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**Abstract**

We consider a fluid-structure interaction system coupling a viscous fluid governed by the compressible Navier-Stokes equations and a rigid body immersed in the fluid and modeled by the Newton's law. In this work, we consider the Navier slip boundary conditions. Our aim is to show the local in time existence and uniqueness of the strong solution to the corresponding problem. The main step of this work is that we use Lagrangian change of variables in order to handle the transport equation and to reduce the problem in the initial domain. However, the specificity here is that the Lagrangian coordinates do not coincide with the Eulerian coordinates at the boundaries since we consider slip boundary conditions. Therefore, it brings some extra nonlinear terms in the boundary conditions. The strategy is based on the study of the linearized system with nonhomogeneous boundary conditions and on the Banach fixed point theorem.

**Keywords:** Compressible Navier-Stokes system; Rigid body motion; Strong solutions.
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# 2 1 Introduction and main result

3 Let $\Omega$ be a bounded domain of $\mathbb{R}^3$. At each time $t$, the fluid domain is denoted by $\mathcal{F}(t)$ and we consider a rigid
4 body $\mathcal{S}(t)$ immersed in the fluid region such that $\mathcal{F}(t) = \Omega\backslash\overline{\mathcal{S}(t)}$. We assume that $\mathcal{F}(t)$ is a connected set with
5 a smooth enough boundary. The problem is described by the following fluid structure interaction system

$$\begin{cases} \partial_t\widetilde{\rho} + \nabla\cdot(\widetilde{\rho}\widetilde{v}) = 0 & \text{in } \mathcal{F}(t), \quad t>0, \\ \widetilde{\rho}(\partial_t\widetilde{v} + (\widetilde{v}\cdot\nabla)\widetilde{v}) - \nabla\cdot(\sigma(\widetilde{v},\widetilde{p})) = 0 & \text{in } \mathcal{F}(t), \quad t>0, \\ m\ell'(t) = -\displaystyle\int_{\partial\mathcal{S}(t)} \sigma(\widetilde{v},\widetilde{p})\widetilde{n}\, d\Gamma & t>0, \\ J\omega'(t) = J\omega(t)\times\omega(t) - \displaystyle\int_{\partial\mathcal{S}(t)} (x-h(t))\times\sigma(\widetilde{v},\widetilde{p})\widetilde{n}\, d\Gamma & t>0, \end{cases} \tag{1.1}$$

6 with the slip boundary conditions

$$\begin{cases} \widetilde{v}_{\widetilde{n}} = 0, \quad [2\mu\mathbb{D}(\widetilde{v})\widetilde{n} + \beta_0\widetilde{v}]_{\widetilde{\tau}} = 0 & \text{on } \partial\Omega, \quad t>0, \\ (\widetilde{v}-\widetilde{v}_{\mathcal{S}})_{\widetilde{n}} = 0, \quad [2\mu\mathbb{D}(\widetilde{v})\widetilde{n} + \beta_1(\widetilde{v}-\widetilde{v}_{\mathcal{S}})]_{\widetilde{\tau}} = 0 & \text{on } \partial\mathcal{S}(t), \quad t>0, \end{cases} \tag{1.2}$$

7 and the initial conditions

$$\begin{cases} \widetilde{\rho}(0,\cdot) = \widetilde{\rho}^0, \quad v(0,\cdot) = \widetilde{v}^0 & \text{in } \mathcal{F}(0), \\ h(0) = 0, \quad \ell(0) = \ell^0, \quad Q(0) = \mathbb{I}_3, \quad \omega(0) = \omega^0. \end{cases} \tag{1.3}$$

The function $\widetilde{\rho}$ represents the density of the fluid while $\widetilde{v}$ denotes the fluid velocity and $\widetilde{p}$ is the fluid pressure. The Cauchy stress tensor is defined by

$$\sigma(\widetilde{v},\widetilde{p}) = \mathbb{S}(\widetilde{v}) - \widetilde{p}\mathbb{I}_3,$$

with

$$\mathbb{S}(\widetilde{v}) = 2\mu\mathbb{D}(\widetilde{v}) + \lambda(\nabla\cdot\widetilde{v})\mathbb{I}_3, \quad \mathbb{D}(\widetilde{v}) = \frac{1}{2}\left(\nabla\widetilde{v} + (\nabla\widetilde{v})^\top\right),$$

where $\mu > 0$ and $\lambda + \mu \geqslant 0$ are constants and represent respectively the dynamic and the bulk viscosities of the fluid. The pressure $\widetilde{p} = \widetilde{p}(\widetilde{\rho})$ is given by the following constitutive law:

$$p(\widetilde{\rho}) = a_\rho\widetilde{\rho}^\gamma, \quad a_\rho > 0,\ \gamma \geqslant 1.$$

The vector $h(t)$ and the matrix $Q(t)$ represent the translation of the center of mass and the rotation of the rigid body. Note that we assume without loss of generality that $h(0) = 0$ and $Q(0) = \mathbb{I}_3$. Then, at time $t$, the domain $\mathcal{S}(t)$ occupied by the structure is described as follows

$$\mathcal{S}(t) = h(t) + Q(t)\mathcal{S},$$

and we denote the associated region occupied by the fluid $\mathcal{F}(t) = \Omega\backslash\overline{\mathcal{S}(t)}$. Then, the domain of the rigid body and the fluid are given as follows at initial time

$$\mathcal{S}(0) = \mathcal{S}, \quad \mathcal{F}(0) = \Omega\backslash\overline{\mathcal{S}} = \mathcal{F}.$$

8 In the sequel, $n$ stands for the unit normal vector at the boundary $\partial\mathcal{F}$. Since $Q$ is an orthogonal matrix, we get

$$Q'(t) = \mathbb{A}(t)Q(t), \quad \mathbb{A}(t)z = \omega(t)\times z,\ z\in\mathbb{R}^3, \tag{1.4}$$

9 where $\omega$ is the angular velocity of the rigid body. Hence, the velocity of the structure is given by

$$\widetilde{v}_{\mathcal{S}}(t,x) = \ell(t) + \omega(t)\times(x-h(t)), \quad \ell(t) = h'(t), \quad t>0, \quad x\in\partial\mathcal{S}(t). \tag{1.5}$$

The constant $m$ and the tensor $J$ represent the mass and the moment of inertia of the rigid body such that

$$J(t) = \int_{\mathcal{S}(t)} \rho_{\mathcal{S}} \left( |x - h(t)|^2 \mathbb{I}_3 - (x - h(t)) \otimes (x - h(t)) \right) \, dx,$$

where $\rho_{\mathcal{S}}$ is the structure density. The constants $\beta_0$ and $\beta_1$ are the friction coefficients verifying $\beta_0 + \beta_1 > 0$. The vector $\widetilde{n}$ is the outer unit normal vector on $\partial\mathcal{F}(t)$. For any vector $z \in \mathbb{R}^3$, we define the normal and the tangential components of $z$ as follows

$$z_{\widetilde{n}} = (z \cdot \widetilde{n}) \cdot \widetilde{n}, \quad z_{\widetilde{\tau}} = (\mathbb{I}_3 - \widetilde{n} \otimes \widetilde{n}) z.$$

We furnish the initial data $(\rho^0, v^0, \ell^0, \omega^0)$ that satisfy the following conditions

$$\operatorname{dist}(\mathcal{S}, \partial\Omega) > 0, \tag{1.6}$$

$$p^0 = a_\rho (\rho^0)^\gamma, \quad v^0 \in [H^3(\mathcal{F})]^3, \quad \rho^0 \in H^3(\mathcal{F}), \quad \min_{\overline{\mathcal{F}}} \rho^0 > 0, \tag{1.7}$$

$$v_{\mathcal{S}}^0 = \ell^0 + \omega^0 \times y, \quad \begin{cases} v_n^0 = 0, & \left[2\mu \mathbb{D}(v^0) n + \beta_0 v^0\right]_\tau = 0 \quad \text{on } \partial\Omega, \\ (v^0 - v_{\mathcal{S}}^0)_n = 0, & \left[2\mu \mathbb{D}(v^0) n + \beta_1 (v^0 - v_{\mathcal{S}}^0)\right]_\tau = 0 \quad \text{on } \partial\mathcal{S}, \end{cases} \tag{1.8}$$

$$\begin{cases} \left( \dfrac{1}{\rho^0} \nabla \cdot \sigma(v^0, p^0) \right) \cdot n = -v^0 \cdot (\nabla n v^0) \quad \text{on } \partial\Omega, \\ \left( \dfrac{1}{\rho^0} \nabla \cdot \sigma(v^0, p^0) \right) \cdot n = \omega^0 \times (v^0 - \ell^0) \cdot n + (v_{\mathcal{S}}^0 - v^0) \cdot \left( \omega^0 \times n + \nabla n (v^0 - v_{\mathcal{S}}^0) \right) \\ + \left( -m^{-1} \displaystyle\int_{\partial\mathcal{S}(0)} \sigma(v^0, p^0) n \, d\Gamma + J^{-1}(0) \left( J(0)\omega^0 \times \omega^0 - \displaystyle\int_{\partial\mathcal{S}(0)} y \times \sigma(v^0, p^0) n \, d\Gamma \right) \times y \right) \cdot n \quad \text{on } \partial\mathcal{S}. \end{cases} \tag{1.9}$$

**Remark 1.1.** *In order to state the main result, we consider $X$ a $C^1$-diffeomorphism that transforms $\overline{\mathcal{F}(0)} = \overline{\mathcal{F}}$ into $\overline{\mathcal{F}(t)}$ that is defined more precisely in Section 2 and we denote by $v$ and $\rho$ the corresponding functions to $\widetilde{v}$ and $\widetilde{\rho}$ using the change of variables $X$*

$$v(t, \cdot) = \widetilde{v}(t, X(t, \cdot)), \quad \rho(t, \cdot) = \widetilde{\rho}(t, X(t, \cdot)).$$

The main theorem of this paper is stated as follows

**Theorem 1.2.** *Let $\rho^0$, $v^0$, $\ell^0$, $\omega^0$ the initial data that satisfy* (1.6), (1.7), (1.8) *and* (1.9). *Then, there exists $T > 0$ such that the system* (1.1), (1.2), (1.3) *admits a unique strong solution with*

$$\rho \in H^1(0, T; H^3(\mathcal{F})) \cap C^1(0, T; H^2(\mathcal{F})) \cap H^2(0, T; H^1(\mathcal{F})),$$

$$v \in L^2(0, T; [H^4(\mathcal{F})]^3) \cap H^2(0, T; [L^2(\mathcal{F})]^3) \cap C(0, T; [H^3(\mathcal{F})]^3) \cap H^1(0, T; [H^2(\mathcal{F})]^3) \cap C^1(0, T; [H^1(\mathcal{F})]^3),$$

$$\ell \in [H^2(0, T)]^3, \quad \omega \in [H^2(0, T)]^3.$$

**Remark 1.3.** *The friction coefficients can be non negative functions such that $\beta_0 + \beta_1 \geqslant \alpha > 0$ and regular enough to be estimated in the fixed point procedure. In [1], the authors consider the incompressible Navier-Stokes system and proved asymptotic results when the friction coefficient tends to 0 or $\infty$ and this is an interesting open problem that could be investigated in the case of fluid-structure interaction system.*

**Remark 1.4.** *We prove the existence and the uniqueness of the solution provided that $T$ is small enough to ensure that the Change of variables $X$ is a $C^1$ diffeomorphism mapping such that*

$$\det(\nabla X) \geqslant \kappa > 0,$$

*where the constant $\kappa$ depends on $d(\mathcal{S}, \partial\Omega)$. Moreover, we need to fix $T$ small enough to get the contraction mapping in Section 4.*

**Remark 1.5.** *To get* (1.9)*, we use the Lagrangian change of variables* $X$ *that is described with details in Section 2. Note that the normal vector* $\widetilde{n}$ *is given by*

$$\widetilde{n}(t, X(t,\cdot)) = \begin{cases} n(X(t,\cdot)) & \text{on } \partial\Omega, \\ Q(t)n(Q(t)^\top(X(t,\cdot) - h(t))) & \text{on } \partial\mathcal{S}. \end{cases}$$

*All this ingredients allow us to take the time derivative of the impermeability conditions in* (1.2) *and deduce* (1.9)*.*

The slip boundary conditions were introduced first by Navier in [61] and they are generally used to model the motion of the fluid at the boundaries eventually for collision problems, see [44, 34, 74, 45]. We mention [51] where the authors give an interesting investigation about the slip and no slip boundary conditions. In the mathematical literature, the survey of this type of coupling problems is tackled. In the case of an incompressible fluid-rigid body systems, there are many interesting works in the literature considering no slip boundary conditions, see for example [17, 67, 22, 23, 24, 31, 32, 41, 71, 46, 72, 73]. In the case of slip boundary conditions, we have [33, 63, 15, 15, 20]. The controllability of the fluid-rigid body problem is studied in [14, 65, 12, 48] considering no slip boundary conditions whereas in the case of Navier boundary conditions, we mention [26]. The stabilization problem is discussed in [3, 2]. In the case of a compressible fluid-solid interaction system with no slip conditions, the global existence of weak solutions are investigated in [25, 29] and in [11] for strong solutions stated in the Hilbert space framework. In the $L^p - L^q$ setting, the study of the interaction fluid-rigid body problem is investigated in [43, 42] assuming no slip boundary conditions. Furthermore, the analysis of the motion of a piston in a viscous gaz modeled by the 1D compressible Navier-Stokes system is studied in [57]. The stabilization problem is investigated in [66] in the Hilbert setting and a global result on stabilization in $L^p - L^q$ framework is done in [55]. Concerning compressible Navier-Stokes equations without structure involving slip boundary conditions, existence of local strong solutions in the $L^p - L^q$ framework is tackled in [60], whereas the existence of global strong solutions is obtained in [68]. In the case of the compressible Navier-Stokes equations set in a moving domain, weak and strong solutions are studied in [49] under slip or no slip boundary conditions. Moreover, in the $L^p - L^q$ setting, the compressible problem in a moving domain is studied in [50]. In [62], the existence of a weak solution to fluid-solid interaction problem is proved considering slip boundary conditions.

Let us mention some works that are devoted to the case of an elastic structure. In the case of an incompressible fluid-beam interaction system with Dirichlet boundary conditions, the existence of weak solutions was investigated in [19] considering a damping in the structure equation and [35] in the case where the structure satisfies the plate equation without any damping. The existence of strong solutions for the fluid-damped beam system was established in [8, 52]. For global in time solutions, we mention [36] where the authors proved no contact between the beam and the domain cavity. In [37], the existence of strong solutions is proved whenever the structure satisfies either a damped beam equation, a linear wave equation or a linear beam equation with inertia of rotation. In [7], the authors considered the incompressible fluid-wave interaction system and showed maximal regularity for the solutions, they proved in particular that the corresponding semigroup of the linearized system is analytic. In [5, 6], the authors obtained Gevrey regularity for a system coupling incompressible Navier Stokes system with a beam equation. For the stabilization problem, we refer to [64, 4]. In the case of boundary conditions involving pressure, regularity results are obtained in [18]. In the case of an incompressible fluid-beam interaction system with Navier boundary conditions, the existence of weak solutions is investigated in [40, 39, 59] whereas the existence of strong solutions is discussed in [27]. In the case of a compressible fluid and an elastic structure, weak solutions are obtained in [10], local existence of strong solutions is obtained in [58] for the coupled system considering a damped beam equation and assuming that the initial displacement of the structure is 0. Maximal-regularity in $L^p - L^q$ framework is obtained in [56] where the structure is modeled by the damped beam equation and in the case where the structure verifies the wave equation, the existence and uniqueness of the strong solution are proved in [54] in the $L^2$ approach.

The objective of this paper is to provide local existence and uniqueness of the strong solution to the system (1.1), (1.2), (1.3) in the Hilbert space framework. In our knowledge this is the first work of strong solutions of a compressible fluid structure system involving slip boundary conditions in the Hilbert setting. Note that it could be very interesting to study the maximal regularity of the system by following [60] and [43] and in that

case, no compatibility condition is required. In section 2, we write the system in a fixed configuration by using the Lagrangian change of variables. Since the system considered here is compressible, the change of variables that is commonly used is the Lagrangian flow to transform the domain $\mathcal{F}(t)$ into $\mathcal{F}$. In order to obtain a local solution, it is crucial to go back in the initial domain so that we can use the classical Banach fixed point theorem. However, in the Dirichlet case (see [66, 42]), we have the following correspondence on the boundary

$$X(t,y) = \begin{cases} h(t) + Q(t)y & y \in \partial\mathcal{S}, \\ y & y \in \partial\Omega, \end{cases}$$

that is not true considering slip boundary conditions. Due to this change of variables, nonlinear terms appear in the whole system even in the boundary conditions that cannot be avoided even in the case without structure (see [68, 60]). In Section 3, we study the linearized system. We first make a suitable lifting at the boundary conditions to handle the nonlinearities coming from the change of variables and we establish a first regularity result for the Linearized system with nonhomogeneous boundary conditions. An important step in this section is to elaborate elliptic estimates for the associated Lamé system. In section 4, we use a Banach fixed point argument and prove Theorem 1.2.

# 2 Change of variables

Let define $X(t,\cdot)$ as the Lagrangian change of variables associated with the fluid velocity $\widetilde{v}$ that reduces $\mathcal{F}(0) = \mathcal{F}$ into $\mathcal{F}(t)$

$$\begin{cases} \partial_t X(t,y) = \widetilde{v}(t, X(t,y)) & y \in \mathcal{F},\ t > 0, \\ X(0,y) = y & y \in \overline{\mathcal{F}}. \end{cases} \tag{2.1}$$

Let recall that $\rho$ and $v$ are the new corresponding functions to $\widetilde{\rho}$ and $\widetilde{v}$ after the change of variables (2.1)

$$\rho(t,\cdot) = \widetilde{\rho}(t, X(t,\cdot)), \quad v(t,\cdot) = \widetilde{v}(t, X(t,\cdot)).$$

We introduce the following notations

$$b_X = \operatorname{Cof} \nabla X, \quad \delta_X = \det \nabla X.$$

After some standard calculations, we have

$$\mathbb{D}(\widetilde{v}) = \mathbb{D}^X(v) = \frac{1}{2\delta_X}\left(\nabla v b_X^\top + (\nabla v b_X^\top)^\top\right), \quad \mathbb{S}(\widetilde{v}) = \mathbb{S}^X(v) = 2\mu\mathbb{D}^X(v) + \lambda\frac{1}{\delta_X} b_X : \nabla v,$$

and

$$\sigma^X(v) = \mathbb{S}^X(v) b_X, \quad \nabla \cdot (\mathbb{S}(\widetilde{v})) = \frac{1}{\delta_X}\nabla \cdot \left(\sigma^X(v)\right).$$

From [21, Theorem 1.1-1, p.13], we get the following relation

$$\widetilde{n}_X(t,\cdot) = \widetilde{n}(t, X(t,\cdot)) = \frac{1}{|b_X(t,\cdot)n|} b_X(t,\cdot)n, \tag{2.2}$$

where $n$ is the unitary exterior normal to $\partial\mathcal{F}$. We deduce

$$\widetilde{\tau}_X^i(t,\cdot) = \widetilde{\tau}^i(t, X(t,\cdot)) = |b_X(t,\cdot)n| b_X^{-\top}(t,\cdot)\tau^i, \quad i = 1,2,$$

with $(\widetilde{\tau}^i)_{i=1,2}$ (resp. $(\tau^i)_{i=1,2}$) designates a basis of the tangent subspace on each point of $\partial\mathcal{F}(t)$ (resp. $\partial\mathcal{F}$). The transport equation $(1.1)_1$ gives

$$\partial_t \rho + \frac{\rho}{\delta_X}\nabla v : b_X = 0. \tag{2.3}$$

Then, the system (1.1), (1.2) and (1.3) is rewritten in the Lagrangian coordinates as follows

$$\begin{cases} \partial_t \rho + \rho^0 \nabla \cdot v = \mathbb{T}(\rho, v) & \text{in } (0,T)\times\mathcal{F}, \\ \rho^0 \partial_t v - \nabla\cdot(\mathbb{S}(v)) = \nabla\cdot(\mathbb{F}_1(\rho,v)) + \mathbb{F}_2(\rho,v) & \text{in } (0,T)\times\mathcal{F}, \\ v_{\mathcal{S}}(y) = \ell + \omega \times y & \text{in } (0,T), \\ m\ell' = \mathbb{F}_3(\rho, v, \ell, \omega) & \text{in } (0,T), \\ J(0)\omega' = \mathbb{F}_4(\rho, v, \ell, \omega) & \text{in } (0,T), \end{cases} \tag{2.4}$$

with the boundary conditions

$$\begin{cases} v\cdot n = \mathbb{B}_{\partial\Omega}(\rho, v, \ell, \omega), & [(2\mu\mathbb{D}(v)n + \beta_0 v)]\cdot\tau^i = (\mathbb{G}_{\partial\Omega}(\rho, v, \ell, \omega))_i & \text{on } (0,T)\times\partial\Omega, \\ (v - v_{\mathcal{S}})\cdot n = \mathbb{B}_{\partial\mathcal{S}}(\rho, v, \ell, \omega), & [(2\mu\mathbb{D}(v)n + \beta_1(v - v_{\mathcal{S}}))]\cdot\tau^i = (\mathbb{G}_{\partial\mathcal{S}}(\rho, v, \ell, \omega))_i & \text{on}(0,T)\times\partial\mathcal{S}, \end{cases} \tag{2.5}$$

where $i = 1, 2$ and the initial conditions

$$\rho(0,\cdot) = \rho^0 \text{ in } \mathcal{F}, \quad v(0,\cdot) = v^0 \text{ in } \mathcal{F}, \quad \ell(0) = \ell^0, \quad \omega(0) = \omega^0. \tag{2.6}$$

The functions $\mathbb{T}$, $\mathbb{F}_1$, $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$, $\mathbb{B}$ and $\mathbb{G}$ are given as follows

$$\mathbb{T}(\rho, v) = \rho^0 \nabla v : \mathbb{I}_3 - \frac{\rho}{\delta_X}\nabla v : b_X \quad \text{on } \mathcal{F}, \tag{2.7}$$

$$\mathbb{F}_1(\rho, v) = \sigma^X(v) - \mathbb{S}(v) - a_\rho \rho^\gamma b_X, \quad \mathbb{F}_2(\rho, v) = (\rho^0 - \rho\delta_X)\partial_t v \quad \text{on } \mathcal{F}, \tag{2.8}$$

$$\mathbb{F}_3(\rho, v, \ell, \omega) = -\int_{\partial\mathcal{S}} (\sigma^X(v) - a_\rho\rho^\gamma b_X) n \, d\Gamma, \tag{2.9}$$

$$\mathbb{F}_4(\rho, v, \ell, \omega) = (J(0) - J)\,\omega' + J\omega\times\omega - \int_{\partial\mathcal{S}} (X - h)\times(\sigma^X(v) - a_\rho\rho^\gamma b_X) n \, d\Gamma, \tag{2.10}$$

$$\mathbb{B}_{\partial\Omega}(v, \ell, \omega) = v\cdot(n - \widetilde{n}_X) \quad \text{on } \partial\Omega, \tag{2.11}$$

$$\mathbb{B}_{\partial\mathcal{S}}(v, \ell, \omega) = (v - v_{\mathcal{S}})\cdot(n - \widetilde{n}_X) + (\omega\times(X - \mathrm{Id}))\cdot\widetilde{n}_X - \omega\times h\cdot\widetilde{n}_X \quad \text{on } \partial\mathcal{S}, \tag{2.12}$$

$$[\mathbb{G}_{\partial\Omega}(v, \ell, \omega)]_i = 2\mu\mathbb{D}(v)n\cdot\tau^i - 2\mu\mathbb{D}^X(v)\widetilde{n}_X\cdot\widetilde{\tau}^i_X + \beta_0 v\cdot(\tau^i - \widetilde{\tau}^i_X), \ i = 1, 2 \text{ on } \partial\Omega, \tag{2.13}$$

$$\begin{aligned}[\mathbb{G}_{\partial\mathcal{S}}(v, \ell, \omega)]_i = {} & 2\mu\mathbb{D}(v)n\cdot\tau^i - 2\mu\mathbb{D}^X(v)\widetilde{n}_X\cdot\widetilde{\tau}^i_X + \beta_1(v - v_{\mathcal{S}})\cdot(\tau^i - \widetilde{\tau}^i_X) \\ & + \beta_1\omega\times(X - \mathrm{Id})\cdot\widetilde{\tau}^i_X - \beta_1\omega\times h\cdot\widetilde{\tau}^i_X, \ i = 1, 2 \text{ on } \partial\mathcal{S}. \end{aligned} \tag{2.14}$$

As in [60], we show that $X$ is $C^1$-diffeomorphism.

**Proposition 2.1.** *Under the boundary conditions* (2.5)*, if $v$ is regular and $T$ is small enough, the application $X$ defined in* (2.1) *is a $C^1$-diffeomorphism and transforms $\overline{\mathcal{F}(0)} = \overline{\mathcal{F}}$ into $\overline{\mathcal{F}(t)}$.*

*Proof.* From (2.1), we see that

$$\nabla X(t,\cdot) = \mathbb{I}_3 + \int_0^t \nabla v(s,\cdot)\, ds, \quad t\in(0,T).$$

If $T$ is small enough and $v$ regular enough, we see that $\nabla X$ is invertible with $\delta_X > 0$ in such a way the transformation $X$ preserves the orientation. Hence $X$ is a one to one mapping. Since $\partial\mathcal{S}$ is smooth, there exists $\eta_{\mathcal{S}}$ a smooth function such that $\eta_{\mathcal{S}} = 0$ in a part of $\partial\mathcal{S}$. Then, the corresponding part in $\partial\mathcal{S}(t)$ is locally expressed as $\xi(t,x) = \eta_{\mathcal{S}}(Q(t)^\top(x - h(t))) = 0$. Let prove that $X(t,\partial\mathcal{S})\subset\partial\mathcal{S}(t)$. Let $y$ be in the neighborhood of $\partial\mathcal{S}$ satisfying $\eta_{\mathcal{S}}(y) = 0$, we obtain

$$\partial_t[\xi(t, X(t,y))] = \nabla\xi(X(t,y))\cdot(\widetilde{v}(t, X(t,y)) - \widetilde{v}_{\mathcal{S}}(t, X(t,y))).$$

Since $\nabla\xi(X(t,\cdot))$ is collinear to $\widetilde{n}_X(t,\cdot)$, therefore from the impermeability boundary condition given in (2.5), we obtain that $\partial_t[\xi(t,X(t,y))]=0$. Thus

$$\xi(t,X(t,y))=\xi(0,X(0,y))=\eta_{\mathcal{S}}(y)=0.$$

Hence, we get $X(t,\partial\mathcal{S})\subset\partial\mathcal{S}(t)$ for $t\geqslant 0$. The same approach can be used to prove that $X(t,\partial\Omega)\subset\partial\Omega$. We obtain $X(t,\partial\mathcal{F})\subset\partial\mathcal{F}(t)$. Moreover, using the classical local inversion theorem, we see that $X(t,\partial\mathcal{F})$ is an open and a closed subset of $\partial\mathcal{F}(t)$. Since, $\partial\mathcal{F}(t)$ is a connected set, it follows that $X(t,\mathcal{F})=\partial\mathcal{F}(t)$. Finally, the injectivity of the mapping $X$ and the connectedness of $\mathcal{F}(t)$ yield that $X(t,\mathcal{F})=\mathcal{F}(t)$. □

# 3 Linearized system

The linearized system is given by

$$\begin{cases}\partial_t\rho+\rho^0\nabla\cdot v=h & \text{in } (0,T)\times\mathcal{F},\\ \rho(0,\cdot)=\rho^0 & \text{in } \mathcal{F},\end{cases}\tag{3.1}$$

$$\begin{cases}\rho^0\partial_t v-\nabla\cdot(\mathbb{S}(v))=\nabla\cdot f_1+f_2 & \text{in } (0,T)\times\mathcal{F},\\ v_{\mathcal{S}}(y)=\ell+\omega\times y & \text{in } (0,T),\\ m\ell'=f_3 & \text{in } (0,T),\\ J(0)\omega'=f_4 & \text{in } (0,T),\end{cases}\tag{3.2}$$

with the boundary conditions

$$\begin{cases}v\cdot n=a, & [(2\mu\mathbb{D}(v)n+\beta_0 v)]\cdot\tau^i=g_i,\ i=1,2 & \text{on}(0,T)\times\partial\Omega,\\ (v-v_{\mathcal{S}})\cdot n=a, & [(2\mu\mathbb{D}(v)n+\beta_1(v-v_{\mathcal{S}}))]\cdot\tau^i=g_i,\ i=1,2 & \text{on}(0,T)\times\partial\mathcal{S},\end{cases}\tag{3.3}$$

and the initial conditions

$$\rho(0,\cdot)=\rho^0 \text{ in } \mathcal{F},\quad v(0,\cdot)=v^0 \text{ in } \mathcal{F},\quad \ell(0)=\ell^0,\quad \omega(0)=\omega^0,\tag{3.4}$$

such that the following compatibility condition is satisfied

$$v^0\cdot n=a(0,\cdot)\ \text{ on } \partial\Omega,\quad (v^0-v^0_{\mathcal{S}})\cdot n=a(0,\cdot)\ \text{ on } \partial\mathcal{S}.\tag{3.5}$$

The aim of this section is to establish a regularity result for (3.2), (3.3) and (3.4). First, we need to give some regularity properties for the elliptic problem (3.6) associated to the Lamé operator.

## 3.1 Elliptic problem

Let consider the stationary system

$$\begin{cases}-\nabla\cdot\mathbb{S}(v)=f & \text{in } \mathcal{F},\\ v\cdot n=a & \text{on } \partial\mathcal{F},\\ [2\mu\mathbb{D}(v)n+\beta v]\cdot\tau^i=g_i & \text{on } \partial\mathcal{F}.\end{cases}\tag{3.6}$$

**Theorem 3.1.** *Let $\beta>0$ and assume that $\partial\mathcal{F}$ is $C^2$ boundary. Let*

$$f\in[L^2(\mathcal{F})]^3,\quad a\in H^{3/2}(\partial\mathcal{F}),\quad g_i\in H^{1/2}(\partial\mathcal{F}),\ i=1,2.$$

*Then, the system (3.6) admits a unique strong solution $v\in[H^2(\mathcal{F})]^3$ such that*

$$\|v\|_{[H^2(\mathcal{F})]^3}\leqslant C\left(\|f\|_{[L^2(\mathcal{F})]^3}+\|a\|_{H^{3/2}(\partial\mathcal{F})}+\|g_1\|_{H^{1/2}(\partial\mathcal{F})}+\|g_2\|_{H^{1/2}(\partial\mathcal{F})}\right).\tag{3.7}$$

*Moreover, if $\partial\mathcal{F}$ is $C^{m+2}$ boundary and*

$$f\in[H^m(\mathcal{F})]^3,\quad a\in H^{m+3/2}(\partial\mathcal{F}),\quad g_i\in H^{m+1/2}(\partial\mathcal{F}),\ i=1,2,$$

*the system (3.6) admits a unique strong solution $v\in[H^{m+2}(\mathcal{F})]^3$ such that*

$$\|v\|_{[H^{m+2}(\mathcal{F})]^3}\leqslant C\left(\|f\|_{[H^m(\mathcal{F})]^3}+\|a\|_{H^{m+3/2}(\partial\mathcal{F})}+\|g_1\|_{H^{m+1/2}(\partial\mathcal{F})}+\|g_2\|_{H^{m+1/2}(\partial\mathcal{F})}\right).\tag{3.8}$$

The rest of this subsection is devoted to the proof of Theorem 3.1.

**Step 1: Weak solutions.** First, we study the existence of week solutions for the system (3.6). Let define the following space

$$\mathbb{V} = \{v \in [H^1(\mathcal{F})]^3;\ v \cdot n = 0 \text{ on } \partial\mathcal{F}\}.$$

Let $f \in \mathbb{V}'$, $a \in H^{1/2}(\partial\mathcal{F})$ and $g \in H^{-1/2}(\partial\mathcal{F})$. We multiply the first equation of (3.6) by $w \in \mathbb{V}$ to obtain

$$\int_{\mathcal{F}} \mathbb{S}(v) : \nabla w \ dy + \int_{\partial\mathcal{F}} \beta v \cdot w \ d\Gamma = \langle f, w\rangle_{\mathbb{V}',\mathbb{V}} + \langle g, w\rangle_{H^{-1/2},H^{1/2}}, \tag{3.9}$$

such that $g$ is a vector satisfying $g \cdot n = 0$ and $g \cdot \tau^i = g_i$ for $i = 1, 2$. We consider a lifting $w_\ell \in [H^1(\mathcal{F})]^3$ satisfying

$$w_\ell \cdot n = a \quad \text{on } \partial\mathcal{F}, \tag{3.10}$$

and the estimate

$$\|w_\ell\|_{[H^1(\mathcal{F})]^3} \leqslant C \|a\|_{H^{1/2}(\partial\mathcal{F})}. \tag{3.11}$$

We set $v = u + w_\ell$. Then, we have the following definition.

**Definition 3.2.** *The function $v$ is a weak solution of the system (3.6) if and only if $u = v - w_\ell$ verifies the following variational formulation*

$$\int_{\mathcal{F}} \mathbb{S}(u) : \nabla w \ dy + \int_{\partial\mathcal{F}} \beta u \cdot w \ d\Gamma = -\int_{\mathcal{F}} \mathbb{S}(w_\ell) : \nabla w \ dy + \langle f, w\rangle_{\mathbb{V}',\mathbb{V}} - \int_{\partial\mathcal{F}} \beta w_\ell \cdot w \ d\Gamma + \langle g, w\rangle_{H^{-1/2},H^{1/2}}, \tag{3.12}$$

*for all $w \in \mathbb{V}$, where $w_\ell$ satisfies (3.10) and (3.11).*

We notice that

$$\int_{\mathcal{F}} \mathbb{S}(u) : \nabla u \ dy = \int_{\mathcal{F}} 2\mu \left|\mathbb{D}(u)\right|^2 \ dy + \int_{\mathcal{F}} \lambda \left|\nabla \cdot u\right|^2 \ dy.$$

Using Lemma A.1, we get for all $u \in \mathbb{V}$

$$\int_{\mathcal{F}} \mathbb{S}(u) : \nabla u \ dy + \int_{\partial\mathcal{F}} \beta \left|u\right|^2 \ d\Gamma \geqslant C\|u\|^2_{[H^1(\mathcal{F})]^3}. \tag{3.13}$$

Hence, we can apply the Lax-Milgram theorem and using (3.11), we deduce the existence of a unique solution $v$ of (3.6) which verifies the estimates

$$\|v\|_{[H^1(\mathcal{F})]^3} \leqslant C\left(\|f\|_{\mathbb{V}'} + \|a\|_{H^{1/2}(\partial\mathcal{F})} + \|g\|_{[H^{-1/2}(\partial\mathcal{F})]^3}\right). \tag{3.14}$$

**Step 2: Strong solutions in $H^2$.** Without loss of generality, we take $a = 0$ in (3.6) since we can always find a suitable lifting of the impermeability condition, see for example [38, p.128, Theorem 2.5.1.1]. The interior regularity can be done as the Dirichlet boundary condition, then we skip the proof.

First, let consider the case, where $\mathcal{F}$ is the half ball $\mathcal{B}(0,1) \cap \mathbb{R}^3_+$. The proof is standard and it is done by using the method of difference quotient as in [28]. Let $f \in [L^2(\Omega)]^3$, $g \in [H^{1/2}(\partial\mathcal{F})]^3$ and consider $v$ the solution of the variational formulation (3.9). Let define the difference quotient

$$D^h_k v(y) = \frac{v(y + he_k) - v(y)}{h},\ k = 1, 2, 3, \quad h \in \mathbb{R}^*.$$

We select $\zeta$, a cut off function such that

$$\zeta \equiv 1 \text{ in } \mathcal{B}(0, 1/2) \cap \mathbb{R}^3_+, \quad \zeta \equiv 0 \text{ in } \mathbb{R}^3 \backslash \mathcal{B}(0, 1),$$

so that $\zeta$ vanishes on the curved boundary of $\mathcal{B}(0,1)\cap\mathbb{R}^3_+$. We take the test function $w=-D_k^{-h}(\zeta^2D_k^hv)\in\mathbb{V}$ and replace it in (3.9). We obtain for $h$ small enough

$$\int_{\mathcal{F}}2\mu\zeta^2|\mathbb{D}D_k^hv|^2\,dy+\int_{\mathcal{F}}\lambda\zeta^2|\nabla\cdot D_k^hv|^2\,dy+\int_{\partial\mathcal{F}}\beta\zeta^2|D_k^hv|^2\,d\Gamma+2\mu\int_{\mathcal{F}}D_k^h\mathbb{D}(v):2\zeta\nabla\zeta D_k^hv\,dy$$
$$+\lambda\int_{\mathcal{F}}2\zeta(\nabla\zeta\cdot D_k^hv)(\nabla\cdot D_h^kv)\,dy=-\int_{\mathcal{F}}f\cdot D_k^{-h}(\zeta^2D_k^hv)\,dy-\int_{\partial\mathcal{F}}g\cdot D_k^{-h}(\zeta^2D_k^hv)\,d\Gamma.$$

Since

$$\|D_k^{-h}(\zeta^2D_k^hv)\|_{[L^2(\mathcal{F})]^3}\leqslant C\left(\|\zeta^2\nabla(D_k^hv)\|_{[L^2(\mathcal{F})]^9}+\|(D_k^hv)\|_{[L^2(\mathcal{F})]^9}\right),\tag{3.15}$$

we get

$$\int_{\mathcal{F}}f\cdot D_k^{-h}(\zeta^2D_k^hv)\,dy\leqslant\varepsilon\|\zeta^2\nabla(D_k^hv)\|^2_{[L^2(\mathcal{F})]^9}+C_\varepsilon\left(\|f\|^2_{[L^2(\mathcal{F})]^3}+\|g\|^2_{[H^{1/2}(\partial\mathcal{F})]^3}\right).$$

In the other hand, let extend the unit normal $n$ over $\mathcal{F}$ and $g$ by a function also denoted by $g\in[H^1(\mathcal{F})]^3$ . We have

$$-\int_{\partial\mathcal{F}}g\cdot D_k^{-h}(\zeta^2D_k^hv)\,d\Gamma=\sum_{i=1}^3\Big(-\int_{\mathcal{F}}\partial_i((n)_ig)\cdot D_k^{-h}(\zeta^2D_k^hv)\,dy+\int_{\mathcal{F}}D_k^h((n)_ig)\cdot\zeta^2(D_k^h\partial_iv)\,dy$$
$$-2\int_{\mathcal{F}}((n)_ig)\cdot\zeta\partial_i\zeta(D_k^hv)\,dy\Big).\tag{3.16}$$

Using (3.15), we get

$$\int_{\partial\mathcal{F}}g\cdot D_k^{-h}(\zeta D_k^hv)\,d\Gamma\leqslant\varepsilon\|\zeta^2\nabla(D_k^hv)\|^2_{[L^2(\mathcal{F})]^9}+C_\varepsilon\left(\|f\|^2_{[L^2(\mathcal{F})]^3}+\|g\|^2_{[H^{1/2}(\partial\mathcal{F})]^3}\right).\tag{3.17}$$

Hence gathering all these estimates, we obtain

$$\|D_k^hv\|_{[H^1(\Omega)]^3}\leqslant C\left(\|f\|_{[L^2(\mathcal{F})]^3}+\|g\|_{[H^{1/2}(\partial\mathcal{F})]^3}\right),$$

for $k=1,2$. Taking $h$ to zero, it yields

$$\|\partial^2_{kj}v\|_{[L^2(\mathcal{F})]^3}\leqslant C\left(\|f\|_{[L^2(\mathcal{F})]^3}+\|g\|_{[H^{1/2}(\partial\mathcal{F})]^3}\right),\ j\in\{1,2,3\},\ k=1,2.$$

Using the first equation in (3.6), we get finally

$$\|(2\mu+\lambda)\partial^2_{33}v\|_{[L^2(\mathcal{F})]^3}\leqslant C\sum_{k=1}^2\|\partial_kv\|_{[H^1(\mathcal{F})]^3}\leqslant C\left(\|f\|_{[L^2(\mathcal{F})]^3}+\|g\|_{[H^{1/2}(\mathcal{F})]^3}\right).$$

Now we consider the case when $\mathcal{F}$ is arbitrary. Let $y_0\in\partial\mathcal{F}$, since $\mathcal{F}$ is $C^2$, then there exist $r>0$ and $\eta:\mathbb{R}^2\longrightarrow\mathbb{R}$ of class $C^2$ such that

$$\mathcal{F}\cap\mathcal{B}(y_0,r)=\{y\in\mathbb{R}^3;\ y_3>\eta(y_1,y_2)\},\quad\partial\mathcal{F}\cap\mathcal{B}(y_0,r)=\{y\in\mathbb{R}^3;\ y_3=\eta(y_1,y_2)\}.$$

Let $Y_\eta:\mathcal{B}\longrightarrow\mathcal{F}\cap\mathcal{B}(y_0,r)$ and $Y_\eta^{-1}:\mathcal{F}\cap\mathcal{B}(y_0,r)\longrightarrow\mathcal{B}$, where $\mathcal{B}=\mathcal{B}(0,s)\cap\{z_3=0\}$ with $s$ small enough. We define

$$Y_\eta(z)=\begin{pmatrix}z_1\\z_2\\z_3+\eta(z_1,z_2)\end{pmatrix},\quad Y_\eta^{-1}(y)=\begin{pmatrix}y_1\\y_2\\y_3-\eta(y_1,y_2)\end{pmatrix}.$$

Let define $v_\eta=v(Y_\eta(\cdot))$ on $\mathcal{B}$. Then, the formulation in (3.9) gives

$$\int_{\mathcal{B}}\sigma_\eta(v_\eta):\nabla w_\eta\,dz+\int_{\partial\mathcal{B}}\beta v_\eta\cdot w_\eta\,d\Gamma=\int_{\mathcal{B}}\delta_\eta f_\eta\cdot w_\eta\,dz+\int_{\partial\mathcal{B}}g_\eta\cdot w_\eta\,d\Gamma,\tag{3.18}$$

where

$$b_\eta = \mathrm{Cof}\,\nabla Y_\eta, \quad \mathbb{D}_\eta(v_\eta) = \frac{1}{2\delta_\eta}\left(\nabla v_\eta(b_\eta)^\top + (\nabla v_\eta(b_\eta)^\top)^\top\right),$$

$$\mathbb{S}_\eta(v) = 2\mu\mathbb{D}_\eta(v) + \frac{\lambda}{\delta_\eta}(b_\eta : \nabla v_\eta)\mathbb{I}_3, \quad \sigma_\eta(v_\eta) = \mathbb{S}_\eta(v_\eta)b_\eta.$$

We emphasize here that for any $w \in \mathbb{V}$, we have $w_\eta \cdot n_\eta = 0$ on $\partial\mathcal{B} \cap \{z_3 = 0\}$. Note that (3.18) is coercive, in fact

$$\begin{aligned}\int_{\mathcal{B}} \sigma_\eta(v_\eta) : \nabla v_\eta \; dz &= \int_{\mathcal{B}} \frac{1}{2\delta_\eta}\mu\left|\nabla v_\eta(b_\eta)^\top + (\nabla v_\eta(b_\eta)^\top)^\top\right|^2 \; dz \\ &+ \int_{\mathcal{B}} \frac{1}{\delta_\eta}\lambda\left|b_\eta : \nabla v_\eta\right|^2 \; dz = \int_{\mathcal{B}} \delta_\eta\left(2\mu\left|\mathbb{D}_\eta(v_\eta)\right|^2\right) \; dy + \int_{\mathcal{B}} \frac{1}{\delta_\eta}\lambda\left|b_\eta : \nabla v_\eta\right|^2 \; dy \\ &= \int_{\mathcal{F}\cap\mathcal{B}(y_0,r)} 2\mu\left|\mathbb{D}(v)\right|^2 \; dy + \int_{\mathcal{F}\cap\mathcal{B}(y_0,r)} \lambda\left|\nabla\cdot v\right|^2 \; dy. \quad (3.19)\end{aligned}$$

Using Lemma A.1, we get for all $v_\eta \in [H^1(\mathcal{B})]^3$ satisfying $v_\eta \cdot n_\eta = 0$ on $\partial\mathcal{B} \cap \{z_3 = 0\}$

$$\int_{\mathcal{B}} \sigma_\eta(v_\eta) : \nabla v_\eta \; dy + \int_{\partial\mathcal{B}} \beta\left|v_\eta\right|^2 \; d\Gamma \geqslant C\|v_\eta\|^2_{[H^1(\mathcal{B})]^3}. \quad (3.20)$$

As the half ball case, we take $w_\eta = -D_k^{-h}(\zeta^2 D_k^h v_\eta)$ for $h$ small enough where $\zeta$ is a cut off function such that $\zeta \equiv 1$ in $\mathcal{B}(0, s/2) \cap \mathbb{R}^3_+$ and vanishes outside $\mathcal{B}$. Note that we have $w_\eta \cdot n_\eta = 0$ on $\mathcal{B} \cap \{z_3 = 0\}$. We get

$$\begin{aligned}\int_{\mathcal{B}} D_k^h(\sigma_\eta(v_\eta)) : \zeta^2\nabla D_k^h v_\eta \; dz &+ \int_{\partial\mathcal{B}\cap\{z_3=0\}} \beta\zeta^2|D_k^h v_\eta|^2 \; d\Gamma \\ &= -\int_{\mathcal{B}} \delta_\eta f_\eta \cdot D_k^{-h}(\zeta^2 D_k^h v_\eta) \; dz - \int_{\partial\mathcal{B}} g_\eta \cdot D_k^{-h}(\zeta^2 D_k^h v_\eta) \; d\Gamma. \quad (3.21)\end{aligned}$$

Repeating the previous steps as in the half ball case and using the coercivity of the operator $\nabla \cdot \sigma_\eta$ in $\mathcal{B}$ and the fact that $D_k^h(b_\eta)$ is bounded, we get

$$\|D_k^h v_\eta\|_{[H^1(\mathcal{B})]^3} \leqslant C\left(\|f_\eta\|_{[L^2(\mathcal{B})]^3} + \|g_\eta\|_{[H^{1/2}(\mathcal{B})]^3}\right). \quad (3.22)$$

We take $h$ to zero and thus we obtain

$$\|\partial^2_{kj}(v_\eta)\|_{[L^2(\mathcal{B})]^3} \leqslant C\left(\|f_\eta\|_{[L^2(\mathcal{B})]^3} + \|g_\eta\|_{[H^{1/2}(\partial\mathcal{B})]^3}\right), \; j \in \{1,2,3\}, \; k = 1,2.$$

Now, we go back to the equation of the system (3.6) that can be written under the change of variables $Y_\eta$

$$-\nabla\cdot(\sigma_\eta(v_\eta)) = \delta_\eta f_\eta, \quad \text{on } \mathcal{B}.$$

After some calculation, we obtain that

$$\|(a_{33}\mathbb{I}_3 + \mathbb{M}_\eta)\partial^2_{33} v\|_{[L^2(\mathcal{B})]^3} \leqslant C\sum_{k=1}^{2}\|\partial^2_k(v_\eta)\|_{[H^1(\mathcal{B})]^3} \leqslant C\left(\|f_\eta\|_{[L^2(\mathcal{B})]^3} + \|g_\eta\|_{[H^{1/2}(\partial\mathcal{B})]^3}\right),$$

where $a_{33} = (\frac{\mu}{\delta_\eta} b_\eta^\top b_\eta)_{33} > 0$ and $\mathbb{M}_\eta$ is a non negative symmetric matrix

$$\mathbb{M}_\eta = \left(\frac{\lambda+\mu}{\delta_\eta}\right) b^3_\eta \otimes b^3_\eta, \quad b^3_\eta = [(b_\eta)_{13} \; (b_\eta)_{23} \; (b_\eta)_{33}]^\top.$$

We deduce that $a_{33}\mathbb{I}_3 + \mathbb{M}_\eta$ is invertible and bounded in $\mathcal{B}$, then we get

$$\|\partial^2_{33} v\|_{[L^2(\mathcal{B})]^3} \leqslant C\left(\|f_\eta\|_{[L^2(\mathcal{B})]^3} + \|g_\eta\|_{[H^{1/2}(\partial\mathcal{B})]^3}\right).$$

Since $\partial\mathcal{F}$ is a compact set, we can use a finite partition to cover the boundary $\partial\mathcal{F}$ by many balls $\mathcal{B}(y_i, r_i)$ and we combine all the estimates to get (3.7). We deduce (3.8) by induction and the proof can be performed as [28, Theorem 5].

## 3.2 Lifting

**Lifting of the impermeability condition**

**Lemma 3.3.** *Let $a \in L^2(0,T;H^{3/2}(\partial\mathcal{F})) \cap H^{3/4}(0,T;L^2(\partial\mathcal{F}))$ such that*

$$a(0,\cdot) \in H^{3/2}(\partial\mathcal{F}) \text{ on } \partial\mathcal{F}.$$

*Then there exists $w \in L^2(0,T;[H^2(\mathcal{F})]^3) \cap H^1(0,T;[L^2(\mathcal{F})]^3)$ such that $w \cdot n = a$ on $\partial\mathcal{F}$ and*

$$\|w\|_{L^2(0,T;[H^2(\mathcal{F})]^3)\cap H^1(0,T;[L^2(\mathcal{F})]^3)} \leqslant C\Big( \|a\|_{L^2(0,T;H^{3/2}(\partial\mathcal{F}))\cap H^{3/4}(0,T;L^2(\partial\mathcal{F}))} + \|a(0,\cdot)\|_{H^{3/2}(\partial\mathcal{F})} \Big), \tag{3.23}$$

*with $C$ a constant independent of $T$.*

*Proof.* From [53, Vol II, chap.4, Theorem 2.3], there exists $w \in L^2(0,T;[H^2(\mathcal{F})]^3) \cap H^1(0,T;[L^2(\mathcal{F})]^3)$ such that $w \cdot n = a$ on $\partial\mathcal{F}$ and

$$\|w\|_{L^2(0,T;[H^2(\mathcal{F})]^3)\cap H^1(0,T;[L^2(\mathcal{F})]^3)} \leqslant C_T \|a\|_{L^2(0,T;H^{3/2}(\partial\mathcal{F}))\cap H^{3/4}(0,T;L^2(\partial\mathcal{F}))}. \tag{3.24}$$

However, the constant $C_T$ in (3.24) may depend on $T$, then, we extend data over $(0,T^*)$ with $T^* > 2T$ as follows:

$$a_{ext}(t) = \begin{cases} a_{ext}(t) = a(t) & t \in (0,T), \\ a_{ext}(t) = a(2T-t) & t \in (T,2T), \\ a_{ext}(t) = a(0,\cdot) & t \in (2T,T^*). \end{cases}$$

Since $a(0,\cdot) \in H^{3/2}(\partial\mathcal{F})$, then $a_{ext} \in L^2(0,T^*;H^{3/2}(\partial\mathcal{F})) \cap H^{3/4}(0,T^*;L^2(\partial\mathcal{F}))$. Then, we obtain

$$\|w\|_{L^2(0,T;[H^2(\mathcal{F})]^3)\cap H^1(0,T;[L^2(\mathcal{F})]^3)} \leqslant C_{T^*} \|a_{ext}\|_{L^2(0,T^*;H^{3/2}(\partial\mathcal{F}))\cap H^{3/4}(0,T^*;L^2(\partial\mathcal{F}))}, \tag{3.25}$$

such that $C_{T^*}$ depends on $T^*$. We notice

$$\|a_{ext}\|_{H^{3/4}(0,T^*;L^2(\partial\mathcal{F}))} \leqslant \|a_{ext} - a(0,\cdot)\|_{H^{3/4}(0,T^*;L^2(\partial\mathcal{F}))} + (T^*)^{1/2} \|a(0,\cdot)\|_{L^2(\partial\mathcal{F})}. \tag{3.26}$$

From Lemma A.2,

$$\begin{aligned} \|a_{ext} - a(0,\cdot)\|_{H^{3/4}(0,T^*;L^2(\partial\mathcal{F}))} &\leqslant C\|a - a(0,\cdot)\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} \\ &\leqslant C\|a\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} + \|a(0,\cdot)\|_{L^2(0,T;L^2(\partial\mathcal{F}))}. \end{aligned} \tag{3.27}$$

with $C$ independent of $T$. In the other hand,

$$\|a(0,\cdot)\|_{L^2(0,T;L^2(\partial\mathcal{F}))} \leqslant T^{1/2}\|a\|_{L^\infty(0,T;L^2(\partial\mathcal{F}))} = T^{1/2}\|a^*\|_{L^\infty(0,1;L^2(\partial\mathcal{F}))} \leqslant CT^{1/2}\|a^*\|_{H^{3/4}(0,1;L^2(\partial\mathcal{F}))},$$

where $a^*(t^*) = a(t^*T)$, $t^* \in (0,1)$. In particular, using the estimate [13, (A.11), p. 36] we get

$$\|a^*\|_{H^{3/4}(0,1;L^2(\partial\mathcal{F}))} \leqslant CT^{-1/2}\|a\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))}.$$

Then, (3.27) gives

$$\|a_{ext} - a(0,\cdot)\|_{H^{3/4}(0,T^*;L^2(\partial\mathcal{F}))} \leqslant C\|a\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))}. \tag{3.28}$$

We get also

$$\|a_{ext}\|_{L^2(0,T^*;H^{3/2}(\partial\mathcal{F}))} \leqslant C\|a\|_{L^2(0,T;H^{3/2}(\partial\mathcal{F}))} + (T^*)^{1/2}\|a(0,\cdot)\|_{H^{3/2}(\partial\mathcal{F})}. \tag{3.29}$$

Combining (3.29), (3.28) and (3.26), we get the estimate (3.23). □

## Lifting of the slip boundary condition

**Lemma 3.4.** *Let $\overline{g}_i \in L^2(0,T;H^{1/2}(\partial\mathcal{F})) \cap H^{1/4}(0,T;L^2(\partial\mathcal{F}))$, $i=1,2$. The non stationary Stokes system*

$$\begin{cases} \partial_t \phi - \nabla\cdot(\mathbb{S}(\phi)) = 0 & \text{in } (0,\infty)\times\mathcal{F}, \\ \phi\cdot n = 0 & \text{on } (0,\infty)\times\partial\mathcal{F}, \\ [2\mu\mathbb{D}(\phi)n + \beta\phi]\cdot\tau^i = \overline{g}_i,\ i=1,2 & \text{on } (0,\infty)\times\partial\mathcal{F}, \\ \phi(0,\cdot) = 0 & \text{in } \mathcal{F}, \end{cases} \tag{3.30}$$

*admits a unique strong solution in $L^2(0,\infty;[H^2(\mathcal{F})]^3)\cap H^1(0,\infty;[L^2(\mathcal{F})]^3)$ and*

$$\|\phi\|_{L^2(0,\infty;[H^2(\mathcal{F})]^3)\cap H^1(0,\infty;[L^2(\mathcal{F})]^3)\cap L^\infty(0,\infty;[H^1(\mathcal{F})]^3)} \leqslant C\left(\|\overline{g}_1\|_{L^2(0,\infty;H^{1/2}(\partial\mathcal{F}))\cap H^{1/4}(0,\infty;L^2(\partial\mathcal{F}))} + \|\overline{g}_2\|_{L^2(0,\infty;H^{1/2}(\partial\mathcal{F}))\cap H^{1/4}(0,\infty;L^2(\partial\mathcal{F}))}\right). \tag{3.31}$$

*Proof.* The proof is based on the transposition method and it is similar to the incompressible case see [70, 69]. Let $\overline{g}_i \in C_0^\infty(0,\infty;H^{1/2}(\partial\mathcal{F}))$ and to simplify the calculations, let $\overline{g}\in C_0^\infty(0,\infty;[H^{1/2}(\partial\mathcal{F})]^3)$ such that $\overline{g}\cdot\tau^i = \overline{g}_i$, $i=1,2$ and $\overline{g}\cdot n = 0$. Let $\phi_g$ the solution of the stationary system with (3.6) with $a=0$ and $g_i = \overline{g}_i$. Setting $\overline{\phi} = \phi - \phi_g$, we see that $\overline{\phi}$ verifies the system

$$\begin{cases} \partial_t\overline{\phi} - \nabla\cdot\big(\mathbb{S}(\overline{\phi})\big) = -\partial_t\phi_g & \text{in } (0,\infty)\times\mathcal{F}, \\ \overline{\phi}\cdot n = 0 & \text{on } (0,\infty)\times\partial\mathcal{F}, \\ \big[2\mu\mathbb{D}(\overline{\phi})n + \beta\overline{\phi}\big]\cdot\tau^i = 0 & \text{on } (0,\infty)\times\partial\mathcal{F}, \\ \overline{\phi}(0,\cdot) = 0 & \text{in } \mathcal{F}. \end{cases}$$

From Theorem 3.1, the associated Lamé operator of the above system is analytic with negative type. Then, there exists a unique strong solution $\overline{\phi}$ that is in $C_0^\infty(0,\infty;[H^2(\mathcal{F})]^3)$ and thus the system (3.30) admits a unique solution $\phi\in C_0^\infty(0,\infty;[H^2(\mathcal{F})]^3)$.

Let show (3.31). Let $f\in L^2(\mathbb{R};[L^2(\mathcal{F})]^3)$ and $T_\infty > 0$ such that $f=0$ on $(T_\infty,\infty)$ the solution of the backward system

$$\begin{cases} \partial_t\psi + \nabla\cdot(\mathbb{S}(\psi)) = f & \text{in } (-\infty,T_\infty)\times\mathcal{F}, \\ \psi\cdot n = 0 & \text{on } (-\infty,T_\infty)\times\partial\mathcal{F}, \\ [2\mu\mathbb{D}(\psi)n + \beta\psi]\cdot\tau^i = 0 & \text{on } (-\infty,T_\infty)\times\partial\mathcal{F}, \\ \psi(T_\infty,\cdot) = 0 & \text{in } \mathcal{F}. \end{cases}$$

We extend $\psi$ (resp. $\phi$ and $\overline{g}$) by zero on $(T_\infty,\infty)$ (resp. on $(-\infty,0)$). The function $\psi$ satisfies the estimate

$$\|\psi\|_{L^2(\mathbb{R};[H^2(\mathcal{F})]^3)\cap H^1(\mathbb{R};[L^2(\mathcal{F})]^3)\cap L^\infty(\mathbb{R};[H^1(\mathcal{F})]^3)} \leqslant C\|f\|_{L^2(\mathbb{R};[L^2(\mathcal{F})]^3}. \tag{3.32}$$

After standard calculations and integration by parts, we get

$$\int_{\mathbb{R}}\int_{\mathcal{F}} \partial_t\phi\cdot f\ dy\ dt = -\int_{\mathbb{R}}\int_{\partial\mathcal{F}} \partial_t\overline{g}\cdot\psi\ d\Gamma\ dt.$$

From the trace theorem, we have that $\psi\in H^{3/4}(\mathbb{R};[L^2(\partial\mathcal{F})]^3)$. Hence, we obtain

$$\int_{\mathbb{R}}\int_{\mathcal{F}} \partial_t\phi\cdot f\ dy\ dt \leqslant C\|\overline{g}\|_{H^{1/4}(\mathbb{R};[L^2(\partial\mathcal{F})]^3)}\|\psi\|_{H^{3/4}(\mathbb{R};[L^2(\partial\mathcal{F})]^3)}. \tag{3.33}$$

Combining (3.33) and (3.32), we get

$$\|\partial_t\phi\|_{L^2(0,\infty;[L^2(\mathcal{F})]^3)} \leqslant C\|\overline{g}\|_{H^{1/4}(0,\infty;[L^2(\partial\mathcal{F})]^3)}. \tag{3.34}$$

Finally, using Theorem 3.1, we obtain (3.31). □

Finally, we consider the following non stationary and nonhomogeneous Lamé system

$$\begin{cases} \rho^0 \partial_t v_\ell - \nabla \cdot (\mathbb{S}(v_\ell)) = 0 & \text{in } (0,T)\times\mathcal{F}, \\ v_\ell \cdot n = a & \text{on } (0,T)\times\partial\mathcal{F}, \\ [2\mu\mathbb{D}(v_\ell)n + \beta v_\ell]\cdot\tau^i = g_i,\ i=1,2 & \text{on } (0,T)\times\partial\mathcal{F}, \\ v_\ell(0,\cdot) = w(0) & \text{in } \mathcal{F}, \end{cases} \tag{3.35}$$

where $w$ is obtained in Lemma 3.3 with $\beta(y) = \beta_0$ if $y \in \partial\mathcal{S}$ and $\beta(y) = \beta_1$ if $y \in \partial\mathcal{F}$. We suppose that $\beta_0 + \beta_1 > 0$.

**Proposition 3.5.** *Let*

$$a \in L^2(0,T;H^{3/2}(\partial\mathcal{F})) \cap H^{3/4}(0,T;L^2(\partial\mathcal{F})), \quad g_i \in L^2(0,T;H^{1/2}(\partial\mathcal{F})) \cap H^{1/4}(0,T;L^2(\partial\mathcal{F})).$$

*The system* (3.35) *admits a unique strong solution* $v_\ell$ *of the system* (3.35) *and there exists a constant* $C$ *non decreasing with respect to* $T$ *such that we have the estimate*

$$\|v_\ell\|_{L^2(0,T;[H^2(\mathcal{F})]^3)\cap H^1(0,T;[L^2(\mathcal{F})]^3)} \leqslant C\Big( \|a\|_{L^2(0,T;H^{3/2}(\partial\mathcal{F}))\cap H^{3/4}(0,T;L^2(\partial\mathcal{F}))} + \|a(0,\cdot)\|_{H^{3/2}(\partial\mathcal{F})} + \|g_1\|_{L^2(0,T;[H^{1/2}(\partial\mathcal{F}))\cap H^{1/4}(0,T;L^2(\partial\mathcal{F})} + \|g_2\|_{L^2(0,T;[H^{1/2}(\partial\mathcal{F}))\cap H^{1/4}(0,T;L^2(\partial\mathcal{F}))} \Big). \tag{3.36}$$

*Proof.* We consider the lifting $\overline{v} = v_\ell - w - \phi$ where $w$ is obtained in Lemma 3.3 and $\phi$ is the solution of (3.30) with

$$\overline{g}_i = g_i - [2\mu\mathbb{D}(w)n + \beta w]\cdot\tau^i,\ i=1,2 \quad t \in (0,T),$$

extended by 0 on $(T,\infty)$. The system (3.35) is reduced to

$$\begin{cases} \rho^0 \partial_t \overline{v} - \nabla \cdot (\mathbb{S}(\overline{v})) = f(w,\phi) & \text{in } (0,T)\times\mathcal{F}, \\ \overline{v} \cdot n = 0 & \text{on } (0,T)\times\partial\mathcal{F}, \\ [2\mu\mathbb{D}(\overline{v})n + \beta\overline{v}]\cdot\tau^i = 0,\ i=1,2 & \text{on } (0,T)\times\partial\mathcal{F}, \\ \overline{v}(0,\cdot) = 0 & \text{in } \mathcal{F}. \end{cases} \tag{3.37}$$

Using the elliptic results elaborated in Theorem 3.1 and [9, Theorem 3.1, p.143], we obtain that $\overline{v}$ satisfies

$$\|\overline{v}\|_{L^2(0,T;[H^2(\mathcal{F})]^3)\cap H^1(0,T;[L^2(\mathcal{F})]^3)\cap L^\infty(0,T;[H^1(\mathcal{F})]^3)} \leqslant Ce^{CT}\|f(w,\phi)\|_{L^2(0,T;[L^2(\partial\mathcal{F})]^3)}. \tag{3.38}$$

Then, using (3.38), (3.31) and (3.23) combined with [13, Lemma A.8] we deduce finally (3.36). □

## 3.3 Linearized system

### 3.3.1 $H^2$ Regularity

We consider the lifting $v_\ell$ that satisfies (2.4). Let $u = v - v_\ell$, the linearized system (3.1), (3.2), (3.3) and (3.4) is reduced as follows

$$\begin{cases} \partial_t \rho + \rho^0 \nabla \cdot u = \overline{h} & \text{in } (0,T)\times\mathcal{F}, \\ \rho(0,\cdot) = \rho^0 & \text{in } \mathcal{F}, \end{cases} \tag{3.39}$$

$$\begin{cases} \rho^0 \partial_t u - \nabla\cdot(\mathbb{S}(u)) = \nabla\cdot\overline{f}_1 + \overline{f}_2 & \text{in } (0,T)\times\mathcal{F}, \\ v_\mathcal{S}(y) = \ell + \omega\times y & \text{in } (0,T), \\ m\ell' = f_3 & \text{in } (0,T), \\ J(0)\omega' = f_4 & \text{in } (0,T), \end{cases} \tag{3.40}$$

with the boundary conditions

$$\begin{cases} u\cdot n = 0, & [(2\mu\mathbb{D}(u)n + \beta_0 u)]\cdot\tau^i = 0 & \text{on}(0,T)\times\partial\Omega, \\ (u - v_\mathcal{S})\cdot n = 0, & [(2\mu\mathbb{D}(u)n + \beta_1(u - v_\mathcal{S}))]\cdot\tau^i = 0 & \text{on}(0,T)\times\partial\mathcal{S}, \end{cases} \tag{3.41}$$

and the initial conditions

$$\rho(0,\cdot)=\rho^0 \text{ in } \mathcal{F}, \quad u(0,\cdot)=u^0 \text{ in } \mathcal{F}, \quad \ell(0)=\ell^0, \quad \omega(0)=\omega^0, \tag{3.42}$$

with the following compatibility condition

$$v_{\mathcal{S}}^0=\ell^0+\omega^0\times y, \quad u^0\cdot n=0 \ \text{ on } \partial\Omega, \quad (u^0-v_{\mathcal{S}}^0)\cdot n=0 \ \text{ on } \partial\mathcal{S}. \tag{3.43}$$

We introduce the spaces

$$\begin{aligned}\mathcal{X}_T=&\big(W^{1,\infty}(0,T;L^2(\mathcal{F}))\cap H^1(0,T;H^1(\mathcal{F}))\big)\\ &\times\big(L^2(0,T;[H^2(\mathcal{F})]^3)\cap H^1(0,T;[L^2(\mathcal{F})]^3)\cap C(0,T;[H^1(\mathcal{F})]^3)\big)\times[H^1(0,T)]^3\times[H^1(0,T)]^3,\end{aligned}$$

where

$$\begin{aligned}\|(\rho,u,\ell,\omega)\|_{\mathcal{X}_T}=&\|\rho\|_{W^{1,\infty}(0,T;L^2(\mathcal{F})}+\|\rho\|_{H^1(0,T;H^1(\mathcal{F}))}+\|u\|_{L^2(0,T;[H^2(\mathcal{F})]^3)}\\ &+\|u\|_{H^1(0,T;[L^2(\mathcal{F})]^3)}+\|u\|_{L^\infty(0,T;[H^1(\mathcal{F})]^3)}+\|\ell\|_{H^1(0,T)}+\|\omega\|_{H^1(0,T)},\end{aligned}$$

and

$$\mathcal{I}_0=\{(u^0,\ell^0,\omega^0)\in[H^1(\mathcal{F})]^3\times\mathbb{R}^3\times\mathbb{R}^3 \text{ satisfying (3.43) }\}.$$

**Proposition 3.6.** *For any*

$$\begin{aligned}(u^0,\ell^0,\omega^0)\in\mathcal{I}_0, \quad \overline{h}\in L^2(0,T;H^1(\mathcal{F}))\cap L^\infty(0,T;L^2(\mathcal{F})), \quad \overline{f}_1\in L^2(0,T;[H^1(\mathcal{F})]^9),\\ \overline{f}_2\in L^2(0,T;[L^2(\mathcal{F})]^3), \quad f_3\in[L^2(0,T)]^3, \quad f_4\in[L^2(0,T)]^3,\end{aligned}$$

*there exists a unique solution of the system* (3.39), (3.40), (3.41) *and* (3.42) *such that*

$$\begin{aligned}\|(\rho,u,\ell,\omega)\|_{\mathcal{X}_T}\leqslant Ce^{CT}\Big(&\|\overline{h}\|_{L^2(0,T;H^1(\mathcal{F}))\cap L^\infty(0,T;L^2(\mathcal{F}))}+\|\overline{f}_1\|_{L^2(0,T;[H^1(\mathcal{F})]^9)}+\|\overline{f}_2\|_{L^2(0,T;[L^2(\mathcal{F})]^3)}\\ &+\|f_3\|_{[L^2(0,T)]^3}+\|f_4\|_{[L^2(0,T)]^3}+\|(u^0,\ell^0,\omega^0)\|_{\mathcal{I}_0}\Big).\end{aligned} \tag{3.44}$$

*Proof.* For $\ell,\ \omega\in\mathbb{R}^3$, we define $\mathbb{N}(\ell,\omega)$ as the solution of the system (3.6) with $a=v_{\mathcal{S}}\cdot n$, $g_i=v_{\mathcal{S}}\cdot\tau^i$ and $f=0$. From Theorem 3.1, we see that $\mathbb{N}\in\mathcal{L}(\mathbb{R}^3\times\mathbb{R}^3,[H^2(\mathcal{F})]^3)$. We introduce the operator

$$\mathbb{A}_F=-\frac{1}{\rho^0}\nabla\cdot\mathbb{S}, \quad \mathcal{D}(\mathbb{A}_F)=\{u\in[H^2(\mathcal{F})]^3;\ u\cdot n=0,\ [2\mu\mathbb{D}(u)n+\beta u]\cdot\tau^i=0 \ \text{ on } \partial\mathcal{F}\},$$

that is self adjoint and by Korn inequality (see Lemma A.1), we deduce that this operator is also coercive, then $\mathbb{A}_F$ is the generator of an analytic semigroup on $[L^2(\mathcal{F})]^3$ (see [9, Theorem 2.12, p.115]). Taking $\widetilde{u}=u-\mathbb{N}(\ell,\omega)$, the fluid system is reduced to

$$\begin{cases}\rho^0\partial_t\widetilde{u}-\nabla\cdot(\mathbb{S}(\widetilde{u}))=f & \text{in } (0,T)\times\mathcal{F},\\ \widetilde{u}\cdot n=0, \quad [(2\mu\mathbb{D}(\widetilde{u})n+\beta\widetilde{u})]\cdot\tau^i=0 & \text{on } (0,T)\times\partial\mathcal{F},\\ \widetilde{u}(0,\cdot)=u^0-\mathbb{N}(\ell^0,\omega^0) & \text{in } \mathcal{F}.\end{cases} \tag{3.45}$$

From [9, Theorem 3.1, p.143], we obtain that $\widetilde{u}$ satisfies

$$\begin{aligned}&\|\widetilde{u}\|_{L^2(0,T;[H^2(\mathcal{F})]^3)\cap H^1(0,T;[L^2(\mathcal{F})]^3)\cap L^\infty(0,T;[H^1(\mathcal{F})]^3)}\leqslant Ce^{CT}\,\|f\|_{L^2(0,T;[L^2(\partial\mathcal{F})]^3)}\\ &\leqslant Ce^{CT}\Big(\|\overline{f}_1\|_{L^2(0,T;[H^1(\mathcal{F})]^9)}+\|\overline{f}_2\|_{L^2(0,T;[L^2(\mathcal{F})]^3)}+\|f_3\|_{[L^2(0,T)]^3}+\|f_4\|_{[L^2(0,T)]^3}+\|(u^0,\ell^0,\omega^0)\|_{\mathcal{I}_0}\Big).\end{aligned} \tag{3.46}$$

Thus, we get

$$\|u\|_{L^2(0,T;[H^2(\mathcal{F})]^3)\cap H^1(0,T;[L^2(\mathcal{F})]^3)\cap L^\infty(0,T;[H^1(\mathcal{F})]^3)} \leqslant Ce^{CT}\bigg(\|\overline{f}_1\|_{L^2(0,T;[H^1(\mathcal{F})]^9)} + \|\overline{f}_2\|_{L^2(0,T;[L^2(\mathcal{F})]^3)} + \|f_3\|_{[L^2(0,T)]^3} + \|f_4\|_{[L^2(0,T)]^3} + \|(u^0,\ell^0,\omega^0)\|_{\mathcal{I}_0}\bigg). \quad (3.47)$$

From (3.39), we get that

$$\rho \in W^{1,\infty}(0,T;L^2(\mathcal{F}))\cap H^1(0,T;H^1(\mathcal{F})).$$

Hence, we deduce that the system (3.40),(3.41), (3.42) admits a unique strong solution that satisfies the energy estimate (3.44). □

Let

$$\begin{aligned}\mathcal{Y}_T = \{(h,f_1,f_2,f_3,f_4,a,g_1,g_2) \in \big(L^2(0,T;H^1(\mathcal{F}))\cap L^\infty(0,T;L^2(\mathcal{F}))\big)\times L^2(0,T;[H^1(\mathcal{F})]^9)\\ \times L^2(0,T;[L^2(\mathcal{F})]^3)\times [L^2(0,T)]^3\times [L^2(0,T)]^3\times \Big(L^2(0,T;H^{3/2}(\partial\mathcal{F}))\cap H^{3/4}(0,T;L^2(\partial\mathcal{F}))\Big)\\ \times \Big(L^2(0,T;H^{1/2}(\partial\mathcal{F}))\cap H^{1/4}(0,T;L^2(\partial\mathcal{F}))\Big)^2,\quad a(0,\cdot)\in H^{3/2}(\partial\mathcal{F})\},\end{aligned}$$

and

$$\begin{aligned}\|(h,f_1,f_2,f_3,f_4,a,g_1,g_2)\|_{\mathcal{Y}_T} = \|h\|_{L^2(0,T;H^1(\mathcal{F}))\cap L^\infty(0,T;L^2(\mathcal{F}))} + \|f_1\|_{L^2(0,T;[H^1(\mathcal{F})]^9)} + \|f_2\|_{L^2(0,T;[L^2(\mathcal{F})]^3)}\\ + \|f_3\|_{[L^2(0,T)]^3} + \|f_4\|_{[L^2(0,T)]^3} + \|a(0,\cdot)\|_{L^2(0,T;H^{3/2}(\partial\mathcal{F}))\cap H^{3/4}(0,T;L^2(\partial\mathcal{F}))} + \|a(0,\cdot)\|_{H^{3/2}(\partial\mathcal{F})}\\ + \|g_1\|_{L^2(0,T;H^{1/2}(\partial\mathcal{F}))\cap H^{1/4}(0,T;L^2(\partial\mathcal{F}))} + \|g_2\|_{L^2(0,T;H^{1/2}(\partial\mathcal{F}))\cap H^{1/4}(0,T;L^2(\partial\mathcal{F}))}.\end{aligned}$$

Then using (3.36), we deduce a regularity results for (3.1), (3.2), (3.3) and (3.4).

**Corollary 3.7.** *Let*

$$(h,f_1,f_2,f_3,f_4,a,g_1,g_2)\in\mathcal{Y}_T,\quad \rho^0\in H^1(\mathcal{F}),\quad v^0\in[H^1(\mathcal{F})]^3,\quad \ell^0,\ \omega^0\in\mathbb{R}^3,$$

*such that* (3.5) *is verified. Then, the system* (3.1), (3.2), (3.3) *and* (3.4) *admits a unique strong solution* $(\rho,v,\ell,\omega)\in\mathcal{X}_T$ *such that*

$$\|(\rho,v,\ell,\omega)\|_{\mathcal{X}_T}\leqslant Ce^{CT}\bigg(\|(h,f_1,f_2,f_3,f_4,a,g_1,g_2)\|_{\mathcal{Y}_T} + \|\rho^0\|_{H^1(\mathcal{F})} + |\ell^0| + |\omega^0| + \|v^0\|_{[H^1(\mathcal{F})]^3}\bigg). \quad (3.48)$$

### 3.3.2 $H^4$ Regularity

Let $(v^0,\ell^0,\omega^0)\in[H^3(\mathcal{F})]^3\times\mathbb{R}^3\times\mathbb{R}^3$ such that

$$v^0_{\mathcal{S}} = \ell^0+\omega^0\times y,\quad \begin{cases} v^0\cdot n = a(0,\cdot), & \big[2\mu\mathbb{D}(v^0)n+\beta v^0\big]_i = g_i(0,\cdot),\ i=1,2 \quad \text{on } \partial\Omega,\\ (v^0-v^0_{\mathcal{S}})\cdot n = a(0,\cdot), & \big[2\mu\mathbb{D}(v^0)n+\beta_1(v^0-v^0_{\mathcal{S}})\big]_i = g_i(0,\cdot),\ i=1,2 \quad \text{on } \partial\mathcal{S},\end{cases} \quad (3.49)$$

and

$$\begin{cases} \Big(\dfrac{1}{\rho^0}\nabla\cdot\mathbb{S}(v^0)+\nabla\cdot(f_1(0,\cdot))+f_2(0,\cdot)\Big)\cdot n = \partial_t a(0,\cdot) & \text{on } \partial\Omega,\\ \Big[\Big(\dfrac{1}{\rho^0}\nabla\cdot\mathbb{S}(v^0)+\nabla\cdot(f_1(0,\cdot))+f_2(0,\cdot)\Big) - \big(m^{-1}f_3(0)+J^{-1}(0)f_4(0)\times y\big)\Big]\cdot n = \partial_t a(0,\cdot) & \text{on } \partial\mathcal{S}.\end{cases} \quad (3.50)$$

Let introduce the spaces

$$\widehat{\mathcal{X}}_T = W^{1,\infty}(0,T;H^2(\mathcal{F})) \cap H^1(0,T;H^3(\mathcal{F})) \cap H^2(0,T;H^1(\mathcal{F})) \\ \times \big(L^2(0,T;[H^4(\mathcal{F})]^3) \cap H^2(0,T;[L^2(\mathcal{F})]^3) \cap C(0,T;[H^3(\mathcal{F})]^3)\big) \times [H^2(0,T)]^3 \times [H^2(0,T)]^3,$$

$$\widehat{\mathcal{I}}_0 = \{(v^0,\ell^0,\omega^0) \in [H^3(\mathcal{F})]^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \text{ satisfying (3.49) and (3.50) }\},$$

where

$$\|(\rho,v,\ell,\omega)\|_{\widehat{\mathcal{X}}_T} = \|\rho\|_{W^{1,\infty}(0,T;H^2(\mathcal{F}))} + \|\rho\|_{H^1(0,T;H^3(\mathcal{F}))} + \|\rho\|_{H^2(0,T;H^1(\mathcal{F}))} + \|v\|_{L^2(0,T;[H^4(\mathcal{F})]^3)} \\ + \|v\|_{L^\infty(0,T;[H^3(\mathcal{F})]^3)} + \|v\|_{H^2(0,T;[L^2(\mathcal{F})]^3)} + \|v\|_{W^{1,\infty}(0,T;[L^2(\mathcal{F})]^1)} + \|v\|_{H^1(0,T;[H^2(\mathcal{F})]^1)} \\ + \|\ell\|_{H^2(0,T)} + \|\ell\|_{W^{1,\infty}(0,T)} + \|\omega\|_{H^2(0,T)} + \|\omega\|_{W^{1,\infty}(0,T)},$$

and

$$\widehat{\mathcal{Y}}_T = \{(h,f_1,f_2,f_3,f_4,a,g_1,g_2) \in \big(L^2(0,T;H^3(\mathcal{F})) \cap L^\infty(0,T;H^2(\mathcal{F})) \cap H^1(0,T;H^1(\mathcal{F}))\big) \\ \times \big(L^2(0,T;[H^3(\mathcal{F})]^9) \cap H^1(0,T;[H^1(\mathcal{F})]^9)\big) \times \big(L^2(0,T;[H^2(\mathcal{F})]^3) \cap H^1(0,T;[L^2(\mathcal{F})]^3)\big) \\ \times [H^1(0,T)]^3 \times [H^1(0,T)]^3 \times \Big(L^2(0,T;H^{7/2}(\partial\mathcal{F})) \cap H^{7/4}(0,T;L^2(\partial\mathcal{F}))\Big) \\ \times \Big(L^2(0,T;H^{5/2}(\partial\mathcal{F})) \cap H^{5/4}(0,T;L^2(\partial\mathcal{F}))\Big)^2; \quad h(0,\cdot) \in H^2(\mathcal{F}),\ f_1(0,\cdot) \in [H^2(\mathcal{F})]^9, \\ f_2(0,\cdot) \in [H^1(\mathcal{F})]^3,\ a(0,\cdot) \in H^{5/2}(\partial\mathcal{F}),\ \partial_t a(0,\cdot) \in H^{3/2}(\partial\mathcal{F}),\ g_1(0,\cdot),\ g_2(0,\cdot) \in H^{3/2}(\partial\mathcal{F})\},$$

where

$$\|(h,f_1,f_2,f_3,f_4,a,g_1,g_2)\|_{\widehat{\mathcal{Y}}_T} = \|h\|_{L^2(0,T;H^3(\mathcal{F}))\cap L^\infty(0,T;H^2(\mathcal{F}))\cap H^1(0,T;H^1(\mathcal{F}))} + \|h(0,\cdot)\|_{H^2(\mathcal{F})} \\ + \|f_1\|_{L^2(0,T;[H^3(\mathcal{F})]^9)\cap H^1(0,T;[H^1(\mathcal{F})]^9)} + \|f_1(0,\cdot)\|_{[H^2(\mathcal{F})]^9} + \|f_2\|_{L^2(0,T;[H^2(\mathcal{F})]^3)\cap H^1(0,T;[L^2(\mathcal{F})]^3} \\ + \|f_2(0,\cdot)\|_{[H^1(\mathcal{F})]^3} + \|f_3\|_{[H^1(0,T)]^3} + \|f_4\|_{[H^1(0,T)]^3} + |f_3(0)| + |f_4(0)| + \|a\|_{L^2(0,T;H^{7/2}(\partial\mathcal{F}))\cap H^{7/4}(0,T;L^2(\partial\mathcal{F}))} \\ + \|a(0,\cdot)\|_{H^{5/2}(\partial\mathcal{F})} + \|\partial_t a(0,\cdot)\|_{H^{3/2}(\partial\mathcal{F})} + \|g_1\|_{L^2(0,T;H^{5/2}(\partial\mathcal{F}))\cap H^{5/4}(0,T;L^2(\partial\mathcal{F}))} \\ + \|g_2\|_{L^2(0,T;H^{5/2}(\partial\mathcal{F}))\cap H^{5/4}(0,T;L^2(\partial\mathcal{F}))} + \|g_1(0,\cdot)\|_{H^{3/2}(\partial\mathcal{F})} + \|g_2(0,\cdot)\|_{H^{3/2}(\partial\mathcal{F})}.$$

**Theorem 3.8.** *Let*

$$(h,f_1,f_2,f_3,f_4,a,g_1,g_2) \in \widehat{\mathcal{Y}}_T, \quad (\rho^0,v^0,\ell^0,\omega^0) \in H^3(\mathcal{F}) \times \widehat{\mathcal{I}}_0,$$

*such that $a(0,\cdot) = 0$ and $g_i(0,\cdot) = 0$, $i = 1,2$. Then, the system (3.1), (3.2), (3.3) and (3.4) admits a unique strong solution in $\widehat{\mathcal{X}}_T$ such that*

$$\|(\rho,v,\ell,\omega)\|_{\widehat{\mathcal{X}}_T} \leqslant Ce^{CT}\Big(\|(h,f_1,f_2,f_3,f_4,a,g_1,g_2)\|_{\widehat{\mathcal{Y}}_T} + \|(\rho^0,v^0,\ell^0,\omega^0)\|_{H^3(\mathcal{F})\times\widehat{\mathcal{I}}_0}\Big). \tag{3.51}$$

*Proof.* Let

$$v^1 = \frac{1}{\rho^0}\nabla\cdot\mathbb{S}(v^0) + \nabla\cdot(f_1(0,\cdot)) + f_2(0,\cdot) \text{ in } \mathcal{F}, \quad \ell^1 = m^{-1}f_3(0), \quad \omega^1 = J^{-1}(0)f_4(0).$$

We differentiate the system (3.2), (3.3) and (3.4) with respect to $t$, then $(\partial_t v,\ell',\omega')$ is the solution of the system

$$\begin{cases} \rho^0\partial_{tt}v - \nabla\cdot(\mathbb{S}(\partial_t v)) = \nabla\cdot\partial_t f_1 + \partial_t f_2 & \text{in } (0,T)\times\mathcal{F}, \\ \partial_t v_{\mathcal{S}}(y) = \ell' + \omega'\times y & \text{in } (0,T), \\ m\ell'' = \partial_t f_3 & \text{in } (0,T), \\ J(0)\omega'' = \partial_t f_4 & \text{in } (0,T), \end{cases}$$

with the boundary conditions

$$\left\{\begin{array}{rll} \partial_t v\cdot n=\partial_t a, & [(2\mu\mathbb{D}(\partial_t v)n+\beta_0\partial_t v)]\cdot\tau^i=\partial_t g_i,\ i=1,2 & \text{on}(0,T)\times\partial\Omega,\\ (\partial_t v-\partial_t v_{\mathcal{S}})\cdot n=\partial_t a, & [(2\mu\mathbb{D}(\partial_t v)n+\beta_1(\partial_t v-\partial_t v_{\mathcal{S}}))]\cdot\tau^i=\partial_t g_i,\ i=1,2 & \text{on}(0,T)\times\partial\mathcal{S},\end{array}\right.$$

and the initial conditions

$$\partial_t v(0,\cdot)=v^1 \text{ in } \mathcal{F},\quad \ell'(0)=\ell^1,\quad \omega'(0)=\omega^1, \tag{3.52}$$

By interpolation, $a\in H^1(0,T;H^{3/2}(\partial\mathcal{F}))\cap H^{7/4}(0,T;L^2(\partial\mathcal{F}))$ and $g\in H^1(0,T;H^{1/2}(\partial\mathcal{F}))\cap H^{5/4}(0,T;L^2(\partial\mathcal{F}))$. Using Corollary 3.7, we obtain that

$$\begin{aligned}\|(\partial_t v,\ell',\omega')\|_{L^2(0,T;[H^1(\mathcal{F})]^3)\cap H^1(0,T;[L^2(\mathcal{F})]^3)\times[H^1(0,T)]^3\times[H^1(0,T)]^3}\leqslant Ce^{CT}\Big(|\ell^1|+|\omega^1|\\ +\|v^1\|_{[H^1(\mathcal{F})]^3}+\|f_1\|_{H^1(0,T;[H^1(\mathcal{F})]^9)}+\|f_2\|_{H^1(0,T;[L^2(\mathcal{F})]^3)}+\|f_3\|_{[H^1(0,T)]^3}+\|f_4\|_{[H^1(0,T)]^3}\\ +\|a\|_{H^1(0,T;H^{3/2}(\partial\mathcal{F}))\cap H^{7/4}(0,T;L^2(\partial\mathcal{F}))}+\|\partial_t a(0,\cdot)\|_{H^{3/2}(\partial\mathcal{F})}+\|g_1\|_{H^1(0,T;H^{1/2}(\partial\mathcal{F}))\cap H^{5/4}(0,T;L^2(\partial\mathcal{F}))}\\ +\|g_2\|_{H^1(0,T;H^{1/2}(\partial\mathcal{F}))\cap H^{5/4}(0,T;L^2(\partial\mathcal{F}))}\Big).\end{aligned} \tag{3.53}$$

Let now estimate $\|a\|_{H^1(0,T;H^{3/2}(\partial\mathcal{F}))}$ taking care of the constants that should not depend on $T$. Since $a(0,\cdot)=0$, we extend $a$ by $a_{ext}$ over $(0,T^*)$ with $T^*>2T$ as the following

$$a_{ext}=\left\{\begin{array}{cc} a(t) & t\in[0,T],\\ a(2T-t) & t\in[T,2T],\\ 0 & t\in[2T,T^*].\end{array}\right.$$

Then, using interpolation argument in $(0,T^*)$, we obtain

$$\begin{aligned}\|a\|^2_{H^1(0,T;H^{3/2}(\partial\mathcal{F}))}&\leqslant\|a_{ext}\|^2_{H^1(0,T^*;H^{3/2}(\partial\mathcal{F}))}\\ &\leqslant C\left(\|a_{ext}\|^2_{L^2(0,T^*;H^{7/2}(\partial\mathcal{F}))}\right)^{3/7}\left(\|a_{ext}\|^2_{H^{7/4}(0,T^*;L^2(\partial\mathcal{F}))}\right)^{4/7}.\end{aligned} \tag{3.54}$$

We notice that

$$\|a_{ext}\|^2_{L^2(0,T^*;H^{7/2}(\partial\mathcal{F}))}=2\|a\|^2_{L^2(0,T;H^{7/2}(\partial\mathcal{F}))}, \tag{3.55}$$

and

$$\begin{aligned}\|a_{ext}\|^2_{H^{7/4}(0,T^*;L^2(\partial\mathcal{F}))}&=\|a_{ext}\|^2_{H^1(0,T^*;L^2(\partial\mathcal{F}))}+\lfloor\partial_t a_{ext}\rfloor^2_{3/4,2,(0,T^*)}\\ &=2\|a\|^2_{H^1(0,T;L^2(\partial\mathcal{F}))}+\lfloor\partial_t a_{ext}-\partial_t a_{ext}(0,\cdot)\rfloor^2_{3/4,2,(0,T^*)}.\end{aligned} \tag{3.56}$$

We estimate $\lfloor\partial_t a_{ext}-\partial_t a_{ext}(0,\cdot)\rfloor^2_{3/4,2,(0,T^*)}$ as it is done in Lemma A.2 to get

$$\lfloor\partial_t a_{ext}-\partial_t a_{ext}(0,\cdot)\rfloor^2_{3/4,2,(0,T^*)}\leqslant C\lfloor\partial_t a-\partial_t a(0,\cdot)\rfloor^2_{3/4,2,(0,T)}=C\lfloor\partial_t a\rfloor^2_{3/4,2,(0,T)}. \tag{3.57}$$

Using (3.54), (3.55), (3.56) and (3.57), we obtain

$$\|a\|^2_{H^1(0,T;H^{3/2}(\partial\mathcal{F}))}\leqslant C\left(\|a\|^2_{L^2(0,T;H^{7/2}(\partial\mathcal{F}))}+\|a\|^2_{H^{7/4}(0,T;L^2(\partial\mathcal{F}))}\right). \tag{3.58}$$

Using the same arguments, we also have

$$\|g_i\|^2_{H^1(0,T;H^{1/2}(\partial\mathcal{F}))}\leqslant C\left(\|g_i\|^2_{L^2(0,T;H^{3/2}(\partial\mathcal{F}))}+\|g_i\|^2_{H^{5/4}(0,T;L^2(\partial\mathcal{F}))}\right),\quad i=1,2. \tag{3.59}$$

In the other hand from Theorem 3.1, we have that

$$\begin{aligned}\|v\|_{L^2(0,T;[H^4(\mathcal{F})]^3)} \leqslant C\Big( &\|f_1\|_{L^2(0,T;[H^3(\mathcal{F})]^9)} + \|f_2\|_{L^2(0,T;[H^2(\mathcal{F})]^3)} + \|\partial_t v\|_{L^2(0,T;[H^2(\mathcal{F})]^3)} \\ &+ \|f_3\|_{[L^2(0,T)]^3} + \|f_4\|_{[L^2(0,T)]^3} + \|a\|_{L^2(0,T;H^{7/2}(\partial\mathcal{F}))} \\ &+ \|g_1\|_{L^2(0,T;H^{5/2}(\partial\mathcal{F}))} + \|g_2\|_{L^2(0,T;H^{5/2}(\partial\mathcal{F}))} + \|\ell\|_{[L^2(0,T)]^3} + \|\omega\|_{[L^2(0,T)]^3}\Big). \quad (3.60)\end{aligned}$$

Gathering (3.53) and (3.60), we obtain (3.51). □

# 4 Fixed-Point and proof of Theorem 1.2

Let us fix $T > 0$ and $R > 0$ and let define

$$\begin{aligned}\mathcal{B}_{R,T} = \{(h, f_1, f_2, f_3, f_4, a, g_1, g_2) \in \widehat{\mathcal{Y}}_T \ ; \quad & h(0,\cdot) = 0, \quad f_1(0,\cdot) = -a_\rho(\rho^0)^\gamma \mathbb{I}_3, \quad f_2(0,\cdot) = 0, \\ f_3 = -\int_{\partial\mathcal{S}} (\mathbb{S}(v^0) - a_\rho(\rho^0)^\gamma) n \, d\Gamma, \quad & f_4 = -\int_{\partial\mathcal{S}} y \times (\mathbb{S}(v^0) - a_\rho(\rho^0)^\gamma) n \, d\Gamma, \\ a(0,\cdot) = 0, \quad \partial_t a(0,\cdot) = \widehat{a}, \quad & g_i(0,\cdot) = 0, \quad \|(h, f_1, f_2, f_3, f_4, a, g_1, g_2)\|_{\widehat{\mathcal{Y}}_T} \leqslant R\},\end{aligned}$$

such that

$$\widehat{a} = \begin{cases} -v^0 \cdot (\nabla n v^0) & \text{on } \partial\Omega, \\ \omega^0 \times (v^0 - \ell^0) \cdot n + (v^0_{\mathcal{S}} - v^0) \cdot \big(\omega^0 \times n + \nabla n (v^0 - v^0_{\mathcal{S}})\big) & \text{on } \partial\mathcal{S}. \end{cases}$$

Let consider the mapping $\mathcal{Z} : \mathcal{B}_{R,T} \longrightarrow \widehat{\mathcal{Y}}_T$ such that

$$\begin{aligned}&\mathcal{Z}(h, f_1, f_2, f_3, f_4, a, g_1, g_2) \\ &\qquad = (\mathbb{T}(\rho, v),\ \mathbb{F}_1(\rho, v),\ \mathbb{F}_2(\rho, v),\ \mathbb{F}_3(\rho, v, \ell, \omega),\ \mathbb{F}_4(\rho, v, \ell, \omega),\ \mathbb{B}(v, \ell, \omega),\ \mathbb{G}_1(v, \ell, \omega),\ \mathbb{G}_2(v, \ell, \omega)),\end{aligned}$$

where $(\rho, v, \ell, \omega)$ is the solution for the linearized system (3.1), (3.2), (3.3) and (3.4) associated to $h$, $f_1$, $f_2$, $a$, $g_1$, and $g_2$. We recall that the nonlinearities $\mathbb{T}$, $\mathbb{F}_1$, $\mathbb{F}_2$, $\mathbb{F}_3$, $\mathbb{F}_4$ are given in (2.7), (2.8), (2.9), (2.10) and

$$\mathbb{B}(v, \ell, \omega) = \begin{cases} \mathbb{B}_{\partial\Omega}(v, \ell, \omega) & \text{on } \partial\Omega \\ \mathbb{B}_{\partial\mathcal{S}}(v, \ell, \omega) & \text{on } \partial\mathcal{S} \end{cases}, \quad \mathbb{G}_i(v, \ell, \omega) = \begin{cases} (\mathbb{G}_{\partial\Omega}(v, \ell, \omega))_i & \text{on } \partial\Omega \\ (\mathbb{G}_{\partial\mathcal{S}}(v, \ell, \omega))_i & \text{on } \partial\mathcal{S} \end{cases} \quad i = 1, 2,$$

where $\mathbb{B}_{\partial\Omega}$, $\mathbb{B}_{\partial\mathcal{S}}$, $\mathbb{G}_{\partial\Omega}$, $\mathbb{G}_{\partial\mathcal{S}}$ are given in (2.11), (2.12) and (2.13), (2.14) respectively. For $R > 0$ large enough, we have that $\mathcal{B}_{R,T}$ is not empty. Indeed, let $v^\star$ be a function in $L^2(0,T;[H^4(\mathcal{F})]^3) \cap H^2(0,T;[L^2(\mathcal{F})]^3)$ such that $v^\star(0) = v^0$. Let $h^\star = 0$, $f_1^\star(t,\cdot) = -a_\rho(\rho^0)^\gamma \mathbb{I}_3$, $g_i^\star = 0$ and define for all $t > 0$

$$a^\star(t) = \begin{cases} -\displaystyle\int_0^t v^\star \cdot (\nabla n v^\star) \, ds & \text{on } \partial\Omega, \\ \displaystyle\int_0^t \omega^0 \times (v^\star - \ell^0) \cdot n + (v^0_{\mathcal{S}} - v^\star) \cdot \big(\omega^0 \times n + \nabla n (v^\star - v^0_{\mathcal{S}})\big) \, ds & \text{on } \partial\mathcal{S}, \end{cases}$$

and

$$f_3^\star(t) = -\int_{\partial\mathcal{S}} (\mathbb{S}(v^0) - a_\rho(\rho^0)^\gamma) n \, d\Gamma, \quad f_4^\star(t) = -\int_{\partial\mathcal{S}} y \times (\mathbb{S}(v^0) - a_\rho(\rho^0)^\gamma) n \, d\Gamma.$$

Then, from (1.7) there exists $R$ such that

$$\|f_1^\star\|_{[H^2(\mathcal{F})]^9} + |f_3^\star| + |f_4^\star| + \|\partial_t a^\star(0,\cdot)\|_{H^{3/2}(\partial\mathcal{F})} \leqslant R/2.$$

Thus, for $T$ small enough, we get that $(h^\star, f_1^\star, f_2^\star, f_3^\star, f_4^\star, a^\star, g^\star)$ belongs to $\mathcal{B}_{R,T}$. Then, $\mathcal{B}_{R,T}$ is non empty. We assume also that

$$\|(\rho^0, v^0, \ell^0, \omega^0)\|_{H^3(\mathcal{F}) \times \widehat{\mathcal{I}}_0} \leqslant R. \quad (4.1)$$

In this section, we prove that $\mathcal{Z}(\mathcal{B}_{R,T}) \subset \mathcal{B}_{R,T}$ for $T$ small enough and that the mapping $\mathcal{Z}|_{\mathcal{B}_{R,T}}$ defines a contraction. In the sequel, $C > 0$ is a generic constant that can change from a line to another and may depend on $R$ but not on $T$. From (3.51), we have the following estimate:

$$\|(\rho, v, \ell, \omega)\|_{\widehat{\mathcal{X}}_T} \leqslant C \left( R + \|(h, f_1, f_2, f_3, f_4, a, g_1, g_2)\|_{\widehat{\mathcal{Y}}_T} \right).$$

Let $(h, f_1, f_2, f_3, f_4, a, g_1, g_2) \in \mathcal{B}_{R,T}$, we need to show that $\mathcal{Z}(h, f_1, f_2, f_3, f_4, a, g_1, g_2) \in \mathcal{B}_{R,T}$. We have

$$\|(\rho, v, \ell, \omega)\|_{\widehat{\mathcal{X}}_T} \leqslant C. \tag{4.2}$$

From (2.1), we have

$$X(t, y) = y + \int_0^t v(t', y)\ dt', \quad t \in (0, T), \quad y \in \mathcal{F}.$$

Since the regularity of $X$ depends on the fluid flow, we have

$$X \in H^1(0, T; [H^4(\mathcal{F})]^3) \cap H^2(0, T; [H^2(\mathcal{F})]^3) \cap C^1([0, T]; [H^3(\mathcal{F})]^3) \cap H^3(0, T; [L^2(\mathcal{F})]^3).$$

It follows that

$$\|X - \mathrm{Id}\,\|_{L^\infty(0,T;[H^4(\mathcal{F})]^3)} \leqslant CT^{1/2}, \quad \|\partial_t X\|_{L^2(0,T;[H^3(\mathcal{F})]^3)} \leqslant CT^{1/2}, \tag{4.3}$$

$$\|b_X\|_{W^{1,\infty}(0,T;H^2(\mathcal{F}))\cap H^2(0,T;H^1(\mathcal{F}))\cap H^1(0,T;H^3(\mathcal{F}))} \leqslant C, \quad \|b_X - \mathbb{I}_3\|_{L^\infty(0,T;H^3(\mathcal{F}))} \leqslant CT^{1/2}, \tag{4.4}$$

$$\|\delta_X\|_{W^{1,\infty}(0,T;H^2(\mathcal{F}))\cap H^2(0,T;H^1(\mathcal{F}))\cap H^1(0,T;H^3(\mathcal{F}))} \leqslant C, \quad \|\delta_X - 1\|_{L^\infty(0,T;H^3(\mathcal{F}))} \leqslant CT^{1/2}. \tag{4.5}$$

Since $\rho$ satisfies (3.1), and $\nabla v \in L^2(0, T; [H^3(\mathcal{F})]^9)$, we deduce

$$\left\|\rho - \rho^0\right\|_{L^\infty(0,T;H^3(\mathcal{F}))} \leqslant CT^{1/2}. \tag{4.6}$$

From (4.1), we also have

$$\|b_X\|_{L^\infty(0,T;H^3(\mathcal{F}))} \leqslant C, \quad \|\delta_X\|_{L^\infty(0,T;H^3(\mathcal{F}))} \leqslant C. \tag{4.7}$$

At the boundary, we have by interpolation

$$\begin{aligned} \|b_X\|_{H^{1/4}(0,T;[L^\infty(\partial\mathcal{F})]^9)\cap H^{7/8}(0,T;[L^\infty(\partial\mathcal{F})]^9)\cap L^\infty(0,T;[L^\infty(\partial\mathcal{F})]^9)} &\leqslant C, \\ \|\delta_X\|_{H^{1/4}(0,T;L^\infty(\partial\mathcal{F}))\cap H^{7/8}(0,T;L^\infty(\partial\mathcal{F}))\cap L^\infty(0,T;L^\infty(\partial\mathcal{F}))} &\leqslant C, \end{aligned} \tag{4.8}$$

and

$$\|b_X\|_{H^{5/4}(0,T;[H^2(\partial\mathcal{F})]^9)} \leqslant C, \quad \|\delta_X\|_{H^{5/4}(0,T;H^2(\partial\mathcal{F}))} \leqslant C, \tag{4.9}$$

$$\|\partial_t b_X\|_{H^{7/8}(0,T;[H^{3/4}(\partial\mathcal{F})]^9)} \leqslant C, \quad \|\partial_t \delta_X\|_{H^{7/8}(0,T;H^{3/4}(\partial\mathcal{F}))} \leqslant C. \tag{4.10}$$

**Remark 4.1.** *In [13, Lemma A.5], we can use interpolation arguments and we infer that the constants in the estimates do not depend on $T$.*

## 4.1 Estimation on $\mathbb{T}$, $\mathbb{F}_1$ and $\mathbb{F}_2$

First, we show that $\mathbb{T} \in L^2(0, T; H^3(\mathcal{F})) \cap L^\infty(0, T; H^2(\mathcal{F}))$, from (4.6), (4.5)

$$\begin{aligned} \|\mathbb{T}(v, \eta, \rho)\|_{L^2(0,T;H^3(\mathcal{F})} \leqslant\ & \|\frac{\rho}{\delta_X} - \rho^0\|_{L^\infty(0,T;H^3(\mathcal{F}))}\, \|\nabla v\|_{L^2(0,T;[H^3(\mathcal{F})]^9)} \\ & + \|\frac{\rho}{\delta_X}\|_{L^\infty(0,T;H^3(\mathcal{F}))}\, \|\nabla v\|_{L^2(0,T;[H^3(\mathcal{F})]^9)}\, \|b_X - \mathbb{I}_3\|_{L^\infty(0,T;[H^3(\mathcal{F})]^9)} \leqslant CT^{1/2}, \end{aligned}$$

$$\|\mathbb{T}(v,\eta,\rho)\|_{L^\infty(0,T;H^2(\mathcal{F}))} \leqslant \|\frac{\rho}{\delta_X}-\rho^0\|_{L^\infty(0,T;H^3(\mathcal{F}))}\|\nabla v\|_{L^\infty(0,T;[H^2(\mathcal{F})]^9)}$$
$$+\|\frac{\rho}{\delta_X}\|_{L^\infty(0,T;H^3(\mathcal{F}))}\|\nabla v\|_{L^\infty(0,T;[H^2(\mathcal{F})]^9)}\|b_X-\mathbb{I}_3\|_{L^\infty(0,T;[H^3(\mathcal{F})]^9)} \leqslant CT^{1/2},$$

and since $\nabla v \in L^\infty(0,T;[H^2(\mathcal{F})]^9)$, we get

$$\|\partial_t\mathbb{T}(v,\eta,\rho)\|_{L^2(0,T;H^1(\mathcal{F}))} \leqslant \Big(\|\frac{\rho}{\delta_X}-\rho^0\|_{L^\infty(0,T;H^3(\mathcal{F}))}$$
$$+\|\frac{\rho}{\delta_X}\|_{L^\infty(0,T;H^3(\mathcal{F}))}\|b_X-\mathbb{I}_3\|_{L^\infty(0,T;[H^3(\mathcal{F})]^9)}\Big)\|\nabla\partial_t v\|_{L^2(0,T;[H^1(\mathcal{F})]^9)}$$
$$+\|\partial_t\left(\frac{\rho}{\delta_X}b_X\right)\|_{L^\infty(0,T;[H^2(\mathcal{F})]^9)}\|\nabla v\|_{L^2(0,T;[H^2(\mathcal{F})]^9)} \leqslant CT^{1/2}.$$

Let prove that $\mathbb{F}_1 \in L^2(0,T;[H^3(\mathcal{F})]^9)\cap H^1(0,T;[H^1(\mathcal{F})]^9)$. Let estimate for example

$$\|\frac{1}{\delta_X}\nabla v(b_X)^\top b_X-\nabla v\|_{L^2(0,T;[H^3(\mathcal{F})]^9)}$$
$$\leqslant \|\frac{1}{\delta_X}-1\|_{L^\infty(0,T;H^3(\mathcal{F}))}\|\nabla v\|_{L^2(0,T;[H^3(\mathcal{F})]^9)}\left\|b_X^\top b_X\right\|_{L^\infty(0,T;[H^3(\mathcal{F})]^9)}$$
$$+\|\frac{1}{\delta_X}\|_{L^\infty(0,T;H^3(\mathcal{F}))}\|\nabla v\|_{L^2(0,T;[H^3(\mathcal{F})]^9)}\left\|b_X^\top-\mathbb{I}_3\right\|_{L^\infty(0,T;[H^3(\mathcal{F})]^9)}\|b_X\|_{L^\infty(0,T;[H^3(\mathcal{F})]^9)}$$
$$+\|\frac{1}{\delta_X}\|_{L^\infty(0,T;H^3(\mathcal{F}))}\|\nabla v\|_{L^2(0,T;[H^3(\mathcal{F})]^9)}\left\|b_X^\top\right\|_{L^\infty(0,T;[H^3(\mathcal{F})]^9)}\|b_X-\mathbb{I}_3\|_{L^\infty(0,T;[H^3(\mathcal{F})]^9)}.$$

Using (4.5), (4.6), (4.2), (4.3), (4.7) and the fact that $H^3(\mathcal{F})\times H^3(\mathcal{F})\hookrightarrow H^3(\mathcal{F})$. We obtain that

$$\|\frac{1}{\delta_X}\nabla v(b_X)^\top b_X-\nabla v\|_{L^2(0,T;[H^3(\mathcal{F})]^9)} \leqslant CT^{1/2}. \tag{4.11}$$

In the other hand, the pressure term is estimated as follows

$$\|p(\rho)b_X\|_{L^2(0,T;[H^3(\mathcal{F})]^9)} \leqslant T^{1/2}\|p(\rho)b_X\|_{L^\infty(0,T;[H^3(\mathcal{F})]^9)} \leqslant CT^{1/2}. \tag{4.12}$$

We obtain also

$$\|\partial_t\left(\frac{1}{\delta_X}\nabla v(b_X)^\top b_X\right)-\partial_t\nabla v\|_{L^2(0,T;[H^1(\mathcal{F})]^9)}$$
$$\leqslant C\Big(\|\partial_t\nabla v\|_{L^2(0,T;[H^1(\mathcal{F})]^9)}\|\frac{(b_X)^\top b_X}{\delta_X}-\mathbb{I}_3\|_{L^\infty(0,T;[H^3(\mathcal{F})]^9)}$$
$$+T^{1/2}\|\nabla v\|_{L^\infty(0,T;[H^2(\mathcal{F})]^9)}\|\partial_t\left(\frac{(b_X)^\top b_X}{\delta_X}\right)\|_{L^\infty(0,T;[H^2(\mathcal{F})]^9)}\Big) \leqslant CT^{1/2}. \tag{4.13}$$

We also have

$$\|\partial_t\left(p(\rho)b_X\right)\|_{L^2(0,T;[H^1(\mathcal{F})]^9)} \leqslant T^{1/2}\Big(\|\partial_t p(\rho)\|_{L^\infty(0,T;H^2(\mathcal{F}))}\|b_X\|_{L^\infty(0,T;[H^3(\mathcal{F})]^9)}$$
$$+\|p(\rho)\|_{L^\infty(0,T;H^3(\mathcal{F}))}\|\partial_t b_X\|_{L^\infty(0,T;[H^2(\mathcal{F})]^9)}\Big) \leqslant CT^{1/2}. \tag{4.14}$$

Then, from (4.11), (4.14) and (4.13) we deduce that $\mathbb{F}_1 \in L^2(0,T;[H^3(\mathcal{F})]^9)\cap H^1(0,T;[H^1(\mathcal{F})]^9)$ and

$$\|\mathbb{F}_1\|_{L^2(0,T;[H^3(\mathcal{F})]^9)\cap H^1(0,T;[H^1(\mathcal{F})]^9)} \leqslant CT^{1/2}.$$

Now, we deal with $\mathbb{F}_2$, we have

$$\|\mathbb{F}_2\|_{L^2(0,T;[H^2(\mathcal{F})]^9)} \leqslant \|\rho\delta_X - \rho^0\|_{L^\infty(0,T;H^2(\mathcal{F}))}\|\partial_t v\|_{L^2(0,T;[H^2(\mathcal{F})]^3)} \leqslant CT^{1/2},$$

and

$$\begin{aligned}\|\partial_t\mathbb{F}_2\|_{L^2(0,T;[L^2(\mathcal{F})]^3)} &\leqslant \|\rho\delta_X - \rho^0\|_{L^\infty(0,T;H^3(\mathcal{F}))}\|\partial_{tt} v\|_{L^2(0,T;[L^2(\mathcal{F})]^3)}\\ &\quad + \|\partial_t\rho\|_{L^\infty(0,T;H^2(\mathcal{F}))}\|\delta_X\|_{L^\infty(0,T;H^3(\mathcal{F}))}\|\partial_t v\|_{L^\infty(0,T;[H^1(\mathcal{F})]^3)}\\ &\quad + \|\rho\|_{L^\infty(0,T;H^3(\mathcal{F}))}\|\partial_t\delta_X\|_{L^\infty(0,T;H^2(\mathcal{F}))}\|\partial_t v\|_{L^\infty(0,T;[H^1(\mathcal{F})]^3)} \leqslant CT^{1/2}. \quad (4.15)\end{aligned}$$

## 4.2 Estimation on $\mathbb{F}_3$ and $\mathbb{F}_4$

From (2.9) and since $\nabla v,\ b_X \in L^\infty(0,T;[L^\infty(\partial\mathcal{F})]^9)$, we have

$$\|\mathbb{F}_3\|_{[L^2(0,T)]^3} \leqslant CT^{1/2}. \quad (4.16)$$

In the other hand, $\partial_t\mathbb{F}_3$ contains mainly terms of the kind

$$\nabla\partial_t v b_X^\top b_X, \quad \partial_t b_X \nabla v b_X^\top b_X, \quad \rho^{\gamma-1}\partial_t\rho b_X, \quad \rho^\gamma \partial_t b_X b_X, \quad (4.17)$$

that need to be estimated in $L^2(0,T;[L^1(\partial\mathcal{S})]^9)$.

For the first term, since $\nabla\partial_t v \in L^\infty(0,T;[L^2(\mathcal{F})]^9) \cap L^2(0,T;[H^1(\mathcal{F})]^9)$, we get by interpolation that

$$\|\nabla\partial_t v\|_{L^{8/3}(0,T;[H^{1/4}(\partial\mathcal{S})]^9)} \leqslant C. \quad (4.18)$$

Then

$$\|\nabla\partial_t v b_X^\top b_X\|_{L^2(0,T;[L^1(\partial\mathcal{S})]^9)} \leqslant CT^{1/8}. \quad (4.19)$$

Concerning the other terms in (4.17), from (4.4), they all can be estimated in $L^\infty(0,T;[L^\infty(\partial\mathcal{S})]^9)$. Thus combining this fact with (4.16) we obtain that

$$\|\mathbb{F}_3\|_{[H^1(0,T)]^3} \leqslant CT^{1/8}. \quad (4.20)$$

Similarly we deal with $\mathbb{F}_4$, since $\nabla v,\ b_X \in L^\infty(0,T;[L^\infty(\partial\mathcal{F})]^9)$, $X \in L^\infty(0,T;L^\infty(\partial\mathcal{F}))$ and the fact that

$$\|h\|_{[L^\infty(0,T)]^3} \leqslant T^{1/2}\|\ell\|_{[H^1(0,T)]^3} \leqslant C. \quad (4.21)$$

In the other hand, we have

$$\omega(t) = \omega^0 + \int_0^t \omega'(s)\ ds.$$

since $\omega' \in L^2(0,T)$, it implies

$$\|\omega\|_{[L^\infty(0,T)]^3} \leqslant C. \quad (4.22)$$

Then,

$$\|\omega\|_{[L^2(0,T)]^3} \leqslant CT^{1/2}. \quad (4.23)$$

Hence, using (4.22) and (4.23), we obtain

$$\|J\omega \times \omega\|_{[L^2(0,T)]^3} \leqslant CT^{1/2}.$$

Finally using (4.21) and (4.3), we get

$$\|\left(J - J(0)\right)\omega'\|_{[L^2(0,T)]^3} \leqslant \|J - J(0)\|_{[L^\infty(0,T)]^9}\|\omega'\|_{L^2(0,T)}CT^{1/2}.$$

Thus

$$\|\mathbb{F}_4\|_{[L^2(0,T)]^3} \leqslant CT^{1/2}. \tag{4.24}$$

In the other hand, $\partial_t \mathbb{F}_4$ involves terms as (4.17) multiplied by $(X-h)$ that is in $L^\infty(0,T;[L^\infty(\partial\mathcal{S})]^3)$ and additionally we have the terms

$$\nabla v b_X^\top b_X(v-\ell), \quad \rho^\gamma b_X(v-\ell).$$

The above terms can also be estimated in $L^\infty(0,T;[L^\infty(\partial\mathcal{S})]^3)$ since

$$\|\ell\|_{[L^\infty(0,T)]^3} \leqslant |\ell^0| + T^{1/2}\|\ell\|_{[H^2(0,T)]^3}.$$

We have also the term $J\omega' \times \omega$ that can be estimated using (4.23) and the fact that $\omega \in W^{1,\infty}(0,T)$ as follows

$$\|J\omega' \times \omega\|_{[L^2(0,T)]^3} \leqslant CT^{1/2}.$$

Also from (4.23), (4.22), (4.3), (4.5) we obtain

$$\|J'\omega \times \omega\|_{[L^2(0,T)]^3} \leqslant CT^{1/2}.$$

We have also

$$\|\left(J - J(0)\right)\omega''\|_{[L^2(0,T)]^3} \leqslant \|J - J(0)\|_{[L^\infty(0,T)]^9}\|\omega''\|_{[L^2(0,T)]^3} \leqslant CT^{1/2},$$

and since $J' \in [L^\infty(0,T)]^9$, we get

$$\|J'\omega'\|_{[L^2(0,T)]^3} \leqslant \|J'\|_{[L^2(0,T)]^9}\|\omega'\|_{[L^\infty(0,T)]^3} \leqslant CT^{1/2}.$$

Thus, from (4.24) and the arguments above, we deduce finally

$$\|\mathbb{F}_4\|_{[H^1(0,T)]^3} \leqslant CT^{1/8}. \tag{4.25}$$

## 4.3 Estimation on $\mathbb{B}_{\partial\Omega}$ and $\mathbb{B}_{\partial\mathcal{S}}$

Now let us estimate the boundary terms $\mathbb{B}_{\partial\Omega}$ and $\mathbb{B}_{\partial\mathcal{S}}$ defined in (2.11) and (2.12). It suffices to estimate

$$(\widetilde{n}_X - n)\cdot v, \quad \omega \times (X - \mathrm{Id})\cdot \widetilde{n}_X.$$

We notice that $\widetilde{n}_X(t,\cdot) = Q(t)n(Q(t)^\top(X(t,\cdot) - h(t)))$. We get

$$\widetilde{n}_X(t,y) - n(y) = \widetilde{n}(t,X(t,y)) - n(y) = (Q(t) - \mathbb{I}_3)n(Q(t)^\top(X(t,y) - h(t))) + n(Q^\top(X(t,y) - h(t))) - n(y).$$

Using Taylor-Lagrange formula, we get

$$\begin{aligned}\widetilde{n}_X(t,y) - n(y) = (Q(t) - \mathbb{I}_3)n(Q(t)^\top(X(t,y) - h(t))) \\ + \int_0^1 \nabla n(y + \theta(Q(t)^\top(X(t,y) - h) - y))\, d\theta \left(Q(t)^\top(X(t,y) - h(t)) - y\right).\end{aligned}$$

Since $Q \in [H^3(0,T)]^9$, $h \in [H^3(0,T)]^3$, we get that

$$\|Q - \mathbb{I}_3\|_{[L^\infty(0,T)]^9} \leqslant CT, \quad \|h\|_{[L^\infty(0,T)]^3} \leqslant CT,$$

and from (4.3), we get that

$$\begin{aligned}\|Q^\top(X-h) - \mathrm{Id}\,\|_{L^\infty(0,T;[H^{7/2}(\partial\mathcal{S})]^3} \leqslant \|Q^\top(X - \mathrm{Id})\|_{L^\infty(0,T;[H^{7/2}(\partial\mathcal{S})]^3} \\ + \|(Q^\top - \mathbb{I}_3)\|_{[L^\infty(0,T)]^9} + \|Q^\top h\|_{[L^\infty(0,T)]^3} \leqslant CT^{1/2}.\end{aligned} \tag{4.26}$$

Since $\eta_{\mathcal{S}}$ is smooth enough, from (4.26) and [16, Theorem 1.1'] we get that

$$\|n(Q^\top(X-h))\|_{L^\infty(0,T;[H^{7/2}(\partial\mathcal{S})]^3)} + \|\nabla n(y+\theta(Q^\top(X-h)-\mathrm{Id}))\|_{L^\infty(0,T;[H^{7/2}(\partial\mathcal{S})]^3)} \leqslant C.$$

Then, using the same argument on the fixed boundary $\partial\Omega$, we obtain that

$$\|\widetilde{n}_X - n\|_{L^\infty(0,T;[H^{7/2}(\partial\mathcal{F})]^3)} \leqslant CT^{1/2}. \tag{4.27}$$

Since $H^{7/2}(\partial\mathcal{F}) \times H^{7/2}(\partial\mathcal{F}) \hookrightarrow H^{7/2}(\partial\mathcal{F})$, we get

$$\|(\widetilde{n}_X - n)\cdot v\|_{L^2(0,T;H^{7/2}(\partial\mathcal{F}))} \leqslant C\|\widetilde{n}_X - n\|_{L^\infty(0,T;[H^{7/2}(\partial\mathcal{F})]^3)}\|v\|_{L^2(0,T;[H^{7/2}(\partial\mathcal{F})]^3)} \leqslant CT^{1/2}. \tag{4.28}$$

We get also from (4.3) and [16, Theorem 1.1']

$$\begin{aligned}&\|\omega\times(X-\mathrm{Id})\cdot\widetilde{n}_X\|_{L^2(0,T;H^{7/2}(\partial\mathcal{S}))}\\ &\qquad\leqslant \|\omega\|_{L^2(0,T)}\|X-\mathrm{Id}\,\|_{L^\infty(0,T;[H^{7/2}(\partial\mathcal{F})]^3)}\|\widetilde{n}_X\|_{L^\infty(0,T;[H^{7/2}(\partial\mathcal{S})]^3)} \leqslant CT^{1/2}.\end{aligned} \tag{4.29}$$

Let estimate $\partial_t\mathbb{B}_{\partial\Omega}$ and $\partial_t\mathbb{B}_{\partial\mathcal{S}}$ in $H^{3/4}(0,T;L^2(\partial\mathcal{F}))$. In the sequel, we commonly make a change of variable to reduce the time interval into $(0,1)$ to make sure that the constant $C$ is independent of $T$. We use the following notation

$$f(t) = f(t^*T) = f^*(t^*), \quad t^* \in (0,1).$$

Since $\partial_t v \in H^{3/4}(0,T;[L^2(\partial\mathcal{F})]^3)$, using (2.2) we get

$$\begin{aligned}&\|(\widetilde{n}_X - n)\cdot\partial_t v\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} = \|\left(\frac{b_X}{|b_X n|} - \mathbb{I}_3\right) n\cdot\partial_t v\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))}\\ &\leqslant \|\left(\frac{b_X}{|b_X n|} - \mathbb{I}_3\right) n\cdot(\partial_t v - (\partial_t v)(0))\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} + \|\left(\frac{b_X}{|b_X n|} - \mathbb{I}_3\right) n\cdot(\partial_t v)(0)\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))}.\end{aligned} \tag{4.30}$$

We estimate the first term in (4.30) as follows

$$\begin{aligned}&\|\left(\frac{b_X}{|b_X n|} - \mathbb{I}_3\right) n\cdot(\partial_t v - (\partial_t v)(0))\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))}\\ &\qquad= T^{-1/4}T^{-1}\|\left(\frac{b^*_X}{|b^*_X n|} - \mathbb{I}_3\right) n\cdot(\partial_{t^*}v^* - (\partial_{t^*}v^*)(0))\|_{H^{3/4}(0,1;L^2(\partial\mathcal{F}))}\\ &\qquad\leqslant CT^{-5/4}\|\partial_{t^*}v^* - (\partial_{t^*}v^*)(0)\|_{H^{3/4}(0,1;[L^2(\partial\mathcal{F})]^3)}\|\left(\frac{b^*_X}{|b^*_X n|} - \mathbb{I}_3\right)\|_{H^{7/8}(0,1;[L^\infty(\partial\mathcal{F})]^9)}\\ &\leqslant CT^{-5/4}T^{5/4}\|\partial_t v - (\partial_t v)(0)\|_{H^{3/4}(0,T;[L^2(\partial\mathcal{F})]^3)}T^{3/8}\|\left(\frac{b_X}{|b_X n|} - \mathbb{I}_3\right)\|_{H^{7/8}(0,T;[L^\infty(\partial\mathcal{F})]^9)} \leqslant CT^{3/8}.\end{aligned} \tag{4.31}$$

Since $\partial_t v \in L^\infty(0,T;[H^{1/2}(\partial\mathcal{F})]^3)$ and $H^{1/2}(\partial\mathcal{F}) \hookrightarrow L^4(\partial\mathcal{F})$, then using (A.3) we have for the second term in (4.30)

$$\begin{aligned}\|\left(\frac{b_X}{|b_X n|} - \mathbb{I}_3\right) n\cdot(\partial_t v)(0)\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} &\leqslant C\|(\partial_t v)(0)\|_{[L^4(\partial\mathcal{F})]^3}\|\frac{b_X}{|b_X n|} - \mathbb{I}_3\|_{H^{3/4}(0,T;[L^4(\partial\mathcal{F})]^9)}\\ &\leqslant CT^{7/8-3/4}\|\frac{b_X}{|b_X n|} - \mathbb{I}_3\|_{H^{7/8}(0,T;[L^\infty(\partial\mathcal{F})]^9)} \leqslant CT^{1/8}.\end{aligned} \tag{4.32}$$

Since $v \in L^\infty(0,T;[L^\infty(\partial\mathcal{F})]^3)$ and $\partial_t\left(\frac{b_X}{|b_X n|}\right) \in L^\infty(0,T;[L^\infty(\partial\mathcal{F})]^9)$, we can write

$$\begin{aligned}&\|\partial_t\left(\frac{b_X}{|b_X n|}\right) n\cdot v\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} \leqslant \|\partial_t\left(\frac{b_X}{|b_X n|}\right) n\cdot(v-v^0)\|_{H^{3/4}(0,T,L^2(\partial\mathcal{F}))}\\ &\quad+\|(\partial_t\left(\frac{b_X}{|b_X n|}\right) n - (\partial_t\left(\frac{b_X}{|b_X n|}\right)(0))n)\cdot v^0\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} + \|(\partial_t\left(\frac{b_X}{|b_X n|}\right) n\cdot v)(0)\|_{H^{3/4}(0,T;[L^2(\partial\mathcal{F})]^3)}.\end{aligned} \tag{4.33}$$

The third term in (4.33) gives

$$\|(v\cdot\partial_t\left(\frac{b_X}{|b_Xn|}\right)n)(0)\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))}\leqslant CT^{1/2}. \tag{4.34}$$

Since $\partial_t\left(\frac{b_X}{|b_Xn|}\right)\in H^{7/8}(0,T;[L^2(\partial\mathcal{F})]^9)$ and $v\in H^{3/4}(0,T;[L^\infty(\partial\mathcal{F})]^3)$, arguing as (4.30) and (4.31), we obtain

$$\|\partial_t\left(\frac{b_X}{|b_Xn|}\right)n\cdot(v-v^0)\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))}+\|(\partial_t\left(\frac{b_X}{|b_Xn|}\right)n-(\partial_t\left(\frac{b_X}{|b_Xn|}\right)n)(0))\cdot v^0\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))}\leqslant CT^{1/8}. \tag{4.35}$$

In the other hand, we notice that

$$\begin{aligned}\|\omega'\times(X-\mathrm{Id})\cdot\frac{b_X}{|b_Xn|}n\|_{H^1(0,T;L^2(\partial\mathcal{S}))}+\|\omega\times v\cdot\frac{b_X}{|b_Xn|}n-(\omega\times v\cdot\frac{b_X}{|b_Xn|}n)(0)\|_{H^1(0,T;L^2(\partial\mathcal{S}))}\\ +\|\omega\times(X-\mathrm{Id})\cdot\partial_t\left(\frac{b_X}{|b_Xn|}\right)n\|_{H^1(0,T;L^2(\partial\mathcal{S}))}\leqslant C.\end{aligned} \tag{4.36}$$

From Lemma A.3

$$\begin{aligned}\|\omega'\times(X-\mathrm{Id})\cdot\frac{b_X}{|b_Xn|}n\|_{H^{3/4}(0,T;L^2(\partial\mathcal{S}))}+\|\omega\times v\cdot\frac{b_X}{|b_Xn|}n-(\omega\times v\cdot\frac{b_X}{|b_Xn|}n)(0)\|_{H^{3/4}(0,T;L^2(\partial\mathcal{S}))}\\ +\|\omega\times(X-\mathrm{Id})\cdot\partial_t\left(\frac{b_X}{|b_Xn|}\right)n\|_{H^{3/4}(0,T;L^2(\partial\mathcal{S}))}\leqslant CT^{1/4}.\end{aligned} \tag{4.37}$$

Since $(\omega\times v\cdot\frac{b_X}{|b_Xn|}n)(0)\in L^\infty(0,T;L^2(\partial\mathcal{S}))$, we get finally from (4.37)

$$\|\partial_t\left(\omega\times(X-\mathrm{Id})\cdot\frac{b_X}{|b_Xn|}n\right)\|_{H^{3/4}(0,T;L^2(\partial\mathcal{S}))}\leqslant CT^{1/4}. \tag{4.38}$$

The last term in (2.12) can be estimated similarly. Gathering the estimates (4.28), (4.31), (4.32), (4.34), (4.35), (4.29) and (4.38) we get

$$\|\mathbb{B}_{\partial\mathcal{S}}\|_{H^{7/4}(0,T;L^2(\partial\mathcal{F}))\cap L^2(0,T;H^{7/2}(\partial\mathcal{F}))}+\|\mathbb{B}_{\partial\Omega}\|_{H^{7/4}(0,T;L^2(\partial\mathcal{F}))\cap L^2(0,T;H^{7/2}(\partial\mathcal{F}))}\leqslant CT^{1/8}. \tag{4.39}$$

## 4.4 Estimation on $\mathbb{G}_{\partial\Omega}$ and $\mathbb{G}_{\partial\mathcal{S}}$

First, we extend $\mathbb{G}_{\partial\Omega}$ and $\mathbb{G}_{\partial\mathcal{S}}$ over $\mathcal{F}$ and we use the same calculation as $\mathbb{F}_1$, we see that $\mathbb{G}_{\partial\Omega},\mathbb{G}_{\partial\mathcal{S}}\in L^2(0,T;[H^3(\mathcal{F})]^2)$. Using the trace theorem, we get

$$\|(\mathbb{G}_{\partial\mathcal{S}})_i\|_{L^2(0,T;H^{5/2}(\partial\mathcal{F}))}+\|(\mathbb{G}_{\partial\Omega})_i\|_{L^2(0,T;H^{5/2}(\partial\mathcal{F}))}\leqslant CT^{1/2},\quad i=1,2. \tag{4.40}$$

Now we estimate $\partial_t(\mathbb{G}_{\partial\Omega})_i$ and $\partial_t(\mathbb{G}_{\partial\mathcal{S}})_i$ in $H^{1/4}(0,T;L^2(\partial\mathcal{F}))$. We notice that

$$2\mu\mathbb{D}(v)n\cdot\tau-2\mu\mathbb{D}^X(v)\widetilde{n}_X\cdot\widetilde{\tau}^i_X=2\mu\mathbb{D}(v)n\cdot\tau-2\mu b_X^{-1}\mathbb{D}^X(v)b_Xn\cdot\tau^i.$$

Let estimate for example

$$\begin{aligned}\|\partial_t\left(b_X^{-1}\frac{1}{\delta_X}\nabla v(b_X)^\top b_Xn\cdot\tau^i\right)-\partial_t\nabla vn\cdot\tau^i\|_{H^{1/4}(0,T;L^2(\partial\mathcal{F}))}\\ \leqslant C\Big(\|\partial_t\nabla v\left(b_X^{-1}\frac{1}{\delta_X}(b_X)^\top b_X-\mathbb{I}_3\right)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^9)}\\ +\|\nabla v\partial_t\left(b_X^{-1}\frac{1}{\delta_X}(b_X)^\top b_X\right)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^9)}\Big).\end{aligned} \tag{4.41}$$

From Lemma A.4, the first term at the right hand side of (4.41) gives

$$\|\nabla\partial_t v(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X-\mathbb{I}_3)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^9)}$$
$$\leqslant CT^{1/8}\|\nabla\partial_t v\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^9)}\|b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X-\mathbb{I}_3\|_{H^{7/8}(0,T;[L^\infty(\partial\mathcal{F})]^9)}. \quad (4.42)$$

Using (4.8) and Slobodeckij seminorm with $s=7/8$, we deduce that

$$\|b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X-\mathbb{I}_3\|_{H^{7/8}(0,T;[L^\infty(\partial\mathcal{F})]^9)\cap L^\infty(0,T;[L^\infty(\partial\mathcal{F})]^9)}\leqslant C. \quad (4.43)$$

Hence,

$$\|\nabla\partial_t v(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X-\mathbb{I}_3)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^9)}\leqslant CT^{1/8}. \quad (4.44)$$

In the other hand, the second term of (4.41) is estimated thanks to Lemma A.4, (4.8), (4.10), Lemma A.4, using the fact that $\nabla v\in H^{7/8}(0,T;[H^{3/4}(\partial\mathcal{F})]^9)$ and $H^{3/4}(\partial\mathcal{F})\hookrightarrow L^8(\partial\mathcal{F})$

$$\|(\nabla v)\partial_t\left(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X\right)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^9)}$$
$$\leqslant CT^{1/8}\|\nabla v-(\nabla v)(0)\|_{H^{7/8}(0,T;[L^2(\partial\mathcal{F})]^9)}\|\partial_t\left(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X\right)\|_{H^{1/4}(0,T;[L^\infty(\partial\mathcal{F})]^9)}$$
$$+\|(\nabla v)(0)\left(\partial_t\left(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X\right)-\partial_t\left(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X\right)(0)\right)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^9)}$$
$$+\|(\nabla v)(0)\partial_t\left(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X\right)(0)\|_{L^2(0,T;[L^2(\partial\mathcal{F})]^9)}. \quad (4.45)$$

Since $\partial_t b_X$, $\partial_t\delta_X$, $b_X$, $b_X^\top$ and $\delta_X$ are in $H^{1/4}(0,T;[L^\infty(\partial\mathcal{F})]^9)\cap L^\infty(0,T;[L^\infty(\partial\mathcal{F})]^9)$, using (4.8) and (4.9), we obtain

$$\|\partial_t\left(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X\right)\|_{H^{1/4}(0,T;[L^\infty(\partial\mathcal{F})]^9)}\leqslant C. \quad (4.46)$$

In the other hand, using Lemma A.3, (4.8) and (4.10), we obtain

$$\|\nabla v(0)\left(\partial_t\left(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X\right)-\partial_t\left(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X\right)(0)\right)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^9)}$$
$$\leqslant T^{5/8}\|\nabla v(0)\left(\partial_t\left(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X\right)-\partial_t\left(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X\right)(0)\right)\|_{H^{7/8}(0,T;[L^2(\partial\mathcal{F})]^9)}\leqslant CT^{5/8}. \quad (4.47)$$

Since $\nabla v,\ \partial_t\left(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X\right)\in L^\infty(0,T;[H^{3/2}(\partial\mathcal{F})]^9)$, we have

$$\|(\nabla v)(0)\partial_t\left(b_X^{-1}\frac{1}{\delta_X}b_X^\top b_X\right)(0)\|_{L^2(0,T;[L^2(\partial\mathcal{F})]^9)}\leqslant CT^{1/2}. \quad (4.48)$$

Then, from (4.45), (4.46), (4.47) and (4.48), we get

$$\|2\mu\mathbb{D}(v)n\cdot\tau-2\mu\mathbb{D}^X(v)\widetilde{n}_X\cdot\widetilde{\tau}_X^i\|_{H^{1/4}(0,T;L^2(\partial\mathcal{F}))}\leqslant CT^{1/8}. \quad (4.49)$$

The third term in (2.13) writes

$$\beta_0 v\cdot(\tau-\widetilde{\tau}_X^i)=\beta_0(\mathbb{I}_3-|b_Xn|b_X^{-1})v\cdot\tau.$$

We get

$$\|\partial_t\left((\mathbb{I}_3-|b_Xn|b_X^{-1})v\right)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^3)}\leqslant\|(\mathbb{I}_3-|b_Xn|b_X^{-1})\partial_tv\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^3)}$$
$$+\|(\partial_t(|b_Xn|b_X^{-1})(v-v^0)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^3)}+\|(\partial_t(|b_Xn|b_X^{-1})-\partial_t(|b_Xn|b_X^{-1})(0))v^0\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^3)}$$
$$+\|\partial_t(|b_Xn|b_X^{-1}v)(0)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^3)}. \quad (4.50)$$

Since $\partial_tv\in H^{1/4}(0,T;[H^1(\partial\mathcal{F})]^3)$ and $\mathbb{I}_3-|b_Xn|b_X^{-1}\in H^{7/8}(0,T;[L^\infty(\partial\mathcal{F})]^9)$, then

$$\|(\mathbb{I}_3-|b_Xn|b_X^{-1})\partial_tv\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^3)}\leqslant CT^{1/8}. \quad (4.51)$$

We have

$$\|\partial_t(|b_Xn|b_X^{-1})(v-v^0)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^3)}$$
$$\leqslant CT^{1/8}\|\partial_t(|b_Xn|b_X^{-1})\|_{H^{1/4}(0,T;[L^\infty(\partial\mathcal{F})]^9)}\|v-v^0\|_{H^{7/8}(0,T;[L^2(\partial\mathcal{F})]^3)}\leqslant CT^{1/8}. \quad (4.52)$$

We have also from Lemma A.3

$$\|(\partial_t(|b_Xn|b_X^{-1})-\partial_t(|b_Xn|b_X^{-1})(0))v^0\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^3)}$$
$$\leqslant T^{5/8}\|(\partial_t(|b_Xn|b_X^{-1})-\partial_t(|b_Xn|b_X^{-1})(0))v^0\|_{H^{7/8}(0,T;[L^2(\partial\mathcal{F})]^3)}\leqslant CT^{5/8}. \quad (4.53)$$

Since $v,\ \partial_tb_X^{-1}\in L^\infty(0,T;[H^{3/2}(\partial\mathcal{F})]^9)$, thus

$$\|(\partial_t(|b_Xn|b_X^{-1})v)(0)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^3)}\leqslant CT^{1/2}. \quad (4.54)$$

Then, combining (4.50), (4.51), (4.52), (4.53) and (4.54), we obtain

$$\|\partial_t\left((\mathbb{I}_3-|b_Xn|b_X^{-1})v\right)\|_{H^{1/4}(0,T;[L^2(\partial\mathcal{F})]^3)}\leqslant CT^{1/8}. \quad (4.55)$$

Using the same arguments as we obtained (4.38), we get

$$\|\partial_t\left(\omega\times(X-\mathrm{Id})\cdot|b_Xn|b_X^{-\top}\tau^i\right)\|_{H^{1/4}(0,T;L^2(\partial\mathcal{S}))}\leqslant CT^{3/4}.$$

Gathering (4.4), (4.55), (4.49) and (4.40), we obtain finally

$$\|(\mathbb{G}_{\partial\mathcal{S}})_i\|_{H^{5/4}(0,T;L^2(\partial\mathcal{F}))\cap L^2(0,T;H^{5/2}(\partial\mathcal{F}))}+\|(\mathbb{G}_{\partial\Omega})_i\|_{H^{5/4}(0,T;L^2(\partial\mathcal{F}))\cap L^2(0,T;H^{5/2}(\partial\mathcal{F}))}\leqslant CT^{1/8}. \quad (4.56)$$

Same calculations implies that $\mathcal{Z}$ defines also a contraction mapping on $\mathcal{B}_{R,T}$. That concludes the proof of Theorem 1.2.

# A Useful Estimates

**Lemma A.1** (Korn Inequality)**.** *Let $\beta>0$. There exists a constant $C>0$ such that*

$$\|\widehat{u}\|^2_{[H^1(\mathcal{F})]^9}\leqslant C\left(\|\mathbb{D}(\widehat{u})\|^2_{[L^2(\mathcal{F})]^9}+\|\beta\widehat{u}_\tau\|^2_{[L^2(\partial\mathcal{F})]^3}\right),$$

*for all $\widehat{u}\in[H^1(\mathcal{F})]^3$ such that $\widehat{u}_n=0$.*

*Proof.* The proof is classical, it can be found in [1]. For the sake of completeness, we sketch the ideas of the proof. We only need to show that there exist a positive constant $C$ such that

$$\|\mathbb{D}(\widehat{u})\|^2_{[L^2(\mathcal{F})]^9}+\|\beta\widehat{u}_\tau\|^2_{[L^2(\partial\mathcal{F})]^3}\geqslant C\|\widehat{u}\|^2_{[L^2(\mathcal{F})]^3}, \quad (A.1)$$

and the result will follow from the classical Korn inequality since $\mathcal{F}$ is Lipschitz domain (see for instance [47]). Suppose that the inequality (A.1) is not true, then for any positive integer $k$ there exists $u_k$ such that

$$\|\mathbb{D}(\widehat{u}_k)\|^2_{[L^2(\mathcal{F})]^9} + \|\beta(\widehat{u}_k)_\tau\|^2_{[L^2(\partial\mathcal{F})]^3} \leqslant \frac{1}{k}\|\widehat{u}_k\|^2_{[L^2(\mathcal{F})]^3}.$$

We normalize by $\|\widehat{u}_k\|_{[L^2(\mathcal{F})]^3}$, in such a way we get a sequence denoted again $(\widehat{u}_k)_k$ that verifies

$$\|\widehat{u}_k\|_{[L^2(\mathcal{F})]^3} = 1.$$

We deduce from the classical Korn inequality that $(\widehat{u}_k)_k$ is bounded in $[H^1(\mathcal{F})]^3$, then up to a subsequence, it converges weakly to $\widehat{u}$ in $[H^1(\mathcal{F})]^3$. Consequently

$$\|\mathbb{D}(\widehat{u})\|^2_{[L^2(\mathcal{F})]^9} + \|\beta\widehat{u}_\tau\|^2_{[L^2(\partial\mathcal{F})]^3} \leqslant \liminf_{k\longrightarrow\infty} \|\mathbb{D}(\widehat{u}_k)\|^2_{[L^2(\mathcal{F})]^9} + \|\beta(\widehat{u}_k)_\tau\|^2_{[L^2(\partial\mathcal{F})]^3} = 0.$$

Then, $\mathbb{D}(\widehat{u}) = 0$ in $\mathcal{F}$. It implies that $\widehat{u}(z) = a + b \times z$ in $\mathcal{F}$, but since $\widehat{u} = 0$ on $\partial\mathcal{F}$, then $a = b = 0$, which contradicts the fact that $\|\widehat{u}_k\|_{[L^2(\mathcal{F})]^3} = 1$. □

**Lemma A.2.** *Let $\mathcal{X}$ a Banach space and $\xi \in H^{3/4}(0,T;\mathcal{X})$ such that $\xi(0) = 0$, then there exists $\xi_{ext} \in H^{3/4}(0,T^*;\mathcal{X})$, where $T^* > 2T$ and*

$$\|\xi_{ext}\|_{H^{3/4}(0,T^*;\mathcal{X})} \leqslant C\|\xi\|_{H^{3/4}(0,T;\mathcal{X})}, \tag{A.2}$$

*with $C$ a constant that does not depend on $T$.*

*Proof.* We define $\xi_{ext}$ as follows

$$\begin{cases} \xi_{ext}(t) = \xi(t), & t \in [0,T], \\ \xi_{ext}(t) = \xi(2T-t), & t \in [T,2T], \\ \xi_{ext}(t) = 0, & t \in [2T,T^*], \end{cases}$$

We have that $\xi_{ext} \in H^{3/4}(0,T^*;\mathcal{X})$. Let us prove the estimate (A.2). Since $\xi(0) = 0$, then it suffices to show the estimate (A.2) using Slobodeckij seminorm with $s = 3/4$.

$$\begin{aligned}[\xi_{ext}]_{s,2,(0,T^*)} = \int_0^{T^*}\int_0^{T^*} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|^2_{\mathcal{X}}}{|t-t'|^{2s+1}}\,dt'dt = \int_0^{2T}\int_0^{2T} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|^2_{\mathcal{X}}}{|t-t'|^{2s+1}}\,dt'dt \\ + \int_0^{2T}\int_{2T}^{T^*} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|^2_{\mathcal{X}}}{|t-t'|^{2s+1}}\,dt'dt + \int_{2T}^{T^*}\int_0^{2T} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|^2_{\mathcal{X}}}{|t-t'|^{2s+1}}\,dt'dt.\end{aligned}$$

Then,

$$\begin{aligned}[\xi_{ext}]_{s,2,(0,T^*)} = \int_0^{2T}\int_0^{2T} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|^2_{\mathcal{X}}}{|t-t'|^{2s+1}}\,dt'dt + \int_0^{2T}\int_{2T}^{T^*} \frac{\|\xi_{ext}(t)\|^2_{\mathcal{X}}}{|t-t'|^{2s+1}}\,dt'dt \\ + \int_{2T}^{T^*}\int_0^{2T} \frac{\|\xi_{ext}(t')\|^2_{\mathcal{X}}}{|t-t'|^{2s+1}}\,dt'dt. \end{aligned} \tag{A.3}$$

Let us deal with the first term in the right hand side of (A.3).

$$\begin{aligned}\int_0^{2T}\int_0^{2T} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|^2_{\mathcal{X}}}{|t-t'|^{2s+1}}\,dt'dt = \int_0^{T}\int_0^{T} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|^2_{\mathcal{X}}}{|t-t'|^{2s+1}}\,dt'dt \\ + \int_0^{T}\int_T^{2T} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|^2_{\mathcal{X}}}{|t-t'|^{2s+1}}\,dt'dt + \int_T^{2T}\int_0^{T} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|^2_{\mathcal{X}}}{|t-t'|^{2s+1}}\,dt'dt \\ + \int_T^{2T}\int_T^{2T} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|^2_{\mathcal{X}}}{|t-t'|^{2s+1}}\,dt'dt. \end{aligned} \tag{A.4}$$

Since the second and the third terms in the right hand side of (A.4) are of the same kind then it suffices to treat one of them using a change of variables as follows

$$\int_0^T \int_T^{2T} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|_{\mathcal{X}}^2}{|t-t'|^{2s+1}}\, dt'dt = \int_0^T \int_0^T \frac{\|\xi(t)-\xi(t')\|_{\mathcal{X}}^2}{|2T-t-t'|^{2s+1}}\, dt'dt$$
$$= \int_0^T \int_0^T \frac{|t-t'|^{2s+1}}{|2T-t-t'|^{2s+1}} \frac{\|\xi(t)-\xi(t')\|_{\mathcal{X}}^2}{|t-t'|^{2s+1}}\, dt'dt. \quad (A.5)$$

Since $t, t' \in (0,T)$, then

$$\left|\frac{t-t'}{2T-t-t'}\right| \leqslant 1.$$

Thus,

$$\int_0^T \int_T^{2T} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|_{\mathcal{X}}^2}{|t-t'|^{2s+1}}\, dt'dt \leqslant \int_0^T \int_0^T \frac{\|\xi(t)-\xi(t')\|_{\mathcal{X}}^2}{|t-t'|^{2s+1}}\, dt'dt = \lfloor \xi \rfloor_{s,2,(0,T)}^2.$$

We obtain that

$$\int_0^{2T} \int_0^{2T} \frac{\|\xi_{ext}(t)-\xi_{ext}(t')\|_{\mathcal{X}}^2}{|t-t'|^{2s+1}}\, dt'dt \leqslant 4\lfloor \xi \rfloor_{s,2,(0,T)}^2. \quad (A.6)$$

Now, we deal with the second term in the right hand side of (A.3), the third term can be treated similarly. Let define $c(t)$ and $d(t)$ such that

$$c(t) = \frac{1}{t}\int_0^t (\xi(t)-\xi(t'))\, dt', \quad d(t) = \int_t^T \frac{c(t')}{t'}\, dt', \quad t \in (0,T).$$

We notice that

$$\xi(t) = c(t) - (d(t) - d(0)).$$

Indeed,

$$\xi(t) = \frac{1}{t}\int_0^t (\xi(t)-\xi(t'))\, dt' + \frac{1}{t}\int_0^t \xi(t')\, dt' = c(t) + \int_0^t \xi(t')\left(\frac{1}{t}-\frac{1}{t'}\right) dt' + \int_0^t \frac{\xi(t')}{t'}\, dt'$$
$$= c(t) + \int_0^t \xi(t')\left(-\int_{t'}^t \frac{1}{\tau^2}\, d\tau\right) dt' + \int_0^t \frac{\xi(t')}{t'}\, dt' = c(t) - \int_0^t \int_0^\tau \frac{\xi(t')}{\tau^2}\, dt'\, d\tau + \int_0^t \frac{\xi(t')}{t'}\, dt'$$
$$= c(t) + \int_0^t \frac{1}{\tau^2}\left[\int_0^\tau \xi(\tau) - \xi(t')\, dt'\right] d\tau = c(t) - (d(t) - d(0)).$$

We want to show that

$$\int_0^T t^{-2s}\|\xi(t)\|_{\mathcal{X}}^2\, dt \leqslant C\lfloor \xi \rfloor_{s,2,(0,T)}^2, \quad (A.7)$$

where $C$ is a constant that does not depend on $T$. We have

$$\int_0^T t^{-2s}\|c(t)\|_{\mathcal{X}}^2\, dt \leqslant \int_0^T \int_0^t \frac{\|\xi(t)-\xi(t')\|_{\mathcal{X}}^2}{t^{2s+1}}\, dt'dt = \int_0^T \int_{t'}^T \frac{\|\xi(t)-\xi(t')\|_{\mathcal{X}}^2}{t^{2s+1}}\, dtdt'$$
$$= \int_0^T \int_{t'}^T \frac{|t-t'|^{2s+1}}{t^{2s+1}} \frac{\|\xi(t)-\xi(t')\|_{\mathcal{X}}^2}{|t-t'|^{2s+1}}\, dtdt'. \quad (A.8)$$

Since $t \in (t',T)$ and $t \in (0,T)$ then

$$0 \leqslant \frac{t-t'}{t} \leqslant 1.$$

W obtain

$$\int_0^T t^{-2s}\|c(t)\|_{\mathcal{X}}^2\ dt \leqslant \int_0^T \int_{t'}^T \frac{\|\xi(t)-\xi(t')\|_{\mathcal{X}}^2}{|t-t'|^{2s+1}}\ dtdt' \leqslant \int_0^T \int_0^T \frac{\|\xi(t)-\xi(t')\|_{\mathcal{X}}^2}{|t-t'|^{2s+1}}\ dtdt' = \lfloor\xi\rfloor_{s,2,(0,T)}^2. \tag{A.9}$$

In the other hand,

$$d(t)-d(0) = -\int_0^t \frac{c(t')}{t'}\ dt'.$$

Then, by Hardy inequality, we get

$$\int_0^T t^{-2s}\|d(t)-d(0)\|_{\mathcal{X}}^2\ dt = \int_0^T \left(t^{-(s-1/2)}\int_0^t \frac{\|c(t')\|_{\mathcal{X}}}{t'}\ dt'\right)^2 \frac{dt}{t} \leqslant \frac{1}{(s-1/2)^2}\int_0^T t^{-2s}\|c(t)\|_{\mathcal{X}}^2\ dt. \tag{A.10}$$

Finally, using (A.9), we obtain

$$\int_0^T t^{-2s}\|d(t)-d(0)\|_{\mathcal{X}}^2\ dt \leqslant \frac{1}{(s-1/2)^2}\lfloor\xi\rfloor_{s,2,(0,T)}^2. \tag{A.11}$$

Then, by (A.11) and (A.9), we deduce (A.7). It remains to estimate the second term in the right hand side of (A.3) by using (A.7).

$$\begin{aligned}\int_0^{2T}\int_{2T}^{T^*} \frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{|t-t'|^{2s+1}}\ dt'dt &= \int_0^{2T}\|\xi_{ext}(t)\|_{\mathcal{X}}^2\int_{2T}^{T^*}\frac{1}{(t'-t)^{2s+1}}\ dt'dt \\ &= \int_0^{2T}\|\xi_{ext}(t)\|_{\mathcal{X}}^2\left[\frac{-1}{2s(t'-t)^{2s}}\right]_{2T}^{T^*}dt \leqslant 2\int_0^{2T}\frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{(2T-t)^{2s}}dt.\end{aligned} \tag{A.12}$$

We write

$$\int_0^{2T}\frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{(2T-t)^{2s}}dt = \int_0^T\frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{(2T-t)^{2s}}dt + \int_T^{2T}\frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{(2T-t)^{2s}}dt = \int_0^T\frac{t^{2s}}{(2T-t)^{2s}}\frac{\|\xi(t)\|_{\mathcal{X}}^2}{t^{2s}}dt + \int_0^T\frac{\|\xi(t)\|_{\mathcal{X}}^2}{t^{2s}}dt.$$

Since $t\in(0,T)$ then $t<2T-t$ and hence

$$\int_0^{2T}\frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{(2T-t)^{2s}}dt \leqslant 2\int_0^T t^{-2s}\|\xi(t)\|_{\mathcal{X}}^2dt. \tag{A.13}$$

From (A.7), we deduce that

$$\int_0^{2T}\frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{(2T-t)^{2s}}dt \leqslant C\lfloor\xi\rfloor_{s,2,(0,T)}^2. \tag{A.14}$$

Thus,

$$\int_0^{2T}\int_{2T}^{T^*}\frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{|t-t'|^{2s+1}}\ dt'dt \leqslant C\lfloor\xi\rfloor_{s,2,(0,T)}^2. \tag{A.15}$$

Gathering (A.15), (A.6), we obtain (A.2). □

We present also in the sequel some technical estimates concerning fractional Sobolev spaces that have been established in [13].

**Lemma A.3.** *Assume $\sigma_2\in(1/2,1]$ and $\sigma_1\in[0,\sigma_2]$. Using the above result, there exists a constant independent of $T$ such that for any $\xi\in H^{\sigma_2}(0,T;\mathfrak{X})$ and $\xi(0)=0$, then*

$$\|\xi\|_{H^{\sigma_1}(0,T,\mathfrak{X})} \leqslant CT^{\sigma_2-\sigma_1}\|\xi\|_{H^{\sigma_2}(0,T,\mathfrak{X})}. \tag{A.16}$$

**Lemma A.4.** *Assume that $\mathfrak{X}_1$, $\mathfrak{X}_2$ and $\mathfrak{X}_3$ are Banach spaces such that*

$$\|fg\|_{\mathfrak{X}_3} \leqslant C \|f\|_{\mathfrak{X}_1} \|g\|_{\mathfrak{X}_2}, \quad \forall f \in \mathfrak{X}_1, \quad \forall g \in \mathfrak{X}_2.$$

*Let us assume $\sigma \in (1/2, 1]$, $s \in [0, 1/2]$, $T_0 > 0$. Then there exists a constant $C$ such that for any $T \leqslant T_0$ we have*

$$\|u_1 u_2\|_{H^s(0,T;\mathfrak{X}_3)} \leqslant C T^{\sigma-s-1/2} \|u_1\|_{H^s(0,T;\mathfrak{X}_1)} \|u_2\|_{H^\sigma(0,T;\mathfrak{X}_2)} + \|u_2(0)\|_{\mathfrak{X}_2} \|u_1\|_{H^s(0,T;\mathfrak{X}_1)}, \tag{A.17}$$

*for all $u_1 \in H^s(0,T;\mathfrak{X}_1)$ and $u_2 \in H^\sigma(0,T;\mathfrak{X}_2)$.*

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