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## Article

# Homotopy of Linearly Ordered Split–Join Chains in Covering Spaces of Foliated $n$ -Manifold Charts

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**Abstract:** Topological spaces can be induced by various algebraic ordering relations such as, linear, partial and the inclusion-ordering of open sets forming chains and chain complexes. In general, the classifications of covering spaces are made by using fundamental groups and lifting. However, the Riesz ordered  $n$ -spaces and Urysohn interpretations of real-valued continuous functions as ordered chains provide new perspectives. This paper proposes the formulation of covering spaces of  $n$ -space charts of a foliated  $n$ -manifold containing linearly ordered chains, where the chains do not form topologically separated components within a covering section. The chained subspaces within covering spaces are subjected to algebraic split–join operations under a bijective function within chain-subspaces to form simply directed chains and twisted chains. The resulting sets of chains form simply directed chain-paths and oriented chain-paths under the homotopy path-products involving the bijective function. It is shown that the resulting embedding of any chain in a leaf of foliated  $n$ -manifold is homogeneous and unique. The finite measures of topological subspaces containing homotopies of chain-paths in covering spaces generate multiplicative and cyclic group varieties of different orders depending upon the types of measures. As a distinction, the proposed homotopies of chain-paths in covering spaces and the homogeneous chain embedding in a foliated  $n$ -manifold do not consider the formation of circular nerves and the Nachbin topological preordering, thereby avoiding symmetry/asymmetry conditions.

**Keywords:** homotopy; algebraic chain; covering spaces; manifold; homeomorphism

**MSC:** 54A20; 54H10; 55P10; 55P15



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## 1. Introduction

An algebraically chained topological space is essentially a connected topological space endowed with algebraic ordering relations. The topological notion of connectedness was first investigated by Bolzano, which was based on continua (i.e., considering algebraic ordering) [1]. The equivalence between the notions of continuum of real space as a chain and the connectedness in Euclidean space was proposed by Bolzano. Note that the continuum of the set of real numbers  $R$  is enriched with an algebraic linear-ordering relation forming a chain such that  $(R, \cdot, +, *)$  becomes an ordered field under the algebraic linear order  $* \in \{<, >\}$ . Recall that the concept of continuum and compact chain can be formulated in a metric space  $(X, d)$ , where  $X$  is connected and compact [2,3]. It was noted by Lefschetz that any compact metric space can admit topological chains as well as chain complexes, where the concept of  $B$  – chains retains the topological notion of local connectedness [4]. Moreover, a continuum  $X$  is chainable by  $\varepsilon$  – chain if it covers the space  $X$  [2]. Later Cantor pointed out that the definition of topological connectedness given by Bolzano could lead to a set of continuums  $\{\aleph_j : j \in \Lambda\}$  such that  $[j \neq k] \Rightarrow [\aleph_j \cap \aleph_k = \emptyset]$ , where  $\Lambda$  is an index set [1]. As a result, Cantor further refined the concept of connectedness by separating the notion of (chain like) continuum of real space from the topological notion of connectedness as follows [5].

**Definition 1.** If  $A \subset X$  is a Euclidean space then  $A$  is connected  $\forall x \forall y \in A$  and  $\forall \varepsilon > 0$  there is a sequence  $\langle p_i \in A \rangle_{i=1}^n$  such that the following three conditions are maintained by a function  $d : A^2 \rightarrow [0, +\infty)$  within the space: (1)  $\forall k \in [1, n-1], d(p_k, p_{k+1}) < \varepsilon$ , (2)  $d(x, p_1) < \varepsilon$  and, (3)  $d(p_n, y) < \varepsilon$ .

Evidently, Cantor is considering that  $(X, d)$  is a metric space (not necessarily compact) and it leads to the notion of a perfect set where  $\{x\} \subset X$  is closed and dense in itself. Later Jordan incorporated the concept of algebraic chains and topologically connected chain-components as follows [1].

**Definition 2.** A topological subspace  $A \subset X$  is a connected component if  $\forall E \subset A, \exists F \subset A$  such that  $E \cap F \neq \varnothing$ . Moreover, in  $A \subset X$  if  $\forall [a, b] \subset A, \exists \varepsilon > 0$  such that  $[e, f \in [a, b]] \Rightarrow [d(e, f) < \varepsilon]$  then  $[a, b]$  is a  $\varepsilon$ -chain component.

Observe that an algebraically chained topological space is essentially a connected space endowed with additional order structures. For example, the Riesz topological space is a partially ordered vector space equipped with a lattice ordered structure. The Riesz topological space invites the concept of sequential convergence and connectedness, which can be presented as follows [6].

**Definition 3.** Let  $X$  be an (Archimedean) Riesz space with the weak order  $e$  and  $g \in X^+$ . The sequence  $\langle f_k \in X \rangle_{k=1}^{+\infty}$  converges relatively  $g$ -uniformly to  $f \in E$  if the sequence  $\langle \varepsilon_j \rangle_{j=1}^{+\infty} \rightarrow 0$  admits  $|f_k - f| \leq \varepsilon_j g$ .

It is straightforward to observe that if a Riesz topological space is sequentially convergent within the space then it is (sequentially) connected. It is important to note that the earlier notions of topological connectedness employed closed sets and the Riesz ordered space moves away from such a notion of connectedness. On the contrary, Riesz proposed an alternative definition of topological connectedness and order types [7].

**Definition 4.** A topological space  $X$  is said to be connected if it cannot be decomposed into  $A, B \subset X$  such that  $(A \cup \partial A) \cap \partial B = \varnothing$  and  $\partial A \cap (B \cup \partial B) = \varnothing$ . Moreover, a space  $X$  can be an  $n$ -fold ordered space.

Evidently, the concept of Riesz connectedness of a topological space considers that  $X \equiv R^n$  and  $A \neq \varnothing, B \neq \varnothing$ , in general. The distinction between Cantor and Riesz is that the Riesz topological connectedness does not employ the notion of metric space. Finally, Hausdorff further generalized the concept of separation of a topological space (without considering chains) as the following: A topological space  $X = A \cup B$  is connected if  $\overline{A} \cap (B = B^0) \neq \varnothing$  and  $(A = A^0) \cap \overline{B} \neq \varnothing$ . Thus, the Hausdorff connectedness does not resort to any additional algebraic order types. On the other hand, Lefschetz had shown that a chainable, topologically connected and metrizable space does not contain any isolated point [4]. Moreover, the embeddings of a Nachbin preordered set  $(A, \leq_{pre})$  generate Alexandroff topology endowed with topological order  $\leq_\tau$  admitting the following property (often called a Nachbin topological property) [8].

**Definition 5.** The full embedding  $i_{em} : (A, \leq_{pre}) \rightarrow X$  of a preordered set forms Alexandroff topological order  $\leq_\tau$  such that  $\forall x, y \in X, [x \leq_\tau y] \Rightarrow [N_x \leq N_y]$ , where  $N_x$  and  $N_y$  denote respective neighborhood filters and the relation  $N_x \leq N_y$  denotes that  $N_x$  is a topologically finer neighborhood filter than  $N_y$ .

Note that a preordering relation is reflexive and transitive without paying any attention to the symmetry/asymmetry property and a preordered set can be separated. Thus, there

are interactions between the concepts of topological chains, ordered spaces, connectedness and the separation of topological spaces.

### 1.1. Motivation

The constructions of topological spaces based on multiple order types are interesting because the resulting topological subspaces can be enriched with additional algebraic properties. In 1905 Riesz proposed that any arbitrary set  $X$  can be equipped with  $n$ -fold different algebraic orders and such a space can be denoted as  $(X, <_n)$  [7]. For example, if  $x, y \in R^n$  such that  $[x <_i y] \Rightarrow [\forall i \in [1, n], a_i < b_i]$ , then  $(X, <_n)$  is enriched with  $n$ -order types where  $x \equiv \langle a_1, a_2, \dots, a_n \rangle$  and  $y \equiv \langle b_1, b_2, \dots, b_n \rangle$ . In general, two sets  $X, Y$  are endowed with same algebraic order if there is a bijection  $f : X \rightarrow Y$  preserving the algebraic order relation. However, this is a stronger condition for retaining ordering relations and it can be relaxed. Earlier, it was shown that two topological hyperconvex spaces can be ordered under a special topological-ordering relation affecting the measurability in such spaces [9]. A topological space  $X$  is Bolzano–Weierstrass space ( $B_w$ -space) if every infinite subset of  $X$  has an accumulation point. In general, the abstract (differentiable) manifold theory is often constructed on  $B_{w*}$  topological spaces and the chains in such a topological space are formulated by employing a family of sets within the corresponding topological spaces [5,10]. Note that such topological chained spaces are not enriched with any additional algebraic ordering relations. It is shown by Corson that every  $B_w$  linear (weak) topological space admits countable chains [11,12]. In the special topological spaces, such as tree space  $T$ , the chains can be admitted and such chains as well as their anti-chains are countable [11]. However, these are specific topological spaces, namely weak topological Banach spaces, containing chains and the results may not be applicable to the general topological spaces. However, the interesting topological property of the chained spaces is that every para-compact space with countable chains is Lindelöf and there are relations between the open covering and topologically connected chains [5,11]. Urysohn, showed that real-valued continuous functions can be considered as ordered chains and moreover, a set of topological objects, called separating chains, can be employed to define a set of continuous real-valued functions [13]. Furthermore, the chained components can be formed within a topological space under an algebraic equivalence relation [14]. These observations provide motivation to ask the following questions towards generalizations: (1) What are the topological properties of covering spaces of chart maps of a foliated  $n$ -manifold containing algebraic chains with different directions and orientations, (2) What are the properties of various chain-paths constructed under bijections in covering spaces in view of path-homotopy theory of algebraic topology, and (3) How to formulate an algebraically ordered chained covering space without resorting to a topological inclusion-order relation of open sets? This paper addresses these questions in relative detail in the views of general topology as well as algebraic topology.

### 1.2. Contributions

The contributions made in this paper can be summarized as follows: Inspired by Riesz  $n$ -fold ordered multidimensional spaces  $(X, <_n)$ , Jordan chain components and the observations by Urysohn about real-valued functions, we propose a set of covering spaces of chart maps of a foliated  $n$ -manifold  $M^n$  where  $\partial M^n = \varphi$ . A connected  $n$ -space chart (topological subspace) generated by  $f : (A \subset M^n) \rightarrow R^n$  is finitely chained under algebraic ordering as  $(B \subset f(A), * = *_n)$ , where  $*_n$  denotes an order relation in an  $n$ -space. The covering maps of the chained  $n$ -space charts are formulated to form two varieties of algebraically chained subspaces: (1) the simply directed chains, and (2) the twisted chains under surjections within the covering spaces. Accordingly, two varieties of homotopy chained-paths are formed within the covering spaces under bijections by following the principles of: (a) path-homotopy of algebraic topology, and (b) combinatorial split-join of multiple homotopy chain-paths on the discrete fiber space. The resulting homotopy chain-paths under split-join in the covering spaces are either simply directed or oriented in

nature. We show that the embedding of a topological chained subspace in  $M^n$  is unique, while maintaining the algebraic order, and any non-homogeneous chain-paths cannot be embedded in  $M^n$ . Interestingly, two split-join homotopy chain-paths do not mutually commute. This is to emphasize that the chains in covering spaces, as proposed in this paper, do not form any nerves homeomorphic to  $S^1$  and the algebraic relation is not an equivalence relation forming any equivalence classes within the covering spaces, maintaining generality. Moreover, we show that all algebraically chained Hausdorff subspaces within a covering space can be considered as a measure-space and it is finitely measurable. As a result, different measures give rise to two varieties of group algebraic structures. It forms a 5th order multiplicative group if the measure in Hausdorff covering space is not unique and it forms a 6th order cyclic group if the integer-valued finite measure is unique in a Hausdorff covering space. Finally, this is to emphasize that the proposed chain-homotopy does not depend on the concepts of separating chains and inclusion-order of open sets in a topological space. Furthermore, we do not follow the formulation of Nachbin preordered set embeddings to form Alexandroff topological order in the covering spaces and it does not affect measurability.

The rest of the paper is organized as follows. The preliminary concepts related to the domain are presented in Section 2. The definitions of split-join chains and homotopy chain-paths are presented in Section 3. Section 4 presents the main results. Finally, Section 5 concludes the paper.

## 2. Preliminaries

In this section, the concepts and definitions of topological chains, chain connected spaces and associated algebraic structures are introduced. The classical topological properties of cover of chained-subspaces, chain-homeomorphism and separations are illustrated. Note that, the index set is denoted as  $\Lambda$  and the sets  $R, Z$  denote the sets of real numbers and integers, respectively. The homeomorphism between  $X$  and  $Y$  is denoted as  $Hom(X, Y)$ . First, we present the definition of a topological chain within a space [5].

**Definition 6.** Let  $\langle C_i \subset X \rangle_{i=1}^n$  be a sequence in a topological space  $(X, \tau_X)$ . If the sequence maintains the property that  $\forall i \in [1, n], C_i \cap C_{i+1} \neq \emptyset$  then  $\langle C_i \rangle_{i=1}^n$  is a topological chain in  $(X, \tau_X)$ .

As a consequence, if  $a \in C_1$  and  $b \in C_n$  then the pair  $(a, b)$  is said to be topologically chain-connected in  $(X, \tau_X)$ . Moreover, a topological space  $(X, \tau_X)$  is a connected space if  $\left(\bigcup_{i \in [1, n]} C_i\right) \subset U_{cov}$  where  $U_{cov} = \{A_k \subset X : k \in \Lambda\}$  is a cover of  $X$  and  $A_k = A_k^o$ . We can further extend this concept towards the formation of nerves in a chained topological space as follows [3].

**Definition 7.** If  $U_{cov}$  is a covering chain in the topological space  $(X, \tau_X)$  then it is a circular chain if the nerve of  $U_{cov}$  is homeomorphic to  $S^1$ .

**Remark 1.** If there is a non-decomposable and non-chainable continuum, which is homeomorphic to  $S^1$  then it is called a pseudo-solenoid, and a planar continuum is always homeomorphic to a pseudo-circle in a topological space [3].

The algebraic relation induced within a topological space and the open covers of the corresponding topological space lead to the formation of two varieties of topological chains, which are defined as follows [3,14].

**Definition 8.** Let  $S$  be an equivalence relation. The equivalence classes  $\{[x_i \in X]_S : i \in \Lambda\}$  generated by  $S$  within a topological space  $(X, \tau_X)$  are called  $S$ -components and a sequence  $\langle x_k \in [x_i] \rangle_{k=1}^n$  is a finite  $S$ -chain. Moreover, if  $X$  is a continuum with open  $\varepsilon$ -cover  $U_{cov(\varepsilon)}$  endowed with a circular nerve homeomorphic to  $S^1$  then  $U_{cov(\varepsilon)}$  forms a  $\varepsilon$ -chain in  $(X, \tau_X)$ .

This leads to the following lemma involving the covering map in a pseudo-solenoid space [3].

**Lemma 1.** *If  $f : X \rightarrow Y$  is a finite covering map and  $X$  is continuum then if  $Y$  is pseudo-solenoid then it implies that  $X$  is also a pseudo-solenoid.*

**Proof.** It is straightforward to observe that the covering map  $f : X \rightarrow Y$  is a surjection preserving the properties given as:  $(f \circ f^{-1})(U \subset Y) = Id_Y$  and  $(f^{-1} \circ f)(V \subset X) = Id_X$ , where  $Id_X$  and  $Id_Y$  are identity functions in the respective topological spaces retaining local homeomorphisms.  $\square$

Note that if we consider that the base space  $Y$  is not a pseudo-solenoid, then the covering map  $f : X \rightarrow Y$  can admit a pseudo-solenoid structure in the covering space  $X$  if and only if the covering map preserves the  $Hom(f(X), S^1)$  property. Recall that a topological space  $(X, \tau_X)$  is called an  $A$  – space if any arbitrary sets  $(A \subset X) = A^o, (B \subset X) = B^o$  the space maintains the topological property that:  $(A \cap B) = (A \cap B)^o$  considering that  $(A \cap B) \neq \varphi$ . If we consider two  $A$  – spaces  $X, Y$  of  $T_0$  – type then a covering homeomorphism between the corresponding chained-spaces can be found, which is presented in the following theorem [15].

**Theorem 1.** *If  $f : X \rightarrow Y$  is a continuous surjection between two  $A$  – spaces then it is a covering if  $\forall A_c \subset Y, \exists B_{c1}, B_{c2} \subset X$  such that  $Hom(f(B_{c1} \cup B_{c2}), A_c)$  is a chain-homeomorphism and  $B_{c1} \cap B_{c2} = \varphi$ .*

It is important to note that, in this case the topological chain-subspaces are finite in nature and the proof of the theorem is given in [15]. The algebraic chains within a topological space are often constructed from the partially ordering relation. Interestingly, in a  $H$  – closed partially ordered topological space, the locally compact subspaces are separated by upper and lower chain-points [16]. This is to emphasize that a linearly ordered  $H$  – closed topological space retains the properties of partially ordered chaining properties of the corresponding topological space. A topological space  $(X, \tau_X)$  is equipped with countable-chain-condition if  $\forall \{A_i : i \in \Lambda\} \subset \tau_X$  the family of sets are countable such that  $A_i \neq \varphi$  and  $[i \neq k] \Rightarrow [A_i \cap A_k = \varphi]$  condition is preserved [17]. The topological chains formulated under inclusion-order of open sets leads to the definition of separating chains [13].

**Definition 9.** *Let us consider the countable set  $X_{ch} = \{A_i \subset X : i \in \Lambda, A_i \in \tau_X\}$  in a topological space  $(X, \tau_X)$ . A set of topological chains  $(X_{ch}, \subset)$  under inclusion-order is called a separating chain if it maintains the following three properties, which are given as:*

- (1)  $[A_i, A_k \in X_{ch} : A_i \subset A_k] \Rightarrow [\exists A_m \in X_{ch} : \overline{A_i} \subseteq A_m \subseteq \overline{A_m} \subseteq A_k],$
- (2)  $\bigcap_{i \in \Lambda} A_i = \varphi$
- (3)  $\bigcup_{i \in \Lambda} A_i = X$

As a result, the topological compactness can be conceptualized by means of separating chains as presented in the following theorem [13].

**Theorem 2.** *A topological space  $(X, \tau_X)$  is pseudo-compact if every separating chain in  $(X_{ch}, \subset)$  contains  $\{\varphi, X\}$ .*

The indexing schemes of separating chains and the proof of the theorem are detailed in [13]. It is important to note that a set of separating chains can help in determining the separation of a topological space. On the other hand, if we consider a partially ordered set  $(X, <_k)$  generating algebraic  $k$  – chains, then a topological space  $(X, \tau_X)$  containing  $k$  – chains satisfies the countable  $k$  – chains property if every  $A, B \subset X$  contains  $k$  – chains of the same size, where  $A \cap B = \varphi$  within the space [18]. Note that the  $k$  – chains property moves away from the inclusion-ordering property of open sets in a topological space.



### 3. Chained Covering Spaces and Split-Joins

In this section, we present a set of definitions related to the formation of chained covering spaces and the algebraic split-join operations on the chains. The formulations of corresponding chain homotopies are also presented.

#### 3.1. Chained Covering Spaces of $n$ -Manifold Charts: Definitions

In this paper, the algebraic ordering relation is generally denoted as  $*$   $\in$   $\{<, >\}$  and we will represent a specific ordering relation chosen from the set as necessary to maintain clarity for specific cases. This is to emphasize that in the remaining sections of the paper the Riesz  $n$ -fold ordering relations in  $R^n$  are preserved as  $*_n \in \{<_n, >_n\}$  when we are considering that the covering spaces are  $n$ -spaces for the chart maps of a manifold  $M^n$  (i.e., in the remainder of the paper we consider that  $*$   $\equiv$   $*_n$  and we write  $<$  in place of  $<_n$  for simplicity). The same goes for the linear ordering relation  $*$   $\equiv$   $>$ . Moreover, the algebraic chain in a topological space  $A$  is denoted as  $(A, *)$  and by following the conventions of homotopy theory of algebraic topology, we will denote the directed homotopy path in an algebraic chain as: (1)  $[A]$  if the path follows monotonic sequence  $\langle x_k \rangle_{k=1}^m$  chosen in  $(A, <)$  such that  $x_1 < x_2 < x_3 < \dots < x_m$  and (2)  $[\bar{A}]$  if the path follows monotonic sequence  $\langle x_{n(k)} \rangle_{k=1}^m$  chosen in  $(A, <)$  such that  $x_{n(1)} > x_{n-1(2)} > x_{n-2(3)} > \dots > x_{2(m-1)} > x_{1(m)}$ . For example, if an algebraic chain in a topological space is given by  $(A = \{x_1, x_2\}, <)$  then  $[A] \equiv \langle x_1, x_2 \rangle$  and  $[\bar{A}] \equiv \langle x_2, x_1 \rangle$ . Similarly, if the chain is given by  $(A = \{x_1, x_2\}, >)$  then  $[A] \equiv \langle x_2, x_1 \rangle$  and  $[\bar{A}] \equiv \langle x_1, x_2 \rangle$ . Furthermore, if  $[A]$  and  $[B]$  are two linear homotopy chain-paths in a topological space, then the homotopy path-composition is represented as  $[A] *_H [B]$  by following the conventions of algebraic topology while preserving the order relation throughout the chain-homotopy.

**Definition 10.** Let  $M^n$  be an  $n$ -manifold such that  $\partial M^n = \varphi$  and  $(U_x \subset R^n, f)$  be a chart map of the open neighborhood  $N_x \subset M^n$  of  $x \in M^n$  generated by  $f : M^n \rightarrow R^n$ . The chart map  $(U_x \subset R^n, f)$  is defined as algebraically chained if there is a finite ordered chain  $(A \subset U_x, *)$ .

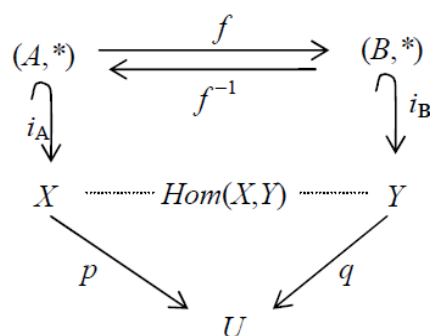
Observe that if the ordered chain  $(A \subset U_x, *)$  is finite then  $A = \bar{A}$  in  $R^n$ , where  $U_x = U_x^0$ . In general, if we consider a topological space  $(X, \tau_X)$  then  $(A \subset X, *)$  is called a topological chained subspace. In an ordered chain  $(A \subset U_x, *)$  in Hausdorff  $(X \equiv R^n, \tau_X)$ , any point  $\{x\} \subset A$  is called a split-set. We can further extend this concept to the covering spaces which is defined as follows.

**Definition 11.** Let us consider a surjective covering map given by  $p : X \rightarrow U_x$  such that  $S_{p(i)} \in p^{-1}(U_x)$  is a section at  $i \in \Lambda$ . The space  $X$  is called as an algebraically (ordered) chained covering space of  $U_x$  if  $p^{-1}(U_x)$  contains a set of chains  $\{(A_{p(i)} \subset S_{p(i)}^0, *) : i \in \Lambda\}$ .

If we consider that  $p^{-1}(U_x)$  is a specific chained covering space such that the set of chains  $\{(A_{p(i)} \subset S_{p(i)}^0, <) : i \in \Lambda\}$  is admitted, then it leads to the concept of twisted chained covering spaces, which is defined as follows.

**Definition 12.** Let us consider a covering map given by  $q : Y \rightarrow U_x$  such that  $X \cap Y = \varphi$  and the chart map  $(U_x \subset R^n, f)$  is algebraically chained as  $(A \subset U_x, <)$ . The covering map  $q(Y)$  is called a twisted chained covering map if it admits the set of chains given as  $\{(B_{q(i)} \subset S_{q(i)}^0, >) : i \in \Lambda, S_{q(i)} \in q^{-1}(U_x)\}$ .

The schematic representation of chained-covering spaces of a chart  $U$  under foliation of an  $n$ -manifold and the inclusion functions  $i_A : (A, *) \rightarrow X$  and  $i_B : (B, *) \rightarrow Y$  is illustrated in Figure 1. Note that the inclusions are not the Nachbin variety and the homeomorphism is an oriented isomorphism.



**Figure 1.** Representation of chained covering spaces.

**Remark 2.** It is important to note that if there is an inclusion function  $i_{em(R)} : U_x \rightarrow R^n$  such that  $(A \subset U_x, *)$  is a finite chain preserving  $A = \overline{A}$  in  $R^n$  then there are corresponding inclusion functions in covering spaces  $i_{em(X)} : S_{p(i)} \rightarrow X$  and  $i_{em(Y)} : S_{q(i)} \rightarrow Y$  preserving the properties that  $A_{p(i)} = \overline{A_{p(i)}}$  and  $B_{q(i)} = \overline{B_{q(i)}}$  in the respective chained covering spaces.

### 3.2. Split-Join Combinatorial Chains: Definitions

Let us consider a general topological chained subspace  $(A, *)$ , where the algebraic ordering relation is  $* \in \{<, >\}$ . If we choose any point  $x \in A$  then  $A_{I(x)} \subset A$  and  $A_{F(x)} \subset A$  are called the *initial segment* and *final segment* of  $(A, *)$  if the following conditions are maintained.

**Condition I:**  $A_{I(x)} \cap A_{F(x)} = \{x\}$  and,

**Condition II:**  $\forall x \in A_{I(x)}, \forall y \in A_{F(x)}, [* \equiv <] \Rightarrow [x < y], [* \equiv >] \Rightarrow [x > y]$ .

We call the specific point  $x \in A$  a *split-join point* of the chain.

**Remark 3.** Every split-join point  $x$  forms a split-set  $\{x\}$  in a Hausdorff topological space.

Note that Condition II follows the concept of topological nets and if  $A$  is uncountable with  $\dim(A) = 1$  then the partitioning of a topological chained subspace  $(A, *)$  has resemblance to the formation of basis elements of standard topology in  $R$ . This preserves the Urysohn interpretations of real-valued continuous functions as chains. However, the multiple general topological chained subspaces contain a much richer combinatorial property of *split-join chain set* formation, which is defined as follows.

**Definition 13.** Let  $(A, *)$  and  $(B, *)$  be two topological chained subspaces such that  $A \cap B = \varnothing$  and  $A, B$  are finite as well as uncountable. If the function  $f : (A_S \subset A, *) \rightarrow (B_S \subset B, *)$  is a bijection then  $\lambda_x[A, B] \equiv A_{I(x)} \cup B_{F(f(x))}$  is called a *split-join chain set* at  $x \in A_S$ .

If we induce the algebraic ordering relation  $* \in \{<, >\}$  within the split-join algebraic chain set  $\lambda_x[A, B]$  then it results in the formation of corresponding *split-join homotopy chain-path*, which is defined as follows.

**Definition 14.** A split-join algebraic chain set  $\lambda_x[A, B]$  is called a *split-join homotopy chain-path* generated under relation  $* \in \{<, >\}$ , which is denoted as  $\lambda_x[[A, B], *]$ , if either  $* \equiv <$  or  $* \equiv >$  in  $\lambda_x[A, B]$ .

**Remark 4.** It is important to note that,  $\lambda_x[[A, B], *]$  denotes that the split-join homotopy chain-path starts in  $(A, *)$  and ends in  $(B, *)$ . However, the split-join homotopy chain-path  $\lambda_x[[B, A], *]$  denotes that the path starts in chain  $(B, *)$  and ends in chain  $(A, *)$ .



### 3.3. Directions and Orientations

There are two possible combinatorial formations of split-join homotopy chain-paths within the topological spaces, which are called *simply directed homotopy chain-path* and *oriented homotopy chain-path*. A split-join homotopy chain-path  $\lambda_x[[A, B], *]$  is called a *simply directed homotopy chain-path* in a topological space if it preserves *any one* of the following properties (either condition C1 or condition C2) under bijective  $f : (A_S \subset A, *) \rightarrow (B_S \subset B, *)$ , where either  $* \equiv <$  or  $* \equiv >$  in  $\lambda_x[A, B]$ .

$$\begin{aligned} \text{C1} : \lambda_x[A, B] &\equiv A_{I(x)} \cup B_{F(f(x))}, \\ \text{C2} : \lambda_x[A, B] &\equiv A_{F(x)} \cup B_{I(f(x))}. \end{aligned} \quad (1)$$

This is to emphasize that any one of the aforesaid conditions needs to be satisfied by an algebraic chain-path in a topological space.

On the other hand, if we consider two topological chained subspaces  $(A, <)$  and  $(B, >)$  then  $\lambda_x[[A, B], *]$  is called as an *oriented homotopy chain-path* in a topological space under relation  $* \in \{<, >\}$  if any one of the following conditions (either condition C3 or condition C4) are satisfied.

$$\begin{aligned} \text{C3} : \lambda_x[A, B] &\equiv A_{I(x)} \cup B_{I(f(x))}. \\ \text{C4} : \lambda_x[A, B] &\equiv A_{F(x)} \cup B_{F(f(x))}. \end{aligned} \quad (2)$$

This is to emphasize that *any one* condition within any one of the aforesaid sets of conditions need to be satisfied to form a single algebraic chain-path within a topological space. Moreover, the simply directed homotopy chain-paths under split-joins are formed when the chains are homogeneous in terms of algebraic relation. On the contrary, the oriented homotopy chain-paths under split-joins are formed when the algebraic relations are not homogeneous in two chains.

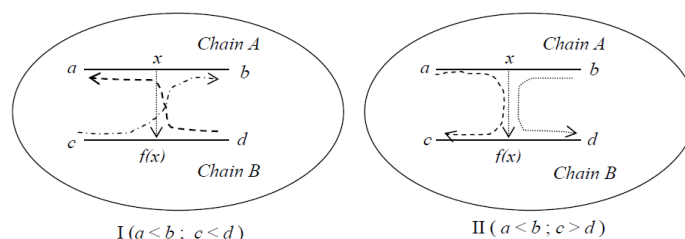
**Example 1.** As an example, we can represent two simply directed homotopy chain-paths derived from Equation (1) under split-join considering two chains  $(A, <)$  and  $(B, <)$  retaining ordering relational homogeneity as:

$$\begin{aligned} \lambda_x[[B, A], <]_1 &\equiv [B_{I(f(x))}] *_H [A_{F(x)}], \\ \lambda_x[[B, A], <]_2 &\equiv [\overline{B_{F(f(x))}}] *_H [\overline{A_{I(x)}}]. \end{aligned} \quad (3)$$

Similarly, the two oriented homotopy chain-paths can be derived from Equation (2) under split-join considering two chains  $(A, <)$  and  $(B, >)$  without preserving homogeneity of algebraic ordering relation as:

$$\begin{aligned} \lambda_x[[A, B], *]_1 &\equiv [A_{I(x)}] *_H [B_{I(f(x))}], \\ \lambda_x[[A, B], *]_2 &\equiv [\overline{A_{F(x)}}] *_H [\overline{B_{F(f(x))}}]. \end{aligned} \quad (4)$$

Geometrically, the algebraic split-join on a set of linearly ordered chains forms a variety of chain-paths under homotopic path-products as illustrated in Figure 2.



**Figure 2.** Sets of chain-paths (I: simply directed; II: oriented).

The oriented homotopic path-products of chain-paths induce specific orientations of topological chained subspaces (paths) within the corresponding chained covering spaces, which is an additional characteristic property. However, the simply directed homotopy chain-paths linearly follow the algebraic ordering relation without showing any additional orientations of topological chain-paths within the spaces.

#### 4. Main Results

The interplays between various types of algebraic chained covering spaces and chart maps of a manifold  $M^n$  exhibit a rich set of topological properties. First, we show that the bijection  $h : (A_{p(i)} \subset S_{p(i)}^o, *) \rightarrow (A_{p(k)} \subset S_{p(k)}^o, *)$  between two algebraic chains in  $S_{p(i)}, S_{p(k)} \in p^{-1}(U_x)$  for some  $\{i, k\} \subset \Lambda$  forms the simply directed homotopy chain-paths within the covering spaces.

**Theorem 3.** *If  $p^{-1}(U_x)$  is a covering space of  $(U_x \subset R^n, f)$  for a manifold  $M^n$  containing two algebraically ordered homogeneous chains  $(A_{p(i)} \subset S_{p(i)}^o, *)$  and  $(A_{p(k)} \subset S_{p(k)}^o, *)$  in  $S_{p(i)}, S_{p(k)} \in p^{-1}(U_x)$ ,  $\{i, k\} \subset \Lambda$  then the bijection  $h : (A_{p(i)} \subset S_{p(i)}^o, *) \rightarrow (A_{p(k)} \subset S_{p(k)}^o, *)$  admits two simply directed homotopy chain-paths  $\lambda_y[[A_{p(k)}, A_{p(i)}], *, *]_1$  and  $\lambda_y[[A_{p(k)}, A_{p(i)}], *, *]_2$  at a split-join point  $y$ .*

**Proof.** Let  $M^n$  be an  $n$ -manifold with a chart map  $(U_x \subset R^n, f)$  and  $p : X \rightarrow U_x$  be a corresponding covering map. Suppose there are two algebraic chains  $(A_{p(i)} \subset S_{p(i)}^o, *)$  and  $(A_{p(k)} \subset S_{p(k)}^o, *)$  such that either  $* \equiv <$  or  $* \equiv >$  is maintaining homogeneity of algebraic ordering relation within  $S_{p(i)}, S_{p(k)} \in p^{-1}(U_x)$  for some  $\{i, k\} \subset \Lambda$ . If we consider a bijection  $h : (A_s \subset A_{p(i)}, *) \rightarrow (B_s \subset A_{p(k)}, *)$  then there is a split-join point  $y \in A_s$  forming a set of sections of homotopy chain-paths within the covering spaces which are given as  $\{A_{I(y)}, A_{F(y)}, B_{I(h(y))}, B_{F(h(y))}\}$  where  $(A_{I(y)} \cup A_{F(y)}) \subset A_{p(i)}$  and,  $(B_{I(h(y))} \cup B_{F(h(y))}) \subset A_{p(k)}$ . It results in the formation of two split-join chain sets given as:

$$\begin{aligned}\lambda_y[A_{p(i)}, A_{p(k)}]_1 &\equiv A_{I(y)} \cup B_{F(h(y))}, \\ &\text{and,} \\ \lambda_y[A_{p(i)}, A_{p(k)}]_2 &\equiv A_{F(y)} \cup B_{I(h(y))}.\end{aligned}$$

Thus, the corresponding two sets of simply directed homotopy chain-paths are admitted within  $p^{-1}(U_x)$  depending upon the algebraic ordering relation, which is given as follows.

$$\begin{aligned}[* \equiv <] &\Rightarrow \\ (\lambda_y[[A_{p(k)}, A_{p(i)}], <]_1 &\equiv [B_{I(h(y))}] *_H [A_{F(y)}]; \\ \lambda_y[[A_{p(k)}, A_{p(i)}], <]_2 &\equiv [\overline{B_{F(h(y))}}] *_H [\overline{A_{I(y)}}]. \\ &\text{and,} \\ [* \equiv >] &\Rightarrow \\ (\lambda_y[[A_{p(k)}, A_{p(i)}], >]_1 &\equiv [\overline{B_{I(h(y))}}] *_H [\overline{A_{F(y)}}]; \\ \lambda_y[[A_{p(k)}, A_{p(i)}], >]_2 &\equiv [B_{F(h(y))}] *_H [A_{I(y)}].\end{aligned}\tag{5}$$

Hence, the covering spaces  $p^{-1}(U_x)$  admits two simply directed homotopy chain-paths  $\lambda_y[[A_{p(k)}, A_{p(i)}], *, *]_1$  and  $\lambda_y[[A_{p(k)}, A_{p(i)}], *, *]_2$  at a split-join point  $y$ .  $\square$

**Remark 5.** *It is easy to observe that a simply directed homotopy chain-path in the covering spaces maintains a path-homotopy property between the two split-join sets irrespective of the algebraic ordering relation. However, the two different simply directed homotopy chain-paths in the covering spaces are not in mutual path-homotopy. This results in the following corollary representing the topological properties of two homotopy chain-paths in view of algebraic topology.*

**Corollary 1.** In a topological space  $(Y = E \cup F, \tau_Y)$  two split-join homotopy chain-paths  $\lambda_y[[A, B], *]$  and  $\lambda_y[[B, A], *]$  do not mutually commute at any split-join point  $y$ , where  $E \cap F \neq \varnothing$  and  $A \cup B \subset Y$ .

**Theorem 4.** If  $p : X \rightarrow U_x$  is an algebraically chained covering map and  $q : Y \rightarrow U_x$  is a twisted chained covering map of  $(U_x \subset R^n, f)$  then the bijection  $h : (A_{p(i)} \subset S_{p(i)}^o, <) \rightarrow (A_{q(i)} \subset S_{q(i)}^o, >)$  forms two oriented homotopy chain-paths in  $S_{p(i)} \cup S_{q(i)}$ .

**Proof.** Let  $(U_x \subset R^n, f)$  be a chart map of an  $n$ -manifold  $M^n$ . Suppose  $p : X \rightarrow U_x$  and  $q : Y \rightarrow U_x$  are two covering maps such that  $X \cap Y = \varnothing$  and  $p^{-1}(U_x)$  generates chained covering spaces, whereas  $q^{-1}(U_x)$  generates the corresponding twisted chained covering spaces. Accordingly, let us consider a bijection  $h : (A_{p(i)} \subset S_{p(i)}^o, <) \rightarrow (A_{q(i)} \subset S_{q(i)}^o, >)$ , where  $S_{p(i)} \in p^{-1}(U_x)$  and  $S_{q(i)} \in q^{-1}(U_x)$ . If we choose a split-join point  $x \in A_{p(i)}$ , then by following the conditions of oriented homotopy chain-paths we can algebraically construct two homotopy chain-paths as follows.

$$\begin{aligned} A_{I(x)}, A_{F(x)} &\subset A_{p(i)}, \\ B_{I(h(x))}, B_{F(h(x))} &\subset A_{q(i)}, \\ \lambda_x[[A_{p(i)}, A_{q(i)}], *]_1 &\equiv [A_{I(x)}] *_H [B_{I(h(x))}], \\ \lambda_x[[A_{p(i)}, A_{q(i)}], *]_2 &\equiv [A_{F(x)}] *_H [B_{F(h(x))}]. \end{aligned} \quad (6)$$

Hence, the bijection  $h : (A_{p(i)} \subset S_{p(i)}^o, <) \rightarrow (A_{q(i)} \subset S_{q(i)}^o, >)$  forms two topologically oriented homotopy chain-paths in  $S_{p(i)} \cup S_{q(i)}$ .  $\square$

Note that Theorem 3 and Theorem 4 give rise to a set of path-homotopies, which are algebraically chained topological subspaces in covering spaces under split-join operations. This set of path-homotopies of chain-paths is given as:

$$W = \left\{ \lambda_y[[A_{p(k)}, A_{p(i)}], *]_1, \lambda_y[[A_{p(k)}, A_{p(i)}], *]_2, \lambda_x[[A_{p(i)}, A_{q(i)}], *]_1, \lambda_x[[A_{p(i)}, A_{q(i)}], *]_2 \right\}.$$

Moreover, the aforesaid theorem indicates that embedding of a homotopy chain-path in a  $M^n$  depends on the nature of covering spaces and this observation leads to the following lemma.

**Lemma 2.** If an algebraically chained topological subspace  $(A \subset U_x, *)$  can be embedded into  $M^n$  by a continuous injective function  $i_{em} : A \rightarrow M^n$  then either the covering map  $p : X \rightarrow U_x$  is an algebraic chain map or a twisted chain map, but not both.

**Proof.** The proof is relatively straightforward, and we prove it by following the method of contradiction. Suppose we consider a covering map  $p^{-1} : U_x \rightarrow X$  which generates both the chained covering spaces and also the twisted covering spaces. Let us consider an algebraic chained topological subspace  $(A \subset U_x, *)$  and a chain embedding under continuous injective function  $i_{em} : A \rightarrow M^n$ . In this case there are two covering sections  $\{i, k\} \subset \Lambda, \{S_{p(i)}, S_{p(k)}\} \subset p^{-1}(U_x)$  such that  $A \subset (p(S_{p(i)}) \cap p(S_{p(k)}))$ . As the covering map  $p : X \rightarrow U_x$  is a surjection, thus if  $* \equiv <$  and  $* \equiv >$  then it leads to the conclusion that  $(A \subset U_x, *)$  is not a chain and  $i_{em} : A \rightarrow M^n$  does not exist, which is a contradiction. Hence, the covering map  $p : X \rightarrow U_x$  can be either an algebraic chain map or a twisted chain map, but not both.  $\square$

Observe that the covering map is a variety of an evenly covering map of  $(U_x \subset R^n, f)$ . We can further enrich the algebraic properties of the chained covering spaces of a chart map  $(U_x \subset R^n, f)$  of  $M^n$ . For example, the set of Hausdorff topological subspaces  $W_{xy} = W \cup \{\{x\}, \{y\}\}$  gives rise to a multiplicative group structure  $G_\lambda = (W_{xy}, \mu_\lambda, *)$

under the corresponding finite real-valued measure  $\mu_\lambda : W_{xy} \rightarrow (0, +\infty)$ . This observation is presented in the following theorem.

**Theorem 5.** *The set of Hausdorff topological subspaces  $W_{xy}$  forms a 5th order multiplicative group  $G_\lambda = (W_{xy}, \mu_\lambda, *)$  in chained covering spaces under the finite real-valued measure  $\mu_\lambda : W_{xy} \rightarrow (0, +\infty)$ .*

**Proof.** Let  $W_{xy} = W \cup \{\{x\}, \{y\}\}$  be a set of chained Hausdorff topological subspaces in the covering spaces containing split-sets and  $\mu_\lambda : W_{xy} \rightarrow (0, +\infty)$  be a real-valued finite measure. Note that the  $\mu_\lambda(W_{xy}) > 0$  condition is maintained by the measure. Suppose we fix  $\mu_\lambda(\{x\}) = \mu_\lambda(\{y\}) = 1$  considering the split-sets of cardinalities equal to 1. Hence, the measure  $\mu_\lambda(W_{xy}) = \{\mu_\lambda(\{x\})\} \cup \mu_\lambda(W)$  forms a 5th order multiplicative group structure  $G_\lambda = (W_{xy}, \mu_\lambda, *)$  if  $\forall r_i \exists r_k \in \mu_\lambda(W)$ , the real-valued measure preserves the condition  $\mu_\lambda(\{x\}) = r_i * r_k$ , where  $\mu_\lambda(W) = \{\mu_\lambda(w \in W)\}$ .  $\square$

This immediately leads to the following lemma considering  $\{0\} \subset \mathbb{Z}^+$ .

**Lemma 3.** *If the cyclic group  $G_z = (W_{xy}, \mu_z, +) \cong (Z_6, +)$  is formed under the finite measure,  $\mu_z : W_{xy} \rightarrow \mathbb{Z}^+ \setminus \{+\infty\}$  then  $\mu_z(\{x\}) \neq \mu_z(\{y\})$ .*

**Proof.** The proof is relatively straightforward. If  $G_z = (W_{xy}, \mu_z, +) \cong (Z_6, +)$  is a 6th order cyclic group then  $\mu_z(W_{xy}) = \{i : i \in [0, 5]\}$  admitting cyclic  $(Z_6, +)$  with generator  $\langle g \rangle \in \{1, 5\}$ . As a result, we can conclude that  $\mu_z(\{x\}) \neq \mu_z(\{y\})$ .  $\square$

**Corollary 2.** *Every  $x \in U_x$  in Hausdorff  $(U_x \subset \mathbb{R}^n, f)$  is finitely measurable under some real-valued measure  $\mu : U_x \rightarrow [0, +\infty)$ .*

**Proof.** Let us consider that  $(X \equiv \mathbb{R}^n, \tau_X)$  is a Hausdorff topological space and the covering spaces  $p^{-1}(U_x), q^{-1}(U_x)$  are measure preserving. If  $\exists x \in U_x$  in  $(U_x \subset \mathbb{R}^n, f)$  such that  $\mu(\{x\}) \rightarrow +\infty$  then  $\forall \{y\} \in p^{-1}(\{x\})$  the following condition will result:  $(\mu \circ p^{-1})(\{x\}) \rightarrow +\infty$ . It leads to two contradictions such as, (1) there are  $S_{p(i)} \in p^{-1}(U_x)$  which are not locally compactable if  $\mu_\lambda = \mu$  and, (2) the generator  $\langle g \rangle$  cannot form a cyclic group algebraic structure if  $\mu_z = \mu$ . Hence, we conclude that the measure  $\mu : U_x \rightarrow [0, +\infty)$  is admitted.  $\square$

## 5. Conclusions

The covering spaces of a topological space endowed with algebraic order relations as well as algebraic operations within subspaces give rise to a set of interesting topological properties. This paper proposes the formulation of path-homotopy of linearly ordered chains within the covering spaces of a foliated  $n$ -manifold chart under the algebraic split-join operation on chains. The split-join algebraic operations form simply directed chains and oriented chains within the covering subspaces. It results in the formation of chained covering subspaces retaining topological homeomorphism between two covering maps, where the homeomorphism is an oriented isomorphism. The homotopic path-products between various chained covering subspaces under bijection, generate the corresponding simply directed or oriented chain-paths, which are finitely measurable in the Hausdorff topological space. We show that the embeddings of split-join chains on an  $n$ -manifold under homotopy path-product need to be homogeneous and unique. It is shown that the non-homogeneous set of chains under the split-join operation and homotopy path-product do not mutually commute. Interestingly, different varieties of finite measures of homotopy chain-paths establish two varieties of groups such as, a fifth-order multiplicative group structure and a sixth-order cyclic group structure. The distinctive properties of the proposed formulation are that it does not include the formation of circular nerves and Nachbin embeddings of preordered sets. Moreover, the algebraic chains are formed without considering inclusion-order of open sets in the respective topological spaces. However, the elements of Riesz ordering in  $n$ -spaces, Jordan chain components, and interpretations of

Urysohn about chains have influenced the formulation. Finally, this is to emphasize that the chain-paths formed under split-join algebraic operations in  $n$ -spaces do not follow the concept of separating chains within the covering spaces under the homotopy path-products of sets of chains.

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