestation u or as a relative.

estimate of the intensity of manifestation u if u is a matter of degree (e.g., fever). Diagnosis problems have been considered for a long time in the fuzzy set literature (e.g., Sanchez [26], Tsukamoto and Terano [28]). The approach that is usually proposed is the following one. The (fuzzy) set of manifestations A is produced by an unknown set of disorders B through R^{-1} , which associates the set of more or less certain manifestations with a disorder v (with $\mu_{R^{-1}(v, u)} = \mu_R(u, v)$). This can be written as

$$B \textcircled{\exists} * R^{-1} = A. \quad (56)$$

The composition $\textcircled{\exists} *$ is used since A is the set of manifestations that are in relation R^{-1} with at least one disorder in B . Equation (56) may have no solution; see [26]. In that case we may look for the best partial explanation of the manifestations, i.e., for the largest subset B such that

$$B \textcircled{\exists} * R^{-1} \subseteq A. \quad (57)$$

Then, the largest solution of (57), in the sense of inclusion, is given by

$$\forall v \in V, \quad \mu_{\hat{B}}(v) = \inf_{u \in U} \mu_{R^{-1}(v, u)} * \rightarrow \mu_A(u) \quad (58)$$

in the general case; see [26], [5]. In (58), we recognize a lower inverse image in the sense of subsection 2.2.3. Note that \hat{B} is also the largest solution of (56) when this equation has (a) solution(s). When A and R are an ordinary subset and an ordinary relation, respectively, (58) can be written as

$$\hat{B} = \{v \in V, \Gamma^{-1}(v) \subseteq A\}. \quad (59)$$

Thus, \hat{B} gathers all the disorders whose manifestations are included in A . Besides when (56) has (a) solution(s) it is interesting to look for the smallest solution in the sense of inclusion. The smallest solution of (57) is obviously the empty set and is of no interest. The set of solutions of (56), when it is not empty, may have no unique smallest solution. The reader is referred to [26] (see also [5]) for an algorithm that yields these smallest solutions. The smallest solution B_v , when it is unique, is particularly interesting since it is the smallest set of disorders that explains all the manifestations in A . However, it is easy to understand that there may exist several minimal sets of disorders that cause all the manifestations in A and that are not comparable with respect to set

inclusion. In the general case, it is interesting to look for these different minimum sets. They can be viewed as several distinct possible explanations of the symptom set A . Fuzzy diagnosis methods [26], [28] and their applications [1, 2] basically compute greatest and least solutions to (57).

By contrast, B^* defined by (18) reduces, when A and R are an ordinary subset and an ordinary relation, to

$$B^* = \{v \in V, A \subseteq \Gamma^{-1}(v)\}$$

Thus, a disorder that belongs to B^* produces all the manifestations in A and perhaps some others. In the same case, B^* defined by (14) reduces to

$$B^* = \{v \in V, A \cap \Gamma^{-1}(v) \neq \emptyset\}.$$

B^* gathers the disorders that explain at least a part of A . Disorders in B^* or in B^* may produce manifestations outside of A . However, A is a set of *observed* manifestations and depending on the case, a manifestation that does not belong to A may either be absent or present but not observed (for instance because special tests are needed to determine it). B^* and B^* are worth considering, especially when (56) has no solution (\hat{B} is not sufficient for explaining all the manifestations in A). B^* is useful if we look for a unique cause of a set of manifestations provided that such a cause exists. A disorder in B^* , when this set is not empty, offers a "parsimonious" covering (see Reggia et al. [24]) of all manifestations in A , since a unique cause is the simplest possible explanation. In any case, B^* includes all the manifestations that, in a group or separately, are able to cause A . It is easy to prove that

$$B^* \supseteq \hat{B} \quad (60)$$

in the general case, provided that $\Gamma^{-1}(v)$ is normalized (i.e., $\forall v, \exists u, \mu_R(u, v) = 1$).

Moreover, the weighted upper and lower images introduced in 2.3 can be useful in discussing a diagnosis problem. Indeed, we can, for instance, look for a disorder that would explain all the *important* manifestations, using B_{w^*} (defined by (35)); or, if the manifestations in A are of equal importance, look for a disorder that would explain *at least most of* them (when $B^* = \emptyset$).

Besides, the notion of lower image can play an important role if we also discuss the prescriptions for curing the disorders. Let S be a (fuzzy) relation defined on $V \times W$ where W is the set of all possible prescriptions. Let $B \subseteq V$

be a set of disorders that are possibly present. Then the subset of W , defined as the lower image of B via S , i.e., $C^* = B \textcircled{V} S$, gathers a set of possible prescriptions that separately are good for any disorder in B . Thus choosing a prescription in C^* , provided it is not empty, minimizes the consequences of the uncertainty concerning the actual disorder.

4. CONCLUSION

In this paper, we have provided a detailed account of the compositional rule of inference with fuzzy relations, based on recent results in the algebra of triangular norms and the solution of fuzzy relational equations. However we have tried to take a step inside the semantics of the various ways of projecting a fuzzy set from one space into another through a fuzzy relation. Two main lines of applications of these techniques have been emphasized: deductive inference with imprecise or vague knowledge on the one hand; and abductive inference with graded causal knowledge on the other hand. They correspond to two different views (resp.: disjunctive and conjunctive) of a fuzzy set.

Strangely enough, although there has been much activity around fuzzy deductive inference in the last 10 years, research on abductive reasoning via fuzzy relation is almost extinct. Significantly, no chapter on this topic is proposed in the recent monograph, albeit excellent, by Di Nola et al. [5] on fuzzy relations, despite the fact that originally, first results in fuzzy relational equations by Sanchez [25] [26] were motivated by applications to diagnosis. By contrast, the usefulness of fuzzy relational equations for deductive reasoning under fuzziness has been pointed out more recently by Di Nola et al. [4], Trillas and Valverde [11], and the authors [22, 6].

This paper has studied the properties and the meaning of the dual form of the generalized modus ponens. It has also indicated how to interpret the results of fuzzy-relation-based diagnosis methods that have been obtained in the past, as well as new ways of approaching this problem. There is a need to revive these diagnosis methods, considering the current activity with respect to abductive reasoning in artificial intelligence, as exemplified by the relational models of Reggia et al. [24]. In the future, we plan to investigate the links between fuzzy relational abductive reasoning and current symbolic or probabilistic models of diagnosis in artificial intelligence.

APPENDIX

A triangular norm $*$ [27] is a two-place function from $[0, 1] \times [0, 1]$ to $[0, 1]$ such that (i) $a * b = b * a$, (ii) $a * (b * c) = (a * b) * c$, (iii) if $a \leq b$ and $c \leq d$,

then $a * c \leq b * d$, and (iv) $1 * a = a$. The greatest triangular norm is min and the least one is defined by

$$a * b = T_w(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1; \\ 0, & \text{otherwise} \end{cases}$$

thus we always have $T_w(a, b) \leq a * b \leq \min(a, b)$. Other noticeable triangular norms are $a * b = a \cdot b$ and $a * b = \max(0, a + b - 1)$; moreover, we have

$$T_w(a, b) \leq \max(0, a + b - 1) \leq a \cdot b \leq \min(a, b).$$

By duality, each triangular norm $*$ is associated with a triangular conorm defined by

$$a \perp b = 1 - (1 - a) * (1 - b).$$

Triangular norms are conjunction operators whereas co-norms are disjunction operators; the main co-norms are, in increasing order,

$$\max(a, b) \leq a + b - a \cdot b \leq \min(1, a + b) \leq \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Let \circ be a two-place function from $[0, 1] \times [0, 1]$ to $[0, 1]$ nondecreasing with respect to its second argument. Let \mathcal{R} denote the transformation defined by

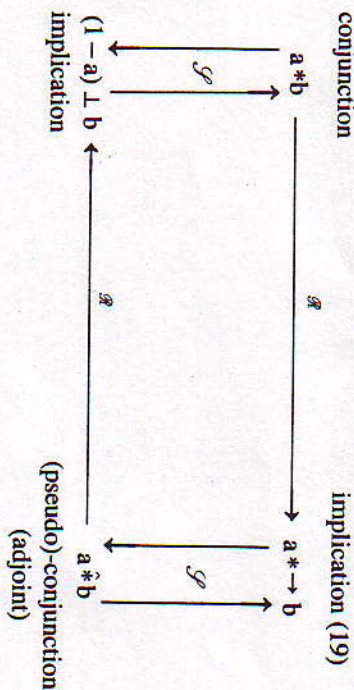
$$\forall (a, b) \in [0, 1]^2, \quad a \mathcal{R} (\circ) b = \sup\{s \in [0, 1], a \circ s \leq b\}$$

and \mathcal{S} be the transformation defined by

$$\forall (a, b) \in [0, 1]^2, \quad a \mathcal{S} (\circ) b = 1 - (a \circ (1 - b)).$$

Then the operations $*$, \cdot , $\hat{*}$, \perp introduced in the paper are related together

according to the following diagram:



where $*$ is a continuous triangular norm. Indeed, the adjoint $\hat{*}$ can be obtained as:

$$\begin{aligned} a \hat{*} b &= 1 - (a * \rightarrow (1 - b)) \\ &= 1 - \sup \{s \in [0, 1], a * s \leq 1 - b\} \\ &= \inf \{(1 - s) \in [0, 1], a * s \leq 1 - b\} \\ &= \inf \{t \in [0, 1], (1 - a) \perp t \geq b\} \end{aligned}$$

Let us calculate $\mathcal{R}(\hat{*})$.

$$\begin{aligned} a \mathcal{R}(\hat{*}) b &= \sup \{r \in [0, 1], a \hat{*} r \leq b\} \\ &= \sup \{r \in [0, 1], \inf \{t \in [0, 1], (1 - a) \perp t \geq r\} \leq b\} \end{aligned}$$

Observe that $\forall r, \{t \in [0, 1], (1 - a) \perp t \geq r\} \neq \emptyset$; indeed $t = 1$ belongs to this set owing to $\forall c, c \perp 1 = 1$. Since, obviously, $\exists t \leq b$ such that $(1 - a) \perp t \geq (1 - a) \perp b$, the supremum in the preceding expression is reached for an r at least equal to $(1 - a) \perp b$. Assume that the supremum is reached for $\hat{r} > (1 - a) \perp b$; then it means that $\exists t_0 \leq b$ such that $(1 - a) \perp t_0 \geq \hat{r} > (1 - a) \perp b$; since owing to the continuity of $*$ and consequently of \perp , the greatest lower bound of the set $\{t \in [0, 1], (1 - a) \perp t \geq \hat{r}\}$ belongs to this set. This contradicts the nondecreasingness of \perp . Hence an $\mathcal{R}(\hat{*}) b = (1 - a) \perp b$. Obviously, \mathcal{R} is involutive.

The diagram does not apply for a discontinuous triangular norm such as T_w . Note that only a part of the characteristic properties of triangular norms are used in the preceding proof.

The following table gives the corresponding operations for the main continuous triangular norms.

	\perp	$*$	$* \rightarrow$	$*$
$(1 - a) \perp b = a \hat{*} \rightarrow b$	$\max(1 - a, b)$	$\begin{cases} 0 & \text{if } a + b \leq 1 \\ b & \text{if } a + b > 1 \end{cases}$	$\begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$	$\begin{cases} 0 & \text{if } a = 0 \\ \max\left(0, \frac{a + b - 1}{a}\right) & \text{if } a \neq 0 \end{cases}$
$a \cdot b$	$1 - a + ab$	$a + b - ab$	$\begin{cases} 1 & \text{if } a = 0 \\ \min\left(1, \frac{b}{a}\right) & \text{if } a \neq 0 \end{cases}$	$\begin{cases} 0 & \text{if } a = 0 \\ \max\left(0, \frac{a + b - 1}{a}\right) & \text{if } a \neq 0 \end{cases}$
	$\min(1, 1 - a + b)$	$\min(1, a + b)$	$\min(1, 1 - a + b)$	$\max(0, a + b - 1)$

Note that the adjoint property is involutive, i.e., $\hat{\odot} = \mathcal{G} \circ \mathcal{H}(\hat{\odot}) = \mathcal{G} \circ \mathcal{H} \circ \mathcal{G}(\hat{\odot}) = \hat{\odot}$, whether $\hat{\odot} = * \text{ or } \hat{*}$.

A continuous triangular norm $*$ satisfies the following inequalities [5]:

$$a^*(a^* \rightarrow b) \leq b \quad (\text{A1})$$

$$a^* \rightarrow (a^* b) \geq b \quad (\text{A2})$$

PROPOSITION. (A1) and (A2) remain valid when $*$ is changed into its adjoint $\hat{*}$, and $* \rightarrow$ into $\hat{*} \rightarrow$.

Proof.

$$(\text{A1}) \Leftrightarrow a^*(a^* \rightarrow (1-b)) \leq 1-b$$

$$\Leftrightarrow (1-a) \perp (a^* b) \geq b \quad (\text{definition of } * \text{ and } \perp)$$

$$\Leftrightarrow a^* \rightarrow (a^* b) \geq b \text{ which is (A2) for } * \text{ and } * \rightarrow$$

$$(\text{A2}) \Leftrightarrow a^* \rightarrow (a^*(1-b)) \geq 1-b$$

$$\Leftrightarrow a^* \rightarrow ((1-a) \perp b) \leq b \quad (\text{definition of } * \text{ and } \perp)$$

$$\Leftrightarrow a^* \rightarrow (a^* \rightarrow b) \leq b \text{ which is (A1) for } * \text{ and } * \rightarrow$$

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