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Existence of strong solutions for a compressible fluid-solid interaction system with Navier slip boundary conditions

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Abstract

We consider a fluid-structure interaction system coupling a viscous fluid governed by the compressible Navier-Stokes equations and a rigid body immersed in the fluid and modeled by the Newton's law. In this work, we consider the Navier slip boundary conditions. Our aim is to show the local in time existence and uniqueness of the strong solution to the corresponding problem. The main step of this work is that we use Lagrangian change of variables in order to handle the transport equation and to reduce the problem in the initial domain. However, the specificity here is that the Lagrangian coordinates do not coincide with the Eulerian coordinates at the boundaries since we consider slip boundary conditions. Therefore, it brings some extra nonlinear terms in the boundary conditions. The strategy is based on the study of the linearized system with nonhomogeneous boundary conditions and on the Banach fixed point theorem.

Keywords: Compressible Navier-Stokes system; Rigid body motion; Strong solutions.

2010 Mathematics Subject Classification. 35Q30, 76N10, 74F10.

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2 1 Introduction and main result

3 Let Ω be a bounded domain of \mathbb{R}^3 . At each time t , the fluid domain is denoted by $\mathcal{F}(t)$ and we consider a rigid
 4 body $\mathcal{S}(t)$ immersed in the fluid region such that $\mathcal{F}(t) = \Omega \setminus \overline{\mathcal{S}(t)}$. We assume that $\mathcal{F}(t)$ is a connected set with
 5 a smooth enough boundary. The problem is described by the following fluid structure interaction system

$$\left\{ \begin{array}{ll} \partial_t \tilde{\rho} + \nabla \cdot (\tilde{\rho} \tilde{v}) = 0 & \text{in } \mathcal{F}(t), \quad t > 0, \\ \tilde{\rho}(\partial_t \tilde{v} + (\tilde{v} \cdot \nabla) \tilde{v}) - \nabla \cdot (\sigma(\tilde{v}, \tilde{p})) = 0 & \text{in } \mathcal{F}(t), \quad t > 0, \\ m \ell'(t) = - \int_{\partial \mathcal{S}(t)} \sigma(\tilde{v}, \tilde{p}) \tilde{n} \, d\Gamma & t > 0, \\ J \omega'(t) = - \int_{\partial \mathcal{S}(t)} (x - h(t)) \times \sigma(\tilde{v}, \tilde{p}) \tilde{n} \, d\Gamma & t > 0, \end{array} \right. \quad (1.1) \quad \boxed{\text{\{fs\}}}$$

6 with the slip boundary conditions

$$\left\{ \begin{array}{ll} \tilde{v}_{\tilde{n}} = 0, \quad [2\mu \mathbb{D}(\tilde{v}) \tilde{n} + \beta_0 \tilde{v}]_{\tilde{\tau}} = 0 & \text{on } \partial \Omega, \quad t > 0, \\ (\tilde{v} - \tilde{v}_{\mathcal{S}})_{\tilde{n}} = 0, \quad [2\mu \mathbb{D}(\tilde{v}) \tilde{n} + \beta_1 (\tilde{v} - \tilde{v}_{\mathcal{S}})]_{\tilde{\tau}} = 0 & \text{on } \partial \mathcal{S}(t), \quad t > 0, \end{array} \right. \quad (1.2) \quad \boxed{\text{\{bc\}}}$$

7 and the initial conditions

$$\left\{ \begin{array}{l} \tilde{\rho}(0, \cdot) = \tilde{\rho}^0, \quad v(0, \cdot) = \tilde{v}^0 \quad \text{in } \mathcal{F}(0), \\ h(0) = 0, \quad \ell(0) = \ell^0, \quad Q(0) = \mathbb{I}_3, \quad \omega(0) = \omega^0. \end{array} \right. \quad (1.3) \quad \boxed{\text{\{ic\}}}$$

The function $\tilde{\rho}$ represents the density of the fluid while \tilde{v} denotes the fluid velocity and \tilde{p} is the fluid pressure. The Cauchy stress tensor is defined by

$$\sigma(\tilde{v}, \tilde{p}) = \mathbb{S}(\tilde{v}) - \tilde{p} \mathbb{I}_3,$$

with

$$\mathbb{S}(\tilde{v}) = 2\mu \mathbb{D}(\tilde{v}) + \lambda (\nabla \cdot \tilde{v}) \mathbb{I}_3, \quad \mathbb{D}(\tilde{v}) = \frac{1}{2} (\nabla \tilde{v} + (\nabla \tilde{v})^\top),$$

where $\mu > 0$ and $\lambda \geq 0$ are constants and represent respectively the dynamic and the bulk viscosities of the fluid. The pressure $\tilde{p} = \tilde{p}(\tilde{\rho})$ is given by the following constitutive law:

$$p(\tilde{\rho}) = a_\rho \tilde{\rho}^\gamma, \quad a_\rho > 0, \quad \gamma \geq 1.$$

The vector $h(t)$ and the matrix $Q(t)$ represent the translation of the center of mass and the rotation of the rigid body. Note that we assume without loss of generality that $h(0) = 0$ and $Q(0) = \mathbb{I}_3$. Then, at time t , the domain $\mathcal{S}(t)$ occupied by the structure is described as follows

$$\mathcal{S}(t) = h(t) + Q(t)y, \quad y \in \mathcal{S},$$

and we denote the associated region occupied by the fluid $\mathcal{F}(t) = \Omega \setminus \overline{\mathcal{S}(t)}$. Then, the domain of the rigid body and the fluid are given as follows at initial time

$$\mathcal{S}(0) = \mathcal{S}, \quad \mathcal{F}(0) = \Omega \setminus \overline{\mathcal{S}} = \mathcal{F}.$$

8 Since Q is an orthogonal matrix, we get

$$Q'(t) = \mathbb{A}(t)Q(t), \quad \mathbb{A}(t)z = \omega(t) \times z, \quad z \in \mathbb{R}^3, \quad (1.4) \quad \boxed{\text{\{rot\}}}$$

9 where ω is the angular velocity of the rigid body. Hence, the velocity of the structure is given by

$$\tilde{v}_{\mathcal{S}}(t, x) = \ell(t) + \omega(t) \times (x - h(t)), \quad \ell(t) = h'(t), \quad t > 0, \quad x \in \partial \mathcal{S}(t). \quad (1.5)$$

The constant m and the matrix J represent the mass and the moment of inertia of the rigid body. The vector \tilde{n} is the outer unit normal vector on $\partial\mathcal{F}(t)$. For any vector $z \in \mathbb{R}^3$, we define the normal and the tangential components of z as follows

$$z_{\tilde{n}} = (z \cdot \tilde{n}) \cdot \tilde{n}, \quad z_{\tilde{\tau}} = (\mathbb{I}_3 - \tilde{n} \otimes \tilde{n})z.$$

We furnish the initial data $(\rho^0, v^0, \ell^0, \omega^0)$ that satisfy the following conditions

$$\text{dist}(\mathcal{S}, \partial\Omega) > 0, \tag{1.6}$$

$$p^0 = a_\rho(\rho^0)^\gamma, \quad v^0 \in [H^3(\mathcal{F})]^3, \quad \rho^0 \in H^3(\mathcal{F}), \quad \min_{\overline{\mathcal{F}}} \rho^0 > 0, \tag{1.7}$$

$$v_S^0 = \ell^0 + \omega^0 \times y, \quad \begin{cases} v_n^0 = 0, & [2\mu\mathbb{D}(v^0)n + \beta_0 v^0]_\tau = 0 & \text{on } \partial\Omega, \\ (v^0 - v_S^0)_n = 0, & [2\mu\mathbb{D}(v^0)n + \beta_1(v^0 - v_S^0)]_\tau = 0 & \text{on } \partial\mathcal{S}, \end{cases} \tag{1.8}$$

$$\begin{cases} \left(\frac{1}{\rho^0} \nabla \cdot \sigma(v^0, p^0) \right) \cdot n = -v^0 \cdot (\nabla n v^0) & \text{on } \partial\Omega, \\ \left(\frac{1}{\rho^0} \nabla \cdot \sigma(v^0, p^0) \right) \cdot n = \left(-m^{-1} \int_{\partial\mathcal{S}(0)} \sigma(v^0, p^0)n \, d\Gamma - J^{-1} \left(\int_{\partial\mathcal{S}(0)} y \times \sigma(v^0, p^0)n \, d\Gamma \right) \times y \right) \cdot n \\ \quad + (\omega^0 \times (v^0 - \ell^0)) \cdot n - (v^0 - v_S^0) \cdot (\nabla n v^0) & \text{on } \partial\mathcal{S}. \end{cases} \tag{1.9}$$

In order to state the main result, we consider X a C^1 -diffeomorphism that transforms $\overline{\mathcal{F}(0)} = \overline{\mathcal{F}}$ into $\overline{\mathcal{F}(t)}$ that is defined more precisely in Section 2 and we denote v and ρ the corresponding functions to this change of variables

$$v(t, \cdot) = \tilde{v}(t, X(t, \cdot)), \quad \rho(t, \cdot) = \tilde{\rho}(t, X(t, \cdot)).$$

The main theorem of this paper is stated as follows

Theorem 1.1. *Let $\rho^0, v^0, \ell^0, \omega^0$ the initial data that satisfy (1.6), (1.7), (1.8) and (1.9). Then, there exists $T > 0$ such that the system (1.1), (1.2), (1.3) admits a unique strong solution with*

$$\rho \in H^1(0, T; H^3(\mathcal{F})) \cap C^1(0, T; H^2(\mathcal{F})) \cap H^2(0, T; H^1(\mathcal{F})),$$

$$v \in L^2(0, T; [H^4(\mathcal{F})]^3) \cap H^2(0, T; [L^2(\mathcal{F})]^3) \cap C(0, T; [H^3(\mathcal{F})]^3) \cap H^1(0, T; [H^2(\mathcal{F})]^3) \cap C^1(0, T; [H^1(\mathcal{F})]^3),$$

$$\ell \in [H^2(0, T)]^3, \quad \omega \in [H^2(0, T)]^3.$$

The slip boundary conditions were introduced first by Navier in [58] and they are generally used to model the motion of the fluid at the boundaries eventually for collision problems, see [43, 33, 72, 44]. In the mathematical literature, the survey of this type of coupling problems is tackled. In the case of an incompressible fluid-rigid body systems, there are many interesting works in the literature considering no slip boundary conditions, see for example [17, 65, 22, 23, 24, 30, 31, 40, 69, 45, 70, 71]. In the case of slip boundary conditions, we have [32, 61, 15, 15, 20]. The controllability of the fluid-rigid body problem is studied in [14, 63, 12, 47] considering no slip boundary conditions whereas in the case of Navier boundary conditions, we mention [26]. The stabilization problem is discussed in [3, 2]. In the case of a compressible fluid-solid interaction system with no slip conditions, the global existence of weak solutions are investigated in [25, 29] and in [11] for strong solutions stated in the Hilbert space framework. In the $L^p - L^q$ setting, the study of the interaction fluid-rigid body problem is investigated in [42, 41] assuming no slip boundary conditions. Furthermore, the analysis of the motion of a piston in a viscous gaz modeled by the 1D compressible Navier-Stokes system is studied in [54]. The stabilization problem is investigated in [64]. Concerning compressible Navier-Stokes equations without structure involving slip boundary conditions, existence of local strong solutions in the $L^p - L^q$ framework is tackled in [57], whereas the existence of global strong solutions is obtained in [66]. In the case of the compressible Navier-Stokes equations set in a moving domain, weak and strong solutions are studied in [48] under slip or no slip boundary conditions. Moreover, in the $L^p - L^q$ setting, the compressible problem in a moving domain is studied in [49]. In [59], the existence of a weak solution to fluid-solid interaction problem is proved considering slip boundary conditions.

1 Let us mention some works that are devoted to the case of an elastic structure. In the case of an incom-
 2 pressible fluid-beam interaction system with Dirichlet boundary conditions, the existence of weak solutions was
 3 investigated in [19] considering a damping in the structure equation and [34] in the case where the structure
 4 satisfies the plate equation without any damping. The existence of strong solutions for the fluid-damped beam
 5 system was established in [8, 50]. For global in time solutions, we mention [35] where the authors proved no
 6 contact between the beam and the domain cavity. In [36], the existence of strong solutions is proved when-
 7 ever the structure satisfies either a damped beam equation, a linear wave equation or a linear beam equation
 8 with inertia of rotation. In [7], the authors considered the incompressible fluid-wave interaction system and
 9 showed maximal regularity for the solutions, they proved in particular that the corresponding semigroup of the
 10 linearized system is analytic. In [5, 6], the authors obtained Gevrey regularity for a system coupling incompress-
 11 ible Navier Stokes system with a beam equation. For the stabilization problem, we refer to [62, 4]. In the case of
 12 boundary conditions involving pressure, regularity results are obtained in [18]. In the case of an incompressible
 13 fluid-beam interaction system with Navier boundary conditions, the existence of weak solutions is investigated
 14 in [39, 38, 56] whereas the existence of strong solutions is discussed in [27]. In the case of a compressible fluid
 15 and an elastic structure, weak solutions are obtained in [10], local existence of strong solutions is obtained in
 16 [55] for the coupled system considering a damped beam equation and assuming that the initial displacement of
 17 the structure is 0. Maximal-regularity in $L^p - L^q$ framework is obtained in [53] where the structure is modeled
 18 by the damped beam equation and in the case where the structure verifies the wave equation, the existence and
 19 uniqueness of the strong solution are proved in [52] in the L^2 approach.

The objective of this paper is to provide local existence and uniqueness of the strong solution to the system
 (1.1), (1.2), (1.3) in the Hilbert space framework. In section 2, we write the system in a fixed configuration
 by using the Lagrangian change of variables. Since the system considered here is compressible, the change of
 variables that is commonly used is the Lagrangian flow to transform the domain $\mathcal{F}(t)$ into \mathcal{F} . In order to obtain
 a local solution, it is crucial to go back in the initial domain so that we can use the classical Banach fixed point
 theorem. However, in the Dirichlet case (see [64, 41]), we have the following correspondence on the boundary

$$X(t, y) = \begin{cases} h(t) + Q(t)y & y \in \partial\mathcal{S}, \\ y & y \in \partial\Omega, \end{cases}$$

20 that is not true considering slip boundary conditions. Due to this change of variables, nonlinear terms appear
 21 in the whole system even in the boundary conditions that cannot be avoided even in the case without structure
 22 (see [66, 57]). In Section 3, we study the linearized system. We first make a suitable lifting at the boundary
 23 conditions to handle the nonlinearities coming from the change of variables and we establish a first regularity
 24 result for the Linearized system with nonhomogeneous boundary conditions. An important step in this section
 25 is to elaborate elliptic estimates for the associated Lamé system. In section 4, we use a Banach fixed point
 26 argument and prove Theorem 1.1.

27 2 Change of variables

chV

28 Let define $X(t, \cdot)$ as the Lagrangian change of variables associated with the fluid velocity \tilde{v} that reduces $\mathcal{F}(0) = \mathcal{F}$
 29 into $\mathcal{F}(t)$

$$\begin{cases} \partial_t X(t, y) = \tilde{v}(t, X(t, y)) & y \in \mathcal{F}, t > 0, \\ X(0, y) = y & y \in \overline{\mathcal{F}}. \end{cases} \quad (2.1) \quad \{\text{lag}\}$$

Let recall that ρ and v are the new corresponding functions to $\tilde{\rho}$ and \tilde{v} after the change of variables (2.1)

$$\rho(t, \cdot) = \tilde{\rho}(t, X(t, \cdot)), \quad v(t, \cdot) = \tilde{v}(t, X(t, \cdot)).$$

We introduce the following notations

$$b_X = \text{Cof } \nabla X, \quad \delta_X = \det \nabla X.$$

After some standard calculations, we have

$$\mathbb{D}(\tilde{v}) = \mathbb{D}^X(v) = \frac{1}{2\delta_X} (\nabla v b_X^\top + (\nabla v b_X^\top)^\top), \quad \mathbb{S}(\tilde{v}) = \mathbb{S}^X(v) = 2\mu\mathbb{D}^X(v) + \lambda\frac{1}{\delta_X} b_X : \nabla v,$$

and

$$\sigma^X(v) = \mathbb{S}^X(v) b_X, \quad \nabla \cdot (\mathbb{S}(\tilde{v})) = \frac{1}{\delta_X} \nabla \cdot (\sigma^X(v)).$$

1 From [21, Theorem 1.1-1, p.13], we get the following relation

$$\tilde{n}_X(t, \cdot) = \tilde{n}(t, X(t, \cdot)) = \frac{1}{|b_X(t, \cdot)n|} b_X(t, \cdot)n, \quad (2.2) \quad \{\text{1725}\}$$

where n is the unitary exterior normal to $\partial\mathcal{F}$. We deduce

$$\tilde{\tau}_X^i(t, \cdot) = \tilde{\tau}^i(t, X(t, \cdot)) = |b_X(t, \cdot)n| b_X^{-\top}(t, \cdot)\tau^i, \quad i = 1, 2,$$

2 with $(\tilde{\tau}^i)_{i=1,2}$ (resp. $(\tau^i)_{i=1,2}$) designates a basis of the tangent subspace on each point of $\partial\mathcal{F}(t)$ (resp. $\partial\mathcal{F}$).

3 The transport equation (1.1)₁ gives

$$\partial_t \rho + \frac{\rho}{\delta_X} \nabla v : b_X = 0. \quad (2.3) \quad \{\text{dens}\}$$

4 Then, the system (1.1), (1.2) and (1.3) is rewritten in the Lagrangian coordinates as follows

$$\left\{ \begin{array}{ll} \partial_t \rho + \rho^0 \nabla \cdot v = \mathbb{T}(\rho, v) & \text{in } (0, T) \times \mathcal{F}, \\ \rho^0 \partial_t v - \nabla \cdot (\mathbb{S}(v)) = \nabla \cdot (\mathbb{F}_1(\rho, v)) + \mathbb{F}_2(\rho, v) & \text{in } (0, T) \times \mathcal{F}, \\ v_S(y) = \ell + \omega \times y & \text{in } (0, T), \\ m\ell' = \mathbb{F}_3(\rho, v, \ell, \omega) & \text{in } (0, T), \\ J\omega' = \mathbb{F}_4(\rho, v, \ell, \omega) & \text{in } (0, T), \end{array} \right. \quad (2.4) \quad \{\text{fs1}\}$$

5 with the boundary conditions

$$\left\{ \begin{array}{ll} v \cdot n = \mathbb{B}(\rho, v, \ell, \omega), & [(2\mu\mathbb{D}(v)n + \beta_0 v)] \cdot \tau^i = (\mathbb{G}(\rho, v, \ell, \omega))_i \quad \text{on } (0, T) \times \partial\Omega, \\ (v - v_S) \cdot n = \mathbb{B}(\rho, v, \ell, \omega), & [(2\mu\mathbb{D}(v)n + \beta_1(v - v_S))] \cdot \tau^i = (\mathbb{G}(\rho, v, \ell, \omega))_i \quad \text{on } (0, T) \times \partial\mathcal{S}, \end{array} \right. \quad (2.5) \quad \{\text{bc1}\}$$

6 where $i = 1, 2$ and the initial conditions

$$\rho(0, \cdot) = \rho^0 \text{ in } \mathcal{F}, \quad v(0, \cdot) = v^0 \text{ in } \mathcal{F}, \quad \ell(0) = \ell^0, \quad \omega(0) = \omega^0. \quad (2.6) \quad \{\text{ic1}\}$$

7 The functions \mathbb{T} , \mathbb{F}_1 , \mathbb{F}_2 , \mathbb{F}_3 , \mathbb{F}_4 , \mathbb{B} and \mathbb{G} are given as follows

$$\mathbb{T}(\rho, v) = \rho^0 \nabla v : \mathbb{I}_3 - \frac{\rho}{\delta_X} \nabla v : b_X \quad \text{on } \mathcal{F}, \quad (2.7) \quad \{\text{defT}\}$$

8

$$\mathbb{F}_1(\rho, v) = \sigma^X(v) - \mathbb{S}(v) - a_\rho \rho^\gamma b_X, \quad \mathbb{F}_2(\rho, v) = (\rho^0 - \rho \delta_X) \partial_t v \quad \text{on } \mathcal{F}, \quad (2.8) \quad \{\text{defF}\}$$

$$\mathbb{F}_3(\rho, v, \ell, \omega) = - \int_{\partial\mathcal{S}} (\sigma^X(v) - a_\rho \rho^\gamma b_X) n \, d\Gamma,$$

$$\mathbb{F}_4(\rho, v, \ell, \omega) = - \int_{\partial\mathcal{S}} (X - h) \times (\sigma^X(v) - a_\rho \rho^\gamma b_X) n \, d\Gamma, \quad (2.9) \quad \{\text{defF34}\}$$

9

$$\mathbb{B}(v, \ell, \omega) = v \cdot (n - \tilde{n}_X) \quad \text{on } \partial\Omega, \quad (2.10) \quad \{\text{defB1}\}$$

10

$$\mathbb{B}(v, \ell, \omega) = (v - v_S) \cdot (n - \tilde{n}_X) + (\omega \times (X - \text{Id})) \cdot \tilde{n}_X - \omega \times h \cdot \tilde{n}_X \quad \text{on } \partial\mathcal{S}, \quad (2.11) \quad \{\text{defB}\}$$

11

$$[\mathbb{G}(v, \ell, \omega)]_i = 2\mu\mathbb{D}(v)n \cdot \tau - 2\mu\mathbb{D}^X(v)\tilde{n}_X \cdot \tilde{\tau}_X^i + \beta_0 v \cdot (\tau - \tilde{\tau}_X^i) \quad \text{on } \partial\Omega, \quad (2.12) \quad \{\text{defG1}\}$$

$$[\mathbb{G}(v, \ell, \omega)]_i = 2\mu\mathbb{D}(v)n \cdot \tau - 2\mu\mathbb{D}^X(v)\tilde{n}_X \cdot \tilde{\tau}_X^i + \beta_1(v - v_S) \cdot (\tau - \tilde{\tau}_X^i) + \beta_1 \omega \times (X - \text{Id}) \cdot \tilde{\tau}_X^i - \beta_1 \omega \times h \cdot \tilde{\tau}_X^i \quad \text{on } \partial\mathcal{S}. \quad (2.13) \quad \{\text{defG2}\}$$

12 As in [57], we show that X is C^1 -diffeomorphism.

Proposition 2.1. *Under the boundary conditions (2.5), if v is regular and T is small enough, the application X defined in (2.1) is a C^1 -diffeomorphism and transforms $\overline{\mathcal{F}(0)} = \overline{\mathcal{F}}$ into $\overline{\mathcal{F}(t)}$.*

Proof. From (2.1), we see that

$$\nabla X(t, \cdot) = \mathbb{I}_3 + \int_0^t \nabla v(s, \cdot) ds, \quad t \in (0, T).$$

If T is small enough and v regular enough, we see that ∇X is invertible with $\delta_X > 0$ in such a way the transformation X preserves the orientation. Hence X is a one to one mapping. Since $\partial\mathcal{S}$ is smooth, there exists $\eta_{\mathcal{S}}$ a smooth function such that $\eta_{\mathcal{S}} = 0$ in a part of $\partial\mathcal{S}$. Then, the corresponding part in $\partial\mathcal{S}(t)$ is locally expressed as $\xi(t, x) = \eta_{\mathcal{S}}(Q(t)^\top(x - h(t))) = 0$. Let prove that $X(t, \partial\mathcal{S}) \subset \partial\mathcal{S}(t)$. Let y be in the neighborhood of $\partial\mathcal{S}$ satisfying $\eta_{\mathcal{S}}(y) = 0$, we obtain

$$\partial_t[\xi(t, X(t, y))] = \nabla\xi(X(t, y)) \cdot (\tilde{v}(t, X(t, y)) - \tilde{v}_{\mathcal{S}}(t, X(t, y))).$$

Since $\nabla\xi(X(t, \cdot))$ is collinear to $\tilde{n}_X(t, \cdot)$, therefore from the impermeability boundary condition given in (2.5), we obtain that $\partial_t[\xi(t, X(t, y))] = 0$. Thus

$$\xi(t, X(t, y)) = \xi(0, X(0, y)) = \eta_{\mathcal{S}}(y) = 0.$$

Hence, we get $X(t, \partial\mathcal{S}) \subset \partial\mathcal{S}(t)$ for $t \geq 0$. The same approach can be used to prove that $X(t, \partial\Omega) \subset \partial\Omega$. We obtain $X(t, \partial\mathcal{F}) \subset \partial\mathcal{F}(t)$. Moreover, using the classical local inversion theorem, we see that $X(t, \partial\mathcal{F})$ is an open and a closed subset of $\partial\mathcal{F}(t)$. Since, $\partial\mathcal{F}(t)$ is a connected set, it follows that $X(t, \partial\mathcal{F}) = \partial\mathcal{F}(t)$. Finally, the injectivity of the mapping X and the connectedness of $\mathcal{F}(t)$ yield that $X(t, \partial\mathcal{F}) = \mathcal{F}(t)$. \square

3 Linearized system

The linearized system is given by

$$\begin{cases} \partial_t \rho + \rho^0 \nabla \cdot v = h & \text{in } (0, T) \times \mathcal{F}, \\ \rho(0, \cdot) = \rho^0 & \text{in } \mathcal{F}, \end{cases} \quad (3.1) \quad \{\text{dens1}\}$$

$$\begin{cases} \rho^0 \partial_t v - \nabla \cdot (\mathbb{S}(v)) = \nabla \cdot f_1 + f_2 & \text{in } (0, T) \times \mathcal{F}, \\ v_{\mathcal{S}}(y) = \ell + \omega \times y & \text{in } (0, T), \\ m\ell' = f_3 & \text{in } (0, T), \\ J\omega' = f_4 & \text{in } (0, T), \end{cases} \quad (3.2) \quad \{\text{fs1}\}$$

with the boundary conditions

$$\begin{cases} v \cdot n = a, & [(2\mu\mathbb{D}(v)n + \beta_0 v)] \cdot \tau^i = g_i & \text{on } (0, T) \times \partial\Omega, \\ (v - v_{\mathcal{S}}) \cdot n = a, & [(2\mu\mathbb{D}(v)n + \beta_1(v - v_{\mathcal{S}}))] \cdot \tau^i = g_i & \text{on } (0, T) \times \partial\mathcal{S}, \end{cases} \quad (3.3) \quad \{\text{bcl}\}$$

and the initial conditions

$$\rho(0, \cdot) = \rho^0 \text{ in } \mathcal{F}, \quad v(0, \cdot) = v^0 \text{ in } \mathcal{F}, \quad \ell(0) = \ell^0, \quad \omega(0) = \omega^0, \quad (3.4) \quad \{\text{icl}\}$$

such that the following compatibility condition is satisfied

$$v^0 \cdot n = a(0, \cdot) \text{ on } \partial\Omega, \quad (v^0 - v_{\mathcal{S}}^0) \cdot n = a(0, \cdot) \text{ on } \partial\mathcal{S}. \quad (3.5) \quad \{\text{comp*}\}$$

The aim of this section is to establish a regularity result for (3.2), (3.3) and (3.4). First, we need to give some regularity properties for the elliptic problem (3.6) associated to the Lamé operator.

1 3.1 Elliptic problem

2 Let consider the stationary system

$$\begin{cases} -\nabla \cdot \mathbb{S}(v) = f & \text{in } \mathcal{F}, \\ v \cdot n = a & \text{on } \partial\mathcal{F}, \\ [2\mu\mathbb{D}(v)n + \beta v] \cdot \tau^i = g_i & \text{on } \partial\mathcal{F}. \end{cases} \quad (3.6) \quad \{\text{parab1s}\}$$

elliptic **Theorem 3.1.** *Let $\beta > 0$ and assume that $\partial\mathcal{F}$ is C^2 boundary. Let*

$$f \in [L^2(\mathcal{F})]^3, \quad a \in H^{3/2}(\partial\mathcal{F}), \quad g_i \in H^{1/2}(\partial\mathcal{F}), \quad i = 1, 2.$$

3 *Then, the system (3.6) admits a unique strong solution $v \in [H^2(\mathcal{F})]^3$ such that*

$$\|v\|_{[H^2(\mathcal{F})]^3} \leq C (\|f\|_{[L^2(\mathcal{F})]^3} + \|a\|_{H^{3/2}(\partial\mathcal{F})} + \|g_1\|_{H^{1/2}(\partial\mathcal{F})} + \|g_2\|_{H^{1/2}(\partial\mathcal{F})}). \quad (3.7) \quad \{\text{12:47}\}$$

Moreover, if $\partial\mathcal{F}$ is C^{m+2} boundary and

$$f \in [H^m(\mathcal{F})]^3, \quad a \in H^{m+3/2}(\partial\mathcal{F}), \quad g_i \in H^{m+1/2}(\partial\mathcal{F}), \quad i = 1, 2,$$

4 *the system (3.6) admits a unique strong solution $v \in [H^{m+2}(\mathcal{F})]^3$ such that*

$$\|v\|_{[H^{m+2}(\mathcal{F})]^3} \leq C (\|f\|_{[H^m(\mathcal{F})]^3} + \|a\|_{H^{m+3/2}(\partial\mathcal{F})} + \|g_1\|_{H^{m+1/2}(\partial\mathcal{F})} + \|g_2\|_{H^{m+1/2}(\partial\mathcal{F})}). \quad (3.8) \quad \{\text{14:38}\}$$

5 The rest of this subsection is devoted to the proof of Theorem 3.1.

Step 1: Weak solutions. First, we study the existence of weak solutions for the system (3.6). Let define the following space

$$\mathbb{V} = \{v \in [H^1(\mathcal{F})]^3; v \cdot n = 0 \text{ on } \partial\mathcal{F}\}.$$

6 Let $f \in \mathbb{V}'$, $a \in H^{1/2}(\partial\mathcal{F})$ and $g \in H^{-1/2}(\partial\mathcal{F})$. We multiply the first equation of (3.6) by $w \in \mathbb{V}$ to obtain

$$\int_{\mathcal{F}} \mathbb{S}(v) : \nabla w \, dy + \int_{\partial\mathcal{F}} \beta v \cdot w \, d\Gamma = \langle f, w \rangle_{\mathbb{V}', \mathbb{V}} + \langle g, w \rangle_{H^{-1/2}, H^{1/2}}, \quad (3.9) \quad \{\text{weak}\}$$

7 such that g is a vector satisfying $g \cdot n = 0$ and $g \cdot \tau^i = g_i$ for $i = 1, 2$. We consider a lifting $w_\ell \in [H^1(\mathcal{F})]^3$
8 satisfying

$$w_\ell \cdot n = a \quad \text{on } \partial\mathcal{F}, \quad (3.10) \quad \{\text{c14bis}\}$$

9 and the estimate

$$\|w_\ell\|_{[H^1(\mathcal{F})]^3} \leq C \|a\|_{H^{1/2}(\partial\mathcal{F})}. \quad (3.11) \quad \{\text{17:13}\}$$

10 We set $v = u + w_\ell$. Then, we have the following definition.

Definition 3.2. *The function v is a weak solution of the system (3.6) if and only if $u = v - w_\ell$ verifies the following variational formulation*

$$\begin{aligned} \int_{\mathcal{F}} \mathbb{S}(u) : \nabla w \, dy + \int_{\partial\mathcal{F}} \beta u \cdot w \, d\Gamma = & - \int_{\mathcal{F}} \mathbb{S}(w_\ell) : \nabla w \, dy + \langle f, w \rangle_{\mathbb{V}', \mathbb{V}} \\ & - \int_{\partial\mathcal{F}} \beta w_\ell \cdot w \, d\Gamma + \langle g, w \rangle_{H^{-1/2}, H^{1/2}}, \end{aligned} \quad (3.12) \quad \{\text{weak2}\}$$

11 for all $w \in \mathbb{V}$, where w_ℓ satisfies (3.10) and (3.11).

We notice that

$$\int_{\mathcal{F}} \mathbb{S}(u) : \nabla u \, dy = \int_{\mathcal{F}} 2\mu |\mathbb{D}(u)|^2 \, dy + \int_{\mathcal{F}} \lambda |\nabla \cdot u|^2 \, dy.$$

1 Using Lemma A.1, we get for all $u \in \mathbb{V}$

$$\int_{\mathcal{F}} \mathbb{S}(u) : \nabla u \, dy + \int_{\partial\mathcal{F}} \beta |u|^2 \, d\Gamma \geq C \|u\|_{[H^1(\mathcal{F})]^3}^2. \quad (3.13) \quad \{\text{keyhey}\}$$

2 Hence, we can apply the Lax-Milgram theorem and using (3.11), we deduce the existence of a unique solution
3 v of (3.6) which verifies the estimates

$$\|v\|_{[H^1(\mathcal{F})]^3} \leq C (\|f\|_{\mathbb{V}'} + \|a\|_{H^{1/2}(\partial\mathcal{F})} + \|g\|_{[H^{-1/2}(\partial\mathcal{F})]^3}). \quad (3.14) \quad \{\text{est}\}$$

4 **Step 2: Strong solutions in H^2 .** Without loss of generality, we take $a = 0$ in (3.6) since we can always find
5 a suitable lifting of the impermeability condition, see for example [37, p.128, Theorem 2.5.1.1]. The interior
6 regularity can be done as the Dirichlet boundary condition, then we skip the proof.

First, let consider the case, where \mathcal{F} is the half ball $\mathcal{B}(0,1) \cap \mathbb{R}_+^3$. The proof is standard and it is done by using the method of difference quotient as in [28]. Let $f \in [L^2(\Omega)]^3$, $g \in [H^{1/2}(\partial\mathcal{F})]^3$ and consider v the solution of the variational formulation (3.9). Let define the difference quotient

$$D_k^h v(y) = \frac{v(y + he_k) - v(y)}{h}, \quad k = 1, 2, 3, \quad h \in \mathbb{R}^*.$$

We select ζ , a cut off function such that

$$\zeta \equiv 1 \text{ in } \mathcal{B}(0, 1/2) \cap \mathbb{R}_+^3, \quad \zeta \equiv 0 \text{ in } \mathbb{R}^3 \setminus \mathcal{B}(0, 1),$$

so that ζ vanishes on the curved boundary of $\mathcal{B}(0,1) \cap \mathbb{R}_+^3$. We take the test function $w = -D_k^{-h}(\zeta^2 D_k^h v) \in \mathbb{V}$ and replace it in (3.9). We obtain for h small enough

$$\begin{aligned} & \int_{\mathcal{F}} 2\mu \zeta^2 |\mathbb{D} D_k^h v|^2 \, dy + \int_{\mathcal{F}} \lambda \zeta^2 |\nabla \cdot D_k^h v|^2 \, dy + \int_{\partial\mathcal{F}} \beta \zeta^2 |D_k^h v|^2 \, d\Gamma + 2\mu \int_{\mathcal{F}} D_k^h \mathbb{D}(v) : 2\zeta \nabla \zeta D_k^h v \, dy \\ & + \lambda \int_{\mathcal{F}} 2\zeta (\nabla \zeta \cdot D_k^h v) (\nabla \cdot D_k^h v) \, dy = - \int_{\mathcal{F}} f \cdot D_k^{-h}(\zeta^2 D_k^h v) \, dy - \int_{\partial\mathcal{F}} g \cdot D_k^{-h}(\zeta^2 D_k^h v) \, d\Gamma. \end{aligned}$$

7 Since

$$\|D_k^{-h}(\zeta^2 D_k^h v)\|_{[L^2(\mathcal{F})]^3} \leq C (\|\zeta^2 \nabla(D_k^h v)\|_{[L^2(\mathcal{F})]^9} + \|(D_k^h v)\|_{[L^2(\mathcal{F})]^9}), \quad (3.15) \quad \{2022\}$$

we get

$$\int_{\mathcal{F}} f \cdot D_k^{-h}(\zeta^2 D_k^h v) \, dy \leq \varepsilon \|\zeta^2 \nabla(D_k^h v)\|_{[L^2(\mathcal{F})]^9}^2 + C_\varepsilon (\|f\|_{[L^2(\mathcal{F})]^3}^2 + \|g\|_{[H^{1/2}(\partial\mathcal{F})]^3}^2).$$

In the other hand, let extend the unit normal n over \mathcal{F} and g by a function also denoted by $g \in [H^1(\mathcal{F})]^3$. We have

$$\begin{aligned} - \int_{\partial\mathcal{F}} g \cdot D_k^{-h}(\zeta^2 D_k^h v) \, d\Gamma &= \sum_{i=1}^3 \left(- \int_{\mathcal{F}} \partial_i((n)_i g) \cdot D_k^{-h}(\zeta^2 D_k^h v) \, dy + \int_{\mathcal{F}} D_k^h((n)_i g) \cdot \zeta^2 (D_k^h \partial_i v) \, dy \right. \\ & \quad \left. - 2 \int_{\mathcal{F}} ((n)_i g) \cdot \zeta \partial_i \zeta (D_k^h v) \, dy \right). \quad (3.16) \quad \{1859\} \end{aligned}$$

8 Using (3.15), we get

$$\int_{\partial\mathcal{F}} g \cdot D_k^{-h}(\zeta^2 D_k^h v) \, d\Gamma \leq \varepsilon \|\zeta^2 \nabla(D_k^h v)\|_{[L^2(\mathcal{F})]^9}^2 + C_\varepsilon (\|f\|_{[L^2(\mathcal{F})]^3}^2 + \|g\|_{[H^{1/2}(\partial\mathcal{F})]^3}^2). \quad (3.17) \quad \{1900\}$$

Hence gathering all these estimates, we obtain

$$\|D_k^h v\|_{[H^1(\Omega)]^3} \leq C (\|f\|_{[L^2(\mathcal{F})]^3} + \|g\|_{[H^{1/2}(\partial\mathcal{F})]^3}),$$

for $k = 1, 2$. Taking h to zero, it yields

$$\|\partial_{k,j}^2 v\|_{[L^2(\mathcal{F})]^3} \leq C (\|f\|_{[L^2(\mathcal{F})]^3} + \|g\|_{[H^{1/2}(\partial\mathcal{F})]^3}), \quad j \in \{1, 2, 3\}, \quad k = 1, 2.$$

Using the first equation in (3.6), we get finally

$$\|(2\mu + \lambda)\partial_{33}^2 v\|_{[L^2(\mathcal{F})]^3} \leq C \sum_{k=1}^2 \|\partial_k v\|_{[H^1(\mathcal{F})]^3} \leq C (\|f\|_{[L^2(\mathcal{F})]^3} + \|g\|_{[H^{1/2}(\mathcal{F})]^3}).$$

Now we consider the case when \mathcal{F} is arbitrary. Let $y_0 \in \partial\mathcal{F}$, since \mathcal{F} is C^2 , then there exist $r > 0$ and $\eta : \mathbb{R}^2 \rightarrow \mathbb{R}$ of class C^2 such that

$$\mathcal{F} \cap \mathcal{B}(y_0, r) = \{y \in \mathbb{R}^3; y_3 > \eta(y_1, y_2)\}, \quad \partial\mathcal{F} \cap \mathcal{B}(y_0, r) = \{y \in \mathbb{R}^3; y_3 = \eta(y_1, y_2)\}.$$

Let $Y_\eta : \mathcal{B} \rightarrow \mathcal{F} \cap \mathcal{B}(y_0, r)$ and $Y_\eta^{-1} : \mathcal{F} \cap \mathcal{B}(y_0, r) \rightarrow \mathcal{B}$, where $\mathcal{B} = \mathcal{B}(0, s) \cap \{z_3 = 0\}$ with s small enough. We define

$$Y_\eta(z) = \begin{pmatrix} z_1 \\ z_2 \\ z_3 + \eta(z_1, z_2) \end{pmatrix}, \quad Y_\eta^{-1}(y) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 - \eta(y_1, y_2) \end{pmatrix}.$$

1 Let define $v_\eta = v(Y_\eta(\cdot))$ on \mathcal{B} . Then, the formulation in (3.9) gives

$$\int_{\mathcal{B}} \sigma_\eta(v_\eta) : \nabla w_\eta \, dz + \int_{\partial\mathcal{B}} \beta v_\eta \cdot w_\eta \, d\Gamma = \int_{\mathcal{B}} \delta_\eta f_\eta \cdot w_\eta \, dz + \int_{\partial\mathcal{B}} g_\eta \cdot w_\eta \, d\Gamma, \quad (3.18) \quad \boxed{\{1555\}}$$

where

$$b_\eta = \text{Cof } \nabla Y_\eta, \quad \mathbb{D}_\eta(v_\eta) = \frac{1}{2\delta_\eta} (\nabla v_\eta(b_\eta)^\top + (\nabla v_\eta(b_\eta)^\top)^\top),$$

$$\mathbb{S}_\eta(v) = 2\mu\mathbb{D}_\eta(v) + \frac{\lambda}{\delta_\eta} (b_\eta : \nabla v_\eta)\mathbb{I}_3, \quad \sigma_\eta(v_\eta) = \mathbb{S}_\eta(v_\eta)b_\eta.$$

We emphasize here that for any $w \in \mathbb{V}$, we have $w_\eta \cdot n_\eta = 0$ on $\partial\mathcal{B} \cap \{z_3 = 0\}$. Note that (3.18) is coercive, in fact

$$\begin{aligned} \int_{\mathcal{B}} \sigma_\eta(v_\eta) : \nabla v_\eta \, dz &= \int_{\mathcal{B}} \frac{1}{2\delta_\eta} \mu |\nabla v_\eta(b_\eta)^\top + (\nabla v_\eta(b_\eta)^\top)^\top|^2 \, dz \\ &+ \int_{\mathcal{B}} \frac{1}{\delta_\eta} \lambda |b_\eta : \nabla v_\eta|^2 \, dz = \int_{\mathcal{B}} \delta_\eta (2\mu |\mathbb{D}_\eta(v_\eta)|^2) \, dy + \int_{\mathcal{B}} \frac{1}{\delta_\eta} \lambda |b_\eta : \nabla v_\eta|^2 \, dy \\ &= \int_{\mathcal{F} \cap \mathcal{B}(y_0, r)} 2\mu |\mathbb{D}(v)|^2 \, dy + \int_{\mathcal{F} \cap \mathcal{B}(y_0, r)} \lambda |\nabla \cdot v|^2 \, dy. \end{aligned} \quad (3.19)$$

2 Using Lemma A.1, we get for all $v_\eta \in [H^1(\mathcal{B})]^3$ satisfying $v_\eta \cdot n_\eta = 0$ on $\partial\mathcal{B} \cap \{z_3 = 0\}$

$$\int_{\mathcal{B}} \sigma_\eta(v_\eta) : \nabla v_\eta \, dy + \int_{\partial\mathcal{B}} \beta |v_\eta|^2 \, d\Gamma \geq C \|v_\eta\|_{[H^1(\mathcal{B})]^3}^2. \quad (3.20)$$

As the half ball case, we take $w_\eta = -D_k^{-h}(\zeta^2 D_k^h v_\eta)$ for h small enough where ζ is a cut off function such that $\zeta \equiv 1$ in $\mathcal{B}(0, s/2) \cap \mathbb{R}_+^3$ and vanishes outside \mathcal{B} . Note that we have $w_\eta \cdot n_\eta = 0$ on $\mathcal{B} \cap \{z_3 = 0\}$. We get

$$\begin{aligned} \int_{\mathcal{B}} D_k^h(\sigma_\eta(v_\eta)) : \zeta^2 \nabla D_k^h v_\eta \, dz + \int_{\partial\mathcal{B} \cap \{z_3=0\}} \beta \zeta^2 |D_k^h v_\eta|^2 \, d\Gamma \\ = - \int_{\mathcal{B}} \delta_\eta f_\eta \cdot D_k^{-h}(\zeta^2 D_k^h v_\eta) \, dz - \int_{\partial\mathcal{B}} g_\eta \cdot D_k^{-h}(\zeta^2 D_k^h v_\eta) \, d\Gamma. \end{aligned} \quad (3.21) \quad \boxed{\{1846\}}$$

- 1 Repeating the previous steps as in the half ball case and using the coercivity of the operator $\nabla \cdot \sigma_\eta$ in \mathcal{B} and
 2 the fact that $D_k^h(b_\eta)$ is bounded, we get

$$\|D_k^h v_\eta\|_{[H^1(\mathcal{B})]^3} \leq C (\|f_\eta\|_{[L^2(\mathcal{B})]^3} + \|g_\eta\|_{[H^{1/2}(\mathcal{B})]^3}). \quad (3.22) \quad \boxed{\{1930\}}$$

We take h to zero and thus we obtain

$$\|\partial_{kj}^2(v_\eta)\|_{[L^2(\mathcal{B})]^3} \leq C (\|f_\eta\|_{[L^2(\mathcal{B})]^3} + \|g_\eta\|_{[H^{1/2}(\partial\mathcal{B})]^3}), \quad j \in \{1, 2, 3\}, \quad k = 1, 2.$$

Now, we go back to the equation of the system (3.6) that can be written under the change of variables Y_η

$$-\nabla \cdot (\sigma_\eta(v_\eta)) = \delta_\eta f_\eta, \quad \text{on } \mathcal{B}.$$

After some calculation, we obtain that

$$\|(a_{33}\mathbb{I}_3 + \mathbb{M}_\eta)\partial_{33}^2 v\|_{[L^2(\mathcal{B})]^3} \leq C \sum_{k=1}^2 \|\partial_k^2(v_\eta)\|_{[H^1(\mathcal{B})]^3} \leq C (\|f_\eta\|_{[L^2(\mathcal{B})]^3} + \|g_\eta\|_{[H^{1/2}(\partial\mathcal{B})]^3}),$$

where $a_{33} = (\frac{\mu}{\delta_\eta} b_\eta^\top b_\eta)_{33} > 0$ and \mathbb{M}_η is a non negative symmetric matrix

$$\mathbb{M}_\eta = \left(\frac{\lambda + \mu}{\delta_\eta} \right) b_\eta^3 \otimes b_\eta^3, \quad b_\eta^3 = [(b_\eta)_{13} \quad (b_\eta)_{23} \quad (b_\eta)_{33}]^\top.$$

We deduce that $a_{33}\mathbb{I}_3 + \mathbb{M}_\eta$ is invertible and bounded in \mathcal{B} , then we get

$$\|\partial_{33}^2 v\|_{[L^2(\mathcal{B})]^3} \leq C (\|f_\eta\|_{[L^2(\mathcal{B})]^3} + \|g_\eta\|_{[H^{1/2}(\partial\mathcal{B})]^3}).$$

- 3 Since $\partial\mathcal{F}$ is a compact set, we can use a finite partition to cover the boundary $\partial\mathcal{F}$ by many balls $\mathcal{B}(y_i, r_i)$ and
 4 we combine all the estimates to get (3.7). We deduce (3.8) by induction and the proof can be performed as [28,
 5 Theorem 5].

6 3.2 Lifting

7 Lifting of the impermeability condition

1442 **Lemma 3.3.** *Let $a \in L^2(0, T; H^{3/2}(\partial\mathcal{F})) \cap H^{3/4}(0, T; L^2(\partial\mathcal{F}))$ such that*

$$a(0, \cdot) \in H^{3/2}(\partial\mathcal{F}) \text{ on } \partial\mathcal{F}.$$

- 8 *Then there exists $w \in L^2(0, T; [H^2(\mathcal{F})]^3) \cap H^1(0, T; [L^2(\mathcal{F})]^3)$ such that $w \cdot n = a$ on $\partial\mathcal{F}$ and*

$$\|w\|_{L^2(0, T; [H^2(\mathcal{F})]^3) \cap H^1(0, T; [L^2(\mathcal{F})]^3)} \leq C \left(\|a\|_{L^2(0, T; H^{3/2}(\partial\mathcal{F})) \cap H^{3/4}(0, T; L^2(\partial\mathcal{F}))} + \|a(0, \cdot)\|_{H^{3/2}(\partial\mathcal{F})} \right), \quad (3.23) \quad \boxed{\{esti\}}$$

9 *with C a constant independent of T .*

- 10 *Proof.* From [51, Vol II, chap.4, Theorem 2.3], there exists $w \in L^2(0, T; [H^2(\mathcal{F})]^3) \cap H^1(0, T; [L^2(\mathcal{F})]^3)$ such
 11 that $w \cdot n = a$ on $\partial\mathcal{F}$ and

$$\|w\|_{L^2(0, T; [H^2(\mathcal{F})]^3) \cap H^1(0, T; [L^2(\mathcal{F})]^3)} \leq C_T \|a\|_{L^2(0, T; H^{3/2}(\partial\mathcal{F})) \cap H^{3/4}(0, T; L^2(\partial\mathcal{F}))}. \quad (3.24) \quad \boxed{\{trace\}}$$

However, the constant C_T in (3.24) may depend on T , then, we extend data over $(0, T^*)$ with $T^* > 2T$ as follows:

$$a_{ext}(t) = \begin{cases} a_{ext}(t) = a(t) & t \in (0, T), \\ a_{ext}(t) = a(2T - t) & t \in (T, 2T), \\ a_{ext}(t) = a(0, \cdot) & t \in (2T, T^*). \end{cases}$$

1 Since $a(0, \cdot) \in H^{3/2}(\partial\mathcal{F})$, then $a_{ext} \in L^2(0, T^*; H^{3/2}(\partial\mathcal{F})) \cap H^{3/4}(0, T^*; L^2(\partial\mathcal{F}))$. Then, we obtain

$$\|w\|_{L^2(0, T; [H^2(\mathcal{F})]^3) \cap H^1(0, T; [L^2(\mathcal{F})]^3)} \leq C_{T^*} \|a_{ext}\|_{L^2(0, T^*; H^{3/2}(\partial\mathcal{F})) \cap H^{3/4}(0, T^*; L^2(\partial\mathcal{F}))}, \quad (3.25) \quad \{\text{1136}\}$$

2 such that C_{T^*} depends on T^* . We notice

$$\|a_{ext}\|_{H^{3/4}(0, T^*; L^2(\partial\mathcal{F}))} \leq \|a_{ext} - a(0, \cdot)\|_{H^{3/4}(0, T^*; L^2(\partial\mathcal{F}))} + (T^*)^{1/2} \|a(0, \cdot)\|_{L^2(\partial\mathcal{F})}. \quad (3.26) \quad \{\text{1502}\}$$

From Lemma A.2,

$$\begin{aligned} \|a_{ext} - a(0, \cdot)\|_{H^{3/4}(0, T^*; L^2(\partial\mathcal{F}))} &\leq C \|a - a(0, \cdot)\|_{H^{3/4}(0, T; L^2(\partial\mathcal{F}))} \\ &\leq C \|a\|_{H^{3/4}(0, T; L^2(\partial\mathcal{F}))} + \|a(0, \cdot)\|_{L^2(0, T; L^2(\partial\mathcal{F}))}. \end{aligned} \quad (3.27) \quad \{\text{1034}\}$$

with C independent of T . In the other hand,

$$\|a(0, \cdot)\|_{L^2(0, T; L^2(\partial\mathcal{F}))} \leq T^{1/2} \|a\|_{L^\infty(0, T; L^2(\partial\mathcal{F}))} = T^{1/2} \|a^*\|_{L^\infty(0, 1; L^2(\partial\mathcal{F}))} \leq CT^{1/2} \|a^*\|_{H^{3/4}(0, 1; L^2(\partial\mathcal{F}))},$$

where $a^*(t^*) = a(t^*T)$, $t^* \in (0, 1)$. In particular, using the estimate [13, (A.11), p. 36] we get

$$\|a^*\|_{H^{3/4}(0, 1; L^2(\partial\mathcal{F}))} \leq CT^{-1/2} \|a\|_{H^{3/4}(0, T; L^2(\partial\mathcal{F}))}.$$

3 Then, (3.27) gives

$$\|a_{ext} - a(0, \cdot)\|_{H^{3/4}(0, T^*; L^2(\partial\mathcal{F}))} \leq C \|a\|_{H^{3/4}(0, T; L^2(\partial\mathcal{F}))}. \quad (3.28) \quad \{\text{1514}\}$$

4 We get also

$$\|a_{ext}\|_{L^2(0, T^*; H^{3/2}(\partial\mathcal{F}))} \leq C \|a\|_{L^2(0, T; H^{3/2}(\partial\mathcal{F}))} + (T^*)^{1/2} \|a(0, \cdot)\|_{H^{3/2}(\partial\mathcal{F})}. \quad (3.29) \quad \{\text{1456}\}$$

5 Combining (3.29), (3.28) and (3.26), we get the estimate (3.23). \square

6 Lifting of the slip boundary condition

7 **Lemma 3.4.** *Let $\bar{g}_i \in L^2(0, T; H^{1/2}(\partial\mathcal{F})) \cap H^{1/4}(0, T; L^2(\partial\mathcal{F}))$, $i = 1, 2$. The non stationary Stokes system*

$$\begin{cases} \partial_t \phi - \nabla \cdot (\mathbb{S}(\phi)) = 0 & \text{in } (0, \infty) \times \mathcal{F}, \\ \phi \cdot n = 0 & \text{on } (0, \infty) \times \partial\mathcal{F}, \\ [2\mu\mathbb{D}(\phi)n + \beta\phi] \cdot \tau^i = \bar{g}_i & \text{on } (0, \infty) \times \partial\mathcal{F}, \\ \phi(0, \cdot) = 0 & \text{in } \mathcal{F}, \end{cases} \quad (3.30) \quad \{\text{lift}\}$$

admits a unique strong solution in $L^2(0, \infty; [H^2(\mathcal{F})]^3) \cap H^1(0, \infty; [L^2(\mathcal{F})]^3)$ and

$$\begin{aligned} &\|\phi\|_{L^2(0, \infty; [H^2(\mathcal{F})]^3) \cap H^1(0, \infty; [L^2(\mathcal{F})]^3) \cap L^\infty(0, \infty; [H^1(\mathcal{F})]^3)} \\ &\leq C \left(\|\bar{g}_1\|_{L^2(0, \infty; H^{1/2}(\partial\mathcal{F})) \cap H^{1/4}(0, \infty; L^2(\partial\mathcal{F}))} + \|\bar{g}_2\|_{L^2(0, \infty; H^{1/2}(\partial\mathcal{F})) \cap H^{1/4}(0, \infty; L^2(\partial\mathcal{F}))} \right). \end{aligned} \quad (3.31) \quad \{\text{estim}\}$$

Proof. The proof is based on the transposition method and it is similar to the incompressible case see [68, 67]. Let $\bar{g}_i \in C_0^\infty(0, \infty; H^{1/2}(\partial\mathcal{F}))$ and to simplify the calculations, let $\bar{g} \in C_0^\infty(0, \infty; [H^{1/2}(\partial\mathcal{F})]^3)$ such that $\bar{g} \cdot \tau^i = \bar{g}_i$, $i = 1, 2$ and $\bar{g} \cdot n = 0$. Let ϕ_g the solution of the stationary system with (3.6) with $a = 0$ and $g_i = \bar{g}_i$. Setting $\bar{\phi} = \phi - \phi_g$, we see that $\bar{\phi}$ verifies the system

$$\begin{cases} \partial_t \bar{\phi} - \nabla \cdot (\mathbb{S}(\bar{\phi})) = -\partial_t \phi_g & \text{in } (0, \infty) \times \mathcal{F}, \\ \bar{\phi} \cdot n = 0 & \text{on } (0, \infty) \times \partial\mathcal{F}, \\ [2\mu\mathbb{D}(\bar{\phi})n + \beta\bar{\phi}] \cdot \tau^i = 0 & \text{on } (0, \infty) \times \partial\mathcal{F}, \\ \bar{\phi}(0, \cdot) = 0 & \text{in } \mathcal{F}. \end{cases}$$

- 1 From Theorem 3.1, the associated Lamé operator of the above system is analytic with negative type. Then,
 2 there exists a unique strong solution $\bar{\phi}$ that is in $C_0^\infty(0, \infty; [H^2(\mathcal{F})]^3)$ and thus the system (3.30) admits a unique
 3 solution $\phi \in C_0^\infty(0, \infty; [H^2(\mathcal{F})]^3)$.

Let show (3.31). Let $f \in L^2(\mathbb{R}; [L^2(\mathcal{F})]^3)$ and $T_\infty > 0$ such that $f = 0$ on (T_∞, ∞) the solution of the backward system

$$\begin{cases} \partial_t \psi + \nabla \cdot (\mathbb{S}(\psi)) = f & \text{in } (-\infty, T_\infty) \times \mathcal{F}, \\ \psi \cdot n = 0 & \text{on } (-\infty, T_\infty) \times \partial\mathcal{F}, \\ [2\mu\mathbb{D}(\psi)n + \beta\psi] \cdot \tau^i = 0 & \text{on } (-\infty, T_\infty) \times \partial\mathcal{F}, \\ \psi(T_\infty, \cdot) = 0 & \text{in } \mathcal{F}. \end{cases}$$

- 4 We extend ψ (resp. ϕ and \bar{g}) by zero on (T_∞, ∞) (resp. on $(-\infty, 0)$). The function ψ satisfies the estimate

$$\|\psi\|_{L^2(\mathbb{R}; [H^2(\mathcal{F})]^3) \cap H^1(\mathbb{R}; [L^2(\mathcal{F})]^3) \cap L^\infty(\mathbb{R}; [H^1(\mathcal{F})]^3)} \leq C \|f\|_{L^2(\mathbb{R}; [L^2(\mathcal{F})]^3)}. \quad (3.32) \quad \{\text{1009}\}$$

After standard calculations and integration by parts, we get

$$\int_{\mathbb{R}} \int_{\mathcal{F}} \partial_t \phi \cdot f \, dy \, dt = - \int_{\mathbb{R}} \int_{\partial\mathcal{F}} \partial_t \bar{g} \cdot \psi \, d\Gamma \, dt.$$

- 5 From the trace theorem, we have that $\psi \in H^{3/4}(\mathbb{R}; [L^2(\partial\mathcal{F})]^3)$. Hence, we obtain

$$\int_{\mathbb{R}} \int_{\mathcal{F}} \partial_t \phi \cdot f \, dy \, dt \leq C \|\bar{g}\|_{H^{1/4}(\mathbb{R}; [L^2(\partial\mathcal{F})]^3)} \|\psi\|_{H^{3/4}(\mathbb{R}; [L^2(\partial\mathcal{F})]^3)}. \quad (3.33) \quad \{\text{1022}\}$$

- 6 Combining (3.33) and (3.32), we get

$$\|\partial_t \phi\|_{L^2(0, \infty; [L^2(\mathcal{F})]^3)} \leq C \|\bar{g}\|_{H^{1/4}(0, \infty; [L^2(\partial\mathcal{F})]^3)}. \quad (3.34) \quad \{\text{1025}\}$$

- 7 Finally, using Theorem 3.1, we obtain (3.31). □

- 8 Finally, we consider the following non stationary and nonhomogeneous Lamé system

$$\begin{cases} \rho^0 \partial_t v_\ell - \nabla \cdot (\mathbb{S}(v_\ell)) = 0 & \text{in } (0, T) \times \mathcal{F}, \\ v_\ell \cdot n = a & \text{on } (0, T) \times \partial\mathcal{F}, \\ [2\mu\mathbb{D}(v_\ell)n + \beta v_\ell] \cdot \tau^i = g_i & \text{on } (0, T) \times \partial\mathcal{F}, \\ v_\ell(0, \cdot) = w(0) & \text{in } \mathcal{F}, \end{cases} \quad (3.35) \quad \{\text{ilame}\}$$

- 9 where w is obtained in Lemma 3.3 with $\beta(y) = \beta_0$ if $y \in \partial\mathcal{S}$ and $\beta(y) = \beta_1$ if $y \in \partial\mathcal{F}$. We suppose that
 10 $\beta_0 + \beta_1 > 0$.

lame **Proposition 3.5.** *Let*

$$a \in L^2(0, T; H^{3/2}(\partial\mathcal{F})) \cap H^{3/4}(0, T; L^2(\partial\mathcal{F})), \quad g_i \in L^2(0, T; H^{1/2}(\partial\mathcal{F})) \cap H^{1/4}(0, T; L^2(\partial\mathcal{F})).$$

The system (3.35) admits a unique strong solution v_ℓ of the system (3.35) and there exists a constant C non decreasing with respect to T such that we have the estimate

$$\begin{aligned} \|v_\ell\|_{L^2(0, T; [H^2(\mathcal{F})]^3) \cap H^1(0, T; [L^2(\mathcal{F})]^3)} &\leq C \left(\|a\|_{L^2(0, T; H^{3/2}(\partial\mathcal{F})) \cap H^{3/4}(0, T; L^2(\partial\mathcal{F}))} \right. \\ &\left. + \|a(0, \cdot)\|_{H^{3/2}(\partial\mathcal{F})} + \|g_1\|_{L^2(0, T; [H^{1/2}(\partial\mathcal{F})] \cap H^{1/4}(0, T; L^2(\partial\mathcal{F}))} + \|g_2\|_{L^2(0, T; [H^{1/2}(\partial\mathcal{F})] \cap H^{1/4}(0, T; L^2(\partial\mathcal{F}))} \right). \end{aligned} \quad (3.36) \quad \{\text{estim1}\}$$

Proof. We consider the lifting $\bar{v} = v_\ell - w - \phi$ where w is obtained in Lemma 3.3 and ϕ is the solution of (3.30) with

$$\bar{g}_i = g_i - [2\mu\mathbb{D}(w)n + \beta w] \cdot \tau^i, \quad t \in (0, T),$$

1 extended by 0 on (T, ∞) . The system (3.35) is reduced to

$$\begin{cases} \rho^0 \partial_t \bar{v} - \nabla \cdot (\mathbb{S}(\bar{v})) = f(w, \phi) & \text{in } (0, T) \times \mathcal{F}, \\ \bar{v} \cdot n = 0 & \text{on } (0, T) \times \partial \mathcal{F}, \\ [2\mu \mathbb{D}(\bar{v})n + \beta \bar{v}] \cdot \tau^i = 0 & \text{on } (0, T) \times \partial \mathcal{F}, \\ \bar{v}(0, \cdot) = 0 & \text{in } \mathcal{F}. \end{cases} \quad (3.37) \quad \{\text{fs3}\}$$

2 Using the elliptic results elaborated in Theorem 3.1 and [9, Theorem 3.1, p.143], we obtain that \bar{v} satisfies

$$\|\bar{v}\|_{L^2(0, T; [H^2(\mathcal{F})]^3) \cap H^1(0, T; [L^2(\mathcal{F})]^3) \cap L^\infty(0, T; [H^1(\mathcal{F})]^3)} \leq C e^{CT} \|f(w, \phi)\|_{L^2(0, T; [L^2(\partial \mathcal{F})]^3)}. \quad (3.38) \quad \{\text{11:38}\}$$

3 Then, using (3.38), (3.31) and (3.23) combined with [13, Lemma A.8] we deduce finally (3.36). \square

4 3.3 Linearized system

5 3.3.1 H^2 Regularity

6 We consider the lifting v_ℓ that satisfies (2.4). Let $u = v - v_\ell$, the linearized system (3.1), (3.2), (3.3) and (3.4)
7 is reduced as follows

$$\begin{cases} \partial_t \rho + \rho^0 \nabla \cdot u = \bar{h} & \text{in } (0, T) \times \mathcal{F}, \\ \rho(0, \cdot) = \rho^0 & \text{in } \mathcal{F}, \end{cases} \quad (3.39) \quad \{\text{dens12}\}$$

$$\begin{cases} \rho^0 \partial_t u - \nabla \cdot (\mathbb{S}(u)) = \nabla \cdot \bar{f}_1 + \bar{f}_2 & \text{in } (0, T) \times \mathcal{F}, \\ v_S(y) = \ell + \omega \times y & \text{in } (0, T), \\ m\ell' = f_3 & \text{in } (0, T), \\ J\omega' = f_4 & \text{in } (0, T), \end{cases} \quad (3.40) \quad \{\text{fs2}\}$$

9 with the boundary conditions

$$\begin{cases} u \cdot n = 0, & [(2\mu \mathbb{D}(u)n + \beta_0 u)] \cdot \tau^i = 0 & \text{on } (0, T) \times \partial \Omega, \\ (u - v_S) \cdot n = 0, & [(2\mu \mathbb{D}(u)n + \beta_1(u - v_S))] \cdot \tau^i = 0 & \text{on } (0, T) \times \partial \mathcal{S}, \end{cases} \quad (3.41) \quad \{\text{bc2}\}$$

10 and the initial conditions

$$\rho(0, \cdot) = \rho^0 \text{ in } \mathcal{F}, \quad u(0, \cdot) = u^0 \text{ in } \mathcal{F}, \quad \ell(0) = \ell^0, \quad \omega(0) = \omega^0, \quad (3.42) \quad \{\text{ic2}\}$$

11 with the following compatibility condition

$$v_S^0 = \ell^0 + \omega^0 \times y, \quad u^0 \cdot n = 0 \text{ on } \partial \Omega, \quad (u^0 - v_S^0) \cdot n = 0 \text{ on } \partial \mathcal{S}. \quad (3.43) \quad \{\text{comp}\}$$

We introduce the spaces

$$\begin{aligned} \mathcal{X}_T &= (W^{1, \infty}(0, T; L^2(\mathcal{F})) \cap H^1(0, T; H^1(\mathcal{F}))) \\ &\quad \times (L^2(0, T; [H^2(\mathcal{F})]^3) \cap H^1(0, T; [L^2(\mathcal{F})]^3) \cap C(0, T; [H^1(\mathcal{F})]^3)) \times [H^1(0, T)]^3 \times [H^1(0, T)]^3, \\ \mathcal{I}_0 &= \{(u^0, \ell^0, \omega^0) \in [H^1(\mathcal{F})]^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \text{ satisfying (3.43)}\}. \end{aligned}$$

Proposition 3.6. *For any*

$$\begin{aligned} (u^0, \ell^0, \omega^0) \in \mathcal{I}_0, \quad \bar{h} \in L^2(0, T; H^1(\mathcal{F})) \cap L^\infty(0, T; L^2(\mathcal{F})), \quad \bar{f}_1 \in L^2(0, T; [H^1(\mathcal{F})]^9), \\ \bar{f}_2 \in L^2(0, T; [L^2(\mathcal{F})]^3), \quad f_3 \in [L^2(0, T)]^3, \quad f_4 \in [L^2(0, T)]^3, \end{aligned}$$

there exists a unique solution of the system (3.39), (3.40), (3.41) and (3.42) such that

$$\begin{aligned} \|(\rho, u, \ell, \omega)\|_{\mathcal{X}_T} \leq C e^{CT} \left(\|\bar{h}\|_{L^2(0, T; H^1(\mathcal{F})) \cap L^\infty(0, T; L^2(\mathcal{F}))} + \|\bar{f}_1\|_{L^2(0, T; [H^1(\mathcal{F})]^9)} + \|\bar{f}_2\|_{L^2(0, T; [L^2(\mathcal{F})]^3)} \right. \\ \left. + \|f_3\|_{[L^2(0, T)]^3} + \|f_4\|_{[L^2(0, T)]^3} + \|(u^0, \ell^0, \omega^0)\|_{\mathcal{I}_0} \right). \quad (3.44) \quad \{\text{estim1}\} \end{aligned}$$

Proof. For $\ell, \omega \in \mathbb{R}^3$, we define $\mathbb{N}(\ell, \omega)$ as the solution of the system (3.6) with $a = v_S \cdot n$, $g_i = v_S \cdot \tau^i$ and $f = 0$. From Theorem 3.1, we see that $\mathbb{N} \in \mathcal{L}(\mathbb{R}^3 \times \mathbb{R}^3, [H^2(\mathcal{F})]^3)$. We introduce the operator

$$\mathbb{A}_F = -\frac{1}{\rho^0} \nabla \cdot \mathbb{S}, \quad \mathcal{D}(\mathbb{A}_F) = \{u \in [H^2(\mathcal{F})]^3; u \cdot n = 0, [2\mu\mathbb{D}(u)n + \beta u] \cdot \tau^i = 0 \text{ on } \partial\mathcal{F}\},$$

- 1 that is self adjoint and by Korn inequality (see Lemma A.1), we deduce that this operator is also coercive, then
- 2 \mathbb{A}_F is the generator of an analytic semigroup on $[L^2(\mathcal{F})]^3$ (see [9, Theorem 2.12, p.115]). Taking $\tilde{u} = u - \mathbb{N}(\ell, \omega)$,
- 3 the fluid system is reduced to

$$\begin{cases} \rho^0 \partial_t \tilde{u} - \nabla \cdot (\mathbb{S}(\tilde{u})) = f & \text{in } (0, T) \times \mathcal{F}, \\ \tilde{u} \cdot n = 0, \quad [(2\mu\mathbb{D}(\tilde{u})n + \beta\tilde{u})] \cdot \tau^i = 0 & \text{on } (0, T) \times \partial\mathcal{F}, \\ \tilde{u}(0, \cdot) = u^0 - \mathbb{N}(\ell^0, \omega^0) & \text{in } \mathcal{F}. \end{cases} \quad (3.45) \quad \boxed{\{1650\}}$$

From [9, Theorem 3.1, p.143], we obtain that \tilde{u} satisfies

$$\begin{aligned} & \|\tilde{u}\|_{L^2(0, T; [H^2(\mathcal{F})]^3) \cap H^1(0, T; [L^2(\mathcal{F})]^3) \cap L^\infty(0, T; [H^1(\mathcal{F})]^3)} \leq C e^{CT} \|f\|_{L^2(0, T; [L^2(\partial\mathcal{F})]^3)} \\ & \leq C e^{CT} \left(\|\bar{f}_1\|_{L^2(0, T; [H^1(\mathcal{F})]^9)} + \|\bar{f}_2\|_{L^2(0, T; [L^2(\mathcal{F})]^3)} + \|f_3\|_{[L^2(0, T)]^3} + \|f_4\|_{[L^2(0, T)]^3} + \|(u^0, \ell^0, \omega^0)\|_{\mathcal{I}_0} \right). \end{aligned} \quad (3.46) \quad \boxed{\{1700\}}$$

Thus, we get

$$\begin{aligned} \|u\|_{L^2(0, T; [H^2(\mathcal{F})]^3) \cap H^1(0, T; [L^2(\mathcal{F})]^3) \cap L^\infty(0, T; [H^1(\mathcal{F})]^3)} & \leq C e^{CT} \left(\|\bar{f}_1\|_{L^2(0, T; [H^1(\mathcal{F})]^9)} + \|\bar{f}_2\|_{L^2(0, T; [L^2(\mathcal{F})]^3)} \right. \\ & \left. + \|f_3\|_{[L^2(0, T)]^3} + \|f_4\|_{[L^2(0, T)]^3} + \|(u^0, \ell^0, \omega^0)\|_{\mathcal{I}_0} \right). \end{aligned} \quad (3.47) \quad \boxed{\{1712\}}$$

From (3.39), we get that

$$\rho \in W^{1, \infty}(0, T; L^2(\mathcal{F})) \cap H^1(0, T; H^1(\mathcal{F})).$$

- 4 Hence, we deduce that the system (3.40), (3.41), (3.42) admits a unique strong solution that satisfies the energy
- 5 estimate (3.44). □

Let

$$\begin{aligned} \mathcal{Y}_T = \{ & (h, f_1, f_2, f_3, f_4, a, g_1, g_2) \in (L^2(0, T; H^1(\mathcal{F})) \cap L^\infty(0, T; L^2(\mathcal{F}))) \times L^2(0, T; [H^1(\mathcal{F})]^9) \\ & \times L^2(0, T; [L^2(\mathcal{F})]^3) \times [L^2(0, T)]^3 \times [L^2(0, T)]^3 \times \left(L^2(0, T; H^{3/2}(\partial\mathcal{F})) \cap H^{3/4}(0, T; L^2(\partial\mathcal{F})) \right) \\ & \times \left(L^2(0, T; H^{1/2}(\partial\mathcal{F})) \cap H^{1/4}(0, T; L^2(\partial\mathcal{F})) \right)^2, \quad a(0, \cdot) \in H^{3/2}(\partial\mathcal{F})\}, \end{aligned}$$

and

$$\begin{aligned} \|(h, f_1, f_2, f_3, f_4, a, g_1, g_2)\|_{\mathcal{Y}_T} & = \|h\|_{L^2(0, T; H^1(\mathcal{F})) \cap L^\infty(0, T; L^2(\mathcal{F}))} + \|f_1\|_{L^2(0, T; [H^1(\mathcal{F})]^9)} + \|f_2\|_{L^2(0, T; [L^2(\mathcal{F})]^3)} \\ & + \|f_3\|_{[L^2(0, T)]^3} + \|f_4\|_{[L^2(0, T)]^3} + \|a(0, \cdot)\|_{L^2(0, T; H^{3/2}(\partial\mathcal{F})) \cap H^{3/4}(0, T; L^2(\partial\mathcal{F}))} + \|a(0, \cdot)\|_{H^{3/2}(\partial\mathcal{F})} \\ & + \|g_1\|_{L^2(0, T; H^{1/2}(\partial\mathcal{F})) \cap H^{1/4}(0, T; L^2(\partial\mathcal{F}))} + \|g_2\|_{L^2(0, T; H^{1/2}(\partial\mathcal{F})) \cap H^{1/4}(0, T; L^2(\partial\mathcal{F}))}. \end{aligned}$$

- 6 Then using (3.36), we deduce a regularity results for (3.1), (3.2), (3.3) and (3.4).

regH2

Corollary 3.7. *Let*

$$(h, f_1, f_2, f_3, f_4, a, g_1, g_2) \in \mathcal{Y}_T, \quad \rho^0 \in H^1(\mathcal{F}), \quad v^0 \in [H^1(\mathcal{F})]^3, \quad \ell^0, \omega^0 \in \mathbb{R}^3,$$

- 7 such that (3.5) is verified. Then, the system (3.1), (3.2), (3.3) and (3.4) admits a unique strong solution
- 8 $(\rho, v, \ell, \omega) \in \mathcal{X}_T$ such that

$$\|(\rho, v, \ell, \omega)\|_{\mathcal{X}_T} \leq C e^{CT} \left(\|(h, f_1, f_2, f_3, f_4, a, g_1, g_2)\|_{\mathcal{Y}_T} + \|\rho^0\|_{H^1(\mathcal{F})} + |\ell^0| + |\omega^0| + \|v^0\|_{[H^1(\mathcal{F})]^3} \right). \quad (3.48) \quad \boxed{\{\text{estimH2}\}}$$

1 3.3.2 H^4 Regularity

2 Let $(v^0, \ell^0, \omega^0) \in [H^3(\mathcal{F})]^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$v_S^0 = \ell^0 + \omega^0 \times y, \quad \begin{cases} v^0 \cdot n = a(0, \cdot), & [2\mu\mathbb{D}(v^0)n + \beta v^0]_i = g_i(0, \cdot) & \text{on } \partial\Omega, \\ (v^0 - v_S^0) \cdot n = a(0, \cdot), & [2\mu\mathbb{D}(v^0)n + \beta_1(v^0 - v_S^0)]_i = g_i(0, \cdot) & \text{on } \partial\mathcal{S}, \end{cases} \quad (3.49) \quad \boxed{\text{\{cc1\}}}$$

3 and

$$\begin{cases} \left(\frac{1}{\rho^0} \nabla \cdot \mathbb{S}(v^0) + \nabla \cdot (f_1(0, \cdot)) + f_2(0, \cdot) \right) \cdot n = \partial_t a(0, \cdot) & \text{on } \partial\Omega, \\ \left[\left(\frac{1}{\rho^0} \nabla \cdot \mathbb{S}(v^0) + \nabla \cdot (f_1(0, \cdot)) + f_2(0, \cdot) \right) - (m^{-1}f_3(0) + J^{-1}f_4(0) \times y) \right] \cdot n = \partial_t a(0, \cdot) & \text{on } \partial\mathcal{S}. \end{cases} \quad (3.50) \quad \boxed{\text{\{cc2\}}}$$

Let introduce the spaces

$$\begin{aligned} \widehat{\mathcal{X}}_T &= W^{1,\infty}(0, T; H^2(\mathcal{F})) \cap H^1(0, T; H^3(\mathcal{F})) \cap H^2(0, T; H^1(\mathcal{F})) \\ &\quad \times (L^2(0, T; [H^4(\mathcal{F})]^3) \cap H^2(0, T; [L^2(\mathcal{F})]^3) \cap C(0, T; [H^3(\mathcal{F})]^3)) \times [H^2(0, T)]^3 \times [H^2(0, T)]^3, \\ \widehat{\mathcal{I}}_0 &= \{(v^0, \ell^0, \omega^0) \in [H^3(\mathcal{F})]^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \text{ satisfying (3.49) and (3.50)}\}, \end{aligned}$$

$$\begin{aligned} \widehat{\mathcal{Y}}_T &= \{(h, f_1, f_2, f_3, f_4, a, g_1, g_2) \in (L^2(0, T; H^3(\mathcal{F})) \cap L^\infty(0, T; H^2(\mathcal{F})) \cap H^1(0, T; H^1(\mathcal{F}))) \\ &\quad \times (L^2(0, T; [H^3(\mathcal{F})]^9) \cap H^1(0, T; [H^1(\mathcal{F})]^9)) \times (L^2(0, T; [H^2(\mathcal{F})]^3) \cap H^1(0, T; [L^2(\mathcal{F})]^3)) \\ &\quad \times [H^1(0, T)]^3 \times [H^1(0, T)]^3 \times (L^2(0, T; H^{7/2}(\partial\mathcal{F})) \cap H^{7/4}(0, T; L^2(\partial\mathcal{F}))) \\ &\quad \times (L^2(0, T; H^{5/2}(\partial\mathcal{F})) \cap H^{5/4}(0, T; L^2(\partial\mathcal{F})))\}^2; \quad h(0, \cdot) \in H^2(\mathcal{F}), \quad f_1(0, \cdot) \in [H^2(\mathcal{F})]^9, \\ &\quad f_2(0, \cdot) \in [H^1(\mathcal{F})]^3, \quad a(0, \cdot) \in H^{5/2}(\partial\mathcal{F}), \quad \partial_t a(0, \cdot) \in H^{3/2}(\partial\mathcal{F}), \quad g(0, \cdot) \in H^{3/2}(\partial\mathcal{F})\}, \end{aligned}$$

where

$$\begin{aligned} \|(h, f_1, f_2, f_3, f_4, a, g_1, g_2)\|_{\widehat{\mathcal{Y}}_T} &= \|h\|_{L^2(0, T; H^3(\mathcal{F})) \cap L^\infty(0, T; H^2(\mathcal{F})) \cap H^1(0, T; H^1(\mathcal{F}))} + \|h(0, \cdot)\|_{H^2(\mathcal{F})} \\ &\quad + \|f_1\|_{L^2(0, T; [H^3(\mathcal{F})]^9) \cap H^1(0, T; [H^1(\mathcal{F})]^9)} + \|f_1(0, \cdot)\|_{[H^2(\mathcal{F})]^9} + \|f_2\|_{L^2(0, T; [H^2(\mathcal{F})]^3) \cap H^1(0, T; [L^2(\mathcal{F})]^3)} \\ &\quad + \|f_2(0, \cdot)\|_{[H^1(\mathcal{F})]^3} + \|f_3\|_{[H^1(0, T)]^3} + \|f_4\|_{[H^1(0, T)]^3} + |f_3(0)| + |f_4(0)| + \|a\|_{L^2(0, T; H^{7/2}(\partial\mathcal{F})) \cap H^{7/4}(0, T; L^2(\partial\mathcal{F}))} \\ &\quad + \|a(0, \cdot)\|_{H^{5/2}(\partial\mathcal{F})} + \|\partial_t a(0, \cdot)\|_{H^{3/2}(\partial\mathcal{F})} + \|g_1\|_{L^2(0, T; H^{5/2}(\partial\mathcal{F})) \cap H^{5/4}(0, T; L^2(\partial\mathcal{F}))} \\ &\quad + \|g_2\|_{L^2(0, T; H^{5/2}(\partial\mathcal{F})) \cap H^{5/4}(0, T; L^2(\partial\mathcal{F}))} + \|g_1(0, \cdot)\|_{H^{3/2}(\partial\mathcal{F})} + \|g_2(0, \cdot)\|_{H^{3/2}(\partial\mathcal{F})}. \end{aligned}$$

Theorem 3.8. *Let*

$$(h, f_1, f_2, f_3, f_4, a, g_1, g_2) \in \widehat{\mathcal{Y}}_T, \quad (\rho^0, v^0, \ell^0, \omega^0) \in H^3(\mathcal{F}) \times \widehat{\mathcal{I}}_0.$$

4 *Then, the system (3.1), (3.2), (3.3) and (3.4) admits a unique strong solution in $\widehat{\mathcal{X}}_T$ such that*

$$\|(\rho, v, \ell, \omega)\|_{\widehat{\mathcal{X}}_T} \leq C e^{CT} \left(\|(h, f_1, f_2, f_3, f_4, a, g_1, g_2)\|_{\widehat{\mathcal{Y}}_T} + \|(\rho^0, v^0, \ell^0, \omega^0)\|_{H^3(\mathcal{F}) \times \widehat{\mathcal{I}}_0} \right). \quad (3.51) \quad \boxed{\text{\{estimH4\}}}$$

Proof. Let

$$v^1 = \frac{1}{\rho^0} \nabla \cdot \mathbb{S}(v^0) + \nabla \cdot (f_1(0, \cdot)) + f_2(0, \cdot) \text{ in } \mathcal{F}, \quad \ell^1 = m^{-1}f_3(0), \quad \omega^1 = J^{-1}f_4(0).$$

We differentiate the system (3.2), (3.3) and (3.4) with respect to t , then $(\partial_t v, \ell', \omega')$ is the solution of (3.2), (3.3) and (3.4) associated to $(\partial_t f_1, \partial_t f_2, \partial_t f_3, \partial_t f_4, \partial_t a, \partial_t g)$ with the initial conditions (v^1, ℓ^1, ω^1) . Using Corollary 3.7, we obtain that

$$\begin{aligned} \|(\partial_t v, \ell', \omega')\|_{L^2(0,T;[H^1(\mathcal{F})]^3) \cap H^1(0,T;[L^2(\mathcal{F})]^3) \times [H^1(0,T)]^3 \times [H^1(0,T)]^3} &\leq C e^{CT} \left(|\ell^1| + |\omega^1| \right. \\ &+ \|v^1\|_{[H^1(\mathcal{F})]^3} + \|f_1\|_{H^1(0,T;[H^1(\mathcal{F})]^9)} + \|f_2\|_{H^1(0,T;[L^2(\mathcal{F})]^3)} + \|f_3\|_{[H^1(0,T)]^3} + \|f_4\|_{[H^1(0,T)]^3} \\ &+ \|a\|_{H^1(0,T;H^{3/2}(\partial\mathcal{F})) \cap H^{7/4}(0,T;L^2(\partial\mathcal{F}))} + \|\partial_t a(0, \cdot)\|_{H^{3/2}(\partial\mathcal{F})} + \|g_1\|_{H^1(0,T;H^{1/2}(\partial\mathcal{F})) \cap H^{5/4}(0,T;L^2(\partial\mathcal{F}))} \\ &\left. + \|g_2\|_{H^1(0,T;H^{1/2}(\partial\mathcal{F})) \cap H^{5/4}(0,T;L^2(\partial\mathcal{F}))} \right). \end{aligned} \quad (3.52) \quad \boxed{\{15:50\}}$$

In the other hand from Theorem 3.1, we have that

$$\begin{aligned} \|v\|_{L^2(0,T;[H^4(\mathcal{F})]^3)} &\leq C \left(\|f_1\|_{L^2(0,T;[H^3(\mathcal{F})]^9)} + \|f_2\|_{L^2(0,T;[H^2(\mathcal{F})]^3)} + \|\partial_t v\|_{L^2(0,T;[H^2(\mathcal{F})]^3)} \right. \\ &+ \|f_3\|_{[L^2(0,T)]^3} + \|f_4\|_{[L^2(0,T)]^3} + \|a\|_{L^2(0,T;H^{7/2}(\partial\mathcal{F}))} \\ &\left. + \|g_1\|_{L^2(0,T;H^{5/2}(\partial\mathcal{F}))} + \|g_2\|_{L^2(0,T;H^{5/2}(\partial\mathcal{F}))} + \|\ell\|_{[L^2(0,T)]^3} + \|\omega\|_{[L^2(0,T)]^3} \right). \end{aligned} \quad (3.53) \quad \boxed{\{15:55\}}$$

1 Gathering (3.52) and (3.53), we obtain (3.51). □

2 4 Fixed-Point and proof of Theorem 1.1

Let us fix $T > 0$ and $R > 0$ and let define

$$\begin{aligned} \mathcal{B}_{R,T} = \{ &(h, f_1, f_2, f_3, f_4, a, g_1, g_2) \in \widehat{\mathcal{Y}}_T ; \quad h(0, \cdot) = 0, \quad f_1(0, \cdot) = -a_\rho(\rho^0)^\gamma \mathbb{I}_3, \quad f_2(0, \cdot) = 0, \\ &f_3 = - \int_{\partial\mathcal{S}} (\mathbb{S}(v^0) - a_\rho(\rho^0)^\gamma) n \, d\Gamma, \quad f_4 = - \int_{\partial\mathcal{S}} y \times (\mathbb{S}(v^0) - a_\rho(\rho^0)^\gamma) n \, d\Gamma, \\ &a(0, \cdot) = 0, \quad \partial_t a(0, \cdot) = \widehat{a}, \quad g_i(0, \cdot) = 0, \quad \|(h, f_1, f_2, f_3, f_4, a, g)\|_{\widehat{\mathcal{Y}}_T} \leq R \}, \end{aligned}$$

such that

$$\widehat{a} = \begin{cases} -v^0 \cdot (\nabla n v^0) & \text{on } \partial\Omega, \\ \omega^0 \times (v^0 - \ell^0) \cdot n - (v^0 - v_S^0) \cdot (\nabla n v^0) & \text{on } \partial\mathcal{S}. \end{cases}$$

Let consider the mapping $\mathcal{Z} : \mathcal{B}_{R,T} \rightarrow \widehat{\mathcal{Y}}_T$ such that

$$\mathcal{Z}(h, f_1, f_2, f_3, f_4, a, g_1, g_2) = (\mathbb{T}(\rho, v), \mathbb{F}_1(\rho, v), \mathbb{F}_2(\rho, v), \mathbb{F}_3(\rho, v, \ell, \omega), \mathbb{F}_4(\rho, v, \ell, \omega), \mathbb{B}(v, \ell, \omega), \mathbb{G}(v, \ell, \omega)),$$

3 where (ρ, v, ℓ, ω) is the solution for the linearized system (3.1), (3.2), (3.3) and (3.4) associated to $h, f_1, f_2, a,$
 4 $g_1,$ and g_2 . We recall that the nonlinearities $\mathbb{T}, \mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_3, \mathbb{B}$ and \mathbb{G} are given in (2.7), (2.8), (2.9), (2.10), (2.11)
 5 and (2.12), (2.13) respectively.

For $R > 0$ large enough, we have that $\mathcal{B}_{R,T}$ is not empty. Indeed, let v^* be a function in $L^2(0, T; [H^4(\mathcal{F})]^3) \cap H^2(0, T; [L^2(\mathcal{F})]^3)$ such that $v^*(0) = v^0$. Let $h^* = 0, f_1^*(t, \cdot) = -a_\rho(\rho^0)^\gamma \mathbb{I}_3, g_i^* = 0$ and define for all $t > 0$

$$a^*(t) = \begin{cases} - \int_0^t v^* \cdot (\nabla n v^*) \, ds & \text{on } \partial\Omega, \\ \int_0^t \omega^0 \times (v^* - \ell^0) \cdot n - (v^* - v_S^0) \cdot (\nabla n v^*) \, ds & \text{on } \partial\mathcal{S}, \end{cases}$$

and

$$f_3^*(t) = - \int_{\partial\mathcal{S}} (\mathbb{S}(v^0) - a_\rho(\rho^0)^\gamma) n \, d\Gamma, \quad f_4^*(t) = - \int_{\partial\mathcal{S}} y \times (\mathbb{S}(v^0) - a_\rho(\rho^0)^\gamma) n \, d\Gamma.$$

Then, from (1.7) there exists R such that

$$\|f_1^*\|_{[H^2(\mathcal{F})]^9} + |f_3^*| + |f_4^*| + \|\partial_t a^*(0, \cdot)\|_{H^{3/2}(\partial\mathcal{F})} \leq R/2.$$

1 Thus, for T small enough, we get that $(h^*, f_1^*, f_2^*, f_3^*, f_4^*, a^*, g^*)$ belongs to $\mathcal{B}_{R,T}$. Then, $\mathcal{B}_{R,T}$ is non empty. We
 2 assume also that

$$\|(\rho^0, v^0, \ell^0, \omega^0)\|_{H^3(\mathcal{F}) \times \widehat{\mathcal{X}}_0} \leq R. \quad (4.1) \quad \{\text{smallness}\}$$

In this section, we prove that $\mathcal{Z}(\mathcal{B}_{R,T}) \subset \mathcal{B}_{R,T}$ for T small enough and that the mapping $\mathcal{Z}|_{\mathcal{B}_{R,T}}$ defines a contraction. In the sequel, $C > 0$ is a generic constant that can change from a line to another and may depend on R but not on T . From (3.51), we have the following estimate:

$$\|(\rho, v, \ell, \omega)\|_{\widehat{\mathcal{X}}_T} \leq C \left(R + \|(h, f_1, f_2, f_3, f_4, a, g_1, g_2)\|_{\widehat{\mathcal{Y}}_T} \right).$$

3 Let $(h, f_1, f_2, f_3, f_4, a, g_1, g_2) \in \mathcal{B}_{R,T}$, we need to show that $\mathcal{Z}(h, f_1, f_2, f_3, f_4, a, g_1, g_2) \in \mathcal{B}_{R,T}$. We have

$$\|(\rho, v, \ell, \omega)\|_{\widehat{\mathcal{X}}_T} \leq C. \quad (4.2) \quad \{\text{v:estim}\}$$

Thanks to Lemma A.4, we can use interpolation arguments. From (2.1), we have

$$X(t, y) = y + \int_0^t v(t', y) dt', \quad t \in (0, T), \quad y \in \mathcal{F}.$$

Since the regularity of X depends on the fluid flow, we have

$$X \in H^1(0, T; [H^4(\mathcal{F})]^3) \cap H^2(0, T; [H^2(\mathcal{F})]^3) \cap C^1([0, T]; [H^3(\mathcal{F})]^3) \cap H^3(0, T; [L^2(\mathcal{F})]^3).$$

4 It follows that

$$\|X - \text{Id}\|_{L^\infty(0, T; [H^4(\mathcal{F})]^3)} \leq CT^{1/2}, \quad \|\partial_t X\|_{L^2(0, T; [H^3(\mathcal{F})]^3)} \leq CT^{1/2}, \quad (4.3) \quad \{\text{phi:esti}\}$$

$$\|b_X\|_{W^{1,\infty}(0, T; H^2(\mathcal{F})) \cap H^2(0, T; H^1(\mathcal{F})) \cap H^1(0, T; H^3(\mathcal{F}))} \leq C, \quad \|b_X - \mathbb{I}_3\|_{L^\infty(0, T; H^3(\mathcal{F}))} \leq CT^{1/2}, \quad (4.4) \quad \{\text{bx:estim}\}$$

$$\|\delta_X\|_{W^{1,\infty}(0, T; H^2(\mathcal{F})) \cap H^2(0, T; H^1(\mathcal{F})) \cap H^1(0, T; H^3(\mathcal{F}))} \leq C, \quad \|\delta_X - 1\|_{L^\infty(0, T; H^3(\mathcal{F}))} \leq CT^{1/2}. \quad (4.5) \quad \{\text{det:esti}\}$$

7 Since ρ satisfies (3.1), and $\nabla v \in L^2(0, T; [H^3(\mathcal{F})]^9)$, we deduce

$$\|\rho - \rho^0\|_{L^\infty(0, T; H^3(\mathcal{F}))} \leq CT^{1/2}. \quad (4.6) \quad \{\text{rho:esti}\}$$

8 From (4.1), we also have

$$\|b_X\|_{L^\infty(0, T; H^3(\mathcal{F}))} \leq C, \quad \|\delta_X\|_{L^\infty(0, T; H^3(\mathcal{F}))} \leq C. \quad (4.7) \quad \{\text{smallness}\}$$

At the boundary, we have by interpolation

$$\|b_X\|_{H^{1/4}(0, T; [L^\infty(\partial\mathcal{F})]^9) \cap H^{7/8}(0, T; [L^\infty(\partial\mathcal{F})]^9) \cap L^\infty(0, T; [L^\infty(\partial\mathcal{F})]^9)} \leq C, \\ \|\delta_X\|_{H^{1/4}(0, T; L^\infty(\partial\mathcal{F})) \cap H^{7/8}(0, T; L^\infty(\partial\mathcal{F})) \cap L^\infty(0, T; L^\infty(\partial\mathcal{F}))} \leq C, \quad (4.8) \quad \{\text{13:00}\}$$

9 and

$$\|b_X\|_{H^{5/4}(0, T; [H^2(\partial\mathcal{F})]^9)} \leq C, \quad \|\delta_X\|_{H^{5/4}(0, T; H^2(\partial\mathcal{F}))} \leq C, \quad (4.9) \quad \{\text{14:07}\}$$

$$\|\partial_t b_X\|_{H^{7/8}(0, T; [H^{3/4}(\partial\mathcal{F})]^9)} \leq C, \quad \|\partial_t \delta_X\|_{H^{7/8}(0, T; H^{3/4}(\partial\mathcal{F}))} \leq C. \quad (4.10) \quad \{\text{14:27}\}$$

1 4.1 Estimation on \mathbb{T} , \mathbb{F}_1 and \mathbb{F}_2

First, we show that $\mathbb{T} \in L^2(0, T; H^3(\mathcal{F})) \cap L^\infty(0, T; H^2(\mathcal{F}))$, from (4.6), (4.5)

$$\begin{aligned} \|\mathbb{T}(v, \eta, \rho)\|_{L^2(0, T; H^3(\mathcal{F}))} &\leq \left\| \frac{\rho}{\delta_X} - \rho^0 \right\|_{L^\infty(0, T; H^3(\mathcal{F}))} \|\nabla v\|_{L^2(0, T; [H^3(\mathcal{F})]^9)} \\ &\quad + \left\| \frac{\rho}{\delta_X} \right\|_{L^\infty(0, T; H^3(\mathcal{F}))} \|\nabla v\|_{L^2(0, T; [H^3(\mathcal{F})]^9)} \|b_X - \mathbb{I}_3\|_{L^\infty(0, T; [H^3(\mathcal{F})]^9)} \leq CT^{1/2}, \end{aligned}$$

$$\begin{aligned} \|\mathbb{T}(v, \eta, \rho)\|_{L^\infty(0, T; H^2(\mathcal{F}))} &\leq \left\| \frac{\rho}{\delta_X} - \rho^0 \right\|_{L^\infty(0, T; H^3(\mathcal{F}))} \|\nabla v\|_{L^\infty(0, T; [H^2(\mathcal{F})]^9)} \\ &\quad + \left\| \frac{\rho}{\delta_X} \right\|_{L^\infty(0, T; H^3(\mathcal{F}))} \|\nabla v\|_{L^\infty(0, T; [H^2(\mathcal{F})]^9)} \|b_X - \mathbb{I}_3\|_{L^\infty(0, T; [H^3(\mathcal{F})]^9)} \leq CT^{1/2}, \end{aligned}$$

and since $\nabla v \in L^\infty(0, T; [H^2(\mathcal{F})]^9)$, we get

$$\begin{aligned} \|\partial_t \mathbb{T}(v, \eta, \rho)\|_{L^2(0, T; H^1(\mathcal{F}))} &\leq \left(\left\| \frac{\rho}{\delta_X} - \rho^0 \right\|_{L^\infty(0, T; H^3(\mathcal{F}))} \right. \\ &\quad \left. + \left\| \frac{\rho}{\delta_X} \right\|_{L^\infty(0, T; H^3(\mathcal{F}))} \|b_X - \mathbb{I}_3\|_{L^\infty(0, T; [H^3(\mathcal{F})]^9)} \right) \|\nabla \partial_t v\|_{L^2(0, T; [H^1(\mathcal{F})]^9)} \\ &\quad + \|\partial_t \left(\frac{\rho}{\delta_X} b_X \right)\|_{L^\infty(0, T; [H^2(\mathcal{F})]^9)} \|\nabla v\|_{L^2(0, T; [H^2(\mathcal{F})]^9)} \leq CT^{1/2}. \end{aligned}$$

Let prove that $\mathbb{F}_1 \in L^2(0, T; [H^3(\mathcal{F})]^9) \cap H^1(0, T; [H^1(\mathcal{F})]^9)$. Let estimate for example

$$\begin{aligned} &\left\| \frac{1}{\delta_X} \nabla v (b_X)^\top b_X - \nabla v \right\|_{L^2(0, T; [H^3(\mathcal{F})]^9)} \\ &\leq \left\| \frac{1}{\delta_X} - 1 \right\|_{L^\infty(0, T; H^3(\mathcal{F}))} \|\nabla v\|_{L^2(0, T; [H^3(\mathcal{F})]^9)} \|b_X^\top b_X\|_{L^\infty(0, T; [H^3(\mathcal{F})]^9)} \\ &\quad + \left\| \frac{1}{\delta_X} \right\|_{L^\infty(0, T; H^3(\mathcal{F}))} \|\nabla v\|_{L^2(0, T; [H^3(\mathcal{F})]^9)} \|b_X^\top - \mathbb{I}_3\|_{L^\infty(0, T; [H^3(\mathcal{F})]^9)} \|b_X\|_{L^\infty(0, T; [H^3(\mathcal{F})]^9)} \\ &\quad + \left\| \frac{1}{\delta_X} \right\|_{L^\infty(0, T; H^3(\mathcal{F}))} \|\nabla v\|_{L^2(0, T; [H^3(\mathcal{F})]^9)} \|b_X^\top\|_{L^\infty(0, T; [H^3(\mathcal{F})]^9)} \|b_X - \mathbb{I}_3\|_{L^\infty(0, T; [H^3(\mathcal{F})]^9)}. \end{aligned}$$

2 Using (4.5), (4.6), (4.2), (4.3), (4.7) and the fact that $H^3(\mathcal{F}) \times H^3(\mathcal{F}) \hookrightarrow H^3(\mathcal{F})$. We obtain that

$$\left\| \frac{1}{\delta_X} \nabla v (b_X)^\top b_X - \nabla v \right\|_{L^2(0, T; [H^3(\mathcal{F})]^9)} \leq CT^{1/2}. \quad (4.11) \quad \boxed{\{17:10\}}$$

3 In the other hand, the pressure term is estimated as follows

$$\|p(\rho) b_X\|_{L^2(0, T; [H^3(\mathcal{F})]^9)} \leq T^{1/2} \|p(\rho) b_X\|_{L^\infty(0, T; [H^3(\mathcal{F})]^9)} \leq CT^{1/2}. \quad (4.12) \quad \boxed{\{17:08\}}$$

We obtain also

$$\begin{aligned} &\|\partial_t \left(\frac{1}{\delta_X} \nabla v (b_X)^\top b_X \right) - \partial_t \nabla v\|_{L^2(0, T; [H^1(\mathcal{F})]^9)} \\ &\leq C \left(\|\partial_t \nabla v\|_{L^2(0, T; [H^1(\mathcal{F})]^9)} \left\| \frac{(b_X)^\top b_X}{\delta_X} - \mathbb{I}_3 \right\|_{L^\infty(0, T; [H^3(\mathcal{F})]^9)} \right. \\ &\quad \left. + T^{1/2} \|\nabla v\|_{L^\infty(0, T; [H^2(\mathcal{F})]^9)} \|\partial_t \left(\frac{(b_X)^\top b_X}{\delta_X} \right)\|_{L^\infty(0, T; [H^2(\mathcal{F})]^9)} \right) \leq CT^{1/2}. \quad (4.13) \quad \boxed{\{18:16\}} \end{aligned}$$

We also have

$$\begin{aligned} \|\partial_t(p(\rho)b_X)\|_{L^2(0,T;[H^1(\mathcal{F})]^9)} &\leq T^{1/2} \left(\|\partial_t p(\rho)\|_{L^\infty(0,T;H^2(\mathcal{F}))} \|b_X\|_{L^\infty(0,T;[H^3(\mathcal{F})]^9)} \right. \\ &\quad \left. + \|p(\rho)\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\partial_t b_X\|_{L^\infty(0,T;[H^2(\mathcal{F})]^9)} \right) \leq CT^{1/2}. \end{aligned} \quad (4.14) \quad \boxed{\{18:15\}}$$

Then, from (4.11), (4.14) and (4.13) we deduce that $\mathbb{F}_1 \in L^2(0,T;[H^3(\mathcal{F})]^9) \cap H^1(0,T;[H^1(\mathcal{F})]^9)$ and

$$\|\mathbb{F}_1\|_{L^2(0,T;[H^3(\mathcal{F})]^9) \cap H^1(0,T;[H^1(\mathcal{F})]^9)} \leq CT^{1/2}.$$

Now, we deal with \mathbb{F}_2 , we have

$$\|\mathbb{F}_2\|_{L^2(0,T;[H^2(\mathcal{F})]^9)} \leq \|\rho\delta_X - \rho^0\|_{L^\infty(0,T;H^2(\mathcal{F}))} \|\partial_t v\|_{L^2(0,T;[H^2(\mathcal{F})]^3)} \leq CT^{1/2},$$

and

$$\begin{aligned} \|\partial_t \mathbb{F}_2\|_{L^2(0,T;[L^2(\mathcal{F})]^3)} &\leq \|\rho\delta_X - \rho^0\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\partial_{tt} v\|_{L^2(0,T;[L^2(\mathcal{F})]^3)} \\ &\quad + \|\partial_t \rho\|_{L^\infty(0,T;H^2(\mathcal{F}))} \|\delta_X\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\partial_t v\|_{L^\infty(0,T;[H^1(\mathcal{F})]^3)} \\ &\quad + \|\rho\|_{L^\infty(0,T;H^3(\mathcal{F}))} \|\partial_t \delta_X\|_{L^\infty(0,T;H^2(\mathcal{F}))} \|\partial_t v\|_{L^\infty(0,T;[H^1(\mathcal{F})]^3)} \leq CT^{1/2}. \end{aligned} \quad (4.15)$$

1 4.2 Estimation on \mathbb{F}_3 and \mathbb{F}_4

2 From (2.9) and since $\nabla v, b_X \in L^\infty(0,T;[L^\infty(\partial\mathcal{F})]^9)$, we have

$$\|\mathbb{F}_3\|_{[L^2(0,T)]^3} \leq CT^{1/2}. \quad (4.16) \quad \boxed{\{1745\}}$$

3 In the other hand, $\partial_t \mathbb{F}_3$ contains mainly terms of the kind

$$\nabla \partial_t v b_X^\top b_X, \quad \partial_t b_X \nabla v b_X^\top b_X, \quad \rho^{\gamma-1} \partial_t \rho b_X, \quad \rho^\gamma \partial_t b_X b_X, \quad (4.17) \quad \boxed{\{1821\}}$$

4 that need to be estimated in $L^2(0,T;[L^1(\partial\mathcal{S})]^9)$.

5 For the first term, since $\nabla \partial_t v \in L^\infty(0,T;[L^2(\mathcal{F})]^9) \cap L^2(0,T;[H^1(\mathcal{F})]^9)$, we get by interpolation that

$$\|\nabla \partial_t v\|_{L^{8/3}(0,T;[H^{1/4}(\partial\mathcal{S})]^9)} \leq C. \quad (4.18) \quad \boxed{\{1813\}}$$

6 Then

$$\|\nabla \partial_t v b_X^\top b_X\|_{L^2(0,T;[L^1(\partial\mathcal{S})]^9)} \leq CT^{1/8}. \quad (4.19)$$

7 Concerning the other terms in (4.17), from (4.4), they all can be estimated in $L^\infty(0,T;[L^\infty(\partial\mathcal{S})]^9)$. Thus
8 combining this fact with (4.16) we obtain that

$$\|\mathbb{F}_3\|_{[H^1(0,T)]^3} \leq CT^{1/8}. \quad (4.20) \quad \boxed{\{1824\}}$$

Similarly we deal with \mathbb{F}_4 , since $\nabla v, b_X \in L^\infty(0,T;[L^\infty(\partial\mathcal{F})]^9)$, $X \in L^\infty(0,T;L^\infty(\partial\mathcal{F}))$ and the fact that

$$\|h\|_{[L^\infty(0,T)]^3} \leq T^{1/2} \|\ell\|_{[H^1(0,T)]^3} \leq C,$$

9 we obtain that

$$\|\mathbb{F}_4\|_{[L^2(0,T)]^3} \leq CT^{1/2}. \quad (4.21) \quad \boxed{\{11:14\}}$$

In the other hand, $\partial_t \mathbb{F}_4$ involves terms as (4.17) multiplied by $(X - h)$ that is in $L^\infty(0,T;[L^\infty(\partial\mathcal{S})]^3)$ and additionally we have the terms

$$\nabla v b_X^\top b_X (v - \ell), \quad \rho^\gamma b_X (v - \ell).$$

The above terms can also be estimated in $L^\infty(0,T;[L^\infty(\partial\mathcal{S})]^3)$ since

$$\|\ell\|_{[L^\infty(0,T)]^3} \leq |\ell^0| + T^{1/2} \|\ell\|_{[H^2(0,T)]^3}.$$

10 Thus, from (4.21), we obtain

$$\|\mathbb{F}_4\|_{[H^1(0,T)]^3} \leq CT^{1/8}. \quad (4.22) \quad \boxed{\{1145\}}$$

4.3 Estimation on \mathbb{B}

Now let us estimate the boundary term \mathbb{B} defined in (2.10) and (2.11). It suffices to estimate

$$(\tilde{n}_X - n) \cdot v, \quad \omega \times (X - \text{Id}) \cdot \tilde{n}_X.$$

We notice that $\tilde{n}_X(t, \cdot) = Q(t)n(Q(t)^\top(X(t, \cdot) - h(t)))$. We get

$$\tilde{n}_X(t, y) - n(y) = \tilde{n}(t, X(t, y)) - n(y) = (Q(t) - \mathbb{I}_3)n(Q(t)^\top(X(t, y) - h(t))) + n(Q(t)^\top(X(t, y) - h(t))) - n(y).$$

Using Taylor-Lagrange formula, we get

$$\begin{aligned} \tilde{n}_X(t, y) - n(y) &= (Q(t) - \mathbb{I}_3)n(Q(t)^\top(X(t, y) - h(t))) \\ &\quad + \int_0^1 \nabla n(y + \theta(Q(t)^\top(X(t, y) - h(t)) - y)) \, d\theta (Q(t)^\top(X(t, y) - h(t)) - y). \end{aligned}$$

Since $Q \in [H^3(0, T)]^9$, $h \in [H^3(0, T)]^3$, we get that

$$\|Q - \mathbb{I}_3\|_{[L^\infty(0, T)]^9} \leq CT, \quad \|h\|_{[L^\infty(0, T)]^3} \leq CT,$$

and from (4.3), we get that

$$\begin{aligned} \|Q^\top(X - h) - \text{Id}\|_{L^\infty(0, T; [H^{7/2}(\partial\mathcal{S})]^3)} &\leq \|Q^\top(X - \text{Id})\|_{L^\infty(0, T; [H^{7/2}(\partial\mathcal{S})]^3)} \\ &\quad + \|(Q^\top - \mathbb{I}_3)\|_{[L^\infty(0, T)]^9} + \|Q^\top h\|_{[L^\infty(0, T)]^3} \leq CT^{1/2}. \end{aligned} \quad (4.23) \quad \boxed{\{2202\}}$$

Since η_S is smooth enough, from (4.23) and [16, Theorem 1.1'] we get that

$$\|n(Q^\top(X - h))\|_{L^\infty(0, T; [H^{7/2}(\partial\mathcal{S})]^3)} + \|\nabla n(y + \theta(Q^\top(X - h) - \text{Id}))\|_{L^\infty(0, T; [H^{7/2}(\partial\mathcal{S})]^3)} \leq C.$$

2 Then, using the same argument on the fixed boundary $\partial\Omega$, we obtain that

$$\|\tilde{n}_X - n\|_{L^\infty(0, T; [H^{7/2}(\partial\mathcal{F})]^3)} \leq CT^{1/2}. \quad (4.24)$$

3 Since $H^{7/2}(\partial\mathcal{F}) \times H^{7/2}(\partial\mathcal{F}) \hookrightarrow H^{7/2}(\partial\mathcal{F})$, we get

$$\|(\tilde{n}_X - n) \cdot v\|_{L^2(0, T; H^{7/2}(\partial\mathcal{F}))} \leq C\|\tilde{n}_X - n\|_{L^\infty(0, T; [H^{7/2}(\partial\mathcal{F})]^3)} \|v\|_{L^2(0, T; [H^{7/2}(\partial\mathcal{F})]^3)} \leq CT^{1/2}. \quad (4.25) \quad \boxed{\{a1\}}$$

We get also from (4.3) and [16, Theorem 1.1']

$$\begin{aligned} \|\omega \times (X - \text{Id}) \cdot \tilde{n}_X\|_{L^2(0, T; H^{7/2}(\partial\mathcal{S}))} \\ \leq \|\omega\|_{L^2(0, T)} \|X - \text{Id}\|_{L^\infty(0, T; [H^{7/2}(\partial\mathcal{F})]^3)} \|\tilde{n}_X\|_{L^\infty(0, T; [H^{7/2}(\partial\mathcal{S})]^3)} \leq CT^{1/2}. \end{aligned} \quad (4.26) \quad \boxed{\{1534\}}$$

Let estimate $\partial_t \mathbb{B}$ in $H^{3/4}(0, T; L^2(\partial\mathcal{F}))$. In the sequel, we commonly make a change of variable to reduce the time interval into $(0, 1)$ to make sure that the constant C is independent of T . We use the following notation

$$f(t) = f(t^*T) = f^*(t^*), \quad t^* \in (0, 1).$$

Since $\partial_t v \in H^{3/4}(0, T; [L^2(\partial\mathcal{F})]^3)$, using (2.2) we get

$$\begin{aligned} \|(\tilde{n}_X - n) \cdot \partial_t v\|_{H^{3/4}(0, T; L^2(\partial\mathcal{F}))} &= \left\| \left(\frac{b_X}{|b_X n|} - \mathbb{I}_3 \right) n \cdot \partial_t v \right\|_{H^{3/4}(0, T; L^2(\partial\mathcal{F}))} \\ &\leq \left\| \left(\frac{b_X}{|b_X n|} - \mathbb{I}_3 \right) n \cdot (\partial_t v - (\partial_t v)(0)) \right\|_{H^{3/4}(0, T; L^2(\partial\mathcal{F}))} + \left\| \left(\frac{b_X}{|b_X n|} - \mathbb{I}_3 \right) n \cdot (\partial_t v)(0) \right\|_{H^{3/4}(0, T; L^2(\partial\mathcal{F}))}. \end{aligned} \quad (4.27) \quad \boxed{\{11:33\}}$$

We estimate the first term in (4.27) as follows

$$\begin{aligned}
 & \left\| \left(\frac{b_X}{|b_X n|} - \mathbb{I}_3 \right) n \cdot (\partial_t v - (\partial_t v)(0)) \right\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} \\
 & \quad = T^{-1/4} T^{-1} \left\| \left(\frac{b_X}{|b_X n|} - \mathbb{I}_3 \right) n \cdot (\partial_{t^*} v^* - (\partial_{t^*} v^*)(0)) \right\|_{H^{3/4}(0,1;L^2(\partial\mathcal{F}))} \\
 & \quad \leq CT^{-5/4} \|\partial_{t^*} v^* - (\partial_{t^*} v^*)(0)\|_{H^{3/4}(0,1;[L^2(\partial\mathcal{F})]^3)} \left\| \left(\frac{b_X}{|b_X n|} - \mathbb{I}_3 \right) \right\|_{H^{7/8}(0,1;[L^\infty(\partial\mathcal{F})]^9)} \\
 & \quad \leq CT^{-5/4} T^{5/4} \|\partial_t v - (\partial_t v)(0)\|_{H^{3/4}(0,T;[L^2(\partial\mathcal{F})]^3)} T^{3/8} \left\| \left(\frac{b_X}{|b_X n|} - \mathbb{I}_3 \right) \right\|_{H^{7/8}(0,T;[L^\infty(\partial\mathcal{F})]^9)} \leq CT^{3/8}. \quad (4.28) \quad \boxed{\text{a2}}
 \end{aligned}$$

Since $\partial_t v \in L^\infty(0,T;[H^{1/2}(\partial\mathcal{F})]^3)$ and $H^{1/2}(\partial\mathcal{F}) \hookrightarrow L^4(\partial\mathcal{F})$, then using (A.3) we have for the second term in (4.27)

$$\begin{aligned}
 & \left\| \left(\frac{b_X}{|b_X n|} - \mathbb{I}_3 \right) n \cdot (\partial_t v)(0) \right\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} \leq C \left\| (\partial_t v)(0) \right\|_{[L^4(\partial\mathcal{F})]^3} \left\| \frac{b_X}{|b_X n|} - \mathbb{I}_3 \right\|_{H^{3/4}(0,T;[L^4(\partial\mathcal{F})]^9)} \\
 & \quad \leq CT^{7/8-3/4} \left\| \frac{b_X}{|b_X n|} - \mathbb{I}_3 \right\|_{H^{7/8}(0,T;[L^\infty(\partial\mathcal{F})]^9)} \leq CT^{1/8}. \quad (4.29) \quad \boxed{\text{a3}}
 \end{aligned}$$

Since $v \in L^\infty(0,T;[L^\infty(\partial\mathcal{F})]^3)$ and $\partial_t \left(\frac{b_X}{|b_X n|} \right) \in L^\infty(0,T;[L^\infty(\partial\mathcal{F})]^9)$, we can write

$$\begin{aligned}
 & \|\partial_t \left(\frac{b_X}{|b_X n|} \right) n \cdot v\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} \leq \|\partial_t \left(\frac{b_X}{|b_X n|} \right) n \cdot (v - v^0)\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} \\
 & \quad + \left\| (\partial_t \left(\frac{b_X}{|b_X n|} \right) n - (\partial_t \left(\frac{b_X}{|b_X n|} \right) (0)) n) \cdot v^0 \right\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} + \left\| (\partial_t \left(\frac{b_X}{|b_X n|} \right) n \cdot v)(0) \right\|_{H^{3/4}(0,T;[L^2(\partial\mathcal{F})]^3)}. \quad (4.30) \quad \boxed{\text{12:34}}
 \end{aligned}$$

1 The third term in (4.30) gives

$$\left\| (v \cdot \partial_t \left(\frac{b_X}{|b_X n|} \right) n)(0) \right\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} \leq CT^{1/2}. \quad (4.31) \quad \boxed{\text{a4}}$$

2 Since $\partial_t \left(\frac{b_X}{|b_X n|} \right) \in H^{7/8}(0,T;[L^2(\partial\mathcal{F})]^9)$ and $v \in H^{3/4}(0,T;[L^\infty(\partial\mathcal{F})]^3)$, arguing as (4.27) and (4.28), we
 3 obtain

$$\left\| \partial_t \left(\frac{b_X}{|b_X n|} \right) n \cdot (v - v^0) \right\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} + \left\| (\partial_t \left(\frac{b_X}{|b_X n|} \right) n - (\partial_t \left(\frac{b_X}{|b_X n|} \right) n)(0)) \cdot v^0 \right\|_{H^{3/4}(0,T;L^2(\partial\mathcal{F}))} \leq CT^{1/8}. \quad (4.32) \quad \boxed{\text{a5}}$$

In the other hand, we notice that

$$\begin{aligned}
 & \|\omega' \times (X - \text{Id}) \cdot \frac{b_X}{|b_X n|} n\|_{H^1(0,T;L^2(\partial\mathcal{S}))} + \|\omega \times v \cdot \frac{b_X}{|b_X n|} n - (\omega \times v \cdot \frac{b_X}{|b_X n|} n)(0)\|_{H^1(0,T;L^2(\partial\mathcal{S}))} \\
 & \quad + \|\omega \times (X - \text{Id}) \cdot \partial_t \left(\frac{b_X}{|b_X n|} \right) n\|_{H^1(0,T;L^2(\partial\mathcal{S}))} \leq C. \quad (4.33) \quad \boxed{\text{1752}}
 \end{aligned}$$

From Lemma A.3

$$\begin{aligned}
 & \|\omega' \times (X - \text{Id}) \cdot \frac{b_X}{|b_X n|} n\|_{H^{3/4}(0,T;L^2(\partial\mathcal{S}))} + \|\omega \times v \cdot \frac{b_X}{|b_X n|} n - (\omega \times v \cdot \frac{b_X}{|b_X n|} n)(0)\|_{H^{3/4}(0,T;L^2(\partial\mathcal{S}))} \\
 & \quad + \|\omega \times (X - \text{Id}) \cdot \partial_t \left(\frac{b_X}{|b_X n|} \right) n\|_{H^{3/4}(0,T;L^2(\partial\mathcal{S}))} \leq CT^{1/4}. \quad (4.34) \quad \boxed{\text{1810}}
 \end{aligned}$$

1 Since $(\omega \times v \cdot \frac{b_X}{|b_X n|} n)(0) \in L^\infty(0, T; L^2(\partial\mathcal{S}))$, we get finally from (4.34)

$$\|\partial_t \left(\omega \times (X - \text{Id}) \cdot \frac{b_X}{|b_X n|} n \right) \|_{H^{3/4}(0, T; L^2(\partial\mathcal{S}))} \leq CT^{1/4}. \quad (4.35) \quad \boxed{\{1816\}}$$

2 The last term in (2.11) can be estimated similarly. Gathering the estimates (4.25), (4.28), (4.29), (4.31), (4.32),
3 (4.26) and (4.35) we get

$$\|\mathbb{B}\|_{H^{7/4}(0, T; L^2(\partial\mathcal{F})) \cap L^2(0, T; H^{7/2}(\partial\mathcal{F}))} \leq CT^{1/8}. \quad (4.36)$$

4 4.4 Estimation on \mathbb{G}

5 First, we extend \mathbb{G} over \mathcal{F} and we use the same calculation as \mathbb{F}_1 , we see that $\mathbb{G} \in L^2(0, T; [H^3(\mathcal{F})]^3)$. Using
6 the trace theorem, we get

$$\|\mathbb{G}_i\|_{L^2(0, T; H^{5/2}(\partial\mathcal{F}))} \leq CT^{1/2}. \quad (4.37) \quad \boxed{\{b1\}}$$

Now we estimate $\partial_t \mathbb{G}_i$ in $H^{1/4}(0, T; L^2(\partial\mathcal{F}))$. We notice that

$$2\mu \mathbb{D}(v) n \cdot \tau - 2\mu \mathbb{D}^X(v) \tilde{n}_X \cdot \tilde{\tau}_X^i = 2\mu \mathbb{D}(v) n \cdot \tau - 2\mu b_X^{-1} \mathbb{D}^X(v) b_X n \cdot \tau^i.$$

Let estimate for example

$$\begin{aligned} & \left\| \partial_t \left(b_X^{-1} \frac{1}{\delta_X} \nabla v (b_X)^\top b_X n \cdot \tau^i \right) - \partial_t \nabla v n \cdot \tau^i \right\|_{H^{1/4}(0, T; L^2(\partial\mathcal{F}))} \\ & \leq C \left(\left\| \partial_t \nabla v \left(b_X^{-1} \frac{1}{\delta_X} (b_X)^\top b_X - \mathbb{I}_3 \right) \right\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^9)} \right. \\ & \quad \left. + \left\| \nabla v \partial_t \left(b_X^{-1} \frac{1}{\delta_X} (b_X)^\top b_X \right) \right\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^9)} \right). \end{aligned} \quad (4.38) \quad \boxed{\{13:51\}}$$

From Lemma A.5, the first term at the right hand side of (4.38) gives

$$\begin{aligned} & \left\| \nabla \partial_t v \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X - \mathbb{I}_3 \right) \right\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^9)} \\ & \leq CT^{1/8} \left\| \nabla \partial_t v \right\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^9)} \left\| b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X - \mathbb{I}_3 \right\|_{H^{7/8}(0, T; [L^\infty(\partial\mathcal{F})]^9)}. \end{aligned} \quad (4.39)$$

7 Using (4.8) and Slobodeckij seminorm with $s = 7/8$, we deduce that

$$\left\| b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X - \mathbb{I}_3 \right\|_{H^{7/8}(0, T; [L^\infty(\partial\mathcal{F})]^9) \cap L^\infty(0, T; [L^\infty(\partial\mathcal{F})]^9)} \leq C. \quad (4.40) \quad \boxed{\{12:44\}}$$

8 Hence,

$$\left\| \nabla \partial_t v \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X - \mathbb{I}_3 \right) \right\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^9)} \leq CT^{1/8}. \quad (4.41)$$

In the other hand, the second term of (4.38) is estimated thanks to Lemma A.5, (4.8), (4.10), Lemma A.5, using the fact that $\nabla v \in H^{7/8}(0, T; [H^{3/4}(\partial\mathcal{F})]^9)$ and $H^{3/4}(\partial\mathcal{F}) \hookrightarrow L^8(\partial\mathcal{F})$

$$\begin{aligned} & \left\| (\nabla v) \partial_t \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X \right) \right\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^9)} \\ & \leq CT^{1/8} \left\| \nabla v - (\nabla v)(0) \right\|_{H^{7/8}(0, T; [L^2(\partial\mathcal{F})]^9)} \left\| \partial_t \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X \right) \right\|_{H^{1/4}(0, T; [L^\infty(\partial\mathcal{F})]^9)} \\ & \quad + \left\| (\nabla v)(0) \left(\partial_t \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X \right) - \partial_t \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X \right) (0) \right) \right\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^9)} \\ & \quad + \left\| (\nabla v)(0) \partial_t \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X \right) (0) \right\|_{L^2(0, T; [L^2(\partial\mathcal{F})]^9)}. \end{aligned} \quad (4.42) \quad \boxed{\{14:42\}}$$

- 1 Since $\partial_t b_X$, $\partial_t \delta_X$, b_X , b_X^\top and δ_X are in $H^{1/4}(0, T; [L^\infty(\partial\mathcal{F})]^9) \cap L^\infty(0, T; [L^\infty(\partial\mathcal{F})]^9)$, using (4.8) and (4.9), we
 2 obtain

$$\|\partial_t \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X \right)\|_{H^{1/4}(0, T; [L^\infty(\partial\mathcal{F})]^9)} \leq C. \quad (4.43) \quad \{1235\}$$

In the other hand, using Lemma A.3, (4.8) and (4.10), we obtain

$$\begin{aligned} & \|\nabla v(0) \left(\partial_t \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X \right) - \partial_t \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X \right) (0) \right)\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^9)} \\ & \leq T^{5/8} \|\nabla v(0) \left(\partial_t \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X \right) - \partial_t \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X \right) (0) \right)\|_{H^{7/8}(0, T; [L^2(\partial\mathcal{F})]^9)} \leq CT^{5/8}. \end{aligned} \quad (4.44) \quad \{1234\}$$

- 3 Since ∇v , $\partial_t \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X \right) \in L^\infty(0, T; [H^{3/2}(\partial\mathcal{F})]^9)$, we have

$$\|(\nabla v)(0) \partial_t \left(b_X^{-1} \frac{1}{\delta_X} b_X^\top b_X \right) (0)\|_{L^2(0, T; [L^2(\partial\mathcal{F})]^9)} \leq CT^{1/2}. \quad (4.45) \quad \{1231\}$$

- 4 Then, from (4.42), (4.43), (4.44) and (4.45), we get

$$\|2\mu\mathbb{D}(v)n \cdot \tau - 2\mu\mathbb{D}^X(v)\tilde{n}_X \cdot \tilde{\tau}_X^i\|_{H^{1/4}(0, T; L^2(\partial\mathcal{F}))} \leq CT^{1/8}. \quad (4.46) \quad \{1229\}$$

The third term in (2.12) writes

$$\beta_0 v \cdot (\tau - \tilde{\tau}_X^i) = \beta_0 (\mathbb{I}_3 - |b_X n| b_X^{-1}) v \cdot \tau.$$

We get

$$\begin{aligned} & \|\partial_t ((\mathbb{I}_3 - |b_X n| b_X^{-1}) v)\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^3)} \leq \|(\mathbb{I}_3 - |b_X n| b_X^{-1}) \partial_t v\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^3)} \\ & + \|(\partial_t (|b_X n| b_X^{-1}) (v - v^0))\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^3)} + \|(\partial_t (|b_X n| b_X^{-1}) - \partial_t (|b_X n| b_X^{-1}) (0)) v^0\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^3)} \\ & + \|\partial_t (|b_X n| b_X^{-1} v) (0)\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^3)}. \end{aligned} \quad (4.47) \quad \{1225\}$$

- 5 Since $\partial_t v \in H^{1/4}(0, T; [H^1(\partial\mathcal{F})]^3)$ and $\mathbb{I}_3 - |b_X n| b_X^{-1} \in H^{7/8}(0, T; [L^\infty(\partial\mathcal{F})]^9)$, then

$$\|(\mathbb{I}_3 - |b_X n| b_X^{-1}) \partial_t v\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^3)} \leq CT^{1/8}. \quad (4.48) \quad \{1252\}$$

We have

$$\begin{aligned} & \|\partial_t (|b_X n| b_X^{-1}) (v - v^0)\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^3)} \\ & \leq CT^{1/8} \|\partial_t (|b_X n| b_X^{-1})\|_{H^{1/4}(0, T; [L^\infty(\partial\mathcal{F})]^9)} \|v - v^0\|_{H^{7/8}(0, T; [L^2(\partial\mathcal{F})]^3)} \leq CT^{1/8}. \end{aligned} \quad (4.49) \quad \{1226\}$$

We have also from Lemma A.3

$$\begin{aligned} & \|(\partial_t (|b_X n| b_X^{-1}) - \partial_t (|b_X n| b_X^{-1}) (0)) v^0\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^3)} \\ & \leq T^{5/8} \|(\partial_t (|b_X n| b_X^{-1}) - \partial_t (|b_X n| b_X^{-1}) (0)) v^0\|_{H^{7/8}(0, T; [L^2(\partial\mathcal{F})]^3)} \leq CT^{5/8}. \end{aligned} \quad (4.50) \quad \{1227\}$$

- 6 Since v , $\partial_t b_X^{-1} \in L^\infty(0, T; [H^{3/2}(\partial\mathcal{F})]^9)$, thus

$$\|(\partial_t (|b_X n| b_X^{-1}) v) (0)\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^3)} \leq CT^{1/2}. \quad (4.51) \quad \{1420\}$$

- 7 Then, combining (4.47), (4.48), (4.49), (4.50) and (4.51), we obtain

$$\|\partial_t ((\mathbb{I}_3 - |b_X n| b_X^{-1}) v)\|_{H^{1/4}(0, T; [L^2(\partial\mathcal{F})]^3)} \leq CT^{1/8}. \quad (4.52) \quad \{1224\}$$

Using the same arguments as we obtained (4.35), we get

$$\|\partial_t (\omega \times (X - \text{Id}) \cdot |b_X n| b_X^{-\top} \tau^i)\|_{H^{1/4}(0,T;L^2(\partial\mathcal{S}))} \leq CT^{3/4}.$$

1 Gathering (4.4), (4.52), (4.46) and (4.37), we obtain finally

$$\|\mathbb{G}_i\|_{H^{5/4}(0,T;L^2(\partial\mathcal{F})) \cap L^2(0,T;H^{5/2}(\partial\mathcal{F}))} \leq CT^{1/8}. \quad (4.53)$$

2 Same calculations implies that \mathcal{Z} defines also a contraction mapping on $\mathcal{B}_{R,T}$. That concludes the proof of
3 Theorem 1.1.

4 A Useful Estimates

korn **Lemma A.1** (Korn Inequality). *Let $\beta > 0$. There exists a constant $C > 0$ such that*

$$\|\widehat{u}\|_{[H^1(\mathcal{F})]^3}^2 \leq C \left(\|\mathbb{D}(\widehat{u})\|_{[L^2(\mathcal{F})]^9}^2 + \|\beta \widehat{u}_\tau\|_{[L^2(\partial\mathcal{F})]^3}^2 \right),$$

5 for all $\widehat{u} \in [H^1(\mathcal{F})]^3$ such that $\widehat{u}_n = 0$.

6 *Proof.* The proof is classical, it can be found in [1]. For the sake of completeness, we sketch the ideas of the
7 proof. We only need to show that there exist a positive constant C such that

$$\|\mathbb{D}(\widehat{u})\|_{[L^2(\mathcal{F})]^9}^2 + \|\beta \widehat{u}_\tau\|_{[L^2(\partial\mathcal{F})]^3}^2 \geq C \|\widehat{u}\|_{[L^2(\mathcal{F})]^3}^2, \quad (A.1) \quad \{\text{12:12}\}$$

and the result will follow from the classical Korn inequality since \mathcal{F} is Lipschitz domain (see for instance [46]). Suppose that the inequality (A.1) is not true, then for any positive integer k there exists u_k such that

$$\|\mathbb{D}(\widehat{u}_k)\|_{[L^2(\mathcal{F})]^9}^2 + \|\beta(\widehat{u}_k)_\tau\|_{[L^2(\partial\mathcal{F})]^3}^2 \leq \frac{1}{k} \|\widehat{u}_k\|_{[L^2(\mathcal{F})]^3}^2.$$

We normalize by $\|\widehat{u}_k\|_{[L^2(\mathcal{F})]^3}$, in such a way we get a sequence denoted again $(\widehat{u}_k)_k$ that verifies

$$\|\widehat{u}_k\|_{[L^2(\mathcal{F})]^3} = 1.$$

We deduce from the classical Korn inequality that $(\widehat{u}_k)_k$ is bounded in $[H^1(\mathcal{F})]^3$, then up to a subsequence, it converges weakly to \widehat{u} in $[H^1(\mathcal{F})]^3$. Consequently

$$\|\mathbb{D}(\widehat{u})\|_{[L^2(\mathcal{F})]^9}^2 + \|\beta \widehat{u}_\tau\|_{[L^2(\partial\mathcal{F})]^3}^2 \leq \liminf_{k \rightarrow \infty} \|\mathbb{D}(\widehat{u}_k)\|_{[L^2(\mathcal{F})]^9}^2 + \|\beta(\widehat{u}_k)_\tau\|_{[L^2(\partial\mathcal{F})]^3}^2 = 0.$$

8 Then, $\mathbb{D}(\widehat{u}) = 0$ in \mathcal{F} . It implies that $\widehat{u}(z) = a + b \times z$ in \mathcal{F} , but since $\widehat{u} = 0$ on $\partial\mathcal{F}$, then $a = b = 0$, which
9 contradicts the fact that $\|\widehat{u}_k\|_{[L^2(\mathcal{F})]^3} = 1$. \square

ext **Lemma A.2.** *Let \mathcal{X} a Banach space and $\xi \in H^{3/4}(0,T;\mathcal{X})$ such that $\xi(0) = 0$, then there exists $\xi_{ext} \in$
11 $H^{3/4}(0,T^*;\mathcal{X})$, where $T^* > 2T$ and*

$$\|\xi_{ext}\|_{H^{3/4}(0,T^*;\mathcal{X})} \leq C \|\xi\|_{H^{3/4}(0,T;\mathcal{X})}, \quad (A.2) \quad \{\text{Hs}\}$$

12 with C a constant that does not depend on T .

Proof. We define ξ_{ext} as follows

$$\begin{cases} \xi_{ext}(t) = \xi(t), & t \in [0, T], \\ \xi_{ext}(t) = \xi(2T - t), & t \in [T, 2T], \\ \xi_{ext}(t) = 0, & t \in [2T, T^*], \end{cases}$$

We have that $\xi_{ext} \in H^{3/4}(0, T^*; \mathcal{X})$. Let us prove the estimate (A.2). Since $\xi(0) = 0$, then it suffices to show the estimate (A.2) using Slobodeckij seminorm with $s = 3/4$.

$$\begin{aligned} [\xi_{ext}]_{s,2,(0,T^*)} &= \int_0^{T^*} \int_0^{T^*} \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt = \int_0^{2T} \int_0^{2T} \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt \\ &\quad + \int_0^{2T} \int_{2T}^{T^*} \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt + \int_{2T}^{T^*} \int_0^{2T} \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt. \end{aligned}$$

Then,

$$\begin{aligned} [\xi_{ext}]_{s,2,(0,T^*)} &= \int_0^{2T} \int_0^{2T} \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt + \int_0^{2T} \int_{2T}^{T^*} \frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt \\ &\quad + \int_{2T}^{T^*} \int_0^{2T} \frac{\|\xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt. \quad (\text{A.3}) \quad \boxed{\{1214\}} \end{aligned}$$

Let us deal with the first term in the right hand side of (A.3).

$$\begin{aligned} \int_0^{2T} \int_0^{2T} \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt &= \int_0^T \int_0^T \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt \\ &\quad + \int_0^T \int_T^{2T} \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt + \int_T^{2T} \int_0^T \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt \\ &\quad + \int_T^{2T} \int_T^{2T} \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt. \quad (\text{A.4}) \quad \boxed{\{1557\}} \end{aligned}$$

Since the second and the third terms in the right hand side of (A.4) are of the same kind then it suffices to treat one of them using a change of variables as follows

$$\begin{aligned} \int_0^T \int_T^{2T} \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt &= \int_0^T \int_0^T \frac{\|\xi(t) - \xi(t')\|_{\mathcal{X}}^2}{|2T - t - t'|^{2s+1}} dt' dt \\ &= \int_0^T \int_0^T \frac{|t - t'|^{2s+1}}{|2T - t - t'|^{2s+1}} \frac{\|\xi(t) - \xi(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt. \quad (\text{A.5}) \end{aligned}$$

Since $t, t' \in (0, T)$, then

$$\left| \frac{t - t'}{2T - t - t'} \right| \leq 1.$$

Thus,

$$\int_0^T \int_T^{2T} \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt \leq \int_0^T \int_0^T \frac{\|\xi(t) - \xi(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt = [a]_{s,2,(0,T)}^2.$$

1 We obtain that

$$\int_0^{2T} \int_0^{2T} \frac{\|\xi_{ext}(t) - \xi_{ext}(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt \leq 4[\xi]_{s,2,(0,T)}^2. \quad (\text{A.6}) \quad \boxed{\{hs1\}}$$

Now, we deal with the second term in the right hand side of (A.3), the third term can be treated similarly. Let define $c(t)$ and $d(t)$ such that

$$c(t) = \frac{1}{t} \int_0^t (\xi(t) - \xi(t')) dt', \quad d(t) = \int_t^T \frac{c(t')}{t'} dt', \quad t \in (0, T).$$

We notice that

$$\xi(t) = c(t) - (d(t) - d(0)).$$

Indeed,

$$\begin{aligned}
 \xi(t) &= \frac{1}{t} \int_0^t (\xi(t) - \xi(t')) dt' + \frac{1}{t} \int_0^t \xi(t') dt' = c(t) + \int_0^t \xi(t') \left(\frac{1}{t} - \frac{1}{t'} \right) dt' + \int_0^t \frac{\xi(t')}{t'} dt' \\
 &= c(t) + \int_0^t \xi(t') \left(- \int_{t'}^t \frac{1}{\tau^2} d\tau \right) dt' + \int_0^t \frac{\xi(t')}{t'} dt' = c(t) - \int_0^t \int_0^\tau \frac{\xi(t')}{\tau^2} dt' d\tau + \int_0^t \frac{\xi(t')}{t'} dt' \\
 &= c(t) + \int_0^t \frac{1}{\tau^2} \left[\int_0^\tau \xi(\tau) - \xi(t') dt' \right] d\tau = c(t) - (d(t) - d(0)).
 \end{aligned}$$

1 We want to show that

$$\int_0^T t^{-2s} \|\xi(t)\|_{\mathcal{X}}^2 dt \leq C [\xi]_{s,2,(0,T)}^2, \quad (\text{A.7}) \quad \boxed{\{1359\}}$$

where C is a constant that does not depend on T . We have

$$\begin{aligned}
 \int_0^T t^{-2s} \|c(t)\|_{\mathcal{X}}^2 dt &\leq \int_0^T \int_0^t \frac{\|\xi(t) - \xi(t')\|_{\mathcal{X}}^2}{t^{2s+1}} dt' dt = \int_0^T \int_{t'}^T \frac{\|\xi(t) - \xi(t')\|_{\mathcal{X}}^2}{t^{2s+1}} dt dt' \\
 &= \int_0^T \int_{t'}^T \frac{|t - t'|^{2s+1} \|\xi(t) - \xi(t')\|_{\mathcal{X}}^2}{t^{2s+1} |t - t'|^{2s+1}} dt dt'. \quad (\text{A.8})
 \end{aligned}$$

Since $t \in (t', T)$ and $t \in (0, T)$ then

$$0 \leq \frac{t - t'}{t} \leq 1.$$

2 We obtain

$$\int_0^T t^{-2s} \|c(t)\|_{\mathcal{X}}^2 dt \leq \int_0^T \int_{t'}^T \frac{\|\xi(t) - \xi(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt dt' \leq \int_0^T \int_0^T \frac{\|\xi(t) - \xi(t')\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt dt' = [\xi]_{s,2,(0,T)}^2. \quad (\text{A.9}) \quad \boxed{\{1455\}}$$

In the other hand,

$$d(t) - d(0) = - \int_0^t \frac{c(t')}{t'} dt'.$$

3 Then, by Hardy inequality, we get

$$\int_0^T t^{-2s} \|d(t) - d(0)\|_{\mathcal{X}}^2 dt = \int_0^T \left(t^{-(s-1/2)} \int_0^t \frac{\|c(t')\|_{\mathcal{X}}}{t'} dt' \right)^2 \frac{dt}{t} \leq \frac{1}{(s-1/2)^2} \int_0^T t^{-2s} \|c(t)\|_{\mathcal{X}}^2 dt. \quad (\text{A.10}) \quad \boxed{\{1500\}}$$

4 Finally, using (A.9), we obtain

$$\int_0^T t^{-2s} \|d(t) - d(0)\|_{\mathcal{X}}^2 dt \leq \frac{1}{(s-1/2)^2} [\xi]_{s,2,(0,T)}^2. \quad (\text{A.11}) \quad \boxed{\{1518\}}$$

Then, by (A.11) and (A.9), we deduce (A.7). It remains to estimate the second term in the right hand side of (A.3) by using (A.7).

$$\begin{aligned}
 \int_0^{2T} \int_{2T}^{T^*} \frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{|t - t'|^{2s+1}} dt' dt &= \int_0^{2T} \|\xi_{ext}(t)\|_{\mathcal{X}}^2 \int_{2T}^{T^*} \frac{1}{(t' - t)^{2s+1}} dt' dt \\
 &= \int_0^{2T} \|\xi_{ext}(t)\|_{\mathcal{X}}^2 \left[\frac{-1}{2s(t' - t)^{2s}} \right]_{2T}^{T^*} dt \leq 2 \int_0^{2T} \frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{(2T - t)^{2s}} dt. \quad (\text{A.12})
 \end{aligned}$$

We write

$$\int_0^{2T} \frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{(2T - t)^{2s}} dt = \int_0^T \frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{(2T - t)^{2s}} dt + \int_T^{2T} \frac{\|\xi_{ext}(t)\|_{\mathcal{X}}^2}{(2T - t)^{2s}} dt = \int_0^T \frac{t^{2s}}{(2T - t)^{2s}} \frac{\|\xi(t)\|_{\mathcal{X}}^2}{t^{2s}} dt + \int_0^T \frac{\|\xi(t)\|_{\mathcal{X}}^2}{t^{2s}} dt.$$

1 Since $t \in (0, T)$ then $t < 2T - t$ and hence

$$\int_0^{2T} \frac{\|\xi_{ext}(t)\|_{\mathfrak{X}}^2}{(2T-t)^{2s}} dt \leq 2 \int_0^T t^{-2s} \|\xi(t)\|_{\mathfrak{X}}^2 dt. \quad (\text{A.13}) \quad \{\text{1543}\}$$

2 From (A.7), we deduce that

$$\int_0^{2T} \frac{\|\xi_{ext}(t)\|_{\mathfrak{X}}^2}{(2T-t)^{2s}} dt \leq C[\xi]_{s,2,(0,T)}^2. \quad (\text{A.14}) \quad \{\text{1544}\}$$

3 Thus,

$$\int_0^{2T} \int_{2T}^{T^*} \frac{\|\xi_{ext}(t)\|_{\mathfrak{X}}^2}{|t-t'|^{2s+1}} dt' dt \leq C[\xi]_{s,2,(0,T)}^2. \quad (\text{A.15}) \quad \{\text{hs2}\}$$

4 Gathering (A.15), (A.6), we obtain (A.2). \square

5 We present also in the sequel some technical estimates concerning fractional Sobolev spaces that have been
6 established in [13].

lem7 **Lemma A.3.** Assume $\sigma_2 \in (1/2, 1]$ and $\sigma_1 \in [0, \sigma_2]$. Using the above result, there exists a constant independent
8 of T such that for any $\xi \in H^{\sigma_2}(0, T; \mathfrak{X})$ and $\xi(0) = 0$, then

$$\|\xi\|_{H^{\sigma_1}(0, T, \mathfrak{X})} \leq CT^{\sigma_2 - \sigma_1} \|\xi\|_{H^{\sigma_2}(0, T, \mathfrak{X})}. \quad (\text{A.16}) \quad \{\text{chv3}\}$$

interp9 **Lemma A.4.** Let $\sigma \in [0, 1]$, $\mu_1 \geq 0$, $\mu_2 \geq 0$ and $\mu = \sigma\mu_1 + (1-\sigma)\mu_2$. Then there exists a constant C
10 independent of T such that for any function $u \in H^1(0, T; H^{\mu_1}(\Omega)) \cap L^2(0, T; H^{\mu_2}(\Omega))$, we have

$$\|u\|_{H^\sigma(0, T; H^\mu(\Omega))} \leq C \|u\|_{H^1(0, T; H^{\mu_1}(\Omega))}^\sigma \|u\|_{L^2(0, T; H^{\mu_2}(\Omega))}^{1-\sigma}. \quad (\text{A.17}) \quad \{\text{chv4}\}$$

11 On the other hand, for $p, q \in [1, +\infty]$ and $\frac{1}{r} = \frac{\sigma}{p} + \frac{(1-\sigma)}{q}$, we have

$$\|u\|_{L^r(0, T; H^\mu(\Omega))} \leq C \|u\|_{L^p(0, T; H^{\mu_1}(\Omega))}^\sigma \|u\|_{L^q(0, T; H^{\mu_2}(\Omega))}^{1-\sigma}, \quad (\text{A.18}) \quad \{\text{chv5}\}$$

12 for $u \in L^p(0, T; H^{\mu_1}(\Omega)) \cap L^q(0, T; H^{\mu_2}(\Omega))$.

prod **Lemma A.5.** Assume that $\mathfrak{X}_1, \mathfrak{X}_2$ and \mathfrak{X}_3 are Banach spaces such that

$$\|fg\|_{\mathfrak{X}_3} \leq C \|f\|_{\mathfrak{X}_1} \|g\|_{\mathfrak{X}_2}, \quad \forall f \in \mathfrak{X}_1, \quad \forall g \in \mathfrak{X}_2.$$

13 Let us assume $\sigma \in (1/2, 1]$, $s \in [0, 1/2]$, $T_0 > 0$. Then there exists a constant C such that for any $T \leq T_0$ we
14 have

$$\|u_1 u_2\|_{H^s(0, T; \mathfrak{X}_3)} \leq CT^{\sigma-s-1/2} \|u_1\|_{H^s(0, T; \mathfrak{X}_1)} \|u_2\|_{H^\sigma(0, T; \mathfrak{X}_2)} + \|u_2(0)\|_{\mathfrak{X}_2} \|u_1\|_{H^s(0, T; \mathfrak{X}_1)}, \quad (\text{A.19}) \quad \{\text{formule}\}$$

15 for all $u_1 \in H^s(0, T; \mathfrak{X}_1)$ and $u_2 \in H^\sigma(0, T; \mathfrak{X}_2)$.

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